物流系统分析
Logistics System Analysis

第 4 周　一到一配送问题（1）–批量问题
One-to-One Distribution–The Lot Size Problem

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Revisit of ideas of CA

- Accurate cost estimates can be obtained without precise, detailed input data,
- Departures from an optimal decision by a moderate%age do not increase cost significantly. Since there is no need to seek the most accurate estimate of the optimum, there may be little use for highly detailed data,
- Detailed data may get in the way of the optimization, actually hindering the search for an optimum,
- Thus, we advocate a two-step solution approach to logistics problems: the first (analytical) step involves little detail and yields broad solution concepts; the second (or fine tuning) step leads to specific solutions, consistent with the ideals revealed by the first — it uses all the relevant detailed information.

本节学习内容

1. One-to-one systems with constant production and consumption rates -> the robustness and accuracy of the results (Daganzo's work)
2. One-to-one systems with variable demand over time -> numerical methods and a continuous approximation (CA) analytical approach that is based on summarized data (Newell's work)
3. Extension of the CA approach to a location problem that has an analogous structure
4. The accuracy of the CA solutions
5. Extension of the CA approach to multidimensional problems with constraints
6. Network design issues.

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Let us now explore the optimization problem for the optimum shipment size, v^*:

$$z = \{Av + \frac{B}{v} : v \leqslant v_{\max}\} \tag{1}$$

- $A = c_h/D'$ 表示单位货物保管费用*, $B = c_f$ 表示每批货物固定运输费用。
- Consider first the case $v_{\max} = \infty$. Then v^* is the value of v which minimizes the convex expression $Av + B/v$. $v^* = \sqrt{B/A}$
- The optimum cost per item is: $z^* = (cost/item)^* = 2\sqrt{AB}$, which is easy to remember as "twice the square root of the product" of the terms in 1
- As a function of c_f, c_h and D', the optimum cost per item increases at a decreasing rate with c_f and c_h and decreases with the item flow D'. There are economies of scale, since higher item flows lead to lesser average cost.

*D' 表示单位时间生产量，每批货物的保管费用为 $Av = c_h H_1 D'/D' + c_i t_m = c_h H_1 + c_i t_m$，$c_h = c_r + c_i$ 为租赁成本和在途等待成本之和。忽略常数项 $c_i t_m$，引入 $H_1 = \bar{H} = v/D'$ 得到表达式

We now examine the sensitivity of the resulting cost to errors in

- the decision variable, v
- the inputs (A or B)
- the functional form of the equation

Robustness in the Decision Variable

- Suppose that instead of v^* , the chosen shipment size is $v^0 = \gamma v$, where γ is a number close to 1, capturing the relative error in v^0. Then, the ratio of the actual to optimum cost z^0/z^* will be a number, γ', greater than 1, satisfying:

$$\gamma' = [A\gamma\sqrt{B/A} + \frac{B}{\gamma\sqrt{B/A}}]/[2\sqrt{AB}] = \frac{1}{2}[\gamma + \frac{1}{\gamma}] \quad (2)$$

- Independent of A and B, this relationship between input and output relative errors holds for all EOQ models.

Robustness in the Decision Variable (cont.)

- If γ is between 0.5 and 2, so that the optimal shipment size is approximated to within a factor of 2, then $\gamma' < 1.25$. If γ is between 0.8 and 1.25, then $\gamma' < 1.025$ $\rightarrow$ A cost within 2.5% of the optimum can be reached if the decision variable is within 25% of optimal.
- If γ is several times larger (or smaller) than 1, then the cost penalty is severe, i.e., $\gamma' \approx \gamma$ (or $\gamma' \approx 1/\gamma$)
- Obviously, while it is important to get reasonably close to the optimal value of the decision variable (say to within 20-40%), from a practical standpoint it may not be imperative to refine the decision beyond this level.

Robustness in the data errors (cont.)

- Let us now assume that one of the cost coefficients A (or B) is not known precisely. If it is believed to be $A' = \delta A$ (or $B' = \delta B$) , for some $\delta \approx 1$, then the optimal decision with this erroneous cost structure is:

$$v'^* = \begin{cases} \sqrt{B/A}\delta^{-1/2} = v^*\delta^{-1/2} & \text{if } A' = \delta A \\ v^*\delta^{1/2} & \text{if } B' = \delta B \end{cases}$$

- Because the actual to optimal shipment size ratio, v'^*/v^*, is either $\delta^{-1/2}$ or $\delta^{1/2}$, the cost penalty paid is as if $\gamma = \delta^{1/2}$. Thus, the resulting cost is even less sensitive to the data than it is to the decision variables

Robustness in the Data Errors (cont.)

- If the input is known to within a factor of 2 ($0.5 \leqslant \delta \leqslant 2$), then $0.7 \leqslant \gamma \leqslant 1.4$ and $\gamma' \leqslant 1.1$. The cost penalty would be about 10%, whereas before it was 25%. The penalty declines quickly as δ approaches 1
- This robustness to data errors is fortunate because the cost coefficients (for waiting cost especially)are rarely known accurately

Robustness in the Model Errors

- A cost penalty is also paid if the EOQ formula itself is inaccurate.
- To illustrate the impact of such functional errors, we assume that the actual cost, a complicated (perhaps unknown) expression, can be bounded by two EOQ expressions; the cost penalty can then be related to the width of the bounds.
- Suppose, for example, that the actual holding cost $z_h(v)$ is not exactly equal to the EOQ term (Av), but it satisfies:

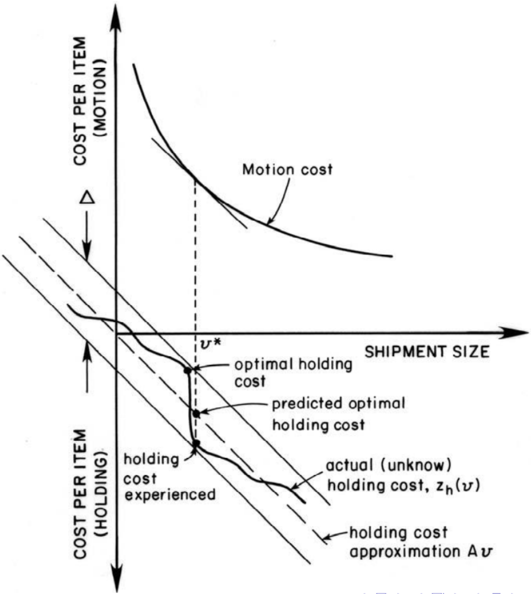
$$Av - \Delta/2 \leqslant z_h(v) \leqslant Av + \Delta/2 \tag{3}$$

  for some small Δ. Such a situation could happen, for example, if storage space could only be obtained in discrete amounts. Because Δ is small, the EOQ lot size v^* is adopted.

Robustness in the Model Errors (cont.)

- The absolute difference between the actual cost $[z_h(v^*) + B/v^*]$ and the predicted EOQ cost z^* cannot exceed $\Delta/2$. It is also easy to see that the difference between the optimal cost with perfect information, $\min\{z_h(v) + B/v\}$, and z^* cannot exceed $\Delta/2$ either. As a result, the difference between the actual and theoretical minimum costs — the cost penalty is bounded by Δ.
- Usually, this penalty will be significantly smaller than the maximum possible
- If Δ is small compared to z^* (e.g., within 10%) the functional form error should be inconsequential. The same conclusion is reached if the motion cost is also inaccurate.
- In general, the EOQ solution will be reasonable if it is accurate to within a small fraction of its predicted optimal cost.

Unusual conditions generating the largest penalty



Error Combinations

多种误差的组合带来的总误差，并不一定是简单叠加

- If errors of the three types exist, one would expect the cost penalty to be greater. Fortunately though, when dealing with errors the whole (the combined penalty) is not as great as the sum of its parts
- Suppose for example that the lot size recipe is not followed very precisely (because, e.g., lots are chosen to be multiples of a box, only certain dispatching times are feasible, etc.) and that as a result 40% discrepancies are expected between the calculated and actual lot sizes. We have already seen that such discrepancies can be expected to increase cost by about 10%.

- Let us assume that one of the inputs (A or B) is suspected to be in error by a factor of 2, which taken alone would also increase cost by about 10%. Would it then be reasonable to expect a 20% cost increase? The answer is no; it should be intuitive that the penalty paid by introducing an input error when the lot size decision does not follow the recipe accurately should be smaller than the penalty paid if the decision follows the recipe.
- In our example, the combined likely increase is 14% [the square root of the sum of the squared errors: $0.14 = (0.1^2 + 0.1^2)^{1/2}$]. Statistical analysis of error propagation through models reveals similar composition laws in more general contexts.

Error Combinations (cont.)

- The previous example illustrated how input and decision errors propagate. Although model errors follow similar laws – the whole is still less than the sum of the parts – for some approximate models the results are surprising. The composed (data and model) error can be actually smaller than the data error alone with the exact model!
- This fortuitous phenomenon has a special significance because it arises when certain discontinuous models with discrete inputs are approximated by continuous functions and data*.

*Daganzo, C.F. (1987) "Increasing model precision can reduce accuracy" Trans.Sci. 21(2), 100-105.

Problems with constraints

前面分析了在忽略 $v < v_{\max}$ 条件时 EOQ 模型的稳健性和误差，同样的结论也适用于带约束的模型和拓展模型。

- The constrained EOQ solution is now presented rather briefly, before turning our attention to the lot size problem with variable demand.
- If we find that $v^* > v_{\max}$ in solving the unconstrained EOQ problem, then the solution is not feasible. Choosing $v = v_{\max}$ is optimal. Hence, the optimal EOQ solution can be expressed as:

$$v^* = \min\{\sqrt{B/A}, v_{\max}\}$$

and the optimal cost per item

$$z^* = \begin{cases} 2\sqrt{AB} & \text{if } \sqrt{B/A} \leqslant v_{\max} \\ Av_{\max} + B/v_{\max} & \text{if } \sqrt{B/A} > v_{\max} \end{cases}$$

Problems with constraints (cont.)

- Note that z^* is an increasing and concave function of A, and also of B.
- As $A = c_h/D'$, z^* is decreasing a function of D' and convex; the economies of scale continue to exist for all ranges of D'.
- The total cost per unit time, $D'z^*$, is proportional to $D'^{1/2}$ until the capacity constraint is reached, and from then on increases linearly with D'. The critical point is $D'_{crit} = (v_{\max})^2 c_h/c_f$*

*固定 B，则总成本为 A 的函数，分段函数的间断点为 $\sqrt{B/A} = v_{\max} \rightarrow A = \frac{B}{v_{\max}^2}$。代入 $A = c_h/D'$，可得 $D'_{crit} = (v_{\max})^2 c_h/c_f$

Optimal EOQ cost as a function various parameters

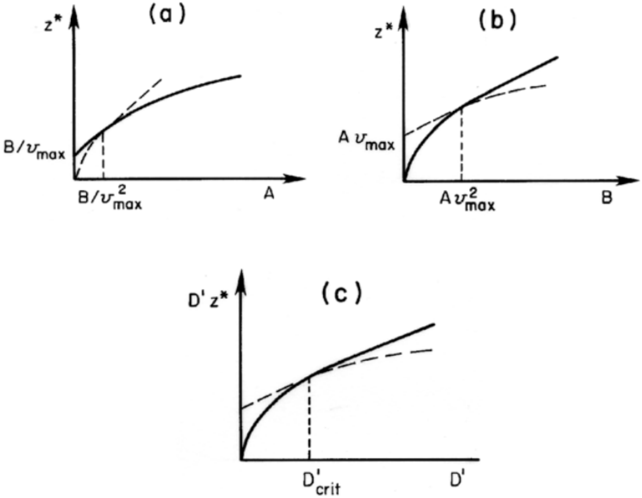


Figure: Optimal EOQ cost as a function various parameters: (a) holding cost per item, A; (固定 B) (b) fixed motion costs, B; (固定 A) and (c) demand rate, D'. Dashed lines are the unused branches of z^*

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Problem with variable demand （变需求下的 EOQ 问题）

接下来考虑在有限时间区间内，消费量以可预测的方式变化的情况下的 EOQ 问题。

- The demand pattern is characterized by a function $D(t)$ that gives the cumulative number of items demanded between times 0 (the beginning of the study period) and t. The time derivative of this function $D'(t)$ represents the variable demand rate.
- We then seek the set of times when shipments are to be received $(t_0 = 0, t_1, \ldots, t_{n-1})$, and the shipment sizes $(v_0, v_1, \ldots, v_{n-1})$, that will minimize the sum of the motion plus holding costs over our horizon, $t \in [0, t_{\max}]$.
- As previously, we also define as **inputs** to our problem a **fixed (motion) cost per vehicle dispatch** c_f, a **holding cost per item-time** $c_h = c_r + c_i$, and a **maximum lot size** $v_{\max}$. With an infinite horizon and a constant demand, $D(t) = D't$, this formulation reduces to the EOQ problem examined in previous sections.

Solution when holding cost $\approx$ rent cost

- If inventory cost is negligible, $c_i \ll c_r$, then holding cost approximately equals rent cost $c_h \approx c_r$. We have already mentioned that rent cost increases with the maximum inventory accumulation*, and that otherwise the cost is rather insensitive to the accumulations at other times. This property of holding cost simplifies the solution to our problem.
- Recall that given a set of n shipments, the motion cost during the period of analysis, $c_f n$, is independent of the shipment times and sizes†. The problem is then to find the sets of shipment times and sizes that will minimize holding cost.

* $c_r \times D' H_1$ 与保存所发生的时间无关
† 移动的总费用与发生时间和批量无关

Solution when holding cost $\approx$ rent cost (cont.)

- A lower bound to the maximum accumulation at the destination is the size of the largest shipment received. (Why?*)
- This lower bound minimized when all the shipments are equal.(Why?†)
- Hence, the largest shipment – and, thus, the maximum accumulation – must exceed or at least equal $D(t_{\max})/n$, the set is an optimal way of sending n shipments with rent cost per unit time: $c_r D(t_{\max})/n$‡.
- Each shipment is just large enough to meet the demand until the next shipment; the consumption between consecutive receiving times, the same in all cases, is $D(t_{\max})/n$§.

*在目的地的积累量的一个下界为最大批量，意味着该批货物到达时刚好无存货

†每个批量相等，意味着最大批量即为平均批量。此时最大批量最小

‡每个批量相等且 n 次配送的总量至少为 $D_{\max}$，则显然每次批量 $\geqslant D(t_{\max})/n$

§每次的送货量刚好能满足两次配送之间的需求量

Solution when holding cost $\approx$ rent cost (cont.)

Clearly the following strategy is optimal:

- Divide the ordinate axis between 0 and $D(t_{\max})$ into n equal segments and find the times t_i for which $D(t)$ equals $(i/n)D(t_{\max})$ for $i = 0, \ldots, n-1$. These are the shipment times,
- Dispatch barely enough to cover the demand until the following shipment.

One must now find the optimal n by minimizing the resulting cost

$$cost/time = c_r[D(t_{\max})/n] + c_f[n/t_{\max}]$$

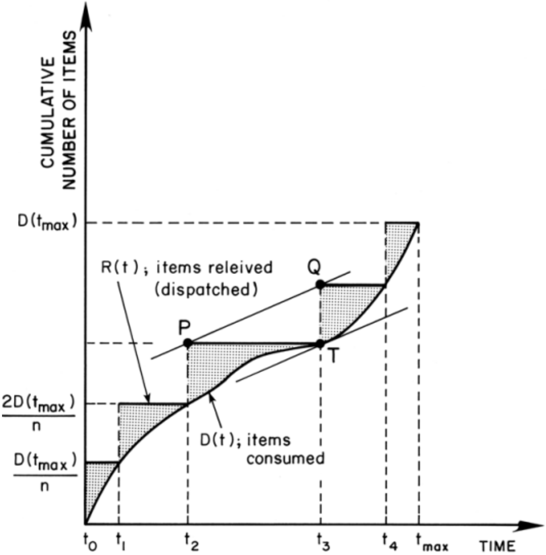
$$cost/item = (\frac{c_r}{\bar{D}'})(\frac{D(t_{\max})}{n}) + c_f[n/D(t_{\max})]$$

where D' is the average consumption rate $\bar{D}' = D(t_{\max})/t_{\max}$

Solution when holding cost $\approx$ rent cost (cont.)

- Note that the formulation is the EOQ expression with $v = D(t_{\max})/n$. The solution now requires that n be an integer (there are constraints on v), but we have already seen that any v close to the unconstrained v^* is near optimal. As a result, unless the time horizon is so short that $n^* = 1$ or 2, the optimal cost per item should be close to the cost with constant demand
- If $v_{\max} < \infty$, the solution procedure does not change. It is still optimal to have equal shipment sizes, but the number of shipments should be large enough to satisfy: $D(t_{\max})/n < v_{\max}$. The solution is still of the same form, with v^{-1} restricted to being an integer multiple of $D(t_{\max})^{-1}$.

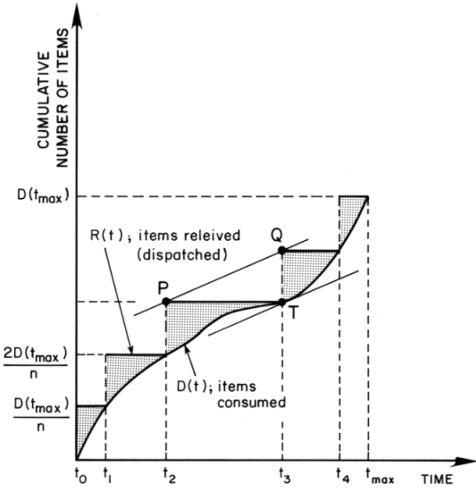
Solution procedure



Solution when rent cost is negligible

另一种情况与之前相反，即租金可忽略，但是在途的保管费用不可忽略。

- This situation occurs when items are so small and expensive, that most of the holding cost arises from the item-hours spent in inventory, and not from the rent for the space to hold them.
- In this case the destination's holding cost should be proportional to the shaded area of right figure

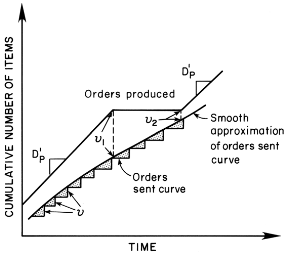


Solution when rent cost is negligible

The combined origin-destination holding cost will also be proportional to the shaded area:

- if the origin holding cost can be ignored <— if the origin produces generic items for so many destinations that the part of its costs that would be prorated (按比例分配) to each destination is negligible.

- if the origin holding cost is proportional to the area. <— if the production strategy at the origin is as described in figure below. The total wait at the origin that can be attributed to the shipping strategy must be similar to that of the destination; i.e., it would also be proportional to the shaded area

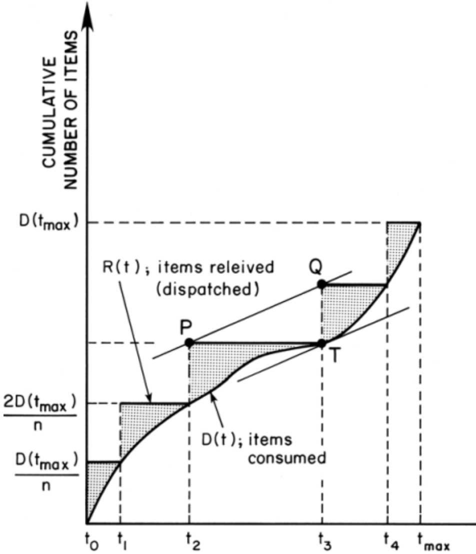


- for typical passenger transportation systems

Solution when rent cost is negligible

- When holding costs are proportional to the shaded area, they are no longer a function of n alone.
- For a set of points $(t_1 \dots t_{n-1})$ to be optimal, each line $\overline{PQ}$ must be parallel to the tangent line to $D(t)$ at the receiving time[a] (point T in the figure)
- We may verify that if this condition is not satisfied, then it is possible to reduce the total shaded area by either advancing or delaying the receiving time by a small amount.

[a]Newell (1971)



Solution when rent cost is negligible (cont.)

- Unfortunately, the smallest shaded area - and thus the waiting cost - no longer can be expressed as a function of n alone, independently of $D(t)$.
- Thus, it seems that a simple expression for the optimal cost cannot be obtained for any $D(t)$

Numerical solution – 滚动时域优化思路

- It can be formulated as a rolling horizon optimization problem in which a shipment time, t_i, is chosen at each stage $(i = 1, \dots, n-1)$, and where the state of the system is the prior shipment time, t_{i-1}. The optimization procedure yields an optimum holding cost for a given n, $z_i^*(n)$, which can be substituted for the first term of the following equation to yield n^*.

$$cost/item = z_i^*(n) + c_f(n/D(t_{\max}))$$

Numerical solution — Newell's method

The following procedure is less laborious and works particularly well if $D(t)$ is smooth, without bends or jumps (refer to figure for the explanation)

1. Choose a point P_1 on the ordinates axis and move across to T_1
2. Draw from P_1 a line parallel to the tangent to $D(t)$ at T_1, and draw from T_1 a vertical line. Label the point of intersection P_2

Steps (i) and (ii) identify a point P_2 from a point P_1. They should be repeated to identify P_3 from P_2, P_4 from P_3, etc., defining in this manner a receiving step curve, $R(t)$. If $R(t)$ does not pass through the end point, $(t_{max}, D(t_{max}))$, the position of P_1 should be perturbed until it does.

Numerical Solution — Newell's method (cont.)

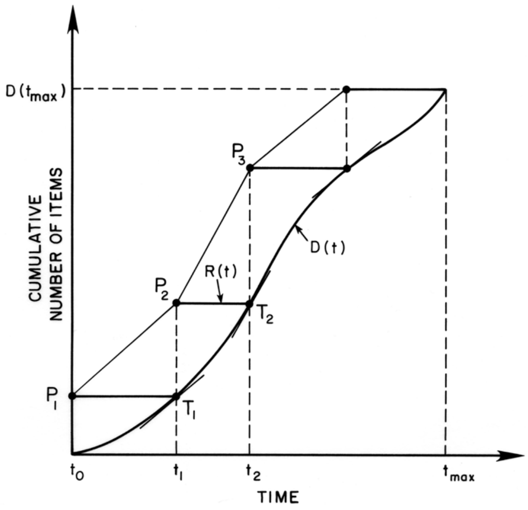


Figure: Construction method for the cumulative number of items shipped versus time

The Continuous Approximation Method

- The CA method replaces the search for $\{t_i\}$ by a search for a continuous function, whose knowledge yields a set of t_i with near minimal cost
- It works well when $D'(t)$ does not change rapidly; i.e., if $D'(t_i) \approx D'(t_{i+1})$ for all i. A byproduct is a simple expression and decomposition principle for the total cost

The Continuous Approximation Method (cont.)

- Let us assume that an optimal solution has been found, and denote by I_i the $i-$th interval between consecutive receiving times: $[t_{i-1}, t_i), i = 1, 2, \ldots$.
- Then divide the total cost during the study period into portions "$cost_i$" corresponding to each interval. That is, "$cost_i$" includes the cost, c_f, of dispatching one shipment plus the product of c_i and the shaded area for interval I_i

$$cost_i = c_f + c_i \times area_i$$

- Clearly, the sum of the prorated costs will equal the total cost. Since $D'(t)$ is continuous, it should be intuitive that there is a point t_i' in each interval I_i for which the area above $D(t)$ satisfies:

$$area_i = \frac{1}{2}(t_i - t_{i-1})^2 D'(t_i')^*$$

*阴影区域的面积等于需求函数在 t_i' 处取值时的，其中横轴长度为 $t_i - t_{i-1}$，纵轴为 $D'(t_i')(t_i - t_{i-1})$

The Continuous Approximation Method (cont.)

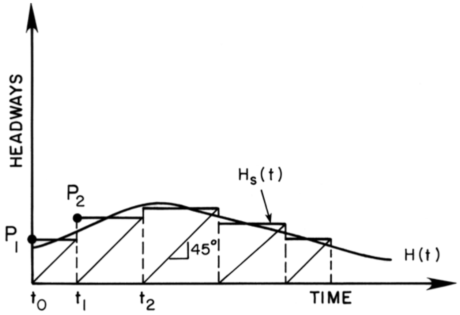
为什么 $[t_{i-1}, t_i)$ 区间内一定存在这样的 t'_i?

- Consider the triangle defined by the horizontal and vertical lines passing through a point P_i in the figure and a straight line passing through T_i with a slope that yields "$area_i$" for the triangle; i.e. slope $D'(t'_i)$.
- Since such a slanted line must intersect $D(t)$ (otherwise the areas above $D(t)$ and above the slanted line could not be equal) there must be a point between T_i and the point of intersection where the two lines have the same slope. The abscissa (横坐标) of this point is t'_i.

Therefore we can write:

$$area_i = \frac{1}{2}(t_i - t_{i-1})^2 D'(t'_i) = \int_{t_{i-1}}^{t_i} \frac{1}{2}(t_i - t_{i-1}) D'(t'_i) \mathrm{d}t$$

The Continuous Approximation Method (cont.)



If we now define $H_s(t)$ as a step function such that $H_s(t) = t_i - t_{i-1}$ if $t \in I_i$ (see the figure above for example), then the cost per interval can be expressed as:

$$cost_i = \int_{t_{i-1}}^{t_i} [\frac{c_f}{H_s(t)} + \frac{c_i H_s(t)}{2} D'(t_i')]\mathrm{d}t.$$

Note that this is an exact expression.

The Continuous Approximation Method (cont.)

If we now approximate $D'(t_i')$ by $D'(t)$ – which is reasonable if $D'(t)$ varies slowly – the total cost over the whole study period can be expressed as the following integral:

$$cost_i = \int_{t_{i-1}}^{t_i} [\frac{c_f}{H_s(t)} + \frac{c_i H_s(t)}{2} D'(t)]\mathrm{d}t.$$

We seek the function $H_s(t)$, which minimizes the equation above. Unfortunately, this is akin to determining the $\{t_i\}$ themselves. A closed form solution can be obtained if $H_s(t)$ is replaced by a smooth function, $H(t)$. That is:

$$cost_i \approx \int_{t_0}^{t_{\max}} [\frac{c_f}{H(t)} + \frac{c_i H(t)}{2} D'(t)]\mathrm{d}t.$$

Now, instead of finding $H_s(t)$, we can find the $H(t)$ which minimizes the new equation - a much easier task - and then choose a set of shipment times (i.e., $H_s(t)$) consistent with $H(t)$.

The Continuous Approximation Method (cont.)

Clearly, the $H(t)$ which minimizes the RHS minimizes the integrand （被积项）at every t; thus:

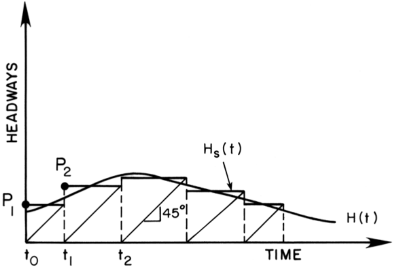
$$H(t) = [2c_f/(c_i D'(t))]^{1/2}.$$

This is the time between dispatches (headway) for the EOQ problem with constant demand $D' = D'(t)$.
A set of shipment times consistent with $H(t)$ can be found easily since $H(t)$ varies slowly with t.

The Continuous Approximation Method (cont.)

The figure suggests how this can be done systematically.

- Starting at the origin (point t_0) draw a 45°line and find a horizontal segment from a point on the vertical axis, such as P_1 in the figure, to the intersection with the 45°line.
- The elevation of P_1 should be such that the area below the segment equals the area below $H(t)$.
- The abscissa of the point of intersection is the next shipment time, t_1. This locates t_1, given t_0.
- The construction is then repeated from t_1 to locate t_2, from t_2, to locate t_3, etc.



In practice one does not need to be quite so precise, since we have already seen that small deviations from optimality have a minor effect.

Now we may calculate the total cost given the optimal $H(t)$.

$$\text{Total cost} \approx \int_{t_0}^{t_{\max}} [2c_i c_f D'(t)]^{1/2} \mathrm{d}t.$$

The integrand of this expression is the optimal EOQ cost per unit time if $D' = D'(t)$. Note that the integrand in the equation can be written as

$$[2c_i c_f / D'(t)]^{1/2} [D'(t) \mathrm{d}t]$$

where the first factor represents the optimal cost per item for an EOQ problem with constant demand, $D'(t)$. The average cost per item (across all the items) is obtained by dividing the total cost by the total number of items $D(t_{\max}) = \int_{t_0}^{t_{\max}} D'(t) \mathrm{d}t$.

The result is:

$$(\text{cost/item})^* = \frac{\int_{t_0}^{t_{\max}} [2c_i c_f D'(t)]^{1/2} \mathrm{d}t}{\int_{t_0}^{t_{\max}} D'(t) \mathrm{d}t}$$

- In practical terms the (cost/item)* expression indicates that the average optimal cost per item can be obtained by averaging the cost of all the items, as if each one of these was given by the EOQ formula with a (constant) demand rate equal to the demand rate at the time when the item is consumed*.
- The total cost expression indicates that, given a partition of $[0, t_{\max}]$ into a collection of short time intervals, the optimum cost can be approximated by the sum of the EOQ costs for each one of the intervals considered isolated from the others‡.
- These equations are so simple that they can be used as building blocks for the study of more complex problems in following lectures.

This is one of the attractive features of the CA approach; it yields cost estimates without having to develop, or even define, a detailed solution to the problem.

*商品的最优平均成本可以通过对所有商品成本的均值求出，而每件成本的成本由 EOQ 公式在该商品被消费时的取值得出。

‡总成本的解释也是类似。假设 $[0, t_{\max}]$ 之间被划分成若干短时间段，最优的总成本可通过每个孤立时间段的 EOQ 成本之和近似

The Continuous Approximation Method (cont.)

- The CA approach can also be used to locate points on any line (time or otherwise) provided that the total cost can be prorated approximately to (short) intervals on the line, while ensuring that the prorated cost to any interval only depends on the characteristics of said interval. In the previous discussion, the integrand in the cost equation $cost_i \approx \int_{t_0}^{t_{\max}} [\frac{c_f}{H(t)} + \frac{c_i H(t)}{2} D'(t)] \mathrm{d}t$ is the prorated cost in $[t, t + \mathrm{d}t)$, which does not depend on the demand rate outside the interval
- The CA approach can also be used to locate points in multidimensional space, when the total cost can be expressed as a sum of neighborhood costs dependent only on their local characteristics. Newell (1973) argues that the CA approach is comparatively more useful then, because in the multidimensional case it is much more difficult for exact numerical methods to deal with the complex boundary conditions that arise. Because the CA approach will be used in forthcoming lectures repeatedly, the next section discusses two additional (one-dimensional) examples.

Readings

- Daganzo. Logistics System Analysis. Ch.3. Page 49-64

Any questions?