A Generalized Description of a 5R, Closed Kinematic Chain, Planar Robot and ROS Implementation
ME 601: Advanced Robotics

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Most robots are described by a series of links and joints, where each link is connected to the previous link by a joint. In such robots, called series robots, the solution to the forward and inverse kinematics is well documented. The kinematics of closed kinematic chain robots, such as the one shown in Figure 1, however, are not as well documented. The objective of this project was to describe a generalized, closed form solution to the forward and inverse kinematics of a closed kinematic chain, 5R robot.

The robot configuration analyzed in this project is illustrated in Figure 1. The robot consists of 6 links: the base joint fixed to the world frame, a right branch consisting of two links, of length R_1 and R_2, and two revolute joints, and a left branch also consisting of two links, of lengths L_1 and L_2, and two revolute joints. The left branch joins the right branch at a distance E from the end effector. The base joints of each branch are separated by a distance b.

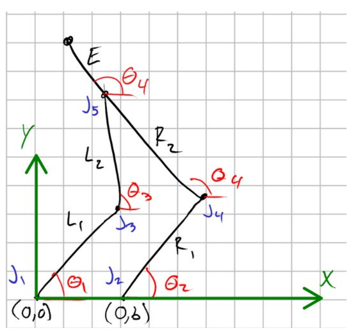

Figure 1: Robot Configuration

Section 1: Forward Kinematics
The forward kinematics of the robot can be found through either the left or right arm. The x and y coordinates of the end effector are given by

$$x = b + R_1\cos\theta_2 + R_2\cos\theta_4 + E\cos\theta_4 \tag{1}$$
$$y = R_1\sin\theta_2 + R_2\sin\theta_4 + E\sin\theta_4 \tag{2}$$

$$x = L_1\cos\theta_1 + L_2\cos\theta_3 + E\cos\theta_4 \tag{3}$$
$$y = L_1\sin\theta_1 + L_2\sin\theta_3 + E\sin\theta_4 \tag{4}$$

Because we have two equations that describe x and y, we can set them equal and isolate the θ_3 terms. Setting (1)=(3) and solving yields

$$\frac{1}{L_2}(b + R_1\cos\theta_2 + R_2\cos\theta_4 - L_1\cos\theta_1) = \cos\theta_3 \tag{5}$$

and from (2)=(4), we get

$$\frac{1}{L_2}(R_1\sin\theta_2 + R_2\sin\theta_4 - L_1\sin\theta_1) = \sin\theta_3\ . \tag{6}$$

Now we can set $(5)^2+(6)^2=1$. Solving this and arranging it in the form

$$A_1\cos\theta_4 + B_1\sin\theta_4 + C_1 = 0 \tag{7}$$

yields coefficients of

$$A_1 = \frac{1}{L_2}(2bR_1 + 2R_1R_2\cos\theta_2 - 2L_1R_2\cos\theta_1)\ ,$$

$$B_1=\frac{1}{L_2}(2R_1R_2\sin\theta_2-2L_1R_2\sin\theta_1)\ ,$$

$$C_1=\frac{1}{L_2}[L_1^2+R_1^2+R_2^2+b^2+2bR_1\cos\theta_2-2bL_1\cos\theta_1-2R_1L_1\cos(\theta_1-\theta_2)]-1\ .$$

Using the tangent half-angle substitution, we can substitute the following:

$$t_4=\tan\frac{\theta_4}{2},\ \sin\theta_4=\frac{2t_4}{1+t_4^2},\ \cos\theta_4=\frac{1-t_4^2}{1+t_4^2}\ .$$

Solving (7) for t_4 results in

$$t_4=\frac{-B_1\pm\sqrt{4B_1^2-4(C_1-A_1)(A_1+C_1)}}{C_1-A_1} \tag{8}$$

Substituting back yields

$$\theta_4=2\tan^{-1}\left(\frac{-B_1\pm\sqrt{4B_1^2-4(C_1-A_1)(A_1+C_1)}}{C_1-A_1}\right)\ . \tag{9}$$

Using (1) and (2), only θ_4 is needed for forward kinematics solution. For completeness, θ_3 is given by

$$\theta_3=\cos^{-1}\left(\frac{x-L_1\cos\theta_1-E\cos\theta_4}{L_2}\right) \tag{10}$$

Section 2: Inverse Kinematics

For the right and left branch, respectively, the x and y position is described by

$$x=b+R_1\cos\theta_2+R_2\cos\theta_4+E\cos\theta_4\ , \tag{11}$$

$$y=R_1\sin\theta_2+R_2\sin\theta_4+E\sin\theta_4\ , \tag{12}$$

$$x=L_1\cos\theta_1+L_2\cos\theta_3+E\cos\theta_4\ , \tag{13}$$

$$y=L_1\sin\theta_1+L_2\sin\theta_3+E\sin\theta_4\ . \tag{14}$$

First, θ_4 will be isolated. Setting (11)=(12) results in

$$\frac{1}{R_2+E}(x-b-R_1\cos\theta_2)=\cos\theta_4\ , \tag{15}$$

$$\frac{1}{R_2+E}(y-R_1\sin\theta_2)=\sin\theta_4\ . \tag{16}$$

Then, $(15)^2+(16)^2=1$. This equation can be arranged in the form

$$A_2\cos\theta_2+B_2\sin\theta_2+C_2=0\ , \tag{17}$$

where

$$A_2=x-b\ ,$$

$$B_2=y\ ,$$

$$C_2=\frac{-1}{2R_1}(x^2+y^2+b^2+R_1^2-R_2^2-E^2-2xb-2R_2E)\ .$$

Using the tangent half-angle substitution yields

$$\theta_2=2\tan^{-1}\left(\frac{-B_2\pm\sqrt{4B_2^2-4(C_2-A_2)(A_2+C_2)}}{C_2-A_2}\right)\ . \tag{18}$$

Recalling (15) and (16), θ_4 is given by

$$\theta_4 = \cos^{-1}\left(\frac{1}{R_2+E}(x-b-R_1\cos\theta_2)\right) = \sin^{-1}\left(\frac{1}{R_2+E}(y-R_1\sin\theta_2)\right). \tag{19}$$

The same process of finding θ_2 can be applied to find θ_1. Starting with (13) and (14), the θ_3 terms can be isolated:

$$\frac{1}{L_2}(x-L_1\cos\theta_1-E\cos\theta_4)=\cos\theta_3 \tag{20}$$

$$\frac{1}{L_2}(y-L_1\sin\theta_1-E\sin\theta_4)=\sin\theta_3 \tag{21}$$

Once again the identity $(20)^2+(21)^2=1$ is applied, and the resulting equation is arranged in the form

$$A_3\cos\theta_1+B_3\sin\theta_1+C_3=0, \tag{22}$$

where

$$A_3=2L_1E\cos\theta_4-2xL_1\ ,$$
$$B_3=2L_1E\sin\theta_4-2yL_1\ ,$$
$$C_3=x^2+y^2+L_1^2-L_2^2+E^2-2xLE\cos\theta_4-2yE\sin\theta_4\ .$$

Again, the solution is given by the tangent half-angle substitution,

$$\theta_1=2\tan^{-1}\left(\frac{-B_3\pm\sqrt{4B_3^2-4(C_3-A_3)(A_3+C_3)}}{C_3-A_3}\right). \tag{23}$$

For completeness, recall the relationship of θ_3 in (20) and (21),

$$\theta_3=\cos^{-1}\left(\frac{1}{L_2}(x-L_1\cos\theta_1-E\cos\theta_4)\right) = \sin^{-1}\left(\frac{1}{L_2}(y-L_1\sin\theta_1-E\sin\theta_4)\right). \tag{24}$$

Section 3: ROS Implementation