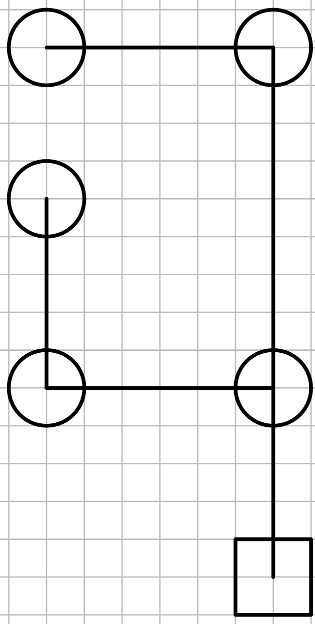
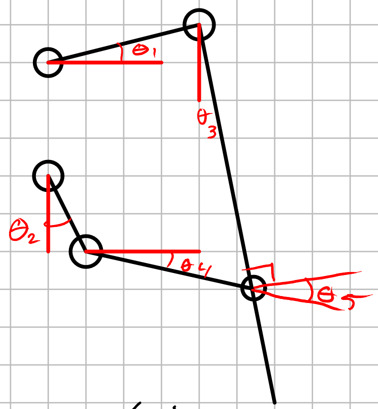
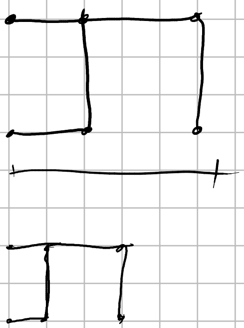
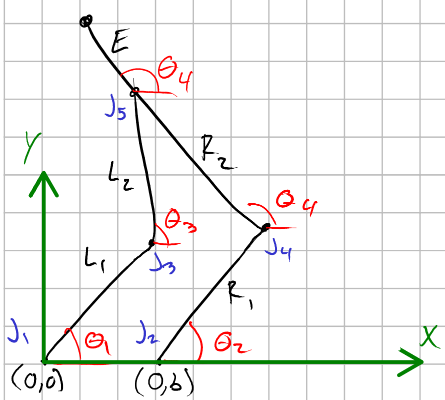




$$u = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$





b is distance between bases of arms

on right arm

$$x = b + R_1\cos\theta_2 + R_2\cos\theta_4 + E\cos\theta_4 \quad (1)$$

$$y = R_1\sin\theta_2 + R_2\sin\theta_4 + E\sin\theta_4 \quad (2)$$

on left arm

$$x = L_1\cos\theta_1 + L_2\cos\theta_3 + E\cos\theta_4 \quad (3)$$

$$y = L_1\sin\theta_1 + L_2\sin\theta_3 + E\sin\theta_4 \quad (4)$$

Set x equations equal and isolate θ_3

$$b + R_1\cos\theta_2 + R_2\cos\theta_4 + \cancel{E\cos\theta_4} = L_1\cos\theta_1 + L_2\cos\theta_3 + \cancel{E\cos\theta_4}$$

$$b + R_1\cos\theta_2 + R_2\cos\theta_4 = L_1\cos\theta_1 + L_2\cos\theta_3$$

$$\frac{1}{L_2}\left(b + R_1\cos\theta_2 + R_2\cos\theta_4 - L_1\cos\theta_1\right) = \cos\theta_3 \quad (5)$$

Set y eqn's equal and isolate θ_3

$$R_1\sin\theta_2 + R_2\sin\theta_4 + \cancel{E\sin\theta_4} = L_1\sin\theta_1 + L_2\sin\theta_3 + \cancel{E\sin\theta_4}$$

$$R_1\sin\theta_2 + R_2\sin\theta_4 = L_1\sin\theta_1 + L_2\sin\theta_3$$

$$\frac{1}{L_2}\left(R_1\sin\theta_2 + R_2\sin\theta_4 - L_1\sin\theta_1\right) = \sin\theta_3 \quad (6)$$

$$⑤^2 + ⑥^2 = 1 \qquad \sin^2 + \cos^2 = 1$$

$$\frac{1}{L_2^2}\left(R_1\sin\theta_2 + R_2\sin\theta_4 - L_1\sin\theta_1\right)^2 + \frac{1}{L_2^2}\left(b + R_1\cos\theta_2 + R_2\cos\theta_4 - L_1\cos\theta_1\right)^2 = 1$$

$$\frac{1}{L_2^2}\left[\left(R_1\sin\theta_2 + R_2\sin\theta_4 - L_1\sin\theta_1\right)^2 + \left(b + R_1\cos\theta_2 + R_2\cos\theta_4 - L_1\cos\theta_1\right)^2\right] = 1$$

$$\begin{aligned}\frac{1}{L_2^2}\Big[& \cancel{L_1^2\sin^2\theta_1 + L_1^2\cos^2\theta_1}\ {+L_1^2} - 2L_1R_1\sin\theta_1\sin\theta_2 - 2L_1R_1\cos\theta_1\cos\theta_2 - 2L_1R_2\sin\theta_1\sin\theta_4 \\ & - 2L_1R_2\cos\theta_1\cos\theta_4 - 2L_1b\cos\theta_1 + \cancel{R_1^2\cos^2\theta_2 + R_1^2\sin^2\theta_2}\ {+R_1^2} \\ & + 2R_1R_2\sin\theta_2\sin\theta_4 + 2R_1R_2\cos\theta_2\cos\theta_4 + 2R_1b\cos\theta_2 \\ & + \cancel{R_2^2\sin^2\theta_4 + R_2^2\cos^2\theta_4}\ {+R_2^2} + 2R_2b\cos\theta_4 + b^2\Big] = 1\end{aligned}$$

$$\frac{1}{L_2^2}\left[\begin{array}{ll} L_1^2 + R_1^2 + R_2^2 + b^2 & + 2bR_1C\theta_2 \\ & + 2bR_2C\theta_4 \\ & - 2bL_1C\theta_1 \\ + 2R_1R_2S\theta_2S\theta_4 & + 2R_1R_2C\theta_2C\theta_4 \\ - 2R_1L_1S\theta_2S\theta_1 & - 2R_1L_1C\theta_2C\theta_1 \\ - 2L_1R_2S\theta_4S\theta_1 & - 2L_1R_2C\theta_4C\theta_1 \end{array}\right] = 1$$

$$\frac{1}{L_2^2}\left[\begin{array}{ll} L_1^2 + R_1^2 + R_2^2 + b^2 & + 2bR_1C\theta_2 \\ & + 2bR_2C\theta_4 \\ & - 2bL_1C\theta_1 \\ + 2R_1R_2S\theta_2S\theta_4 & + 2R_1R_2C\theta_2C\theta_4 \\ - 2R_1L_1S\theta_2S\theta_1 & - 2R_1L_1C\theta_2C\theta_1 \\ - 2L_1R_2S\theta_4S\theta_1 & - 2L_1R_2C\theta_4C\theta_1 \end{array}\right] = 1$$

$$\frac{1}{L_2^2}\left[\begin{array}{ll} L_1^2 + R_1^2 + R_2^2 + b^2 & + 2bR_1C\theta_2 \\ & + 2bR_2C\theta_4 \\ & - 2bL_1C\theta_1 \\ + 2R_1R_2S\theta_2S\theta_4 & + 2R_1R_2C\theta_2C\theta_4 \\ - 2R_1L_1S\theta_2S\theta_1 & - 2R_1L_1C\theta_2C\theta_1 \\ - 2L_1R_2S\theta_4S\theta_1 & - 2L_1R_2C\theta_4C\theta_1 \end{array}\right] = 1$$

$$\begin{aligned}&\frac{1}{L_2^2}\left(L_1^2 + R_1^2 + R_2^2 + b^2 + 2bR_1C\theta_2 - 2bL_1C\theta_1 - 2R_1L_1\left(S\theta_2S\theta_1 + C\theta_2C\theta_1\right)\right) - 1 \\ &+ \frac{1}{L_2^2}\left(2bR_2 + 2R_1R_2C\theta_2 - 2L_1R_2C\theta_1\right)C\theta_4 + \frac{1}{L_2^2}\left(2R_1R_2S\theta_2 - 2L_1R_2S\theta_1\right)S\theta_4 = 0\end{aligned}$$

$$A = \frac{1}{L_2^2}\left(2bR_2 + 2R_1R_2C\theta_2 - 2L_1R_2C\theta_1\right)$$

$$B = \frac{1}{L_2^2}\left(2R_1R_2S\theta_2 - 2L_1R_2S\theta_1\right)$$

$$C = \frac{1}{L_2^2}\left[L_1^2 + R_1^2 + R_2^2 + b^2 + 2bR_1C\theta_2 - 2bL_1C\theta_1 - 2R_1L_1\cos(\theta_1 - \theta_2)\right] - 1$$

$$A\cos\theta_4 + B\sin\theta_4 + C = 0$$

using tan half-angle transformation:

$$t_4 = \tan\frac{\theta_4}{2}, \quad \sin\theta_4 = \frac{2t_4}{1+t_4^2}, \quad \cos\theta_4 = \frac{1-t_4^2}{1+t_4^2}$$

$$A\left(\frac{1-t_4^2}{1+t_4^2}\right) + B\left(\frac{2t_4}{1+t_4^2}\right) + C = 0$$

$$\frac{A(1-t_4^2) + B\,2t_4 + C(1+t_4^2)}{1+t_4^2} = 0$$

$$\frac{A - t_4^2A + 2Bt_4 + C + Ct_4^2}{1+t_4^2} = 0$$

$$\frac{(C-A)t_4^2 + 2Bt_4 + (A+C)}{1+t_4^2} = 0$$

bottom is always >0, Solve numerator with quadratic formula

$$t_4 = \frac{-2B \pm \sqrt{4B^2 - 4(C-A)(A+C)}}{2(C-A)}$$

$$t_4 = \frac{-B \pm \sqrt{B^2 - (C-A)(A+C)}}{(C-A)}$$

$$\theta_4 = 2\arctan\left(\frac{-B \pm \sqrt{B^2 - (C-A)(A+C)}}{(C-A)}\right)$$

Sub into 1 and 2 to find

x_{ee}, y_{ee} with given θ_1, θ_2

$(x_1, y_1) = (0, 0)$

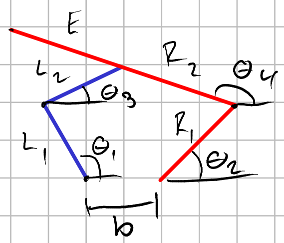
$(x_2, y_2) = (b, 0)$

$(x_3, y_3) = (L_1 \cos\theta_1, L_1 \sin\theta_1)$

$(x_4, y_4) = (b + R_1 \cos\theta_2, R_1 \sin\theta_2)$

$(x_5, y_5) = (b + R_1 \cos\theta_2 + R_2 \cos\theta_4, R_1 \sin\theta_2 + R_2 \sin\theta_4)$

For elbows out



right side

$x = b + R_1 \cos\theta_2 + R_2 \cos\theta_4 + E \cos\theta_4$

$y = R_1 \sin\theta_2 + R_2 \sin\theta_4$

Left Side

$x = L_1 \cos\theta_1 + L_2 \cos\theta_3 + E \cos\theta_4$

$y = L_1 \sin\theta_1 + L_2 \sin\theta_3 + E \sin\theta_4$

Same as previous derivation. FK Done!

don't forget about θ_3

from ① find θ_3

$x = L_1 \cos\theta_1 + L_2 \cos\theta_3 + E \cos\theta_4$

$$\theta_3 = \cos^{-1}\left(\frac{x - L_1 \cos\theta_1 - E \cos\theta_4}{L_2}\right)$$

Now for IK

right side $\quad x_{ee} = b + R_1 \cos\theta_2 + R_2 \cos\theta_4 + E \cos\theta_4$ ①

$\quad y_{ee} = R_1 \sin\theta_2 + R_2 \sin\theta_4 + E \sin\theta_4$ ②

Left Side $\quad x_{ee} = L_1 \cos\theta_1 + L_2 \cos\theta_3 + E \cos\theta_4$ ③

$\quad y_{ee} = L_1 \sin\theta_1 + L_2 \sin\theta_3 + E \sin\theta_4$ ④

$$\frac{1}{R_2 + E}(x_{ee} - b - R_1 \cos\theta_2) = \cos\theta_4 \quad ⑤ \quad \text{from ①}$$

$$\frac{1}{R_2 + E}(y_{ee} - R_1 \sin\theta_2) = \sin\theta_4 \quad ⑥ \quad \text{from ②}$$

$⑤^2 + ⑥^2 = 1$

$$\left(\frac{1}{R_2+E}\right)^2 (x_{ee} - b - R_1 \cos\theta_2)^2 + \left(\frac{1}{R_2+E}\right)^2 (y_{ee} - R_1 \sin\theta_2)^2 = 1$$

$$\frac{1}{(R_2+E)^2}\left[\left(x^2 + b^2 + R_1^2 \cos^2\theta_2 - 2xb + 2bR_1 \cos\theta_2 - 2xR_1 \cos\theta_2\right) + \left(y^2 - 2yR_1 \sin\theta_2 + R_1^2 \sin^2\theta_2\right)\right] = 1$$

$$\frac{1}{(R_2+E)^2}\left[\left(x^2+b^2+R_1^2\cos^{\cancel{2}}\theta_2-2xb+2bR_1\cos\theta_2-2xR_1\cos\theta_2\right)+\left(y^2-2yR_1\sin\theta_2+R_1^2\cancel{\sin^2\theta_2}\right)\right]=1$$

$$+R_1^2$$

$$\left(\frac{1}{R_2+E}\right)^2\left[x^2+b^2+R_1^2+y^2-2xb+2bR_1\cos\theta_2-2xR_1\cos\theta_2-2yR_1\sin\theta_2\right]=1$$

$$x^2+b^2+R_1^2+y^2-2xb+2bR_1\cos\theta_2-2xR_1\cos\theta_2-2yR_1\sin\theta_2=R_2^2+2R_2E+E^2$$

$$x^2+b^2+R_1^2+y^2-2xb-R_2^2-2R_2E-E^2=-2bR_1\cos\theta_2+2xR_1\cos\theta_2+2yR_1\sin\theta_2$$

$$+2R_1(x\cos\theta_2-b\cos\theta_2)+2yR_1\sin\theta_2$$

$$\frac{\left(x^2+b^2+R_1^2+y^2-2xb-R_2^2-2R_2E-E^2\right)}{2R_1}=(x-b)\cos\theta_2+y\sin\theta_2$$

arrange in the form $A_2\cos\theta_2+B_2\sin\theta_2+C_2=0$ (7)

$A_2=x-b$

$B_2=y$

$C_2=-\frac{1}{2R_1}\left(x^2+b^2+R_1^2+y^2-2xb-R_2^2-2R_2E-E^2\right)$

using tan half angle substitution:

$$t_2=\tan\frac{\theta_2}{2},\quad \sin\theta_2=\frac{2t_2}{1+t_2^2},\quad \cos\theta_2=\frac{1-t_2^2}{1+t_2^2}$$

$$A\left(\frac{1-t_2^2}{1+t_2^2}\right)+B\left(\frac{2t_2}{1+t_2^2}\right)+C=0$$

$$(1-t_2^2)A+2Bt_2+(1+t_2^2)C=0$$

$$(C-A)t_2^2+2Bt_2(A+C)=0$$

$$t_2=\frac{-2B\pm\sqrt{4B^2-4(A+C)(C-A)}}{2(C-A)}=\frac{-B\pm\sqrt{B^2-(A+C)(C-A)}}{(C-A)}$$

$$\theta_2=2\tan^{-1}\left(\frac{-B_2\pm\sqrt{B_2^2-(A_2+C_2)(C_2-A_2)}}{(C_2-A_2)}\right)$$

recall 5 and 6

$$\frac{1}{R_2+E}\left(x_{ee}-b-R_1\cos\theta_2\right) = \cos\theta_4 \quad (5)$$

$$\frac{1}{R_2+E}\left(y_{ee}-R_1\sin\theta_2\right) = \sin\theta_4 \quad (6)$$

then $\theta_4 = \cos^{-1}\left(\frac{1}{R_2+E}\left(x_{ee}-b-R_1\cos\theta_2\right)\right) = \sin^{-1}\left(\frac{1}{R_2+E}\left(y_{ee}-R_1\sin\theta_2\right)\right)$

recall 3 and 4

$$x = L_1\cos\theta_1 + L_2\cos\theta_3 + E\cos\theta_4 \quad (3)$$

$$y = L_1\sin\theta_1 + L_2\sin\theta_3 + E\sin\theta_4 \quad (4)$$

$$\frac{1}{L_2}\left(x - L_1\cos\theta_1 - E\cos\theta_4\right) = \cos\theta_3 \quad (8)$$

$$\frac{1}{L_2}\left(y - L_1\sin\theta_1 - E\sin\theta_4\right) = \sin\theta_3 \quad (9)$$

$$(8)^2 + (9)^2 = 1$$

$$\frac{1}{L_2^2}\left[\left(x - L_1\cos\theta_1 - E\cos\theta_4\right)^2 + \left(y - L_1\sin\theta_1 - E\sin\theta_4\right)^2\right] = 1$$

$$\left(x^2 + L_1^2\cos^2\theta_1 + E^2\cos^2\theta_4 + y^2 + L_1^2\sin^2\theta_1 + E^2\sin^2\theta_4 - 2xL_1\cos\theta_1 - 2xE\cos\theta_4 + 2L_1E\cos\theta_1\cos\theta_4 - 2yL_1\sin\theta_1 - 2yE\sin\theta_4 + 2L_1E\sin\theta_1\sin\theta_4\right) = L_2^2$$

$$x^2+y^2+L_1^2+E^2 - 2xL_1\cos\theta_1 - 2xE\cos\theta_4 - 2yL_1\sin\theta_1 - 2yE\sin\theta_4 + 2L_1E\cos\theta_1\cos\theta_4 + 2L_1E\sin\theta_1\sin\theta_4 - L_2^2 = 0$$

$$A\cos\theta_1 + B\sin\theta_1 + C = 0 \quad (10)$$

$$C = x^2+y^2+L_1^2-L_2^2+E^2-2xE\cos\theta_4-2yE\sin\theta_4$$

$$A = 2L_1E\cos\theta_4 - 2xL_1$$

$$B = 2L_1E\sin\theta_4 - 2yL_1$$

use tan Half Angle again

$$\theta_1 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{B^2-(A+C)(C-A)}}{(C-A)}\right)$$

for completeness, find θ_3, θ_4

recall (7) and (8)

$$\frac{1}{L_2}(x - L_1\cos\theta_1 - E\cos\theta_4) = \cos\theta_3 \quad (7)$$

$$\frac{1}{L_2}(y - L_1\sin\theta_1 - E\sin\theta_4) = \sin\theta_3 \quad (8)$$

$$\theta_3 = \cos^{-1}\left(\frac{1}{L_2}(x - L_1\cos\theta_1 - E\cos\theta_4)\right) = \sin^{-1}\left(\frac{1}{L_2}(y - L_1\sin\theta_1 - E\sin\theta_4)\right)$$

Summarizing,

for θ_2 (7) $A_2\cos\theta_2 + B_2\sin\theta_2 + C_2 = 0$

$A_2 = x - b$

$B_2 = y$

$C_2 = \frac{1}{2R_1}(x^2 + b^2 + R_1^2 + y^2 - 2xb - R_2^2 - 2R_2E - E^2)$

$\theta_2 = 2\tan^{-1}\left(\frac{-B_2 \pm \sqrt{B_2^2 - (A_2+C_2)(C_2-A_2)}}{(C_2 - A_2)}\right)$

for θ_1 (10) $A_1\cos\theta_1 + B_1\sin\theta_1 + C_1 = 0$

$A_1 = 2L_1E\cos\theta_4 - 2xL_1$

$B_1 = 2L_1E\sin\theta_4 - 2yL_1$

$C_1 = x^2 + y^2 + L_1^2 - L_2^2 + E^2 - 2xE\cos\theta_4 - 2yE\sin\theta_4$

$\theta_1 = 2\tan^{-1}\left(\frac{-B_1 \pm \sqrt{B_1^2 - (A_1+C_1)(C_1-A_1)}}{(C_1 - A_1)}\right)$

IK Done!

Jacobian

find $J = J_p^{-1} J_t$

$$J_p \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_t \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

given $A\cos\theta + B\sin\theta + C = 0$

$$\frac{\partial}{\partial t}\left(A\cos\theta + B\sin\theta + C\right) = 0$$

$$\frac{\partial A}{\partial t}\cos\theta + \frac{\partial B}{\partial t}\sin\theta + \frac{\partial C}{\partial t} = 0$$

$+ A\cos\theta_2\dot{\theta}$ $\quad + B\sin\theta\,\dot{\theta}$

$- A\sin\theta$ $\quad + B\cos\theta$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{y}}{\partial \dot{\theta}_1} & \frac{\partial \dot{y}}{\partial \dot{\theta}_2} \end{pmatrix}$$

Perturb $\hat{\theta}_1$ by ε