THE GRADIENT STRUCTURE TENSOR

Anders U. Waldeland, 11.12.18

Structure tensors provide a description of the local structure in images using a tensor field. For a 3D image, this yields a 3×3 tensor matrix for each voxel. Knutsson (1989) showed that this tensor could be optimally obtained using six spatially oriented quadrature filters. The gradient structure tensor (GST) (Bigun and Granlund (1987)), is a simplified implementation of the structure tensor, where the tensor is estimated using three gradient filters.

For a 3D seismic cube, the GST is computed using gradient estimates along the three image dimensions ($\mathbf{x} = [x, y, t]^T$). The gradient tensor:

$$\overline{\mathbf{T}} \equiv \overline{\mathbf{g}\mathbf{g}^T}, \tag{1}$$

is the estimated covariance matrix of the gradient vector field:

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_x(\mathbf{x}) \\ g_y(\mathbf{x}) \\ g_t(\mathbf{x}) \end{bmatrix}. \tag{2}$$

The gradients (g_x, g_y, g_t) along the three axes (x, y, t) are obtained by convolving the seismic data cube $(I(\mathbf{x}))$ with the derivative of the 3D Gaussian function $G(\mathbf{x}, \sigma)$:

$$g_i = I(\mathbf{x}) * \frac{\partial}{\partial \mathbf{x}_i} G(\mathbf{x}, \sigma_g). \tag{3}$$

where σ_g is the gradient scale[1] (full width at half maximum of the Gaussian function). The GST is then computed as the smoothed outer product of the gradient vector:

$$\overline{\mathbf{T}} = \begin{bmatrix} \overline{g_x^2} & \overline{g_y g_x} & \overline{g_t g_x} \\ \overline{g_x g_y} & \overline{g_y^2} & \overline{g_t g_y} \\ \overline{g_x g_t} & \overline{g_y g_t} & \overline{g_t^2} \end{bmatrix}, \tag{4}$$

where $^{-}$ is the smoothing operator. The smoothed tensor elements are computed using the Gaussian window function:

$$\overline{\mathbf{T}}_{ij} = \mathbf{T}_{ij} * G(\mathbf{x}, \sigma_T), \tag{5}$$

where σ_T is the smoothing scale[2].

Slope estimation

The local structure in the image can be analyzed by considering the eigenvalues ($\lambda_1 \geq \lambda_2 \geq \lambda_3$) and corresponding eigenvectors $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ of $\overline{\mathbf{T}}$. Since $\overline{\mathbf{T}}$ is the estimated covariance

[1] The GST is sensitive to structures at different scales depending on σ_g. When applied to seismic data, we want the GST to be sensitive to reflections. This can be achieved by setting σ_g to match half the typical thickness of the reflections.

[2] The parameter σ_T controls the spatial smoothing. By having a large σ_T, the GST becomes more robust against noise. By having a smaller σ_T, more detailed information about the structure may be obtained.

matrix of the gradient vector field, the eigenvectors span the axes of the covariance ellipsoid. This means that $\mathbf{v}_1$ will have the same direction as the locally dominant direction of the gradient vector field. It is also possible to derive attributes from the eigenstructure (e.g. as done in Kempen et al. (1999); Randen et al. (2000)).

By considering the components of $\mathbf{v}_1 = [v_{1x}, v_{1y}, v_{1t}]$, the slopes (estimates of the derivatives) are given as:

$$q_x = \widehat{\frac{\partial t}{\partial x}} = \frac{v_{1x}}{v_{1t}}, \tag{6}$$

$$q_y = \widehat{\frac{\partial t}{\partial y}} = \frac{v_{1y}}{v_{1t}}. \tag{7}$$

The eigenvector analysis is carried out for all locations in space, which produces the slope-fields.

Curvature estimation

The quadratic gradient structure tensor (QST) (van de Weijer et al. (2001); Bakker (2002)) is a method for estimating local curvature. The main assumption behind this method is that we are considering a locally quadratic surface on the form:

$$S(\mathbf{x}) \approx \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}\mathbf{x} + c = 0, \tag{8}$$

where $\mathbf{b}$ is the unit normal vector to the surface and $\mathbf{A}$ is a symmetric 3×3 matrix with at least two non-zero eigenvalues. When observing the surface in the reflector-oriented coordinate system (u, v, w) (the u-axis is normal to the surface) the surface becomes:

$$S(\mathbf{x}) \approx \frac{1}{2}\kappa_1 v + \frac{1}{2}\kappa_2 w + u \tag{9}$$

where κ_1 and κ_2 are the principal curvatures of the surface. Further, it is shown in Bakker (2002) that these curvatures can be estimated by considering the transform that deforms S into a plane. This results in the expressions for obtaining the curvatures in the reflector-oriented coordinate system:

$$\kappa_1 = \frac{\overline{v g_u g_v}}{\overline{v^2 g_u^2}}, \tag{10}$$

$$\kappa_2 = \frac{\overline{w g_u g_w}}{\overline{w^2 g_u^2}}, \tag{11}$$

where the g_u, g_v and g_w are the gradients along the axes in the reflector-oriented coordinate system.

In Bakker (2002), the vectors spanning this coordinate system are given by the eigenvectors of the GST:

$$\begin{aligned} \mathbf{u} &= [x_u, y_u, t_u]^T \mathrel{\widehat{=}} \mathbf{v}_1, \\ \mathbf{v} &= [x_v, y_v, t_v]^T \mathrel{\widehat{=}} \mathbf{v}_2, \\ \mathbf{w} &= [x_w, y_w, t_w]^T \mathrel{\widehat{=}} \mathbf{v}_3. \end{aligned} \tag{12}$$

In that case, the extracted curvatures will be the principal curvatures of the surface. We are, however, seeking the derivatives with respect to x and y. Therefore, we do not use the eigenvectors to span this coordinate system but force **v** and **w** to be oriented along the x- and y- axes.

Further more, Bakker (2002) shows that estimates of the second order derivatives are obtained by accounting for the rotation of the reflector-oriented coordinate system. Assuming that the two approaches for defining the reflector-oriented coordinate system coincide (that the principal curvature is the same as the derivative with respect to x), the curvatures (estimates of the second order derivatives) are given as:

$$\kappa_x = \widehat{\frac{\partial t^2}{\partial^2 x}} = \kappa_1 \left(1 + \left(\frac{\partial t}{\partial x}\right)^2\right)^{\frac{3}{2}}, \tag{13}$$

$$\kappa_y = \widehat{\frac{\partial t^2}{\partial^2 y}} = \kappa_2 \left(1 + \left(\frac{\partial t}{\partial y}\right)^2\right)^{\frac{3}{2}}. \tag{14}$$

Efficient implementation of QST

We recognize that the reflector-oriented coordinate system is dependent on the eigenvectors, which are spatially variant and therefore not realizable with convolutions directly. Bakker (2002) propose the following solution to this problem.

The expressions for obtaining curvatures using the QST (eq. 10 and 11):

$$\kappa_1 = \frac{\overline{v g_u g_v}}{\overline{v^2 g_u^2}}, \qquad \kappa_2 = \frac{\overline{w g_u g_w}}{\overline{w^2 g_u^2}},$$

are dependent on the local reflector-oriented coordinate system (u, v, w) and the gradients along these axes (g_u, g_v, g_w). The problem with this is that the reflector-oriented coordinate system is spatially variant. This means that we cannot compute the gradient vector field using global convolutions directly. Following the derivations in Bakker (2002), equation 10 and 11 can be rewritten to be independent of the gradients (g_u, g_v, g_w), given that the unit vectors for this local coordinate system are known:

$$\begin{aligned} \mathbf{u} &= [x_u, y_u, t_u]^T, \\ \mathbf{v} &= [x_v, y_v, t_v]^T, \\ \mathbf{w} &= [x_w, y_w, t_w]^T. \end{aligned} \tag{15}$$

The curvatures are then given as:

$$\kappa_1 = \frac{\overline{v g_u g_v}}{\overline{v^2 g_u^2}} = \frac{\mathbf{v}^T \cdot \left(\overline{\mathbf{xT}} \circ \mathbf{u}\mathbf{v}^T\right)}{\mathbf{v}^T \cdot \left(\overline{\mathbf{XT}} \circ \mathbf{u}\mathbf{u}^T\right) \cdot \mathbf{v}}, \tag{16}$$

$$\kappa_2 = \frac{\overline{w g_u g_w}}{\overline{w^2 g_u^2}} = \frac{\mathbf{w}^T \cdot \left(\overline{\mathbf{xT}} \circ \mathbf{u}\mathbf{w}^T\right)}{\mathbf{w}^T \cdot \left(\overline{\mathbf{XT}} \circ \mathbf{u}\mathbf{u}^T\right) \cdot \mathbf{w}}, \tag{17}$$

where $\overline{\mathbf{xT}}$ is a three dimensional matrix with elements given as:

$$\overline{\mathbf{xT}}_{ikl} = \overline{i g_k g_l}, \quad i, k, l \in \{x, y, z\}, \tag{18}$$

and $\overline{\mathbf{XT}}$ is a four dimensional matrix with elements given as:

$$\overline{\mathbf{XT}}_{ijkl} = \overline{ijg_kg_l}, \;\; i,j,k,l \in \{x,y,z\}. \tag{19}$$

These matrix elements ($\overline{\mathbf{xT}}_{ikl}$ and $\overline{\mathbf{XT}}_{ijkl}$) are computed by convolving the components of the gradient vector field with respect to the global x, y, t coordinate system (eq. 3) with a smoothing window (bar notation). To give an example we consider $\overline{\mathbf{XT}}_{1223}$ where $i = x$, $j = y$, $k = y$ and $l = t$. This element is computed as:

$$\overline{\mathbf{XT}}_{1223} = \overline{xyg_yg_t} = G(\mathbf{x}, \sigma_T)x(\mathbf{x})y(\mathbf{x}) * g_y(\mathbf{x})g_t(\mathbf{x}) \tag{20}$$

where $G(\mathbf{x}, \sigma_T)x(\mathbf{x})y(\mathbf{x})$ is an element-wise multiplication between the smoothing window function and windows containing the spatial coordinates for x and y (symmetric around zero). Since the window kernels are separable they can be computed using 1D convolutions, which offers a significant speed up. The $\circ$ operator performs a sum over to the first two indices after an element wise multiplication. To explain this, we consider:

$$\mathbf{A} = \mathbf{B} \circ \mathbf{C}. \tag{21}$$

If $\mathbf{B}$ is four-dimensional and $\mathbf{C}$ is two-dimensional, the elements in $\mathbf{A}$ are given as:

$$\mathbf{A}_{kl} = \sum_{i,j} \mathbf{B}_{ijkl}\mathbf{C}_{ij}. \tag{22}$$

If $\mathbf{B}$ is three-dimensional and $\mathbf{C}$ is two-dimensional, the elements in $\mathbf{A}$ are given as:

$$\mathbf{a}_{k} = \sum_{i,j} \mathbf{B}_{ijk}\mathbf{C}_{ij}. \tag{23}$$

REFERENCES

Bakker, P., 2002, Image structure analysis for seismic interpretation: Phd thesis, Technische Universiteit Delft.

Bigun, J., and G. H. Granlund, 1987, Optimal Orientation Detection of Linear Symmetry: Proceedings of the IEEE First International Conference on Computer Vision, **54**, 433–438.

Kempen, G. M. P. V., N. van den Brink, L. J. van Vliet, M. V. Ginkel, P. W. Verbeek, and H. Blonk, 1999, The application of a Local Dimensionality Estimator to the analysis of 3-D microscopic network structures: Proceedings of the 11th Scandinavian Conference on Image Analysis, Kangerlussuaq, Greenland, 447–455.

Knutsson, H., 1989, Representing local structure using tensors: Proceedings of 6th Scandinavian Conference on Image Analysis, Oulu, Finland, Linköping University Electronic Press, 244–251.

Randen, T., E. Monsen, C. Signer, A. Abrahamsen, J. O. Hansen, T. Sæter, J. Schlaf, and L. Sonneland, 2000, Three-Dimensional Texture Attributes for Seismic Data Analysis: Presented at the 70th SEG Annual Meeting, Expanded Abstracts.

van de Weijer, J., L. J. van Vliet, P. W. Verbeek, and M. van Ginkel, 2001, Curvature Estimation in Oriented Patterns Using Curvilinear Models Applied to Gradient Vector Fields: IEEE Transactions on Pattern Analysis and Machine Intelligence, **23**, 1035–1042.