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1. You have 2 parents, 4 grandparents, 8 great-grandparents, and so forth. If all of your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations, counting your parent's generation as number 1? Hint: What kind of sequence is this? Use the sum formula for that sequence to solve the problem. Show your work. (3 marks).

SOLUTION 1.

The number of ancestors of $(n+1)$-th generation is twice bigger than the number of ancestors of n-th generation since every relative have/had 2 parents and all of them are different. This is a geometric progression with common factor $q = 2$ and initial value $b_1 = 2$ (father and mother). Then $b_n = b_1 q^{n-1} = 2\times 2^{n-1} = 2^n$. Then the number of all ancestors for the past 40 genrations is simply the sum

$$S_{40} = b_1 + b_2 + \cdots + b_{40} = 2^1 + 2^2 + \cdots + 2^{40} = \frac{2^{41} - 1^{41}}{2 - 1} = 2^{41} - 1 = 2199023255551.$$

2. Give a proof by contradiction to show that there does not exist a constant c such that for all integers n≥1, $(n+1)^2 - n^2 < c$ (3 marks)

SOLUTION 2.

Assume there exists c such that $(n+1)^2 - n^2 < c$ for all integers $n \geq 1$. If $c < 0$, then substituting $n = 1$ into inequality makes it false since

$$(1+1)^2 - 1^2 = 2^2 - 1^2 = 3 > 0 > c.$$

Alternatively, if $c \geq 0$ then substituting an integer $\lceil c \rceil$ into inequality makes it false too since

$$\begin{aligned}(\lceil c \rceil + 1)^2 - \lceil c \rceil^2 = (\lceil c \rceil^2 + 2\lceil c \rceil + 1) - \lceil c \rceil^2 = 2\lceil c \rceil + 1 = \lceil c \rceil + (\lceil c \rceil - c) + c + 1 \\ \geq 0 + 0 + c + 1 > c.\end{aligned}$$

In both cases there is a counterexample. Hence, such a constant c doesn't exist.

3. Given the fact that ⌈x⌉ < x + 1, give a proof by contradiction that if n items are placed in m boxes then at least one box must contain at least ceiling(n/m) items. (3 marks)

SOLUTION 3.

Assume there are positive integers n, m such that all of m boxes contains less than $\lceil n/m \rceil$ items. Number of items in a box is an integer, and the biggest possible integer less than $\lceil n/m \rceil$ is one less: $\lceil n/m \rceil - 1$. Then the number of all items, n, equaling the sum of items in each of m boxes, is no greater than $m(\lceil n/m \rceil - 1) < m(n/m + 1 - 1) = n$. This is a contradiction, hence the assumption was wrong and at least one box must contain at least $\lceil n/m \rceil$ items.

4. Use mathematical induction to prove the following statement is true for all integers n≥2. Clearly identify the base case, the induction hypothesis and the induction step you are using in your proof. (3 marks)

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

SOLUTION 4.

The base case is $n = 2$. The statement is true in the base case:

$$\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{3}{4} = \frac{2+1}{2\times 2} = \frac{n+1}{2n}.$$

The induction hypothesis is that for some integer $n = k \geq 2$ the statement is true, i.e.

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}.$$

The induction step is to prove the statement for $n = k + 1$:

$$\begin{aligned}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k} \times \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{(k+1)^2 - 1}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)} = \frac{n+1}{2n}.\end{aligned}$$

Then, by mathematical induction, the statement is true for all integers $n \geq 2$.

5. Use the Euclidian Algorithm (outlined in Epp pages 220 – 224) to hand-calculate the greatest common denominator (gcd) of 832 and 10933 (2 marks)

SOLUTION 5.

$$\begin{aligned}&\gcd(832{,}10933) = \gcd\big(832, (832\times 13 + 117)\big) = \\ &\gcd(832{,}117) = \gcd\big((117\times 7 + 13), 117\big) = \\ &\gcd(13{,}117) = \gcd\big(13, (13\times 9)\big) = 13.\end{aligned}$$

The greatest common denominator of 832 and 10933 is 13.

6. Prove, by contraposition, that if (n(n-1) + 3(n-1) - 2) is even then n is odd. Assume only the definition of odd/even. (3 marks)

SOLUTION 6.

The contrapositive of the statement "if $(n(n-1)+3(n-1)-2)$ is even then n is odd" is the statement "if n is not odd then $(n(n-1)+3(n-1)-2)$ is not even". These two statements are equivalent. Let's simplify the contrapositive: "if n is even then (n^2+2n-5) is odd" (Integer is either even or odd, so if it is not even, then it is odd, and if it is not odd, then it is even). Integer n is even if there is such k that $n=2k$, and it is odd if there is such k that $n=2k+1$. Hence, if n is even, then there is k such that $n=2k$, then $(n^2+2n-5)=((2k)^2+2(2k)-5)=(4k^2+4k-5)=2(2k^2+2k-3)+1=2m+1$ for $m=(2k^2+2k-3)$, then (n^2+2n-5) is odd. Thus, we proved the statement.

7. Use mathematical induction to prove that $\sum^{n}_{i=1}$ (5i-4)=n(5n−3)/2. Clearly identify the base case, the induction Hypothesis and the induction step you are using in your proof. (3 marks)

SOLUTION 7.

The base case is $n=1$. Really,

$$\sum_{i=1}^{1}(5i-4)=(5\times 1-4)=1=\frac{1(5\times 1-3)}{2}=\frac{n(5n-3)}{2}.$$

The induction hypothesis is that for some integer $n=k\geq 1$ the statement is true, i.e.

$$\sum_{i=1}^{k}(5i-4)=\frac{k(5k-3)}{2}.$$

The induction step is to prove the statement for next integer, which is $n=k+1$:

$$\begin{aligned}\sum_{i=1}^{n+1}(5i-4)=\sum_{i=1}^{n}(5i-4)+(5n-4)=\sum_{i=1}^{k}(5i-4)+(5(k+1)-4)=\frac{k(5k-3)}{2}+(5k+1)\\ =\frac{5k^2-3k}{2}+\frac{10k+2}{2}=\frac{5k^2+7k+2}{2}=\frac{(k+1)(5k+2)}{2}\\ =\frac{(k+1)(5(k+1)-3)}{2}=\frac{n(5n-3)}{2}.\end{aligned}$$

Then, by mathematical induction, the statement is true for all integers $n\geq 1$.