12/11/23

6.6 Homework

1. $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ $\Big\}$ $A^TA\vec{x} = A^Tb$

$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 6 & 14 & 11 \end{array}\right]$

$X^TX = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}, \quad X^Ty = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$ $\Big)$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (X^TX)^{-1}X^Ty \Rightarrow (X^TX)^{-1} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix}$

$\Rightarrow \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 11 \end{bmatrix} = \begin{bmatrix} .9 \\ .4 \end{bmatrix}$

$\Rightarrow y = .9 + .4x$

3. $X^TX = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}, \quad X^Ty = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (X^TX)^{-1}X^Ty$

$= \frac{1}{20}\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} .1 \\ 1.3 \end{bmatrix}$

$\Rightarrow y = .1 + 1.3x$

5. $\hat{x} = (A^TA)^{-1}A^Tb$

$y = \beta_0 + \beta_1 x \Rightarrow$ least-squares line

$\Rightarrow (x_1, y_1) \ldots, (x_n, y_n) \Rightarrow y_1 = \beta_0 + \beta_1 x_1, \ldots$
$y_n = \beta_0 + \beta_1 x_n$

$\Rightarrow X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$

$\Rightarrow X\beta = y \Rightarrow$

THM 14. $\Rightarrow X^TX$ = invertible if cols of X = linearly independent, so $X\beta = y$ has only one least-squares solution

$$col_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad col_2 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

col_1 and col_2 are independent if they not multiples of each other, only possible if at least 2 entries of col_2 are different

8a.

$$y = .5132x - .03348x^2 + .001016x^3$$