Homework 02

MATH316

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Section 1.7

Tip

Assigned: 4, 5

1.7.4

Problem:

Give an example of two 2×2 matrices A and B such that $AB \neq BA$.

Solution:

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Calculate AB:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1 \cdot 0 + 2 \cdot 1) & (1 \cdot 1 + 2 \cdot 0) \\ (0 \cdot 0 + 1 \cdot 1) & (0 \cdot 1 + 1 \cdot 0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Calculate BA:

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (0 \cdot 1 + 1 \cdot 0) & (0 \cdot 2 + 1 \cdot 1) \\ (1 \cdot 1 + 0 \cdot 0) & (1 \cdot 2 + 0 \cdot 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Compare:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Final Answer:

$$AB \neq BA$$

1.7.5

Problem:

Give an example of two 2×2 matrices A and B such that $AB = BA$.

Solution:

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Calculate AB:

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} (1 \cdot 2 + 0 \cdot 0) & (1 \cdot 0 + 0 \cdot 3) \\ (0 \cdot 2 + 1 \cdot 0) & (0 \cdot 0 + 1 \cdot 3) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Calculate BA:

$$BA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (2 \cdot 1 + 0 \cdot 0) & (2 \cdot 0 + 0 \cdot 1) \\ (0 \cdot 1 + 3 \cdot 0) & (0 \cdot 0 + 3 \cdot 1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Compare:

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Final Answer:

$$AB = BA$$

Section 1.8

Assigned: 2, 5

1.8.2

Problem:

Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$$

or show that the inverse does not exist.

Solution:

To find the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse exists if and only if $\det(A) \neq 0$, and the inverse is given by the formula:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 1: Calculate the determinant of A
The determinant of matrix A is:

$$\det(A) = (5)(-3) - (0)(0) = -15$$

Since $\det(A) = -15$, which is non-zero, the matrix is invertible.

Step 2: Apply the inverse formula
Using the formula for the inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{-15} \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

Multiply the scalar $\frac{1}{-15}$ with each element of the matrix:

$$A^{-1} = \begin{bmatrix} \frac{-3}{-15} & \frac{0}{-15} \\ \frac{0}{-15} & \frac{5}{-15} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{-1}{3} \end{bmatrix}$$

Final Answer:

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{-1}{3} \end{bmatrix}$$

1.8.5

Problem:

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

using row reduction, or show that the inverse does not exist.

Solution:

We will augment the matrix A with the identity matrix and perform row operations to reduce A to the identity matrix. The result on the right will be the inverse of A, if it exists.
Start with the augmented matrix:

$$[A|I] = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

Step 1: Make the pivot in the first row, first column, a 1.
Since the first element is already a 1, no changes are needed.

Step 2: Eliminate the elements below the pivot.

- Add row 1 to row 2 to eliminate the -1 in the second row, first column:
$$R_2 \to R_2 + R_1$$
Result:
$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$$
- Subtract row 1 from row 3 to eliminate the 1 in the third row, first column:
$$R_3 \to R_3 - R_1$$
Result:
$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 5 & 5 & -1 & 0 & 1 \end{bmatrix}$$

Step 3: Make the pivot in the second row, second column, a 1.

- Divide row 2 by -1 to make the pivot a 1:

$$R_2 \to \frac{R_2}{-1}$$

Result:

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 5 & 5 & -1 & 0 & 1 \end{bmatrix}$$

Step 4: Eliminate the elements above and below the pivot in the second column.

- Add 2 times row 2 to row 1 to eliminate the -2 in the first row, second column:

$$R_1 \to R_1 + 2R_2$$

Result:

$$\begin{bmatrix} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 5 & 5 & -1 & 0 & 1 \end{bmatrix}$$

- Subtract 5 times row 2 from row 3 to eliminate the 5 in the third row, second column:

$$R_3 \to R_3 - 5R_2$$

Result:

$$\begin{bmatrix} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 5 & 1 \end{bmatrix}$$

Step 5: Check if the matrix is invertible.
At this point, the third row of the matrix has become zero in the left side of the augmented matrix, which indicates that the matrix does not have full rank. Since we cannot get a 1 in the third row, third column, the matrix is not invertible.

Final Answer:

The matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ is not invertible because the third row becomes zero during the row-reduction process.

Section 1.9

Assigned: 1, 5, 17

1.9.1

Problem:

Given the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

compute the determinant by hand and determine whether or not the matrix is invertible.

Solution:

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\det(A) = ad - bc$$

For the given matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, we have:

- $a = 2$
- $b = 1$
- $c = 2$
- $d = 2$

Calculate the determinant:

$$\det(A) = (2)(2) - (1)(2) = 4 - 2 = 2$$

Since the determinant is not zero, A is invertible.

Final Answer:

- The determinant of A is 2.
- The matrix is invertible.

1.9.5

Problem:

Given the matrix

$$A = \begin{bmatrix} -3 & 1 & 0 & 5 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & -7 & 11 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

compute the determinant by hand and determine whether or not the matrix is invertible.

Solution:

Since the matrix A is upper triangular, the determinant is simply the product of the diagonal elements.
The diagonal elements of the matrix are -3, 2, -7, and 6. So, the determinant is:

$$\det(A) = (-3) \times (2) \times (-7) \times (6)$$

Let's compute it step by step:

$$\det(A) = (-3) \times (2) = -6$$

$$\det(A) = -6 \times (-7) = 42$$

$$\det(A) = 42 \times (6) = 252$$

Since the determinant is not zero, A is invertible.

Final Answer:

- The determinant of A is 252.
- The matrix is invertible.

1.9.17

Problem:

Suppose that A^2 is not invertible. Can you determine if A is invertible or not? Explain.

Solution:

If A^2 is not invertible, we can conclude that A is also not invertible.

Explanation:

- A matrix A is invertible if and only if its determinant is non-zero, meaning $\det(A) \neq 0$.
- The determinant of A^2 is related to the determinant of A by the formula:
$$\det(A^2) = (\det(A))^2$$
- If A^2 is not invertible, then $\det(A^2) = 0$. From the equation $\det(A^2) = (\det(A))^2$, the only way for this to be true is if $\det(A) = 0$.

Thus, if $\det(A^2) = 0$, it must be that $\det(A) = 0$, meaning A is not invertible.

Final Answer:

If A^2 is not invertible, then A is also not invertible.

Section 1.10

Assigned: 1, 5, 9, 11

1.10.1

Problem:

Given the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

compute the eigenvalues and any corresponding real eigenvectors.

Solution:

Step 1: Find the Eigenvalues
The eigenvalues of a matrix are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

where λ represents the eigenvalues and I is the identity matrix.
The matrix $A - \lambda I$ is:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

Calculate the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = (5-\lambda)(3-\lambda) - (1)(0) = (5-\lambda)(3-\lambda)$$

The characteristic equation is:

$$(5-\lambda)(3-\lambda) = 0$$

Solving this equation gives the eigenvalues:

$$\lambda_1 = 5 \quad \text{and} \quad \lambda_2 = 3$$

Step 2: Find the Eigenvectors
Find the eigenvectors corresponding to each eigenvalue.

Eigenvalue $\lambda_1 = 5$:
Substitute $\lambda_1 = 5$ into the matrix $A - \lambda I$:

$$A - 5I = \begin{bmatrix} 5-5 & 1 \\ 0 & 3-5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

To find the eigenvector corresponding to $\lambda_1 = 5$, solve the equation:

$$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the system of equations:

$$x_2 = 0$$

There is no condition on x_1, so x_1 can be any real number. Thus, the eigenvector corresponding to $\lambda_1 = 5$ is any scalar multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Eigenvalue $\lambda_2 = 3$:
Substitute $\lambda_2 = 3$ into the matrix $A - \lambda I$:

$$A - 3I = \begin{bmatrix} 5-3 & 1 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

To find the eigenvector corresponding to $\lambda_2 = 3$, solve the equation:

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the system of equations:

$$2x_1 + x_2 = 0$$

From this, we get $x_2 = -2x_1$. Thus, the eigenvector corresponding to $\lambda_2 = 3$ is any scalar multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Final Answer:

- The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 3$.
- The eigenvector corresponding to $\lambda_1 = 5$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- The eigenvector corresponding to $\lambda_2 = 3$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

1.10.5

Problem:

Given the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

compute the eigenvalues and any corresponding real eigenvectors.

Solution:

Step 1: Find the Eigenvalues
The eigenvalues of a matrix are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

where λ represents the eigenvalues and I is the identity matrix.
The matrix $A - \lambda I$ is:

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Now, calculate the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = (2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$

The determinant of the 2×2 matrix is:

$$(2-\lambda)(2-\lambda) - (1)(0) = (2-\lambda)^2$$

Thus, the characteristic equation becomes:

$$(2-\lambda)^3 = 0$$

Solving this equation gives the eigenvalue:

$$\lambda_1 = 2$$

Step 2: Find the Eigenvector
Substitute $\lambda_1 = 2$ into the matrix $A - \lambda I$:

$$A - 2I = \begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

To find the eigenvector corresponding to $\lambda_1 = 2$, solve the equation:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the system of equations:

$$x_2 = 0$$

$$x_3 = 0$$

There is no condition on x_1, so x_1 can be any real number. Thus, the eigenvector corresponding to $\lambda_1 = 2$ is any scalar multiple of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Final Answer:

- The eigenvalue of A is $\lambda_1 = 2$.
- The eigenvector corresponding to $\lambda_1 = 2$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.