Homework 1

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1.2.8

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 5 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{aligned} x_1 - 3x_4 &= 5 \\ x_3 - 2x_4 &= 4 \end{aligned} \rightarrow \begin{aligned} x_1 &= 5 + 3x_4 \\ x_2 &= x_2 \\ x_3 &= 4 + 2x_4 \\ x_4 &= x_4 \end{aligned}$$

$$\boxed{\vec{x} = \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} x_4}$$

The system is consistent.

1.2.9

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{aligned} x_1 - 2x_2 + 4x_4 &= -1 \\ x_3 + 3x_4 &= 2 \\ x_5 &= -5 \end{aligned}$$

$$\begin{aligned} x_1 &= -1 + 2x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= 2 - 3x_4 \\ x_4 &= x_4 \\ x_5 &= -5 \end{aligned}$$

$$\boxed{\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ -5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} x_4}$$

The system is consistent.

1.2.11

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right] \rightarrow$$

No further operations required. Row two can be interpreted as $0=3$ which is false. This means that this system is inconsistent.

1.2.17

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & h & 6 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & -h & 0 \end{array}\right] \rightarrow \begin{array}{r} x_1 + hx_2 = 3 \\ -hx_2 = 0 \end{array}$$

h must equal 0 for the system to be consistent. If $h=0$, then the solution set would be: $\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$

* infinite solutions

1.3.1

$$A=\begin{bmatrix} 1 & -3 & 2 \\ -4 & 1 & 0 \end{bmatrix}, \quad x=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\boxed{\text{Ax is not defined because A has three columns but x has only two rows.}}$

Ax is not defined because A has three columns but x has only two rows.

1.3.9

$b=[0\ 7\ 4]^T$, $a_1=[-1\ 2\ 1]^T$, $a_2=[3\ 1\ 1]^T$, $a_3=[1\ 5\ 3]^T$

$$A=\begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \quad b=\begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix}$$

If b is a linear combination of a_1, a_2, a_3, then: $\vec{b}=a_1x_1+a_2x_2+a_3x_3$, which can be represented as an augmented matrix.

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 2 & 1 & 5 & 7 \\ 1 & 1 & 3 & 4 \end{array}\right] \begin{array}{c} R_3 \leftrightarrow R_1 \\ 2R_3+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 7 & 7 & 7 \\ -1 & 3 & 1 & 0 \end{array}\right]$$

R_1+R_3

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 7 & 7 & 7 \\ 0 & 4 & 4 & 4 \end{array}\right] \begin{array}{c} \frac{1}{7}R_2 \\ \frac{1}{4}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] R_3-R_2 \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-R_2+R_1 \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \vec{x}=\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}+\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}x_3$$

Yes b is a linear combination of a_1, a_2, and a_3. There are more than one set of weights that will work since x_3 is a free variable.

1.3.11

If b is a linear combination of A then $\vec{b} = x_1 a_1 + x_2 a_2 + x_3 a_3$, which can be represented as an augmented matrix.

$$\left[\begin{array}{ccc|c} 4 & 5 & -1 & 13 \\ 3 & 1 & 2 & -4 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & \frac{5}{4} & -\frac{1}{4} & \frac{13}{4} \\ 3 & 1 & 2 & -4 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & \frac{5}{4} & -\frac{1}{4} & \frac{13}{4} \\ 0 & -\frac{11}{4} & \frac{11}{4} & -\frac{55}{4} \end{array}\right]$$

$$\xrightarrow{\frac{1}{-\frac{11}{4}}R_2} \left[\begin{array}{ccc|c} 1 & \frac{5}{4} & -\frac{1}{4} & \frac{13}{4} \\ 0 & 1 & -1 & 5 \end{array}\right] \xrightarrow{R_1 - \frac{5}{4}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 5 \end{array}\right]$$

$$\begin{aligned} x_1 + x_3 &= -3 \\ x_2 - x_3 &= 5 \\ x_3 &= x_3 \end{aligned} \rightarrow \vec{x} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

Vector b is a linear combination of the columns of A. There are multiple weights that can express b as a combination of the columns of A.

1.3.17

$$A\vec{x} = 0 \quad \begin{bmatrix} 4 & 5 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In augmented form: $\left[\begin{array}{ccc|c} 4 & 5 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$

In R.R.E.F form: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \begin{aligned} x_1 + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \\ x_3 &= x_3 \end{aligned}$$

In parametric form: $\boxed{\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3}$

1.4.1

$A = \begin{bmatrix} 4 & -1 \\ 1 & -4 \end{bmatrix}$ The equation $A\vec{x} = b$ is consistent for every choice of $b \in \mathbb{R}^m$ if the matrix A has full row rank. We can check this by finding the determinant and checking that its non-zero. The invertible matrix theorem states that if $\det(A) \neq 0$ then A is invertible which also means the matrix spans the m dimension.

$\det(A) = (4)(-4) - (1)(-1) = -16 + 1 = -15$
$-15 \neq 0$ so $A\vec{x} = b$ is consistent for every choice of $b \in \mathbb{R}^2$

1.4.2

$A = \begin{bmatrix} 4 & -1 \\ -12 & 3 \end{bmatrix}$ this follows the same reasoning for the problem above.

$\det(A) = (4)(3) - (-12)(-1) = 0$

Since the $\det(A) = 0$, $A\vec{x} = b$ is not consistent for every $b \in \mathbb{R}^2$

1.4.9

No, it is not possible for $A\vec{x}=b$ to be consistent for every $b \in \mathbb{R}^m$ if $m>n$ because to be consistent for every $b \in \mathbb{R}^m$ A needs to have full row rank but when $m>n$ this is not possible because the max rank of A will be equal to to the smaller of m and n so when $m>n$, A cannot have rank m meaning it does not have full rank.

1.4.14

$b=\begin{bmatrix}1\\-11\\14\end{bmatrix}$ $A=\begin{bmatrix}2&1\\-1&3\\4&-2\end{bmatrix}$ If b is in the span of A then the following equation holds true $A\vec{x}=b$ which can be represented as

$$\left[\begin{array}{cc|c}2&1&1\\-1&3&-11\\4&-2&14\end{array}\right] \rightarrow \text{R.R.E.F.} \rightarrow \left[\begin{array}{cc|c}1&0&2\\0&1&-3\\0&0&0\end{array}\right] = \begin{array}{l}x_1=2\\x_2=-3\end{array}$$

So, yes b lies in the span of A. The weights that allow b to be written as a linear combination of A are $x_1=2$ and $x_2=-3$

$$b=2\begin{bmatrix}2\\-1\\4\end{bmatrix}-3\begin{bmatrix}1\\3\\-2\end{bmatrix}$$

1.5.3

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

A has full row rank because the rows are clearly linearly independent of each other because neither row is a scalar multiple of one another. Since A as full row rank the system $A\vec{x} = b$ will be consistent for every $b \in \mathbb{R}^2$.

1.5.7

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A$

This is a 4x3 matrix and because there are more equations than variables A cannot have full row rank meaning A can not span $\mathbb{R}^4$ but instead spans $\mathbb{R}^3$ because its row rank is 3.

1.5.13

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

If b lies in the span of A then the following there are some weights that make $Ax = b$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \rightarrow 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

So, yes b lies in the span of A and can be made using the weights $\langle x_1, x_2, x_3 \rangle = \langle 6, -2, 0 \rangle$.

1.5.21 $Ax=0$ can be represented as:

$$A=\left[\begin{array}{ccc|c}1&-1&2&0\\4&-2&6&0\\-7&3&-10&0\end{array}\right]\Rightarrow RREF\rightarrow\left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

$$= \begin{aligned}x_1+x_3&=0\\x_2-x_3&=0\\x_3&=x_3\end{aligned} \rightarrow \text{parametric form} \rightarrow \boxed{\vec{x}=\begin{bmatrix}-1\\1\\1\end{bmatrix}x_3}$$

1.5.23

$$A=\begin{bmatrix}1&2&0&1\\3&1&5&-7\\4&-1&10&-13\end{bmatrix}\quad b=\begin{bmatrix}1\\3\\5\end{bmatrix}$$ $Ax=b$ can be represented:

$$\left[\begin{array}{cccc|c}1&2&0&1&1\\3&1&5&-7&3\\4&-1&10&-13&5\end{array}\right]\Rightarrow RREF\Rightarrow\left[\begin{array}{cccc|c}1&0&0&-5&-1\\0&1&0&3&1\\0&0&1&1&1\end{array}\right]$$

$$= \begin{aligned}x_1-5x_4&=-1\\x_2+3x_4&=1\\x_3+x_4&=1\\x_4&=x_4\end{aligned} \rightarrow \text{parametric form } (\vec{x}=\vec{x}_p+\vec{x}_n): \boxed{\vec{x}=\begin{bmatrix}-1\\1\\1\\\ \end{bmatrix}+\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}+}$$

1.5.26 *same logic as above

$$\left[\begin{array}{ccc|c}1&-1&2&5\\4&-2&6&16\\-7&3&-10&-27\end{array}\right]\rightarrow RREF\Rightarrow\left[\begin{array}{ccc|c}1&0&1&3\\0&1&-1&-2\\0&0&0&0\end{array}\right]$$

$$= \begin{aligned}x_1+x_3&=3\\x_2-x_3&=-2\\x_3&=x_3\end{aligned} \rightarrow \text{Parametric form } \vec{x}=\vec{x}_p+\vec{x}_n: \boxed{\vec{x}=\begin{bmatrix}3\\-2\\0\end{bmatrix}+\begin{bmatrix}-1\\1\\1\end{bmatrix}+}$$

1.6.1

$S = \{\vec{v}_1, \vec{v}_2\}$, $\vec{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -9 \\ 6 \end{bmatrix}$

$S = \begin{bmatrix} 3 & -9 \\ -2 & 6 \end{bmatrix}$

v_1 an v_2 are not linearly independent because v_2 can me made by scaling v_1 by $-3 \rightarrow \underbrace{\begin{bmatrix} 3 \\ -2 \end{bmatrix}}_{v_1}(-3) = \underbrace{\begin{bmatrix} -9 \\ 6 \end{bmatrix}}_{v_3}$

1.6.5

$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

$S = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix}$ $\det(S) = (-1)\begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= (-1)(3-5) - 3(6-5) + 1(2-1)$

$= (-1)(-2) - 3(1) + 1(1)$

$= 2 - 3 + 1 = 0$

Since $\det(S) = 0$ the vectors v_1, v_2, v_3 are linearly independent by the invertible matrix theorem.

1.6.10

No, it is not possible for three vectors to span $\mathbb{R}^5$ because to span $\mathbb{R}^n$ you need at least n linearly independent vectors. So to span $\mathbb{R}^5$ you need 5 linearly independent vectors at the minimum.

1.6.12

No, it is not possible for S to be linearly independent if it spans $\mathbb{R}^3$ and contains four vectors because at most three vectors can be linearly independent in $\mathbb{R}^3$ (think identity matrix) making the fourth vector dependent.

1.6.14

when col. of A gauranteed not to span $\mathbb{R}^m$

- If $n < m$: fewer col. than the dimension of the space.

when col. of A gauranteed to be lin. dependent.

- If $n > m$: more col. rows means at least one col is a lin. combo. of the others because there are more vectors than the dimension of the space making the col.'s lin. dependent.

  (Think how I_n spans $\mathbb{R}^n$)

1.6.15

A set is lin. dependent if at least one of the vectors in the set is a scalar multiple of another and since any vector scaled by zero is the zero vector and if the zero vector is in the set the set is lin. dependent.

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is lin. dependent if there are weights not all zero such that:

$c_1\vec{v}_1 + c_2\vec{v}_2, \ldots + c_n\vec{v}_n = \vec{0}$

Let set S contain only the zero vector $\{\vec{0}\}$ and let weight c_1 be any real number:

$c_1\vec{0} = 0$ so any set containing $\vec{0}$ is dependent because you can let all the weights be 0 except the weight for $\vec{0}$ and you will get 0.