~~Homework #1.7: 4, 5, 7~~

#4 Give an example of 2x2 matrices A & B for which $AB \neq BA$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\Downarrow$

$AB \neq BA$

Matrix multiplication is in general not communitive since it's a non-Abelian group.

1 points for trying
2 points for picking A,B
2 points for verifying

#5) Give an example of 2x2 matrices A and B for which $AB = BA$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $BA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Downarrow$

$AB = AB$.

1 points for trying
2 points for picking A,B
2 points for verifying

It's communitive when they are diagonalizable

if You let either A or B be I, it will satisfy the equation.

#7) $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. if possible, find a matrix B s.t $BA = I_2$ — 2X2 identity matrix.
if B exists, determine whether $BA = AB$.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

To find B s.t $BA = I_2$ is the same as saying find A^{-1}.

so $B = A^{-1}$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{5} \end{array}\right] \Rightarrow B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

Since $A \cdot A^{-1} = A^{-1} \cdot A = I$, $BA = AB$

1 point for trying
2 points for getting the correct A inverse
2 points for verifying AB = BA

Homework #1.8: 2, 5, 16.

#2) find inverse of $\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ or show does not exist.

1 point for trying
2 points for right method of finding inverse.
(you can use the formula for this one too)
2 points for the correct answer

$$\left[\begin{array}{cc|cc} 5 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

inverse

#5) find inverse of $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ or show doesn't exist.

1 point for trying
2 points for the right approach
2 points for claiming not invertible

$$\det\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} = 1\cdot\det\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} + 1\cdot\det\begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$$

$= 1\cdot(-5) + 1\cdot 5 = 0 \Rightarrow$ not invertible.

#16) Let θ be any real number. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. compute A^T and A^TA. what do you observe?

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we observe that $A^TA = I$, meaning $A^T = A^{-1}$. then we know A is orthogonal

1 point for trying
1 point for A transpose
1 point for calculating ATA the right way
1 point for the correctness of ATA
1 point for the relationship observation

Homework #1.9: 1, 5, 8, 17

#1 compute determinant. State whether or not the matrix is invertible.

$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$. $\det A = 2 \neq 0$, so it's invertible

1 point for trying
2 points for correct approach
1 point for correct answer
1 point for stating invertible

#5 $A = \begin{bmatrix} -3 & 1 & 0 & 5 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & -7 & 11 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

1 point for trying
2 points for correct method
2 points for correct answer
1 point for stating it's invertible

$\det A = 6 \cdot \det \begin{bmatrix} -3 & 1 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & -7 \end{bmatrix} = 6 \cdot (-7) \cdot \det \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$

$= 6 \cdot (-7) \cdot (-6) = 252 \neq 0$, so it's invertible.

#8 For which value(s) of h is $\begin{bmatrix} 1 & 2 \\ -3 & h \end{bmatrix}$ invertible? explain in at least 2 different ways.

1 point for trying

a: $\det \begin{bmatrix} 1 & 2 \\ -3 & h \end{bmatrix} = h+6 \Rightarrow$ to make $h+6 \neq 0$, $\boxed{h \neq -6}$ determinant $\neq 0$

3 points for 2 correct ways

1 point for the right answer

b: $\begin{bmatrix} 1 & 2 \\ -3 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 6+h \end{bmatrix} \Rightarrow 6+h \neq 0$, $\boxed{h \neq -6}$. pivots in every row/column

#17 suppose A^2 is not invertible. Can you determin if A is invertible or not? explain.

$A^2 = A \cdot A$ if A^2 is not invertible, we know $\det(A^2)$

$= \det(A \cdot A)$

$= (\det A) \cdot (\det A) = 0$

1 point for trying
1 point for claiming we can determine it
3 points for the correct approach

Then we have $\det A = 0$

Therefore $\boxed{\text{A is not invertible}}$.

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Homework 1.10: 1, 5, 9, 11

#1) compute eigenvalues and corresponding real eigenvectors.

$A=\begin{bmatrix}5&1\\0&3\end{bmatrix}$ $A-\lambda I=\begin{bmatrix}5-\lambda&1\\0&3-\lambda\end{bmatrix}$

1 point for trying
2 points for correct eigenvalues
2 points for correct eigenvectors

$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda&1\\0&3-\lambda\end{bmatrix}=(5-\lambda)(3-\lambda)=0$

$\Downarrow$

$\lambda=5,3$

when $\boxed{\lambda=5}$. $A-\lambda I=\begin{bmatrix}0&1\\0&-2\end{bmatrix}$.

since $(A-\lambda I)\cdot\vec{v}=0$, solve $\left[\begin{array}{cc|c}0&1&0\\0&-2&0\end{array}\right]\sim\left[\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right]$

$\begin{cases}v_1=v_1\\v_2=0\end{cases}\Rightarrow v=v_1\boxed{\begin{bmatrix}1\\0\end{bmatrix}}\to$ eigenvector

when $\boxed{\lambda=3}$. $A-\lambda I=\begin{bmatrix}2&1\\0&0\end{bmatrix}$

$\left[\begin{array}{cc|c}2&1&0\\0&0&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&\frac{1}{2}&0\\0&0&0\end{array}\right]\Rightarrow\begin{cases}v_1=-\frac{1}{2}v_2\\v_2=v_2\end{cases}\Rightarrow\vec{v}=v_2\boxed{\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}}$

$\downarrow$
eigenvector

#5) $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$ $A-\lambda I=\begin{bmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{bmatrix}$

1 point for trying2 points for corre

$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{bmatrix}=(2-\lambda)(2-\lambda)(2-\lambda)=0$

$\Downarrow$

$\lambda=2$

$\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = v_1 \\ v_2 = 0 \\ v_3 = 0 \end{array}$$

$$v = v_1 \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$ eigenvector

#9) 2×2 matrix A has eigenvalues 5 and -1. corresponding eigenvectors are $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ compute Ax where $x = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

$A\vec{u} = \lambda \vec{u}$ $\quad A\vec{v} = \lambda \vec{v}$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{l} b = 0 \\ d = 5 \end{array}$$

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} a = -1 \\ c = 0 \end{array}$$

$$\Rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 20 \end{bmatrix}}$$

1 point for trying
2 points for correct approach (you can use the formula in the textbook too)
2 points for correct answer

#11) $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ determine eigenvalues and eigenvectors. Does $\mathbb{R}^3$ have a linear independent spaning set that consists of eigenvectors of A?

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}$$

1 point for correct approach finding eigenvalues
1 point for correct approach finding eigen vectors
2 points for correct eigenvalue and eigen vectors
1 point for stating Yes

$$\det(A - \lambda I) = 1 \cdot \det\begin{bmatrix} 1 & 1 \\ -2-\lambda & 1 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & 1 \end{bmatrix} + (-2-\lambda) \cdot \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$= (3+\lambda) - (-3-\lambda) + (-2-\lambda)[(-2-\lambda)^2 - 1] = -\lambda^3 - 6\lambda^2 - 9\lambda = 0$$

$$\Downarrow$$

$$\lambda(\lambda+3)^2 = 0$$

$$\boxed{\lambda = 0, -3}$$

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~~when~~ $\boxed{\lambda=0}$: $A-\lambda I=\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right] \sim \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$

$\Rightarrow \vec{V_1}=V_3$
$\vec{V_2}=V_3$
$\vec{V_3}=V_3$

$\vec{V}=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\vec{V_3}\boxed{\begin{bmatrix}1\\1\\1\end{bmatrix}}$ eigenvector

when $\boxed{\lambda=-3}$: $A-\lambda I=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$

$\Rightarrow V_1=-V_2-V_3$
$V_2=V_2$
$V_3=V_3$

$\vec{V}=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=V_2\boxed{\begin{bmatrix}-1\\1\\0\end{bmatrix}}+V_3\boxed{\begin{bmatrix}-1\\0\\1\end{bmatrix}}$

eigenvector

Yes)
eigenbasis: $\begin{bmatrix}1&-1&-1\\1&1&0\\1&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ so linear independant.

$\Downarrow$

span $\mathbb{R}^3$

Homework #1-11 : 3, 17, 29

#3 determine whether or not set H is a subspace of V.

If H is, show it satisfies 3 properties.

If not, show counterexample.

1 point for trying
3 points for verifying 3 properties correctly
1 point for the right conclusion

$V = \mathbb{R}^3$, $H = \left\{ t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}$. Yes

we'll show $\vec{0}$ is in H. since $0 \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{0} \in H$ ✓

We'll then show H is closed under addition.

Let $t_1, t_2 \in \mathbb{R}$. then we know $t_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \in H$ and $t_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \in H$.

Therefore $t_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2t_1 + 2t_2 \\ 0 + 0 \\ -t_1 - t_2 \end{bmatrix} = (t_1 + t_2) \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

since $t_1, t_2 \in \mathbb{R}$, $t_1 + t_2 \in \mathbb{R}$, we're convinced that $(t_1 + t_2) \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \in H$ ✓

Finally, we'll show H is closed under scalar multiplication.

Let $r \in \mathbb{R}$. Then we have $r \cdot t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = rt \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

Since $rt \in \mathbb{R}$, we know that $rt \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \in H$. ✓

#17 $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ describe all eigenspaces of A.

$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = (\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda = 1, 3$

when $\boxed{\lambda = 1}$: $A - \lambda I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_1 = v_2$
$v_2 = v_2$
$\Rightarrow v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2 \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ $\Rightarrow$ eigenspace $= \left\{ t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

eigenvector

when $\boxed{\lambda=3}$: $A-\lambda I=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix} \Rightarrow \left[\begin{array}{cc|c}-1 & -1 & 0\\ -1 & -1 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$

$\Rightarrow \begin{matrix} v_1=-v_2 \\ v_2=v_2 \end{matrix} \Rightarrow v=\begin{bmatrix}v_1\\ v_2\end{bmatrix}=v_2\boxed{\begin{bmatrix}-1\\ 1\end{bmatrix}}$ → eigenvector

2 points for right approach of finding eigenvalues and eigen vectors
2 points for correct eigenvalues and eigen vectors
1 point for eigenspace

eigenspace $=\left\{t\begin{bmatrix}-1\\ 1\end{bmatrix}: t\in\mathbb{R}\right\}$

#29) consider $y'=3y-3$
explain why any function of the form $y=Ce^{3t}+1$ is a solution.
Is the set of all these solutions a subspace of the vector space of continuous functions?

Plug in $y=Ce^{3t}+1$ to $y'=3y-3$:

$3Ce^{3t}=3(Ce^{3t}+1)$, so it's a solution.

To be a subspace, $y=0$ has to be in the solution set.

Plug in $y=0$ to $y'=3y-3$:

$0=-3$ ✗, so $y=0\notin$ solution!

Therefore, it's not a subspace of that vector space.

1 point for trying
2 points for verifying the solution by plugging back in
1 point for stating No
1 point for explaining why

Homework # 1.12 : 1, 9, 13, 14

#1 In vector space V, determine a basis for subspace H and State the dimension of H

1 point for trying
2 points for correct basis
2 points for dimension

$V = \mathbb{R}^3$, $H = \left\{ t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is a basis for H since it's linear in dependent and span H.

$\dim(H) = 1$

#9 Is the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^2$? Justify.

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Pivots in each row & column so they are linear independent and span $\mathbb{R}^2$

$\Downarrow$

S is a basis for $\mathbb{R}^2$

1 point for trying
1 point for Yes
3 points for correct explanation

#13 Can a set with 3 vectors be a basis for $\mathbb{R}^4$?

No. Because to be able to span $\mathbb{R}^4$, we have to have at least 4 vectors.

2 points for No
3 points for explanation

#14 Can a set with 7 vectors be a basis for $\mathbb{R}^6$?

No. Because to be a basis for $\mathbb{R}^6$, which has dimension 6, we need 6 independent vectors. 7 vectors make it linear dependant.

2 points for No
3 points for explaining