Homework 03

MATH316

See code

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Section 2.2

Assigned: 1, 9, 13, 23

2.2.1

Problem:

Consider the differential equation $y'' = 4y$.

a. What is the order of this equation?

b. Show via substitution that the function $y = e^{2t}$ is a solution to this equation.

c. Are there any other functions of the form $y = e^{rt}$ $(r \neq 2)$ that are also solutions to the equation? If so, which? Justify your answer.

Solution:

Part (a)

The order of the equation is the highest derivative, so in this case the order is 2.

Part (b)

To verify that $y = e^{2t}$ is a solution to $y'' = 4y$, we compute the derivatives:

- First derivative:
  $y' = 2e^{2t}$
- Second derivative:
  $y'' = 4e^{2t}$

Substitute into the equation:

$$4e^{2t} = 4 \cdot e^{2t}$$

Both sides match, confirming that $y = e^{2t}$ is a solution.

Part (c)

For solutions of the form $y = e^{rt}$:

- First derivative:
  $y' = re^{rt}$
- Second derivative:
  $y'' = r^2e^{rt}$

Substitute into the equation:

$$r^2e^{rt} = 4e^{rt}$$

Divide by e^{rt} (assuming $e^{rt} \neq 0$):

$$r^2 = 4$$

Solve for r:

$$r = \pm 2$$

Thus, the two solutions are $y = e^{2t}$ and $y = e^{-2t}$.

Final Answer:

- **(a):** The order is 2.

- **(b):** $y = e^{2t}$ is a solution.
- **(c):** The other solution is $y = e^{-2t}$.

2.2.9

Problem:

Solve the differential equation:

$$y' = t + \cos t$$

Solution:

To find the family of solutions, we integrate both sides with respect to t.

1. **Integrate the right-hand side:**

$$y = \int (t + \cos t)\, dt$$

2. **Use linearity of integration:**

$$y = \int t\, dt + \int \cos t\, dt$$

3. **Evaluate the integrals:**

$$\int t\, dt = \frac{t^2}{2} \quad \text{and} \quad \int \cos t\, dt = \sin t$$

4. **Combine the results and add the constant of integration:**

$$y = \frac{t^2}{2} + \sin t + C$$

Final Answer:

$$y = \frac{t^2}{2} + \sin t + C$$

2.2.13

Problem:

Solve the differential equation:

$$y' = t \sin t$$

Solution:

We integrate both sides with respect to t to find the general solution.

1. **Set up the integral:**

$$y = \int t \sin t \, dt$$

2. **Use integration by parts:**

   Let:

   - $u = t$, so $du = dt$
   - $dv = \sin t \, dt$, so $v = -\cos t$

   Now apply the formula for integration by parts:

$$\int u \, dv = uv - \int v \, du$$

   Substituting the values:

$$y = -t \cos t - \int (-\cos t) \, dt$$

3. **Simplify the integral:**

$$y = -t\cos t + \int \cos t\, dt$$

$$\int \cos t\, dt = \sin t$$

4. **Combine the results:**

$$y = -t\cos t + \sin t + C$$

where C is the constant of integration.

Final Answer:

$$y = -t\cos t + \sin t + C$$

2.2.23

Problem:

Solve the initial value problem:

$$y' = te^{-t^2}, \quad y(0) = -1$$

Solution:

1. **Integrate both sides** to find the general solution.

$$y = \int te^{-t^2}\, dt$$

2. **Use substitution:**

Let $u = -t^2$, so that:

$$du = -2t\,dt \quad \Rightarrow \quad -\frac{1}{2}du = t\,dt$$

Substituting into the integral:

$$y = \int e^u \left(-\frac{1}{2}\,du\right)$$

$$y = -\frac{1}{2}\int e^u\,du$$

3. **Integrate:**

$$y = -\frac{1}{2}e^u + C$$

Substituting back $u = -t^2$:

$$y = -\frac{1}{2}e^{-t^2} + C$$

4. **Apply the initial condition $y(0) = -1$.**

At $t = 0$:

$$-1 = -\frac{1}{2}e^0 + C$$

Since $e^0 = 1$, the equation becomes:

$$-1 = -\frac{1}{2} + C$$

Solve for C:

$$C = -1 + \frac{1}{2} = -\frac{1}{2}$$

5. **Write the particular solution:**

$$y = -\frac{1}{2}e^{-t^2} - \frac{1}{2}$$

Final Answer:

$$y = -\frac{1}{2}e^{-t^2} - \frac{1}{2}$$

Section 2.3

Assigned: 1, 3, 7, 15

2.3.1

Problem:

Classify the equation $y' + 7y = e^t$ as linear or nonlinear.

Solution:

A differential equation is **linear** if the unknown function $y(t)$ and its derivatives:

- Appear only to the first power (no exponents).
- Are not multiplied by each other.

The given equation is:

$$y' + 7y = e^t$$

- y' and y appear to the first power.
- There are no products or powers involving y and its derivatives.
- e^t is a function of t only, and it does not affect the linearity with respect to $y(t)$.

Final Answer:

The equation is **linear**.

2.3.3

Problem:

Classify the equation $\cos y' + \sin y = t^2$ as linear or nonlinear.

Solution:

A differential equation is **linear** if the unknown function $y(t)$ and its derivatives:

- Appear only to the first power (no exponents).
- Are not multiplied by each other.
- All coefficients can be functions of t, but must not involve $y(t)$ or its derivatives nonlinearly.

Let's analyze the given equation:

$$\cos y' + \sin y = t^2$$

- $\cos y'$ involves a trigonometric function applied to the derivative y', which makes it **nonlinear**.
- $\sin y$ applies a trigonometric function to the unknown function y, further making the equation **nonlinear**.

Final Answer:

The equation is **nonlinear**.

2.3.7

Problem:

Solve the differential equation:

$$y' + y = 0$$

Solution:

This is a first-order, linear, homogeneous differential equation. To solve it, we can use **separation of variables**.

1. **Rewrite the equation:**

$$y' = -y$$

2. **Separate the variables:**

$$\frac{dy}{y} = -dt$$

3. **Integrate both sides:**

$$\int \frac{1}{y}\, dy = \int -1\, dt$$

   The integrals give:

$$\ln|y| = -t + C$$

   where C is the constant of integration.

4. **Solve for y:**

   Exponentiate both sides to get rid of the logarithm:

$$|y| = e^{-t+C} = e^C e^{-t}$$

   Let $C_1 = e^C$, where $C_1 > 0$. Therefore:

$$y = \pm C_1 e^{-t}$$

   The general solution can be written as:

$$y = Ce^{-t}$$

where C is any real constant.

Final Answer:

$$y = Ce^{-t}$$

2.3.15

Problem:

Solve the differential equation:

$$y' + 2y = 2t$$

Solution:

1. **Rewrite the equation in standard form.**

$$y' + 2y = 2t$$

2. **Find the integrating factor.**

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

3. **Multiply both sides by the integrating factor.**

$$e^{2t}y' + 2e^{2t}y = 2te^{2t}$$

The left side becomes:

$$\frac{d}{dt}\left(e^{2t}y\right) = 2te^{2t}$$

4. **Integrate both sides.**

We need to solve:

$$e^{2t}y = \int 2te^{2t}\,dt$$

Using **integration by parts**:

- $u = 2t$, so $du = 2\,dt$
- $dv = e^{2t}\,dt$, so $v = \frac{1}{2}e^{2t}$

$$\int 2te^{2t}\,dt = 2t \cdot \frac{1}{2}e^{2t} - \int \frac{1}{2}e^{2t} \cdot 2\,dt$$

$$= te^{2t} - \frac{1}{2}e^{2t} + C$$

5. **Solve for y.**

Now we have:

$$e^{2t}y = te^{2t} - \frac{1}{2}e^{2t} + C$$

Divide both sides by e^{2t}:

$$y = t - \frac{1}{2} + Ce^{-2t}$$

Final Answer:

$$y = t - \frac{1}{2} + Ce^{-2t}$$

Section 2.4

Assigned: 3, 8, 15

2.4.3

Problem:

The evaporation rate of moisture from a sheet hung on a clothesline is proportional to the sheet's moisture content. If half of the moisture evaporates in the first 30 minutes, how long will it take for 95% of the moisture to evaporate?

Solution:

Step 1: Solve the differential equation.

The general solution for exponential decay is:

$$M(t) = M_0 e^{-kt}$$

where M_0 is the initial amount of moisture.

Step 2: Use the information that half the moisture evaporates in 30 minutes.

When $t = 30$, the remaining moisture is half of the initial moisture M_0:

$$M(30) = \frac{M_0}{2}$$

Substitute into the general solution:

$$\frac{M_0}{2} = M_0 e^{-30k}$$

Divide both sides by M_0:

$$\frac{1}{2} = e^{-30k}$$

Take the natural logarithm of both sides:

$$\ln\left(\frac{1}{2}\right) = -30k$$

Since $\ln\left(\frac{1}{2}\right) = -\ln 2$, we have:

$$-\ln 2 = -30k$$

Solve for k:

$$k = \frac{\ln 2}{30}$$

Step 3: Find the time for 95% evaporation.

If 95% of the moisture evaporates, 5% remains. So:

$$M(t) = 0.05M_0$$

Substitute into the general solution:

$$0.05M_0 = M_0 e^{-kt}$$

Divide both sides by M_0:

$$0.05 = e^{-kt}$$

Take the natural logarithm of both sides:

$$\ln(0.05) = -kt$$

Substitute $k = \frac{\ln 2}{30}$:

$$\ln(0.05) = -\frac{\ln 2}{30} \cdot t$$

Solve for t:

$$t = \frac{30 \ln(0.05)}{-\ln 2}$$

Step 4: Simplify the expression.

Using the fact that $\ln(0.05) = \ln\left(\frac{1}{20}\right) = -\ln 20$, we get:

$$t = \frac{30 \ln 20}{\ln 2}$$

Using $\ln 20 \approx 2.9957$ and $\ln 2 \approx 0.6931$:

$$t = \frac{30 \times 2.9957}{0.6931} \approx \frac{89.871}{0.6931} \approx 129.6 \,\text{minutes}.$$

Final Answer:

It will take approximately **129.6 minutes** for 95% of the moisture to evaporate.

2.4.8

Problem:

Brine is entering a 25-m^3 tank at flow rate of 0.5 m^3/min and at a concentration of 6 g/m^3. The uniformly mixed solution exits the tank at a rate of 0.25 m^3/min. Assume that initially there are 5 m^3 of solution in the tank at a concentration of 25 g/m^3.

a. State an IVP that is satisfied by the amount of salt $A(t)$ in grams in the tank at time t.

b. Solve the IVP stated in (a). For what values of t is this problem valid? Why?

c. At exactly what time will the least amount of salt be present in the tank? How much salt will there be at that time?

d. Plot a direction field for the IVP stated in (a), including a plot of the solution. Discuss why this direction field and the solution make sense in the physical context of the problem.

Solution:

(a) State an IVP that is satisfied by the amount of salt $A(t)$ in grams in the tank at time t.

Set up a differential equation to represent the amount of salt $A(t)$ in the tank at time t.

- **Flow of brine into the tank:**
    - Inflow rate: 0.5 m^3/min
    - Concentration of inflow: 6 g/m^3
    - **Salt inflow rate:**

$$\text{Inflow rate of salt} = (0.5\,\text{m}^3/\text{min}) \cdot (6\,\text{g}/\text{m}^3) = 3\,\text{g}/\text{min}$$

- **Flow of solution out of the tank:**
  - Outflow rate: 0.25 m^3/min
  - **Concentration of outflow:** Since the solution is uniformly mixed, the concentration of the outflow is:

    $$\frac{A(t)}{V(t)}$$

    where $V(t)$ is the volume of the solution in the tank at time t.

- **Volume change in the tank:**
  Initially, there are 5 m^3 in the tank, and the inflow adds 0.5 m^3/min while the outflow removes 0.25 m^3/min.
  So the volume at time t is:

  $$V(t) = 5 + (0.5 - 0.25)t = 5 + 0.25t\,\text{m}^3$$

- **Outflow rate of salt:**

  $$\text{Outflow rate of salt} = 0.25 \cdot \frac{A(t)}{V(t)}$$

- **Differential equation:** The change in salt amount over time is given by:

  $$\frac{dA}{dt} = \text{Salt inflow rate} - \text{Salt outflow rate}$$

  Substituting:

  $$\frac{dA}{dt} = 3 - 0.25 \cdot \frac{A(t)}{5 + 0.25t}$$

- **Initial condition:**
  At $t = 0$, the concentration in the tank is 25 g/m^3 over 5 m^3, so the initial amount of salt is:

  $$A(0) = 25 \cdot 5 = 125\,\text{g}$$

The IVP is:

$$\frac{dA}{dt} = 3 - 0.25 \cdot \frac{A(t)}{5 + 0.25t}, \quad A(0) = 125$$

(b) Solve the IVP. For what values of t is this problem valid?

Solve this **linear differential equation** using an integrating factor.

1. **Rewrite the equation:**

$$\frac{dA}{dt} + \frac{0.25}{5 + 0.25t} A(t) = 3$$

2. **Find the integrating factor:**

$$\mu(t) = e^{\int \frac{0.25}{5+0.25t}\, dt}$$

Let $u = 5 + 0.25t$, so $du = 0.25\, dt$. The integral becomes:

$$\mu(t) = e^{\ln(u)} = u = 5 + 0.25t$$

3. **Multiply both sides by the integrating factor:**

$$(5 + 0.25t)\frac{dA}{dt} + 0.25A(t) = 3(5 + 0.25t)$$

4. **Rewrite as a derivative:**

$$\frac{d}{dt}\left(A(t)(5 + 0.25t)\right) = 3(5 + 0.25t)$$

5. **Integrate both sides:**

$$A(t)(5 + 0.25t) = \int 3(5 + 0.25t)\, dt$$

$$A(t)(5 + 0.25t) = 3\left(5t + \frac{0.25}{2}t^2\right) + C$$

$$A(t)(5 + 0.25t) = 15t + 0.375t^2 + C$$

6. **Solve for $A(t)$:**

$$A(t) = \frac{15t + 0.375t^2 + C}{5 + 0.25t}$$

7. **Apply the initial condition $A(0) = 125$:**

$$125 = \frac{15(0) + 0.375(0)^2 + C}{5 + 0.25(0)} = \frac{C}{5}$$

$$C = 125 \cdot 5 = 625$$

8. **Substitute $C = 625$ into the solution:**

$$A(t) = \frac{15t + 0.375t^2 + 625}{5 + 0.25t}$$

The solution is valid as long as the volume in the tank does not exceed 25 m^3, meaning:

$$5 + 0.25t \leq 25 \quad \Rightarrow \quad t \leq 80\,\text{min.}$$

(c) At exactly what time will the least amount of salt be present in the tank? How much salt will there be at that time?

We know that:

$$\frac{dA}{dt} = 3 - 0.25 \cdot \frac{A(t)}{5 + 0.25t}$$

and:

$$A(t) = \frac{15t + 0.375t^2 + 625}{5 + 0.25t}$$

We want to find the time t when the amount of salt in the tank is minimized. This occurs when:

$$\frac{dA}{dt} = 0$$

Step 1: Substitute $A(t)$ into $\frac{dA}{dt}$

Substitute the expression for $A(t)$ into the differential equation:

$$0 = 3 - 0.25 \cdot \frac{15t + 0.375t^2 + 625}{(5 + 0.25t)^2}$$

Step 2: Solve for t

Rearrange the equation and solve for t. This yields:

$$t = \frac{-60 \pm 10\sqrt{114}}{3}$$

Evaluating the solutions:

$$t_1 \approx -55.59 \quad \text{and} \quad t_2 \approx 15.59$$

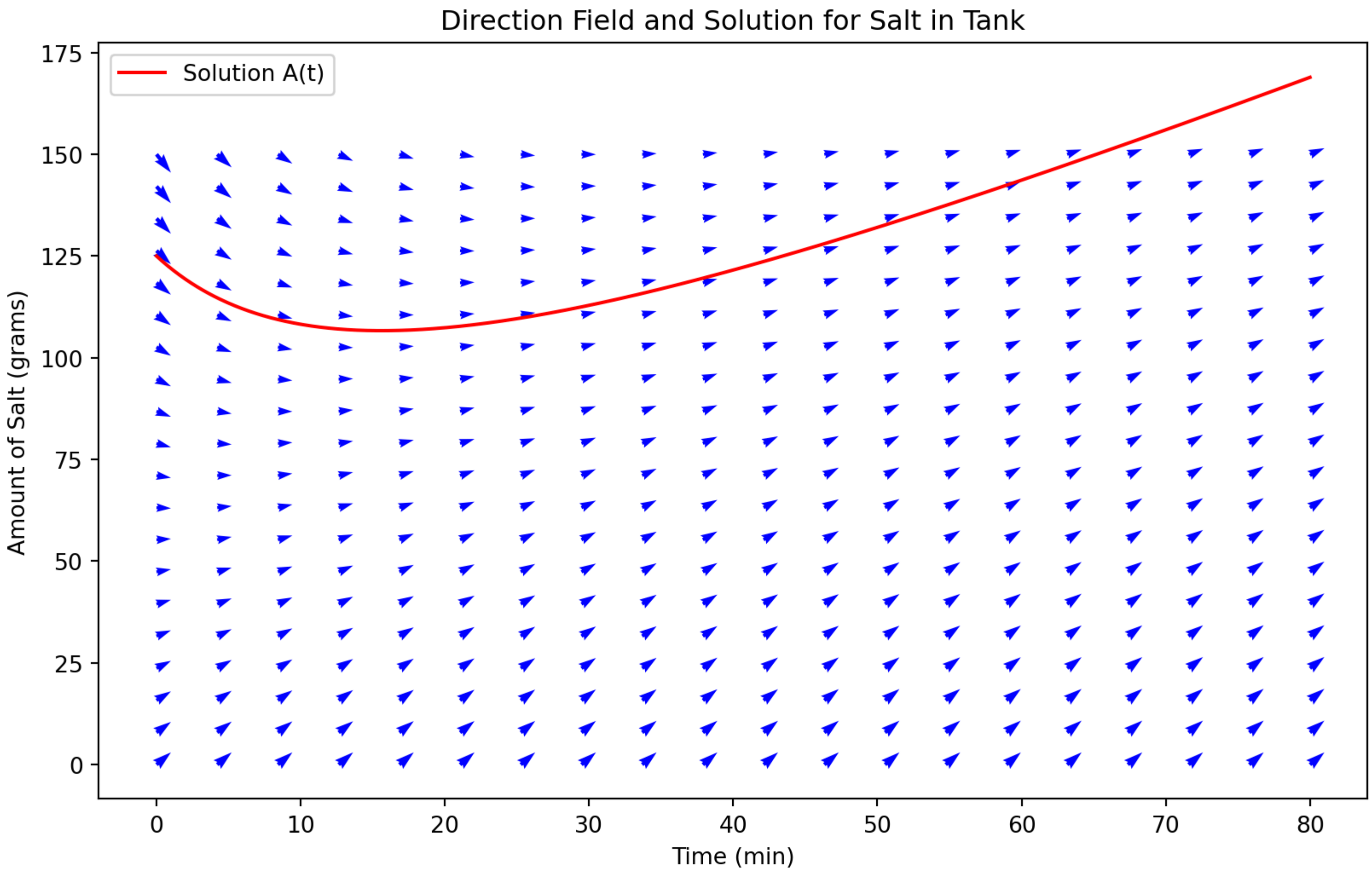
Since negative time is not physically meaningful, we discard t_1.

$$A(15.59) = \frac{949.95}{8.8975} \approx 106.75 \, \text{grams}$$

At $t \approx 15.59$ minutes, the amount of salt in the tank will be approximately **106.75 grams**.

(d) Plot a direction field and solution curve.

▶ Show code



Final Answer:

- **(a):** The IVP is $\frac{dA}{dt} = 3 - 0.25 \cdot \frac{A(t)}{5+0.25t}, \quad A(0) = 125.$
- **(b):** The solution is $A(t) = \frac{15t+0.375t^2+625}{5+0.25t}$, valid for $t \leq 80\,\text{min}.$
- **(c):** The least amount of salt is approximately **106.75 grams**, which occurs at $t \approx 15.59\,\text{min}.$
- **(d):** The direction field plot shows how the salt content evolves over time, with the solution curve representing the salt content decreasing to a minimum of about 106.75 grams before increasing again.

2.4.15

Solution:

This problem can be solved using **Newton's Law of Cooling**:

$$T(t) = T_{\text{out}} + (T_0 - T_{\text{out}})e^{-kt},$$

where:

- $T(t)$ is the temperature at time t,
- $T_0 = 68°\text{F}$ is the initial indoor temperature,
- $T_{\text{out}} = 4°\text{F}$ is the constant outdoor temperature,
- k is the cooling constant,
- t is the time in **hours** after the furnace fails (starting at 10 pm).

Find the Cooling Constant k

At 2 am (4 hours after the furnace failure), the indoor temperature is $60°\text{F}$. Substituting into the formula:

$$60 = 4 + (68 - 4)e^{-4k}.$$

Simplify:

$$60 = 4 + 64e^{-4k} \quad \Rightarrow \quad 56 = 64e^{-4k}.$$

Solve for e^{-4k}:

$$e^{-4k} = \frac{56}{64} = \frac{7}{8}.$$

Take the natural logarithm of both sides:

$$-4k = \ln\left(\frac{7}{8}\right)$$

$$k = \frac{\ln\left(\frac{7}{8}\right)}{-4} = \frac{\ln(7) - \ln(8)}{-4} = \frac{\ln(8) - \ln(7)}{4}$$

Find the Time When $T(t) = 32°\text{F}$

We want to find the time t when the indoor temperature reaches $32^\circ\mathrm{F}$. Substituting into the formula:

$$32 = 4 + 64e^{\frac{\ln(8)-\ln(7)}{4}\cdot(-t)}.$$

Subtract 4 from both sides:

$$28 = 64e^{\frac{\ln(7)-\ln(8)}{4}\cdot t}.$$

Divide both sides by 64:

$$e^{\frac{\ln(7)-\ln(8)}{4}\cdot t} = \frac{28}{64} = \frac{7}{16}$$

Take the natural logarithm of both sides:

$$\frac{\ln(7)-\ln(8)}{4}\cdot t = \ln\left(\frac{7}{16}\right)$$

Using the logarithm properties:

$$\ln\left(\frac{7}{16}\right) = \ln(7) - \ln(16)$$

where:

$$\ln(16) = 4\ln(2)$$

we substitute:

$$\ln\left(\frac{7}{16}\right) = \ln(7) - 4\ln(2)$$

Thus:

$$\frac{\ln(7)-\ln(8)}{4}\cdot t = \ln(7) - 4\ln(2)$$

Solve for t:

$$t = \frac{4\left(\ln(7) - 4\ln(2)\right)}{\ln(7)-\ln(8)} \approx 24.76$$

Determine the Exact Time

The furnace failed at **10 pm**, so the temperature will drop to $32°\mathrm{F}$ approximately:

$$10\,\mathrm{pm} + 24.76\,\mathrm{hours}.$$

Convert 0.76 hours to minutes:

$$0.76 \times 60 = 45.6\,\mathrm{minutes}.$$

The indoor temperature will reach $32°\mathrm{F}$ at approximately **10:45 pm the following day**.

Final Answer:

The homeowner will need to start worrying about the pipes freezing at after **24.76 hours** or at approximately **10:45 pm the following day**.