Homework 04

MATH316

See code

AUTHOR
Nathan Lunceford

Section 2.5

Assigned: 15, 19, 23, 29, 37, 41

2.5.15

Problem:

Solve the differential equation $y' = 10y$.

Solution:

1. Rewrite the equation:

$$\frac{dy}{dt} = 10y$$

2. Separate the variables y and t:

$$\frac{1}{y}\,dy = 10\,dt$$

3. Integrate both sides:

$$\int \frac{1}{y}\,dy = \int 10\,dt$$

4. The result of the integration is:

$$\ln|y| = 10t + C$$

5. Exponentiate both sides to solve for y:

$$|y| = e^{10t+C} = e^C \cdot e^{10t}$$

6. Let $A = e^C$:

$$y = Ae^{10t}$$

Final Answer:

The general solution is:

$$y = Ae^{10t}$$

where A is the constant of integration.

2.5.19

Problem:

Solve the differential equation $t^2y' + y^2 = 1$.

Solution:

1. Rewrite the equation:

$$t^2\frac{dy}{dt} + y^2 = 1$$

2. Rearrange the equation to isolate $\frac{dy}{dt}$:

$$t^2\frac{dy}{dt} = 1 - y^2$$

3. Separate the variables y and t:

$$\frac{dy}{1-y^2} = \frac{dt}{t^2}$$

4. Now use partial fraction decomposition on $\frac{1}{1-y^2}$:

$$\frac{1}{1-y^2} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$$

5. Substitute the partial fraction decomposition back into the equation:

$$\frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy = \frac{dt}{t^2}$$

6. Integrate both sides:

   - Left-hand side:

$$\frac{1}{2}\left(\int \frac{1}{1-y}\, dy + \int \frac{1}{1+y}\, dy\right) = \frac{1}{2}\left(-\ln|1-y| + \ln|1+y|\right)$$

   Simplifying:

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|$$

   - Right-hand side:

$$\int \frac{1}{t^2}\, dt = -\frac{1}{t}$$

7. Combine the integrals:

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = -\frac{1}{t} + C$$

8. Multiply through by 2:

$$\ln\left|\frac{1+y}{1-y}\right| = -\frac{2}{t} + 2C$$

9. Exponentiate both sides to remove the logarithm:

$$\left|\frac{1+y}{1-y}\right| = e^{-\frac{2}{t}+2C}$$

10. Let $A = e^{2C}$, which simplifies the equation to:

$$\frac{1+y}{1-y} = Ae^{-\frac{2}{t}}$$

11. Solve for y:

    Cross-multiply:

$$1 + y = (1-y)Ae^{-\frac{2}{t}}$$

    Expand the right-hand side:

$$1 + y = Ae^{-\frac{2}{t}} - Ae^{-\frac{2}{t}}y$$

    Now gather terms involving y on one side:

$$y + Ae^{-\frac{2}{t}}y = Ae^{-\frac{2}{t}} - 1$$

    Factor out y on the left-hand side:

$$y(1 + Ae^{-\frac{2}{t}}) = Ae^{-\frac{2}{t}} - 1$$

    Solve for y:

$$y = \frac{Ae^{-\frac{2}{t}} - 1}{1 + Ae^{-\frac{2}{t}}}$$

Final Answer:

The general solution is:

$$y = \frac{Ae^{-\frac{2}{t}} - 1}{1 + Ae^{-\frac{2}{t}}}$$

where A is the constant of integration.

2.5.23

Problem:

Solve the differential equation:

$$y - t\frac{dy}{dt} = 6 - 3t^2\frac{dy}{dt}$$

Solution:

1. **Rearrange the equation:**

   Start by moving all terms involving $\frac{dy}{dt}$ to one side:

   $$y - t\frac{dy}{dt} = 6 - 3t^2\frac{dy}{dt}$$

   Rearranging the terms involving $\frac{dy}{dt}$:

   $$y = 6 + \frac{dy}{dt}\left(3t^2 - t\right)$$

2. **Solve for $\frac{dy}{dt}$:**

   Isolate $\frac{dy}{dt}$:

   $$y - 6 = \frac{dy}{dt}(3t^2 - t)$$

Now, solve for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{y-6}{3t^2 - t}$$

3. **Separate the variables:**

   Separate the variables y and t:

$$\frac{dy}{y-6} = \frac{dt}{t(1-3t)}$$

4. **Integrate both sides:**

   Now we will integrate both sides:

   - The left-hand side:

$$\int \frac{1}{y-6}\, dy = \ln|y-6|$$

   - The right-hand side can be handled by using partial fraction decomposition on $\frac{1}{t(1-3t)}$:

     We decompose:

$$\frac{1}{t(1-3t)} = \frac{A}{t} + \frac{B}{1-3t}$$

     Multiply both sides by $t(1-3t)$:

$$1 = A(1-3t) + Bt$$

     Expand:

$$1 = A - 3At + Bt$$

     Group terms involving t:

$$1 = A + t(B - 3A)$$

     Equating coefficients gives the system:

$$A = 1 \quad \text{and} \quad B - 3A = 0$$

From $A = 1$, we have $B = 3$.

The partial fraction decomposition is:

$$\frac{1}{t(1-3t)} = \frac{1}{t} + \frac{3}{1-3t}$$

Now integrate both terms:

$$\int \frac{1}{t}\,dt + \int \frac{3}{1-3t}\,dt = \ln|t| - \ln|1-3t|$$

5. **Combine the integrals:**

   Combine the results from both sides:

$$\ln|y-6| = \ln|t| - \ln|1-3t| + C$$

6. **Simplify the logarithms:**

   Simplify the right-hand side using properties of logarithms:

$$\ln|y-6| = \ln\left|\frac{t}{1-3t}\right| + C$$

   Exponentiate both sides to remove the logarithms:

$$|y-6| = e^C\left|\frac{t}{1-3t}\right|$$

   Let $A = e^C$, so:

$$y - 6 = A\frac{t}{1-3t}$$

7. **Solve for y:**

$$y = 6 + A\frac{t}{1-3t}$$

Final Answer:

The solution to the differential equation is:

$$y = 6 + A\frac{t}{1-3t}$$

where A is the constant of integration.

2.5.29

Problem:

Solve the initial value problem $y' = 10y$, with the initial condition $y(0) = 3$.

Solution:

We know from **Problem 2.5.15** that the general solution to the differential equation $y' = 10y$ is:

$$y = Ae^{10t}$$

1. Use the initial condition $y(0) = 3$ to find A:

$$y(0) = Ae^{10\cdot 0} = A = 3$$

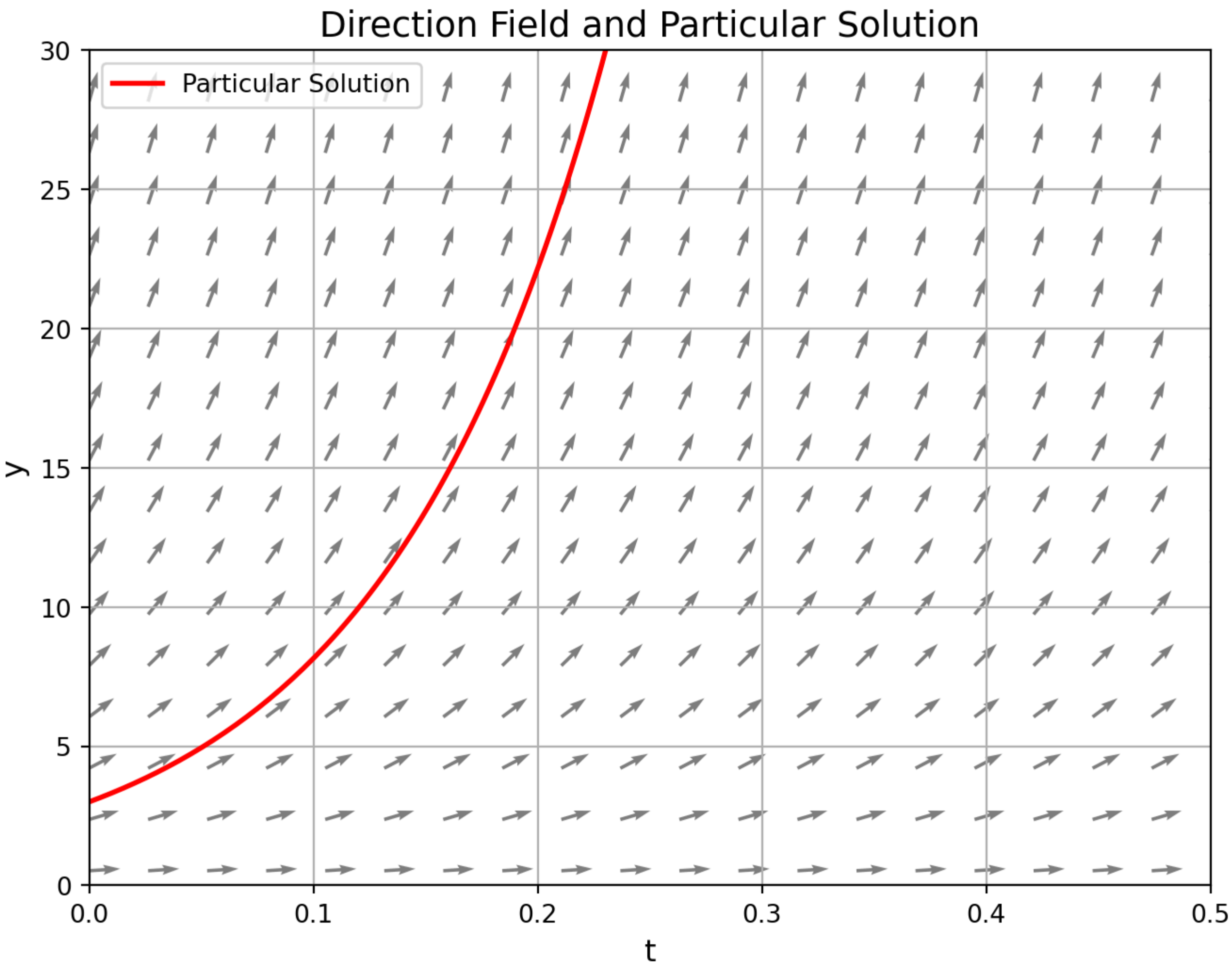
   Therefore, $A = 3$.

2. The particular solution is:

$$y = 3e^{10t}$$

Plotting the Direction Field and Solution:

▶ Show code



Final Answer:

The solution to the initial value problem is:

$$y = 3e^{10t}$$

This solution represents exponential growth with an initial value of 3 at $t = 0$, as reflected in the direction field and solution curve plot.

2.5.37

Problem:

Solve the initial value problem:

$$y - t\frac{dy}{dt} = 6 - 3t^2\frac{dy}{dt}, \quad y(1) = 5$$

Solution:

We know from **Problem 2.5.23** that the general solution to this differential equation is:

$$y = 6 + A\frac{t}{1-3t}$$

Use the initial condition $y(1) = 5$ to find the value of A.

1. **Substitute the initial condition $t = 1$ and $y = 5$ into the general solution:**

$$5 = 6 + A\frac{1}{1 - 3 \cdot 1}$$

2. **Simplify the equation:**

$$5 = 6 + A\frac{1}{-2}$$

   This simplifies to:

$$5 = 6 - \frac{A}{2}$$

3. **Solve for A:**

   Subtract 6 from both sides:

$$-1 = -\frac{A}{2}$$

Multiply both sides by -2:

$$A = 2$$

4. **Substitute $A = 2$ into the general solution:**

$$y = 6 + 2\frac{t}{1 - 3t}$$

The particular solution to the initial value problem is:

$$y = 6 + \frac{2t}{1 - 3t}$$

Plotting the Direction Field and Solution:

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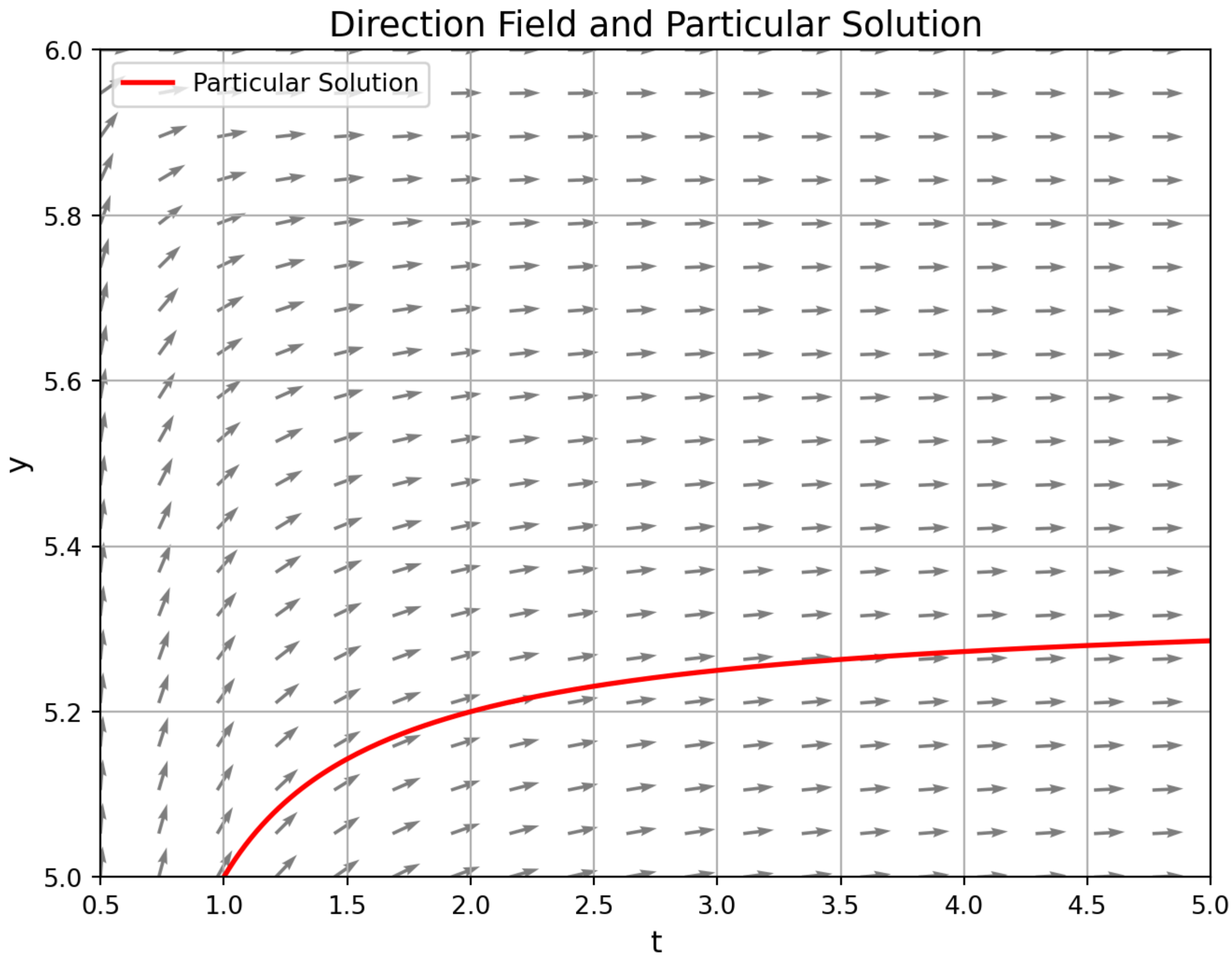

Final Answer:

The solution to the initial value problem is:

$$y = 6 + \frac{2t}{1 - 3t}$$

2.5.41

Problem:

Solve the initial value problem:

$$(y+t)y'+y=t, \quad y(0)=1$$

with the initial condition $y(0)=1$, and then plot an appropriate direction field and sketch your solution.

Solution:

Rearrange the equation

The given equation is:

$$(y+t)\frac{dy}{dt}+y=t$$

Rewrite it as:

$$\frac{dy}{dt}=\frac{t-y}{y+t}$$

Cross multiply:

$$(t-y)dt=(y+t)dy$$

This can be rewritten as:

$$(y-t)dt+(y+t)dy=0$$

Which is now in the standard form of an exact equation:

$$M(t,y)dt+N(t,y)dy=0$$

Identify $M(t,y)$ and $N(t,y)$

From the rewritten form of the equation:

$$(y-t)dt + (y+t)dy = 0$$

we can identify the functions $M(t,y)$ and $N(t,y)$ as:

$$M(t,y) = y - t \quad \text{and} \quad N(t,y) = y + t$$

Test for exactness

To check if the equation is exact, verify the condition for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Compute the partial derivatives:

- $M(t,y) = y - t$, so $\frac{\partial M}{\partial y} = 1$
- $N(t,y) = y + t$, so $\frac{\partial N}{\partial t} = 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, the equation is exact.

Find the potential function

Since the equation is exact, there exists a potential function $\Phi(t,y)$ such that:

$$\frac{\partial \Phi}{\partial t} = M(t,y) = y - t \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = N(t,y) = y + t$$

Integrate $M(t,y)$ with respect to t

Integrate $M(t,y) = y - t$ with respect to t:

$$\Phi(t,y) = \int (y-t)\,dt = yt - \frac{t^2}{2} + h(y)$$

where $h(y)$ is a function of y only.

Differentiate $\Phi(t,y)$ with respect to y

Differentiate $\Phi(t, y)$ with respect to y and set it equal to $N(t, y) = y + t$:

$$\frac{\partial \Phi}{\partial y} = t + h'(y) = y + t$$

Simplifying:

$$h'(y) = y$$

Integrate $h'(y)$

Integrating $h'(y) = y$ with respect to y:

$$h(y) = \frac{y^2}{2} + C$$

The potential function is:

$$\Phi(t, y) = yt - \frac{t^2}{2} + \frac{y^2}{2} + C$$

General solution

The general solution of an exact differential equation is obtained by setting the potential function $\Phi(t, y)$ equal to a constant:

$$yt - \frac{t^2}{2} + \frac{y^2}{2} = K$$

Apply the initial condition

We are given $y(0) = 1$. Substituting $t = 0$ and $y = 1$ into the equation:

$$1(0) - \frac{0^2}{2} + \frac{1^2}{2} = K$$

$$K = \frac{1}{2}$$

Final implicit solution

Substitute $K = \frac{1}{2}$ back into the equation:

$$yt - \frac{t^2}{2} + \frac{y^2}{2} = \frac{1}{2}$$

Multiply through by 2 to simplify:

$$2yt - t^2 + y^2 = 1$$

This is the implicit solution of the differential equation:

$$2yt - t^2 + y^2 = 1$$

Solve for y using the quadratic formula

Solve this quadratic equation for y. Rearrange the equation:

$$y^2 + 2ty + (-t^2 - 1) = 0$$

This is a quadratic equation of the form $ay^2 + by + c = 0$, where:

- $a = 1$
- $b = 2t$
- $c = -t^2 - 1$

Use the quadratic formula to solve for y:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute the values of a, b, and c:

$$y = \frac{-2t \pm \sqrt{(2t)^2 - 4(1)(-t^2 - 1)}}{2}$$

Simplifying:

$$y = \frac{-2t \pm \sqrt{4t^2 - 4(-t^2 - 1)}}{2}$$

$$y = \frac{-2t \pm \sqrt{4t^2 + 4 + 4t^2}}{2}$$

$$y = \frac{-2t \pm \sqrt{8t^2 + 4}}{2}$$

Factor out 4 from inside the square root:

$$y = \frac{-2t \pm \sqrt{4(2t^2+1)}}{2}$$

$$y = \frac{-2t \pm 2\sqrt{2t^2+1}}{2}$$

$$y = -t \pm \sqrt{2t^2+1}$$

Apply the initial condition

We are given $y(0) = 1$. Substituting $t = 0$ into the equation:

$$1 = -0 \pm \sqrt{2(0)^2 - 1}$$

$$1 = \pm\sqrt{1}$$

$$1 \neq -1$$

So our only solution is:

$$y = -t + \sqrt{2t^2 - 1}$$

Plotting the Direction Field and Solution:

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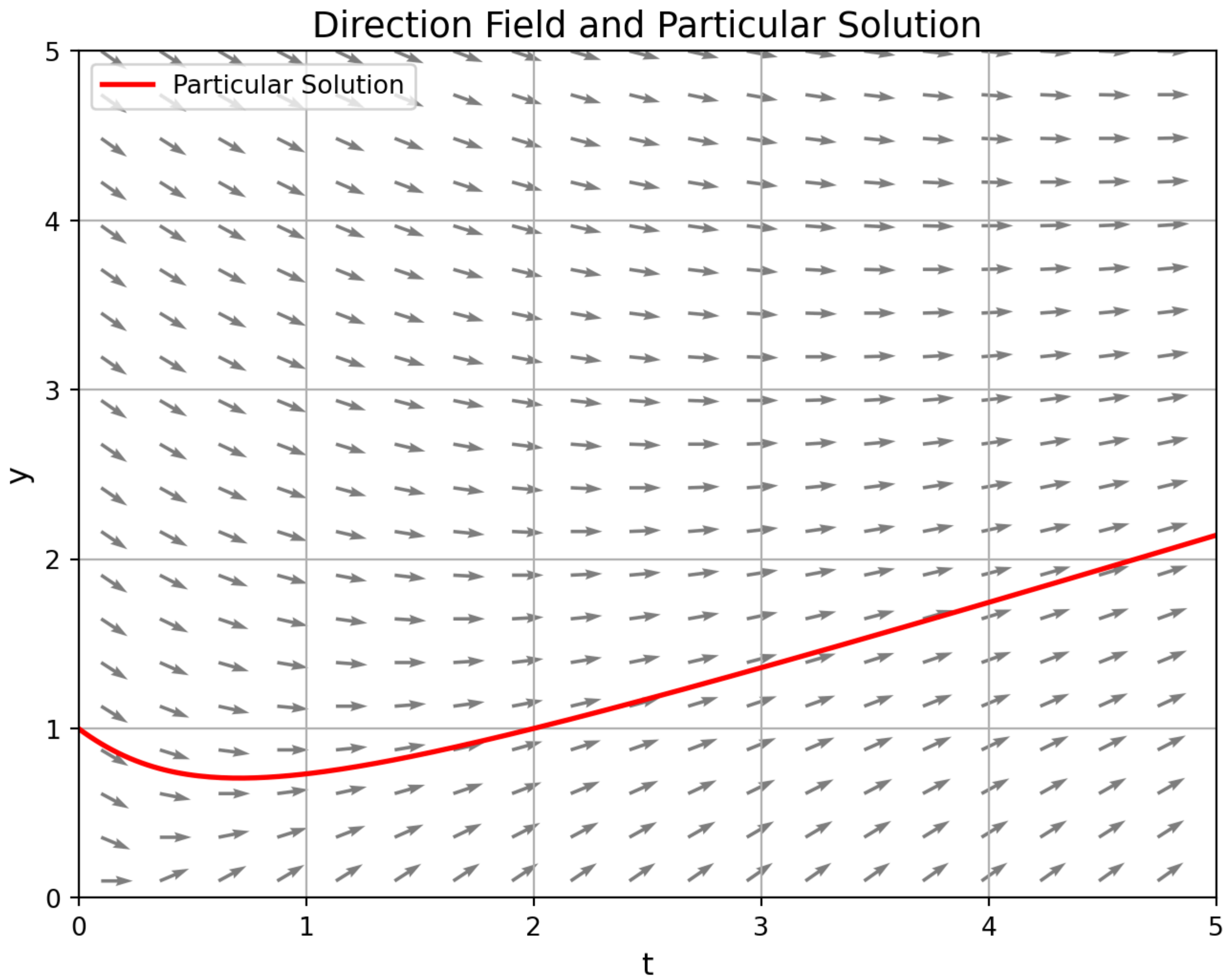


Final Answer:

The solution to the initial value problem is:

$$y = -t + \sqrt{2t^2 - 1}$$

Section 2.6

Assigned: 5, 7, 11

2.6.5

Problem:

Solve the initial value problem:

$$y' + 2ty = 0, \quad y(0) = -2$$

using Euler's method with $h = 0.1$ on the interval $[0, 1]$. Additionally, find the exact solution and compare the values and plots of the approximate and exact solutions.

Solution:

Exact solution

The given equation is:

$$y' + 2ty = 0$$

This is a first-order linear differential equation, and it can be solved by using an integrating factor. The general form of a linear differential equation is:

$$y' + p(t)y = 0$$

where $p(t) = 2t$. The integrating factor $\mu(t)$ is:

$$\mu(t) = e^{\int p(t)\,dt} = e^{t^2}$$

Multiplying both sides of the equation by e^{t^2}:

$$e^{t^2}y' + 2te^{t^2}y = 0$$

This simplifies to:

$$\frac{d}{dt}\left(e^{t^2}y\right) = 0$$

Integrating both sides:

$$e^{t^2} y = C$$

The general solution is:

$$y = Ce^{-t^2}$$

Apply the initial condition $y(0) = -2$:

$$-2 = Ce^0 \quad \Rightarrow \quad C = -2$$

The particular solution is:

$$y = -2e^{-t^2}$$

Euler's method

Euler's method uses the formula:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where $f(t, y) = -2ty$ from the original equation $y' = -2ty$.

We are given $h = 0.1$ and the initial condition $y(0) = -2$. We will approximate the solution for $t \in [0, 1]$.

- Step size: $h = 0.1$
- Initial condition: $y(0) = -2$
- Differential equation: $y' = -2ty$

Comparison of approximate and exact solutions

Table:

▼ Show code

```
import pandas as pd

# Given initial condition and step size
h = 0.1
t_vals_euler = np.arange(0, 1.1, h)
```

```python
y_euler = np.zeros(len(t_vals_euler))
y_euler[0] = -2  # y(0) = -2

# Define the function for the differential equation y' = -2ty
def f(t, y):
    return -2 * t * y

# Euler's method iteration
for i in range(1, len(t_vals_euler)):
    y_euler[i] = y_euler[i-1] + h * f(t_vals_euler[i-1], y_euler[i-1])

# Exact solution
def exact_solution(t):
    return -2 * np.exp(-t**2)

# Create a table for Euler's method and the exact solution
t_vals_exact_small = t_vals_euler  # Use the same t-values for comparison
y_exact_small = exact_solution(t_vals_exact_small)

# Create a DataFrame to compare Euler's method with the exact solution
data = {
    't': t_vals_euler,
    'Euler Approximation': y_euler,
    'Exact Solution': y_exact_small
}
df_comparison = pd.DataFrame(data)

table_markdown = df_comparison.to_markdown(index=False)
# print(table_markdown)
```

t	Euler Approximation	Exact Solution
0	-2	-2
0.1	-2	-1.9801
0.2	-1.96	-1.92158
0.3	-1.8816	-1.82786
0.4	-1.7687	-1.70429

t	Euler Approximation	Exact Solution
0.5	-1.62721	-1.5576
0.6	-1.46449	-1.39535
0.7	-1.28875	-1.22525
0.8	-1.10832	-1.05458
0.9	-0.930992	-0.889716
1	-0.763413	-0.735759

Graph:

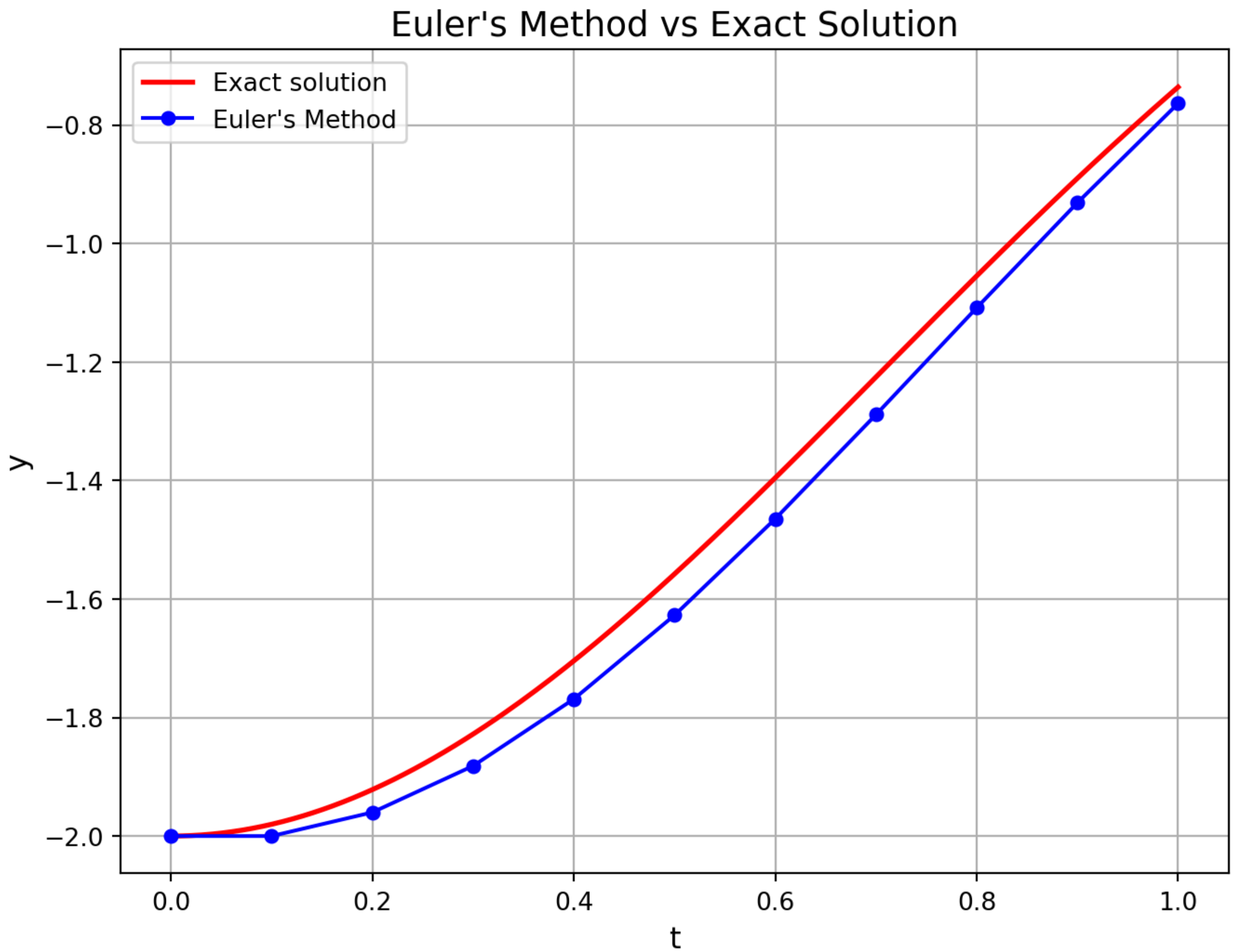
▼ Show code

```
t_vals_exact = np.linspace(0, 1, 200)
y_exact = exact_solution(t_vals_exact)

# Plot Euler's Method approximation and exact solution
plt.figure(figsize=(8, 6))
plt.plot(t_vals_exact, y_exact, 'r', label="Exact solution", linewidth=2)
plt.plot(t_vals_euler, y_euler, 'bo-', label="Euler's Method", markersize=5)

# Add labels, title, and legend
plt.title("Euler's Method vs Exact Solution", fontsize=14)
plt.xlabel("t", fontsize=12)
plt.ylabel("y", fontsize=12)
plt.grid(True)
plt.legend()

# Display the plot
plt.show()
```



Final Answer:

The **exact solution** to the differential equation is:

$$y = -2e^{-t^2}$$

The graph compares Euler's method with the exact solution. Euler's method provides a reasonable approximation but slightly underestimates the true values as t increases.

2.6.7

Problem:

Solve the initial value problem using Euler's method with $h = 0.1$ on the interval $[0, 1]$:

$$y' - y = 0, \quad y(0) = 2$$

In addition, find the exact solution and compare the values and plots of the approximate and exact solutions.

Solution:

Exact Solution

The given equation is:

$$y' - y = 0$$

This is a separable differential equation. Rearrange it:

$$\frac{dy}{dt} = y$$

Separate the variables:

$$\frac{1}{y}\,dy = dt$$

Integrate both sides:

$$\int \frac{1}{y}\,dy = \int dt$$

$$\ln|y| = t + C$$

Exponentiate both sides to solve for y:

$$|y| = e^{t+C} = e^C \cdot e^t$$

Let $A = e^C$, so:

$$y = Ae^t$$

Use the initial condition $y(0) = 2$ to find A:

$$2 = Ae^0 \quad \Rightarrow \quad A = 2$$

The particular solution is:

$$y = 2e^t$$

Euler's Method

Euler's method uses the formula:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where $f(t, y) = y$ (from the original equation $y' = y$).

Given:

- Step size $h = 0.1$
- Initial condition $y(0) = 2$
- Differential equation: $y' = y$

Comparison of approximate and exact solutions

Table

▼ Show code

```
# Step size and initial conditions
h = 0.1
t_vals_euler = np.arange(0, 1.1, h)
y_euler = np.zeros(len(t_vals_euler))
y_euler[0] = 2  # Initial condition y(0) = 2

# Define the differential equation y' = y
def f(t, y):
    return y
```

```python
# Euler's method iteration
for i in range(1, len(t_vals_euler)):
    y_euler[i] = y_euler[i-1] + h * f(t_vals_euler[i-1], y_euler[i-1])

# Exact solution
def exact_solution(t):
    return 2 * np.exp(t)

# Exact solution values for comparison
t_vals_exact = np.linspace(0, 1, 200)
y_exact = exact_solution(t_vals_exact)

# Create a DataFrame for comparison
y_exact_small = exact_solution(t_vals_euler)
df_comparison = pd.DataFrame({
    't': t_vals_euler,
    'Euler Approximation': y_euler,
    'Exact Solution': y_exact_small
})

table_markdown = df_comparison.to_markdown(index=False)
# print(table_markdown)
```

t	Euler Approximation	Exact Solution
0	2	2
0.1	2.2	2.21034
0.2	2.42	2.44281
0.3	2.662	2.69972
0.4	2.9282	2.98365
0.5	3.22102	3.29744
0.6	3.54312	3.64424
0.7	3.89743	4.02751

t	Euler Approximation	Exact Solution
0.8	4.28718	4.45108
0.9	4.7159	4.91921
1	5.18748	5.43656

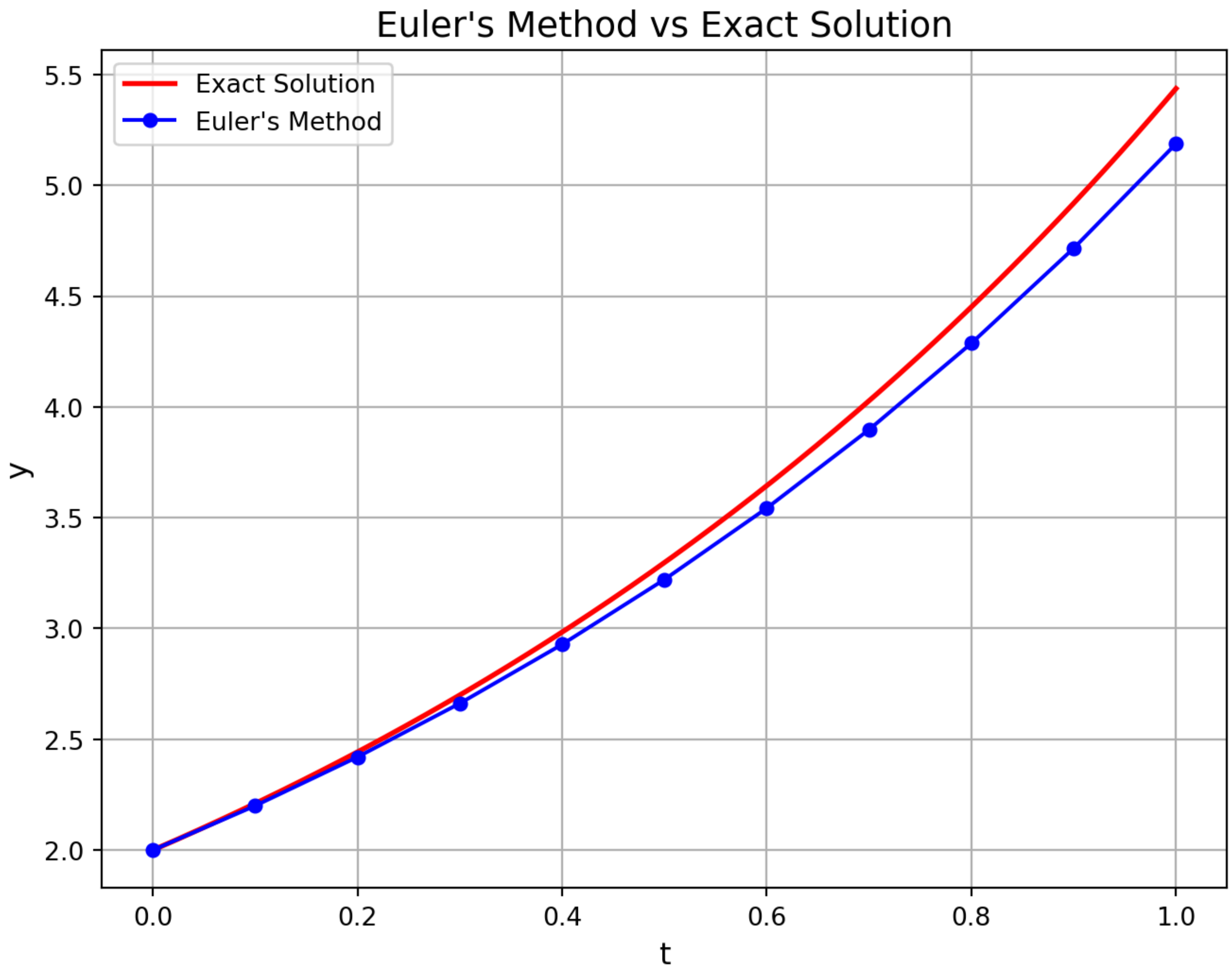
Graph:

▼ Show code

```
# Plot Euler's Method approximation and exact solution
plt.figure(figsize=(8, 6))
plt.plot(t_vals_exact, y_exact, 'r', label="Exact Solution", linewidth=2)
plt.plot(t_vals_euler, y_euler, 'bo-', label="Euler's Method", markersize=5)

# Add labels, title, and legend
plt.title("Euler's Method vs Exact Solution", fontsize=14)
plt.xlabel("t", fontsize=12)
plt.ylabel("y", fontsize=12)
plt.grid(True)
plt.legend()

# Display the plot
plt.show()
```



Final Answer:

The **exact solution** to the differential equation is:

$$y = 2e^t$$

The graph compares Euler's method with the exact solution. Euler's method provides a reasonable approximation but slightly underestimates the true values as t increases.

2.6.11

Problem:

Solve the initial value problem using Euler's method with $h = 0.1$ on the interval $[0, 1]$:

$$(y')^2 - 2y^2 = t, \quad y(0) = 2$$

In addition, explain why it is not possible to solve the IVP exactly by established methods.

Solution:

Rearrange the Differential Equation

We are given:

$$(y')^2 - 2y^2 = t$$

First, solve for y':

$$(y')^2 = t + 2y^2$$

Taking the square root of both sides (assuming the positive root):

$$y' = \sqrt{t + 2y^2}$$

Euler's Method

Euler's method uses the formula:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where $f(t, y) = \sqrt{t + 2y^2}$, and the initial condition is $y(0) = 2$.

Given:

- Step size $h = 0.1$
- Initial condition $y(0) = 2$
- Differential equation: $y' = \sqrt{t + 2y^2}$

Calculate the approximate solution using Euler's method.

Table

▶ Show code

t	Euler Approximation
0	2
0.1	2.28284
0.2	2.60723
0.3	2.97865
0.4	3.40344
0.5	3.8889
0.6	4.4434
0.7	5.07655
0.8	5.79934
0.9	6.62435
1	7.56597

Graph

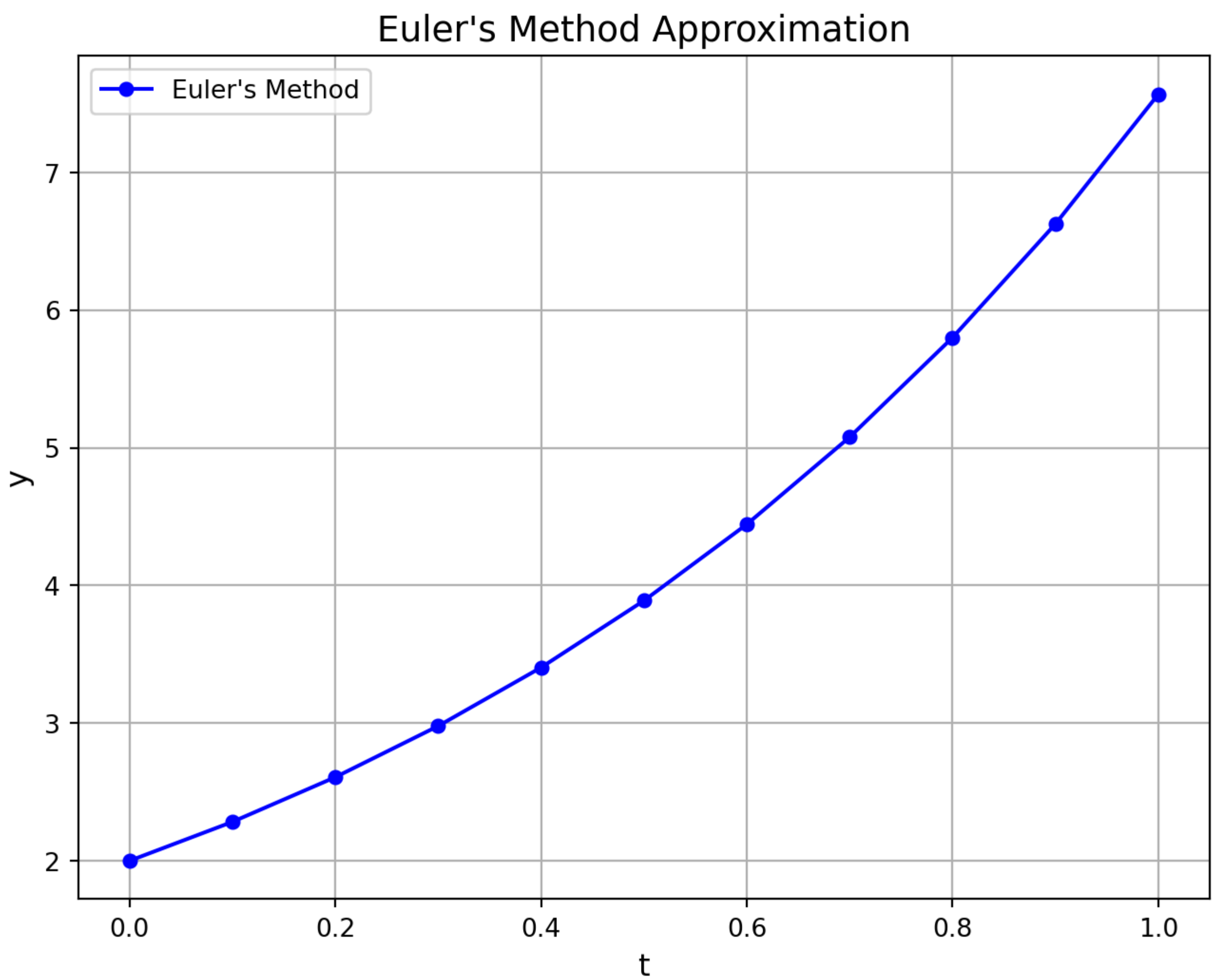
▼ Show code

```
plt.figure(figsize=(8, 6))
plt.plot(t_vals_euler, y_euler, 'bo-', label="Euler's Method", markersize=5)

# Add labels, title, and legend
plt.title("Euler's Method Approximation", fontsize=14)
plt.xlabel("t", fontsize=12)
```

```
plt.ylabel("y", fontsize=12)
plt.grid(True)
plt.legend()

# Display the plot
plt.show()
```



Final Answer:

The equation involves $(y')^2$, making it non-linear and not separable. As a result, standard methods like separation of variables or integrating factors cannot be applied, so numerical methods such as Euler's method are needed to approximate the solution.