Jacobi Method Convergence

Theorem 2.10 (p. 107)

Theorem 2.10 (p. 107): If the $n \times n$ matrix A is strictly diagonally dominant, then (1) A is a nonsingular matrix, and (2) for every vector b and every starting guess, the Jacobi Method applied to $Ax = b$ converges to the (unique) solution.

Theorem A.7 (p. 588): If the $n \times n$ matrix A has spectral radius $\rho(A) < 1$, and b is arbitrary, then, for any vector x_0, the iteration $x_{k+1} = A\,x_k + b$ converges. In fact, there exists a unique x_* such that $\lim_{k\to\infty} x_k = x_*$ and $x_* = Ax_* + b$.

Definition The **spectral radius** $\rho(A)$ of a square matrix A is the maximum magnitude of its eigenvalues.

Definition The **infinity** or max norm of a vector $x \in \mathbb{R}^n$ is $||x||_\infty = \max_{1\le i\le n} |x_i|$.

Proof:

Recall that the Jacobi Method for solving $A\,x = b$ is $x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b$, where $A = L + D + U$, L is the lower triangular part of A, D is the diagonal part of A, and U is the upper triangular part of A.

We will apply Theorem A.7 by showing that the spectral radius of $-D^{-1}(L+U)$ is less than 1, or $\rho(D^{-1}(L+U)) < 1$. For notational convenience, let $R = L+U$ denote the non-diagonal part of the matrix A. Then we must show that $\rho(D^{-1}R) < 1$.

(1) Given any vector x, we can create a scaled version of x, say v, as $v = \frac{x}{c}$. What value of c will guarantee that $||v||_\infty = 1$?

(2) Let λ represent an arbitrary eigenvalue of $D^{-1}R$ with corresponding eigenvector v. Then $D^{-1}R\,v = \lambda\,v$, or $Rv = \lambda Dv$ (why?). We'll look at each side of this equation in turn.

Suppose we scale the eigenvector v such that $||v||_\infty = 1$. Then $|v_i| \le 1$ for every index i, $1 \le i \le n$, and $|v_m| = 1$ for at least one index m, $1 \le m \le n$. Using this index m, explain why the absolute value of the m^{th} row of Rv is

$$|r_{m,1}v_1 + r_{m,2}v_2 + \cdots + r_{m,m-1}v_{m-1} + r_{m,m+1}v_{m+1} + \cdots + r_{m,n}v_n|.$$

(3) Now, explain why the absolute value of the m^{th} row of λDv is $|\lambda||d_{m,m}|$.

Combining steps (2) and (3), we can write

$$|\lambda||d_{m,m}| = |r_{m,1}v_1 + r_{m,2}v_2 + \cdots + r_{m,m-1}v_{m-1} + r_{m,m+1}v_{m+1} + \cdots + r_{m,n}v_n|. \tag{1}$$

(4) Explain why $|r_{m,1}v_1 + r_{m,2}v_2 + \cdots + r_{m,m-1}v_{m-1} + r_{m,m+1}v_{m+1} + \cdots + r_{m,n}v_n| \leq \sum_{j \neq m} |r_{m,j}|$

(5) Explain why $\sum_{j \neq m} |r_{m,j}| < |d_{m,m}|$.

(6) Use the results from Steps (4) and (5) with Equation 1 to show that $|\lambda||d_{m,m}| < |d_{m,m}|$. What does this say about $|\lambda|$?

(7) Since λ is an arbitrary eigenvalue, then $|\lambda_{\max}| < 1$. In other words, the spectral radius $\rho(D^{-1}R) < 1$. Thus, by Theorem A.7, the Jacobi Method (iteration with $A = D^{-1}R$) converges for any starting point x_0. Let $x_* = \lim_{k \to \infty} x_k$, and show that x_* is the solution to $A\,x = b$, so A must be nonsingular. This completes the proof of Theorem 2.10.