Convergence of Newton's Method

Theorem 1.11 (p. 53)

<u>Theorem 1.11</u> (p. 53): Let f be twice continuously differentiable and $f(r) = 0$. If $f'(r) \neq 0$, then Newton's Method is locally and quadratically convergent to r. The error e_i at step i satisfies

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} = M,$$

where

$$M = \left| \frac{f''(r)}{2f'(r)} \right|.$$

<u>Proof:</u>

(1) Newton's Method is $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$. This has the form of Fixed-Point Iteration (i.e., $x_{i+1} = g(x_i)$), where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Show that $g'(r) = 0$.

What does this imply about the convergence of Newton's Method? (*Hint: see Theorem 1.6*).

(2) To complete our proof, we need to show that $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = \left|\frac{f''(r)}{2f'(r)}\right|$.

Begin by writing out the first <u>two</u> terms of the Taylor series for $f(r)$ centered at x_i, along with the remainder term (see p. 21, with $x = r$, $x_0 = x_i$, and $c = c_i$).

(3) Manipulate your result so it looks like this *(note: the LHS of this equation is the RHS of Newton's Method)*:

$$x_i - \frac{f(x_i)}{f'(x_i)} = r + \frac{f''(c_i)}{2f'(x_i)}(r - x_i)^2.$$

In order for these expressions to be defined, we must have $f'(x_i) \neq 0$. By assumption, we have $f'(r) \neq 0$. What allows us to conclude that the same is true for $f'(x_i)$?

(4) Let $e_i = |r - x_i|$ (so $e_{i+1} = |r - x_{i+1}|$). Use these together with the definition of Newton's method in step (1) and your result from step (3) to obtain

$$e_{i+1} = \left| \frac{f''(c_i)}{2f'(x_i)} \right| e_i^2.$$

(5) Dividing both sides by e_i^2 and taking the limit as $i \to \infty$ will complete the proof. However, evaluating the limit on the right-hand side requires some care. Justify each of the following steps in this process:

$$\begin{aligned}
\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} &= \lim_{i\to\infty} \left| \frac{f''(c_i)}{2f'(x_i)} \right| \\
&= \left| \lim_{i\to\infty} \frac{f''(c_i)}{2f'(x_i)} \right| && \text{rationale:} \\
&= \left| \frac{\lim_{i\to\infty} f''(c_i)}{\lim_{i\to\infty} 2f'(x_i)} \right| && \text{rationale:} \\
&= \left| \frac{f''(\lim_{i\to\infty} c_i)}{2f'(\lim_{i\to\infty} x_i)} \right| && \text{rationale:} \\
&= \left| \frac{f''(\lim_{i\to\infty} c_i)}{2f'(r)} \right| && \text{rationale:} \\
&= \left| \frac{f''(r)}{2f'(r)} \right| && \text{rationale:}
\end{aligned}$$