2. Apply Multivariate Newton's Method to the system (4.35) for the three circles in Computer Problem 1. Use initial vector $(x_0, y_0, K_0) = (0, 0, 0)$.

3. Find the point (x, y) and distance K that minimizes the sum of squares distance to the circles with radii increased by K, as in Example 4.23 (a) circles with centers $(-1, 0), (1, 0), (0, 1), (0, -2)$ and all radii 1 (b) circles with centers $(-2, 0), (3, 0), (0, 2), (0, -2)$ and all radii 1.

4. Carry out the steps of Computer Problem 3 with the following circles and plot the results
(a) centers $(-2, 0), (2, 0), (0, 2), (0, -2)$, and $(2, 2)$, with radii 1, 1, 1, 1, 2 respectively
(b) centers $(1, 1), (1, -1), (-1, 1), (-1, -1), (2, 0)$ and all radii 1.

5. Use the Gauss–Newton Method to fit a power law to the height–weight data of Example 4.10 without linearization. Compute the RMSE.

6. Use the Gauss–Newton Method to fit the blood concentration model (4.21) to the data of Example 4.11 without linearization.

7. Use the Levenberg–Marquardt Method with $\lambda = 1$ to fit a power law to the height–weight data of Example 4.10 without linearization. Compute the RMSE.

8. Use the Levenberg–Marquardt Method with $\lambda = 1$ to fit the blood concentration model (4.21) to the data of Example 4.11 without linearization.

9. Apply Levenberg–Marquardt to fit the model $y = c_1 e^{-c_2(t-c_3)^2}$ to the following data points, with an appropriate initial guess. State the initial guess, the regularization parameter λ used, and the RMSE. Plot the best least squares curve and the data points.
(a) $(t_i, y_i) = \{(-1, 1), (0, 5), (1, 10), (3, 8), (6, 1)\}$
(b) $(t_i, y_i) = \{(1, 1), (2, 3), (4, 7), (5, 12), (6, 13), (8, 5), (9, 2)(11, 1)\}$

10. Further investigate Example 4.25 by determining the initial guesses from the grid $0 \le c_1 \le 10$ with a grid spacing of 1, and $0 \le c_2 \le 1$ with a grid spacing of 0.1, $c_3 = 1$, for which the Levenberg–Marquardt Method converges to the correct least squares solution. Use the MATLAB `mesh` command to plot your answers, 1 for a convergent initial guess and 0 otherwise. Make plots for $\lambda = 50$, $\lambda = 1$, and the Gauss–Newton case $\lambda = 0$. Comment on the differences you find.

11. Apply Levenberg–Marquardt to fit the model $y = c_1 e^{-c_2 t} \cos(c_3 t + c_4)$ to the following data points, with an appropriate initial guess. State the initial guess, the regularization parameter λ used, and the RMSE. Plot the best least squares curve and the data points. This problem has multiple solutions with the same RMSE, since c_4 is only determined modulo 2π.
(a) $(t_i, y_i) = \{(0, 3), (2, -5), (3, -2), (5, 2), (6, 1), (8, -1), (10, 0)\}$
(b) $(t_i, y_i) = \{(1, 2), (3, 6), (4, 4), (5, 2), (6, -1), (8, -3)\}$

Reality Check 4 GPS, Conditioning, and Nonlinear Least Squares

The global positioning system (GPS) consists of 24 satellites carrying atomic clocks, orbiting the earth at an altitude of 20,200 km. Four satellites in each of six planes, slanted at 55° with respect to the poles, make two revolutions per day. At any time, from any point on earth, five to eight satellites are in the direct line of sight. Each satellite has a simple mission: to transmit carefully synchronized signals from predetermined positions in space, to be picked up by GPS receivers on earth. The receivers use the information, with some mathematics (described shortly), to determine accurate (x, y, z) coordinates of the receiver.

At a given instant, the receiver collects the synchronized signal from the ith satellite and determines its transmission time t_i, the difference between the times the signal was transmitted and received. The nominal speed of the signal is the speed of light, $c \approx 299792.458$ km/sec. Multiplying transmission time by c gives the distance of the satellite from the receiver, putting the receiver on the surface of a sphere centered at the satellite position and with radius ct_i. If three satellites are available, then three spheres are known, whose intersection consists of two points, as shown in Figure 4.16. One intersection point is the location of the receiver. The other is normally far from the earth's surface and can be safely disregarded. In theory, the problem is reduced to computing this intersection, the common solution of three sphere equations.

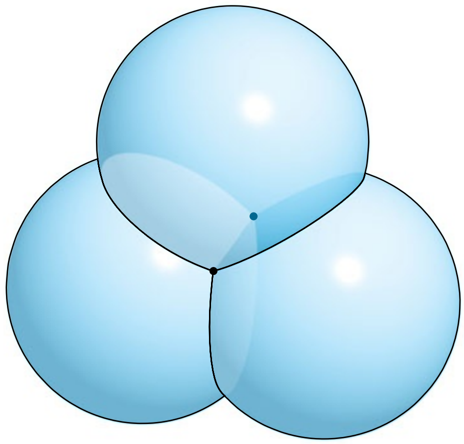

Figure 4.16 Three Intersecting Spheres. Generically, only two points lie on all three spheres.

However, there is a major problem with this analysis. First, although the transmissions from the satellites are timed nearly to the nanosecond by onboard atomic clocks, the clock in the typical low-cost receiver on earth has relatively poor accuracy. If we solve the three equations with slightly inaccurate timing, the calculated position could be wrong by several kilometers. Fortunately, there is a way to fix this problem. The price to pay is one extra satellite. Define d to be the difference between the synchronized time on the (now four) satellite clocks and the earth-bound receiver clock. Denote the location of satellite i by (A_i, B_i, C_i). Then the true intersection point (x, y, z) satisfies

$$\begin{aligned}
r_1(x, y, z, d) &= \sqrt{(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2} - c(t_1 - d) = 0 \\
r_2(x, y, z, d) &= \sqrt{(x - A_2)^2 + (y - B_2)^2 + (z - C_2)^2} - c(t_2 - d) = 0 \\
r_3(x, y, z, d) &= \sqrt{(x - A_3)^2 + (y - B_3)^2 + (z - C_3)^2} - c(t_3 - d) = 0 \\
r_4(x, y, z, d) &= \sqrt{(x - A_4)^2 + (y - B_4)^2 + (z - C_4)^2} - c(t_4 - d) = 0
\end{aligned} \tag{4.37}$$

to be solved for the unknowns x, y, z, d. Solving the system reveals not only the receiver location, but also the correct time from the satellite clocks, due to knowing d. Therefore, the inaccuracy in the GPS receiver clock can be fixed by using one extra satellite.

Geometrically speaking, four spheres may not have a common intersection point, but they will if the radii are expanded or contracted by the right common amount. The system (4.37) representing the intersection of four spheres is the three-dimensional analogue of (4.35), representing the intersection point of three circles in the plane.

The system (4.37) can be seen to have two solutions (x, y, z, d). The equations can be equivalently written

$$\begin{aligned}(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2 &= [c(t_1 - d)]^2\\(x - A_2)^2 + (y - B_2)^2 + (z - C_2)^2 &= [c(t_2 - d)]^2\\(x - A_3)^2 + (y - B_3)^2 + (z - C_3)^2 &= [c(t_3 - d)]^2\\(x - A_4)^2 + (y - B_4)^2 + (z - C_4)^2 &= [c(t_4 - d)]^2.\end{aligned} \tag{4.38}$$

Note that by subtracting the last three equations from the first, three *linear* equations are obtained. Each linear equation can be used to eliminate a variable x, y, z, and by substituting into any of the original equations, a quadratic equation in the single variable d results. Therefore, system (4.37) has at most two real solutions, and they can be found by the quadratic formula.

Two further problems emerge when GPS is deployed. First is the conditioning of the system of equations (4.37). We will find that solving for (x, y, z, d) is ill-conditioned when the satellites are bunched closely in the sky.

The second difficulty is that the transmission speed of the signals is not precisely c. The signals pass through 100 km of ionosphere and 10 km of troposphere, whose electromagnetic properties may affect the transmission speed. Furthermore, the signals may encounter obstacles on earth before reaching the receiver, an effect called multipath interference. To the extent that these obstacles have an equal impact on each satellite path, introducing the time correction d on the right side of (4.37) helps. In general, however, this assumption is not viable and will lead us to add information from more satellites and consider applying Gauss–Newton to solve a least squares problem.

Consider a three-dimensional coordinate system whose origin is the center of the earth (radius $\approx$ 6370 km). GPS receivers convert these coordinates into latitude, longitude, and elevation data for readout and more sophisticated mapping applications using global information systems (GIS), a process we will not consider here.

Suggested activities:

1. Solve the system (4.37) by using Multivariate Newtons Method. Find the receiver position (x, y, z) near earth and time correction d for known, simultaneous satellite positions $(15600, 7540, 20140)$, $(18760, 2750, 18610)$, $(17610, 14630, 13480)$, $(19170, 610, 18390)$ in km, and measured time intervals 0.07074, 0.07220, 0.07690, 0.07242 in seconds, respectively. Set the initial vector to be $(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$. As a check, the answers are approximately $(x, y, z) = (-41.77271, -16.78919, 6370.0596)$, and $d = -3.201566 \times 10^{-3}$ seconds.

2. Write a MATLAB program to carry out the solution via the quadratic formula. Hint: Subtracting the last three equations of (4.37) from the first yields three linear equations in the four unknowns $x\vec{u}_x + y\vec{u}_y + z\vec{u}_z + d\vec{u}_d + \vec{w} = 0$, expressed in vector form. A formula for x in terms of d can be obtained from

$$0 = \det[\vec{u}_y|\vec{u}_z|x\vec{u}_x + y\vec{u}_y + z\vec{u}_z + d\vec{u}_d + \vec{w}],$$

noting that the determinant is linear in its columns and that a matrix with a repeated column has determinant zero. Similarly, we can arrive at formulas for y and z, respectively, in terms of d, that can be substituted in the first quadratic equation of (4.37), to make it an equation in one variable.

3. If the MATLAB Symbolic Toolbox is available (or a symbolic package such as Maple or Mathematica), an alternative to Step 2 is possible. Define symbolic variables by using the `syms` command and solve the simultaneous equations with the Symbolic Toolbox command `solve`. Use `subs` to evaluate the symbolic result as a floating point number.

4. Now set up a test of the conditioning of the GPS problem. Define satellite positions (A_i, B_i, C_i) from spherical coordinates (ρ, ϕ_i, θ_i) as

$$A_i = \rho \cos\phi_i \cos\theta_i$$
$$B_i = \rho \cos\phi_i \sin\theta_i$$
$$C_i = \rho \sin\phi_i,$$

where $\rho = 26570$ km is fixed, while $0 \le \phi_i \le \pi/2$ and $0 \le \theta_i \le 2\pi$ for $i = 1, \ldots, 4$ are chosen arbitrarily. The ϕ coordinate is restricted so that the four satellites are in the upper hemisphere. Set $x = 0$, $y = 0$, $z = 6370$, $d = 0.0001$, and calculate the corresponding satellite ranges $R_i = \sqrt{A_i^2 + B_i^2 + (C_i - 6370)^2}$ and travel times $t_i = d + R_i/c$.

We will define an error magnification factor specially tailored to the situation. The atomic clocks aboard the satellites are correct up to about 10 nanoseconds, or 10^{-8} second. Therefore, it is important to study the effect of changes in the transmission time of this magnitude. Let the backward, or input error be the input change in meters. At the speed of light, $\Delta t_i = 10^{-8}$ second corresponds to $10^{-8}c \approx 3$ meters. Let the forward, or output error be the change in position $||(\Delta x, \Delta y, \Delta z)||_\infty$, caused by such a change in t_i, also in meters. Then we can define the dimensionless

$$\text{error magnification factor} = \frac{||(\Delta x, \Delta y, \Delta z)||_\infty}{c||(\Delta t_1, \ldots, \Delta t_m)||_\infty},$$

and the condition number of the problem to be the maximum error magnification factor for all small Δt_i (say, 10^{-8} or less).

Change each t_i defined in the foregoing by $\Delta t_i = +10^{-8}$ or -10^{-8}, not all the same. Denote the new solution of the equations (4.37) by $(\overline{x}, \overline{y}, \overline{z}, \overline{d})$, and compute the difference in position $||(\Delta x, \Delta y, \Delta z)||_\infty$ and the error magnification factor. Try different variations of the Δt_i's. What is the maximum position error found, in meters? Estimate the condition number of the problem, on the basis of the error magnification factors you have computed.

5. Now repeat Step 4 with a more tightly grouped set of satellites. Choose all ϕ_i within 5 percent of one another and all θ_i within 5 percent of one another. Solve with and without the same input error as in Step 4. Find the maximum position error and error magnification factor. Compare the conditioning of the GPS problem when the satellites are tightly or loosely bunched.

6. Decide whether the GPS error and condition number can be reduced by adding satellites. Return to the unbunched satellite configuration of Step 4, and add four more. (At all times and at every position on earth, 5 to 12 GPS satellites are visible.) Design a Gauss–Newton iteration to solve the least squares system of eight equations in four variables (x, y, z, d). What is a good initial vector? Find the maximum GPS position error, and estimate the condition number. Summarize your results from four unbunched, four bunched, and eight unbunched satellites. What configuration is best, and what is the maximum GPS error, in meters, that you should expect solely on the basis of satellite signals? ✔