Lagrange's Equations for Particles

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1 A Short Overview

In these notes we derive Lagrange's equations in generalized coordinates q with velocities $\dot{q}$. The basic idea is that Lagrange's equations are a *covariant* form of Newton's equation $F = ma$; the Lagrange equations can be viewed as a way to write Newton's law in an arbitrary coordinate system. In fact this represents an underlying principle that Einstein generalized in the theory of relativity. If Newton's equations hold for Cartesian coordinates, then Lagrange's equations hold in any coordinate system related to Cartesian coordinates by a non-singular transformation.

In more detail, we consider a system with N degrees of freedom, described by a configuration space with coordinates $q = (q_1, q_2, \ldots, q_N)$. Define the N-component (candidate to be a vector) $\mathbb{L} = (\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_N)$ with

$$\boxed{\mathbb{L}_i = \frac{d}{dt}\frac{\partial\mathcal{L}(q,\dot{q},t)}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}(q,\dot{q},t)}{\partial q_i}\ , \quad \text{where } i = 1, \ldots, N}\ .$$

The Lagrange equations are

$$\mathbb{L} = 0\ . \tag{1}$$

1.1 Lagrange's Equations

Here it the plan to derive to derive (1) from $M\vec{a} = \vec{F}$. Once we have done that, we take the Lagrange equations as basic. Physics is described by the Lagrangian, and this Lagrangian may be different from the Lagrangian of Newton. In fact Lagrangians describe special-relativistic physics, as well as non-relativistic physics. They can also describe motion on geodesics in curved space, namely general-relativistic physics. We list four major step for the usual form of Lagrange equations, and three more steps in case there are constraints.

I. Suppose that we take q to be the Cartesian coordinates $q = (x_1, x_2, \ldots, x_N)$. In this case we set $\mathbb{L} = \mathbb{N}$ (for Newton). We check that with Lagrangian $\mathcal{L} = T - V$, the equation $\mathbb{N} = 0$ is identical to Newton's equations of motion $m_i\ddot{x}_i = -\frac{\partial V}{\partial x_i}$, namely $M\vec{a} = \vec{F}$.

II. Consider a change of coordinates $x \to q$. We show that the vector $\mathbb{L}$ transforms *covariantly*. This means that there is a matrix J that relates $\mathbb{L}$ linearly to $\mathbb{N}$, namely

$$J\mathbb{L} = \mathbb{N}\ .$$

The matrix J is the Jacobian of the transformation $x \to q$.

III. If J is an invertible $N \times N$ matrix, then $\mathbb{L} = 0$ if and only if Newton's equations $\mathbb{N} = 0$ holds. Thus Lagrange equations in general coordinates are a consequence of Lagrange equations in Cartesian coordinates.

IV. Suppose we have two invertible transformations $x \to q$ and $x \to q'$ given by Jacobians $J_{x\to q}$ and $J_{x\to q'}$. Then the Jacobian $J_{q\to q'}$ is given by

$$J_{q\to q'} = J_{x\to q}^{-1} J_{x\to q'} \quad , \tag{2}$$

and it satisfies

$$J_{q\to q'} \mathbb{L}' = \mathbb{L} \; . \tag{3}$$

It is in this sense that the Lagrange equations are *covariant* under a general transformation of coordinates $q \to q'$. The equations in the two coordinate systems are linearly related by the Jacobian matrix of the transformation between the two coordinate systems.

1.2 Lagrange's Equations with Constraints

The invertibility of the Jacobian matrix J means that J has maximal rank. We can generalize this picture by assuming that there are N variables q related to n variables x, and that $N > n$. Thus the redundant variables q have to satisfy some relations among themselves; these relations are "constraints." In this case, $J_{x\to q}$ is no longer a square matrix, but it is an $n \times N$ matrix. Let us assume as before, that J has maximum rank, which in this case is n. We find as before that

$$J\mathbb{L} = \mathbb{N} \; . \tag{4}$$

Now $k = N - n$ plays an important role, because the Jacobian J has k linearly-independent unit vectors $v^{(j)}$ that are null vectors for the matrix J. They satisfy

$$Jv^{(j)} = 0 \; , \quad \text{for} \quad j = 1, \ldots, k \; . \tag{5}$$

V. The relation $J\mathbb{L} = \mathbb{N} = 0$ now means that the Lagrange equations take the form

$$\mathbb{L} = \sum_{j=1}^{k} \lambda_j v^{(j)} \; . \tag{6}$$

The unknown coefficients λ_j are called Lagrange multipliers. These multipliers and the vectors $v^{(j)}$ are additional unknowns that need to be determined in order to solve the Lagrange equations.

VI. If the coordinates q satisfy a constraint $f(q) = 0$, and we express the coordinates q as functions of x, namely $q = q(x)$, then $v = \nabla_q f(q)$ is a null vector for J. Such a gradient is interpreted in physics as a "force of constraint" in the Lagrange equations.

VII. Different constraint functions $f^{(j)}(q)$, for $j = 1, 2, \ldots, k$, are said to be *independent*, if their gradient vectors are linearly independent. These give a set of forces of constraint.

2 A Paradigm Shift

Before we go further, let's talk about the significance of the Lagrangian method. At a pedestrian level, Lagrangian mechanics provides a computational approach to understand a variety of force laws arising from Newton's $F = ma$. Let's illustrate this in a simple case. Suppose the acceleration $a = \frac{d}{dt}v$ is the time rate of change of velocity v, and the momentum is $p = mv$, so $\frac{d}{dt}p = ma$ and Newton's equation is $F = \frac{d}{dt}p$. Lagrangian mechanics gives us a routine way to derive specific forms of Newton's equations for different coordinate systems and different physical situations.

Stepping back to a conceptual overview, the shift in emphasis from Newton's law to the Lagrangian formulation actually marks a profound change. This modifies the way we think about physical laws, and will not be apparent if we only focus on computation. Lagrangian mechanics marks a shift from describing fundamental physics by particular equations, to regarding the form of a Lagrangian as the basic idea.

The fundamental idea of Lagrangian mechanics is *covariance*. The physical law remains the same in different choices of coordinates! For example choose configuration space coordinates $q = (q_1, \ldots, q_N)$, that depend on a time parameter t, describing velocities $\dot{q}_i(t) = \frac{dq_i(t)}{dt}$, which we denote by $\dot{q}$. Assume also that you are given a Lagrangian $\mathcal{L}(q(t), \dot{q}(t))$ (evaluated at some time). Then the form of the Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{7}$$

does not change if we change from coordinate system q to the coordinate system $Q(t) = Q(q(t))$. This is unlike Newton's equations, which have a different form in Cartesian coordinates from how they appear in polar coordinates (where centripetal and Coriolis terms appear). In other words Lagrangian mechanics is "covariant," a principle that pervades relativity theory in physics and also much of modern mathematics. This thinking also led Einstein to the equations of general relativity. So we start these notes with a discussion of what happens when one changes coordinates.

These changes in thinking are what is called a *paradigm shift*. Although Newton's equations remain central to understanding time evolution in physics, the focus in understanding "how the universe works" focused on "what is the form of the Lagrangian" as a way to conceptualize the laws of nature. The resulting Lagrange equations $\mathfrak{F} = \frac{d}{dt}\mathfrak{p}$ have a similar form to Newton's equations. Here force F and momentum p in Newton's law are replaced by a "generalized" force $\mathfrak{F}$ and "generalized" momentum $\mathfrak{p}$, both derived from a Lagrangian $\mathcal{L}$. The major point is that the secrets of nature are all encoded in an intuitive Lagrangian function $\mathcal{L}$ to conceptualize the physical ideas.

Lagrange's formulation has much wider applicability than just particle motion in non-relativistic mechanics. There is a Lagrangian formulation for the motion of particles moving relativistically. Lagrangian mechanics applies to optics. Lagrangians theory applies to the evolution of fields, such as for the electromagnetic field (Maxwell fields), matter fields, etc.

The study of the Lagrangian is closely tied to another important approach, Hamiltonian mechanics, which is just as important as Lagrangian mechanics. Both of them are central to understanding classical physics, as well as lying at the basis of both the formulation of quantum theory, as well as its detailed understanding. They are also central to understanding the relations between symmetry and conservation laws that we study in this course in classical physics, and which appear also in a similar fashion in the context of quantum theory.

Lagrange's equations also fit naturally into the whole area of variational principles (action principles) in physics, which we will explore at length later in the term. These variational principles not only lead to the derivation of equations of motion, but to the formulation of other ideas, such as the principle of least action or the principle of least time. Variational principles also relate the Lagrangian and the Hamiltonian approaches.

While we will not cover all the topics mentioned here in this course, you should be happy to know that you will encounter the same concepts we learn in this course in the every other course or book about physics. As Lagrangian and Hamiltonian ideas permeate all of modern physics, we will have a great deal of fun in this course!

3 Change of Coordinates

A very special aspect of physics concerns a *change of variables* and its relation to how one formulates the laws of physics. In particular, one often looks for *covariant* laws that have the same form in different coordinate systems. One might mention that Einstein liked this notion a lot, and his work on relativity (both special and general) revolves around this concept, which is closely tied to *symmetry* in physics.

However covariance emerged much earlier in classical mechanics. The form of Lagrange's equations remains the same in a wide variety of coordinate systems, as a consequence that Lagrange equations transform from one coordinate system to another according to a transformation law determined by the Jacobian matrix that relates these coordnate systems! We will see in these notes that the equations have a certain *covariance* under change from one coordinate system to another.

3.1 The Jacobian

We begin the discussion of transformations from a coordinate system x (which here we take to be Cartesian) to a new (or generalized) coordinate system q. We denote this $x \to q$ or $q = q(x)$. A given function of the coordinate, such as the potential $V(q)$, could also be expressed as a function of x by $\widetilde{V}(x) = V(q(x))$, with the tilde to emphasize that V and $\widetilde{V}$ are different functions. Here we assume that the transformation $x \to q$ is invertible at almost all points, with the inverse $q \to x$, and the identity transformation $q = q(x) = q(x(q))$ or $x = x(q) = x(q(x))$.

For example, if x and q are one-dimensional, then the derivative characterizes the invertibility of the transformation. One can write the transformation of an increment as,

$$dx = \frac{dx}{dq} dq \tag{8}$$

The assumption of invertibility and the chain rule ensures that

$$1 = \frac{dx(q(x))}{dx} = \frac{dx}{dq}\frac{dq}{dx} \,. \quad \text{So} \quad \frac{dq}{dx} = \frac{1}{\frac{dx}{dq}} \,. \tag{9}$$

The Jacobian matrix gives the similar transformation for the volume element in N dimensions. For a transformation $q \to x$ yielding $x = x(q)$, let us denote the Jacobian matrix to be $J_{q\to x}$. In particular we write in the case of N variables $x = (x_1, x_2, \ldots, x_N)$ and N variables $q = (q_1, q_2, \ldots, q_N)$, the matrix $J_{q\to x}$ is

$$J_{q\to x} = \frac{\partial(x_1,\ldots,x_N)}{\partial(q_1,\ldots,q_N)} = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \cdots & \frac{\partial x_N}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \cdots & \frac{\partial x_N}{\partial q_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_1}{\partial q_N} & \frac{\partial x_2}{\partial q_N} & \cdots & \frac{\partial x_N}{\partial q_N} \end{pmatrix} . \tag{10}$$

The individual matrix entries are

$$(J_{q\to x})_{ij} = \frac{\partial x_j}{\partial q_i} , \tag{11}$$

where the partial derivative $\frac{\partial x_i}{\partial q_j}$ means that one varies the coordinate $x_i = x(q)_i$ by changing q_j, while keeping fixed the coordinates q_k, for $k \neq j$.

The Jacobian $J_{x\to q}$ of the inverse transformation $x \to q$ has the form

$$J_{x\to q} = \frac{\partial(q_1,\ldots,q_N)}{\partial(x_1,\ldots,x_N)} , \quad \text{with entries} \quad (J_{x\to q})_{ij} = \frac{\partial q_j}{\partial x_i} , \tag{12}$$

or

$$J_{x\to q} = \frac{\partial(q_1,\ldots,q_N)}{\partial(x_1,\ldots,x_N)} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_N}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_N}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial q_1}{\partial x_N} & \frac{\partial q_2}{\partial x_N} & \cdots & \frac{\partial q_N}{\partial x_N} \end{pmatrix} . \tag{13}$$

In this case the individual matrix entries are

$$(J_{x\to q})_{ij} = \frac{\partial q_j}{\partial x_i} , \tag{14}$$

where the partial derivative $\frac{\partial q_i}{\partial x_j}$ means that one varies the coordinate $q_i = q(x)_i$ by changing x_j, while keeping fixed the coordinates x_k, for $k \neq j$.

3.2 Composition of Transformations and Multiplication of Jacobians

Suppose that we have two invertible transformations $q \to q''$ and $q'' \to q'$. Then we can compose them to obtain the transformation $q \to q' = q \to q'' \to q'$ given by $q'(q) = q'(q''(q))$.

Proposition 3.1. *The Jacobian $J_{q\to q'}$ of the composition of two transformations $q \to q' = q \to q'' \to q'$ is the product of Jacobian matrices of the individual transformations,*

$$J_{q\to q'} = J_{q\to q''} J_{q''\to q'} . \tag{15}$$

Also

$$J_{q'\to q} = (J_{q\to q'})^{-1} . \tag{16}$$

Proof. This identity follows from the chain rule for differentiation. Assuming there are N components in all three coordinate systems q, x, q', one can rewrite (15) as

$$
\begin{aligned}
(J_{q\to q'})_{ij} &= \left(\frac{\partial\,(q'_1,\ldots,q'_N)}{\partial(q_1,\ldots,q_N)}\right)_{ij} = \frac{\partial q'_j}{\partial q_i} = \sum_{k=1}^{N} \frac{\partial q'_j}{\partial q''_k}\frac{\partial q''_k}{\partial q_i} \\
&= \sum_{k=1}^{N} \left(\frac{\partial\,(q''_1,\ldots,q''_N)}{\partial(q_1,\ldots,q_N)}\right)_{ik} \left(\frac{\partial\,(q'_1,\ldots,q'_N)}{\partial(q''_1,\ldots,q''_N)}\right)_{kj} = \sum_{k=1}^{N} (J_{q\to q''})_{ik}(J_{q''\to q'})_{kj}\ ,
\end{aligned} \tag{17}
$$

as claimed in (15). Since $J_{q\to q} = I$ is the identity matrix, the relation (16) follows. □

3.3 Scalars, Vectors, Tensors

One can classify functions $f(q)$ by the way they change under a specific transformation of coordinates, or possibly under a family of transformations taken together–like a family of translations or a family of rotations of the coordinate system.

Scalar: Consider the function $S(q)$ depending on two coordinates $q = (q_1, q_2)$. Under the transformation $q = q(q')$, or $q' \to q$ there is a function S' such that

$$S(q(q')) = S'(q')\ . \tag{18}$$

This is a scalar if $S = S'$.

Consider the rotation of coordinates

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q'_1\cos\phi - q'_2\sin\phi \\ q'_1\sin\phi + q'_2\cos\phi \end{pmatrix}\ . \tag{19}$$

Then

$$S(q) = q_1^2 + q_2^2 = ,{q'_1}^2 + {q'_2}^2\ . \tag{20}$$

So for this case the scalar function S is actually *invariant* under rotations, namely

$$S' = S\ . \tag{21}$$

Vector: Suppose that q has N variables, $q = (q_1, q_2, \ldots, q_N)$. One defines a vector $\vec{v}$, or simply v, as a set of N functions $v = (v(q)_1, \ldots, v(q)_N)$ which transform under $q \to q'$ by the action of the Jacobian matrix on the components. There are two natural transformation laws for vector quantities: *covariant and contravariant.*[1] By convention the first kind is covariant, which we give in matrix form and also component form:

$$\textbf{Covariant Vector:}\quad v'(q') = J_{q'\to q}\, v(q)\ , \tag{22}$$

[1]This convention to call a vector covariant or contravariant is not universal. In some books one uses the opposite convention from the one that we use here. The important point is to be consistent. Warning: *Here we use V to denote a covariant vector; this has no relation to our use of the same symbol V to denote a potential function in other parts of the notes!*

or

$$v'(q')_i = \sum_{j=1}^{N} \frac{\partial q_j}{\partial q'_i}\, v(q)_j \ , \quad \text{for } i = 1, \ldots, N \ . \tag{23}$$

Note that for the covariant transformation, the summation is over the numerator in the components of the Jacobian matrix.

Likewise, the second kind of vector $w = w(q)$ is called contravariant. With J^{T} denoting the transpose of the matrix J, the transformation law is

$$\textbf{Contravariant Vector:} \quad w'(q') = J^{\mathrm{T}}_{q \to q'} w(q) \ , \quad \text{or} \quad w'(q')_i = \sum_{j=1}^{N} \frac{\partial q'_i}{\partial q_j}\, w(q)_j \ . \tag{24}$$

The two transformation laws differ by whether the Jacobian is for $q' \to q$ or for the inverse transformation $q \to q'$, and in one case there is a transpose. For the contravariant transformation, the summation is over the denominator in the components of the Jacobian matrix.

If v is a covariant vector and w is a contravariant vector, define their *contraction* by $\langle w, v\rangle = \sum_{k=1}^{N} w_k(q) v_k(q)$. Also let $\langle w', v'\rangle = \sum_{k=1}^{N} w'_k(q') v'_k(q')$.

Proposition 3.2. *The contraction $\langle w, v\rangle$ of a contravariant vector w with a covariant vector v is a scalar.*

Proof. Explicitly,

$$\langle w', v'\rangle = \left\langle J^{\mathrm{T}}_{q \to q'} w, J_{q' \to q} v\right\rangle = \langle w, J_{q \to q'} J_{q' \to q} v\rangle = \langle w, v\rangle \ . \tag{25}$$

In the last identity, we use (16). □

Example 1: Covariant Vector Let $S(q) = S'(q')$, be a scalar under the transformation $q \to q$. We claim that the gradient $V = \nabla S$ of S is a covariant vector. It has with components

$$v_i = \frac{\partial S(q)}{\partial q_i} \ , \quad \text{and also} \quad v'_i = \frac{\partial S'(q')}{\partial q'_i} \ . \tag{26}$$

In fact, using the chain rule for differentiation we check the transformation law. It is

$$v'_i = \frac{\partial S'(q')}{\partial q'_i} = \frac{\partial S(q(q'))}{\partial q'_i} = \sum_{j=1}^{N} \frac{\partial q_j}{\partial q'_i} \frac{\partial S(q)}{\partial q_j} = \sum_{j=1}^{N} \frac{\partial q_j}{\partial q'_i}\, v_j = (J_{q' \to q} v)_i \ , \tag{27}$$

which is the covariant rule (22)–(23). Sometimes, one omits the scalar function and refers to ∇ itself as a covariant vector; in differential geometry this is called a *tangent* vector.

Example 2: Contravariant Vector On the other hand, the differential $w = dq$ with components $w_i = dq_i$ and w' with components $w'_i = dq'_i$ defines a contravariant vector. This has the transformation law

$$w'_i = dq'_i = \sum_{k=1}^{N} \frac{\partial q'_i}{\partial q_k}\, dq_k = \sum_{k=1}^{N} (J_{q\to q'})^{\mathrm{T}}_{ik}\, dq_k = (J^{\mathrm{T}}_{q\to q'} w)_i \; . \tag{28}$$

This is the contravariant law (24). One calls the differential dq a *covector*, or in differential geometry a *cotangent* vector.

One can generalize this to k-component indices, in which the corresponding sets of quantities are called a *tensor* or rank k, and each index transforms under a change of coordinates either covariantly or contravariantly. One example of a tensor is the collection of products of components of k vectors and covectors. For example if $\{v_i\}$ and $\{w_i\}$ are the components of covariant vectors, then $\{v_i w_j\}$ are components of a covariant tensor of rank 2, etc.

3.4 When is a Coordinate Change Invertible?

One Coordinate The simplest case to visualize is the case of a single real coordinate $q \in \mathbb{R}$ is a function of the single coordinate $x \in \mathbb{R}$. The coordinate change $q(x)$ is invertible if and only if $q(x)$ is strictly monotonic, namely strictly increasing or decreasing. For then only one value of x gives a particular value of $q(x)$. We can specify that $x(q)$ is precisely that x.

Think of this in terms of the graph of a function in the plane with coordinates $(x, q(x))$. The inverse function would be the graph that you obtain by rotating the plane by 180 degrees about the line in the plane at 45° to the axes. This rotation interchanges the two axes. However this inverse function is single-valued only if the original function is strictly monotonic. A sufficient condition[2] yielding strict monotonicity of $q(x)$ is the requirement that

$$\frac{dq}{dx} \neq 0 \; . \tag{29}$$

Many Coordinates At first glance, the situation seems much more complicated with many coordinates. However, there is an elementary condition which reduces to (29) in the case of one variable that is sufficient to ensure that $q(x)$ has an inverse function $x(q)$. This inverse function satisfies $x(q(x')) = x'$ for all x'. This condition is the invertibility of the Jacobian matrix.

Any matrix, like J, has an inverse matrix, if and only if it has non-vanishing determinant. The matrix J has an inverse if and only if it has a non-zero determinant. Thus a natural assumption is that

$$\det J_{x\to q} = \det \frac{\partial q}{\partial x} = \det \frac{\partial(q_1, \ldots, q_N)}{\partial(x_1, \ldots, x_N)} \neq 0 \; . \tag{30}$$

The *inverse function theorem* shows that $x(q)$ exists. It is exactly what we need:

[2] This condition is sufficient, but not necessary. If $q = x^3$, then $\frac{dq}{dx} = 0$ at the inflection point $x = 0$, while $q(x)$ is strictly monotonic. The inverse function is $x = q^{1/3}$, and one needs to specify the branch of the cube root to yield $q^{1/3}$ to be monotonic.

Theorem.[3] *If $q(x)$ is differentiable and* $\det J \neq 0$ *in the neighborhood of a particular point $x = a$, then the inverse coordinate transformation $x(q)$ exists in a neighborhood of $q(a)$ and satisfies $x(q(x')) = x'$ in this neighborhood.*

3.5 Change of Volume Element

For one coordinate $q = q(x)$ in terms of x, the expression of the line element dq in terms of the line element dx is given by the derivative,

$$dq = \frac{dq}{dx}\, dx \ . \tag{31}$$

The corresponding transformation of the volume element for N coordinates is given by the transformation of the Jacobian,

$$dq_1\, dq_2 \cdots dq_N = \det J_{x\to q}\ dx_1\, dx_2\ \cdots dx_N \ . \tag{32}$$

The inverse relation is

$$dx_1\, dx_2 \cdots dx_N = \det J_{q\to x}\ dq_1\, dq_2\ \cdots dq_N \ . \tag{33}$$

3.6 The Example Polar Coordinates in Dimensions 2 and 3

A simple example is the transformation from Cartesian coordinates in the plane to plane polar coordinates. Choose

$$q = (r, \theta) \ , \quad \text{and} \quad x = (x_1, x_2) \ .$$

As long as $0 \leqslant r < \infty$ and $0 \leqslant \theta < 2\pi$, there is a single-valued transformation $q(x)$ from Cartesian coordinates to polar coordinates. It is defined for $x \neq 0$ by

$$r = +\sqrt{x_1^2 + x_2^2} \ , \quad \theta = \arctan \frac{x_2}{x_1} \ .$$

There the inverse transformation is

$$x_1 = r\cos\theta \ , \quad x_2 = r\sin\theta \ .$$

Then the Jacobian $J_{q\to x}$ is,

$$J_{q\to x} = \frac{\partial x}{\partial q} = \frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \ , \qquad \text{and} \ \ \det J_{q\to x} = r \ .$$

[3] See for example, Michael Spivak, *Calculus on Manifolds*, page 35, Benjamin Cummings, 1965, or H. K. Nickerson, D. C. Spencer and N. E. Steenrod, *Advanced Calculus*, Chapters IX and X, Dover, 2011.

Also the Jacobian of the inverse transformation is[4]

$$J_{x\to q} = \frac{\partial(r,\theta)}{\partial(x_1,x_2)} = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial \theta}{\partial x_1} \\ \frac{\partial r}{\partial x_2} & \frac{\partial \theta}{\partial x_2} \end{pmatrix} = J_{q\to x}^{-1} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}, \qquad \text{and} \ \det J_{q\to x} = \frac{1}{r} \ .$$

We conclude that $J_{x\to q}$ is a non-singular (invertible) matrix for $r \neq 0$. This agrees with the direct analysis above. The volume element $dx_1\,dx_2$ in Cartesian coordinates, when expressed in terms of polar coordinates becomes

$$dx_1\,dx_2 = \det J_{q\to x}\ dr\,d\theta = r\,dr\,d\theta \ . \tag{34}$$

A second example is the similar transformation in three-dimensional space, from Cartesian coordinates to spherical polar coordinates. (We used those coordinates in the notes on Rutherford scattering.) In that case

$$x_1 = r\sin\theta\cos\varphi \ , \quad x_2 = r\sin\theta\sin\varphi \ , \quad x_3 = r\cos\theta \ . \tag{35}$$

This gives

$$J_{q\to x} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix} , \tag{36}$$

so

$$\begin{aligned} \det J_{q\to x} &= \sin\theta\cos\varphi\ \det\begin{pmatrix} r\cos\theta\sin\theta & -r\sin\theta \\ r\sin\theta\cos\varphi & 0 \end{pmatrix} - r\cos\theta\cos\varphi\ \det\begin{pmatrix} \sin\theta\sin\varphi & \cos\theta \\ r\sin\theta\cos\varphi & 0 \end{pmatrix} \\ &\quad -r\sin\theta\sin\varphi\ \det\begin{pmatrix} \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\sin\varphi & -r\sin\theta \end{pmatrix} \\ &= r^2\sin\theta\left((\sin^2\theta\cos^2\varphi) + (\cos^2\theta\cos^2\varphi) + (\sin^2\theta\sin^2\varphi + \cos^2\theta\sin^2\varphi)\right) \\ &= r^2\sin\theta \ . \end{aligned} \tag{37}$$

Hence the volume element is

$$dx_1\,dx_2\,dx_3 = \det J_{q\to x}\ dr\,d\theta\,d\varphi = r^2\sin\theta\,dr\,d\theta\,d\varphi \ . \tag{38}$$

4 Lagrange's Equations

At this point we see the paradigm shift in regard to classical mechanics. One takes the Lagrange equations as the starting point, rather than Newton's equations. And one generalizes the choice $\mathcal{L} = T - V$ to other situations, aside from non-relativistic particle mechanics. In fact the Lagrange equations also apply in many other situations, including ones involving relativity, or with equations

[4]Here could have used $\frac{d}{du}\arctan u = \frac{1}{1+u^2}$ to compute $J_{x\to q}$ directly. Given a function $u(v)$, the inverse function $v(u)$ satisfies $u(v(u)) = u$. Thus differentiating with respect to u shows that $\frac{du}{dv}\frac{dv}{du} = 1$ or $\frac{dv}{du} = \frac{1}{\frac{du}{dv}}$. For $u = \tan v$ and $v = \arctan u$ one has $\frac{du}{dv} = \frac{1}{\cos^2 v}$, so $\frac{dv}{du} = \cos^2 v = \frac{1}{1+\tan^2 v} = \frac{1}{1+u^2}$.

such as Maxwell's equations for electromagnetic fields. One describes different physics by specifying the appropriate Lagrangian $\mathcal{L}$, from the field of optics on the one hand, to field theory for particle physics on the other! The same ideas occur both in classical and in quantum physics.

Consider the Lagrangian $\mathcal{L}(q, \dot{q}, t)$. We suppose that we can express the Lagrangian in terms of Cartesian coordinates $x, \dot{x}$ using (51) as

$$\mathcal{L}(q, \dot{q}, t) = \mathcal{L}(q(x), \dot{q}(x, \dot{x}), t) = \mathcal{L}_N(x, \dot{x}, t) \ . \tag{39}$$

Here we use the subscript N to denote the Newtonian form of the Lagrangian. We assume that the function $\mathcal{L}_N$ is the Lagrangian in Cartesian coordinates x for a system of particles moving in a potential $V(x)$ that can be expressed in terms of the instantaneous position x. The potential does not depend on the instantaneous velocity $\dot{x}$. Then Newton's equations hold in Cartesian coordinates, and Lagrange's equations also hold as they agree with Newton's, see §4.1.

4.1 Newton's Equations in Cartesian Coordinates

We consider here the simplest case of Lagrange's equations. We consider a set of non-relativistic particles described by Cartesian coordinates $\vec{x}_1, \vec{x}_2, \ldots$. In order to simplify our notation, let us not label both particles and the components of their position, but just pick a set of coordinates $x_1, x_2, \ldots, x_N$, so if there are n particles moving in 3-space, then $N = 3n$. The first three x's are the x, y, z (Cartesian) coordinates of particle 1 in three-space, etc. We let x stand for the collection of all the particle coordinates $x_1, \ldots, x_N$. We assign m_i to be a mass associated with the coordinate x_i, so for our example $m_1 = m_2 = m_3$ denotes the mass of the first particle. We denote the instantaneous velocities of the coordinates by $\dot{x}$, which stands for the collection of velocity coordinates $\dot{x}_1, \ldots, \dot{x}_N$. A trajectory will be a curve $x(t)$ parameterized by time t. The velocity along the trajectory is $\dot{x}(t) = dx(t)/dt$.

We assume that the motion along a trajectory evolves according to Newton's law $F = ma$. Explicitly, we assume that

$$m_i \ddot{x}_i = F_i \ ,$$

where the forces F with components $F_1, \ldots F_N$ are *conservative*. This means that the forces are the negative gradient of a single, given potential function $V(x)$ that does not depend on the velocities,

$$F_i(x) = -\frac{\partial V(x)}{\partial x_i} \ .$$

In summary, Newton's equations can be written

$$m_i \ddot{x}_i + \frac{\partial V}{\partial x_i} = 0 \ , \ \text{ for } \ i = 1, \ldots, N \ . \tag{40}$$

Given V for a particular problem, we can solve these equations, either exactly, approximately, or computationally, to obtain information about the resulting motion. One could also write this as

$$\dot{p}_i = F_i \ , \tag{41}$$

where p_i is the component of a momentum, and F_i the corresponding component of force.

In this simple situation, the kinetic energy T of the system of particles is

$$T = T(\dot{x}) = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{x}_j^2 \ ,$$

while the potential energy is

$$V = V(x) \ .$$

The total energy $E = T + V$ is a conserved quantity for the motion. In fact using (40), one has

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = \sum_{j=1}^{N} m_j \ddot{x}_j \dot{x}_j + \sum_{j=1}^{N} \frac{\partial V}{\partial x_j} \dot{x}_j = \sum_{j=1}^{N} \left(m_j \ddot{x}_j + \frac{\partial V(x)}{\partial x_j} \right) \dot{x}_j = 0 \ . \tag{42}$$

Joseph-Louis Lagrange was born in 1736. He discovered a beautiful reformulation of Newton's mechanics. He found that for the Lagrangian[5] $\mathcal{L}_N = T - V$, Newton's equations have the simple form

$$\frac{d}{dt} \frac{\partial \mathcal{L}_N}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}_N}{\partial x_i} = 0 \ . \tag{43}$$

Here we consider the position x and the velocity $\dot{x}$ as independent variables. In order to check (43), we calculate

$$\frac{\partial \mathcal{L}_N}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i \ , \quad \text{and} \quad \frac{d}{dt} \frac{\partial \mathcal{L}_N}{\partial \dot{x}_i} = m_i \, \ddot{x}_i \ .$$

Also

$$\frac{\partial \mathcal{L}_N}{\partial x_i} = - \frac{\partial V}{\partial x_i} \ .$$

Thus

$$\frac{d}{dt} \frac{\partial \mathcal{L}_N}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}_N}{\partial x_i} = m_i \ddot{x}_i + \frac{\partial V}{\partial x_i} \ ,$$

showing that in Cartesian coordinates Newton's equations for a set of particles (40) agree with Lagrange's equations (43).

4.2 Formulation of the Transformation Law

Let us define two sets of N functions, which we put together as "vectors." The first set is the "Newton" vector $\mathbb{N}$ with components

$$\mathbb{N}_i = \frac{d}{dt} \frac{\partial \mathcal{L}_N(x, \dot{x}, t)}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}_N(x, \dot{x}, t)}{\partial x_i} \ , \quad \text{where} \ \ i = 1, \ldots, N \ .$$

The Newton equations for each component of $x(t)$ on the trajectory of a point are

$$\mathbb{N} = 0 \ . \tag{44}$$

[5]The name Lagrangian was coined by Hamilton, much later

The second set of functions is the "Lagrange" vector $\mathbb{L}$ with components

$$\mathbb{L}_i = \frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_i}\;, \quad \text{where} \;\; i = 1,\ldots,N\;.$$

The Lagrange equations of motion along trajectories in the generalized coordinates $q(t)$ described by a Lagrangian $\mathcal{L}(q,\dot{q})$ are the equations

$$\mathbb{L} = 0\;. \tag{45}$$

We claim that the vectors obey the following relation under an invertible transformation $x \to q$:

Theorem 4.1. *Assume that under the transformation $x \to q$ that*

$$\mathcal{L}(q,\dot{q},t) = \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t) = \mathcal{L}_N(x,\dot{x},t)\;. \tag{46}$$

Then the vectors $\mathbb{L}$ and $\mathbb{N}$ are related by

$$\boxed{\mathbb{N} = J_{x\to q}\,\mathbb{L}}\;. \tag{47}$$

Consequently, if $\det J_{x\to q} \neq 0$, then Newton's equations in Cartesian coordinates ensure that Lagrange equations hold in the q coordinate system. Given two such coordinate transformations, we have the covariance law

$$J_{q\to q'}\mathbb{L}' = \mathbb{L}\;, \tag{48}$$

stated as (3) in the introduction.

We denote the rest of this section to prove Theorem 4.1. In practice one only requires that $\det J$ be non-singular for generic points in coordinate space. One then has Lagrange equations at these points, and one can extend them to all points by continuity. This shows that the Lagrange equations hold in the generalized coordinate system q. Using Proposition 3.1, we see that the statement (2) holds and therefore so does (3).

4.3 From Newton to Lagrange

Now we consider the change of coordinates for x to q, namely at a fixed time we write $q = q(x)$. If we wish to follow a trajectory in time, we do this by seeing what the transformation does to the trajectory $x(t)$ and let $q(t) = q(x(t))$. One may wish to have a more general transformation, in which the new coordinate system might be different at each time along the trajectory. (For example, we could rotate the coordinate system around an axis at a fixed rate. Then we would write the transformation from $x$to q as a function $q(x(t),t)$. This would really complicate the notation; so we suppress the extra variable but indicate any dependence of the transformation on time as a non-zero partial derivative $\frac{\partial q(x(t))}{\partial t}$. Thus the total time derivative of $\dot{q} = \frac{dq}{dt}$ of q along a trajectory can be written (using the chain rule for differentiation) as

$$\dot{q} = \frac{dq}{dt} = \sum_{i=1}^{N} \frac{\partial q(x(t))}{\partial x_i(t)}\,\dot{x}_i(t) + \frac{\partial q(x(t))}{\partial t}\;. \tag{49}$$

In terms of the individual component coordinates we write

$$\dot{q}_j = \frac{dq_j}{dt} = \sum_{i=1}^{N} \frac{\partial q_j(x(t))}{\partial x_i(t)}\, \dot{x}_i(t) + \frac{\partial q_j(x(t))}{\partial t} \,. \tag{50}$$

In order to simplify notation we write this as

$$\dot{q}_j = \frac{dq_j}{dt} = \sum_{i=1}^{N} \frac{\partial q_j}{\partial x_i}\, \dot{x}_i + \frac{\partial q_j}{\partial t} \,. \tag{51}$$

Although q_j is a function of x, but not $\dot{x}$, the derivative $\dot{q}_j$ depends on both x as well as $\dot{x}$.

The Lagrangian $\mathcal{L}$ for a set of particles with coordinates $q = (q_1, q_2, \ldots, q_N)$ is a function of the coordinates and the corresponding velocities $\dot{q} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N)$ at a particular instant of time. As time varies, the Lagrangian function follows a possible trajectory $q(t)$. Along a trajectory we have $\mathcal{L} = \mathcal{L}(q(t), \dot{q}(t), t)$. We claim that the Lagrange equations describe physical trajectories, and they satisfy the Lagrange equations of motion. These equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \,, \;\text{ for }\; i = 1, \ldots, N \,. \tag{52}$$

One calls the *generalized momentum* for the coordinate q_i

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \,. \tag{53}$$

Likewise the generalized force for the coordinate direction i is

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} \,. \tag{54}$$

Then the Lagrange equation has the form

$$\dot{p}_i = F_i \,, \;\text{ for }\; i = 1, \ldots, N \,, \tag{55}$$

or *rate of change of generalized momentum equals the generalized force.*

4.4 Following a Trajectory

Here we reduce the derivation of Lagrange's equations to the derivation of the covariance of a certain vector under invertible changes from Cartesian coordinates x to generalized coordinate q. At a given instant of time t, we consider the coordinates $q, \dot{q}$ to be independent variables.

Let us follow a point q on a physical trajectory $q(t)$. The time dependence $q(t)$ of the point q is identical with the time dependence $x(t)$ of the same point in Cartesian coordinates, namely $q(t) = q(x(t))$. Differentiate (51) with respect to $\dot{x}_i$, remembering that both the components of x and of $\dot{x}$ are independent variables. From this we infer the

$$\textbf{Rule of Cancellation of Dots:} \qquad \boxed{\frac{\partial \dot{q}_j}{\partial \dot{x}_i} = \frac{\partial q_j}{\partial x_i}} \,. \tag{56}$$

Furthermore it follows from (51) that we have the

$$\textbf{Rule of Second Derivatives:}\quad \boxed{\frac{\partial \dot{q}_j}{\partial x_i} = \sum_{k=1}^{N} \frac{\partial^2 q_j}{\partial x_k \partial x_i}\dot{x}_k + \frac{\partial^2 q_j}{\partial t \partial x_i} = \frac{d}{dt}\frac{\partial q_j}{\partial x_i}}\,. \tag{57}$$

4.5 Details of the Proof of Theorem 4.1

In order to establish (47), we calculate the terms in $\mathbb{N}_i$. But we express $\mathbb{N}$ in terms of $\mathbb{L}$ by using the transformation $q(x)$ and the relation (39). Then using the chain rule,

$$\begin{aligned}\frac{\partial \mathcal{L}_N(x,\dot{x},t)}{\partial \dot{x}_i} &= \frac{\partial \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t)}{\partial \dot{x}_i} = \sum_{j=1}^{N} \frac{\partial \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t)}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \dot{x}_i} \\ &= \sum_{j=1}^{N} \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j}\frac{\partial q_j}{\partial x_i}\,.\end{aligned}$$

In the last equality we use identity (56) to cancel the dots. Hence differentiating this expression gives

$$\frac{d}{dt}\frac{\partial \mathcal{L}_N(x,\dot{x},t)}{\partial \dot{x}_i} = \sum_{j=1}^{N} \left(\left(\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j}\right)\frac{\partial q_j}{\partial x_i} + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j}\frac{d}{dt}\frac{\partial q_j}{\partial x_i}\right)\,. \tag{58}$$

Also

$$\begin{aligned}\frac{\partial \mathcal{L}_N(x,\dot{x},t)}{\partial x_i} &= \frac{\partial \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t)}{\partial x_i} = \sum_{j=1}^{N}\left(\frac{\partial \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t)}{\partial q_j}\frac{\partial q_j}{\partial x_i} + \frac{\partial \mathcal{L}(q(x),\dot{q}(x,\dot{x}),t)}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_i}\right) \\ &= \sum_{j=1}^{N}\left(\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_j}\frac{\partial q_j}{\partial x_i} + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_i}\right)\,. \end{aligned}\tag{59}$$

The identity (57) shows that the final sums in (58) are identical to the last terms in (59). Therefore subtracting (59) from (58) shows that

$$\begin{aligned}\mathbb{N}_i &= \frac{d}{dt}\frac{\partial \mathcal{L}_N(x,\dot{x},t)}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}_N(x,\dot{x},t)}{\partial x_i} \\ &= \sum_{j=1}^{N}\left(\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_j}\right)\frac{\partial q_j}{\partial x_i} \\ &= \sum_{j=1}^{N} \mathbb{L}_j \frac{\partial q_j}{\partial x_i} = \sum_{j=1}^{N} \mathbb{L}_j\,(J_{x\to q})_{ij} = (J_{x\to q}\,\mathbb{L})_i\,.\end{aligned}$$

This is just the claimed relation in terms of components for the vector relation (47), so it completes the proof of Lagrange's equations.

4.6 Example: Plane Polar Coordinates

Take $\mathcal{L} = T - V$ to have the form

$$\mathcal{L}(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r,\theta) \ . \tag{60}$$

Then the r-momentum is $p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$, and the θ-momentum is $p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = L = mr^2\dot{\theta}$. The corresponding Lagrange equations are

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \ , \quad \text{and} \quad \dot{L} = -\frac{\partial V}{\partial \theta} \ . \tag{61}$$

5 The Euler-Lagrange Equations

In this section we relate our previous derivation of Lagrange's equations to the Euler-Lagrange equations. These are the Lagrange equations considered in the calculus of variations of the action. First we show that the Lagrange equations at each time are equivalent to an integral version of the Lagrange equations. Then we show that the integral version is the same as the vanishing of the first variation of the action.

5.1 An Integral Form of Lagrange's Equations

We have defined the vector $\mathbb{L}(t)$ depending on the trajectory $q(t)$ evaluated at an instant of time t. And at that instant we assume that $\mathbb{L}(t)$ depends only on the coordinate q and its velocity $\dot{q}$. We then found that if q is given by an invertible (but possibly time-dependent) transformation from Cartesian coordinates x, namely $q(t) = q(x(t),t)$, then Newton's equations in Cartesian coordinates are equivalent to Lagrange's equations of motion $\mathbb{L}(t) = 0$ for each time t.

Here we want to reformulate this condition that Lagrange's equations hold as a set of equations in integral form.e Suppose we have N degrees of freedom. Lt $\eta(t) = \{\eta_1(t), \eta_2(t), \ldots, \eta_N(t)\}$ denote a set of N functions of time. Let us define the pairing of $\mathbb{L}(t)$ with $\eta(t)$ on the time interval $[T_1, T_2]$ by

$$\langle \mathbb{L}, \eta \rangle_{[T_1,T_2]} = \sum_{j=1}^{N} \int_{T_1}^{T_2} \mathbb{L}_j(t)\,\eta_j(t)dt \ . \tag{62}$$

Definition 5.1. *Then Lagrange equations hold pointwise on the time interval $t \in [T_1, T_2]$ if*

$$\mathbb{L}(t) = 0 \ , \ \textit{for all } t \in [T_1, T_2] \ . \tag{63}$$

Lagrange's equations hold in integral form, if for any smooth function of time $\eta(t)$, for which $\eta(T_1) = \eta(T_2) = 0$, one has

$$\langle \mathbb{L}, \eta \rangle_{[T_1,T_2]} = 0 \ . \tag{64}$$

Proposition 5.2. *Assume that $\mathbb{L}(t)$ depends continuously on t. Then Lagrange equations hold pointwise, if and only if Lagrange's equations hold in integral form.*

Proof. If Lagrange's equations hold pointwise, namely $\mathbb{L}(t) = 0$, then clearly the integral (64) vanishes for all continuous $\eta(t)$ and the integral form of Lagrange's equations hold. So we only need to establish the converse: that (64) ensures (63).

Given a point $t = t_0$ in the interior of the time interval $t_0 \in [T_1, T_2]$, choose $0 < \epsilon$ sufficiently small so that an ϵ-neighborhood of t_0 is still inside the interval, namely $T_1 < t_0 - \epsilon < t_0 + \epsilon < T_2$. Suppose that $\mathbb{L}(t) \neq 0$. Since $\mathbb{L}(t)$ is continuous, we can choose ϵ sufficiently small so that $\mathbb{L}(t)$ is strictly positive or strictly negative for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$. If we choose $\eta(t)$ so that

$$\eta_i(t) \begin{cases} > 0 \;, & \text{for } t \in (t_0 - \epsilon, t_0 + \epsilon) \\ = 0 \;, & \text{for } t \notin [t_0 - \epsilon, t_0 + \epsilon] \end{cases} \;,$$

then the integral $\langle \mathbb{L}, \eta \rangle_{[T_1,T_2]}$ is strictly positive or strictly negative. This is a contradiction. So (63) holds at t_0. But t_0 is arbitrary, so (63) holds for all $t \in (T_1, T_2)$, and by continuity also at the endpoints. □

We will now introduce the action $S_{T_1,T_2}(q)$ for a trajectory $q(t)$ going from point Q_1 at time T_1 to the point Q_2 at time T_2. As a function of the trajectory q we will define a directional derivative $(D_\eta S_{T_1,T_2})(q)$ of $S_{T_1,T_2}(q)$ in the direction η at the point q. The end result will be

Proposition 5.3 (Hamilton's Principle). *The derivative of the action $S(q)$ in direction η is proportional to $\langle \mathbb{L}, \eta \rangle$. We require that $\eta(T_1) = \eta(T_2) = 0$, so the endpoints Q_1 and Q_2 of the trajectory that we vary are fixed. In this case,*

$$\boxed{(D_\eta \, S_{T_1,T_2})(q) = - \langle \mathbb{L}, \eta \rangle_{[T_1,T_2]}} \;. \tag{65}$$

This shows that the Lagrange equations $\langle \mathbb{L}, \eta \rangle_{[T_1,T_2]} = 0$ in the time interval $[T_1, T_2]$ are equivalent to the directional derivative of the action $S(q)$ vanishing in all directions η at the trajectory q, with the requirement that the endpoints of the trajectory are fixed. One says that the trajectory q is a *critical point* of the action; a critical point is a point at which the first derivative of the action vanishes in all directions. The rest of this section will be devoted to understanding Proposition 5.3.

5.2 The Action

Suppose that we are given a Lagrangian $\mathcal{L}(q(t), \dot{q}(t), t)$. We let q denote the collection of N generalized coordinates $q = (q_1, \ldots, q_N)$. We also use the variable q to denote a continuous path $q(t)$ in configuration space between two fixed times, a starting time T_1 and and ending time T_2. Then the endpoints are $Q_1 = q(T_1)$ and $Q_2 = q(T_2)$, for fixed Q_1, Q_2. This path need not be the solution to any equation; it is just an arbitrary trajectory between these endpoints. Only special paths do obey Lagrange's equations. Sometimes one writes $q(\,\cdot\,)$ to denote a variable that represents a path, rather than $q(t)$ which is the value on the path at a specific time t.

Then the action $S_{T_1,T_2}(q)$ depends on an entire path $q(t)$ is the integral of the Lagrangian over this path:

$$S_{T_1,T_2}(q) = \int_{T_1}^{T_2} \mathcal{L}(q(t), \dot{q}(t), t) \, dt \;. \tag{66}$$

Since this notation is cumbersome, we generally abbreviate it by understanding that the end points are fixed at times T_1 and T_2 and the variable of the action is the entire path $q(t)$ for t in the interval $[T_1, T_2]$. Thus we write

$$S(q) = \int_{T_1}^{T_2} \mathcal{L}\, dt\ . \tag{67}$$

5.3 Hamilton's Principle & Critical Points of the Action

The main thing that we study in mechanics is the variation of the action. So it is natural to introduce the derivative of the action $S(q)$ at the point (namely at the path) q in the direction η, which is the infinitesimal variation. If $q_1(t)$ and $q_2(t)$ are two different paths with the same endpoints, their difference must vanish at T_1 and T_2. Thus it is natural to vary the path $q(t)$ by adding a path $\eta(t)$ for which $\eta(T_1) = \eta(T_2) = 0$. We say that η "vanishes at the endpoints." If ϵ is any real number, then $\epsilon\eta$ is a path $\epsilon\eta(t)$ which also vanishes for $t = T_1$ and $t = T_2$. Thus $q + \epsilon\eta$ is an allowed variation of q. We use the notation $(D_\eta S)(q)$ to indicate this directional derivative, defined as follows:

Definition 5.4. *The directional derivative $(D_\eta S)(q)$ of the action $S(q)$ in the direction η at the path q is*

$$(D_\eta S)(q) = \lim_{\epsilon\to 0} \frac{S(q+\epsilon\eta) - S(q)}{\epsilon}\ . \tag{68}$$

The path Q is a critical point of $S(q)$, if $(D_\eta S)(Q) = 0$ for all η that vanishes at the endpoints.

Hamilton's principle relates a solution to Lagrange's equations, to a path $Q(t)$ that is an extreme point of the action:

Theorem 5.5 (Hamilton's Principle). *Suppose the action $S(q)$ is given by a Lagrangian $\mathcal{L}$ for the time interval $t \in [T_1, T_2]$. Then a path Q satisfies Lagrange's equations for $\mathcal{L}$ in the time interval $[T_1, T_2]$, if and only if Q is a critical point of the action $S(q)$.*

5.4 The Directional Derivative and Hamilton's Principle

Hamilton's principle says that Lagrange's equations for $q(t)$ are equivalent to q being a stationary point for the action under variations that fix the endpoints $q(t_1) = Q_1$ and $q(t_2) = Q_2$. We have stated this as Proposition 5.3.

The natural way to formulate Hamilton's principle is to define the directional derivative of $S(q)$. Before dealing with $S(q)$ itself, let us recall how one generalizes the derivative $f'(q)$ of a function $f(q)$ of one variable $q \in \mathbb{R}$ to higher dimensions. If $f(q)$ is a real-valued function of $q \in \mathbb{R}^N$, then one can define the directional derivative $(D_n f)(q)$ of f in the direction of a unit vector $n \in \mathbb{R}^N$ by

$$(D_{\vec{n}} f)(q) = \lim_{\epsilon\to 0} \frac{f(q+\epsilon\vec{n}) - f(q)}{\epsilon}\ . \tag{69}$$

In fact this just equals the component of the gradient ∇f in the direction n, as

$$(D_{\vec{n}}f)(q) = \sum_{j=1}^{N} \frac{\partial f(q)}{\partial q_j} n_j = (\vec{n} \cdot \nabla f)(q)\ . \tag{70}$$

We define the directional derivative of a function of a function in exactly the same way. We will define this derivative as a directional derivative D_η. means that the directional derivative vanishes in all directions η with fixed endpoints, namely for any function $\eta(t)$ satisfying the Dirichlet boundary conditions (**??**). We then see that a solution $Q(t)$ to the Lagrange equations (**??**) is a trajectory $Q(t)$ that is a stationary point (critical point) of the action, with fixed endpoints.

Consider $\mathcal{I} = [T_1, T_2]$ to be a given time interval. We also need to define the directional derivative of the action, as well as a critical point of the action for variations of the field with fixed boundary values.

Proof of Proposition 5.3. We calculate the directional derivative $(D_\eta S)(q)$ in terms of the Lagrangian. Assuming we can interchange the order of integration and differentiation in the action, one can express the derivative in terms of the Lagrangian as

$$\begin{aligned}(D_\eta S)(q) &= \int_{T_1}^{T_2} \lim_{\epsilon\to 0} \left(\frac{L(q(t)+\epsilon\eta(t), \dot{q}(t)+\epsilon\dot{\eta}(t), t) - L(q(t),\dot{q}(t),t)}{\epsilon} \right) dt \\ &= \int_{T_1}^{T_2} \left(\frac{d}{d\epsilon} L(q(t)+\epsilon\eta(t), \dot{q}(t)+\epsilon\dot{\eta}(t), t) \right)\bigg|_{\epsilon=0} dt\ .\end{aligned} \tag{71}$$

Use the chain rule express the integrand as

$$\begin{aligned}&\left(\frac{d}{d\epsilon} L(q(t)+\epsilon\eta(t), \dot{q}(t)+\epsilon\dot{\eta}(t), t) \right)\bigg|_{\epsilon=0} \\ &\quad = \sum_{i=1}^{N} \left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_i(t)}\, \eta_i(t) + \frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)}\, \dot{\eta}_i(t) \right) \\ &\quad = \sum_{i=1}^{N} \left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_i(t)}\, \eta_i(t) + \frac{d}{dt}\left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)}\, \eta_i(t) \right) - \left(\frac{d}{dt}\, \frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)} \right) \eta_i(t) \right)\ .\end{aligned}$$

So inserting this into (71) one has

$$\begin{aligned}(D_\eta S)(q) &= \sum_{i=1}^{N} \int_{T_1}^{T_2} \left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_i(t)} - \frac{d}{dt}\, \frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)} \right) \eta_i(t) dt \\ &\qquad + \sum_{i=1}^{N} \frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)}\, \eta_i(t) \bigg|_{T_1}^{T_2}\ .\end{aligned}$$

The final term vanishes due to the Dirichlet boundary conditions on $\eta_i(t)$, so

$$(D_\eta S)(q) = \sum_{i=1}^{N} \int_{T_1}^{T_2} \left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_i(t)} - \frac{d}{dt}\, \frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)} \right) \eta_i(t) dt\ . \tag{72}$$

We claim that $(D_\eta S)(Q) = 0$ for all $\eta(t)$ vanishing at the endpoints, if and only if $Q(t)$ satisfies Lagrange's equations. In fact it follows immediately that if each component of $Q(t)$ satisfies Lagrange's equations, then $(D_\eta S)(Q) = 0$.

In order to see that the converse is true, suppose that $Q(t)$ is a trajectory for which $(D_\eta S)(Q) = 0$ for all $\eta(t)$ vanishing at T_1, T_2. Let us focus on a point t_0 strictly between the endpoints, $T_1 < t_0 < T_2$. Let $0 \leqslant \varphi(t) \leqslant 1$ be a smooth function that equals 1 in a neighborhood of t_0 and which vanishes in a neighborhood of T_1 and T_2. Now choose a fixed value of j, and take $\eta_i(t) = \varphi(t)\delta_{ij}\left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_i(t)} - \frac{d}{dt}\left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_i(t)}\right)\right)$. With this choice

$$(D_\eta S)(Q) = \int_{T_1}^{T_2} \varphi(t) \left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_j(t)} - \frac{d}{dt}\frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_j(t)}\right)^2 dt\bigg|_{q=Q} . \tag{73}$$

The integrand is point-wise positive, and equals

$$\left(\frac{\partial L(q(t),\dot{q}(t),t)}{\partial q_j(t)} - \frac{d}{dt}\frac{\partial L(q(t),\dot{q}(t),t)}{\partial \dot{q}_j(t)}\right)^2\bigg|_{q=Q} , \tag{74}$$

in a neighborhood of t_0. Thus the j^{th} component of $(D_\eta S)(Q) = 0$. But j is arbitrary, so this ensures that $(D_\eta S)(Q) = 0$ in a neighborhood of t_0. Since the point t_0 is arbitrary, as is the choice of j, (74) must vanish everywhere on the open interval (T_1, T_2). To satisfy Lagrange's equations the trajectory $Q(t)$ must have a time derivative. So $Q(t)$ is continuous and satisfies Lagrange's equations everywhere. □

6 Redundant Coordinates and Constraints

6.1 Equations of Constraint

Sometimes it is useful to describe a mechanics problem using some redundant coordinates. This is also called a system of coordinates with *constraints*. In this case we take a set of N generalized coordinates $q = (q_1, \ldots, q_N)$ that is strictly greater, $N > n$, than the number n of independent coordinates $x = (x_1, \ldots, x_n)$. In this section of the notes, we do not assume that x denotes a set of Cartesian coordinates. Rather we only assume that x denotes a set of *independent, generalized coordinates* for which the Lagrange function is $\mathcal{L}_N(x, \dot{x})$ and the Lagrange equations hold in the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}_N}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}_N}{\partial x_i} , \quad \text{for } i = 1, \ldots, n . \tag{75}$$

In such a situation, we cannot conclude that Lagrange's equations (52) hold, at least not without modification. The reason is that our derivation of the equations used the fact that one can go back and forth between the coordinates q and the coordinates x. More specifically, we used the invertibility of the $N \times N$ Jacobian matrix $J_{x\to q}$. In the present situation, as noted in the introduction, extra terms arise in the equations. One can interpret these terms as giving forces of constraint;

they ensure that the motion described by the solution to the equations lies in the subspace that is allowed by the physics.

One cannot write the coordinate x as a function of the redundant coordinates q, because there are values of q that do not correspond to any x. For example, if x-space is a line in a plane with coordinates q, then one cannot map a point q not lying on the line into a point x. Since the difference in the dimensions of q-space and x-space is $k = N - n$, one needs to give $k = N - n$ conditions on the coordinates q. One generally gives k functions, called *constraint* functions, such as $f^{(1)}(q), f^{(2)}(q), \ldots, f^{(k)}(q)$, that vanish on x-space. Thus the equations

$$f^{(1)}(q) = 0 \;, \quad f^{(2)}(q) = 0 \;, \ldots, \quad f^{(k)}(q) = 0 \;, \tag{76}$$

are called the *equations of constraint*.

When the constraint equations are of this form only depending on the coordinates (or possibly depending on the coordinates and an explicit dependence on the time) the constraints are called *holonomic*. In particular, holonomic constraint functions do not depend on the instantaneous velocities $\dot{q}$. In these notes we consider holonomic constraints.

We end up with the following Lagrange equations in that situation. In order to simplify notation, we suppress the explicit time dependence of $\mathcal{L}$. We obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f^{(j)}(q)}{\partial q_i} \;, \;\text{ where }\; i = 1, \ldots, N \;, \tag{77}$$

and the k equations of constraint, which are

$$f^{(j)}(q) = 0 \;, \;\text{ for }\; j = 1, \ldots, k \;. \tag{78}$$

6.2 An Inclined Plane

In this example, we consider a "point" particle moving under the influence of gravity in one dimension. One can take a single generalized coordinate x to be the distance x that the particle has moved. This is a Cartesian coordinate system and corresponds to the $n = 1$. Suppose that the particle is actually moving in a friction-less manner along a line in the (q_1, q_2) plane. Here q_1, q_2 denotes a second Cartesian coordinate system in which we imbed the inclined plane. Assume that q_2 denotes the vertical direction, and the plane is inclined at an angle α to the horizontal direction q_1, and in a direction so q_1 increases with time. Let the $N = 2$ coordinate system q and the $n = 1$ coordinate system x have coinciding origins.

We can easily express a Lagrangian for the particle in terms of the x coordinate as

$$\mathcal{L}_N(x, \dot{x}) = T - V = \frac{1}{2}\, m\dot{x}^{\,2} + mgx \sin\alpha \;. \tag{79}$$

The Lagrange equation of motion is

$$m\ddot{x} = mg \sin\alpha \;,$$

which we can easily solve.

Let us define the Lagrangian,

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right) - mg\, q_2 \,, \tag{80}$$

which describes the vertical and horizontal coordinates of a point particle under the influence of gravity. Furthermore, suppose that one can express q as a function of a parameter x lying on a line such that

$$q_1 = x \cos\alpha \;, \quad \text{and} \quad q_2 = -x \sin\alpha \;. \tag{81}$$

Here we fix the angle α, so x parameterizes a point on a plane inclined by angle α to the horizontal. Then the corresponding velocities in q and x space satisfy the relations,

$$\dot{q}_1 = \dot{x} \cos\alpha \;, \quad \text{and} \quad \dot{q}_2 = -\dot{x} \sin\alpha \;. \tag{82}$$

Using (81)–(82), we express $\mathcal{L}(q, \dot{q})$ as a function $\mathcal{L}_N(x, \dot{x})$. One finds that for x on the allowed line in q-space,

$$\mathcal{L}(q, \dot{q}) = \mathcal{L}(q(x), \dot{q}(x, \dot{x})) = \frac{1}{2} m\dot{x}^2 + mgx \sin\alpha = \mathcal{L}_N(x, \dot{x}) \;. \tag{83}$$

This agrees with $\mathcal{L}_N(x, \dot{x})$ in (79) for which we know that Lagrange's equations hold. So our choice of $\mathcal{L}$ is the natural one.

Now we come to the tricky point: While the relation (83) holds for all x on the allowed line, it does not hold in general. We **cannot** write $\mathcal{L}(q, \dot{q}) = \mathcal{L}_N(x(q), \dot{x}(q, \dot{q}))$ for all q in the present case, although this was possible in the context of §4 where there are no constraints.

6.3 A Non-Square Jacobian Matrix

Let us generalize from this example to study the general case where there are more q coordinates than x coordinates. We gain insight by looking at the Jacobian. Assume that we have N generalized coordinates q and $n < N$ generalized coordinates x. In this case we do not assume that the coordinates x are Cartesian coordinates. Rather we do assume that the coordinates x are generalized coordinates for which we have given a Lagrangian $\mathcal{L}_N(x, \dot{x})$, and that the Lagrange equations (52) hold in these coordinates. We assume that we have a well-defined, differentiable transformation $q = q(x)$. Define the $N \times n$ Jacobian matrix

$$J_{x \to q} = \frac{\partial(q_1, q_2, \ldots, q_N)}{\partial(x_1, x_2, \ldots, x_n)} \;, \quad \text{with entries} \quad (J_{x \to q})_{ij} = \frac{\partial q_j}{\partial x_i} \;. \tag{84}$$

This Jacobian matrix and its transpose has the form,

$$J_{x \to q} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_N}{\partial x_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial q_1}{\partial x_n} & \frac{\partial q_2}{\partial x_n} & \cdots & \frac{\partial q_N}{\partial x_n} \end{pmatrix} , \qquad J_{x \to q}^{\mathrm{T}} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_n} \\ \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_2}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial q_N}{\partial x_1} & \cdots & \frac{\partial q_N}{\partial x_n} \end{pmatrix} .$$

As $n < N$, the maximal rank of $J_{x \to q}$ is n. Therefore there are at least $k = N - n \geqslant 1$ linearly-independent null vectors $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ of the Jacobian matrix $J_{x \to q}$. These null vectors $v^{(j)}$ satisfy

$$J_{x \to q} v^{(j)} = 0 \ , \quad \text{for all } q \ , \quad \text{and for } j = 1, \ldots, k \ . \tag{85}$$

In this case, the null space of $J_{x \to q}$ is spanned by the k linearly-independent null vectors $v^{(j)}$.

Comments on the rank of the matrix $\boldsymbol{J_{x \to q}}$. The domain of J is $\mathbb{R}^N$ while its range in $J\mathbb{R}^N$ is a subset of $\mathbb{R}^n$. The integer $\text{Rank}(J_{x \to q})$ is the dimension of its range. Let us suppress the subscript on J and write $\text{Rank}(J) = \dim(J\mathbb{R}^N)$. As $n \leqslant N$, clearly $\text{Rank}(J) \leqslant n$. We say that J has maximal rank if $\text{Rank}(J) = n$ for all x. This replaces the condition that J be invertible, for if $n = N$, the conditions coincide. If the $n \times N$ matrix J has maximal rank n, and $n \leqslant N$, then the null space of J has dimension $k = N - n$.

Note that an $\ell \times \ell$ matrix H with entries H_{ij} is hermitian if $H_{ij} = \overline{H_{ji}}$. Any $\ell \times \ell$ hermitian matrix H has an orthonormal basis of eigenvectors e_j with real eigenvalues λ_j. They satisfy $He_j = \lambda_j e_j$ for $j = 1, \ldots, \ell$. In fact we claim that

$$\boxed{\text{Rank}(J) = \text{Rank}(J^*J) = \text{Rank}(JJ^*) = \text{Rank}(J^*) = \text{Rank}(J^{\mathrm{T}})} \ . \tag{86}$$

Here J^* is the hermitian adjoint of J, and J^{T} is the transpose of J. For real matrices (like our J) they agree, $J^* = J^{\mathrm{T}}$. The $n \times n$ matrix J^*J and the $N \times N$ matrix JJ^* are both hermitian matrices.

To justify the claim (86), we show the following basic result:

Proposition 6.1. *The matrices J^*J and JJ^* have the same non-zero eigenvalues, also counting multiplicity. Thus J and J^* have the same rank.*

Proof. Consider $X = \{e_i\}$ for $i = 1, \ldots, r$ as the subset of an orthonormal eigenvector basis for J^*J consisting of all those vectors e_i with the corresponding eigenvalues $\lambda_j \neq 0$. Then we claim that the r vectors $f_i = Je_i$ are non-zero eigenvectors of JJ^* with the same eigenvalues λ_i. In fact

$$JJ^* f_i = JJ^*Je_i = J\lambda_i e_i = \lambda_i f_i \ .$$

And we have assumed $\lambda_i \neq 0$. Furthermore these f_i are linearly independent, for if $\sum_i c_i f_i = 0$, then $J \sum_i c_i f_i = \sum_i c_i \lambda_i e_i = 0$. However this is the case only for $\lambda_i c_i = 0$ for all i. As $\lambda_i \neq 0$, we conclude that $c_i = 0$. Thus there is a 1 to 1 correspondence between linearly-independent eigenvectors with non-zero eigenvalues for J^*J and for JJ^*. In fact the number of these eigenvectors is the rank of J^*J and of JJ^*. So these ranks both equal r. Furthermore the vectors f_i span the range of J. Therefore the rank of J also equals r. By interchanging J and J^*, also the rank of J^* equals r.

But also the eigenvectors $e_i \in X$ span the orthogonal complement of the null space of J. So $r + \dim \text{Null}(J) = N$. In other words

$$\dim \text{Null}(J) = N - r.$$

For $n \leqslant N$, the maximal rank of J^*J is n, so for J of maximal rank, $\dim \text{Null}(J) = N - n = k$. $\square$

6.3.1 Assumptions for Lagrange Equations with Constraints

We make the following assumptions:

A1. We are given a Lagrangian $\mathcal{L}(q, \dot{q})$, and a differentiable transformation $q = q(x)$, such that the Lagrange equations derived from the Lagrangian $\mathcal{L}_N(x, \dot{x}) = \mathcal{L}(q(x), \dot{q}(x, \dot{x}))$ hold.

A2. The $n \times N$ Jacobian matrix $J = \frac{\partial(q_1,\ldots,q_N)}{\partial(x_1,\ldots,x_n)}$ has maximal rank n.

A3. We are given $k = N - n$ constraint functions $f^{(j)}(q)$ which vanish on the range of x, or

$$f^{(j)}(q(x)) = 0\ , \quad \text{for all } \ x\ , \quad \text{and for all } \ j = 1, \ldots, k\ . \tag{87}$$

A4. The constraint functions $f^{(j)}$ are *independent* for generic points q. (We explain below what this means.)

6.3.2 Forces of Constraint

If $f^{(j)}(q)$ is a function of constraint, the gradient vectors $v^{(j)}$ play a special role. They are

$$v^{(j)} = \nabla_q f^{(j)} = \begin{pmatrix} \frac{\partial f^{(j)}(q)}{\partial q_1} \\ \frac{\partial f^{(j)}(q)}{\partial q_2} \\ \vdots \\ \frac{\partial f^{(j)}(q)}{\partial q_N} \end{pmatrix}\ . \tag{88}$$

We claim that each such vector $v^{(j)}$ satisfies the equation

$$J v^{(j)} = 0\ . \tag{89}$$

To check this, note that each constraint function satisfies the constraint condition (87); namely $f^{(j)}(q(x)) = 0$ for all x. Therefore any derivative of $f^{(j)}(q(x))$ with respect of any component of x vanishes; as an identity, it is the case that $\partial f^{(j)}(q(x))/\partial x_\ell = 0$ for each $\ell = 1, \ldots, n$ and for all x. Using the chain rule to write out these derivatives, we find that

$$\frac{\partial f^{(j)}(q(x))}{\partial x_\ell} = \sum_{i=1}^{N} \frac{\partial f^{(j)}(q(x))}{\partial q_\ell} \frac{\partial q_\ell}{\partial x_i} = \left(J\, v^{(j)}\right)_\ell = 0\ , \quad \text{for } \ \ell = 1, \ldots, n\ . \tag{90}$$

Let us now return to our assumption **A4** above that we called the "independence of the constraints." In fact this assumption means that the different null-vector solutions $v^{(j)} = \nabla_q f^{(j)}$ to the equation $J v^{(j)} = 0$ provide k linearly independent solutions to this equation. This means that the k different gradient vectors $\nabla_q f^{(j)}$ are linearly independent,

$$\sum_{j=1}^{k} c_j\, \nabla_q f^{(j)} = 0\ , \quad \text{only if all the coefficients } \ c_j = 0\ . \tag{91}$$

Assuming further that the dimension of the vector space of null vectors of J is k, then every null vector v satisfying $Jv = 0$ can be expanded as a linear combination of the solutions $v^{(j)} = \nabla f^{(j)}$ given by the constraints. In other words, there are a set of constants λ_j such that an arbitrary null vector v for J can be written as

$$v = \sum_{j=1}^{k} \lambda_j \, v^{(j)} \; . \tag{92}$$

In this context, let us call the coefficients λ_j *Lagrange multipliers*. The reason is that is how they will arise in the context of Lagrange's equations.

We want to have the relation (92), so now we *add an additional assumption* to our analysis so that this is the case. This assumption replaces the assumption in §4 that the $N \times N$ Jacobian matrix J is invertible. The assumption involves two parts, which taken together ensure that k constraint functions give a basis of null vectors for J.

6.4 Lagrange's Equations with Constraints

In the above situation, there are $N + k$ equations and $N + k$ unknowns to these equations. The equations are the N Lagrange equations and N constraint equations. The unknowns are the N coordinates $q_1(t), q_2(t), \ldots, q_N(t)$, along with k constants $\lambda_1, \ldots, \lambda_k$, called *Lagrange multipliers*. There are also $N+k$ equations which determine the $N+k$ unknowns. The terms $\lambda_k \nabla_q f^{(k)} = \lambda_k v^{(k)}$ can be interpreted as *forces of constraint*.

6.5 Deriving Lagrange's Equations with Constraints

In order to establish the validity of the equations (77) as the Lagrange equations with constraints, we begin by defining $\mathbb{N}$ and $\mathbb{L}$ as in §4 and following the argument in §4.5. We wish to show that the transformation from $\mathbb{N}$ to $\mathbb{L}$ is covariant. In fact

$$\mathbb{N} = J_{x \to q} \, \mathbb{L} \; .$$

The details of deriving this relation are identical to the proof of the same relation (47) in §4. The only difference is that one needs to keep track that the number N of components of q is different from the number n of components of x.

Furthermore $\mathbb{N} = 0$ on a physical trajectory, as the Lagrange equations for $\mathcal{L}_N(x, \dot{x})$ hold. The difference between the previous case without constraints and the present case is: the Jacobian J is not invertible. However the assumption that J has maximal rank means that every null vector of J is a linear combination of the null vectors arising from the independent constraints. Thus the Lagrange equations hold in the form (77) in the q coordinates. The Lagrange multipliers are unknowns, and we solve the $N + k$ equations for the λ_j's, as well as for the trajectories $q(t)$.

6.6 Example: Moving Down the Inclined Plane

Let us take the example of §6.2, with the Lagrangian $\mathcal{L}(q,\dot q)=\frac{1}{2}\,m\,(\dot q_1^2+\dot q_2^2)-mg\,q_2$ given in (80). On the inclined plane, we have the relations

$$q_1 = x\cos\alpha\ , \quad q_2=-x\sin\alpha\ , \quad f(q)=q_2+q_1\tan\alpha=0\ . \tag{93}$$

Off the inclined plane, we cannot write $x=x(q)$, as a function of $q=(q_1,q_2)$, because most points (q_1,q_2) do not correspond to a point on the inclined plane. In other words the transformation $x\to q$ is not in general invertible; it is only invertible for q satisfying the constraint. Note that

$$(\nabla_q f)\,(q)=\begin{pmatrix}\tan\alpha\\ 1\end{pmatrix}\ , \quad \text{and}\quad \nabla_q\mathcal{L}\,(q,\dot q)=\begin{pmatrix}0\\ -mg\end{pmatrix}\ . \tag{94}$$

There are three equations to solve, namely the two Lagrange equations (77) and the single constraint equation (78). There are also three unknowns, namely $(q_1(t),q_2(t),\alpha)$. The equations are

$$\begin{aligned} m\ddot q_1 &= \lambda\tan\alpha\ , && \textbf{Equation 1}\\ m\ddot q_2+mg &= +\lambda\ , && \textbf{Equation 2}\\ q_2+q_1\tan\alpha &= 0\ , && \textbf{Constraint Equation} \end{aligned} \tag{95}$$

Written in terms of the Lagrangian equation $\mathbb{L}=\lambda(\nabla_q f)(q)$, or in terms of components,

$$\mathbb{L}=m\frac{d^2}{dt^2}\begin{pmatrix}q_1(t)\\ q_2(t)\end{pmatrix}+m\begin{pmatrix}0\\ g\end{pmatrix}=\lambda\begin{pmatrix}\tan\alpha\\ 1\end{pmatrix}\ . \tag{96}$$

From these relations one can solve for the motion. One infers that

$$\begin{pmatrix}\ddot q_1(t)\\ \ddot q_2(t)\end{pmatrix}=g\sin\alpha\begin{pmatrix}\cos\alpha\\ -\sin\alpha\end{pmatrix}\ , \qquad \text{and}\ \ \lambda=mg\cos^2\alpha\ . \tag{97}$$

One can interpret the terms $\lambda\tan\alpha$ and λ in (95) as the x_1 and x_2 components of a "force of constraint," keeping the particle on the surface of the inclined plane.

6.7 Example: A Cylinder that Rolls without Slipping on the Incline

Consider the equation for a cylinder of mass M, of radius a, and of moment of inertia I, that rolls without slipping on an inclined plane making angle α to the horizontal. We will describe the motion by using two generalized coordinates x and θ. As in the previous example, x describes the displacement of the center of mass along the inclined plane. The angle θ gives the angle of rotation of the cylinder about its axis. The corresponding Lagrangian for this system is

$$\mathcal{L}(x,\theta,\dot x,\dot\theta)=\frac{1}{2}M\dot x^2+\frac{1}{2}I\dot\theta^2+Mgx\sin\alpha\ . \tag{98}$$

Taking $\theta = 0$ for $x = 0$ the constraint is

$$f(q) = x - a\theta \; . \tag{99}$$

In this example, as in the previous example, $N = 2$ and $n = 1$, so there is one Lagrange multiplier λ. The $N + n = 3$ equations (2 Euler-Lagrange equations and 1 constraint equation) have the form

$$M\ddot{x} = Mg\sin\alpha + \lambda \; , \qquad I\ddot{\theta} = -\lambda a \; , \qquad \text{and } \; x = a\theta \; . \tag{100}$$

We now solve for $x(t), \theta(t), \lambda$.

We use that the moment of inertia for a cylinder is $I = \frac{1}{2}Ma^2$. In fact

$$I = M\frac{\int_0^{2\pi} d\theta \int_0^a r^2\, rdr}{\int_0^{2\pi} d\theta \int_0^a rdr} = M\frac{(2\pi)(\frac{1}{4}a^4)}{(2\pi)(\frac{1}{2}a^2)} = \frac{1}{2}Ma^2 \; . \tag{101}$$

The third equation in (100) yields $\ddot{x} = a\ddot{\theta}$, and taken together with the second equation shows

$$\lambda = -\frac{I}{a^2}\ddot{x} \; . \tag{102}$$

Putting this into the first equation in (100) we have

$$\ddot{x} = \left(1 + \frac{I}{Ma^2}\right)^{-1} g\sin\alpha = \frac{2}{3}g\sin\alpha \; . \tag{103}$$

The equation for $\ddot{x}(t)$ can be integrated using the initial conditions to yield

$$x(t) = x(0) + t\dot{x}(0) + \frac{t^2}{3}g\sin\alpha \; . \tag{104}$$

Also $\ddot{\theta} = a^{-1}x$, so

$$\theta(t) = \theta(0) + t\dot{\theta}(0) + \frac{t^2}{3a}g\sin\alpha \; . \tag{105}$$

The constants $x(0), \dot{x}(0), \theta(0), \dot{\theta}(0)$ determine the solution. We can also compute the generalized force of constraint given by the Lagrange multiplier λ,

$$\lambda = -\frac{1}{1 + \frac{Ma^2}{I}}Mg\sin\alpha = -\frac{1}{3}Mg\sin\alpha \; . \tag{106}$$