CONSERVED CURRENTS AND CONTINUUM LAGRANGE EQUATIONS

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OCTOBER 31, 2024

Abstract. The method of deriving equations of motion from a Lagrangian or an action applies not only to ordinary differential equations depending on time. It also can be used for certain partial differential equations. These include wave equations, the Dirac equation, Maxwell's equations, the Yang-Mills equations, etc. An analog of Noether's theorem leads to a connection between symmetries of the Lagrangian and the existence of "conserved currents."

Contents

I. Lagrange Equations for Continuum Systems

The Lagrange equations for a continuum "field" $\varphi(x)$ are the focus of these notes. A scalar field φ is a function on space-time $\mathbb{R}^{s+1}$, which we parameterize by points $x = (x_0, \vec{x}) = (x_0, x_1, \ldots, x_s)$. Let $d = s+1$ denote the space-time dimension. In order to simplify notation and to be compatible with special relativity, we take $x_0 = ct$, where c is the velocity of light. While in some sections we take $d = 4$, for the usual dimension of space-time, we could equally well consider other dimensions. We sometimes call the space-time a *world sheet*.

The field φ takes values in a *target space*. Here we consider the target space $\mathbb{R}^n$, though one often considers a target space that is $\mathbb{C}^n$, or even a target manifold. In the case the target space is $\mathbb{R}^n$, we denote the field by a real, n-dimensional vector

$$\varphi(x) = \{\varphi_i(x)\} \ , \ \text{ for } \ i = 1, \ldots, n \ , \quad x \in \mathbb{R}^d \ .$$

In addition, generally the field $\varphi(x)$ is the solution to a differential equation, the equation of motion for the field. Here we study equations that can be derived from a Lagrangian, in a fashion we now investigate. Such Lagrangians describe many equations of mathematical physics.

The equations of motion for a field that are analogous. For an n-component field, the equations are a consequence of assuming the form of a Lagrangian density L. They have the form

$$\boxed{\sum_{\mu=0}^{s} \frac{\partial}{\partial x_\mu} \left(\frac{\partial L\left(\varphi(x), \partial\varphi(x)\right)}{\partial(\partial_\mu \varphi_i(x))} \right) = \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))} \ , \ \text{ for } \ i = 1, \ldots, n} \ . \tag{I.1}$$

II. Basic Properties

Let us suppose that we are given a Lagrangian density $L(\varphi(x), \partial\varphi(x))$, where we denote the gradient of the field by $\partial\varphi(x)$, with components $\partial_\mu \varphi_i(x) = \frac{\partial \varphi_i(x)}{\partial x_\mu}$. The Lagrangian is the integral of the Lagrangian density over the spatial variables,

$$\mathcal{L}(\varphi, \partial\varphi) = \int_{\mathbb{R}^s} L(\varphi(x), \partial\varphi(x))\, d\vec{x} \ . \tag{II.1}$$

Similarly, the action S is the integral of the Lagrangian density over space-time,

$$S(\varphi) = \int L(\varphi(x), \partial\varphi(x))\, dx \ . \tag{II.2}$$

II.1. **Lagrange Equations for Particles: Review.** In the particle case, we wrote the canonical momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. For N degrees of freedom q_i,

$$p_i(t) = \frac{\partial \mathcal{L}(q(t), \dot{q}(t))}{\partial(\dot{q}_i(t))} \ , \ \text{ and } \ F_i(q(t), \dot{q}(t)) = \frac{\partial \mathcal{L}(q(t), \dot{q}(t))}{\partial(q_i(t))} \ .$$

Then Lagrange's equations of motion are

$$\boxed{\frac{dp_i(t)}{dt} = F_i(q(t), \dot{q}(t))}\ . \tag{II.3}$$

II.2. **Lagrange Equations for Fields.** The equations of motion for a field that are analogous. For an n-component field $\varphi(x) = \{\varphi_i(x) : i = 1, \ldots, n\}$, the equations of motion have the form (I.1). One can also interpret this in terms of canonical momenta and forces. In the field case, one has several components of a *canonical momentum* field $\pi(x)$, labelled by the different coordinate directions, not only for the time coordinate. So in space-time one also has a canonical momentum arising from each spatial coordinate. We need a double index to label the components of $\pi(x)$, namely (ν, i), where ν labels the space-time coordinates and i labels the components of the scalar field. Let

$$\pi_{\nu,i}(x) = \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial\left(\partial_\nu \varphi_i(x)\right)}\ . \tag{II.4}$$

One can also introduce the "force" field

$$F_i(\varphi(x), \partial\varphi(x)) = \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))}\ .$$

In terms of these fields, the Lagrange equations of motion are

$$\boxed{\sum_{\mu=0}^{s} \frac{\partial \pi_{\mu,i}(x)}{\partial x_\mu} = F_i(\varphi(x), \partial\varphi(x))\ , \ \text{ for } \ i = 1, \ldots, n}\ . \tag{II.5}$$

These are just the equations (I.1), the analog of the particle Lagrange equation (II.3). We now derive these equations by varying the action.

III. Derivation of Lagrange's Equations

We now see that solution to Lagrange equations (I.1), (II.5) are fields $\varphi(x)$ that are critical (stationary) points of an action functional.

We work in a compact region of space-time B, with boundary ∂B. Consider the action $S_B(\varphi)$ defined as

$$S_B(\varphi) = \int_B L\, d\vec{x}dt = \int_B L(\varphi(x), \partial\varphi(x))\, d\vec{x}dt\ . \tag{III.1}$$

We need to define the directional derivative $(D_\eta S)(\varphi)$ of the action S in the direction η at the point φ. We need to specify the meaning of varying the field $\varphi \to \varphi + \eta$ with fixed boundary values. Also we need to define a critical point of the action $S(\varphi)$.

Definition III.1. *The directional derivative $\left(D_\eta S_B\right)(\varphi)$ of the function $S_B(\varphi)$ in the direction η at the point φ in field space is*

$$\left(D_\eta S_B\right)(\varphi) = \lim_{\epsilon \to 0} \frac{S_B(\varphi + \epsilon\eta) - S_B(\varphi)}{\epsilon}\ . \tag{III.2}$$

Definition III.2. *The field $\varphi(x)$ has fixed boundary values on the boundary ∂B of a space-time domain B, in case each variation vanishes on the boundary, namely $\eta(x) = 0$ for all $x \in \partial B$. One can also say that η has vanishing boundary conditions.*

Note that $\eta(x)$ has vanishing boundary conditions on ∂B, if and only if $\epsilon\,\eta(x)$ has vanishing boundary conditions on ∂B, for any constant ϵ.

Definition III.3. *The function φ is a* critical point *of $S_B(\varphi)$ with fixed boundary values, in the case that $(D_\eta S_B)(\varphi) = 0$ for every $\eta(x)$ with vanishing boundary conditions on ∂B.*

Proposition III.4. *The function $\varphi(x)$ is a solution to the Lagrange equations* (I.1) *in a simply-connected space-time region B, if and only if $\varphi(x)$ is a critical point of the functional $S_B(\varphi)$ with fixed boundary values.*

Proof. We calculate the directional derivative $(D_\eta S)(\varphi)$ in terms of the Lagrangian density. Assuming we can interchange the order of integration and differentiation, one can express the derivative in terms of the Lagrangian density as

$$(D_\eta S_B)(\varphi) = \int_B \left(\frac{d}{d\epsilon}\, L(\varphi(x) + \epsilon\eta(x), \partial\varphi(x) + \epsilon\partial\eta(x)) \right)\bigg|_{\epsilon=0} d\vec{x}dt \;. \tag{III.3}$$

Use the chain rule to obtain an expression for the integrand as

$$\begin{aligned} &\left(\frac{d}{d\epsilon}\, L(\varphi(x) + \epsilon\eta(x), \partial\varphi(x) + \epsilon\partial\eta(x)) \right)\bigg|_{\epsilon=0} && \text{(III.4)} \\ &\quad= \sum_{i=1}^{n} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))}\, \eta_i(x) + \sum_{\mu=0}^{s} \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu\varphi_i(x))}\, \partial_\mu\eta_i(x) \right) \\ &\quad= \sum_{i=1}^{n} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))} - \sum_{\mu=0}^{s} \frac{\partial}{\partial x_\mu} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu\varphi_i(x))} \right) \right) \eta_i(x) \\ &\qquad\qquad + \sum_{i=1}^{n} \sum_{\mu=0}^{s} \frac{\partial}{\partial x_\mu} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu\varphi_i(x))}\, \eta_i(x) \right) \;. && \text{(III.5)} \end{aligned}$$

Consider the final double sum in (III.5); we show that its integral over B vanishes. In fact this integral vanishes for each fixed value of the index i. To see that this is the case, note that for each i the integrand has the form

$$\sum_{\mu=0}^{s} \frac{\partial K_\mu(x)}{\partial x_\mu} \;,$$

where

$$K_\mu(x) = \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu\varphi_i(x))}\, \eta_i(x) \;.$$

Use Green's theorem to transform the volume integral of the divergence of the vector $K(x)$ over B into an integral over the boundary ∂B of B of the normal component of K. We write this as

$$\int_B \left(\sum_{\mu=0}^{s} \frac{\partial K_\mu(x)}{\partial x_\mu} \right) d\vec{x}dt = \int_{\partial B} \sum_{\mu=0}^{s} K_\mu(x) d\sigma_\mu \ . \tag{III.6}$$

Here σ_μ denotes a surface area element. Since η has zero boundary values on ∂B, and since K_μ is proportional to $\eta_i(x)$, this surface integral (III.6) vanishes for each i. Hence it also vanishes when summed over i.

This show that the expression (III.3) for the directional derivative $(D_\eta S_B)(\varphi)$ simplifies to

$$(D_\eta S_B)(\varphi) = \int_B \sum_{i=1}^{n} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))} - \sum_{\mu=0}^{s} \frac{\partial}{\partial x_\mu} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu \varphi_i(x))} \right) \right) \eta_i(x) \, d\vec{x}dt \ . \tag{III.7}$$

Clearly if the Lagrange equations (I.1) hold, then $(D_\eta S)(\varphi)$ vanishes for any η. On the other hand, if φ is a critical point with fixed boundary conditions, then $(D_\eta S)(\varphi)$ vanishes for every η (which vanishes on the boundary ∂B). From this we conclude that the Lagrange equations hold: for at any interior point x, we take $\eta_i(x)$ to vanish for all i except $i = j$, and for that value of i take

$$\eta_j(x) = \chi(x) \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\varphi_i(x))} - \sum_{\mu=0}^{s} \frac{\partial}{\partial x_\mu} \left(\frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu \varphi_i(x))} \right) \right) \ .$$

Here we choose χ so that $0 \leqslant \chi(x)$, and also $\chi(x) = 0$ outside a small neighborhood of x lying totally inside B. Hence the integrand in (III.7) is point wise positive, and therefore must vanish. The Lagrange equations then hold in a neighborhood of each interior point. □

IV. Conserved Charge Replaced by Conserved Currents

Review: In the case of a particle system, Noether's theorem gave a condition for the existence of a quantity Q that we called a "charge" which does not change in time. In more detail, if there is a continuous family of transformations $q \mapsto q_\epsilon$, with $q_\epsilon = q$ the identity transformation, such that along a trajectory $q(t)$ satisfying Lagrange's equations, there is a function $G(q, \dot{q})$ that satisfies

$$\frac{d\mathcal{L}(q_\epsilon, \dot{q}_\epsilon)}{d\epsilon} = \frac{dG(q_\epsilon, \dot{q}_\epsilon)}{dt}\bigg|_{\epsilon=0} \ , \tag{IV.1}$$

then we found that

$$Q(t) = \sum_j p_{\epsilon,j}(t) \, \frac{dq_{\epsilon,j}(t)}{d\epsilon} - G(q_\epsilon, \dot{q}_\epsilon)\bigg|_{\epsilon=0} \qquad \textbf{(Noether for particle mechanics)} \tag{IV.2}$$

does not depend on time t along a solution to Lagrange's equations. In other words, the charge Q is conserved, namely $\frac{dQ(t)}{dt} = 0$.

New: In this section we explore how to generalize this notion to a continuous system. We are led to the notion of a *conserved current*. The basic idea is that we replace the charge Q by the charge Q_Σ in a spatial region Σ, and express this quantity as the integral of a charge density in Σ.

That is a 4-vector $J(t,\vec{x})$ that satisfies

$$\boxed{\sum_{\mu=0}^{3} \frac{\partial J_\mu}{\partial x_\mu} = 0} . \quad \textbf{Conserved Current} \tag{IV.3}$$

One interprets the zero component $J_0(t,\vec{x}) = \rho(t,\vec{x})$ as a charge density, and the other three components $\vec{J}(t,\vec{x})$ for $\mu = 1,2,3$ as a current, describing the flow of charge. For the remainder of this section we elaborate on this idea.

IV.1. **Local Charge and its Conservation.** For a continuum system, one can generalize this notion of charge to a local notion. Given a spatial region Σ, we can ask: "What is the charge Q_Σ in the region Σ of space?" The interpretation of *conservation* then needs to be generalized as well. Since charge might flow into or out of the region Σ, one says that the charge Q_Σ is conserved, if and only if Q_Σ charges when charge flows across the boundary of Σ, into or out of the region Σ. This defines a *local* notion of charge conservation, that generalizes the notion of global charge conservation we introduced for particle systems. Global charge conservation would re-enter the picture to say that in case $\Sigma = \mathbb{R}^3$, then the charge $Q_{\mathbb{R}^3}$ is independent of time.

In order to formulate this mathematically, we need to introduce the notion of both a *charge density* $\rho(t,\vec{x})$ describing the charge in a region, as well as a *current* vector $\vec{J}(t,\vec{x})$ to describe the flow of charge. The charge $Q_\Sigma(t)$ in the spatial region Σ equals the integral of the charge density,

$$Q_\Sigma(t) = \int_\Sigma \rho(t,\vec{x}) d\vec{x} \; . \tag{IV.4}$$

Furthermore the instantaneous flow of charge into the region Σ is given by

$$-\int_{\partial\Sigma} \vec{J}(t,\vec{x}) \cdot \vec{n}\, d\sigma \; . \tag{IV.5}$$

Here $\partial\Sigma$ denotes the boundary surface of Σ, and $\vec{n}$ denotes the outward normal to this surface. The integral $\int \vec{J} \cdot \vec{n}\, d\sigma$ denotes integration of the component $\vec{J} \cdot \vec{n}$ of the current vector $\vec{J}$ that is normal to the boundary $\partial\Sigma$ of Σ, using the surface area element denoted by $d\sigma$ to integrate over $\partial\Sigma$. The minus sign is necessary to interpret the flow as into, rather than out of, the region Σ. Hence we reformulate "conservation of charge" to mean that

$$\frac{\partial}{\partial t} \int_\Sigma \rho(t,\vec{x})\, d\vec{x} = -\int_{\partial\Sigma} \vec{J}(t,\vec{x}) \cdot \vec{n}\, d\sigma \; , \tag{IV.6}$$

for every region Σ. Using the divergence theorem for integration in 3-space, we can write

$$\int_{\partial\Sigma} \vec{J}(t,\vec{x}) \cdot \vec{n}\, d\sigma = \int_{\Sigma} \nabla \cdot \vec{J}(t,\vec{x})\, d\vec{x}\ . \tag{IV.7}$$

Thus one can rewrite (IV.6) as

$$\frac{\partial}{\partial t} \int_{\Sigma} \rho(t,\vec{x})\, d\vec{x} + \int_{\Sigma} \nabla \cdot \vec{J}(t,\vec{x})\, d\vec{x} = 0\ , \tag{IV.8}$$

or

$$\int_{\Sigma} \left(\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial J_j}{\partial x_j} \right) d\vec{x} = 0\ . \tag{IV.9}$$

As this relation holds for all Σ, it means that the average of the integrand over any region Σ must vanish. We now argue that the integrand itself must vanish for any current that varies continuously in $(\vec{x}, t)$. For suppose that x is a given point at which the integrand $\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial J_j}{\partial x_j}$ is non-zero. Choose Σ to be a sphere centered at x, with a sufficiently small radius, that the integrand has a fixed sign throughout Σ. The vanishing of the integral (IV.9) thus means that the integrand (which has a fixed sign on Σ) must be zero at every point in Σ. Since x was chosen arbitrarily, this shows that the integrand itself must vanish everywhere, and

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial J_j}{\partial x_j} = 0\ . \tag{IV.10}$$

Hence we have our condition relating a conserved current to the notion of conservation of charge.

IV.2. **Current as a Four-Vector.** The question we then address is: can one find a Noether's theorem to describe this situation? Namely is there a symmetry of the system described by a Lagrangian density, that leads to a formula for a conserved current J, in the way that a symmetry of a particle Lagrangian led us to the conserved charge (IV.2)?

Let $x = (x_0, \vec{x}) = (t, \vec{x})$, so we write the 4-vector current as

$$J = (\rho, \vec{J}) = (J_0, \vec{J})\ . \tag{IV.11}$$

Then the local conservation law for a conserved current can be written as (IV.3). In this notation the charge Q_Σ can be written

$$Q_\Sigma = \int_{\Sigma} J_0(\vec{x}, t)\, d\vec{x}\ . \tag{IV.12}$$

For $\Sigma = \mathbb{R}^3$, the total charge is

$$Q = \int_{\mathbb{R}^3} J_0(\vec{x}, t)\, d\vec{x}\ . \tag{IV.13}$$

Assuming that $\vec{J}$ converges sufficiently fast to zero as $|\vec{x}\,| \to \infty$, then the conservation law (IV.3) ensures that

$$\frac{d}{dx_0}\, Q = 0 \; . \tag{IV.14}$$

V. Noether Currents

The relationship between symmetry of a Lagrangian and the existence of a conserved current was first studied by the great algebraist Emmy Noether, who worked at the beginning of the 20^{th} century in Erlangen, Göttingen, and Bryn Mayr. In spite of mathematical physics lying outside the mainstream of her scientific activity, she left a splendid and lasting legacy in that field. You might be interested in looking at physicist Nina Byers' historical account of Noether, `http://arxiv.org/pdf/physics/9807044v2.pdf`.

In the case that the Lagrangian is given by a space-time density, we can derive the existence of a conserved vector J as a consequence of a continuous symmetry of the Lagrangian. We do this in a way exactly analogous to the derivation of Noether's theorem in the case of a Lagrangian describing particles.

We now consider a family of such fields $\varphi(\epsilon, x)$ where ϵ parameterizes a family of transformations of the field. By convention we take $\varphi(\epsilon = 0, x) = \varphi(x)$ to denote the field without a transformation (namely $\epsilon = 0$ is the parameter for the identity transformation). Define the transformed Lagrangian density L_ϵ by

$$L_\epsilon(\varphi(x), \partial\varphi(x)) = L(\varphi(\epsilon, x), \partial\varphi(\epsilon, x)) \; , \tag{V.1}$$

and the corresponding Lagrangian by

$$\mathcal{L}_\epsilon = \int_{\mathbb{R}^s} L_\epsilon \, d\vec{x} \; . \tag{V.2}$$

Let $\varphi(x) = \{\varphi_i(x)\}$, where $i = 1, \ldots, n$ indexes the components of φ as introduced in §II. Also take $x = (x_0, x_1, \ldots, x_s) \in \mathbb{R}^{s+1}$.

Proposition V.1 (Noether's Theorem). *Let $\varphi(x)$ be a solution to Lagrange equations for the Lagrangian density $L = L(\varphi(x), \partial\varphi(x))$. Suppose that $L \mapsto L_\epsilon$ is a symmetry of L, in the sense that there are functions $G_\nu(x) = G_\nu(\varphi(x), \partial\varphi(x))$, such that*

$$\frac{d}{d\epsilon}\, L_\epsilon \bigg|_{\epsilon=0} = \sum_{\nu=0}^{s} \frac{\partial G_\nu}{\partial x_\nu} \; . \tag{V.3}$$

Then the current defined as

$$J_\nu(x) = \left(\sum_{i=1}^{n} \pi_{\nu,i}(x)\, \frac{d\varphi_i(\epsilon, x)}{d\epsilon} \Bigg|_{\epsilon=0} \right) - G_\nu(x) \tag{V.4}$$

is conserved, and $\rho(x) = J_0(x)$ is the density of a conserved Noether charge.

Proof. Calculate $\frac{dL_\epsilon}{d\epsilon}$ using the chain rule. We find that

$$\begin{aligned}
\frac{dL_\epsilon}{d\epsilon} &= \sum_i \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\varphi_i(\epsilon,x))} \frac{d\varphi_i(\epsilon,x)}{d\epsilon} + \sum_{\nu=0}^{s} \frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \frac{d(\partial_\nu\varphi_i(\epsilon,x))}{d\epsilon} \right) \\
&= \sum_i \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\varphi_i(\epsilon,x))} \frac{d\varphi_i(\epsilon,x)}{d\epsilon} + \sum_{\nu=0}^{s} \frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \frac{\partial}{\partial x_\nu}\frac{d\varphi_i(\epsilon,x)}{d\epsilon} \right) \\
&= \sum_i \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\varphi_i(\epsilon,x))} - \sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \right) \right) \frac{d\varphi_i(\epsilon,x)}{d\epsilon} \\
&\quad + \sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \sum_i \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \frac{\partial\varphi_i(\epsilon,x)}{\partial\epsilon} \right) .
\end{aligned}$$

The Lagrange equations of motion (I.1) ensure that for any solution $\varphi_i(\epsilon,x)$, one has

$$\sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \right) = \frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\varphi_i(\epsilon,x))} . \tag{V.5}$$

Therefore for a solution to Lagrange's equations corresponding terms in $dL_\epsilon/d\epsilon$ cancel, and the derivative at $\epsilon = 0$ simplifies to

$$\begin{aligned}
\left.\frac{dL_\epsilon}{d\epsilon}\right|_{\epsilon=0} &= \sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \sum_i \left. \left(\frac{\partial L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))}{\partial(\partial_\nu\varphi_i(\epsilon,x))} \frac{d\varphi_i(\epsilon,x)}{d\epsilon} \right) \right|_{\epsilon=0} \\
&= \sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \sum_i \left. \left(\pi_{\nu,i}(x) \frac{d\varphi_i(\epsilon,x)}{d\epsilon} \right) \right|_{\epsilon=0} \\
&= \sum_{\nu=0}^{s} \frac{\partial G_\nu}{\partial x_\nu} .
\end{aligned} \tag{V.6}$$

The last equality follows from the assumption in the proposition concerning the existence of G_ν. Subtracting the last two expressions in (V.6) shows that

$$\sum_{\nu=0}^{s} \frac{\partial}{\partial x_\nu} \left(\sum_i \left. \left(\pi_{\nu,i}(x) \frac{d\varphi_i(\epsilon,x)}{d\epsilon} \right) \right|_{\epsilon=0} - G_\nu \right) = 0 . \tag{V.7}$$

The form of the conserved current J then follows for

$$J_\nu = \sum_i \pi_{\nu,i} \left. \frac{d\varphi_i(\epsilon,x)}{d\epsilon} \right|_{\epsilon=0} - G_\nu \tag{V.8}$$

as claimed. □

VI. Target-Space Symmetry and Space-Time Symmetry

For the remainder of these notes, we go into detail for a few interesting examples of symmetries and corresponding Noether currents. To simplify our discussion we begin by studying transformations of a scalar field. The fields we consider are functions on space-time, rather than particle trajectories that depend only on time. This lead to a far wider class of symmetries that one may consider for fields, rather than for Lagrangians for point particles. Thus we have a wealth of new possibilities to apply Noether's theorem. We will distinguish two distinct types of symmetries:

1. **Space-Time Symmetry, denoted ST.** These symmetries arise from consideration of transformations $x \mapsto x'$ on space-time, and $\varphi(x) \mapsto \varphi(x')$ on fields. We will call these *space-time symmetries*. If we want to distinguish the origin of the symmetry, and we designate the conserved vectors as J_{ST}, the conserved charge as Q_{ST}, etc.
2. **Target-Space Symmetry, denoted TS.** Another second type of transformation involves maps among the components of the field, which we denote $\varphi(x) \mapsto \varphi'(x)$. We call *target-space symmetries*. If we want to distinguish the origin of the symmetry, and we designate the conserved vectors as J_{TS}, the conserved charge as Q_{TS}, etc.

For "scalar" quantities it is easy to disentangle transformations on space-time from those in the target space. However for vectors, spinors, tensors, etc., one needs to combine the two into symmetries into a transformation of the form $\varphi(x) \mapsto \varphi'(x')$.

VII. The Energy-Momentum Density Tensor

The first example of a Noether current arises from a Lagrangian density that is translation invariant in time and space. We consider an n-component, scalar field $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$, for $x \in \mathbb{R}^4$. In relation to the general Noether's theorem of Proposition V.1, we take the special case $s = 3$ in this example that we discuss throughout the rest of these notes. This means that we study the four translations $x \mapsto x + \epsilon\, \widehat{e}_\mu$, where ϵ is a real parameter and $\widehat{e}_\mu$ is a unit vector in the μ^{th} direction of 4-dimensional space-time, $\mu = 0, 1, 2, 3$.

The four different translations in space-time give rise to four different conserved current vectors. Let us call the components of the four conserved vectors $J_\nu^{(0)}(x), J_\nu^{(1)}(x), J_\nu^{(2)}(x)$, and $J_\nu^{(3)}(x)$. We collect the components of the four conserved vectors together into a 4×4 matrix with entries $T_{\nu\mu}(x) = J_\nu^{(\mu)}(x)$. The matrix $T_{\nu\mu}(x)$ is called the *energy-momentum density* tensor.

The four conserved charges will be identified with the total energy of the field H (or Hamiltonian) and the total momentum vector of the field $\vec{P}$, which has components P_j, for $j = 1, 2, 3$. Explicitly

$$H = \int T_{00}(x)\, d\vec{x}\ , \ \text{ and } \ P_j = -\int T_{0j}(x)\, d\vec{x}\ . \tag{VII.1}$$

Correspondingly one calls $T_{00}(x)$ the energy density and $-T_{0j}(x)$ the momentum density.

Note: Here we have introduced a minus sign in the definition of P_j in order to correspond with the natural situation for the solution $\cos\omega(x-t)$ to the two-dimensional wave equation $\Box\cos\omega(x-t)=0$ (with $c=1$). This wave moves in to the right, of increasing x. So it is natural that it has a positive momentum. The minus sign in $\vec{P}$ ensures this. This minus sign is an indication that a better formula (and one which is correct when taking relativity into account) would involve the metric $g_{\mu\nu}$ on space-time in the definition of $T_{\mu\nu}(x)$. Then the Minkowski metric would introduce the minus sign into the momentum automatically. But we do not enter into that discussion here.

Proposition VII.1. *Suppose the Lagrangian density $L = L(\varphi(x), \partial\varphi(x))$ does not depend explicitly on x. Define the transformation $g_\epsilon^{(\nu)} : x \mapsto x + \epsilon\,\widehat{e}_\nu$. These four transformations gives rise to the set of conserved Noether currents $J^{(\nu)}(x)$ whose components $J_\mu^{(\nu)}(x) = T_{\mu\nu}(x)$ comprise the energy-momentum density tensor*

$$T_{\mu\nu}(x) = \left(\sum_i \pi_{\mu,i}(x)\,\frac{\partial\varphi_i(x)}{\partial x_\nu}\right) - \delta_{\mu\nu}\,L\ . \tag{VII.2}$$

The energy-momentum density satisfies

$$\sum_{\mu=0}^{3}\frac{\partial T_{\mu\nu}(x)}{\partial x_\mu} = 0\ , \quad \text{for } \nu = 0,1,2,3\ . \tag{VII.3}$$

In other words, the energy-momentum tensor defines four different conserved current vectors $J^{(\nu)}(x)$, for $\nu = 0,1,2,3$ with components

$$J_\mu^{(\nu)} = T_{\mu\nu}(x)\ . \tag{VII.4}$$

Here $J^{(0)}$ describes local energy conservation, and $J^{(\nu)}$ for $\nu = 1,2,3$ describe conservation of the ν^{th} component of momentum.

Proof. Define $\varphi(\epsilon,x) = \varphi(g_\epsilon^{(\nu)}x)$ and $L_\epsilon(\varphi(x),\partial\varphi(x)) = L(\varphi(\epsilon,x),\partial\varphi(\epsilon,x))$. Note L does not depend explicitly on x; it only depends on x through the implicit dependence of $\varphi(x)$ on x. We therefore infer that for this choice of L_ϵ,

$$\frac{\partial L_\epsilon(\varphi(x),\partial\varphi(x))}{\partial\epsilon} = \frac{\partial L_\epsilon(\varphi(x),\partial\varphi(x))}{\partial x_\nu}\ .$$

Let us define

$$G_\mu^{(\nu)}(\varphi(x),\partial\varphi(x)) = \delta_{\nu\mu}\,L_\epsilon(\varphi(x),\partial\varphi(x))\ .$$

Then

$$\frac{\partial L_\epsilon(\varphi(x),\partial\varphi(x))}{\partial\epsilon} = \sum_{\mu=0}^{3}\frac{\partial G_\mu^{(\nu)}(\varphi(x),\partial\varphi(x))}{\partial x_\mu}\ . \tag{VII.5}$$

Hence the Lagrangian L_ϵ satisfies the hypotheses of Proposition V.1. (In fact the identity (V.3) also holds for $\epsilon \neq 0$.) Furthermore for the transformation $g_\epsilon^{(\nu)}$ one has

$$\frac{\partial}{\partial \epsilon}\varphi(\epsilon, x) = \frac{\partial}{\partial \epsilon}\varphi(x + \epsilon\,\widehat{e}_\nu) = \partial_\nu \varphi(\epsilon, x) \ .$$

We therefore conclude that the conserved current given by Proposition V.1 is

$$J_\mu^{(\nu)}(x) = \sum_i \left(\pi_{\mu,i}(x)\,\partial_\nu\varphi_i(x)\right) - \delta_{\mu\nu}\,L \ , \tag{VII.6}$$

which is (VII.2) as claimed. As $T_{\mu\nu}(x) = J_\mu^{(\nu)}(x)$, the conservation laws (VII.3) follow. $\square$

VII.1. **Example: A Non-Linear Wave Equation.** Let us work out the energy density and the momentum density for a particular wave equation, namely system of non-linear wave equations for an n-component field $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$. Consider

$$\square\,\varphi_i(x) + \frac{\partial V(\varphi(x))}{\partial \varphi_i(x)} = 0 \ , \ \text{ for } \ x = (x_0, \vec{x}) \ \text{ with } \ \vec{x} \in \mathbb{R}^4 \ .$$

Here $\square$ is the wave operator

$$\square = \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \ .$$

The given potential function $V(\xi)$ maps the variable $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ to the real numbers $\mathbb{R}$. Denote the partial derivatives of V by the short-hand $V_i(\xi) = \partial V(\xi)/\partial\xi_i$. Thus we can use

$$V_i(\varphi(x)) = \frac{\partial V(\varphi(x))}{\partial \varphi_i(x)} \ ,$$

to write the equations of motion in the form,

$$\square\,\varphi_i(x) + V_i(\varphi(x)) = 0 \ , \quad \text{for} \quad i = i, \ldots, n \ . \tag{VII.7}$$

In the special case $V(\xi) = \frac{1}{2}m^2\sum_{j=1}^n \xi_j^2$, the resulting equation is the Klein-Gordon linear wave equation. In another special case with $V = 0$, the equation reduces to the linear wave equation $\square\,\varphi_i(x) = 0$. For a general V, the Lagrangian density is

$$L(\varphi(x), \partial\varphi(x)) = \frac{1}{2}\sum_{i=1}^{n}\left((\partial_0\varphi_i(x))^2 - \sum_{j=1}^{3}(\partial_j\varphi_i(x))^2 \right) - V(\varphi(x)) \ . \tag{VII.8}$$

Given this Lagrangian density, one has

$$\pi_{\mu,i}(x) = \frac{\partial L(\varphi(x), \partial\varphi(x))}{\partial(\partial_\mu\varphi_i(x))} = \begin{cases} \dfrac{\partial\varphi_i(x)}{\partial x_0} \ , & \text{if } \ \mu = 0 \\[2ex] -\dfrac{\partial\varphi_i(x)}{\partial x_j} \ , & \text{if } \ \mu = 1, \ldots, 3 \end{cases} \ . \tag{VII.9}$$

VII.1.1. *The Hamiltonian and Energy Density.* Given these calculations, we read off from (VII.2) that the energy density is

$$T_{00}(x) = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial\varphi_i(x)}{\partial x_0}\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{3}\left(\frac{\partial\varphi_i(x)}{\partial x_j}\right)^2 + V(\varphi(x))\ . \tag{VII.10}$$

This yields the total energy in terms of an integral over an energy density $H(x) = T_{00}(x)$, namely

$$H = \int_{x_0=0} H(x)\, d\vec{x} = \int_{x_0=0}\left(\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial\varphi_i(x)}{\partial x_0}\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{3}\left(\frac{\partial\varphi_i(x)}{\partial x_j}\right)^2 + V(\varphi(x))\right) d\vec{x}\ . \tag{VII.11}$$

Since H is conserved, we can take the integral over the $x_0 = 0$ initial surface, or over any other time slice.

VII.1.2. *The Momentum and Momentum Density.* Likewise (VII.2) shows that the density for the j^{th} component of the momentum P_j is $T_{0j}(x)$, which one can also write as $P_j(x)$. This is just

$$P_j(x) = T_{0j}(x) = \sum_{i=1}^{n}\pi_{0,i}\frac{\partial}{\partial x_j}\,\varphi_i(x)\ .$$

Note that this expression for the momentum density is independent of V. Correspondingly the total momentum is

$$P_j = \int_{x_0=0} P_j(x)\, d\vec{x} = \sum_{i=1}^{n}\int_{x_0=0}\frac{\partial\varphi_i(x)}{\partial x_0}\,\frac{\partial\varphi_i(x)}{\partial x_j}\, d\vec{x}\ . \tag{VII.12}$$

VIII. Rotational Symmetry

In this section we investigate the consequence of rotational symmetry. In this section, we specialize the Lagrangian density for the wave equation given in (VII.8) to the case of a two-dimensional target $n = 2$ and four space-time dimensions. Thus consider a very special case of a two-component field $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, with $x = (x_0, x_1, x_2, x_3)$ a space-time vector in a 4-dimensional space-time.

We distinguish two different types of rotations: the first rotation that one might consider is rotation of the coordinates in the (x_1, x_2) plane. So when one talks of rotational symmetry one should be very clear about what is being rotated. The different sorts of rotation give rise to different conserved currents. In both cases we can generalize this to higher dimensional world sheets or higher dimensional targets.

In the following example the Lagrangian has both target-space rotational symmetry and also space-time rotational symmetry. We choose a symmetric, quartic potential. So

$$V(\xi) = \frac{1}{2}m^2\xi^2 + \lambda\left(\xi^2\right)^2\ , \quad \text{where} \quad \xi^2 = \xi_1^2 + \xi_2^2\ . \tag{VIII.1}$$

and

$$L(\varphi(x), \partial\varphi(x)) = \frac{1}{2} \sum_{i=1}^{2} \left(\left(\frac{\partial \varphi_i(x)}{\partial x_0} \right)^2 - \sum_{j=1}^{3} \left(\frac{\partial \varphi_i(x)}{\partial x_j} \right)^2 \right) - \frac{1}{2}\, m^2 \sum_{i=1}^{2} \varphi_i(x)^2 - \lambda \left(\sum_{i=1}^{2} \varphi_i(x)^2 \right)^2 . \tag{VIII.2}$$

VIII.1. **Rotation in the Target Space.** Define rotation in the target-space by angle ϵ as the transformation

$$\varphi(\epsilon, x) = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x)\cos\epsilon - \varphi_2(x)\sin\epsilon \\ \varphi_1(x)\sin\epsilon + \varphi_2(x)\cos\epsilon \end{pmatrix} = \begin{pmatrix} \varphi_1(\epsilon, x) \\ \varphi_2(\epsilon, x) \end{pmatrix} .$$

This transformation yields a conserved current $J_{\mathrm{TS}} = (J_0, J_1, J_2, J_4)$ whose value we find from Noether's theorem as follows.

First notice that each term in the Lagrangian density, after summation over the index i, is left invariant by the target space rotation. For example,

$$\sum_{i=1}^{2} \varphi_i(\epsilon, x)^2 = (\varphi_1(x)\cos\epsilon - \varphi_2(x)\sin\epsilon)^2 + (\varphi_1(x)\sin\epsilon + \varphi_2(x)\cos\epsilon)^2 = \sum_{i=1}^{2} \varphi_i(x)^2 .$$

As a consequence, the Lagrange density remains invariant under this transformation,

$$L_\epsilon(\varphi(x), \partial\varphi(x)) = L(\varphi(\epsilon, x), \partial\varphi(\epsilon, x)) = L(\varphi(x), \partial\varphi(x)) .$$

Thus

$$\frac{d}{d\epsilon}\, L_\epsilon(\varphi(x), \partial\varphi(x)) = 0 . \tag{VIII.3}$$

And in particular, the Lagrangian $\mathcal{L}_\epsilon = \int L_\epsilon(\varphi(x), \partial\varphi(x))\, d\vec{x} = \mathcal{L}$ is also invariant.

Thus we can apply Noether's Theorem (Proposition V.1) with $G_\mu(x) = 0$, to give the conserved current J_{TS} for rotations in the target space, with components

$$J_{\mathrm{TS},\mu}(x) = \sum_{i=1}^{2} \left(\frac{\partial L(\varphi(\epsilon, x), \partial\varphi(\epsilon, x))}{\partial(\partial_\mu \varphi_i(\epsilon, x))}\, \frac{\partial \varphi_i(\epsilon, x)}{\partial\epsilon} \right) \Bigg|_{\epsilon=0} = -\pi_{\mu,1}(x)\, \varphi_2(x) + \pi_{\mu,2}(x)\, \varphi_1(x) . \tag{VIII.4}$$

In particular, the conserved charge density is

$$J_{\mathrm{TS}}(x) = \begin{pmatrix} J_{\mathrm{TS},0}(x) \\ J_{\mathrm{TS},1}(x) \\ J_{\mathrm{TS},2}(x) \\ J_{\mathrm{TS},3}(x) \end{pmatrix} = \begin{pmatrix} \pi_{0,2}(x)\, \varphi_1(x) - \pi_{0,1}(x)\, \varphi_2(x) \\ \pi_{1,2}(x)\, \varphi_1(x) - \pi_{1,1}(x)\, \varphi_2(x) \\ \pi_{2,2}(x)\, \varphi_1(x) - \pi_{2,1}(x)\, \varphi_2(x) \\ \pi_{3,2}(x)\, \varphi_1(x) - \pi_{3,1}(x)\, \varphi_2(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial\varphi_2(x)}{\partial x_0}\, \varphi_1(x) - \frac{\partial\varphi_1(x)}{\partial x_0}\, \varphi_2(x) \\ \frac{\partial\varphi_1(x)}{\partial x_1}\, \varphi_2(x) - \frac{\partial\varphi_2(x)}{\partial x_1}\, \varphi_1(x) \\ \frac{\partial\varphi_1(x)}{\partial x_2}\, \varphi_2(x) - \frac{\partial\varphi_2(x)}{\partial x_2}\, \varphi_1(x) \\ \frac{\partial\varphi_1(x)}{\partial x_3}\, \varphi_2(x) - \frac{\partial\varphi_2(x)}{\partial x_3}\, \varphi_1(x) \end{pmatrix} .$$

In writing the charge density $J_{\mathrm{TS},0}(x)$, let us for shorthand drop the index $\nu = 0$ in the canonical momenta $\pi_{\nu,i}$, as only $\nu = 0$ occurs; in this expression we take $\pi_i = \partial\varphi_i/\partial x_0$. The corresponding charge for the spatial domain B is

$$\begin{aligned} Q_{\mathrm{TS,B}}(x_0) &= \int_B J_{\mathrm{TS},0}(x)\, d\vec{x} \\ &= \sum_{i,j=1}^{2} \epsilon_{ij} \int_B \varphi_i(x)\, \pi_j(x)\, d\vec{x}\ . \end{aligned} \tag{VIII.5}$$

VIII.2. **Fields with n Compoents.** In case the field $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ has n components, in place of 2, let us assume that the potential $V(\varphi(x))$ in the Lagrangian density is a function of $\sum_{i=1}^n \varphi_i(x)^2$. Then a similar argument shows that a target space rotation in each of the $\binom{n}{2}$ planes $(\varphi_\alpha(x), \varphi_\beta(x))$ leads to the conserved currents labelled by this (α, β) plane. We have

$$J_{\mathrm{TS}}^{(\alpha\beta)}(x) = \begin{pmatrix} J_{\mathrm{TS},0}^{(\alpha\beta)}(x) \\ J_{\mathrm{TS},1}^{(\alpha\beta)}(x) \\ J_{\mathrm{TS},2}^{(\alpha\beta)}(x) \\ J_{\mathrm{TS},3}^{(\alpha\beta)}(x) \end{pmatrix} = \begin{pmatrix} \pi_{0,\beta}(x)\, \varphi_\alpha(x) - \pi_{0,\alpha}(x)\, \varphi_\beta(x) \\ \pi_{1,\beta}(x)\, \varphi_\alpha(x) - \pi_{1,\alpha}(x)\, \varphi_\beta(x) \\ \pi_{2,\beta}(x)\, \varphi_\alpha(x) - \pi_{2,\alpha}(x)\, \varphi_\beta(x) \\ \pi_{3,\beta}(x)\, \varphi_\alpha(x) - \pi_{3,\alpha}(x)\, \varphi_\beta(x) \end{pmatrix}\ ,$$

with the corresponding conserved charges from taking $B = \mathbb{R}^3$. Namely the conserved charges arising from invariance under rotations in the (α, β)-plane of the target space are:

$$Q_{\mathrm{TS},B}^{(\alpha\beta)}(x_0) = \int_{\mathbb{R}^3} J_{\mathrm{TS},0}^{(\alpha\beta)}(x) d\vec{x} = \int_{\mathbb{R}^3} \left(\varphi_\alpha(x)\, \pi_\beta(x) - \varphi_\beta(x) \pi_\alpha(x)\right)\, d\vec{x}\ . \tag{VIII.6}$$

VIII.3. **Rotations in Space-Time.** Let us return now to discuss the case $n = 2$ and consider the specific space-time rotation

$$\varphi(\epsilon, x) = \varphi(x(\epsilon))\ , \tag{VIII.7}$$

where $x \mapsto x(\epsilon)$ is the rotation in the (x_1, x_2) spatial plane,

$$\begin{pmatrix} x(\epsilon)_0 \\ x(\epsilon)_1 \\ x(\epsilon)_2 \\ x(\epsilon)_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon & 0 \\ 0 & \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1\cos\epsilon - x_2\sin\epsilon \\ x_1\sin\epsilon + x_2\cos\epsilon \\ x_3 \end{pmatrix}\ .$$

This rotation defines the Lagrangian density

$$L_\epsilon(\varphi(x), \partial\varphi(x)) = L(\varphi(x(\epsilon)), \partial\varphi(x(\epsilon)))\ . \tag{VIII.8}$$

While the Lagrangian density is not invariant under this transformation, the total Lagrangian remains invariant,

$$\mathcal{L}_\epsilon = \int L_\epsilon d\vec{x} = \int L d\vec{x} = \mathcal{L}\ .$$

We now see that the transformation (VIII.7) yields a conserved Noether current that we denote $\vec{J}_{\rm ST}=(J_{{\rm ST},0},J_{{\rm ST},1},J_{{\rm ST},2},J_{{\rm ST},2})$. In order to find this current, note that

$$\left.\frac{dx(\epsilon)_0}{d\epsilon}\right|_{\epsilon=0}=0\;,\quad \left.\frac{dx(\epsilon)_1}{d\epsilon}\right|_{\epsilon=0}=-x_2\;,\quad \left.\frac{dx(\epsilon)_2}{d\epsilon}\right|_{\epsilon=0}=x_1\;,\quad \text{and}\quad \left.\frac{dx(\epsilon)_2}{d\epsilon}\right|_{\epsilon=0}=0\;.$$

As a consequence, the infinitesimal rotation of the field in the (x_1,x_2) plane of the world sheet equals

$$\left.\frac{d}{d\epsilon}\,\varphi(x(\epsilon))\right|_{\epsilon=0}=\sum_{\mu=0}^{2}\frac{\partial\varphi(x(\epsilon))}{\partial x(\epsilon)_\mu}\left.\frac{dx(\epsilon)_\mu}{d\epsilon}\right|_{\epsilon=0}=\left(x_1\,\frac{\partial}{\partial x_2}-x_2\,\frac{\partial}{\partial x_1}\right)\varphi(x)\;. \tag{VIII.9}$$

The operator

$$\ell_{12}=\left(x_1\,\frac{\partial}{\partial x_2}-x_2\,\frac{\partial}{\partial x_1}\right)\;, \tag{VIII.10}$$

is the familiar form of an infinitesimal rotation in the (x_1,x_2) plane. We write

$$\left.\frac{d}{d\epsilon}\,\varphi(x(\epsilon))\right|_{\epsilon=0}=\ell_{12}\,\varphi(x)\;. \tag{VIII.11}$$

Similarly one has

$$\begin{aligned}\left.\frac{d}{d\epsilon}\,L_\epsilon(\varphi(x),\partial\varphi(x))\right|_{\epsilon=0} &= \left.\left(\sum_{\nu=0}^{3}\frac{\partial L_\epsilon(\varphi(x),\partial\varphi(x))}{\partial x_\nu(\epsilon)}\,\frac{dx_\nu(\epsilon)}{d\epsilon}\right)\right|_{\epsilon=0}\\ &= x_1\,\frac{\partial L(\varphi(x),\partial\varphi(x))}{\partial x_2}-x_2\,\frac{\partial L(\varphi(x),\partial\varphi(x))}{\partial x_1}\\ &= \ell_{12}\,L(\varphi(x),\partial\varphi(x))\;.\end{aligned}$$

This motivates one to define the four-component quantity $G(x)$ with components $G_\mu(x)$, as

$$G(x)=\begin{pmatrix}G_0(x)\\G_1(x)\\G_2(x)\\G_3(x)\end{pmatrix}=\begin{pmatrix}0\\-x_2\,L(\varphi(x),\partial\varphi(x))\\x_1\,L(\varphi(x),\partial\varphi(x))\\0\end{pmatrix}\;.$$

Then

$$\left.\frac{d}{d\epsilon}\,L_\epsilon(\varphi(x),\partial\varphi(x))\right|_{\epsilon=0}=\sum_{\nu=0}^{3}\frac{\partial G_\nu(x)}{\partial x_\nu}\;.$$

Apply Noether's Theorem (Proposition V.1) to give the conserved current $\vec{J}_{\rm ST}(x)$ associated with rotations on the world sheet, with components

$$J_{{\rm ST},\mu}(x)=\sum_{i=1}^{2}\pi_{\nu,i}(x)\left.\frac{d\varphi_i(x(\epsilon))}{d\epsilon}\right|_{\epsilon=0}-G_\mu(x)=\sum_{i=1}^{2}\pi_{\mu,i}(x)\,\ell_{12}\,\varphi_i(x)-G_\mu(x)\;.$$

Putting in the expressions computed above, we see that for the Lagrangian (VIII.2) the conserved current equals

$$\vec{J}_{\mathrm{ST}}(x) = \begin{pmatrix} J_{\mathrm{ST},0}(x) \\ J_{\mathrm{ST},1}(x) \\ J_{\mathrm{ST},2}(x) \\ J_{\mathrm{ST},3}(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} \pi_{0,i}(x)\, \ell_{12}\, \varphi_i(x) \\ \left(\sum_{i=1}^{2} \pi_{1,i}(x)\, \ell_{12}\, \varphi_i(x)\right) + x_2\, L(\varphi(x), \partial\varphi(x)) \\ \left(\sum_{i=1}^{2} \pi_{2,i}\, \ell_{12}\, \varphi_i(x)\right) - x_1\, L(\varphi(x), \partial\varphi(x)) \\ \sum_{i=1}^{2} \pi_{3,i}(x)\, \ell_{12}\, \varphi_i(x) \end{pmatrix} . \qquad \text{(VIII.12)}$$

In this case, the conserved charge corresponding to the symmetry of rotation in the (x_1, x_2) plane of the world sheet equals

$$Q_{\mathrm{ST}} = \int_{x_0=0} J_{\mathrm{ST},0}(x)\, d\vec{x} = \int_{x_0=0} \sum_{i=1}^{2} \pi_i(x)\, \ell_{12}\, \varphi_i(x)\, d\vec{x} . \qquad \text{(VIII.13)}$$

In (VIII.13) we use the notation introduced in the previous section, namely $\pi_i(x) = \pi_{0,i}(x)$.

One can interpret this conserved quantity Q_{ST} as the angular momentum $\mathfrak{L}_{12}$ of the field,

$$\mathfrak{L}_{12} = \int_{x_0=0} \mathfrak{L}_{12}(x)\, d\vec{x} , \quad \text{where} \quad \mathfrak{L}_{12}(x) = \sum_{i=1}^{2} \pi_i(x)\, \ell_{12}\, \varphi_i(x) . \qquad \text{(VIII.14)}$$

Here the angular momentum density of the field $\mathfrak{L}_{12}(x)$ equals $J_{\mathrm{ST},0}(x)$. This can be expressed as a sum of moments of the momentum density of the field. In fact, recalling that the momentum density $T_{0j}(x)$ found in (**??**), we see that

$$\mathfrak{L}_{12}(x) = x_1\, P_2(x) - x_2\, P_1(x) . \qquad \text{(VIII.15)}$$

It is clear that these considerations generalize in a straightforward way to rotations in the planes (x_2, x_3) and (x_3, x_1). Each such rotation gives rise to a conserved Noether current and to a corresponding conserved charge. The three charges can be labeled by the rotation plane, or by the orthogonal direction. They are $\mathfrak{L}_1 = \mathfrak{L}_{23}$, $\mathfrak{L}_2 = \mathfrak{L}_{31}$, and $\mathfrak{L}_3 = \mathfrak{L}_{12}$, which we put together into the vector

$$\vec{\mathfrak{L}} = \int_{x_0=0} \vec{\mathfrak{L}}(x)\, d\vec{x} = \begin{pmatrix} \mathfrak{L}_1 \\ \mathfrak{L}_2 \\ \mathfrak{L}_3 \end{pmatrix} = \begin{pmatrix} \int_{x_0=0} \mathfrak{L}_1(x)\, d\vec{x} \\ \int_{x_0=0} \mathfrak{L}_2(x)\, d\vec{x} \\ \int_{x_0=0} \mathfrak{L}_3(x)\, d\vec{x} \end{pmatrix} . \qquad \text{(VIII.16)}$$

In order to find a formula for this density, we follow the calculation leading to (VIII.15). We have the density of momentum density $\vec{P}(x)$ with components $P_j(x) = T_{0j}(x)$. Then the angular momentum density vectors is

$$\vec{\mathfrak{L}}(x) = \vec{x} \times \vec{P}(x) = \begin{pmatrix} x_2 P_3(x) - x_3 P_2(x) \\ x_3 P_1(x) - x_1 P_3(x) \\ x_1 P_2(x) - x_2 P_1(x) \end{pmatrix} . \qquad \text{(VIII.17)}$$

IX. Lorentz Boosts

The sample Lagrangian density (VIII.2) also displays a symmetry under Lorentz boosts along either the x_1 axis on the world sheet, or along the x_2 axis. Define Lorentz boost by the transformation

$$\varphi(\epsilon, x) = \varphi(x(\epsilon)) \;, \tag{IX.1}$$

where $x \mapsto x(\epsilon)$ is the boost along the x_1 axis by the velocity $c \tanh \epsilon$, namely

$$\vec{x}(\epsilon) = \begin{pmatrix} x(\epsilon)_0 \\ x(\epsilon)_1 \\ x(\epsilon)_2 \\ x(\epsilon)_3 \end{pmatrix} = \begin{pmatrix} \cosh\epsilon & \sinh\epsilon & 0 & 0 \\ \sinh\epsilon & \cosh\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_0 \cosh\epsilon + x_1 \sinh\epsilon \\ x_0 \sinh\epsilon + x_1 \cosh\epsilon \\ x_2 \\ x_3 \end{pmatrix} \;.$$

This boost defines the Lagrangian density

$$L_\epsilon(\varphi(x), \partial\varphi(x)) = L(\varphi(x(\epsilon)), \partial\varphi(x(\epsilon))) \;. \tag{IX.2}$$

In this case neither the Lagrangian density is invariant under this transformation, nor is the total Lagrangian invariant. However the total action does remain invariant, as a consequence of the Jacobian of the Lorentz transformation being one, so

$$S_\epsilon = \int L_\epsilon \, d\vec{x}\, dt = \int L \, d\vec{x}\, dt = S \;. \tag{IX.3}$$

Note that

$$\left.\frac{dx(\epsilon)_0}{d\epsilon}\right|_{\epsilon=0} = x_1 \;, \quad \left.\frac{dx(\epsilon)_1}{d\epsilon}\right|_{\epsilon=0} = x_0 \;, \quad \left.\frac{dx(\epsilon)_2}{d\epsilon}\right|_{\epsilon=0} = 0 \;, \text{ and } \left.\frac{dx(\epsilon)_3}{d\epsilon}\right|_{\epsilon=0} = 0 \;.$$

As a consequence, the infinitesimal rotation of the field in the (x_1, x_2) plane of the world sheet equals

$$\left.\frac{d}{d\epsilon}\,\varphi(x(\epsilon))\right|_{\epsilon=0} = \sum_{\mu=0}^{3} \frac{\partial\varphi(x(\epsilon))}{\partial x(\epsilon)_\mu} \left.\frac{dx(\epsilon)_\mu}{d\epsilon}\right|_{\epsilon=0} = \left(x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_1} \right) \varphi(x) \;. \tag{IX.4}$$

The operator

$$b_{01} = \left(x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0} \right) \;, \tag{IX.5}$$

is the familiar form of an infinitesimal Lorentz boost in the (x_0, x_1) plane, given by a change in velocity of the coordinate axes along the x_1 coordinate direction. We write

$$\left.\frac{d}{d\epsilon}\,\varphi(x(\epsilon))\right|_{\epsilon=0} = b_{01}\,\varphi(x) \;. \tag{IX.6}$$

Similarly one has

$$\begin{aligned}\frac{d}{d\epsilon}\, L_\epsilon(\varphi(x),\partial\varphi(x))\Big|_{\epsilon=0} &= \left(\sum_{\nu=0}^{3} \frac{\partial L_\epsilon(\varphi(x),\partial\varphi(x))}{\partial x_\nu(\epsilon)}\, \frac{dx_\nu(\epsilon)}{d\epsilon}\right)\Bigg|_{\epsilon=0}\\ &= x_0\, \frac{\partial L(\varphi(x),\partial\varphi(x))}{\partial x_1} + x_1\, \frac{\partial L(\varphi(x),\partial\varphi(x))}{\partial x_0}\\ &= b_{01}\, L(\varphi(x),\partial\varphi(x))\ .\end{aligned}$$

This motivates one to define the four-component quantity

$$G(x) = \begin{pmatrix} G_0(x)\\ G_1(x)\\ G_2(x)\\ G_3(x)\end{pmatrix} = \begin{pmatrix} x_1\, L(\varphi(x),\partial\varphi(x))\\ x_0\, L(\varphi(x),\partial\varphi(x))\\ 0\\ 0\end{pmatrix}\ ,$$

for which

$$\frac{d}{d\epsilon}\, L_\epsilon(\varphi(x),\partial\varphi(x))\Big|_{\epsilon=0} = \sum_{\nu=0}^{3} \frac{\partial G_\nu(x)}{\partial x_\nu}\ .$$

Thus we can apply Noether's Theorem (Proposition V.1) to give the conserved current $\vec{J}(x)$ associated with boosts along the first coordinate axis, with components

$$J_\mu(x) = \sum_{i=1}^{2} \pi_{\mu,i}(x)\, \frac{d\varphi_i(x(\epsilon))}{d\epsilon}\Bigg|_{\epsilon=0} - G_\mu(x) = \sum_{i=1}^{2} \pi_{\mu,i}(x)\, b_{01}\, \varphi_i(x) - G_\mu(x)\ .$$

Putting in the expressions computed above, we see that for the Lagrangian (VIII.2) the conserved current equals

$$\begin{aligned}\vec{J}(x) &= \begin{pmatrix} J_0(x)\\ J_1(x)\\ J_2(x)\\ J_3(x)\end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{2} \pi_{0,i}(x)\, b_{01}\, \varphi_i(x)\right) - x_1\, L(\varphi(x),\partial\varphi(x))\\ \left(\sum_{i=1}^{2} \pi_{1,i}(x)\, b_{01}\, \varphi_i(x)\right) - x_0\, L(\varphi(x),\partial\varphi(x))\\ \sum_{i=1}^{2} \pi_{2,i}\, b_{01}\, \varphi_i(x)\\ \sum_{i=1}^{2} \pi_{3,i}\, b_{01}\, \varphi_i(x)\end{pmatrix}\\ &= \begin{pmatrix} x_1 H(x) + x_0 P_1(x)\\ \left(\sum_{i=1}^{2} \pi_{1,i}(x)\, b_{01}\, \varphi_i(x)\right) - x_0\, L(\varphi(x),\partial\varphi(x))\\ \sum_{i=1}^{2} \pi_{2,i}\, b_{01}\, \varphi_i(x)\\ \sum_{i=1}^{2} \pi_{3,i}\, b_{01}\, \varphi_i(x)\end{pmatrix}\ . \qquad \text{(IX.7)}\end{aligned}$$

In the last equality, we use the relations for the energy density $H(x)$ in (VII.10) and for the momentum density $P_1(x)$ in (**??**).

So this shows that there is a conserved quantity (let us denote it by M_1) associated with an infinitesimal boost along the x_1 axis. It has the density

$$M_1(x) = J_0(x) = x_1\, H(x) + x_0\, P_1(x)\ . \qquad \text{(IX.8)}$$

Here $H(x)$ is the density of the energy, and $P_1(x)$ is the density of the first component of the momentum.

In case we calculate the conserved quantity as an integral over the $x_0 = 0$ surface, then the momentum term drops out, and the conserved charge is a first moment of the energy density,

$$M_1 = \int_{x_0=0} x_1 \, H(x) \, d\vec{x} \; . \tag{IX.9}$$

A similar argument shows that the conserved charge corresponding to an infinitesimal Lorentz boost along the second coordinate axis is

$$M_2 = \int_{x_0=0} x_2 \, H(x) \, d\vec{x} \; . \tag{IX.10}$$

In case of three-dimensional space, one would have a set of three conserved charges, one associated with each coordinate direction. We write this as a vector $\vec{M}$ of conserved charges with components M_j. The above argument shows that the density of this conserved quantity is

$$\boxed{M_j(x) = x_j \, H(x) + x_0 \, P_j(x) \; , \;\; \text{for} \;\; j = 1, 2, 3} \; . \tag{IX.11}$$

This gives the global conserved quantities M_j for the surface $x_0 = 0$ as the integrals of the corresponding densities. The terms proportional to x_0 vanish on this surface, so the globally conserved quantities M_j can be expressed as first moments of the energy density,

$$\boxed{M_j = \int_{x_0=0} x_j \, H(x) \, d\vec{x} \; , \;\; \text{for} \;\; j = 1, 2, 3} \; . \tag{IX.12}$$