PRINCIPLE OF LEAST ACTION: DOES PHYSICAL MOTION GIVE THE *LEAST* ACTION? AN EXAMPLE: THE SIMPLE HARMONIC OSCILLATOR

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I. An Overview of the Answer

We study the simple harmonic oscillator, since we can solve the problem exactly. Let us consider a one-dimensional oscillator with angular frequency $\omega > 0$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \ . \tag{1}$$

We study motion on the time interval $[T_1, T_2]$ and fix the values of $q(t)$ at the two endpoints: the time interval is $T = T_2 - T_1$, and we require $q(T_1) = Q_1$ and $q(T_2) = Q_2$. Let us denote $Q(t)$ to be the solution to Lagrange's equations for the oscillator with these end-point conditions.

I.1. **A Partial Answer.** Does the physical orbit $Q(t)$ minimize the action $S_{[T_1,T_2]}(q)$ for the simple harmonic oscillator? We first show

$$S_{[T_1,T_2]}(q) = S_{[T_1,T_2]}(Q) + S_{[T_1,T_2]}(\eta) \,. \tag{2}$$

Then we find a particular $\eta(t)$ such that for $T > \pi/\omega$, one has $S(\eta) < 0$. This shows that the action $S_{[T_1,T_2]}(q)$ for the simple harmonic oscillator is not minimized by $S_{[T_1,T_2]}(Q)$ for large time intervals T. In fact we give an example where the principle of least action breaks down for $T > \pi/\omega$.

I.2. **A Complete Answer.** We show in these notes that for the oscillator, the answer to question "is the principle of least action correct?" is: The solution $Q(t)$ minimizes the action $S_{[T_1,T_2]}(q)$ for the simple harmonic oscillator, if and only if the magnitude T of the time interval satisfies

$$\boxed{T \leqslant \frac{\pi}{\omega}} \,. \tag{3}$$

For the oscillator we can find an exact formula for $S(\eta)$ for arbitrary η, in terms of an eigenvalue problem. In §IV we relate this to determining the eigenvalues of a linear, self-adjoint transformation $T = -\frac{d^2}{dt^2}$ on the interval $[T_1, T_2]$. This transformation T is not a matrix, but a linear, self-adjoint differential operator. It acts on the Hilbert space $L^2([T_1, T_2], dt)$ of square-integrable functions on the interval. Nevertheless, this eigenvalue problem is similar to the eigenvalue problem for a hermitian matrix. In other words, the eigenfunctions $e_j(t)$ of T that satisfy

$$Te_j(t) = \lambda_j e_j(t) \,, \text{ with } \eta_j(T_1) = \eta_j(T_2) = 0 \,, \tag{4}$$

are a basis for the space $L^2([T_1, T_2], dt)$. Thus one can expand any η as a sum of normalized eigenfunctions $\eta(t) = \sum_j c_j e_j(t)$.

For this oscillator problem, one can calculate $e_j(t)$ and the eigenvalues λ_j in closed form. Then one can show that the action satisfies

$$\begin{aligned} S_{[T_1,T_2]}(q) &= S_{[T_1,T_2]}(Q) + S_{[T_1,T_2]}(\eta) \\ &= S_{[T_1,T_2]}(Q) + \frac{m}{2} \sum_{j=1}^{\infty} c_j^2 \left(\left(\frac{j\pi}{T}\right)^2 - \omega^2 \right) \,, \end{aligned} \tag{5}$$

where $c_j = \int_{T_1}^{T_2} e_j(t)\eta(t)dt$. In other words,

$$S_{[T_1,T_2]}(\eta) = \frac{m}{2} \sum_{j=1}^{\infty} c_j^2 \left(\left(\frac{j\pi}{T}\right)^2 - \omega^2 \right) \,. \tag{6}$$

Thus we have a formula for the value of $S_{[T_1,T_2]}(\eta)$ that depends only on our choice the coefficients c_j, which are up to us to choose. Each c_j^2 is multiplied by a corresponding eigenvalue of $-\frac{d^2}{dt^2} - \omega^2$ and by $\frac{m}{2}$. The smallest eigenvalue λ_1 is negative in case $T > \pi/\omega$. If we choose $c_1 = 1$ and all the other $c_j = 0$ for $j > 1$, then (5) shows that $S_{[T_1,T_2]}(\eta) = \frac{m}{2}\left(\frac{\pi^2}{T^2} - \omega^2\right)$ is negative for the small $T > \pi/\omega$ as claimed above. Any other η will have a larger contribution to $S_{[T_1,T_2]}(q)$, so this is the answer for the minimum! If $T \leqslant \pi/\omega$, then $S_{[T_1,T_2]}(\eta) \geqslant 0$. We now show that these claims are true!

II. Details of the Simple Harmonic Oscillator

II.1. **Review.** Given a Lagrangian $\mathcal{L}(q, \dot{q}, t)$, for one degree of freedom, the Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \; . \tag{7}$$

One often studies solutions to the initial-value problem for this equation; one specifies a position $q(T_1) = Q_1$ and velocity $\dot{q}(T_1) = \dot{Q}_1$ at a particular initial time T_1, and asks for the solution $q(t)$ to the equations for a later time $t > T_1$ that agrees with the initial position and velocity. On the other hand, in the derivation of Lagrange's equations by the variational princple we studied the action $S_{[T_1,T_2]}(q)$, defined for a particular time interval $t \in [T_1, T_2]$ with $T_1 < T_2$. This action is

$$S_{[T_1,T_2]}(q) = \int_{T_1}^{T_2} \mathcal{L}(q(t), \dot{q}(t))dt \; . \tag{8}$$

The action is a function of an arbitrary trajectory $q(t)$ from time T_1 to time T_2, whether or not that trajectory satisfies Lagrange's equations. In this case we have specified both the starting point of the trajectory $q(T_1) = Q_1$ and the ending point of the trajectory $q(T_2) = Q_2$, rather than the initial velocity. In studying variation of the action, we did not require that the trajectory have a particular initial velocity, but rather we fix the starting and ending points of the trajectory. One could also consider the situation for Lagrange's equations where one specifies the endpoints of a trajectory, rather than initial conditions.

Hamilton's principle relates these two situations. It is the statement that any solution $q(t)$ to the Lagrange equations (7) which starts at $q(T_1) = Q_1$ and ends at $q(T_2) = Q_2$, is a stationary point of the action (8). And conversely, any trajectory $q(t)$ starting from $Q_1 = q(T_1)$ and ending at $Q_2 = q(T_2)$ that is a stationary point of the action (8) is a solution to Lagrange's equations (7).

In more detail, a stationary point of the action $S_{[T_1,T_2]}(q)$ is defined to be a trajectory $q(t)$ for which the first directional derivative $D_\eta S_{[T_1,T_2]}(q)$ vanishes for all variations $\eta(t)$ of $q(t)$ for which $\eta(T_1) = \eta(T_2) = 0$. One calls this variation with vanishing endpoints; in such a situation, the varied trajectory $\tilde{q}(t) = q(t) + \eta(t)$ always starts at Q_1 and ends at Q_2. We defined the variational derivative in direction η at the trajectory q as

$$D_\eta S_{[T_1,T_2]}(q) = \lim_{\epsilon\to 0} \frac{S_{[T_1,T_2]}(q+\epsilon\eta) - S_{[T_1,T_2]}(q)}{\epsilon} \tag{9}$$

In some physics books, Hamilton's principle is called the *principle of least action*. This is meant to assert that a solution $q(t)$ to Lagrange's equations minimizes the action $S_{[T_1,T_2]}(q)$. In these notes we investigate this statement for the simple harmonic oscillator.

II.2. **Oscillator Basics.** Consider a one-dimensional, simple harmonic oscillator with mass m, angular frequency ω, and Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \ . \tag{10}$$

The action for a trajectory $q(t)$ on the time interval $t \in [T_1, T_2]$ is $S_{[T_1,T_2]}(q) = \int_{T_1}^{T_2} \mathcal{L}\, dt$, namely

$$S_{[T_1,T_2]}(q) = \frac{1}{2}m\int_{T_1}^{T_2} \dot{q}(t)^2 dt - \frac{1}{2}m\omega^2 \int_{T_1}^{T_2} q(t)^2 dt \ . \tag{11}$$

The Euler-Lagrange equation of motion is

$$\ddot{q}(t) = -\omega^2 q(t) \ . \tag{12}$$

We assume that the angular frequency $\omega > 0$.

II.3. **The Solution with Given Initial Conditions.** Here we denote a solution to the equation of motion by $Q(t)$. We use the upper-case $Q(t)$, in order to distinguish clearly a solution to Lagrange's equations from a general trajectory $q(t)$ that may or may not solve the equation (12).

The initial conditions for position and velocity at one time, say T_1, uniquely determine the solution to (12). Let us assume that $Q(T_1) = Q_1$ and $\dot{Q}(T_1) = \dot{Q}_1$ at time $t = T_1$. The corresponding solution is

$$Q(t) = Q_1 \cos\left(\omega(t - T_1)\right) + \frac{\dot{Q}_1}{\omega} \sin\left(\omega(t - T_1)\right) \ . \tag{13}$$

In the zero-frequency limit, the solution does not oscillate; rather it becomes the free-particle motion,

$$Q(t) = Q_1 + (t - T_1)\dot{Q}_1 \ , \quad \text{in the limit } \omega = 0 \ . \tag{14}$$

One easily checks that (13) satisfies the equation of motion and the initial conditions. In fact there is only one such solution: for assume that there is a second one $\widetilde{Q}(t)$. then $R(t) = Q(t) - \widetilde{Q}(t)$ also satisfies the equation of motion and has $R(T_1) = \dot{R}(T_1) = 0$. This means that

$$\frac{d^n R(t)}{dt^n} = -\omega^2 \frac{d^{n-2}R(t)}{dt^{n-2}} \ , \tag{15}$$

for all n. So

$$\left.\frac{d^n R(t)}{dt^n}\right|_{t=T_1} = 0\,, \quad \text{for all } n\,. \tag{16}$$

One can show that the solution to the oscillator is analytic in time, so vanishing of all the time derivates at a particular times ensures that $R(t) \equiv 0$ for all t. Thus $\widetilde{Q}(t) = Q(t)$.

subsectionThe Action of the Solution: We calculate the action $S_{[T_1,T_2]}(Q)$ of the solution (13). Insert the formula for $Q(t)$ into the expression for the action of the solution given initial data (13). Then,

$$\begin{aligned}
\int_{T_1}^{T_2} \mathcal{L} dt &= \frac{1}{2} m\omega^2 \left(Q_1^2 - \frac{\dot{Q}_1}{\omega^2} \right) \int_{T_1}^{T_2} \left(\sin^2(\omega(t-T_1)) - \cos^2(\omega(t-T_1)) \right) dt \\
&\quad - m\omega\, Q_1 \dot{Q}_1 \int_{T_1}^{T_2} \sin 2\omega(t-T_1) dt \\
&= \frac{1}{4} m\omega \left(\frac{\dot{Q}_1}{\omega^2} - Q_1^2 \right) \sin 2\omega(T_2-T_1) + \frac{1}{2}\, m\dot{Q}_1 Q_1 \left(\cos 2\omega(T_2-T_1) - 1 \right)\,.
\end{aligned}$$

Thus (with $T = T_2 - T_1$),

$$S_{[T_1,T_2]}(Q) = \frac{1}{4}\, m\omega \left(\frac{\dot{Q}_1^2}{\omega^2} - Q_1^2 \right) \sin 2\omega T + \frac{1}{2}\, m\dot{Q}_1 Q_1 \left(\cos 2\omega T - 1 \right)\,. \tag{17}$$

As a function of the size of the time interval T, the action oscillates in sign with period $\frac{\pi}{\omega}$. For small T the first term dominates; so the sign of the action depends on the sign of $\left(\dot{Q}_1^2 - \omega^2 Q_1^2 \right)$.

II.4. **The Solution with Fixed Endpoints.** Let us convert the solution with given initial conditions to a solution with given endpoints. We choose initial and final points Q_1, Q_2 and wish to require that $Q(T_1) = Q_1$ and $Q(T_2) = Q_2$. Let us see what is possible for the actual solution (13).

So to begin, set $Q(T_2) = Q_2$ in (13) and solve this relation for the initial velocity $\dot{Q}_1$; then replace this function of Q_1 and Q_2 in (13). If

$$Q_2 = Q_1 \cos \omega T + \frac{\dot{Q}_1}{\omega} \sin \omega T\,, \tag{18}$$

then one can solve for $\dot{Q}_1$, only if $\omega T \neq n\pi$ for integer n. Since $T > 0$ and $\omega > 0$, we require that

$$\frac{\omega T}{\pi} \neq 1, 2, \ldots \tag{19}$$

If (19) is the case, then

$$\frac{\dot{Q}_1}{\omega} = \frac{Q_2 - Q_1 \cos\omega T}{\sin\omega T} , \tag{20}$$

and

$$Q(t) \;=\; Q_1 \cos(\omega(t-T_1)) + \frac{Q_2 - Q_1\cos(\omega T)}{\sin(\omega T)} \sin(\omega(t-T_1)) . \tag{21}$$

The forbidden values $\omega T = \pi, 2\pi, \ldots$ either are values for which the motion is *periodic* with period T, for then

$$Q\left(T_2\right) = Q\left(T_1\right) \ , \ \text{ and } \ \dot{Q}\left(T_2\right) = \dot{Q}\left(T_1\right) \ , \qquad \text{the case when } n \text{ is even,} \tag{22}$$

or else values for which the motion is said to satisfy *anti-periodic boundary conditions*, namely

$$Q\left(T_2\right) = -Q\left(T_1\right) \ , \ \text{ and } \ \dot{Q}\left(T_2\right) = -\dot{Q}\left(T_1\right) \ , \qquad \text{the case when } n \text{ is odd.} \tag{23}$$

In either case there is no way to determine the initial velocity $\dot{Q}_1$ just from knowing Q_1.

Let us return to an elementary example to compute the action of the trajectory in terms of the endpoints. Let us suppose that we start the system at time $T_1 = 0$ and position $Q_1 = Q(T_1) = 0$. Then the formula (20) reduces to

$$\frac{\dot{Q}_1}{\omega} = \frac{Q_2}{\sin\omega T_2} ,$$

and the expression (17) becomes

$$S_{[0,T_2]}(Q) = \frac{1}{4}\, m\omega\, \frac{Q_2^2}{\sin^2\omega T_2} \sin 2\omega T_2 = \frac{m\omega}{2\tan\omega T_2}\, Q_2^2 . \tag{24}$$

II.5. **Perturbation of the Action for a Physical Orbit.** We observe that the perturbed action away from a physical orbit Q for action $S_{[T_1,T_2]}(q)$ for the oscillator has a very special property. It is additive, because the action is quadratic in q. If $Q(t)$ is a solution to the equation of motion, and if $\eta(t)$ vanishes at the endpoints, then

$$\boxed{S_{[T_1,T_2]}(Q+\eta) = S_{[T_1,T_2]}(Q) + S_{[T_1,T_2]}(\eta)} . \tag{25}$$

Warning: this simple relation holds for the oscillator, but not for most actions!

To demonstrate the relation (25), note that $S_{[T_1,T_2]}(Q+\epsilon\eta)$ has a power series in ϵ that terminates at degree 2. And the second order term is just $S_{[T_1,T_2]}(\eta)$. Thus

$$\begin{aligned} S_{[T_1,T_2]}(Q+\epsilon\eta) &= S_{[T_1,T_2]}(Q) + \epsilon\, \frac{d}{d\epsilon} S_{[T_1,T_2]}(Q+\epsilon\eta)\Big|_{\epsilon=0} + S_{[T_1,T_2]}(\eta) \\ &= S_{[T_1,T_2]}(Q) - \langle \mathbb{L}(Q), \eta\rangle_{[T_1,T_2]} + S_{[T_1,T_2]} \\ &= S_{[T_1,T_2]}(Q) + S_{[T_1,T_2]}(\eta) . \end{aligned} \tag{26}$$

Here we use the fact that $\langle \mathbb{L}(Q), \eta\rangle_{[T_1,T_2]} = 0$ characterizes the solution Q to Lagrange's equation. Thus (25) is valid for the expansion about the physical trajectory.

III. Does Physical Motion Minimize the Action?

Let us suppose we consider Hamilton's principle for a simple harmonic oscillator with angular frequency ω. We look for a critical point of the action $S_{[T_1,T_2]}(q)$, with fixed endpoints. In order to ensure that the endpoints (rather than initial conditions) determine the trajectory, we only consider time intervals that satisfy the condition (19). *It is interesting that we will discover that this same condition relates to whether physical motion minimizes the action.*

III.1. **An η for which $S_{[T_1,T_2]}(\eta)$ is negative:** Once we know that $S(q)$ has the property (25), we have a very simple test whether the physical trajectory Q yields the least action $S_{[T_1,T_2]}(Q)$ for $S(q)$. We only need to determine whether or not it is the case that

$$(27) \qquad 0 \leqslant S_{[T_1,T_2]}(\eta) \ , \quad \text{whenever} \quad \eta(T_1) = \eta(T_2) = 0 \ . \quad \textbf{Test for Least Action}$$

The explicit form for $S_{[T_1,T_2]}(\eta)$ is

$$(28) \qquad S_{[T_1,T_2]}(\eta) = \frac{m}{2}\left(\int_{T_1}^{T_2} \dot{\eta}(t)^2\, dt - \omega^2 \int_{T_1}^{T_2} \eta(t)^2\, dt\right) , \qquad \text{with } \eta(T_1) = \eta(T_2) = 0 \ .$$

The first term in (28) is positive; the second term in (28) is negative. So the question is *which term wins*? In order to answer this question, we start by repeating the argument leading to (25). Write $\dot{\eta}^2 = \frac{d}{dt}(\eta\dot{\eta}) - \eta\ddot{\eta}$, and substitute this into the first term. The boundary term vanishes, and (28) becomes

$$(29) \qquad S_{[T_1,T_2]}(\eta) = \frac{m}{2}\int_{T_1}^{T_2} \eta(t)\left(-\frac{d^2}{dt^2} - \omega^2\right)\eta(t) dt \ , \qquad \text{with } \eta(T_1) = \eta(T_2) = 0 \ .$$

Let us choose for $\eta(t)$ the expression

$$(30) \qquad \eta(t) = \sin\left(\frac{\pi(t - T_1)}{T}\right) , \quad \text{so } -\frac{d^2}{dt^2}\eta(t) = \frac{\pi^2}{T^2}\eta(t) \ .$$

Clearly $\eta(T_1) = \eta(T_2) = 0$. Putting this $\eta(t)$ into (29) shows that,

$$(31) \qquad S_{[T_1,T_2]}(\eta) = \frac{m}{2}\left(\frac{\pi^2}{T^2} - \omega^2\right)\int_{T_1}^{T_2} \eta(t)^2 dt \ .$$

Thus whenever $T > \dfrac{\pi}{\omega}$, we know that the variation of the action (31) is negative. Thus for $T > \dfrac{\pi}{\omega}$, the physical solution $Q(t)$ does not minimize the action.

IV. The General Solution: An Eigenvalue Problem

With finite dimensional matrices, we know there is a relation between a variational approach and the eigenvalue problem for a Hermitian matrix T. Let us denote vectors by q with components q_i or by η with components η_i. Also let $\langle q, \eta\rangle$ denote the standard inner product $\langle q, \eta\rangle = \sum_i \overline{q_i}\, \eta_i$. The real-valued function $F(q)$, defined for vectors q by

$$(32) \qquad F(q) = \frac{\langle q, Tq\rangle}{\langle q, q\rangle} \ ,$$

is stationary at $q = Q$ if Q is an eigenvector of T with eigenvalue $\lambda = \frac{\langle Q, TQ\rangle}{\langle Q, Q\rangle}$.

The directional derivative of $F(q)$ in the direction η is

$$
\begin{aligned}
(D_\eta F)(q) &= \lim_{\epsilon\to 0} \frac{F(q+\epsilon\eta) - F(q)}{\epsilon} \\
&= \frac{\langle \eta, Tq\rangle + \langle q, T\eta\rangle}{\langle q, q\rangle} - \frac{\langle q, Tq\rangle\left(\langle \eta, q\rangle + \langle q, \eta\rangle\right)}{\langle q, q\rangle^2} \;. \qquad (33)
\end{aligned}
$$

Replace q by the unit vector $f = \frac{q}{\langle q, q\rangle^{\frac{1}{2}}}$, so using the reality of $\langle f, Tf\rangle$,

$$
\begin{aligned}
(D_\eta F)(f) &= \langle \eta, (Tf - \langle f, Tf\rangle f)\rangle + \langle (Tf - \langle f, Tf\rangle f), \eta\rangle \\
&= \langle \eta, (Tf - \langle f, Tf\rangle f)\rangle + \overline{\langle \eta, (Tf - \langle f, Tf\rangle f)\rangle} \;. \qquad (34)
\end{aligned}
$$

For q to be a stationary point Q, this derivative must vanish for all η. Therefore at a stationary point,

$$Tf = \langle f, Tf\rangle f \;. \qquad (35)$$

In other words, at a stationary point f is a unit eigenvector of T with eigenvalue $\lambda = \langle f, Tf\rangle$.

Since any $N\times N$ Hermitian matrix has an orthonormal basis $e^{(1)}, \ldots, e^{(N)}$ of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_N$, one can expand any vector q into a sum of such vectors, $q = \sum_{j=1}^N q_j e^{(j)}$, where $q_j = \left\langle e^{(j)}, q\right\rangle$. Then for an arbitrary unit vector f,

$$F(f) = \sum_{j=1}^N |c_j|^2 \lambda_j \;, \quad \text{where} \quad \sum_{j=1}^N |c_j|^2 = 1 \;. \qquad (36)$$

Hence $F(f)$ is minimized by the minimum eigenvalue of T. and it is maximized by the maximum eigenvalue of T.

If we now order the eigenvectors so that the eigenvalues are increasing,

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \;, \qquad (37)$$

then at the stationary point $e^{(1)}$, the function $F(e^{(1)}) = \lambda_1$ has $N-1$ increasing directions. At the stationary point $e^{(k)}$, the function $F(e^{(k)}) = \lambda_k$ has $N-k$ increasing directions and $k-1$ decreasing directions. And at the maximum $F(e^{(N)}) = \lambda_N$ the function has $N-1$ decreasing directions.

The analysis of $T = -\frac{d^2}{dt^2}$ is the infinite dimensional generalization of the case for a Hermitian matrix. The fact that T is hermitian depends on the boundary condition $f(0) = f(T) = 0$. In this case T does have an orthonormal basis of eigenvectors, namely the functions

$$e^{(j)} = \left(\frac{2}{T}\right)^{\frac{1}{2}} \sin\left(\frac{j\pi}{T}(t - T_1)\right) \;, \qquad (38)$$

with eigenvalues

$$\lambda_j = \left(\frac{j\pi}{T}\right)^2 \;, \quad \text{for} \quad j = 1, 2, \ldots . \qquad (39)$$

The eigenvalue λ_1 is the minimum value of $F(f)$ for $\langle f, f\rangle = \int_{T_1}^{T_2} |f(t)|^2 dt = 1$, and there is no maximum eigenvalue nor a maximum value of $F(f)$.

V. Analysis with Fourier Series (Optional)

The rest of these notes are more advanced. I recommend reading this section in case you know the mathematics of quantum theory: in that case you will find it very interesting. The formula (29) for the variation of the action has the form of an "expectation" of a linear transformation $-\frac{d^2}{dt^2} - \omega^2$ in the state η, namely

$$S_{[T_1,T_2]}(\eta) = \left\langle \eta, \left(-\frac{d^2}{dt^2} - \omega^2\right)\eta \right\rangle . \tag{40}$$

Let us explain this in the context of orbits that are square integrable functions of time, namely elements of $L^2[T_1, T_2]$. Suppose that we consider the function $\eta(t)$, vanishing at the endpoints, as an element in the Hilbert space $L^2[T_1, T_2]$ of square integrable functions on the interval $[T_1, T_2]$. The set of all square-integrable functions for which $\eta(T_1) = \eta(T_2) = 0$, is a linear subspace of L^2. For two different functions $\eta_1(t)$ and $\eta_2(t)$, both vanishing at the endpoints, $\eta_1(T_1) = \eta_1(T_2) = \eta_2(T_1) = \eta_2(T_2) = 0$, we can for real numbers λ_1, λ_2 form a new function $\eta(t) = \lambda_1\eta_1(t) + \lambda_2\eta_2(t)$. This also satisfies the boundary condition $\eta(T_1) = \eta(T_2) = 0$. Thus the space of real functions vanishing at the endpoints is a "space of real vectors." This is a subspace of the space of the bigger space of vectors made of functions $\eta(t)$ on the interval $[T_1, T_2]$ where we do not require that the functions vanish at the endpoints.

We can introduce the scalar product between the real vectors η_1 and η_2 as

$$\langle \eta_1, \eta_2\rangle = \int_{T_1}^{T_2} \eta_1(t)\, \eta_2(t) dt . \tag{41}$$

Also we can define the transformation

$$K = -\frac{d^2}{dt^2}$$

that takes a twice differentiable vector η into the vector $T\eta$ defined by

$$(K\eta)(t) = -\ddot{\eta}(t) .$$

V.1. **Symmetry of K:** The transformation K is symmetric when defined on twice-differentiable functions η that vanish at the endpoints. It is always the case that $\eta_1\, \ddot{\eta}_2 =$

$\frac{d}{dt}(\eta_1\,\dot\eta_2) - \dot\eta_1\,\dot\eta_2$. Therefore

$$\begin{aligned}\langle \eta_1, K\eta_2\rangle &= -\int_{T_1}^{T_2} \eta_1(t)\,\ddot\eta_2(t)dt \\ &= \int_{T_1}^{T_2} \dot\eta_1(t)\dot\eta_2(t)dt - \eta_1(t)\dot\eta_2(t)|_{T_1}^{T_2} \\ &= \int_{T_1}^{T_2} \dot\eta_1(t)\dot\eta_2(t)dt = \langle \dot\eta_1, \dot\eta_2\rangle\ . \end{aligned} \tag{42}$$

Now repeat the same argument, but use $\dot\eta_1\,\dot\eta_2 = \frac{d}{dt}(\dot\eta_1\,\eta_2) - \ddot\eta_1\,\eta_2$. Thus

$$\langle \eta_1, K\eta_2\rangle = -\int_{T_1}^{T_2} \ddot\eta_1(t)\eta_2(t)dt = \langle K\eta_1, \eta_2\rangle\ . \tag{43}$$

Note that in deriving (43), the boundary terms arising from integration by parts in (43) vanish as we assume $\eta(T_1) = \eta(T_2) = 0$.

We interpret K as a linear transformation on the space of twice differentiable vectors (not necessarily satisfying the vanishing boundary conditions). In other words K acts like a real, symmetric matrix on a space of vectors, taking one vector into another. The transformation K is linear in the sense that $K(\eta_1 + \eta_2) = K\eta_1 + K\eta_2$. Similarly we interpret $\langle \eta_1, K\eta_2\rangle$ as a matrix element of the transformation K. The diagonal matrix element (or expectation) $\langle \eta, K\eta\rangle$ of K occurs for $\eta_1 = \eta_2 = \eta$, and

$$\langle \eta, K\eta\rangle = \int_{T_1}^{T_2} \dot\eta(t)^2 dt\ . \tag{44}$$

Comparing (44) with (28)–(29), we see that the action $S_{[T_1,T_2]}(\eta)$ is proportional to the expectation of $K - \omega^2$, namely

$$S_{[T_1,T_2]}(\eta) = \frac{1}{2} m \left\langle \eta, \left(K - \omega^2\right)\eta\right\rangle\ . \tag{45}$$

V.2. **Relation of $S_{[T_1,T_2]}(\eta)$ to Eigenvalues and Eigenvectors:** Suppose that the vector η is an eigenvector for K with eigenvalue λ (a real number). This means that

$$K\eta = \lambda\eta\ . \tag{46}$$

Consequently

$$\langle \eta, K\eta\rangle = \lambda\,\langle \eta, \eta\rangle\ . \tag{47}$$

In the case that η is an eigenvector of K, with eigenvalue λ,

$$S_{[T_1,T_2]}(\eta) = \frac{m}{2}\left(\lambda - \omega^2\right)\,\langle \eta, \eta\rangle\ . \tag{48}$$

Hence if $\lambda < \omega^2$, the corresponding eigenvector η yields $S_{[T_1,T_2]}(\eta) < 0$. In that case, $S_{[T_1,T_2]}(Q + \eta) = S_{[T_1,T_2]}(Q) + S_{[T_1,T_2]}(\eta) < S_{[T_1,T_2]}(Q)$. In this case, the trajectory $Q(t)$ does *not* minimize the action.

V.3. **All the Eigenfunctions of K:** Understanding *all* the eigenvalues of K with vanishing boundary conditions, means solving the differential equation

$$-\ddot{\eta}(t)=\lambda\eta(t) \quad \text{with} \quad \eta(T_1)=\eta(T_2)=0\ . \tag{49}$$

This is called the *Dirichlet problem* or the problem of *Dirichlet boundary conditions* for $-\frac{d^2}{dt^2}$ with vanishing boundary conditions at the endpoints $t=T_1,T_2$. Let $T=T_2-T_1$ denote the time difference. The normalized eigenfunctions for this Dirichlet problem are

$$\eta_j(t)=\left(\frac{2}{T}\right)^{1/2}\sin\left(\frac{\pi j(t-T_1)}{T}\right)\ , \qquad \text{where } j=1,2,3,\ldots\ , \tag{50}$$

and the corresponding eigenvalues are

$$\lambda_j=\left(j\frac{\pi}{T}\right)^2\ . \tag{51}$$

For a given j, there is only one function (up to a multiple) satisfying the equation and the boundary condition. For different values of j, the λ_j are different, so the eigenfunctions are mutually orthogonal. The minimum eigenvalue comes from $j=1$ and this is η_1.

V.4. **Expansion in Eigenfunctions:** Suppose we consider a linear combination η of the eigenfunctions η_j. Namely inspect a sum of the form

$$\eta=\sum_{j=1}^{\infty}c_j\eta_j\ . \tag{52}$$

As the eigenfunctions η_j are orthogonal and normalized, the coefficients c_j can be found from η by the relation $c_j=\langle\eta,\eta_j\rangle$. If the set of eigenvectors $\{\eta_j\}$ are actually a basis, then any η can be expanded in this fashion. (We do not address that here.)

In any case, the orthonormality also shows

$$S_{[T_1,T_2]}(\eta)=\frac{m}{2}\sum_{j=1}^{\infty}c_j^2\left(\lambda_j-\omega^2\right)=\frac{m}{2}\sum_{j=1}^{\infty}c_j^2\left(\left(j\frac{\pi}{T}\right)^2-\omega^2\right)\ . \tag{53}$$

Those terms in the sum for which $\lambda_j>\omega^2$ are positive. The smallest eigenvalue is λ_1, and λ_j increases with j^2, so at most a finite number of terms can be negative. They are the terms for which $\lambda_j<\omega^2$.

V.5. **Remark: Relation to a Variational Problem:** The normalized action functional (real valued function)

$$F(\eta)=\frac{\langle\eta,K\eta\rangle}{\langle\eta,\eta\rangle} \tag{54}$$

plays a major role in the study of K. The eigenvectors η of K are critical points of $F(\eta)$. The eigenvalues are $F(\eta)$ when η is an eigenvector. As in the notes on matrices, this functional is used to study the eigenvalue for the Dirichlet problem and to prove that the eigenfunctions are a basis.

V.6. **Remark: Relation of "Least Action" to "Quantum Theory of a Particle in a Box":** For those of you who have studied quantum theory, you might recognize the transformation K. In quantum theory the variable is not the time t. Rather the variable is the position x of a particle on an interval $Q_1 \leqslant x \leqslant Q_2$. Then the study of $K = -\frac{d^2}{dx^2}$ as a transformation on the real subspace of $L^2[Q_1, Q_2]$ with the Dirichlet boundary condition $\eta(Q_1) = \eta(Q_2) = 0$ is called "the study of the Schrödinger Hamiltonian for a *particle in a box*." (Here we use units for which the constants m and $\hbar$ equal 1.)

The minimal eigenvalue of the kinetic energy K is

$$\lambda_1 = \pi^2 (Q_2 - Q_1)^{-2} \,. \tag{55}$$

This is called the *zero-point energy* of a particle in a box of length $\ell = Q_2 - Q_1$. The corresponding eigenfunction is

$$\psi(x) = \sin \lambda_1^{1/2} (x - Q_1) = \sin \left(\frac{\pi (x - Q_1)}{Q_2 - Q_1} \right) \,. \tag{56}$$

This function corresponds to the function (30), proportional to $\eta_1(x)$, considered earlier, but with the variable x replacing the variables t. The fact that the zero-point energy $\lambda_1 \propto \ell^{-2} > 0$ is important in the interpretation of quantum theory.