Symmetry Yields Conservation: Noether's Theorem

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There is an intimate relation *Symmetry* $\Rightarrow$ *Conservation Law* from a symmetry to a conservation law. This is summarized by "Noether's theorem." Here we explain this in the context of classical mechanics, but the same principles arise in quantum physics and also in the study of differential equations. In order to go as well in the other direction and recover *Conservation Law* $\Rightarrow$ *Symmetry*, we need to wait for the Hamiltonian formulation of classical mechanics.

Figure 1: The Noether residence at Hauptstrasse 23, Erlangen, along with its plaque.

Amalia (Emmy) Noether was one of the leading mathematicians of the 20$^{\text{th}}$ century. Emmy's father was a well known algebraic geometer at the University of Erlangen. In 2015 I visited that university to give a colloquium. On that occasion I had the opportunity to take these photos of her home on the main street of the city, a short walk from the university. To the left of the door there is a plaque celebrating Emmy, with a ring symbolizing "Noetherian rings." I took a close-up in the second photo: this shows the inscription, as well as the street number 23.

Emmy Noether earned her doctorate in 1907 at the age of 25 in Erlangen. After that she had an unpaid position in the department, until Hilbert invited her to Göttingen. In spite of her famous work in algebra and also in mathematical physics, she never became a professor in Germany. She emigrated to the US in 1933, where she was immediately appointed Professor of Mathematics at Bryn Mawr College. Unfortunately, two years later Emmy Noether died at the age of 53, as the aftermath of an unsuccessful surgery.

Basic Concepts and Assumptions. Consider a Lagrangian $\mathcal{L}(q, \dot{q}, t)$ where q denotes N generalized coordinates $q = (q_1, \ldots, q_N)$, along with a trajectory $q(t)$ that satisfying the N Lagrange equations. Let p denote the momentum, with components $p_i = \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}$.

Consider a family of trajectories $q_\epsilon(t)$ indexed by a real parameter ϵ. The coordinates of q_ϵ are $q_{\epsilon,i}(t)$ obtained from $q(t)$ by an invertible coordinate transformation, and we assume that $q_{\epsilon=0}(t) = q(t)$. Thus we consider the family of deformed trajectories $q(t) \to q_\epsilon(t)$ that reduce to the identity transformation at $\epsilon = 0$. Furthermore suppose that $\mathcal{L}_\epsilon$ is defined by

$$\mathcal{L}_\epsilon(q, \dot{q}, t) = \mathcal{L}(q_\epsilon, \dot{q}_\epsilon, t) \; . \tag{1}$$

Definition 1. *If $\mathcal{L}_\epsilon = \mathcal{L}$, then the Lagrangian $\mathcal{L}$ has a symmetry under the family of transformations $q(t) \to q_\epsilon(t)$.*

We now introduce the *Noether charge*

$$\mathfrak{Q}(t) = \left(\sum_{i=1}^{N} p_{\epsilon,i}(t) \frac{dq_{\epsilon,i}(t)}{d\epsilon} \right) \Bigg|_{\epsilon=0} \; , \tag{2}$$

Theorem 2 (**Noether Theorem-A).** *Suppose that $\mathcal{L}$ has a symmetry under the family of transformations $q(t) \to q_\epsilon(t)$ that are differentiable in ϵ. Also let $q_{\epsilon=0}(t) = Q(t)$ satisfies Lagrange's equations. Then the quantity $\mathfrak{Q}(t)$ is independent of time t when evaluated on a solution to the equations of motion. In other words, $\mathfrak{Q}$ is conserved.*

Remark 3. In these notes we use the boldface symbol $\mathfrak{Q}$ to denote this "charge;" our notation should not be confused with the Q used in earlier notes to denote a solution to Lagrange's equations! Unfortunately we do not have sufficiently many symbols to go around. Some version of the letter "q" is commonly used both to denote a coordinate, as well as to a denote a charge.

Proof. Since $\mathcal{L}$ has the stated symmetry, $\frac{d\mathcal{L}_\epsilon}{d\epsilon} = \frac{d\mathcal{L}}{d\epsilon} = 0$. Calculate this in terms of the definition (1):

$$0 = \frac{d\mathcal{L}_\epsilon(q,\dot{q},t)}{d\epsilon} = \frac{d\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{d\epsilon} = \sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial q_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon} + \frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\frac{d\dot{q}_{\epsilon,i}}{d\epsilon}\right) . \tag{3}$$

Now we use the relation,

$$\frac{d\dot{q}_{\epsilon,i}(t)}{d\epsilon} = \frac{d}{d\epsilon}\frac{dq_{\epsilon,i}(t)}{dt} = \frac{d}{dt}\frac{dq_{\epsilon,i}(t)}{d\epsilon} . \tag{4}$$

Substituting this into the last term in (3),

$$\begin{aligned} 0 &= \sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial q_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon} + \frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\frac{d}{dt}\frac{dq_{\epsilon,i}}{d\epsilon}\right) \\ &= \sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial q_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon} + \frac{d}{dt}\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon}\right) - \left(\frac{d}{dt}\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\right)\frac{dq_{\epsilon,i}}{d\epsilon}\right) \\ &= \sum_{i=1}^{N}\left(\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial q_{\epsilon,i}} - \frac{d}{dt}\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\right)\frac{dq_{\epsilon,i}}{d\epsilon} + \frac{d}{dt}\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon}\right)\right) . \end{aligned} \tag{5}$$

Let us evaluate this at the point $\epsilon = 0$, and use our assumption that $q_{\epsilon=0}(t) = Q(t)$ satisfies the Lagrange equations. Then for each i,

$$\left(\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial q_{\epsilon,i}} - \frac{d}{dt}\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\right)\Bigg|_{\epsilon=0} = \frac{\partial\mathcal{L}(Q,\dot{Q},t)}{\partial Q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}(Q,\dot{Q},t)}{\partial \dot{Q}_i} = 0 .$$

As a consequence, the last term in (5) vanishes, namely

$$\begin{aligned} \frac{d\mathfrak{Q}}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial\mathcal{L}(q_\epsilon,\dot{q}_\epsilon,t)}{\partial \dot{q}_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon}\right)\Bigg|_{\epsilon=0} = \frac{d}{dt}\left(\sum_{i=1}^{N}p_{\epsilon,i}\frac{dq_{\epsilon,i}}{d\epsilon}\right)\Bigg|_{\epsilon=0} \\ &= \frac{d}{dt}\left(\sum_{i=1}^{N}p_i\left.\frac{dq_{\epsilon,i}}{d\epsilon}\right|_{\epsilon=0}\right) = 0 . \end{aligned} \tag{6}$$

This means that the quantity $\mathfrak{Q}$ defined in (2) is conserved. □

Example 4 (Equal Translations and Conservation of Total Momentum). Consider N particles moving in 3-space. Let the Cartesian coordinates of the j^{th}-particle be $\vec{x}^{\,(j)}$, with components $x_\alpha^{(j)}$, where $j = 1, 2, \ldots, N$, and $\alpha = 1, 2, 3$. If the Lagrangian is unchanged under equal translations $\vec{x}_\epsilon^{(j)} = \vec{x}^{(j)} + \epsilon\vec{a}$ of every particle, then Noether's theorem says that the component $\vec{P} \cdot \vec{a}$ total linear momentum is conserved.

This is the case in the following situation: suppose that

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = T - V \ , \tag{7}$$

where

$$T = \sum_{j=1}^{N} \frac{1}{2} m_j (\dot{\vec{x}}^{(j)})^2 \ , \quad \text{and} \quad V = \sum_{j \neq j'} V_{jj'}(\vec{x}^{\,(j)} - \vec{x}^{\,(j')}) \ . \tag{8}$$

In other words, the potential is the sum of two-body forces depending only on the relative positions of each of the bodies. As T expressed in terms of $\dot{\vec{x}}_\epsilon^{(j)} = \dot{\vec{x}}^{(j)}$ is independent of ϵ, and $\vec{x}_\epsilon^{(j)} - \vec{x}_\epsilon^{(j')} = \vec{x}^{\,(j)} - \vec{x}^{\,(j')}$ is independent of ϵ, we infer that $\mathcal{L}_\epsilon = \mathcal{L}$. Also $\frac{d\vec{x}_\epsilon^{(j)}}{d\epsilon} = \vec{a}$. So the conserved Noether charge is

$$\mathfrak{Q} = \sum_{j=1}^{N} \sum_{\alpha=1}^{3} p_\alpha^{(j)} a_\alpha = \left(\sum_{j=1}^{N} \vec{p}^{\,(j)} \right) \cdot \vec{a} = \vec{P} \cdot \vec{a} \ . \tag{9}$$

Here $\vec{P} = \sum_{j=1}^{N} \vec{p}^{\,(j)}$ is the total momentum, and the conserved quantity $\mathfrak{Q}$ is the component of the total momentum in the direction $\vec{a}$. In our example this is true for any $\vec{a}$, so each component of the total momentum is conserved. Thus the total momentum vector $\vec{P}$ is conserved.

Note that the symmetry $\mathcal{L}_\epsilon = \mathcal{L}$ also can be seen to arise from the independence of $\mathcal{L}$ on the center of mass coordinate $\vec{R} = \sum_{j=1}^{N} \frac{m_j}{M} \vec{x}^{\,(j)}$, with M the total mass. Thus invariance of the Lagrangian under translation of the center of mass coordinate entails conservation of total momentum. In fact, if we express the Lagrangian in terms of $\vec{R}$ and other coordinates relative to $\vec{R}$, then the i^{th} component of the total momentum is

$$P_i = \frac{\partial \mathcal{L}}{\partial R_i} \ . \tag{10}$$

So Lagrange's equations say if $\mathcal{L}$ does not depend on $\vec{R}$, then $\vec{P}$ is conserved. In other words, this instance of Noether's theorem follows directly from changing coordinates in the Lagrangian.

One can modify this argument to consider equal translations of certain particles. If the corresponding Lagrangian is unchanged, then Noether's theorem yields conservation of the sum of the corresponding momenta.

Example 5 (Rotational Symmetry Yields Angular Momentum Conservation). Suppose that we have a single particle moving in a plane, described by Cartesian coordinates (x_1, x_2), with the motion determined by the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{\vec{x}}^2 - V(\vec{x}).$$

Let q_ϵ be defined by a rotation in the Cartesian plane

$$\begin{pmatrix} q_{\epsilon,1} \\ q_{\epsilon,2} \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

and assume that V is rotationally invariant, so $\mathcal{L}_\epsilon = \mathcal{L}$. Note that $\vec{q}_0 = \vec{x}$ is the Cartesian coordinate, and assume that $\vec{x}(t)$ satisfies Lagrange's equations. Then the conserved quantity (2) given by Noether's theorem is

$$\mathfrak{Q} = -p_1x_2 + p_2x_1 = x_1p_2 - x_2p_1 , \tag{11}$$

which is the angular momentum around the origin.

Theorem 6 (Noether Theorem-B). *Suppose that $q \to q_\epsilon$ is a family of transformations that are differentiable in ϵ, and that $q_0(t) = Q(t)$ satisfies Lagrange's equations in time. Furthermore suppose that there is a function $F(q((t), \dot{q}(t))$ such that*

$$\left.\frac{d\mathcal{L}_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \frac{dF}{dt} .$$

Then

$$\mathfrak{Q}(t) = \left.\left(\sum_{i=1}^{N} p_{\epsilon,i}\frac{dq_{\epsilon,i}}{d\epsilon}\right)\right|_{\epsilon=0} - F \tag{12}$$

is conserved (namely independent of time).

Proof. One can follow the proof of Theorem 2, except in place of (6) one obtains

$$\frac{d}{dt}\left(\sum_{i=1}^{N} p_i \left.\frac{dq_{\epsilon,i}}{d\epsilon}\right|_{\epsilon=0}\right) = \frac{dF}{dt} . \tag{13}$$

This shows that (12) is conserved. $\square$

Example 7 (Time-Translation Symmetry Yields Energy Conservation). Let $q_\epsilon(t) = q(t+\epsilon)$ and suppose that $\mathcal{L}$ *does not depend explicitly on time.* (In other words, the only time dependence of $\mathcal{L}(q(t), \dot{q}(t))$ arises from the dependence of the Lagrangian on the trajectory $q(t)$. Then

$$\frac{d\mathcal{L}_\epsilon}{d\epsilon} = \frac{d\mathcal{L}_\epsilon}{dt} \; . \tag{14}$$

So setting $\epsilon = 0$, one has

$$\left.\frac{d\mathcal{L}_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \frac{d\mathcal{L}}{dt} \; . \tag{15}$$

Thus we can take $F = \mathcal{L}$ and

$$\mathfrak{Q} = \sum_{i=1}^{N} p_i \left.\frac{dq_{\epsilon,i}}{d\epsilon}\right|_{\epsilon=0} - \mathcal{L} = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} \tag{16}$$

is conserved.

This conserved quantity is special; it is called the *Hamiltonian.* And it is given a special symbol H. The Hamiltonian is given by the formula

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} \; , \tag{17}$$

whether or not it is conserved. In general

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \; . \tag{18}$$

In summary, if $\mathcal{L}$ does not depend explicitly on t, then the Hamiltonian H is conserved.

Note that if $\mathcal{L} = T - V$, where $T = \frac{1}{2}\sum_{j,j'=1}^{N} T_{jj'} \dot{q}_j \, \dot{q}_{j'}$, where each coefficient $T_{jj'} = T_{jj'}(q)$ is a function of q, but not $\dot{q}$. Then

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} = T + V \; . \tag{19}$$

In many cases the Hamiltonian equals the energy. So in physics the energy is often defined to be the Hamiltonian.