Physics 151 Pset Suggestions

September 4, 2024

1 Pset 1

Recall the Lagrangian vector from class (compact form and in components)

$$\mathbb{L} = \frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} - \nabla_q \mathcal{L}$$
$$\mathbb{L}_j = \frac{d}{dt}\partial_{\dot{q}_j}\mathcal{L} - \nabla_q \mathcal{L}$$

To emphasize dependence on q (or x), we occasionally write

$$\mathbb{L}_q = \frac{d}{dt}\nabla_{\dot{q}}\mathcal{L} - \nabla_q \mathcal{L}$$
$$\mathbb{L}_x = \frac{d}{dt}\nabla_{\dot{x}}\mathcal{L} - \nabla_x \mathcal{L}$$

Also recall that the Jacobian matrix $J_{a\to b}$ from $a \in \mathbb{R}^m$ to $b \in \mathbb{R}^n$ is

$$(J_{a\to b})_{ij} = \begin{bmatrix} \nabla_a b_1 & \cdots & \nabla_a b_n \end{bmatrix} = \frac{\partial b_j}{\partial a_i}$$

so that the chain rule can be written as (compactly and in components)

$$\nabla_a f(b) = J_{a\to b}\nabla_b f(b)$$
$$\frac{\partial}{\partial a_j} f(b(a_1, \cdots, a_m)) = \sum_k \frac{\partial f(b)}{\partial b_k}\frac{\partial b_k}{\partial a_j} = \sum_k (J_{a\to b})_{jk}(\nabla_b f(b))_k$$

This question asks you to show that the Lagrangian vector transforms covariantly:

(a) Given a coordinate transform $x \to q$ so that $q_j = q_j(x_1, \cdots, x_n)$, it determines a transform $(x, \dot{x}) \to (q, \dot{q})$ of the derivatives as well. Show that $J_{\dot{x}\to\dot{q}} = J_{x\to q}$.

Hint: the transformation $x \to q$ is independent of $\dot{x}$, so one can show that $\dot{x}, \dot{q}$ are related by a linear transform $\dot{q} = J_{x\to q}\dot{x}$ or, in components, $\dot{q}_j = \sum_k (J_{x\to q})_{jk}\dot{x}_k$.

(b) Show that $J_{x\to\dot{q}} = \frac{d}{dt}J_{x\to q}$.

(c) Explain why $J_{\dot{x}\to q} = 0$.

(d) The previous parts give us the full Jacobian of $(x, \dot{x}) \to (q, \dot{q})$

$$J_{(x,\dot{x})\to(q,\dot{q})} = \begin{bmatrix} J_{x\to q} & J_{x\to\dot{q}} \\ J_{\dot{x}\to q} & J_{\dot{x}\to\dot{q}} \end{bmatrix} = \begin{bmatrix} J_{x\to q} & \frac{d}{dt}J_{x\to q} \\ 0 & J_{x\to q} \end{bmatrix}$$

Next show that $\nabla_{\dot{x}}\mathbb{L}_q = J_{\dot{x}\to\dot{q}}\nabla_{\dot{q}}$

(e) Show that $\nabla_x\mathbb{L} = (J_{x\to q}\nabla_q + J_{x\to\dot{q}}\nabla_{\dot{q}})\,\mathbb{L}$

(f) Covariance follows from the previous parts: show that $\mathbb{L}_x = J_{x\to q}\mathbb{L}_q$.

1.1 Solutions

(a) By the chain rule

$$\dot{q}_j = \frac{d}{dt}q_j(x) = \sum_k \frac{\partial q_j}{\partial x_k}\frac{d}{dt}x_k = (J_{x\to q})_{kj}\dot{x}_k$$