Graph Theory Homework 2

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Submission instructions

• Copy and number the questions into your Jupyter Python Notebook. For submission, you only need to submit a zipped file named "StudentName_ID.zip" containing: (1) your Jupyter Python Notebook file: 'StudentName_ID.ipynb', which includes all comments and codes of the homework exercises below, and (2) the graph data to load when running the Jupyter Python Notebook.

• For those who are not used to Python notebooks, I have attached below a step-by-step suggestions on how to set up Jupyter Notebooks prepared by Dr Damien J. Duff (see last 2 pages).

• If you have any questions, you can contact me via kamard@itu.edu.tr

1 Bellman-Ford shortest path algorithm[20%]

The Bellman-Ford algorithm is a generalization of Dijkstra's algorithm to directed graphs with no negative cycles. Given a weighted matrix $\mathbf{A}$ of a directed graph $\mathbf{G}$, an element $\mathbf{A}(i,j)$ denotes the weight of the edge *from* node i *to* node j.

- **(1.1)** Explain the key steps of Bellman-Ford algorithm (one-by-one). [5%]
- **(1.2)** Write a function called *BellmanFordAlgo* that takes (i) $\mathbf{A}$ and (2) a starting node ID as inputs, and outputs an array as in Graph Theory Blink 5.4[a], where each row represents a node in the graph $\mathbf{G}$ and comprising three columns (nodes, shortest distance from source input node, previous node). [10%]
  Use of external libraries is not allowed. Code it up from scratch.
- **(1.3)** Run your function in the Jupyter Notebook on the input adjacency matrix $\mathbf{A}$ (see **Fig.** fig:1) and display the output array. You can find the data inside 'Exercise_1_data' folder. [5%]

2 Global efficiency, diffusion efficiency and graph morphospace [40%]

This part is related to lectures 5 and 6, in particular Graph Theory Blink 6.6:
`https://www.youtube.com/watch?v=FX3Dp4ZIgzQ&list=PLug43ldmRSo3MV-Jgjr30E5SpwNKkjTvJ&index=28`

[a] `https://www.youtube.com/watch?v=FwxKQC-iYUg&list=PLug43ldmRSo3MV-Jgjr30E5SpwNKkjTvJ&index=21`

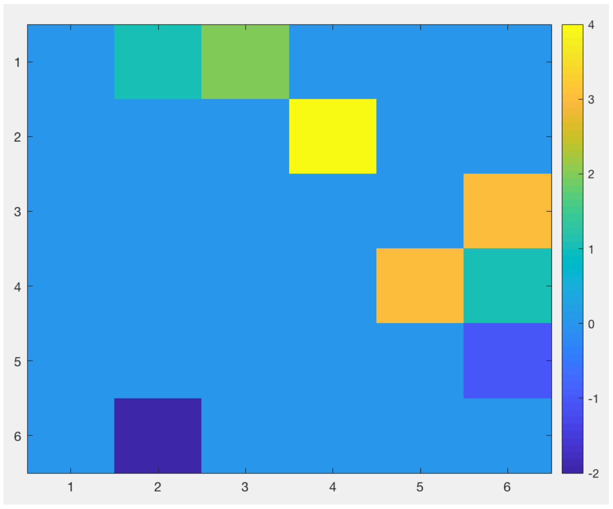


Fig. 1: *The weighted adjacency matrix* **A** *of the directed graph* **G**.

Check the notes below the videos for more helpful information including: `http://basira-lab.com/wp-content/uploads/2019/11/GT_lectures_5_6_2019.pdf`

- **(2.1)** Write a function *globalEfficiency*, which inputs a graph adjacency matrix **A** and outputs the global efficiency value. [5%]
- **(2.2)** Write a function *diffusionEfficiency*, which inputs a graph adjacency matrix **A** and outputs the diffusion efficiency value. [5%]
- **(2.3)** Create a scatter plot where x-axis represents threshold value $\alpha \in [0 : 0.1 : 0.9]$ (0.1 denotes the threshold step size) and the y-axis represents the global efficiency E_{glob} of brain graph adjacency matrices 'Exercise_2_data/ brainGraph1.mat' and 'Exercise_2_data/ brainGraph2.mat' (**Fig.** fig:2). By thresholding each adjacency matrix at different threshold values $\alpha \in [0 : 0.1 : 0.9]$, examine how its global and diffusion efficiencies change. Use two different colors to compare the change in global efficiency across both brain graphs. [5%]
- **(2.4)** In a second figure, plot the diffusion efficiency E_{diff} of both matrices against α. [5%]
- **(2.5)** What conclusions can you derive from previous plots? Compare diffusion and global efficiencies within a single graph and across both graphs. [5%]
- **(2.6)** Plot a morphospace[b] [1] for the set of thresholded brain graphs at $\alpha \in [0 : 0.1 : 0.9]$, where the x-axis denotes E_{diff} and y-axis denotes E_{glob}. Use two different colors to compare the two thresholded graph sets derived from each brain graph, respectively. [10%]

[b] `https://www.youtube.com/watch?v=FX3Dp4ZIgzQ&list=PLug43ldmRSo3MV-Jgjr30E5SpwNKkjTvJ&index=28`

– **(2.7)** Discuss the information flow efficiency for both brain graphs based on your morphospace plot. [5%]

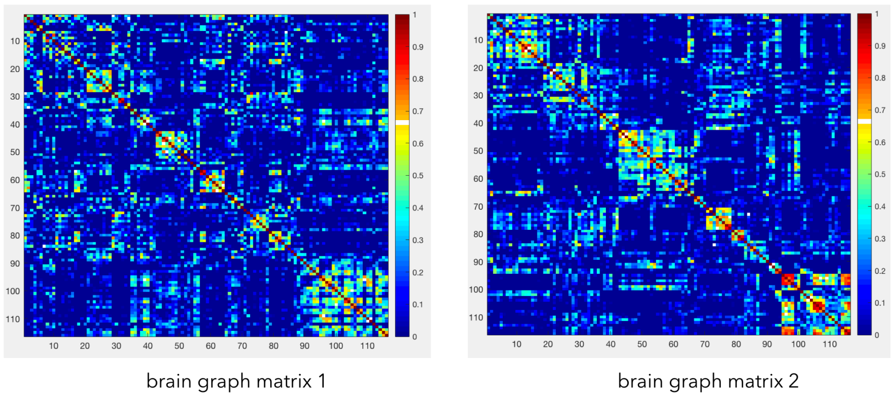


Fig. 2: *The weighted functional connectivity matrices of two brain graphs.*

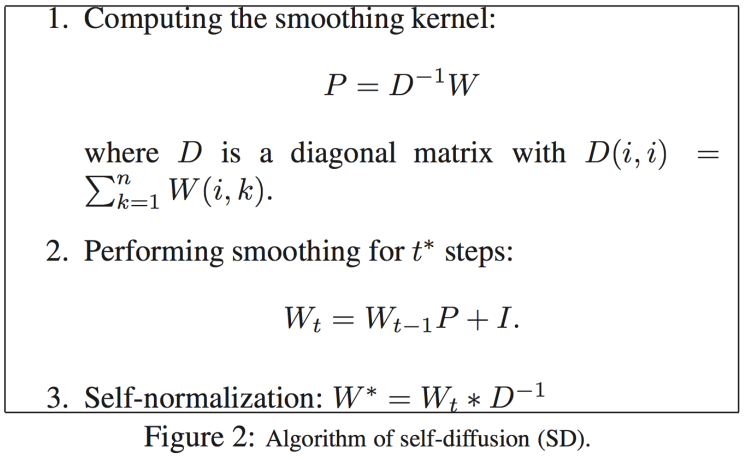
3 Graph self-diffusion for image segmentation [2] [40%]

This part is related to lecture 7: https://www.youtube.com/watch?v=h-ru40T6SGU&list=PLug43ldmRSo3MV-Jgjr30E5SpwNKkjTvJ&index=29 and the research paper on 'Affinity learning via self-diffusion for image segmentation and clustering' by Wang et al. [2].

– **(3.1)** Write a function called *selfDiffuse* that takes (i) a weighted graph adjacency matrix (i.e., similarity matrix) $\mathbf{W}$ and outputs the diffused matrix $\mathbf{W}^\star$ (check algorithm fig:3.). Make sure that your algorithm automatically sets the optimal number of diffusion iterations $t^\star$ as explained in the paper [2]. [10%]
– **(3.2)** Run *selfDiffuse* on both brain graph adjacency matrices 'Exercise_2_data/ brainGraph1.mat' and 'Exercise_2_data/ brainGraph2.mat'. For each brain graph, visualize both original and diffused matrices. [10%]
– **(3.3)** Given the 2 images taken from the Berkeley Segmentation Data Set[c], generate the segmentation maps of each image using the Normalized Cut Python code https://github.com/marktao99/python/blob/master/CVP/samples/ncut.py. Display each original image and its corresponding output segmentation map. [10%]

[c] https://www2.eecs.berkeley.edu/Research/Projects/CS/vision/grouping/resources.html

– **(3.4)** For each image, change the Normalized Cut Python by applying *self-Diffuse* to the similarity matrix $\mathbf{W}$, then use $\mathbf{W}^\star$ to perform the normalized cut and output the image segmentation map. For each image, display 5 different normalized cut segmentation maps when varying the diffusion threshold from $t = 1$ to $t = 2 \times t^\star$ (similar to **Fig** 1 in [2] and **Fig** fig:4 below). [10%]



1. Computing the smoothing kernel:

$$P = D^{-1}W$$

where D is a diagonal matrix with $D(i,i) = \sum_{k=1}^{n} W(i,k)$.

2. Performing smoothing for t^* steps:

$$W_t = W_{t-1}P + I.$$

3. Self-normalization: $W^* = W_t * D^{-1}$

Figure 2: Algorithm of self-diffusion (SD).

Fig. 3: *Graph self-diffusion algorithm to implement [2].*

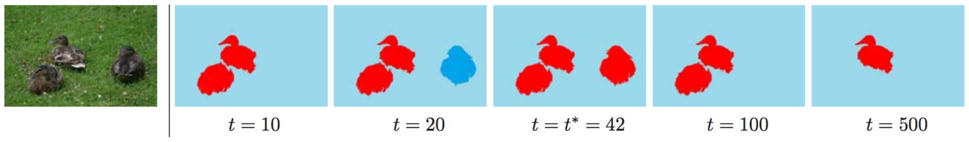


Fig. 4: *A qualitative comparison of segmentation results via self-diffusion w.r.t. different number of iterations [2].*

References

1. Goni, J., Avena-Koenigsberger, A., de Mendizabal, N.V., van den Heuvel, M.P., Betzel, R.F., Sporns, O.: Exploring the morphospace of communication efficiency in complex networks. PLoS One **8** (2013) e58070
2. Wang, B., Tu, Z.: Affinity learning via self-diffusion for image segmentation and clustering. 2012 IEEE Conference on Computer Vision and Pattern Recognition (2012) 2312–2319