Signal Processing Homework #2

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1. The following information is known about an LTI system:
   (i) The system is causal.
   (ii) When the input is
   $$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1],$$
   then the z-transform of the output is
   $$Y(z) = \frac{1 - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}.$$
   a. Find the z-transform of $x[n]$.
   b. What are the possible choices for the ROC of $Y(z)$?
   c. What are the possible choices for a linear constant-coefficient difference equation used to describe the system?
   d. What are the possible choices for the impulse response of the system?

2. Consider the moving average filter
   $$h[n] = \frac{1}{M+1}\sum_{k=0}^{M}\delta[n-k].$$
   What is the group delay of the filter in terms of samples with respect to M? If the sampling period is T_s, what is the group delay in terms of seconds?

3. The sampling period is 600 Hz. Design a length-$(2N+1)$ band-pass filter such that $f_l = 100$ Hz and $f_h = 150$ Hz. Calculate the coefficients from the inverse DTFT of the ideal band-pass filter. What are the values of $h[-1], h[0], h[1]$?

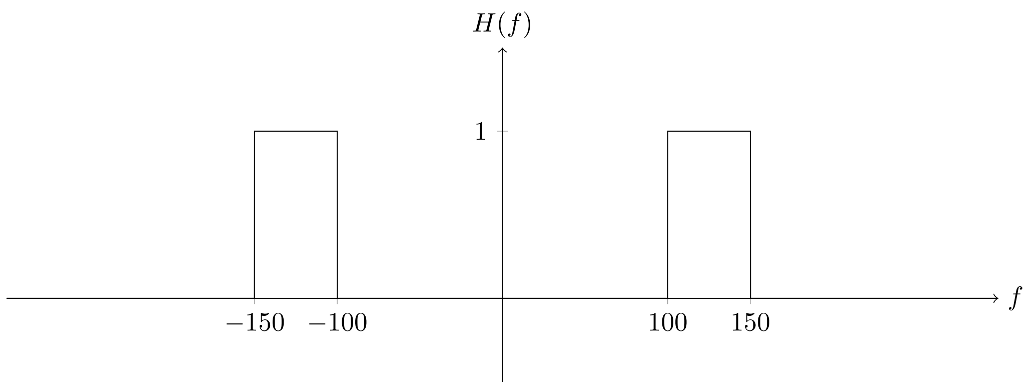

Figure 1: Ideal band-pass filter frequency response

4. The impulse response of an LTI system is $h[n]$. What is the output of the input
   $$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$
   in terms of $h[n]$?

Solutions

1. a. z-transform of $x[n]$ is

$$X(z) = -\frac{1}{3}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{4}{3}\cdot\frac{1}{1-2z^{-1}}, \qquad \frac{1}{2} < |z| < 2$$
$$= \boxed{\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}}$$

b. There 3 possible choice of ROC for $Y(z)$ since there are two poles at $z=2$ and $z=\frac{1}{2}$. The ROCs are

ROC I	$2 < \lvert z\rvert$
ROC II	$\frac{1}{2} < \lvert z\rvert < 2$
ROC III	$\lvert z\rvert < \frac{1}{2}$

The transfer function of the system is

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2}, \qquad \text{ROC: All z-plane except } z = 0$$

Since the output,

$$Y(z) = H(z)X(z)$$

ROC Y contains intersection of ROC of X and ROC of H. Hence, ROC Y is ROC II

$$\boxed{\text{ROC Y: } \frac{1}{2} < |z| < 2}$$

c. Since $H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2}$, the difference equation is

$$Y(z) = H(z)X(z) = (1-z^{-2})X(z)$$
$$\boxed{y[n] = x[n] - x[n-2].}$$

d. The impulse response is

$$\boxed{h[n] = \delta[n] - \delta[n-2].}$$

2. The DTFT of the moving average filter is

$$H(e^{j\omega}) = \frac{1}{M+1}\left(1 + e^{-j\omega} + \cdots + e^{-j(M-1)\omega} + e^{-jM\omega}\right)$$
$$= \frac{1}{M+1}\left(e^{j\frac{M}{2}\omega} + e^{j(\frac{M}{2}-1)\omega} + \cdots + e^{-j(\frac{M}{2}-1)+e^{-j\frac{M}{2}\omega}\omega}\right)e^{-j\frac{M}{2}\omega}$$
$$= \frac{2}{M+1}\left(\cos(M/2\omega) + \cos((M/2-1)\omega) + \cdots\right)e^{-j\frac{M}{2}\omega}$$

Then the phase response is

$$\angle H(e^{j\omega}) = -\frac{M}{2}\omega$$

and the group delay is

$$\tau(\omega) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \boxed{\frac{M}{2}\text{ samples.}}$$

If the sampling period is T_s, then the group delay is

$$\tau_s(\omega) = \boxed{\frac{MT_s}{2}\text{ seconds.}}$$

3. The sampling period $T_s = 1/600$ seconds. Normalized frequencies are

$$\omega_l = \Omega_l T_s = \frac{2\pi 100}{600} = \frac{\pi}{3}, \qquad \omega_h = \Omega_l T_s = \frac{2\pi 150}{600} = \frac{\pi}{2},$$

The DTFT of the band-pass filter can be seen in Figure 2. Let's define ideal low-pass filter as

$$H_L(e^{j\omega}, \omega_c) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}.$$

Then the band-pass filter can be re-written as

$$H_B(e^{j\omega}, \omega_l, \omega_c) = H_L(e^{j\omega}, \omega_h) - H_L(e^{j\omega}, \omega_l).$$

From the inverse DTFT of the H_B, the impulse response of the linear filter is

$$h_B[n] = \frac{\sin(\omega_h n)}{\pi n} - \frac{\sin(\omega_l n)}{\pi n}$$

Since, the impulse response is infinite length, we truncate the ideal impulse response as

$$\boxed{h[n] = \begin{cases} h_B[n], & -N \le n \le N \\ 0, & \text{otherwise} \end{cases}}$$

The impulse response values are

$$h[1] = \frac{\sin(\pi/2)}{\pi} - \frac{\sin(\pi/3)}{\pi} = 0.0426$$
$$h[-1] = \frac{\sin(-\pi/2)}{-\pi} - \frac{\sin(-\pi/3)}{-\pi} = 0.0426$$

The $h[0]$ is undefined. The value is calculated from the limit

$$h[0] = \lim_{n \to 0} \left[\frac{\sin(\pi/2n)}{\pi n} - \frac{\sin(\pi/3n)}{\pi n} \right],$$

using L'Hopital rule

$$h[0] = \frac{cos(\pi/2 \cdot 0) \cdot \pi/2}{\pi} - \frac{cos(\pi/3 \cdot 0) \cdot \pi/3}{\pi} = \frac{1}{2} - \frac{1}{3} = \boxed{0.1667.}$$

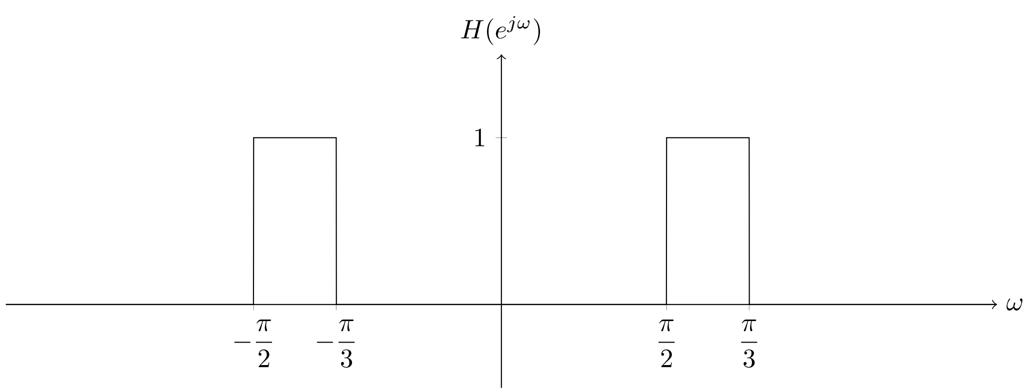

Figure 2: Ideal band-pass filter frequency response

4. Since the system is time invariant

Input		**Output**
$\delta[n]$	$\rightarrow$	$h[n]$
$\delta[n-1]$	$\rightarrow$	$h[n-1]$
$\delta[n-2]$	$\rightarrow$	$h[n-2]$

and from the linearity property

$$\delta[n] + 2\delta[n-1] + 3\delta[n-2] \rightarrow \boxed{h[n] + 2h[n-1] + 3h[n-2].}$$