Relations as Executable Specifications

Taming Partiality and Non-determinism Using Invariants

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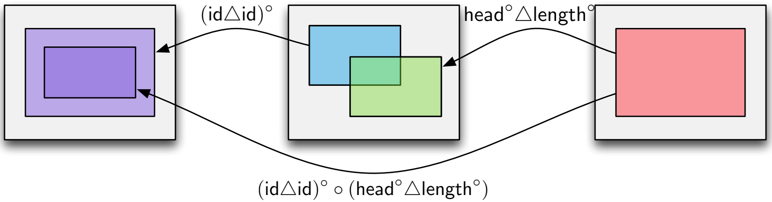
Introduction

- *Relational calculus* provides a more natural way to specify programs;
- Many programs are *partial* and *non-deterministic*;
- A *point-free* (PF) version has been used in a variety of computer science areas;
- Such specifications are not amenable for execution.

Motivating Example

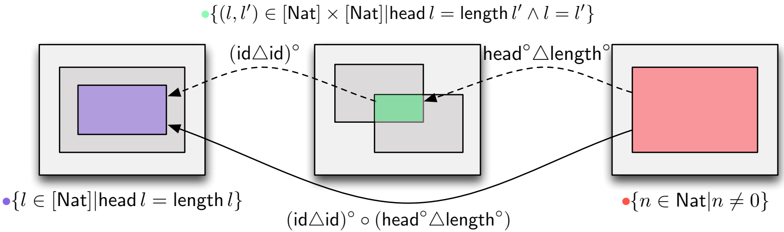
$$(\mathsf{id} \vartriangle \mathsf{id})^\circ \circ (\mathsf{head}^\circ \vartriangle \mathsf{length}^\circ) : \mathsf{Nat} \to [\mathsf{Nat}]$$

- Calculates a list with length n and the same n at its head;
- Not *total* nor *functional*;
- Very inefficient given a naive semantics;
- head° could be generating all lists by increasing length and never reach n.



Motivating Example

- We can *predict* the behavior of partial expressions by calculating the exact *domain* and *range*;
- They can also be used to *narrow* non-deterministic executions.



Taming Partiality and Non-determinism

- We propose a PF relational framework where data-types are enhanced with *invariants*;
- The simplicity of the PF calculus allows us to develop practical type-inference and type-checking algorithms;
- Those invariants can then used to run the specifications more efficiently.

PF Relational Calculus

Identity	$\mathsf{id} : A \to A$
Top	$\top : A \to B$
Bottom	$\bot : A \to B$
Converse	$\cdot^\circ : (A \to B) \to (B \to A)$
Composition	$\cdot \circ \cdot : (B \to C) \to (A \to B) \to (A \to C)$
Intersection	$\cdot \cap \cdot : (A \to B) \to (A \to B) \to (A \to B)$
Union	$\cdot \cup \cdot : (A \to B) \to (A \to B) \to (A \to B)$
Split	$\cdot \vartriangle \cdot : (A \to B) \to (A \to C) \to (A \to B \times C)$
Projections	$\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$
Either	$\cdot \triangledown \cdot : (B \to A) \to (C \to A) \to (B + C \to A)$
Injections	$i_1 : A \to (A + B)$ and $i_2 : B \to (A + B)$
Constants	$! : A \to 1$ and $\underline{\cdot} : B \to (A \to B)$
Conditional	$\cdot ? : (A \to A) \to (A \to A + A)$

PF Relational Calculus

- The calculus possesses simple equational rules;
- They can be harnessed in a rewrite system that simplifies expressions.

$$
\begin{array}{ll}
R \circ \mathrm{id} = R & R \circ \top = \rho R \circ \top \\
\underline{k} \circ R = \underline{k} \circ \delta R & R \circ \bot = \bot \\
\mathrm{id}^\circ = \mathrm{id} & \top^\circ = \top \\
(R \circ S)^\circ = S^\circ \circ R^\circ & (R^\circ)^\circ = R \\
\pi_1 \vartriangle \pi_2 = \mathrm{id} & i_1 \triangledown i_2 = \mathrm{id} \\
\pi_1 \circ (R \vartriangle S) = R \circ \delta S & (R \triangledown S) \circ i_1 = R \\
\pi_1 \circ \pi_2^\circ = \top & i_1^\circ \circ i_2 = \bot \\
(R \vartriangle S) \circ f = (R \circ f) \vartriangle (S \circ f) & U \circ (R \triangledown S) = U \circ R \triangledown U \circ S \\
\pi_1^\circ \circ R = R \vartriangle \top & R \circ i_1^\circ = R \triangledown \bot
\end{array}
$$

Membership Semantics

$$
\begin{array}{ll}
a' \; [\![\mathrm{id}]\!] \; a & = a \equiv a' \\
a \; [\![R^\circ]\!] \; b & = b \; [\![R]\!] \; a \\
b \; [\![S \circ R]\!] \; a & = \exists \; c . \; b \; [\![S]\!] \; c \wedge c \; [\![R]\!] \; a \\
b \; [\![R \cap S]\!] \; a & = b \; [\![R]\!] \; a \wedge b \; [\![S]\!] \; a \\
b \; [\![R \cup S]\!] \; a & = b \; [\![R]\!] \; a \vee b \; [\![S]\!] \; a \\
(b, c) \; [\![R \vartriangle S]\!] \; a & = b \; [\![R]\!] \; a \wedge c \; [\![S]\!] \; a \\
a' \; [\![\pi_1]\!] \; (a, b) & = a \equiv a' \\
b' \; [\![\pi_2]\!] \; (a, b) & = b \equiv b' \\
a \; [\![R \triangledown S]\!] \; (\mathrm{Left} \; b) & = a \; [\![R]\!] \; b \\
a \; [\![R \triangledown S]\!] \; (\mathrm{Right} \; c) & = a \; [\![S]\!] \; c \\
(\mathrm{Left} \; a') \quad [\![i_1]\!] \; a & = a \equiv b \\
(\mathrm{Left} \; a') \quad [\![i_2]\!] \; b & = \mathrm{False} \\
(\mathrm{Right} \; b') \; [\![i_1]\!] \; a & = \mathrm{False} \\
(\mathrm{Right} \; b') \; [\![i_2]\!] \; b & = a \equiv b \\
b \; [\![\top]\!] \; a & = \mathrm{True} \\
b \; [\![\bot]\!] \; a & = \mathrm{False} \\
1 \; [\![!]\!] \; a & = \mathrm{True} \\
b' \; [\![\underline{b}]\!] \; a & = b \equiv b'
\end{array}
$$

Execution Semantics

$$
\begin{array}{lll}
[\![\mathrm{id}]\!] & a & = \{a\} \\
[\![R^\circ]\!] & b & = \{a \mid a \leftarrow A, a\ [\![R^\circ]\!]\ b\} \\
[\![S \circ R]\!] & a & = \{b \mid c \leftarrow [\![R]\!]\ a, b \leftarrow [\![S]\!]\ c\} \\
[\![R \cap S]\!] & a & = \{b \mid b \leftarrow [\![R]\!]\ a, b\ [\![S]\!]\ a\} \\
[\![R \cup S]\!] & a & = [\![R]\!]\ a \cup [\![S]\!]\ a \\
[\![R \vartriangle S]\!] & a & = \{(b, c) \mid b \leftarrow [\![R]\!]\ a, c \leftarrow [\![S]\!]\ a\} \\
[\![\pi_1]\!] & (a, b) & = \{a\} \\
[\![\pi_2]\!] & (a, b) & = \{b\} \\
[\![R \triangledown S]\!] & (\mathrm{Left}\ b) & = [\![R]\!]\ b \\
[\![R \triangledown S]\!] & (\mathrm{Right}\ c) & = [\![S]\!]\ c \\
[\![i_1]\!] & a & = \{\mathrm{Left}\ a\} \\
[\![i_2]\!] & b & = \{\mathrm{Right}\ b\} \\
[\![\top]\!] & a & = B \\
[\![\bot]\!] & a & = \{\} \\
[\![!]\!] & a & = \{1\} \\
[\![\underline{b}]\!] & a & = \{b\}
\end{array}
$$

Predicates as Coreflexives

- *Domain* δR and *range* ρR are predicates that can be defined as coreflexives;
- *Coreflexives* $\Phi : A \to A$ are relations smaller than the identity;
- If a satisfies the predicate Φ then $a \llbracket \Phi \rrbracket a$;
- For pairs $A \times B$ related by $R : A \to B$, the invariant is lifted as $[R] : A \times B \to A \times B$;
- Invariants on coproducts are simply the coproduct $\Phi + \Psi$;
- $R : \Phi \to \Psi$ denotes $\delta R = \Phi$ and $\rho R = \Psi$.

Inferring Checkable Invariants

- Domain and range can be directly computed as $\delta R = R^\circ \circ R \cap \mathsf{id}$ and $\rho R = R \circ R^\circ \cap \mathsf{id}$;
- May result in inefficient membership tests (due to composition);
- By expanding the definition, we define an equivalent definition with most compositions removed;
- Others will fall in the special case $b\ (f^\circ \circ R \circ g)\ a \equiv (f\ b)\ R\ (g\ a)$;
- The rewriting system further simplifies the formula and issues a warning if problematic expressions remain.

Example

$$(\mathsf{id} \vartriangle \mathsf{id})^\circ \circ (\mathsf{head}^\circ \vartriangle \mathsf{length}^\circ) : \mathsf{Nat} \to [\mathsf{Nat}]$$

$$\begin{aligned}
&\rho((\mathsf{id} \vartriangle \mathsf{id})^\circ \circ (\mathsf{head}^\circ \vartriangle \mathsf{length}^\circ)) \\
&\quad = \{\textit{Range definition}\} \\
&\rho((\mathsf{id} \vartriangle \mathsf{id})^\circ \circ \rho(\mathsf{head}^\circ \vartriangle \mathsf{length}^\circ)) \\
&\quad = \{\textit{Range definition}\} \\
&\rho((\mathsf{id} \vartriangle \mathsf{id})^\circ \circ [\mathsf{length}^\circ \circ \mathsf{head}]) \\
&\quad = \{\textit{Range definition}\} \\
&\delta([\mathsf{length}^\circ \circ \mathsf{head}] \circ (\mathsf{id} \vartriangle \mathsf{id})) \\
&\quad = \{\textit{Domain definition}, \textit{Simplifications} : \textit{PF Laws}\} \\
&(\mathsf{head}^\circ \circ \mathsf{length}) \cap \mathsf{id}
\end{aligned}$$

$$(\mathsf{id} \vartriangle \mathsf{id})^\circ \circ (\mathsf{length}^\circ \vartriangle \mathsf{head}^\circ) : \mathsf{in}_{\mathsf{N}} \circ (\bot + \mathsf{id}) \circ \mathsf{out}_{\mathsf{N}} \to (\mathsf{head}^\circ \circ \mathsf{length}) \cap \mathsf{id}$$

Optimizing Non-deterministic Executions

- After determining the domain and range, they can be propagated down to *primitives*,
- This reduces the generation of irrelevant intermediate values;

$$
\begin{aligned}
&[\![\mathrm{id} : \Phi \to \Psi]\!]\; a = [\![\Psi]\!]\; a \\
&[\![\pi_1 : [U] \to \Psi]\!]\; (a, b) = [\![\Psi]\!]\; a \\
&[\![R \circ S : \Phi \to \Psi]\!]\; a = \{\, b \mid \begin{array}{l} c \leftarrow [\![S : \Phi \to \delta R]\!]\; a, \\ b \leftarrow [\![R : \rho S \to \Psi]\!]\; c \,\} \end{array} \\
&[\![R \cap S : \Phi \to \Psi]\!]\; a = \{\, b \mid b \leftarrow [\![R : \Phi \cap \delta S \to \rho(\Psi \circ S \circ \underline{a})]\!]\; a \,\}
\end{aligned}
$$

Recursive Relations with Invariants

- We support the well-know recursion patterns of *catamorphisms* (folds) and *anamorphisms* (unfolds);
- Execution is not problematic, as it is performed by unfolding their definitions;
- However, there is no known normal form for invariants over recursive types;
- The rewrite system tries to simplify the generic domain/range expression.

Recursive Relations: Example

$$\text{unzip} : [A \times B] \rightarrow [A] \times [B]$$

$$
\begin{array}{l}
\rho\text{unzip} \\
\quad = \{\textit{Range definition}\} \\
(\text{unzip} \circ \text{unzip}^{\circ}) \ \cap \ \text{id} \\
\quad = \{\textit{Simplifications} : \text{unzip } \textit{is functional}, \textit{Liftify} : \textit{range of } \text{unzip } \textit{is a product}\} \\
[\pi_2 \circ \text{unzip} \circ \text{unzip}^{\circ} \circ \pi_1^{\circ}] \\
\quad = \{\textit{Catamorphism fusion} : \pi_1 \circ g = \text{nil} \triangledown (\text{cons} \circ (\pi_1 \times \text{id})) \circ F\ \pi_1\} \\
[(\!|\text{nil} \triangledown (\text{cons} \circ (\pi_1 \times \text{id}))|\!) \circ (\!|\text{nil} \triangledown (\text{cons} \circ (\pi_2 \times \text{id}))|\!)^{\circ}] \\
\quad = \{\textit{Definitions} : \text{map}\} \\
[(\text{map } \pi_1) \circ (\text{map } \pi_2)^{\circ}] \\
\quad = \{\textit{Simplifications} : \text{map } \textit{converse}, \text{map } \textit{fusion } (\textit{see below})\} \\
[\text{map } (\pi_1 \circ \pi_2^{\circ})] \\
\quad = \{\textit{Simplifications} : \textit{PF Laws}\} \\
[\text{map } \top]
\end{array}
$$

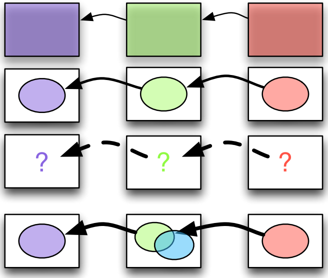
$$\text{unzip} : \text{id} \rightarrow [\text{map } \top]$$

Bidirectional Transformations

- *Lenses* are one of the most famous bidirectional transformation (BX) frameworks;
- A lens $S \trianglerighteq V$ between sources S and more abstract views V consists of transformations $\mathsf{Get} : S \to V$ and $\mathsf{Put} : V \times S \to S$;
- It is said to be *well-behaved* if $\mathsf{Get} \circ \mathsf{Put} \subseteq \pi_1$ (acceptability) and $\mathsf{Put} \circ (\mathsf{Get} \vartriangle \mathsf{id}) \subseteq \mathsf{id}$ (stability).

Bidirectional Transformations

- In principle, it is possible to lift any functional expression to a well-behaved lens;
- Existing frameworks either:
  - Have maximum updatability but disregard some operators;
  - Refine the type-system to allow operators with smaller updatability;
  - Support any operator but disregard updatability;
- We refine the type-system and guarantee maximum updatability.



Generic Non-deterministic Lenses

- Using relational calculus we can define a generic Put that is the largest relation that satisfies the properties;
- A transformation $\mathsf{get} : A \to B$ can be lifted to a well-behaved non-deterministic lens $\lfloor \mathsf{get} \rfloor : \delta \mathsf{get} \trianglerighteq \rho \mathsf{get}$, with $\mathsf{Put} = (\pi_2 \triangledown (\mathsf{get}^\circ \circ \pi_1)) \circ [\mathsf{get}^\circ]?$;
- Emerges naturally from the lens laws:
  - $[\mathsf{get}^\circ]\,?\,(v, s)$ tests if v was changed, returning either the original s (acceptability), or any source s' such that $s' = \mathsf{get}\ v$ (stability);
  - Maximum updatability: $\delta \mathsf{Put} = \rho \mathsf{get} \times \delta \mathsf{get}$.

Generic Non-deterministic Lenses

- Type-checking over δget and ρget directly could be undecidable and due to the central role of the converse, Put can not be directly executed;
- Both these issues can be addressed by the optimizations already presented;
- Forward transformations are functional so problematic cases are very limited:
  - Regarding type-checking, only particular ranges of splits are possibly undecidable;
  - The backward transformation can be efficiently executed;
- Recursive expressions are also supported as they preserve the functionality of their algebras.

Generic Non-deterministic Lenses: Example

$$\lfloor \pi_1 \vartriangle \mathrm{id} \rfloor : \mathrm{id} \trianglerighteq [\pi_1^\circ]$$

- The range is $[\pi_1^\circ] : A \times (A \times B) \to A \times (A \times B)$, so Put only takes views $(a, (b, c))$ where $a \equiv b$;
- When the view is updated, $(\pi_1 \vartriangle \mathrm{id})^\circ$ will run, and π_1° could generate all pairs until reaching (a, c);
- With our optimization, the output of π_1 is restricted to have c in the second element.

Conclusions

- We have presented mechanisms for the efficient execution PF relational expressions over data-types with invariants;
- Regarding BX, we identify an open problem in the composition of lenses;
- By modeling lenses in this framework we were able to implement an expressive PF BX language with maximum updatability;

- Researching possible normal forms for invariants over recursive types;
- Exploring mechanisms for a better control of the non-determinism through user-defined quality measures.