ECE 661: Homework 1

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Problem 1

What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$.

All the points that lie on the Z axis except the origin. Mathematically, the set of points $(0, 0, k)$ where k is a non-zero real number.

Problem 2

Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer.

No. Lets say there is a circle centered at the origin with an infinite radius. All the points that lie on this infinite circle are the points at infinity.

Mathematically, the points at infinity are represented by $P - (x, y, 0)$ where x and y are real numbers. The corresponding two dimensional point is given by $\frac{x}{0}, \frac{y}{0}$. It is basically the point at infinity in a direction determined by x and y. Since x and y are real numbers, there will be more than one points at infinity that are different.

Problem 3

Argue that the matrix rank of a degenerate conic can never exceed 2.

A degenerate conic is represented by two straight lines, $l = (l_1, l_2, l_3)$ and $m = (m_1, m_2, m_3)$. The corresponding homogeneous representation is given by $C = C_1 + C_2$, where $C_1 = lm^T$ and $C_2 = ml^T$ are 3×3 matrices. The rows are C_1 are multiples of $m = [m_1, m_2, m_3]$ and the rows of C_2 are multiples of $l = [l_1, l_2, l_3]$. Therefore, the rank of C_1 and C_2 is 1.

Note that $rank(A + B) \leq rank(A) + rank(B)$ where are A and B are 2D matrices. Hence the rank of C can never exceed 2.

Problem 4

Derive in just 3 steps the intersection of two lines l_1 and l_2 with l_1 passing through the points $(0, 0)$ and $(2, 6)$ and with l_2 passing through the points $(-6, 8)$ and $(-3, 2)$. How many steps would take you if the second line passed through $(-10, -3)$ and $(10, 3)$.

The line (l) passing through the points $p = [p_1, p_2, p_3]$ and $q = [q_1, q_2, q_3]$ is given by $l = p_1 \times p_2$. Now,

$$l_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}2\\6\\1\end{bmatrix} = \begin{bmatrix}-6\\2\\0\end{bmatrix}$$

$$l_2 = \begin{bmatrix}-6\\8\\1\end{bmatrix} \times \begin{bmatrix}-3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\3\\12\end{bmatrix}$$

Now we calculate the intersection point x:

$$x = l_1 \times l_2 = \begin{bmatrix}24\\72\\-30\end{bmatrix}$$

If l_2 passes through $(-10, -3)$ and $(10, 3)$:

$$l_2 = \begin{bmatrix}-10\\-3\\1\end{bmatrix} \times \begin{bmatrix}10\\3\\1\end{bmatrix} = \begin{bmatrix}-6\\20\\0\end{bmatrix}$$

In this case, both lines l_1 and the new l_2 pass through origin. Hence, we can directly say that the intersection point is at the **origin**. We calculated the intersection point through **one step**.

Problem 5

Consider that there are two lines. The first line is passing through points $(0, 0)$ and $(2, -2)$. The second lines is passing through points $(-3, 0)$ and $(0, -3)$. Find the intersection between these two lines. Comment on your answer.

Problem 5 continued on next page...

The line (l) passing through the points $p = [p_1, p_2, p_3]$ and $q = [q_1, q_2, q_3]$ is given by $l = p_1 \times p_2$. Now,

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} == \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

Now we calculate the intersection point x:

$$x = l_1 \times l_2 = \begin{bmatrix} 18 \\ -18 \\ 0 \end{bmatrix}$$

The point of intersection, x, is at infinity showing that the two lines l_1 and l_2 are parallel.

Problem 6

As you know, when a point x is on a conic, the tangent to the conic at that point is given by $l = Cx$. That rises the question of what Cx corresponds to when x is, say, outside the conic. As you'll see later in class, when x is outside the conic, Cx is the line that joins the two points of contact if you draw tangents to C from the point x. This line is referred to as the *polar line*. Now consider for our conic a circle of radius 1 that is centered at the coordinates (5,5) and let x be the origin of the $\mathbb{R}^2$ physical plane. Where does the polar line intersect the x and y axes in this case?

Problem 6 continued on next page...

The equation of the circle centered at (5, 5) with unit radius is given by:

$$(x-5)^2 + (y-5)^2 - 1 = 0 \quad \longrightarrow \quad x^2 + y^2 - 10x - 10y + 49 = 0$$

The homogeneous representation of the circle as a 3×3 matrix is given by:

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & -10/2 \\ 0 & 1 & -10/2 \\ -10/2 & -10/2 & 49 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix}$$

Now the polar line is given by:

$$l = Cx = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 & 49 \end{bmatrix}$$

Now lets calculate where does l intersect with the x axis i.e. $y = 0 \to [0, 1, 0]$ and y axis i.e. $x = 0 \to [1, 0, 0]$.
Intersection with x axis (u):

$$u = x_{axis} \times l = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -5 \\ -5 \\ 49 \end{bmatrix} = \begin{bmatrix} 49 \\ 0 \\ 5 \end{bmatrix}$$

Intersection with x axis (v):

$$v = y_{axis} \times l = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -5 \\ -5 \\ 49 \end{bmatrix} = \begin{bmatrix} 0 \\ -49 \\ -5 \end{bmatrix}$$

Problem 7

Find the intersection of two lines whose equations are given by $x = 1$ and $y = 1$?

The equation of the line (l_1) $x = 1$ is given by $l_1 = [1, 0, -1]$.
The equation of the line (l_2) $y = 1$ is given by $l_2 = [0, 1, -1]$.
The point of intersection (x) of l_1 and l_2 is given by: $x = l_1 \times l_2$

$$x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = [1, 1, 1]$$