CS230-HW3Sol

1. **Modeste 5 pts**

   Since the statement is a biconditional, we have to prove two parts: if p is odd, then p^3 is odd, and if p^3 is odd, then p is odd.

   a) Prove if p is odd, then p^3 is odd. Let p be an odd number. Then $p = 2k+1$, for some integer k. So $p^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Since $4k^3 + 6k^2 + 3k$ is an integer because k is an integer, then p^3 is odd.

   b) Prove if p^3 is odd, then p is odd. If we use a direct proof for this portion, then we will have $p^3 = 2k+1$ and $p = (2k+1)^{1/3}$. This causes an issue as p is required to be an integer. That means we have to prove this using a different proof technique: contrapositive. The statement we will prove is if p is even, then p^3 is even.

   Let p be even. Then p is a multiple of 2: $p = 2k$, for some integer k. So $p^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3$ is an integer, then p^3 is even. Thus, if p is even, then p^3 is even, and if p^3 is odd, then p is odd.

   Since both implications have been proven true, then p is odd if and only if p^3 is odd.

2. **Modeste 6 pts**

   Let x and y be non-zero rational numbers and z be irrational. We need to prove that $x + yz$ is irrational. It is not possible to use a direct proof because there is no way to represent an irrational number in any other way like we can do for other kinds of numbers such as rational or even numbers. We will prove this using a proof by contradiction.

   Let x and y be non-zero rational numbers and z be irrational. Assume that $x + yz$ is a rational number. Since x and y are non-zero rational numbers, then $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are all some integer but not zero. Similarly, $x + yz = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Then $x + yz = \frac{p}{q} \Rightarrow \frac{a}{b} + \frac{c}{d}z = \frac{p}{q} \Rightarrow \frac{c}{d}z = \frac{bp-aq}{bq} \Rightarrow z = \frac{(bp-aq)d}{bcq}$. For $\frac{(bp-aq)d}{bcq}$ to be a rational number, both the numerator and denominator need to be integers and the denominator not equal to zero. Since $b \neq 0$, $c \neq 0$, and $q \neq 0$, then $bcq \neq 0$. Also, both the numerator and denominator are integers because all of the variables being multiplied and subtracted are integers. This means that z is equal to some rational number, which contradicts the assumption that z is irrational. Thus, the assumption that $x + yz$ is a rational number is false. Therefore, if x and y are non-zero rational numbers and z is irrational, then $x + yz$ is irrational.

3. **Ying 6 pts**

   Have p represent the statement "$mn > 35$" and q represent "$m \geq 6$ or $n \geq 8$". Then the question is $p \to q$. The contrapositive is $\neg q \to \neg p$, which translates to "$m < 6$ and $n < 8$" and "$mn \leq 35$" for $\neg q$ and $\neg p$, respectively.

   Let $m < 6$ and $n < 8$. We need to show that $mn \leq 35$. Since $m < 6$ and $n < 8$ are both positive integers, then $m \leq 5$ and $n \leq 7$. This means the largest value m can be is 5 and the largest value n can be is 7. Then the largest value mn can be is 35, so $mn \le 35$, which is what we needed to show. Thus, if $m < 6$ and $n < 8$, then $mn \le 35$. Therefore, if $mn > 35$, then $m \ge 6$ or $n \ge 8$.

4. **Ying 6 pts**

   Have p represent "32 total students" and q be "at least 5 freshmen or at least 8 sophomores or at least 10 juniors or at least 7 seniors". We will prove the that if there are 32 total students in the organization, then there are at least 5 freshmen or at least 8 sophomores or at least 10 juniors or at least 7 seniors by using the contrapositive. The negation of p would be "there are not 32 total students" and the negation of q is that there are "less than 5 freshmen, less than 8 sophomores, less than 10 juniors, and less than 7 seniors". This means that the maximum number of freshmen there can be is 4, the maximum number of sophomores is 7, the maximum number of juniors is 9, and the maximum number of seniors is 6. Then the maximum number of total students in the organization is $4+7+9+6 = 26$. So it is not possible for the total number of students in the organization to be 32, which would cause the contrapositive to be false. Thus if there are less than 5 freshmen, less than 8 sophomores, less than 10 juniors, and less than 7 seniors, then the total number of students in the organization is not equal to 32 is true. Therefore if there are 32 total students in the organization, then there at least 5 freshmen or at least 8 sophomores or at least 10 juniors or at least 7 seniors.

5. **Ling 6 pts**

   Let $p \ge 3$ or $p \le -7$. We need to prove that $(p+2)^2 \ge 25$. Since it isn't known which of the two possible conditions for p is true, both need to be examined individually.

   Case 1: Have $p \ge 3$. Then $p+2 \ge 5 \Rightarrow (p+2)^2 \ge 25$.

   Case 2: Have $p \le -7$. Then $p+2 \le -5 \Rightarrow |p+2| \ge 5 \Rightarrow (p+2)^2 \ge 25$.

   Both cases reach the conclusion that $(p+2)^2 \ge 25$. Therefore if $p \ge 3$ or $p \le -7$, then $(p+2)^2 \ge 25$.

6. **Jonathan 6 pts**

   Assume the square root of 5 is rational for a proof by contradiction. Since the square root of 5 is rational, then $\sqrt{5} = \frac{p}{q}$ where p and q are some non-zero integers and have no common factors ($\frac{p}{q}$ cannot be reduced). Then $(\sqrt{5})^2 = 5 = (\frac{p}{q})^2 = \frac{p^2}{q^2}$. So, $5q^2 = p^2$.

   Since every integer is either odd or even, but not both, then p^2, q^2, q, and p are each odd or even. If q^2 was even, then p^2 would have to be even since an odd number multiplied by an even number is even ($5q^2 = 5(2k) = 2(5k) = p^2$, where $q^2 = 2k$ for some integer k). Also, q and p would both be even if q^2 is even (if a is odd, a^2 is odd). Since we assumed p and q have no common factors, then neither p nor q can be even.

   This means $q = 2k+1$ and $p = 2j+1$ for some integers k and j. Then $5q^2 = p^2 \Rightarrow 5(2k+1)^2 = (2j+1)^2 \Rightarrow 5(4k^2+4k+1) = 4j^2+4j+1 \Rightarrow 20k^2+20k+4+1 = 4j^2+4j+1 \Rightarrow 20k^2+20k+4 = 4j^2+4j \Rightarrow 5k^2+5k+1 = j^2+j \Rightarrow 5k(k+1)+1 = j(j+1)$. It is important to note that when two consecutive numbers are multiplied, the product is always even because one of the two numbers will be even and the other will be odd. This means $j(j+1)$ is always even and $5k(k+1)+1$ is always odd (since $5k(k+1)$ is a even number multiplied with an odd number plus 1). This a contradiction since an odd and an even number can never equal each other. Thus the assumption at the beginning that the square root of 5 is rational is false. Therefore the square root of 5 is irrational.

7. **Ling 5 pts**

   Have $x = 5$ and $y = 1/2$. Then $x^y = 5^{1/2} = \sqrt{5}$. Both x and y are rational numbers and x^y is irrational. This is a constructive proof since we have found, or constructed, a x and a y that satisfies the statement.