Com S 230 : Discrete Computational Structures

Spring Semester, 2021

Sample Problems for Exam 2

1. Short Answers [**24 Points**]

   (a) [**12 Pts**] Is the relation $\{(a, b) \mid a > 3b\}$ over $\mathbb{R}^+$ (i) reflexive, (ii) anti-reflexive, (iii) symmetric, (iv) anti-symmetric, and (v) transitive? Is it (i) an equivalence relation, (ii) a partial order, or (iii) a strict partial order?

   (b) [**6 Pts**] Give an *inductive definition* for a_i of the sequence $5, 12, 26, 54, 110, \ldots$, where a_0 is the first term.

   (c) [**6 Pts**] Give an *inductive definition* for S, the set of ordered pairs (a, b) where a and b are integers, and $a = b$.

2. Equivalence Relations [**20 Points**]

Consider the following relation R on $\mathbb{R}$:

$$(a, b) \in R \text{ if and only if } a = b + n/2 \text{ for some integer } n.$$

(a) [**8 pts**] Prove that R is an equivalence relation.

(b) [**6 pts**] Define $[0]$ and $[1/4]$ by enumeration and by set builder notation.

(c) [**6 pts**] Describe the equivalence classes formally. In other words, for each $x \in \mathbb{R}$, define $[x]$ using set-builder notation. Now, define an interval $[0, y)$ where $y \in \mathbb{R}^+$ such that $[x]$ is distinct for each $x \in [0, y)$.

3. Mathematical Induction [**14 Points**]

Consider the statement

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{2^n - 1}{2^n}$$

Prove this statement, by induction, for all positive integers n.

(a) [**4 pts**] State the *base case* and prove it.

(b) [**4 pts**] State the *inductive hypothesis* (the assumption), and the statement you need to prove.

(c) [**6 pts**] Prove your statement.

4. Strong Induction [**16 Points**]

$P(n)$ says that a postage of $5n$ cents can be formed using just 10-cent and 15-cent stamps. We prove that $P(n)$ is true for $n \geq 2$, *i.e.*, any amount of 10-cents or more postage that is a multiple of 5 can be made using only 10 cent or 15 cent stamps.

(a) To prove $P(n)$ for all $n \geq 2$ by *regular* induction, we need to prove:

i. [**2 Points**] $P(2)$, to complete the basis step.

ii. [**6 Points**] $P(k) \rightarrow P(k+1)$ for all $k \geq 2$.

(b) To prove $P(n)$ for all $n \geq 2$ by *strong* induction, we need to prove:

i. [**3 Points**] $P(2)$ and $P(3)$, to complete the basis step.

ii. [**5 Points**] For all $k \geq 3$, if $P(j)$ is true for $2 \leq j \leq k$, then $P(k+1)$ is true.

5. Countable and Uncountable Sets [**16 Points**]

   (a) [**8 Pts**] Prove that $\mathcal{Q}^+$, the set of positive rational numbers, is countable, using a dovetailing argument.

   (b) [**8 Pts**] Let $\mathcal{F}$ be the set of functions with domain $\mathcal{Z}^+$ and co-domain $\{0, 1\}$. Prove that $\mathcal{F}$ is uncountable, using a diagonalization argument.

6. State Machines and Preserved Invariants [**18 Points**]

A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+0,+4), (+1,+3), (+2,+2). Let S be the set of points that the robot can get to, where each state is an ordered pair.

(a) [**4 Pts**] Give an an inductive definition of S. The basis step is $(0, 0) \in S$.

(b) [**8 Pts**] Prove, by structural induction, that if $(a, b) \in S$, then $a + b$ is divisible by 4. This is the Preserved Invariant.

(c) [**6 Pts**] Consider the statement: if $a, b \in \mathcal{N}$, and $a + b$ is divisible by 4, then $(a, b) \in S$. Show, by counter-example, that this is not true. Modify the set of steps that the robot can take to make the statement true. You do not need to prove your construction.

7. Inductive Definitions of Sets [**12 Points**]

   Consider this inductive definition for a set of binary strings S.

   Base Case: 01 is in S.

   Inductive Case: if x is in S, then $0x1$ is in S.

   For example, 01, 0011 and 000111 are all in S. Now, prove, *by structural induction*, that if x is in S, then x is of the form 0^n1^n, for some positive integer n.

   (a) State the base case and prove it.

   (b) State the inductive hypothesis and prove it.