CS230-HW4Sol

1. **Ling 6 pts**

   We will prove if $A \neq B$, then $A \times B \neq B \times A$ by using a proof by contrapositive: if $A \times B = B \times A$, then $A = B$. To prove $A = B$, then if $x \in A$, then $x \in B$ and if $y \in B$, then $y \in A$. Have $x \in A$ and $y \in B$. Then $(x, y) \in A \times B$ by definition of Cartesian product. Since $A \times B = B \times A$ and $(x, y) \in A \times B$, then $(x, y) \in B \times A$. So $x \in B$ and $y \in A$. Thus $A = B$. Therefore, if $A \neq B$, then $A \times B \neq B \times A$.

2. **Ling 4 pts**

   We will prove $(A \cup B) - C = (A - C) \cup (B - C)$ using a series of equivalences.

$$
\begin{aligned}
x \in (A \cup B) - C &\text{ iff } x \in A \cup B \wedge x \notin C && \text{definition of set difference}\\
&\text{ iff } (x \in A \vee x \in B) \wedge x \notin C && \text{definition of union}\\
&\text{ iff } (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C) && \text{distributive law of logic}\\
&\text{ iff } (x \in A - C) \vee (x \in B - C) && \text{definition of set difference}\\
&\text{ iff } x \in (A - C) \cup (B - C) && \text{definition of set union}
\end{aligned}
$$

3. **Jonathan 8 pts**

   a) Consider $A = \{1\}, B = \{2\}$, and $C = \{1, 2\}$. Then $A \cup C = B \cup C = C = \{1, 2\}$, so $A \cup C \subseteq B \cup C$, but A is not a subset of B.

   b) Consider $A = \{1\}, B = \{2\}$, and $C = \emptyset$. Then $A \cap C = B \cap C = C = \emptyset$, so $A \cap C \subseteq B \cap C$, but A is not a subset of B since $1 \in A$ but $1 \notin B$.

4. **Jonathan 8 pts**

   Let $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$. Assume, for contradiction, A is not a subset of B, in other words, there is some x where $x \in A$ but $x \notin B$. Since $x \in A$, then $x \in A$ or $x \in C$, so $x \in A \cup C$. This means $x \in B \cup C$, by definition of subset. Since we assumed that $x \notin B$ and $x \in B \cup C$, then x has to be an element of C. Then $x \in A \cap C$ since $x \in A$ and $x \in C$. So $x \in B \cap C$ as well. By definition of intersection, then $x \in B$ and $x \in C$. This contradicts the assumption that $x \notin B$. Thus the assumption that A is not a subset of B is false. Therefore if $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$, then $A \subseteq B$.

5. **Ying 8 pts**

   To prove $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$, we must show that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$ and $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

   $\subseteq$: Let $x \in (A \cup B) - (A \cap B)$. Prove $x \in (A - B) \cup (B - A)$. By definition of set difference, $x \in A \cup B$ and $x \notin A \cap B$. Since $x \in A \cup B$, then $x \in A$ or $x \in B$, by definition of set union.

Case 1: Have $x \in A$. We need to show that $x \notin B$. Assume, for contradiction, $x \in B$. Then $x \in A \cap B$, by definition of set intersection. This contradicts $x \notin A \cap B$ from above. Thus $x \notin B$. Since $x \in A$ and $x \notin B$, then $x \in A - B$. Thus $x \in (A - B) \cup (B - A)$.

Case 2: This case is symmetrical to case 1. It is the same steps as case 1 but with A and B switched.

Thus $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

$\supseteq$: Let $x \in (A - B) \cup (B - A)$. Prove $x \in (A \cup B) - (A \cap B)$. By definition of union, $x \in A - B$ or $X \in B - A$.

Case 1: Have $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \notin B$, then $x \notin A \cap B$ because for x to be in $A \cap B$, $x \in A$ and $x \in B$. Since $x \in A$, then $x \in A \cup B$ because $A \subseteq A \cup B$. By the definition of set difference, since $x \in A \cup B$ and $x \notin A \cap B$, then $x \in (A \cup B) - (A \cap B)$.

Case 2: Case 2 is symmetrical to case 1.

Thus $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. Therefore $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

6. **Ying 4 pts**

   The function f is one to one if $\forall x \in \mathbb{Z}^+ \forall x \in \mathbb{Z}^+ (f(x) = f(y) \to x = y$. Let $x, y \in \mathbb{Z}^+$ and $f(x) = f(y)$. Then $f(x) = f(y) \Rightarrow 5x + 9 = 5y + 9 \Rightarrow 5x = 5y \Rightarrow x = y$. Therefore f is one to one.

   To prove f is not onto, then we should be able to find a positive integer that is not mapped to under f. Consider $1 \in \mathbb{Z}^+$. For f to map some positive integer to 1, then $f(x) = 1 \Rightarrow 5x + 9 = 1 \Rightarrow x = \frac{-8}{5}$. Since the number that f would map to x is not in the domain $\mathbb{Z}^+$, then f is not onto.

7. **Modeste 4 pts**

   For a function to be onto, everything in the codomain must be mapped to by some element in the domain under the function: $\forall c \in \mathbb{Z} \exists (m, n) \in \mathbb{Z} \times \mathbb{Z}, f(m, n) = c$. For any $c \in \mathbb{Z}$, then $(0, c)$ would map to c using the function f: $f(0, c) = 0 + c + 0 * c = 0 + c + 0 = c$. Then every integer has some ordered pair of integers that f maps to it.

   To show that f is not one to one, we need to find two different inputs that map to the same place under f. Consider $(0, 2)$ and $(2, 0)$. $f(0, 2) = 0 + 2 + 0 * 2 = 2 + 0 + 2 * 0 = f(2, 0) = 2$, but $(0, 2) \neq (2, 0)$. Therefore f is not one to one.

8. **Modeste 8 pts**

   a) Let x and y be two elements in the domain of g. Prove that $g(x) = g(y) \Rightarrow x = y$. Have $g(x) = g(y)$, then we can apply the function f to both sides since $g(x)$ and $g(y)$ are elements in the domain of f and f is a total function. So $f(g(x)) = f(g(y))$. Since $f \circ g$ is one to one and $f(g(x)) = f(g(y))$, then $x = y$. Thus $g(x) = g(y) \Rightarrow f(g(x)) = f(g(y)) \Rightarrow x = y$. Therefore g is one to one.

   b) g does not have to be onto. That means we can construct a counterexample where $f \circ g$ is onto but there is some element in B that isn't mapped to by g to disprove the statement. Let $A = \{a\}, B = \{1, 2\}$, and $C = \{c\}$. Have $f(1) = f(2) = c$ and $g(a) =$

1. Then $f \circ g$ is onto since every element in C is mapped to by an element in A using $f \circ g$, namely, $f \circ g(a) = c$. But g is not onto because no element in A maps to $2 \in B$.