# Partial Orders

A relation $R$ on a set $S$ is a partial order if it is reflexive, anti-symmetric and transitive.

eg. $\geq$ is a partial order on $\mathbb{Z}$

$a \geq a$ for all $a \in \mathbb{Z}$, so reflexive

$a \geq b \wedge b \geq a \rightarrow a = b$, so anti-symmetric.

$a \geq b \wedge b \geq c \rightarrow a \geq c$, so transitive.

eg. "divides" (denoted by $|$) is a partial order on $\mathbb{Z}^+$.

$a | a$ for all $a \in \mathbb{Z}^+$, so reflexive

$a | b \wedge b | a \rightarrow a = b$, so anti-symmetric

$a | b \wedge b | c \rightarrow a | c$, so transitive.

eg. $\subseteq$ is a partial order on $\mathcal{P}(\mathbb{N})$

$A \subseteq A$ for all $A \in \mathcal{P}(\mathbb{N})$, so reflexive

$A \subseteq B \wedge B \subseteq A \rightarrow A = B$, so anti-symmetric

$A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$, so transitive.

**Def** A <u>total order</u> $R \subseteq A \times A$ is a partial order where for every $a, b \in A$, either $(a,b) \in R$ or $(b,a) \in R$.

**Def** $a$ and $b$ are <u>incomparable</u> if $(a,b) \notin R$ and $(b,a) \notin R$.

eg. $\leq$ is a total order since $\forall a, b \in \mathbb{Z}$, $a \leq b$ or $b \leq a$.

eg. 'divides' is not a total order

$$
\begin{tikzcd}
 & 2 \arrow[r] \arrow[dr] & 4 \\
1 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & 6 \\
 & 5 &
\end{tikzcd}
$$

<u>Note:</u> 2 and 3 are <u>incomparable</u> since $2 \nmid 3$ and $3 \nmid 2$.

eg. Let $R$ be a relation over $\mathbb{Z} \times \mathbb{Z}$ where $(a,b) \, R \, (c,d)$ iff $a \leq c \wedge b \leq d$.

$R$ is not total since $(1,2)$ and $(2,1)$ are incomparable.

eg. Let $R'$ be a relation over $\mathbb{Z} \times \mathbb{Z}$
where $(a, b) R (c, d)$ iff
$$a \lneq c \lor (a = c \land b \leq d)$$

$R'$ is a total order (lexicographic order)
$(1, 2) R (2, 1)$.

<u>Def</u> A <u>strict partial order</u> is
anti-reflexive and transitive.

<u>Theorem</u> A strict partial order is
anti-symmetric.

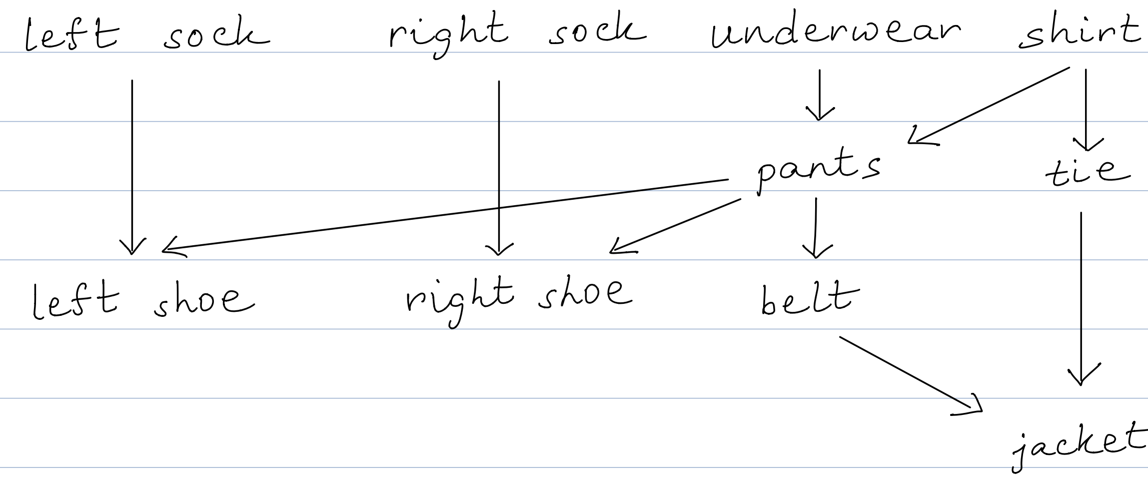
<u>Proof</u>: Suppose $aRb$ and $bRa$. By
transitivity, $aRa$. This contradicts
the fact that $R$ is anti-reflexive.
So $\neg(aRb \land bRa)$. Therefore, $R$ is
anti-symmetric.

QED.

$<$ and $\subsetneq$ are strict partial orders.

A strict partial order can be represented
by a DAG (directed acyclic graph).

eg. a pre-requisite dependancy flowchart eg.




We can solve scheduling problems by relying on a partial order.

## Topological Sorting

come up with a total order of the items that is consistent with the partial order.

① underwear, shirt, pants, left sock, right sock, belt, tie, left shoe, right shoe, jacket.

② left sock, shirt, tie, underwear, right sock, pants, left shoe, belt, jacket, right shoe.