Com S 230 Discrete Computational Structures

Fall Semester 2019

Final Exam

Tuesday, December 17, 2019

Time: 2 hours

Name: ____________________________________

ID: ________________________

The exam is **closed book. No notes or calculators are allowed.** Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. *Always explain your answers!* Good luck and have a great winter break!

Note: For the computational problems, leaving answers in the form of $P(n, r)$ or $C(n, r)$ will be acceptable, in fact preferred! **Just make sure you write the formulas for both on the next page, to show us that you know them!** *Show all intermediate steps.*

1	2	3	4	5	6	7	8	Total
16	14	14	8	18	12	14	24	120

For full credit, write in the formulae:

$P(n, r) =$

$C(n, r) =$

1. Combinatorial Techniques I [**16 Points**]

   You have 5 friends to whom you want to distribute 24 sugar cookies.

   (a) [**4 Pts**] How many ways can you distribute the cookies among your 5 friends?

   (b) [**8 Pts**] What if you need to give (i) at least 5 cookies to Joe and at least 7 cookies to Sam? (ii) at least 5 cookies to Joe and at most 6 cookies to Sam?

   (c) [**4 Pts**] You serve the cookies on a tray and your friends pick out as many as they want until they are all gone. How many cookies would you need to serve to make sure that *at least* one friend gets 6 cookies?

2. Combinatorial Techniques II [**14 Points**]

   You enter a bakery which sells 5 varieties of cookies. You are going to purchase a cookie for each of your 12 closest friends.

   (a) [**6 Pts**] How many ways can you give cookies to your 12 friends, if you want to give *exactly* 4 oatmeal raisin cookies? *exactly* 5 sugar cookies?

   (b) [**4 Pts**] What if you want to give *exactly* 4 oatmeal raisin cookies **and** *exactly* 5 sugar cookies?

   (c) [**4 Pts**] What if you want to give *exactly* 4 oatmeal raisin cookies **or** *exactly* 5 sugar cookies?

3. Combinatorial Techniques III [**14 Points**]

   (a) [**6 Pts**] How many ways can we ship 10 different books in 5 packages with 2 books in each if the packages are going (i) to different addresses? (ii) to the same address?

   (b) [**8 Pts**] How many ways can we pick a committee of four from six CS faculty and four Math faculty if there has to be at least one CS and at least one Math faculty member on the committee? Give two ways to solve the problem.

4. Counting Arguments [**8 Points**]

   Prove that $k(n-k)\binom{n}{k} = n(n-1)\binom{n-2}{k-1}$, where $1 \leq k < n$, using a counting or combinatorial argument.

   Hint: Describe a problem that can be counted in two ways, and the lhs and rhs of the equation gives the two different ways.

5. Graphs and Trees [**18 Points**]

(a) [**6 Pts**] Let G be a simple, undirected graph with 18 vertices. Four vertices have degree 4, three vertices have degree 3, six vertices have degree 2, and the rest have degree 1. How many edges does G have? Justify your answer.

(b) [**6 Pts**] Prove that if G is a tree then G is acyclic but adding any edge $(u, v) \notin E$ creates a cycle.

(c) [**6 Pts**] Prove that if G is acyclic and $|E| = |V| - 1$ then G is a tree. You may assume the following: *If G is a tree, then* $|E| = |V| - 1$.

6. Recursive Definitions and Structural Induction I [**12 Points**]

   Consider this inductive definition for a set of real numbers S.

   Base Case: $1 \in S$.

   Inductive Step: if $x \in S$, then $2x \in S$ and $3x \in S$ and $x/2 \in S$ and $x/3 \in S$.

   Let $A = \{2^k 3^m \mid k, m \in \mathbb{Z}\}$.

   Prove that $S \subseteq A$.

   (a) [**4 Pts**] State what you need to prove for the *base case* and prove it.

   (b) [**8 Pts**] State what you assume and what you need to prove for the *inductive step.* and prove it.

7. Recursive Definitions and Structural Induction II [**14 Points**]

Consider this recursive definition of a *complete ternary tree (CTT)* of height h.

- A *complete ternary tree* of height 0 is a single vertex.
- A *complete ternary tree* of height $h + 1$ is a tree whose root node has **three** children, and each child is the root of a subtree which is a *complete ternary tree* of height h.

(a) [**4 Pts**] Give a recursive definition of $leaves(T)$, the number of leaves in tree T, where T is a CTT, as defined above. *In other words, define* $leaves(T)$ *in terms of* $leaves(T_1)$, $leaves(T_2)$ *and* $leaves(T_3)$, *where* T_1, T_2 *and* T_3 *are the subtrees of* T.

If T is the singleton vertex, $leaves(T) = 1$. *This is the base case of your definition.*

(b) [**10 Pts**] Now prove, by induction on h, that a CTT of height h has 3^h leaves. *Use the recursive definitions above. Justify each step.*

i. [**4 Pts**] First, prove the basis step, where $h = 0$.

ii. [**6 Pts**] Then, prove the inductive step.

8. Countable and Uncountable Sets [**24 Points**]

(a) [**5 Pts**] Prove that if A is countable and $A \cup B$ is uncountable,then B is uncountable.

(b) [**5 Pts**] Show that $\mathbb{Q}^+$, the set of positive rational numbers, is countable.

(c) **[6 Pts]** Prove that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable, using a diagonalization argument.

(d) **[4 Pts]** In this problem, we are going to define a bijection f from $\mathcal{P}(\mathbb{Z}^+)$ to the set of bit strings of infinite length. Let us define (1) $f(\emptyset)$ to be the bit string $000\ldots$, and (2) $f(\mathbb{Z}^+)$ to be the bit string $111\ldots$. What is $f(\{2x \mid x \in \mathbb{Z}^+\})$? In other words, what does f map the set of positive even integers to? Now, define $f(S)$ for all sets $S \subseteq \mathbb{Z}^+$.

(e) **[4 Pts]** Now, argue why this implies that the set of bit strings of infinite length is uncountable.