CS230-HW6Sol

1. **Modeste 14 pts**

   a) $f(x) = 5x$
   Given $x = 1,2,3,..., \in \mathbb{Z}^+$, $f(x) = 5,10,15,\ldots$.

   b) $f(x) = \begin{cases} 5x/2, & \text{if } x \text{ is even} \\ -5(x+1)/2, & \text{if } x \text{ is odd} \end{cases}$
   Given $x = 0,1,2,3,\ldots, \in \mathbb{N}$, $f(x) = 0,-5,5,-10,10,15,-15,\ldots$.

   c) $f : \mathbb{N} \to \{0,1,2,3\} \times \mathbb{N}$
   $f(n) = (n \bmod 4\ , \lfloor \frac{n}{4} \rfloor)$.

2. **Jonathan 14 pts**

   a) This set is countable. To prove that the set is countable, a detailed enumeration needs to be provided for the elements.

	.0	$.\bar{5}$	.5	.55	.555	$\ldots$
0	-	$.\bar{5}$	.5	.55	.555	
5	5	$5.\bar{5}$	5.5	5.55	5.555	
55	55	$55.\bar{5}$	55.5	55.55	55.555	
555	555	$555.\bar{5}$	555.5	555.55	555.555	
$\vdots$						$\ddots$

   Consider the table shown. All elements in the set are accounted for including the case where there are an infinite number of 5's past the decimal point, $.\bar{5}$. The column for an infinite number of 5's need to be included as a column before a finite number of fives so that it will be reached since there is a countably infinite possible number of finite 5's after the decimal point. Then the enumeration using dovetailing becomes $5, .\bar{5}, 55, 5.\bar{5}, .5, 555, 55.\bar{5}, 5.5, .55, \ldots$.

   b) This set is uncountable. This can be proven using diagonalization when considering only those elements that are in $(0,1)$.

   Assume the given set is countable. This means the elements of the set are completely enumerable. The elements can be listed in some order as $s_1, s_2, s_3, \ldots$. Each $s_i = .t_{i1}t_{i2}t_{i3}\ldots$ where $t_{ij} \in \{1,3,5\}$.

	t_{i1}	t_{i2}	t_{i3}	$\ldots$
s_1	t_{11}	t_{12}	t_{13}	
s_2	t_{21}	t_{22}	t_{23}	
s_3	t_{31}	t_{32}	t_{33}	
$\vdots$				$\ddots$

This table should contain all the elements of set and their actual value.

Consider $x = .x_1x_2x_3\ldots$, where $x_i \in \{1,3,5\}$ and $x_i = \begin{cases} 1 & \text{if } t_{ii} = 5 \\ 3 & \text{if } t_{ii} = 1 \\ 5 & \text{if } t_{ii} = 3 \end{cases}$

x is clearly an element of the set since it is only made of 1's, 3's, and 5's. Then x must be equal to some s_i in the table. However, x_i will not equal t_{ii} in each s_i since if $t_{ii} = 1$, then $x_i = 3$, if $t_{ii} = 3$, then $x_i = 5$, and if $t_{ii} = 5$, then $x_i = 1$. Since each t_{ii} will fall under one of those cases, then $x_i \neq t_{ii}$ and $x \neq s_i$. Thus x is not in the table and is not complete, a contradiction. Thus the set is uncountable.

3. **Ling 6 pts** Suppose, for contradiction, the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is countable. Then the functions can be enumerated as $\{f_0, f_1, f_2, ...\}$. We use the table below to show the diagonalization argument.

	0	1	2	$\ldots$
f_0	n_{00}	n_{01}	n_{02}	
f_1	n_{10}	n_{11}	n_{12}	
f_2	n_{20}	n_{21}	n_{22}	
f_3	n_{30}	n_{31}	n_{32}	
$\vdots$				$\ddots$

In the table, $f_0(0) = n_{00}$, $f_0(1) = n_{01}$, $f_1(0) = n_{10}$ and so on, where n_{xy} are all natural numbers. For example, function $f(x) = 2x, x \in \mathbb{N}$ will have a line in the table as $0, 2, 4, 6, ...$ Next, we define a function $g(i) = f_i(i) + 1$ for $i \in \mathbb{N}$. So we have:

$$g \neq f_0 \text{ since } g(0) = f_0(0) + 1 \neq f_0(0)$$
$$g \neq f_1 \text{ since } g(1) = f_1(1) + 1 \neq f_1(1)$$
$$g \neq f_2 \text{ since } g(2) = f_2(2) + 1 \neq f_2(2)$$
$$\vdots$$

Therefore, $g \notin \{f_0, f_1, f_2, ...\}$. But $g(x)$ is clearly a function from $\mathbb{N}$ to $\mathbb{N}$, which means $g \in \{f_0, f_1, f_2, ...\}$. So we have a contradiction. Thus, the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

4. **Ying 6 pts** Suppose there are a countably infinite number of sets. These sets can be ordered $S_1, S_2, S_3, \ldots$ where each S_i is also countably infinite. Then the elements of each set can be ordered $S_i = \{s_{i1}, s_{i2}, s_{i3}, \ldots\}$. These elements can be placed into a table:

	s_{i1}	s_{i2}	s_{i3}	$\cdots$
S_1	s_{11}	s_{12}	s_{13}	
S_2	s_{21}	s_{22}	s_{23}	
S_3	s_{31}	s_{32}	s_{33}	
$\vdots$				$\ddots$

From this table, the elements of each set can be combined using dovetailing with diagonals from the bottom left, so the first diagonal is s_{11}, the second is s_{21}, s_{12}, etc. Then the set that contains all the elements present in the table is equivalent to a countably infinite union of finite sets, which has been proven to be countably infinite.