Binary Relations

Def Let A and B be non-empty sets.
A binary relation R from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$.

$(a, b) \in R$, denoted aRb, or a is related to b.

eg. $S = \{$set of college students$\}$
$C = \{$set of courses offered this semester$\}$

$R = \{ (x, y) \mid$ student x takes course y this semester, $x \in S$, $y \in C \}$

For $x \in S$, $\{ y \mid (x, y) \in R \}$
is the set of courses taken by student x

For, $y \in C$, $\{ x \mid (x, y) \in R \}$
is the set of students taking course y

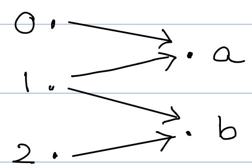
Note: A relation is a generalization of a function.

given $f: A \to B$,

$$R_f = \{ (x,y) \mid x \in A,\ y \in B,\ y = f(x) \}$$

$$= \{ (x, f(x)) \mid x \in A \}$$

A relation can be described by a graph.



$$R = \{ (0,a), (1,a), (1,b), (2,b) \}$$

<u>Def</u> A <u>relation on set S</u> is a binary relation $R \subseteq S \times S$.

eg. $A = \{1,2,3,4\}$

$$R = \{ (a,b) \mid a \text{ divides } b,\ a,b \in A \}$$

So, $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}$

Relations on $\mathbb{Z}$

$$\left.\begin{aligned}
R_1 &= \{ (a,b) \mid a \le b \} \\
R_2 &= \{ (a,b) \mid a \ge b \} \\
R_3 &= \{ (a,b) \mid a = b+1 \} \\
R_4 &= \{ (a,b) \mid a+b \le 5 \} \\
R_5 &= \{ (a,b) \mid a \le 3,\ b \le 3 \}
\end{aligned}\right\} \text{infinite sets}$$

Which ones contain

$(1,1),\ (1,2),\ (-1,1),\ (2,3)$?

R_1 R_2 R_3 R_4 R_5

How many relations are there on a set A of n elements?

Each element of $\mathcal{P}(A \times A)$ is a relation.

a counting problem:

$$|A| = n, \quad |A \times A| = n^2, \quad |\mathcal{P}(A \times A)| = 2^{n^2}$$

Properties of Relations on set A

Def A relation R is reflexive if $(a,a) \in R$ for all $a \in A$.

Let $A = \{1,2,3,4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,2), (3,1), (3,3), (4,4)\}$

R_2 is reflexive, R_1 is not.

Below, we consider relations on set $\mathbb{Z}$,

$R_{div} = \{ (a,b) \mid a \text{ divides } b, \quad a,b \in \mathbb{Z} \}$

reflexive : $\leq$, $\geq$, $=$, R_{div} $\qquad a \leq a$

not reflexive : $<$, $>$, $\neq$, $\emptyset$ $\qquad a \nless a$

<u>Def</u> A relation R is <u>anti-reflexive</u> if $(a,a) \notin R$ for all $a \in A$.

anti-reflexive : $<$, $>$, $\emptyset$, $\neq$

not anti-reflexive : $\leq$, $\geq$, $=$, R_{div}

$R = \{ (1,1), (2,2) \}$ - neither reflexive nor anti-reflexive

<u>Def</u> A relation R is <u>symmetric</u> if $(a,b) \in R$ implies $(b,a) \in R$ for all $a, b \in A$.

$$\forall a \in A, \forall b \in A \left[(a,b) \in R \rightarrow (b,a) \in R \right]$$

symmetric : $=$, $\neq$ $\qquad a \neq b \rightarrow b \neq a$

not symmetric : $<$, $\leq$, $>$, $\geq$, R_{div}

Def A relation R is <u>anti-symmetric</u> if $(a,b) \in R$ and $(b,a) \in R$ only if $a = b$, for all $a, b \in A$.

$$\forall a \in A, \forall b \in A \left[(a,b) \in R \wedge (b,a) \in R \rightarrow a = b \right]$$

anti-symmetric: $<$, $\leq$, $>$, $\geq$, $=$, R_{div}

not anti-symmetric: $\neq$

$\{(1,1), (2,2), (3,3)\}$ - symmetric
anti-symmetric

$\{(1,2), (2,1), (2,3)\}$

$(2,3) \in R \wedge (3,2) \notin R \rightarrow$ not symmetric

$(1,2), (2,1) \in R \wedge 1 \neq 2 \rightarrow$ not anti-symmetric

Def A relation R is <u>transitive</u> if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, for all $a, b, c \in A$.

$$\forall a, b, c \in A, \ (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$$

$R = \{(1,1), (1,2), (2,3), (1,3), (2,1)\}$

Is it transitive? No. since $(2,2) \notin R$

transitive: $=$, $<$, $>$, $\leq$, $\geq$, R_{div}

$a \leq b \wedge b \leq c \rightarrow a \leq c$ yes

not transitive: $\neq$

$1 \neq 2 \wedge 2 \neq 1 \rightarrow 1 \neq 1$ no

Note: $\emptyset$ is anti-reflexive, symmetric, anti-symmetric, transitive!

$\mathbb{Z} \times \mathbb{Z}$ is reflexive, symmetric, transitive

Relation	R	AR	S	AS	T
$<$		×		×	×
$\leq$	×			×	×
$=$	×		×	×	×
$>$		×		×	×
$\geq$	×			×	×
$\neq$		×	×		
R_{div}	×			×	×
$\emptyset$		×	×	×	×
$\mathbb{Z} \times \mathbb{Z}$	×		×		×

Relations on set $\mathbb{Z}$