Com S 230 Discrete Computational Structures

Spring Semester 2021

Final Exam

Tuesday, May 4, 2021

Time: 2 hours

Name: ____________________________________

The exam is **open book, open notes, open Canvas course notes. No calculators! No other materials are allowed, including browsing on internet. Your signature on Canvas pledges that you have not violated these conditions.**

Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. Try to stay as brief and to the point as possible. Good luck!

Note: For the computational problems, leaving answers in the form of $P(n, r)$ or $C(n, r)$ is not acceptable. You may leave your answer in factorial, product (or sum) form, however, without multiplying (or adding) all the terms out. *Show all intermediate steps.*

1	2	3	4	5	6	7	8	Total
14	16	18	16	8	18	12	18	120

1. Short Answers [**14 Points**]

   (a) [**6 Pts**] Prove that if A and B are countably infinite, then $A \times B$ is countably infinite.

   (b) [**8 Pts**] Define the relation $\approx$ on the set $\mathbb{Z} \times \mathbb{Z}$, where $(a, b) \approx (c, d)$ if and only if $a + b = c + d$. Define [(0,0)] and [(1,5)] using enumeration or set-builder notation. Now, describe the different equivalence classes. Are the number of equivalence classes finite, countably infinite or uncountably infinite? Is each equivalence class finite, countably infinite or uncountably infinite?

2. Counting Functions and Relations [**16 Points**]

   Let S and T be sets where $|S| = 5$, $|T| = 6$. *Show your work.*

   (a) [**4 Pts**] How many different one-to-one functions are possible from S to T?

   (b) [**4 Pts**] How many different onto functions are possible from T to S?

   (c) [**4 Pts**] How many different binary relations are possible on T?

   (d) [**4 Pts**] How many equivalence relations on set T are possible where each equivalence class is of size 2?

3. Combinatorial Techniques I [**18 Points**]

   A group of 20 freshmen are choosing their major between Math, Computer Science, Computer Engineering, Software Engineering and EE.

   (a) [**6 Pts**] How many ways can the freshmen pick majors so that there are (i) exactly 6 Math majors? (ii) exactly 4 CS majors?

   (b) [**4 Pts**] How many ways can the freshmen pick majors so that there are exactly 6 Math majors and exactly 4 CS majors?

   (c) [**4 Pts**] How many ways can the freshmen pick majors so that there are 6 Math majors, 4 CS majors, 3 SE majors and 7 EE majors?

   (d) [**4 Pts**] How many ways can the freshmen pick majors so that there are exactly 6 Math majors **or** exactly 4 CS majors?

4. Combinatorial Techniques II [**16 Points**]

   We have bought 30 new computers to put into the 6 CS research labs.

   (a) [**4 Pts**] How many ways can we distribute the computers into the labs?

   (b) [**8 Pts**] What if you need to place (i) *at least* 6 computers in the software lab and *at least* 8 computers in the AI lab? (ii) *at least* 6 computers in the software lab and *at most* 7 computers in the AI lab?

   (c) [**4 Pts**] How many computers would we need to buy to make sure that *at least* one lab gets *at least* 8 computers no matter how they are distributed?

5. Counting Arguments [**8 Points**]

   Give a counting argument to prove the following:

$$P(30,3)C(27,5) = C(30,8)P(8,3) = C(30,5)P(25,3)$$

   Hint: Come up with a problem of picking a committee with officers from a group of size 30 in three different ways.

6. Graph Properties [**18 Points**]

(a) [**6 Pts**] Let G be a simple, undirected graph with 20 edges. Four vertices have degree 4, three vertices have degree 3, four vertices have degree 2, and the rest have degree 1. How many vertices in G have degree 1? Justify your answer.

(b) [**6 Pts**] Let G be a forest with 24 vertices and 5 connected components. How many edges does it have? Justify your answer.

(c) [**6 Pts**] Prove that *if* G is connected and $|E| = |V| - 1$ then G is a tree. You may assume the following: *If* G *is a tree, then* $|E| = |V| - 1$.

7. Inductive Definitions and Structural Induction [**12 Points**]

   Consider the recursive definition of a *full binary tree (FBT)*.

   Basis Step: A single vertex is an FBT.

   Recursive Step: If T_1 and T_2 are both FBTs, then the tree T is an FBT, where T consists of a root node, a left subtree T_1 and a right subtree T_2.

   (a) [**4 Pts**] Give a recursive definition of $leaves(T)$, the number of leaves in tree T, and $internal(T)$, the number of internal nodes (all the nodes that are not leaves) in tree T, where T is a FBT.

   The basis steps are done for you. Give the recursive steps.

   Basis Steps: If T is a single vertex, $leaves(T) = 1$ and $internal(T) = 0$.

   (b) [**8 Pts**] Now prove, by structural induction, that for any FBT T, $leaves(T) = internal(T) + 1$, using the recursive definitions above. First, prove the basis step. Then, prove the inductive step.

8. State Machines and Preserved Invariants [**18 Points**]

A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+0,+3), (+1,+2), (+2,+1). Let S be the set of points that the robot can get to, where each state is an ordered pair.

(a) [**4 Pts**] Give an an inductive definition of S. The basis step is $(0, 0) \in S$.

(b) [**8 Pts**] Prove, by structural induction, that if $(a, b) \in S$, then $a, b \in \mathbb{N}$ and $a + b$ is divisible by 3. This is the Preserved Invariant.

(c) [**6 Pts**] Consider the converse: if $a, b \in \mathbb{N}$, and $a + b$ is divisible by 3, then $(a, b) \in S$. Show, by counter-example, that this is not true. Modify the inductive definition of S to make the converse true. You do not need to prove your construction.