Generalized Permutations & Combinations

I Circular Permutations

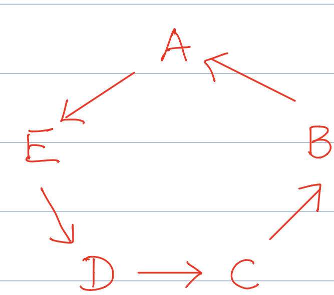
Earlier, we have looked at linear permutations, where we order objects in a line.
Now, we look at circular permutations, where we place objects in a circle.

eg. Suppose there are 5 people A, B, C, D, E.
How many ways can we line them up?
$P(5,5) = 5! = 120$

What if we make them march in order in a circle?
So, if the queue was A B C D E with A at the beginning and E at the end.
A now follows E around the circle,
i.e. we no longer distinguish the head of queue!

so, ABCDE
BCDEA
CDEAB
DEABC
EABCD

all correspond to $\longrightarrow$



How many circular permutations are there?

For each circular permutation, there are 5 linear permutations, so

$$\frac{P(5,5)}{5} = \frac{5!}{5} = 4! = 4 \cdot 3 \cdot 2 = 24$$

<u>Theorem</u> The no. of circular r-permutations of a set of size n is

$$\frac{P(n,r)}{r} = \frac{n!}{r\,(n-r)!}$$

<u>Proof</u>: Each circular r-permutation generates r linear r-permutations since there are r ways to choose the beginning of the list.

So, r (# circular r-permutations) = $P(n,r)$

$\Rightarrow$ # circular r-permutations $= \dfrac{P(n,r)}{r}$

QED.

This is an example of the Division Rule.

Division Rule

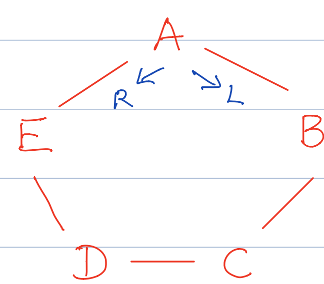
If each way to do procedure 1 generates k ways to do procedure 2, then

$$\# \text{ ways to do proc } 1 = \frac{\# \text{ ways to do proc } 2}{k}$$

Note: We used the Division Rule to count the no. of r- combinations of a set of size n.

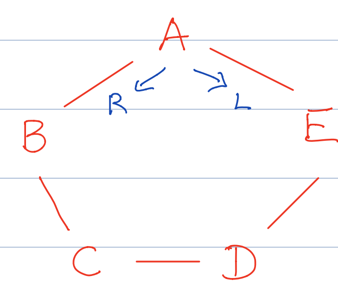
$$\# \text{ r-combinations} = \frac{\# \text{ r-permutations}}{r!}$$

Looking back at circular permutations, note that we do distinguish between



(1)

and



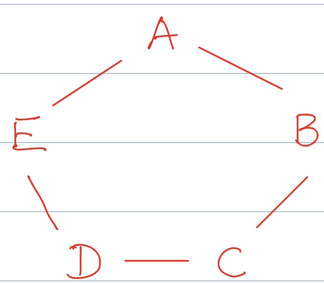
(2)

Suppose these are 5 people sitting around a table. In (1), B is to left of A and E is to right. In (2), E is to left of of A and B is to right.

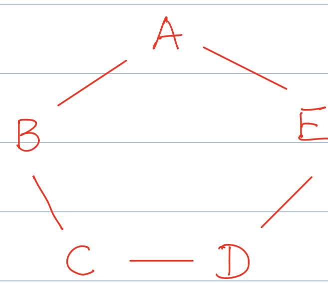
How many ways can we string a necklace of 5 beads from 10 beads of different colors?

$$\frac{P(10,5)}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5} = 6048 \text{ circular 5-permutations}$$

But, here the necklaces



and



are the same since we can turn one over to get the other!

So, no. of necklaces $= \dfrac{\#\text{ circular perms}}{2}$

$$= \frac{6048}{2} = 3024.$$

eg. no. of simple cycles of size 4 in a complete graph of size 10

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 2} = 630$$

II Permutations with Repetitions

When repetitions are allowed in the permutation, this becomes an easy product rule problem!

eg. How many words of length n possible using the English alphabet?

$$\underbrace{26 \times 26 \times \cdots \times 26}_{n} = 26^n$$

26 choices for each letter

eg. How many ways can we pick oranges, apples and pears so that we can pick one fruit for Monday, for Tuesday and for Wednesday, and we can pick the same fruit multiple times.

$$\underset{M}{\underset{\uparrow}{3}} \times \underset{T}{\underset{\uparrow}{3}} \times \underset{W}{\underset{\uparrow}{3}} = 27$$

3 choices each day

Theorem The no. of r-permutations of a set of n objects, with repetition allowed, is n^r.