CS230-HW5Sol

1. **Ying 10 pts**

   a) Reflexive: Let $x \in \mathbb{R}$. Then $xx \geq 0$ is true, so $(x, x) \in R_1$.

   Not Anti-reflexive: Consider $0 \in \mathbb{R}$, then $0 * 0 \geq 0$. So $(0,0) \in R_1$ and R_1 is not anti-reflexive.

   Symmetric: Let $(x, y) \in R_1$. Since $(x, y) \in R_1$, then $xy \geq 0 \Rightarrow yx \geq 0$ and $(y, x) \in R_1$.

   Not Anti-symmetric: Consider $(1,2), (2,1) \in R_1$. Then $2 * 1 \geq 0$ and $1 * 2 \geq 0$, but $1 \neq 2$ and R_1 is not anti-symmetric.

   Not Transitive: Consider $x = 1, y = 0, z = -1$. Then $(1,0), (0,-1) \in R_1$, but $1 * -1 < 0$ and $(1,-1) \notin R_1$. Thus R_1 is not transitive.

   b) Not Reflexive: Consider $1 \in \mathbb{R}$. Then $1 \neq 2 * 1$, so $(1,1) \notin R_2$. Thus R_2 is not reflexive.

   Not anti-reflexive: Consider $0 \in \mathbb{R}$. Then $0 = 2 * 0 = 0$, so $(0,0) \in R_2$. Thus R_2 is not anti-reflexive.

   Not Symmetric: Consider $(2,1) \in R_2$. Then $x = 2, y = 1$, and $x = 2y$, but $y \neq 2x \Rightarrow 1 = 2 * 2$. So $(y, x) \notin R_2$ and R_2 is not symmetric.

   Anti-symmetric: Have $(x, y) \in R_2$ and $(y, x) \in R_2$. Then $x = 2y$ and $y = 2x$. Using substitution, then $x = 4x$. This is only possible if $x = 0 = y$. If $x \neq y$, then either $x = 2y$ or $y = 2x$ must be false, and the the implication is therefore true.

   Not Transitive: Consider $(4,2), (2,1) \in R_2$. Then $4 = 2 * 2$ and $2 = 2 * 1$, but $4 \neq 2 * 1$, so $(4,1) \notin R_2$. Thus R_2 is not transitive.

2. **Ling 8 pts**

   a) Reflexive: Let $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^2$. Then $ab = ab$, so $((a,b),(a,b)) \in R_3$.

   Symmetric: Let $((a,b),(c,d)) \in R_3$. Then $ad = bc \Rightarrow bc = ad \Rightarrow cb = da$. So $((c,d),(a,b)) \in R_3$.

   Transitive: Let $((a,b),(c,d)),((c,d),(e,f)) \in R_3$. Then $ad = bc$ and $cf = de$. These equalities can be rewritten as $a/b = c/d$ and $c/d = e/f$, respectively. So $a/b = e/f \Rightarrow af = be$ and $((a,b),(e,f)) \in R_3$.

   b) Consider $f(x, y) = x/y$. Then $f(a,b) = f(c,d)$ iff $a/b = c/d$ iff $ad = bc$ iff $((a,b),(c,d)) \in R_3$.

   c) For $(x, y) \in [(1,1)]$, then $((1,1),(x,y)) \in R_3$. Then $1y = 1x \Rightarrow y = x$. Thus $[(1,1)] = \{(x,x) | x \in \mathbb{Z}^+\}$.

   d) As shown in part (b), $((a,b),(c,d)) \in R_5$ iff $f(a,b) = f(c,d)$. This implies that there exists an equivalence class for each distinct value of $f(x, y)$, where $x, y \in \mathbb{Z}^+$. Since the range of f is $\mathbb{Q}^+$, for each $q \in \mathbb{Q}^+$, we have a unique class $\{(x,y) | f(x,y) = q, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\} = \{(x,y) | x/y = q, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$.
   We can also describe the classes as follows. By definition of rationals, for each

$q \in \mathbb{Q}^+$, there exist $m, n \in \mathbb{Z}^+$ such that $q = m/n$, where m and n have no common factors. So, for each such pair m, n there is a unique equivalence class $[m, n] = \{(am, an) | a \in \mathbb{Z}^+\}$.
The number of equivalence classes is therefore countably infinite, and each class contains a countably infinite number of elements.

3. **Ying 8 pts**

   a) R_4 is an equivalence relation, which makes it reflexive, symmetric, and transitive.

   Reflexive: Let a be a five digit number. The first two digits of a are the same as the first two digits of a, so $(a, a) \in R_4$.

   Symmetric: Let $(a, b) \in R_4$. Then the first two digits of a are the same as the first two digits of b. So the first two digits of b are the same as the first two digits of b. Thus $(b, a) \in R_4$.

   Transitive: Let $(a, b), (b, c) \in R_4$. Have a_1 be the first digit of a, a_2 be the second digit of a, and similar for b and c. Since $(a, b) \in R_4$, then $a_1 = b_1$ and $a_2 = b_2$. Since $(b, c) \in R_4$, then $b_1 = c_1$ and $b_2 = c_2$. Then $a_1 = c_1$ and $a_2 = c_2$, so $(a, c) \in R_4$.

   Equivalence classes: $[a] = \{a_1 a_2 d_3 d_4 d_5 | d_i \in \{0, 1, \dots, 9\}\}$. The equivalence class for a five digit number a is the set of all five digits numbers that have the same first two digits $a_1 a_2$ as a and the last three are any possible digit.

   Refinement: Consider $(a, b) \in S$ iff the first three digits of a and b are the same. S is clearly an equivalence relation following similar logic to R_4. For the partitions of defined by S to be a refinement of the partitions of R_4, each partition of S is a subset of a partition of R_4. $[a]_S = \{a_1 a_2 a_3 d_4 d_5 | d_i \in \{0, 1, \dots, 9\}\} \subseteq [a]_{R_4}$.

   b) R_5 is not an equivalence relation because R_5 is not transitive. Consider 12345, 16789, and 56789. $(12345, 16789) \in R_5$ since the two numbers have the same first digit. $(16789, 56789) \in R_5$ since the two numbers have the second digit in common. But $(12345, 56789) \notin R_5$ since neither have the same digit in the same place. Thus R_5 is not transitive.

4. **Jonathan 12 pts**

   a) Reflexive: $(f, f) \in R_6$ since $f(0) = f(0)$ and $f(1) = f(1)$.

   Symmetric: Let $(f, g) \in R_6$. Then $f(0) = g(0)$ and $g(1) = f(1)$. So $g(0) = f(0)$ and $g(1) = f(1)$. Thus $(g, f) \in R_6$.

   Transitive: Let $(f, g), (g, h) \in R_6$. Then $f(0) = g(0), f(1) = g(1), g(0) = h(0)$, and $g(1) = h(1)$. So $f(0) = h(0)$ and $f(1) = h(1)$. Thus $(f, h) \in R_6$.

   Equivalence Classes: $[f] = \{g | g(0) = x, g(1) = y, x, y \in \mathbb{Z}^+\}$, where $f(0) = x$ and $f(1) = y$. Then each $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ gives rise to a distinct equivalence class.

   b) Reflexive: $(f, f) \in R_7$ since $f(x) - f(x) = 0$ and $0 \in \mathbb{Z}$.

   Symmetric: Let $(f, g) \in R_7$. Then $f(x) - g(x) = C$, for some $C \in \mathbb{Z}$ and any $x \in \mathbb{Z}$. So $g(x) - f(x) = -C$ for all $x \in \mathbb{Z}$ and $-C \in \mathbb{Z}$. Thus $(g, f) \in R_7$.

Transitive: Let $(f,g),(g,h) \in R_7$. Then $f(x)-g(x) = C$ and $g(x)-h(x) = D$ for some $C,D \in \mathbb{Z}$. These two equations can be added together to give $f(x)-g(x)+g(x)-h(x) = C+D \Rightarrow f(x)-h(x) = C+D$ and $C+D \in \mathbb{Z}$. Thus $(f,h) \in R_7$.

Equivalence Classes: $[f] = \{f(x)+C | C \in \mathbb{Z}\}$

5. **Modeste 12 pts**

a) Reflexive: Have $x \in \mathbb{R}^+$. Then $x/x = 1$ and $1 \in \mathbb{Z}$. Thus R_8 is reflexive.

Anti-symmetric: Let $(x,y),(y,x) \in R_8$. Then $x/y = a$ and $y/x = b$ for some $a,b \in \mathbb{Z}$. So $y = xb \Rightarrow x/(xb) = a \Rightarrow 1/b = a$. The only numbers a and b where this is possible is if $a = b = 1$. So $x/y = 1 \Rightarrow x = y$. Thus R_8 is anti-symmetric.

Transitive: Let $(x,y),(y,z) \in R_8$. Then $x/y = a$ and $y/z = b$ for some $a,b \in \mathbb{Z}$. So $y = bz \Rightarrow x/(bz) = a \Rightarrow x/z = ab$ and $ab \in \mathbb{Z}$. Thus $(x,z) \in R_8$ and R_8 is transitive.

b) Reflexive: Have $x \in \mathbb{R}^+$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus R_9 is reflexive.

Symmetric: Let $(x,y) \in R_9$. Then $x - y = a$ for some $a \in \mathbb{Z}$. So $y - x = -a$ with $-a \in \mathbb{Z}$ and $(y,x) \in R_9$. Thus R_9 is symmetric.

Transitive: Let $(x,y),(y,z) \in R_9$. Then $x - y = a$ and $y - z = b$ for some $a,b \in \mathbb{Z}$. So $y = b + z \Rightarrow x - (b+z) = a \Rightarrow x - z = a + b$ and $a + b \in \mathbb{Z}$. Thus $(x,z) \in R_9$ and R_9 is transitive.

Equivalence Classes: R_9 is an equivalence relation. Then $[2] = \mathbb{Z}$ since any integer subtracted from 2 will be an integer. $[\pi] = \{\pi + k | k \in \mathbb{Z}\}$. For any real number, $[a] = \{a + k | k \in \mathbb{Z}\}$