Com S 230 : Discrete Computational Structures

Spring Semester, 2021

Exam 2

Monday, April 12, 2021

Time: 90 minutes

Name: ______________________________

The exam is **open book, open notes, open Canvas course notes. No other materials are allowed, including browsing on internet. You may not collaborate with others! Your signature on Canvas pledges that you have not violated these conditions.**

Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. Try to stay as brief and to the point as possible. Good luck!

The exam is set for 90 minutes but you are given two hours to complete it, including the time you need to upload your exam to Canvas. Please give yourself sufficient time!

1	2	3	4	5	6	Total
24	22	18	16	16	24	100

1. Short Answers [**24 Points**]

   (a) [**12 Pts**] Is the relation $\{(a, b) \mid a/b \geq 2\}$ over the set of positive reals (i) reflexive, (ii) anti-reflexive, (iii) symmetric, (iv) anti-symmetric, and (v) transitive? Is it (i) an equivalence relation, (ii) a partial order, or (iii) a strict partial order? Justify your answers.

   (b) [**6 Pts**] Give an *inductive definition* for a_i of the sequence $2, -6, 18, -54, 162, \ldots$, where a_0 is the first term.

   (c) [**6 Pts**] Define the set $S = \{5^k 7^m \mid k, m \in \mathcal{Z}\}$ inductively.

2. Equivalence Relations [**22 Points**]

Consider the following relation R on $\mathbb{R}$:

$$(a, b) \in R \text{ if and only if } a = b + 3n \text{ for some integer } n.$$

(a) [**8 pts**] Prove that R is an equivalence relation.

(b) [**6 pts**] Define $[0]$ and $[1]$ by enumeration and by set builder notation.

(c) [**8 pts**] Describe the equivalence classes formally. In other words, for each $x \in \mathbb{R}$, define $[x]$ using set-builder notation. Now, define an interval $[0, y)$ where $y \in \mathbb{R}^+$ such that $[x]$ is distinct for each $x \in [0, y)$. Is each equivalence class countable or uncountable? Are the number of equivalence classes countable or uncountable?

3. Countable and Uncountable Sets [**18 Points**]

(a) [**6 pts**] If A is countable and $A \cup B$ is uncountable, prove that B is uncountable.

(b) [**6 pts**] Prove that $\mathbf{Z} \times \mathbf{Z}$ is countable.

(c) [**6 pts**] Prove that the set S of all infinite length binary strings is uncountable as follows. Suppose, for contradiction, S is countable. Then, $S = \{s_1, s_2, s_3, \ldots\}$ where $s_i = s_{i1}s_{i2}\ldots$. Now, define infinite binary string $r = r_1r_2\ldots$ such that $r \notin S$, giving a contradiction.

4. Mathematical Induction [**16 Points**]

Consider the statement

$$1^3 + \cdots + n^3 = n^2(n+1)^2/4$$

Prove this statement, by induction, for all positive integers n.

(a) [**4 pts**] State the *base case* and prove it.

(b) [**4 pts**] State the *inductive hypothesis* (the assumption), and the statement you need to prove.

(c) [**8 pts**] Prove your statement.

5. Strong Induction [**16 Points**]

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 5-cent stamps. Prove that $P(n)$ is true for all $n \geq 8$, using the steps below.

(a) Prove $P(n)$ for all $n \geq 8$ by *regular* induction.

i. [**2 Pts**] Prove $P(8)$ to complete the basis step.

ii. [**6 Pts**] State clearly what you need to prove for the inductive step and prove it.

(b) Now, prove $P(n)$ for all $n \geq 8$ by *strong* induction.

i. [**4 Pts**] Prove the basis step. How many base cases do you have?

ii. [**4 Pts**] State clearly what you need to prove for the inductive step and prove it.

6. Structural Induction [**24 Points**]

Consider this inductive definition for a set of strings S over the alphabet $\{a, b\}$.

Base Case: $\epsilon \in S$.

Inductive Case: if x is in S, then $axbb$ is in S.

Let $A = \{a^i b^{2i} \mid i \in \mathbf{N}\}$. We prove that $S = A$.

Note: a^i is short form for a string of i a's. So, abb is also written as ab^2, and $aabbbb$ is also written as a^2b^4. Both abb and $aabbbb$ are in A but aab is not.

(a) [**12 Pts**] Prove that $A \subseteq S$, *using mathematical induction.* In other words, prove that if $x = a^i b^{2i}$ for some natural number i, then $x \in S$.

i. [**4 Pts**] State what you prove for the *base case* and prove it.

ii. [**8 Pts**] State what you prove for the *inductive step* and prove it.

(b) **[12 Pts]** Prove that $S \subseteq A$, *using structural induction*. In other words, prove that if $x \in S$, then $x = a^i b^{2i}$ for some natural number i.

i. **[4 Pts]** State what you prove for the *base case* and prove it.

ii. **[8 Pts]** State what you prove for the *inductive step* and prove it.