CS230-HW7Sol

1. **Jonathan 5 pts**

   Base Case: Let $n = 1$. Then $1 + 3 = 4 < 5 = 5 * 1^2$.

   Induction Step: Assume $k + 3 < 5k^2$ for some $k \geq 1$ (Induction Hypothesis/IH).

   Prove $(k + 1) + 3 < 5(k + 1)^2$.

$$\begin{aligned}
(k+1)+3 &= k+3+1 & \\
&< 5k^2+1 & \text{by IH} \\
&< 5k^2+10k+5 & 10k+5>1 \\
&= 5(k^2+2k+1) & \\
&= 5(k+1)^2 &
\end{aligned}$$

   Because there are inequalities used, then $(k+1)+3 < 5(k+1)^2$. Since both the base case and induction step are true, then $k+3 < 5k^2$ for $k \geq 1$.

2. **Modeste 5 pts**

   Basis: $n = 1$, $1 * 1! = 1 = 2 - 1 = 2! - 1$

   Induction Step: Assume for $n = k$ the equality holds:

   $1 * 1! + 2 * 2! + \ldots + k * k! = (k + 1)! - 1$

   Prove $1 * 1! + 2 * 2! + \ldots + k * k! + (k + 1)(k + 1)! = (k + 2)! - 1$

$$\begin{aligned}
1*1!+2*2!+\ldots+k*k!+(k+1)(k+1)! &= (k+1)!-1+(k+1)(k+1)! & \text{by IH} \\
&= (k+1)!(k+1+1)-1 & \text{factor out } (k+1)! \\
&= (k+2)!-1 & x!(x+1)=(x+1)!
\end{aligned}$$

3. **Ying 5 pts**

   Base: $n = 1$, $\frac{1}{1*2} = \frac{1}{2}$

   Induction Step: Assume for $n = k$ the equality holds (IH): $\frac{1}{1*2} + \frac{1}{2*3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

   Prove for $n = k + 1$: $\frac{1}{1*2} + \frac{1}{2*3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$

$$\begin{aligned}\frac{1}{1*2}+\frac{1}{2*3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+1+1)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \text{by IH}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+1+1}\end{aligned}$$

Since the base case for $n = 1$ and the induction step are both true, then the equality is true $\forall n \in \mathbb{Z}^+$

4. **Ying 5 pts**

   Base: $n = 0$, $4^{2*0} - 1 = 1 - 1 = 0$. Since 0 is divisible by any number, then the statement is true.

   Induction Step: Assume the statement is true for $n = k$: 15 divides $4^{2k} - 1$, or equivalently $4^{2k} - 1 = 15m \equiv 4^{2k} = 15m + 1$ for some $m \in \mathbb{N}$.

   Prove $4^{2(k+1)} - 1$ is a multiple of 15.

$$\begin{aligned}4^{2(k+1)} - 1 &= 4^{2k+2} - 1\\ &= 4^2 * 4^{2k} - 1 && x^{y+z} = x^y * x^z\\ &= 16(15m+1) - 1 && \text{by IH}\\ &= 16 * 15m + 16 - 1\\ &= 16 * 15m + 15\\ &= 15(16m+1) && \text{factor 15 from both}\end{aligned}$$

   Since $16m + 1 \in \mathbb{N}$, then $4^{2(k+1)} - 1$ is a multiple of 15. Therefore $\forall n \in \mathbb{N}$, 15 divides $4^{2n} - 1$.

5. **Ling 9 pts**

   a) Base case: 18¢ = 7¢ + 7¢ + 4¢

      Induction step: assume postage of k¢ can be formed using 4¢ and 7¢ for $k \geq 18$.

      We prove that $(k+1)$¢ can be 4¢ and 7¢.

      Case 1: Suppose at least one 7 ¢ stamp was used to form k¢ postage. Then, replace a 7¢ stamp with two 4¢ stamps to obtain $(k+1)$¢ postage: $k - 7 + 4 + 4 = k + 1$¢.

Case 2: Suppose no 7 ¢ stamp was used to form k¢ postage. So only 4¢ stamps were used, implying k is a multiple of 4. Since $k \geq 18$, it follows that $k \geq 20$ (the smallest multiple of 4 greater than 18). So at least five 4¢ stamps were used. Replace five 4¢ stamps with three 7¢ stamps to get $(k+1)$¢, since $k-5\times 4+3\times 7 = k+1$.

b) Base case:

$$\begin{aligned} 18¢ &= 7¢ + 7¢ + 4¢ \\ 19¢ &= 7¢ + 4¢ + 4¢ + 4¢ \\ 20¢ &= 4¢ + 4¢ + 4¢ + 4¢ + 4¢ \\ 21¢s &= 7¢ + 7¢ + 7¢ \end{aligned}$$

Induction step: Let $k \geq 21$, we assume that for all l where $18 \leq l \leq k$, postage of l¢ can be formed using 4¢ and 7¢.

We prove that $(k+1)$¢ postage can be formed.

Since $k \geq 21$, therefore $k-3 \geq 18$, so $(k-3)$¢ postage is possible (by IH).

Add a 4¢ stamp to $(k-3)$¢ postage to form $(k+1)$¢ postage, since $(k-3)+4 = k+1$.

6. **Jonathan 6 pts**

Base Case: Let $n = 1$.

The left hand side of the equality becomes $\overline{\left(\bigcap_{i=1}^{1} A_i\right)} = \overline{A_1}$.

The right hand side of the equality becomes $\bigcup_{i=1}^{1} \overline{A_i} = \overline{A_1}$.

The equality for the base case holds.

Induction Step: Assume $\overline{\left(\bigcap_{i=1}^{k} A_i\right)} = \bigcup_{i=1}^{k} \overline{A_i}$ is true for $k \in \mathbb{Z}^+$ (Induction Hypothesis).

Prove $\overline{\left(\bigcap_{i=1}^{k+1} A_i\right)} = \bigcup_{i=1}^{k+1} \overline{A_i}$

$$\begin{aligned} \overline{\left(\bigcap_{i=1}^{k+1} A_i\right)} &= \overline{\left(\bigcap_{i=1}^{k} A_i \cap A_{k+1}\right)} && \text{pull } k+1 \text{ term out of } \cap \\ &= \overline{\left(\bigcap_{i=1}^{k} A_i\right)} \cup \overline{A_{k+1}} && \text{DeMorgan's of two sets} \\ &= \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} && \text{by IH} \\ &= \bigcup_{i=1}^{k+1} \overline{A_i} && \text{combine terms} \end{aligned}$$