CS230-HW8Sol

1. **Modeste 8 pts**

   Basis Step: $n = 1$, P(1): $f_0 - f_1 + f_2 = f_{2-1} - 1$. Now, $f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$, and $f_{2-1} - 1 = f_1 - 1 = 1 - 1 = 0$

   Inductive Step: Assume P(k): $f_0 - f_1 + f_2 - \ldots - f_{2k-1} + f_{2k} = f_{2k-1} - 1$

   Prove P(k+1): $f_0 - f_1 + f_2 - \ldots - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2k+2} = f_{2(k+1)-1} - 1$

$$\begin{aligned}
f_0 - f_1 + f_2 - \ldots - f_{2k-1} + f_{2k}) - f_{2k+1} + f_{2k+2} &= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2} && \text{IH} \\
&= f_{2k-1} - 1 - f_{2k+1} + f_{2k} + f_{2k+1} \\
&= f_{2k-1} - 1 + f_{2k} \\
&= f_{2k-1} + f_{2k} - 1 \\
&= f_{2k+1} - 1 && \text{def of } f \\
&= f_{2(k+1)-1} - 1
\end{aligned}$$

2. **Modeste 12 pts**

   a) We prove that for all $i \in \mathbb{Z}^+, 4^i \in S$.
   Base case: $i = 1$. We prove that $4^1 \in S$.
   Proof: $4^1 = 4$ and $4 \in S$ by the base case of the recursive definition of S. Therefore, for $i = 1$, $4^i \in S$.
   Inductive step: Assume that $4^k \in S$ for some $k \in \mathbb{Z}^+$. We prove that $4^{k+1} \in S$.
   Proof: $4^{k+1} = (4^k)4$. By our inductive hypothesis, $4^k \in S$ and by the base case of the recursive definition of S, $4 \in S$. Then by the inductive step of the recursive definition of S, $(4^k)4 \in S$. Therefore $4^{k+1} \in S$.
   Therefore, by the principle of mathematical induction, for all $i \in \mathbb{Z}^+$, $4^i \in S$.

   b) We prove that for all $x \in S, x = 4^i$ for some $i \in \mathbb{Z}^+$.
   Base case: Since $4 \in S$ by the basis step of the inductive definition of S, we prove that $4 \in A$.
   Proof: This is true since $4 = 4^1$ and $1 \in \mathbb{Z}^+$. Therefore, $4 \in A$ and the base case holds.
   Inductive step: Consider $s \in S$ and $t \in S$. We assume that $s \in A$ and $t \in A$. By the inductive step of the inductive definition of S, $st \in S$. We prove that $st \in A$.
   Proof: Since $s, t \in A$, $s = 4^j$ and $t = 4^k$ for some $j, k \in \mathbb{Z}^+$. Therefore $st = (4^j)(4^k) = 4^{j+k}$.
   Since $j + k \in \mathbb{Z}^+$, $st \in A$.
   Therefore, by structural induction, $S \subseteq A$.

3. **Ying 5 pts**

   Base: $1 \in S$.

   Induction: If $x \in S$, then $2x, 3x, 5x, x/2, x/3$, and $x/5 \in S$.

4. **Ying 10 pts**

   Using the above statement as our inductive hypothesis, we assume that after k steps, we are in state 0 iff k is divisible by 4. Now, if k is divisible by 4, then we are in state 0 after k steps, so we can deduce that $k+1$ is not divisible by 4 and that we are in state 1 after $k+1$ steps, as required. On the other hand, if k is not divisible by 4, we cannot tell whether $k+1$ is divisible by 4 or what state we are in after $k+1$ steps. We need to strengthen the induction hypothesis to distinguish k based on the value of k mod 4 as follows:

   For all $n \geq 0$, after n steps the state machine is in state n mod 4.

   Base case At $n = 0$, the state machine is in state 0, and 0 mod 4 = 0.

   Inductive step Assume our strengthened hypothesis is true after k steps. We will show it remains true at $k+1$ steps. There are 4 cases to consider.

   Case 1: $k \bmod 4 = 0$ The state machine is in state 0 after k steps by IH, so it will be in state 1 after $k+1$ steps. Since k is divisible by 4, $k+1$ has remainder 1 when divided by 4, so $(k+1) \bmod 4 = 1$. So the strengthened hypothesis remains true at $k+1$.

   Case 2: $k \bmod 4 = 1$ The state machine is in state 1 after k steps by IH, so it will be in state 2 after $k+1$ steps. Since k leaves remainder 1 when divided by 4, $k+1$ has remainder 2 when divided by 4, so $(k+1) \bmod 4 = 2$. So the strengthened hypothesis remains true at $k+1$.

   Case 3: $k \bmod 4 = 2$ The state machine is in state 2 after k steps by IH, so it will be in state 3 after $k+1$ steps. Since k leaves remainder 2 when divided by 4, $k+1$ has remainder 3 when divided by 4, so $(k+1) \bmod 4 = 3$. So the strengthened hypothesis remains true at $k+1$.

   Case 4: $k \bmod 4 = 3$ The state machine is in state 3 after k steps by IH, so it will be in state 0 after $k+1$ steps. Since k leaves remainder 3 when divided by 4, $k+1$ is divisible by 4, so $(k+1) \bmod 4 = 0$. So the strengthened hypothesis remains true at $k+1$.

5. **Ling 10 pts**

   The state machine has states (x, y) where x and y are integers, and every state (x, y) has three outgoing transitions, to $(x-1, y+3)$, to $(x+2, y-2)$, and to $(x+4, y)$.
   Preserved Invariant: if the robot is in state (x, y), then $x - y$ is a multiple of 4.
   Base Case: The robot starts in (0,0). 0-0=0, and 0 is a multiple of 4.
   Inductive step: Assume the robot is in state (x, y), where $x - y$ is a multiple of 4. After one step, there are three states the robot could be in.
   Case 1: The robot moved to $(x-1, y+3)$. Then $(x-1)-(y+3) = x - y - 4$. Since $x - y$ is a multiple of 4 (by IH), $x - y - 4$ is a multiple of 4.
   Case 2: The robot moved to $(x+2, y-2)$. Then $(x+2)-(y-2) = x - y + 4$. Since $x - y$ is a multiple of 4 (by IH), $x - y + 4$ is a multiple of 4.
   Case 3: The robot moved to $(x+4, y)$. Then $(x+4) - y = x - y + 4$. Since $x - y$ is a multiple of 4 (by IH), $x - y + 4$ is a multiple of 4.
   Thus, the invariant is preserved by the transitions.

Since the robot can only move to states (x, y) where $x - y$ is a multiple of 4, and $2 - 0$ is not a multiple of 4, the robot can never move to $(2, 0)$.

6. **Jonathan 15 pts**

   a) Base: $(0,0) \in L'$.
   Recursive: if $(a, b) \in L'$ then
   $(a+1, b+1) \in L'$, $(a-1, b-1) \in L'$, $(a+4, b) \in L'$, and $(a-4, b) \in L'$.

   b) $L' \subseteq L$ means that every ordered pair (a, b) produced in definition (a) has the property that $a - b$ is divisible by 4, i.e., $(a, b) \in L$. By the base case of the definition, $(0,0) \in L'$. Since $0-0 = 4 \times 0$, it follows that $(0,0) \in L$. For the recursive step, assume that ordered pair $(a, b) \in L'$ is such that $(a, b) \in L$, i.e., $a - b$ is divisible by 4. The recursive step allows us to place $(a+1, b+1)$, $(a-1, b-1)$, $(a+4, b)$ and $(a-4, b)$ in L'. We prove that each of these are in L. Now, $(a+1)-(b+1) = a-b$ which is divisible by 4, so $(a+1, b+1) \in L$. Similarly, $(a-1)-(b-1) = a-b$ which is divisible by 4, so $(a-1, b-1) \in L$. Since $a - b$ is divisible by 4, there exists some integer k such that $a-b = 4k$. Now, $(a+4)-b = (a-b)+4 = 4(k+1)$ for some integer k, so $(a+4)-b$ is divisible by 4, implying that $(a+4, b) \in L$. Finally, $(a-4)-b = (a-b)-4 = 4(k-1)$ for the same k so $(a-4)-b$ is divisible by 4, implying that $(a-4, b) \in L$. So in every case, the recursive step produces ordered pairs that satisfy membership in L.

   c) if $(m, n) \in L$ then $m - n = 4k$ for some integer k, so $m = n + 4k$. So any element of L will have form $(n+4k, n) \in L$. We show that $(n+4k, n) \in L'$, i.e., $(n+4k, n)$ is reachable by the inductive definition in (a). There are four cases to consider.

      i. if $n \geq 0$ and $k \geq 0$, we move from $(0,0)$ to (n, n) by using the rule $(a+1, b+1)$, n times, and then move from (n, n) to $(n+4k, n)$ by using the rule $(a+4, b)$, k times.

      ii. if $n \geq 0$ and $k < 0$, we move from $(0,0)$ to (n, n) by using the rule $(a+1, b+1)$, n times, then move from (n, n) to $(n+4k, n)$ by using the rule $(a-4, b)$, $-k$ times.

      iii. if $n < 0$ and $k \geq 0$, we move from $(0,0)$ to (n, n) by using the rule $(a-1, b-1)$, $-n$ times, and then move from (n, n) to $(n+4k, n)$ by using the rule $(a+4, b)$, k times.

      iv. if $n < 0$ and $k < 0$, we move from $(0,0)$ to (n, n) by using the rule $(a-1, b-1)$, $-n$ times, then move from (n, n) to $(n+4k, n)$ by using the rule $(a-4, b)$, $-k$ times.