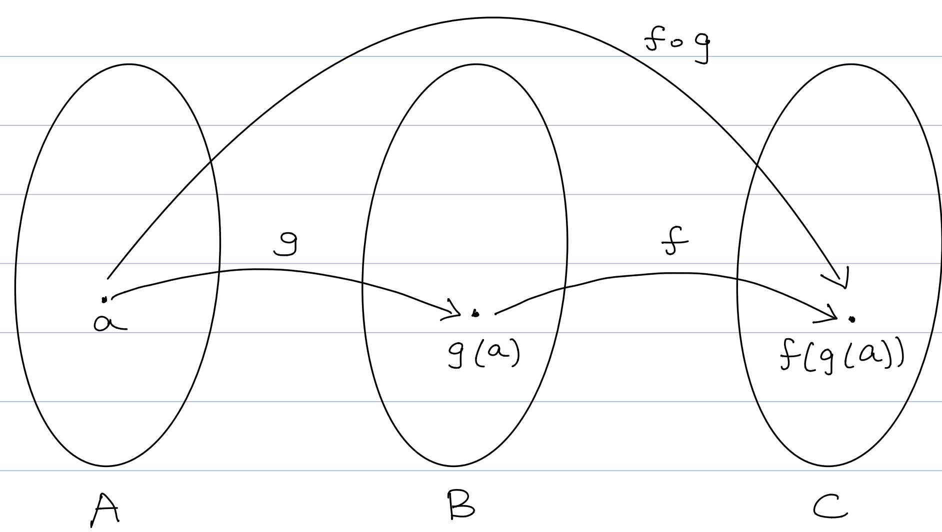
Def Let g be a function from A to B and f be a function from B to C.
The composition of f and g, denoted by the function $f \circ g$ from A to C, is

$$(f \circ g)(a) = f(g(a))$$

eg. $f(x) = x^2$, $\quad f: \mathbb{Z} \to \mathbb{Z}$
$\quad g(x) = x - 2$, $\quad g: \mathbb{Z} \to \mathbb{Z}$

$f \circ g\,(x) = (x-2)^2 \qquad f \circ g : \mathbb{Z} \to \mathbb{Z}$

$g \circ f\,(x) = x^2 - 2 \qquad g \circ f : \mathbb{Z} \to \mathbb{Z}$



eg. $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$

$g(x) = -x$, $g: \mathbb{R} \to \mathbb{R}$

$f \circ g(x) = x^2$ domain $(f \circ g) = \mathbb{R}$

range $(f \circ g) = \mathbb{R}^+ \cup \{0\}$

$g \circ f(x) = -x^2$ domain $(g \circ f) = \mathbb{R}$

range $(g \circ f) = \mathbb{R}^- \cup \{0\}$

range

eg. $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$ $\mathbb{R}^{\geq 0}$

$g(x) = \sqrt{x}$, $g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ $\mathbb{R}^{\geq 0}$

$h(x) = -x$, $h: \mathbb{R} \to \mathbb{R}$ $\mathbb{R}$

$f \circ g(x) = (\sqrt{x})^2 = x$ $f \circ g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$

range $(f \circ g) = \mathbb{R}^+ \cup \{0\}$

$$\underset{\text{domain}}{\mathbb{R}^{\geq 0}} \xrightarrow{g} \underset{\text{range}}{\mathbb{R}^{\geq 0}} \subseteq \underset{\text{domain}}{\mathbb{R}} \xrightarrow{f} \underset{\text{range}}{\mathbb{R}^{\geq 0}}$$

$g \circ f(x) = \sqrt{x^2} = |x|$ $g \circ f: \mathbb{R} \to \mathbb{R}$

range $(g \circ f) = \mathbb{R}^+ \cup \{0\}$

$$\underset{\text{domain}}{\mathbb{R}} \xrightarrow{f} \underset{\text{range}}{\mathbb{R}^{\geq 0}} \subseteq \underset{\text{domain}}{\mathbb{R}^{\geq 0}} \xrightarrow{g} \underset{\text{range}}{\mathbb{R}^{\geq 0}}$$

$g \circ h(x)$ is undefined!

$$1 \xrightarrow{h} -1 \xrightarrow{g} ?$$

$g(h(1)) = g(-1)$ but $g(-1)$ is undefined

problem: $h(1) = -1 \notin \text{domain}(g)$

$$\underset{\text{domain}}{\mathbb{R}} \xrightarrow{h} \underset{\text{range}}{\mathbb{R}} \not\subseteq \underset{\text{domain}}{\mathbb{R}^{\geq 0}} \xrightarrow{g} \underset{\text{range}}{\mathbb{R}^{\geq 0}}$$

More generally, problem: $\text{range}(h) \not\subseteq \text{domain}(g)$

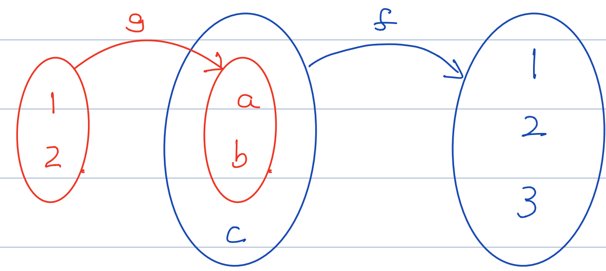
condition required: $\text{range}(h) \subseteq \text{domain}(g)$

Note: $g \cdot h(x)$ is well-defined if we restrict $\text{domain}(h) = \mathbb{R}^{\leq 0}$!

(as suggested by a student in class)

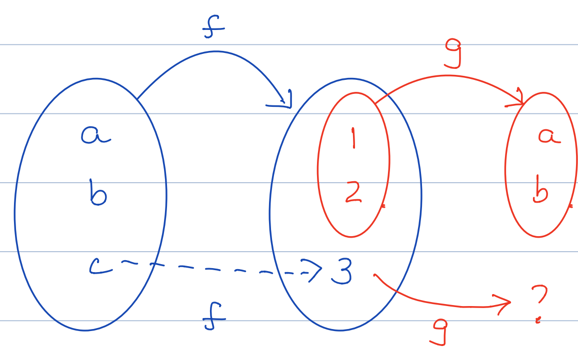
eg.

$f: \{a, b, c\} \to \{1, 2, 3\}$ $\qquad$ $g: \{1, 2\} \to \{a, b\}$



$f \circ g$ is clearly well-defined since

$\text{range}(g) \subseteq \text{co-domain}(g) \subseteq \text{domain}(f)$



$g \circ f$ may not be well-defined since

$\text{co-domain}(f) \not\subseteq \text{domain}(g)$

eg. $g(f(c)) = ?$ $\qquad$ $c \xrightarrow{f} 3 \xrightarrow{g} ?$

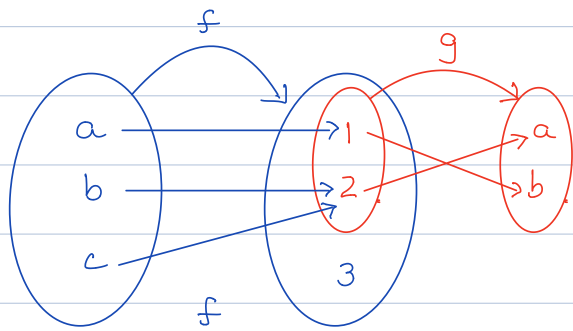
$g \circ f$ is well-defined if

$\text{range}(f) \subseteq \text{domain}(g)$

eg. $f(a)=1$, $f(b)=2$, $f(c)=2$

$g(1)=b$, $g(2)=a$

$\left.\begin{array}{l} \text{range}(f)=\{1,2\} \\ \text{domain}(g)=\{1,2\} \end{array}\right\} \Rightarrow \text{range}(f) \subseteq \text{domain}(g)$



Here, $g \circ f$ is well-defined

$g(f(a)) = g(1) = b$

$g(f(b)) = g(2) = a$

$g(f(c)) = g(2) = a$

Problems

1. $f: P(\mathbb{N}) \times P(\mathbb{N}) \to P(\mathbb{N})$, $f(A,B) = A \cup B$.

a) is f onto? Yes.

Let $C \in P(\mathbb{N})$. We prove there exists $A, B \in P(\mathbb{N})$ such that $f(A,B) = A \cup B = C$.

Given $C \in P(\mathbb{N})$, we can pick

$A = \emptyset$, $B = C$ so, $A \cup B = C$

$A = C$, $B = \emptyset$ so, $A \cup B = C$

$A = B = C$ so, $A \cup B = C$.

b) is f one-to-one? No.

The examples above show f is not one-to-one.

For any C, $f(\emptyset, C) = f(C, \emptyset) = C$

but $(\emptyset, C) \neq (C, \emptyset)$.

More specific example:

$f(\emptyset, \{1\}) = f(\{1\}, \emptyset) = \{1\}$

but $(\emptyset, \{1\}) \neq (\{1\}, \emptyset)$.

eg. $f(m,n) = (m+2,\ n-3)$

$f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$

(a) f is one-to-one

$$f(m_1, n_1) = f(m_2, n_2)$$
$$\Rightarrow (m_1+2,\ n_1-3) = (m_2+2,\ n_2-3)$$
$$\Rightarrow m_1+2 = m_2+2 \ \wedge\ n_1-3 = n_2-3$$
$$\Rightarrow m_1 = m_2 \ \wedge\ n_1 = n_2$$
$$\Rightarrow (m_1, n_1) = (m_2, n_2)$$

(b) f is onto.

Given $(x,y) \in \mathbb{R} \times \mathbb{R}$ find (m,n) such that $f(m,n) = (x,y)$

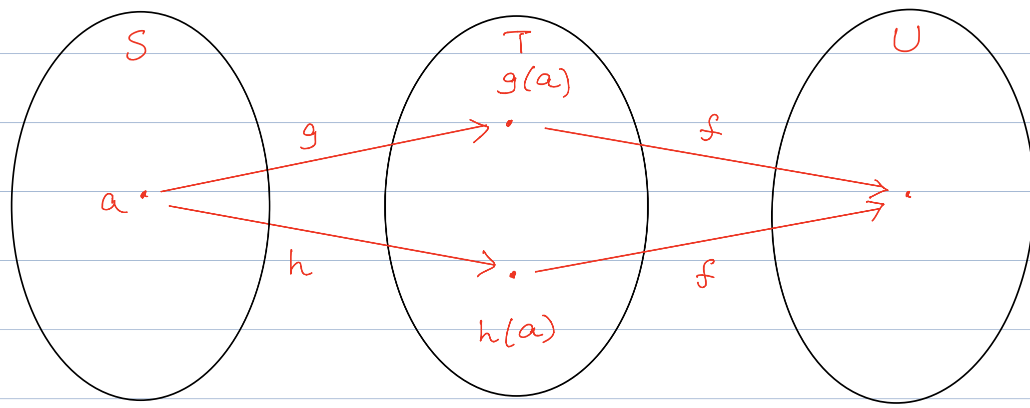
$$f(m,n) = (x,y)$$
$$\Rightarrow (m+2,\ n-3) = (x,y)$$
$$\Rightarrow m+2 = x,\quad n-3 = y$$
$$\Rightarrow m = x-2,\quad n = y+3$$
$$\Rightarrow (m,n) = (x-2,\ y+3)$$

c) $f^{-1}(x,y) = (x-2,\ y+3)$.

Let $f: T \to U$, $g: S \to T$ and $h: S \to T$

a) If $f \circ g = f \circ h$ then $g = h$. True or false?

False



Here, $f \circ g = f \circ h$ but $g \neq h$. Note f is not 1-1.

eg. $f(x) = |x|$ $\quad g(x) = x$ $\quad h(x) = -x$

$f: \mathbb{R} \to \mathbb{R}$ $\quad g: \mathbb{R} \to \mathbb{R}$ $\quad h: \mathbb{R} \to \mathbb{R}$

Here, $f \circ g\,(x) = |x|$ and $f \circ h\,(x) = |-x|$

Since $|x| = |-x|$, thus $f \circ g\,(x) = f \circ h\,(x)$,

However, $g(x) \neq h(x)$ since $x \neq -x$.

b) If f is one-to-one, then $f \circ g = f \circ h$ implies $g = h$.

Proof: Let $x \in S$. We prove that $g(x) = h(x)$.

$$f \circ g = f \circ h \implies f(g(x)) = f(h(x))$$

$$\implies g(x) = h(x) \text{ since } f \text{ is } 1\text{-}1.$$

So, for all $x \in S$, $g(x) = h(x)$.

QED