COMS 331: Theory of Computing, Spring 2023
Homework Assignment 1

Due at 10:00PM, Wednesday, February 1, on Gradescope.

Note: In this class, 0 is a natural number, i.e. $0 \in \mathbb{N}$.

Problem 1. Prove or disprove: If $A = \{0^n 1^n \mid n \in \mathbb{N}\}$, then $A^* = A$.

Problem 2. Prove or disprove: If $B = \{x \in \{0,1\}^* \mid \#(0,x) = \#(1,x)\}$, then $B^* = B$.

Note: The notation $\#(0,x)$ is used to denote the number of 0's in x. Likewise, $\#(1,x)$ is used to denote the number of 1's in x.

Problem 3. Prove: For every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

The demonstration that all of mathematics can be carried out within the framework of set theory includes the following "definition" of the natural numbers. First, the number 0 is defined to be $\varnothing$, the empty set. Next, for each previously defined natural number n, the number $n+1$ is defined to be the set $n \cup \{n\}$.

Problem 4. (a) Write out the numbers 1, 2, and 3, defined as above.

(b) Prove: For every $n \in \mathbb{N}$, $n = \{k \in \mathbb{N} \mid k < n\}$.

Problem 5. Prove: If $A = \{0,1\}$ and $B \subseteq \{0,1\}^*$, then

$$A^* = B^* \Rightarrow A \subseteq B.$$

Problem 6. Exhibit languages $A, B \subseteq \{0,1\}^*$ such that $A^* = B^*$ and $\{0,1\} \subseteq A \subsetneq B$.

Problem 7. Define an (infinite) binary sequence $s \in \{0,1\}^\infty$ to be *prefix-repetitive* if there are infinitely many strings $w \in \{0,1\}^*$ such that $ww \sqsubseteq s$.

Prove: If the bits of a sequence $s \in \{0,1\}^\infty$ are chosen by independent tosses of a fair coin, then

$$Prob[s \text{ is prefix-repetitive}] = 0.$$

Note: $x \sqsubseteq y$ means that x is a prefix of y where x is a string and y is a string or sequence.