COMS 331: Theory of Computing, Spring 2023
Homework Assignment 8

Due at 10:00PM, Wednesday, April 12, on Gradescope.

Given a language $B \subseteq \{0,1\}^*$, define the language $\forall B \subseteq \{0,1\}^*$ by

$$\forall B = \{x \in \{0,1\}^* \mid (\forall w \in \{0,1\}^*)\langle x, w \rangle \in B\}.$$

Problem 51. Let $B \subseteq \{0,1\}^*$. Prove: If B is co-c.e., then $\forall B$ is co-c.e.

Problem 52. Prove: For every co-c.e. language $A \subseteq \{0,1\}^*$, there is a decidable language $B \subseteq \{0,1\}^*$ such that $A = \forall B$.

Given a function $f : \mathbb{N} \to \mathbb{N}$, define the following three subsets of $\mathbb{N} \times \mathbb{N}$:
The <u>graph</u> of f is

$$Graph(f) = \{(n, f(n)) \mid n \in \mathbb{N}\}.$$

The <u>upper</u> <u>graph</u> of f is

$$Graph^+(f) = \{(n, k) \in \mathbb{N} \times \mathbb{N} \mid k > f(n)\}.$$

The <u>lower</u> <u>graph</u> of f is

$$Graph^-(f) = \{(n, k) \in \mathbb{N} \times \mathbb{N} \mid k < f(n)\}.$$

Problem 53. Use a picture of $\mathbb{N} \times \mathbb{N}$ to explain what the sets $Graph(f)$, $Graph^+(f)$, and $Graph^-(f)$ are if $f(n) = n + 2$ for all $n \in \mathbb{N}$.

Recall the rapidly growing function $G : \mathbb{N} \to \mathbb{N}$ from the class lectures.

Problem 54. Answer the following questions and prove that your answers are correct

(a) Is $Graph(G)$ decidable?

(b) Is $Graph(G)$ c.e.?

(c) Is $Graph(G)$ co-c.e.?

Problem 55. Answer the following questions and prove that your answers are correct

(a) Is $Graph^+(G)$ decidable?

(b) Is $Graph^+(G)$ c.e.?

(c) Is $Graph^+(G)$ co-c.e.?

Problem 56. Answer the following questions and prove that your answers are correct

(a) Is $Graph^-(G)$ decidable?

(b) Is $Graph^-(G)$ c.e.?

(c) Is $Graph^-(G)$ co-c.e.?