COMS 331: Theory of Computing, Spring 2023
Homework Assignment 1

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Due at 10:00PM, Wednesday, February 1, on Gradescope.

Note: In this class, 0 is a natural number, i.e. $0 \in \mathbb{N}$.

Problem 1. Prove or disprove: If $A = \{0^n 1^n \mid n \in \mathbb{N}\}$, then $A^* = A$.

A $= \{0^n 1^n | n \in N\}$
A $= \{\epsilon, 01, 0011, 000111, ...\}$
For each n $\in$ N, there are n 0's followed by n 1's. A 0 never appears after a 1 in a given string. A^* defined as $\bigcup_{n=0}^{\infty} A^n = A^0 \cup A^1 \cup A^2 \cup ....$
$A^2 = \{xy | x \in A, y \in A\}$ and let x = 01 and y = 0011. Then xy = 010011 is an element of A^2. A^2 is in A^*. But A was defined as never having 0's after 1's. The string xy has a 0 after a 1, so $xy \notin A$. Since $xy \in A^* \wedge xy \notin A, A^* \neq A$.■

Problem 2. Prove or disprove: If $B = \{x \in \{0,1\}^* \mid \#(0,x) = \#(1,x)\}$, then $B^* = B$.

Note: The notation $\#(0,x)$ is used to denote the number of 0's in x. Likewise, $\#(1,x)$ is used to denote the number of 1's in x.

The standard enumeration of $\{0,1\}^*$ is $\{\epsilon, 0, 1, 00, 01, 10, 11, 000, ...\}$ which contains all possible combinations of 0's and 1's. B contains strings in $\{0,1\}^*$ that have an equal number of 0's and 1's. B^* defined as $\bigcup_{n=0}^{\infty} B^n = B^0 \cup B^1 \cup B^2 \cup ....$
It is true that $B^0 = \{\epsilon\}$ and $\{\epsilon\} \in B.B^1 = B$. Show that for subsequent B^n where n¿1, all values in B^n are in B. Let $B^n = B_1...B_n = \{b_1...b_n | b_1, ..., b_n \in B\}$. All b_x in this definition have an equal number of 0's and 1's. By this definition, all strings in B^n will have an equal number of 0's and 1's because when concatenating the strings there are never more or less 0's than there are 1's.
For example, if you have x 0's and x 1's, and you add y 0's and y 1's accordingly to each, you will have x+y=z 0's and x+y=z 1's and z=z. So, all values in B^n where n¿1 are in B. Thus, every string in B^* has an equal number of 0's and 1's which matches the definition of B, so $B^* = B$.■

Problem 3. Prove: For every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

Proof by induction:
BASE CASE: for n=1, $\sum_{k=1}^{1} \frac{1}{k^2} \leq 2 - \frac{1}{1} = 1 \leq 2 - 1 = 1 \leq 1$ which holds true.
INDUCTION HYPOTHESIS: assume that $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$ holds for n.
INDUCTION STEP: let n be a positive integer and assume the induction hypothesis is true. Then show that $\sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$. Then $\frac{1}{(n+1)^2} + \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$ and we can substitute $2 - \frac{1}{n+1}$ with $2 - \frac{1}{n} + \frac{1}{(n+1)^2}$.
This substitution works because $\frac{1}{n} + \frac{1}{(n+1)^2} = \frac{(n+1)^2}{n(n+1)^2} + \frac{n}{n(n+1)^2} = \frac{n+(n+1)^2}{n(n+1)^2} = \frac{n^2+2n+n+1}{n(n+1)^2} = \frac{n^2+n}{n(n+1)^2} + \frac{2n+1}{n(n+1)^2} = \frac{n(n+1)}{n(n+1)^2} + \frac{2n+1}{n(n+1)^2} = \frac{1}{n+1} + \frac{2n+1}{n(n+1)^2}$ so $2 - (\frac{1}{n+1} + \frac{2n+1}{n(n+1)^2}) = 2 - \frac{1}{n+1} - \frac{2n+1}{n(n+1)^2}$ which can be substituted for $2 - \frac{1}{n+1}$ because the $(-\frac{2n+1}{n(n+1)^2})$ term makes the expression less than $2 - \frac{1}{n+1}$ because n is positive.
So the full inequality looks like $\frac{1}{(n+1)^2} + \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n+1} - \frac{2n+1}{n(n+1)^2} \leq 2 - \frac{1}{n+1}$. Now replace $2 - \frac{1}{n+1}$ with $2 - \frac{1}{n} + \frac{1}{(n+1)^2}$. Since it was proved that the expression continue to satisfy the inequality, so we have $\frac{1}{(n+1)^2} + \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$.
After subtracting $\frac{1}{(n+1)^2}$ from both sides, the expression remaining is the induction hypothesis which is true. ■

The demonstration that all of mathematics can be carried out within the framework of set theory includes the following "definition" of the natural numbers. First, the number 0 is defined to be $\varnothing$, the empty set. Next, for each previously defined natural number n, the number $n+1$ is defined to be the set $n \cup \{n\}$.

Problem 4. (a) Write out the numbers 1, 2, and 3, defined as above.

$1 = (0+1) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$
$2 = (1+1) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$
$3 = (2+1) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$

(b) Prove: For every $n \in \mathbb{N}$, $n = \{k \in \mathbb{N} \mid k < n\}$.

Proof by induction:
BASE CASE: for n=0, $0 = \{k \in N | k < 0\} = \emptyset$, by the definition $0 = \emptyset$ is true.
INDUCTION HYPOTHESIS: assume that $n = \{\in N | k < n\}$ holds for n.
INDUCTION STEP: let $n \in N$ and assume the induction hypothesis is true. Then show that $n+1 = \{k \in N | k < n+1\}$ which is $n+1 = \{k \in N | k < n\} \cup \{n\}$ and by the induction hypothesis, $n+1 = n \cup \{n\}$ which is true. ■

Problem 5. Prove: If $A = \{0, 1\}$ and $B \subseteq \{0, 1\}^*$, then

$$A^* = B^* \Rightarrow A \subseteq B.$$

By definition, Σ^* defined as $\bigcup_{n=0}^{\infty} \Sigma^n = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup ....$
$A^1 = \{0, 1\}$ so $\{0, 1\} \subseteq A^*$. Since $A^* = B^*$, $\{0, 1\} \subseteq B^*$. For some $x \in B^n$, $|x| = n$. We know that $\{0, 1\} \subseteq B^*$, $|0| = 1$ and $|1| = 1$, so $\{0, 1\} \subseteq B^1$. It is also true that $B^1 = B$, so $\{0, 1\} \subseteq B$. Thus, $A^* = B^*$ implies $A \subseteq B$. ■

Problem 6. Exhibit languages $A, B \subseteq \{0,1\}^*$ such that $A^* = B^*$ and $\{0,1\} \subseteq A \subsetneq B$.

$A = \{0,1\}, B = \{0,1,11\}$
Justification: It is clear that $\{0,1\} \subseteq \{0,1\} \subsetneq \{0,1,11\}$. $A^* = B^*$ because every string that has the string 11 in it from language B will be a duplicate of a string made by duplicate 1's from language A, and the 11 in language B is the only difference from language A. By definition $A^* = \bigcup_{n=0}^{\infty} A^n = A^0 \cup A^1 \cup A^2 \cup ...$
$A^1 = \{0,1\}, A^2 = \{00, 01, 10, 11\}$, and $B^1 = \{0,1,11\}$. Even though B^1 differs from A^1 because it has string 11, that string containing 11 will always be found in another A^x. In this case, the 11 from B^1 is found in A^2 and languages don't have duplicate strings so $A^* = B^*$.

Problem 7. Define an (infinite) binary sequence $s \in \{0,1\}^\infty$ to be *prefix-repetitive* if there are infinitely many strings $w \in \{0,1\}^*$ such that $ww \sqsubseteq s$.

Prove: If the bits of a sequence $s \in \{0,1\}^\infty$ are chosen by independent tosses of a fair coin, then

$$Prob[s \text{ is prefix-repetitive}] = 0.$$

Note: $x \sqsubseteq y$ means that x is a prefix of y where x is a string and y is a string or sequence.

Let the bits of the sequence $s \in \{0,1\}^\infty$ be chosen by independent tosses of a fair coin. Let $w \in \{0,1\}^*$. The probability of w being a prefix of $s = Prob[w \sqsubseteq s] = \frac{1}{2^n}$, where $n = |w|$. This is because there are 2 options (0 or 1) for each character in w, so there is a $\frac{1}{2}$ chance that a single character in w will be the same as the corresponding character in s. The probability is $\frac{1}{2^n}$ because we multiply $\frac{1}{2}$ by itself n times, once for each character. So $Prob[ww \sqsubseteq s] = \frac{1}{2^n} * \frac{1}{2^n}$, and following this pattern, $Prob[\text{s is prefix repetitive}] = Prob[w^\infty \sqsubseteq s] = \frac{1}{2^n} * \frac{1}{2^n} * ....$
The $\lim_{n\to\infty}(\frac{1}{2^n})^\infty \approx (\frac{1}{2^\infty})^\infty \approx 0^\infty = 0$. Thus, $Prob[\text{s is prefix repetitive}] = 0$.■