COMS 331: Theory of Computing, Spring 2023
Homework Assignment 5

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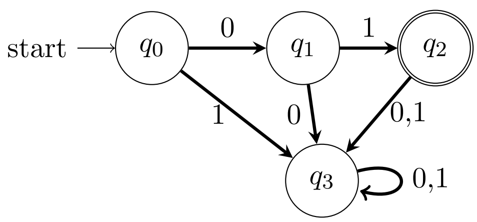
Due at 10:00PM, Wednesday, March 1, on Gradescope.

Problem 30.
The statement is false. Let A = $\{01\}$ and B = $\{0^n1^n | n \in N\}$. Then A∩B = $\{01\}$, which is regular, shown by the DFA below. But, B is not regular as proved in lecture, which is a contradiction to the original claim. Therefore, the claim "if A and A ∩B are regular, then B is regular" is false. ■

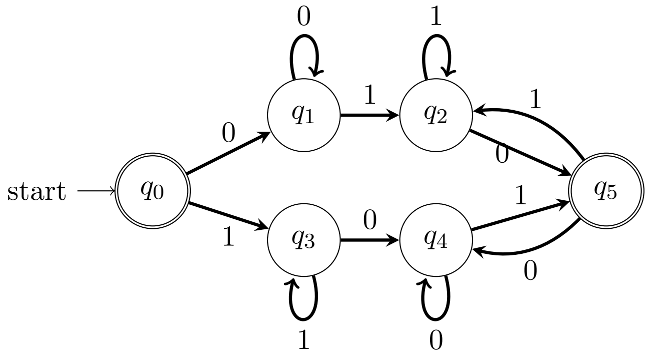


Problem 31.
The statement is true. It is given that A is regular. Assume that B is regular. Then A∩B is regular by the closure properties of regular languages. The statement in the question says that A is regular and A∩B is not regular. By the closure properties of regular languages, B is not regular. ■

Problem 32.
The statement is false. Let A = $\{01\}$ and B = $\{0^n1^n | n \in N\}$. A is regular by the DFA from problem 30. B is not regular according to lecture. A∩B = $\{01\}$ is regular because A∩B = A, and we showed that A is regular. This is a contradiction to the original claim, thus proving it false. ■

Problem 33.
When looking at the first A-extension of x (${A_x}^{(1)}$), we can see that if x is already in A, the first extension is ϵ. Also, we can see that if x only contains 0's ($x = 0^n$ for $n \in N$), then ${A_x}^{(1)} = 1^n$. The previous statement is also true if 0 and 1 are switched around. We know that N is an infinite set, so there are infinitely many strings x = 0, 00, 000, ... such that the corresponding ${A_x}^{(1)} = 1, 11, 111, ...$ Thus, $|A^{(1)}| = \infty$, which proves that A is not regular because it has infinite ordinal extensions. Infinite ordinal extensions implies unbounded ordinal extensions, and by the theorem taught in lecture, if A has unbounded ordinal extensions, then A is not regular. ■

Problem 34.
To prove the language is regular, we can construct a DFA that decides the language:



Therefore, since there is a DFA that decides the language, the language is regular.

Problem 35a.
Let $x \in \{0,1\}^*$. The following are equivalence classes of $\equiv_A$:
1. x is already in the language A
2. x starts with a 1 or has alternating 0's and 1's, or starts with a number of consecutive 0's and then is followed by a greater amount of consecutive 1's.
3. x starts with a number of 0's followed by a lesser number of 1's.
4. x only has 0's (0, 00, 000,... are each their own equivalence class.)

Problem 35b.
To prove the answer to (a) is correct, we must show that each equivalence class has a different z, based on the definition of a canonical equivalence relation. If two equivalence classes have the same z, then they are in the same equivalence class. We can use A-extensions of x to determine the value of z for the equivalence classes. Let the following numbers correspond with the numbers in part a.

1. $z = \epsilon$. When $x \in \{0,1\}^*$ is already in the language A, there only exists one A-extension, which is ${A_x}^{(1)} = \epsilon$. So, all x,y already in the language A are in the same equivalence class becasue $x\epsilon \in \iff y\epsilon \in A$, so $x \equiv_A y$.

2. All the strings described in 2 are in their own equivalence class because there does not exist an A-extension for those strings, meaning that the value of z is different for those strings.

3. For these strings, only the first A-extension exists. Let d be the positive difference in the number of 0's and the number of 1's for $x, y \in \{0,1\}^*$ when x, y starts with a number 0's followed by a lesser number of 1's. Then, x and y are in the same equivalence class because ${A_x}^{(1)} = 1^d$. Thus, z would be 1^d accordingly. If $x = 001, y = 00011$, then z=1. If $x = 001, y = 00011$, then z=11.

4. When x only has 0's, each string with only 0's is in its own equivalence class. This is because the values of z are different for each specified string. An example would be if $x = 0$, then ${A_x}^{(1)} = 1, {A_x}^{(2)} = 011, {A_x}^{(3)} = 00111, \ldots.$ If y=00, then ${A_x}^{(1)} = 11, {A_x}^{(2)} = 0111, {A_x}^{(3)} = 001111, \ldots.$ The values of z correspond with the A-extensions, which we can see are different so $xz \in A$ does not imply $yz \in A$ and $yz \in A$ does not imply $xz \in A$, thus x and y are in different equivalence classes.

Problem 35c.
The answer to a implies that A is not regular because for the fourth group of equivalence classes, there are infinite equivalence classes. There are infinitely many 0^n for $n \in N$, and since each 0^n

is its own equivalence class, shown using ${A_x}^{(1)}$, then A is not regular by infinite ordinal extensions ($|A^{(1)}| = \infty$).

Problem 36.
We know A,B $\subseteq \{0,1\}^*$ are regular languages, so there is some $M_A = (Q_A, \Sigma_A, \delta_A, s_A, F_A)$ such that L(M_A)=A, and there is some $M_B = (Q_B, \Sigma_B, \delta_B, s_B, F_B)$ such that L(M_B)=B. Then we must define some $M_C = (Q_C, \Sigma_C, \delta_C, s_C, F_C)$ such that L(M_C) $= \begin{bmatrix} A \\ B \end{bmatrix}$, which we will do using product construction. $Q_C = Q_A \times Q_B$, $\Sigma_C = \{ \begin{bmatrix} a \\ b \end{bmatrix} | a \in \Sigma_A, b \in \Sigma_B\}$, $s_C = s_A \times s_B$, and $F_C = F_A \times F_B$ by product construction. Now, we define the transition function δ_C as $\delta_C(qp, \begin{bmatrix} a \\ b \end{bmatrix}) = \delta_A(q,a)\delta_B(p,b)$, where $q \in Q_A, p \in Q_B$ and $qp \in Q_C$. This transition function is correct because if you are in state qp and read in $\begin{bmatrix} a \\ b \end{bmatrix}$, then the next state to transition to must be determined by the transition functions δ_A and δ_B accordingly, which is valid since $Q_C = Q_A \times Q_B$. $\delta_A(q,a)$ determines what state to transition to after reading a b in state p. By the definition of Q_C and product construction, the combination of those 2 states is what $\delta(qp, \begin{bmatrix} a \\ b \end{bmatrix})$ is. For example, let $\delta)_A(q,a) = r$ and by definition, $r \in Q_A$, and let $\delta_B(p,b) = t$ and by definition, $t \in Q_B$. Then since $Q_C = Q_A \times Q_B$, and $rt \in Q_C$, then $\delta_C(qp, \begin{bmatrix} a \\ b \end{bmatrix}) = rt$. This definition of M_C proves that $\begin{bmatrix} A \\ B \end{bmatrix}$ is regular.