COMS 331: Theory of Computing, Spring 2023
Homework Assignment 9

Due at 10:00PM, Wednesday, April 19, on Gradescope.

Problem 57. Prove that there is a constant $c \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$|C(s_n) - C(s_{n+1})| \leq c.$$

Define the *tower function* $T : \mathbb{N} \longrightarrow \mathbb{N}$ by the recursion

$$T(0) = 0$$
$$T(n+1) = 2^{T(n)}$$

for all $n \in \mathbb{N}$.

Problem 58. Prove that there exist infinitely many strings $x \in \{0,1\}^*$ such that

$$T(C(x)) < |x|.$$

Problem 59. Prove: If $A \subseteq \{0,1\}^*$ is decidable, then there is a constant $c \in \mathbb{N}$ such that, for all $x \in A$,

$$C(x) \leq \log\left(1 + |A \cap \{0,1\}^{\leq |x|}|\right) + c.$$

Problem 60. Let $A \subseteq \{0,1\}^*$, and let t be a real number with $0 < t < 1$. Prove: For every $n \in \mathbb{N}$ such that $|A \cap \{0,1\}^n| > 2^{tn}$, there exists $x \in A$ such that $|x| = n$ and $C(x) \geq tn$.

Recall the diagonal halting problem

$$K = \{k \in \mathbb{N} \mid M_k(k) \downarrow\}.$$

Problem 61. For each $n \in \mathbb{N}$, define the string

$$z_n = b_0 b_1 ... b_{n-1} \in \{0,1\}^n$$

by

$$b_k = \begin{cases} 1 & \text{if } k \in K \\ 0 & \text{if } k \notin K \end{cases}$$

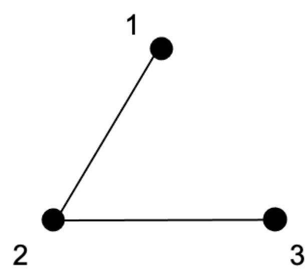
for all $0 \leq k < n$. Prove that there is a constant $c \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$C(z_n) \leq 3 \log(n+1) + c.$$

Problem 62. Prove that there is a constant $c_b \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ and every string $v \in \{00, 01, 10\}^n$,

$$C(v) \leq n(\log 3) + 2\log(n+1) + c_b.$$

A (*finite, undirected*) graph is an ordered pair $G = (V, E)$, where V is a non empty finite set of vertices, and E is a finite set of 2-element subsets of V, called edges. For example,



denotes the graph $G = (V, E)$, where $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$.

Without loss of generality, we can assume that the vertex set of a graph is always of the form $V_n = \{1, 2, ..., n\}$, where $n \in \mathbb{Z}^+$. We can also, for each $n \in \mathbb{N}$, define the standard enumeration $e_1, e_2, ..., e_{\binom{n}{2}}$ of all 2-element subsets of V_n to be the enumeration first in order of the least element of the 2-element subset, then in order of the greater element. Thus, for example

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$$

is the standard enumeration $e_1, ..., e_6$ of the $\binom{4}{2} = 6$ 2-element subsets of V_4. A graph $G = (V_n, E)$ is thus completely specified by the binary string

$$x_G = b_1 b_2 ... b_{\binom{n}{2}} \in \{0,1\}^{\binom{n}{2}}$$

defined by

$$b_k = \begin{cases} 1 & \text{if } e_k \in E \\ 0 & \text{if } e_k \notin E \end{cases}$$

for all $1 \leq k \leq \binom{n}{2}$. We call x_G the standard encoding of the graph G.

A Graph G is complete if each pair of vertices is connected by an edge.

Problem 63. Prove that there is a constant $c \in \mathbb{N}$ such that, for every complete graph $G = (V, E)$,

$$C(x_G) \leq \log(n+1) + c.$$