# COMS 331: Theory of Computing, Spring 2023
# Homework Assignment 10

Due at 10:00PM, Wednesday, April 26, on Gradescope.

**Problem 64.** Let $q, r \in \mathbb{Q}$ be rational numbers with $0 \leq q < r \leq 1$. Prove that, for all sufficiently large $n \in \mathbb{N}$, there exists $x \in \{0,1\}^n$ such that

$$qn < C(x) < rn.$$

A lossless data compression scheme is a pair $(f, g)$ of computable functions $f, g : \{0,1\}^* \longrightarrow \{0,1\}^*$ such that, for all $x \in \{0,1\}^*$,

$$g(f(x)) = x.$$

(Intuitively, $f$ is the "compressor", and $g$ is the "decompressor" that recovers $x$ from its compression $f(x)$.)

**Problem 65.** Prove: For every lossless data compression scheme $(f, g)$ and every $n \in \mathbb{N}$, there is a string $x \in \{0,1\}^n$ such that $|f(x)| \geq |x|$.

**Problem 66.** Prove: For every lossless data compression scheme $(f, g)$, there is a constant $c_{(f,g)} \in \mathbb{N}$ such that, for all $x \in \{0,1\}^*$,

$$C(x) \leq |f(x)| + c_{(f,g)}.$$

**Problem 67.** Prove: For every lossless data compression scheme $(f, g)$, there exist infinitely many strings $x \in \{0,1\}^*$ such that

$$C(x) < |f(x)|.$$

Recall the definitions related to graphs from the Homework 9.

A path in a graph $G = (V, E)$ is a sequence $p = (v_0, v_1, ...v_\ell)$ of vertices $v_i \in V$ such that, for every $0 \leq i < \ell$, $\{v_i, v_{i+1}\} \in E$. The length of this path $p$ is

$$\text{length}(p) = \ell$$

Note that $(v_0)$ is a path of length 0, $(v_0, v_1)$ is a path of length 1, etc.

If $G = (V, E)$ is a graph and $s, t \in V$, then a *path from* $s$ *to* $t$ in $G$ is a path $p = (v_0, ..., v_\ell)$ in $G$ such that $v_0 = s$ and $v_\ell = t$.

If $G = (V, E)$ is a graph and $s, t \in V$, then the *distance from* $s$ *to* $t$ in $G$ is

$$d_G(s,t) = \min\{\text{length}(p) \mid p \text{ is a path from } s \text{ to } t \text{ in } G\},$$

where $\min \varnothing = \infty$, i.e., $d_G(s,t) = \infty$ if there is no path from $s$ to $t$ in $G$.

The diameter of a graph $G = (V, E)$ is

$$\text{diam}(G) = \max\{d_G(s,t) \mid s, t \in V\}.$$

**Problem 68.** Prove that there is a constant $c \in \mathbb{N}$ such that, for every graph $G = (V, E)$ with $\text{diam}(G) > 2$,

$$C(x_G) \leq \binom{n}{2} - (2 - \log 3)n + 8\log(n+1) + c.$$

**Problem 69.** Prove that, for all sufficiently large $n \in \mathbb{N}$, every graph $G = (V_n, E)$ with

$$C(x_G) \geq \binom{n}{2} - \frac{2}{5}n$$

has diameter 2.

**Problem 70.** For each $n \in \mathbb{N}$, suppose that we choose a graph $G = (V_n, E)$ at random by using independent tosses of a fair coin to decide whether each of $e_1, ...e_{\binom{n}{2}}$ is an element of E. Prove that, for all sufficiently large $n \in \mathbb{N}$,

$$\text{Prob}[\text{diam}(G) = 2] > 1 - 2^{-\frac{2}{5}n}.$$