Homework 5

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Problem 3.105
Part a:
A hypergeometric distribution, as the probability of being chosen on a trial is dependent on the outcome of previous trials.

Part b:
N = 8, r = 5, n = 3
We are interested in P(2)+P(3) which is
P(2)+P(3) = $\frac{\binom{5}{2}\binom{8-5}{3-2}}{\binom{8}{3}} + \frac{\binom{5}{3}\binom{8-5}{3-3}}{\binom{8}{3}} = 5/7$

Part c:
$\mu = 3 * (5/8) = 1.875$
$\sigma = 3 * (5/8) * (3/8) * (5/7) = 0.5022$

Problem 3.106
N = 10, n = 5, r = 4
P(Y=y) = $\frac{\binom{4}{y}\binom{6}{5-y}}{\binom{10}{5}}$ for y = 0, 1, 2, 3, 4
The mean and variance of the hypergeometric distribution are: E(Y) = $\frac{5*4}{10} = 2$
Var(Y) = $5 * (4/10) * (6/10) * (5/9) = 0.667$
T=50Y (total repair cost of the defective machine)
E(T) = 50 * 2 = 100 dollars
Var(T) = 50^2Var(Y) = 2500*0.667 = 1666.67 dollars

Problem 3.121
Part a:
P(Y=4) = $(e^{-2} * 2^4)/4!$ = 0.090223522

Part b:
P(Y≥4) = 1-[P(Y=0)+P(Y=1)+P(Y=2)+P(Y=3)] = 1 - 0.85712346 = 0.14287654

Part c:
P(Y<4) = P(Y=0)+P(Y=1)+P(Y=2)+P(Y=3) = 0.135335283 + 0.270670566 + 0.270670566 + 0.180447044 = 0.857123459

Part d:
P(Y≥4 | Y≥2) = $\frac{P(Y \geq 4}{P(Y \geq 2}$
= 0.14287654 / (1 - P(Y=0)-P(Y=1)) = 0.14287654 / 0.59399415 = 0.240535264

Problem 3.122
Part a:
P(X≤3) = P(X=0)+P(X=1)+P(X=2)P(X=3)
$= \frac{e^{-7}7^0}{0!} + \frac{e^{-7}7^1}{1!} + \frac{e^{-7}7^2}{2!} + \frac{e^{-7}7^3}{3!}$ = 0.0009 + 0.0064 + 0.0223 + 0.0521 = 0.0817

Part b:
P(X≥2) = 1 - 0.0073 = 0.9927

Part c:
P(X=5) = $\frac{e^{-7}7^5}{5!} = (0.000912 * 16807)/120 = 0.1277$

Problem 3.132
The number of automobiles entering a mountain tunnel per two-minute period, Y Poisson(λ=1)
P(Y=y) = $\frac{e^{-1}1^y}{y!}$ for y =0,1,2...
P(Y>3) = 1 - P(Y=0) - P(Y=1) - P(Y=2) = 1 - $e^{-1} - e^{-1} - 0.5e^{-1}$
= 1-0.9197 = 0.0803
The above probability of producing a hazardous situation is very low, the Poisson model is adequate for the problem.

Problem 3.147
The mass function of a geometric distribution is:
$$P(X = x) = \begin{cases} (1-p)^{x-1}p & \text{x = 1, 2, ...} \\ 0 & \text{otherwise} \end{cases}$$
The moment-generating function is $M_x(t) = E(e^{tx})$
$= \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p = p * \sum_{x=0}^{\infty} e^{tx}(1-p)^{x-1} = pe^t \sum_{x=1}^{\infty}[e^t(1-p)]^{x-1}$
$= \frac{pe^t}{1-e^t(1-p)}$
Let us call 1 - p = q. Then
$M_x(t) = \frac{pe^t}{1-e^tq)}$

Problem 3.148
M'(t) = $\frac{pe^t}{1-[e^t(1-p)]} + \frac{pe^{2t}(1-p)}{(1-[e^t(1-p)])^2}$
M"(t) = $\frac{pe^t}{1-[e^t(1-p)]} + \frac{3pe^{2t}(1-p)}{(1-[e^t(1-p)])^2} + \frac{2pe^{3t}(1-p)^2}{(1-[e^t(1-p)])^3}$
E(Y) = M'(0) = $\frac{p}{p} + \frac{p(1-p)}{p^2} = 1 + \frac{p(1-p)}{p^2} = \frac{p^2+p-p^2}{p^2} = 1/p$
E(Y^2) = M"(0) = $\frac{pe^0}{1-[e^0(1-p)]} + \frac{3pe^{2*0}(1-p)}{(1-[e^0(1-p)])^2} + \frac{2pe^{3*0}(1-p)^2}{(1-[e^0(1-p)])^3} = \frac{2-p}{p^2}$
Var(Y) = M"(0) - (M'(0))^2 = $\frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

Problem 3.158
Given that Y is a rv with moment generating function m(t):
W = aY+b
$= m_w(t) = E(e^{tW}) = E(e^{t(aY+b)}) = E(e^{tb}e^{taY}) = e^{tb}E(e^{(ta)Y}) = e^{tb}m(at)$

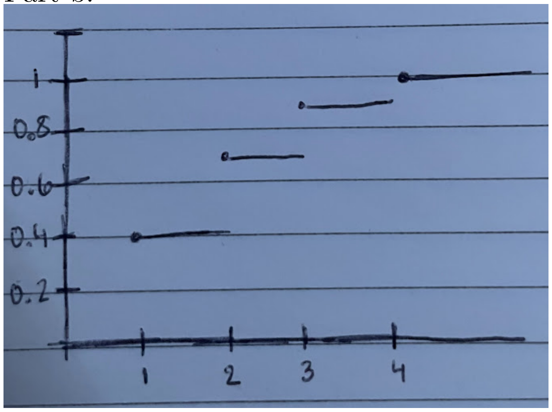
Problem 3.167
Part a:
Let Y be a rv with $\mu = 11$ and $\sigma^2 = 9$
Find the lower bound for P($6 < Y < 16$)
11 - k (3) = 6
3k = 11 - 6

k = 5/3
P(6 < Y < 16 ≤ 1-1/(5/3)^2)
P(6 < Y < 16 ≤ 1-1/(25/9)
P(6 < Y < 16 ≤ 16/25 is the lower bound

Part b:
$\frac{1}{k^2} = 0.09$
$k^2 = 1/0.09$
k = 10/3
C = 3k = 3(10/3) = 10

Problem 4.1
Part a:
F(1) = P($Y \leq 1$) = 0.4
F(2) = P(Y≤2) = P(Y=1)+P(Y=2) = 0.4 + 0.3 = 0.7
F(3) = 0.4 + 0.3 + 0.2 = 0.9
F(4) = 0.4 + 0.3 + 0.2 + 0.1 = 1

$$F(Y) = \begin{cases} 0.4 & \text{for } y < 2 \\ 0.7 & \text{for } 2 \leq y < 3 \\ 0.9 & \text{for } 3 \leq y < 4 \\ 1 & \text{for x} \geq 4 \end{cases}$$

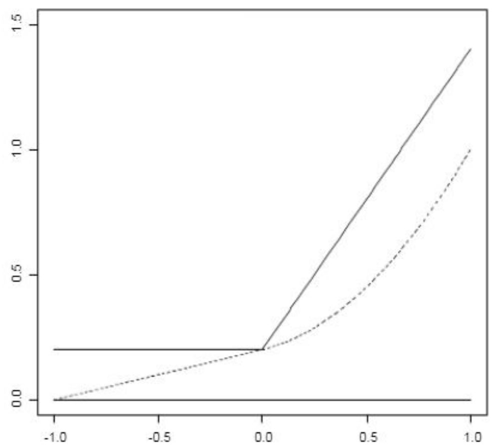
Part b:


Problem 4.18
Part a:
$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$
$\int_{-\infty}^{-1} f_Y(y)\,dy + \int_{-1}^{0} f_Y(y)\,dy + \int_{0}^{1} f_Y(y)\,dy + \int_{1}^{\infty} f_Y(y)\,dy = 1$
$\int_{-\infty}^{-1} 0\,dy + \int_{-1}^{0} 0.2\,dy + \int_{0}^{1}(0.2 + cy)\,dy + \int_{1}^{\infty} 0\,dy = 1$
$0 + 0.2[y]_{-1}^{0} + 0.2[y]_{0}^{1} + c[\frac{y^2}{2}]_{0}^{1} + 0 = 1$
0.2(0+1) + 0.2(1-0) + c(0.5 - 0) = 1
0.2 + 0.2 + c/2 = 1
0.4 + c/2 = 1
c/2 = 1 - 0.4
c = 2*0.6 = 1.2

Part b:
$F_Y(y) = \int_{-\infty}^{y} f_U(u)\,du = \int_{-\infty}^{y} 0\,du = 0$ $F_Y(y) = \int_{-\infty}^{y} f_U(u)\,du = \int_{-\infty}^{-1} f_U(u)\,du + \int_{-1}^{y} f_U(u)\,du = \int_{-\infty}^{-1} 0\,du + \int_{-1}^{y} 0.2\,du$
$= 0 + 0.2[u]_{-1}^{y} = 0.2(y+1) = 0.2+0.2y$
$F_Y(y) = \int_{-\infty}^{y} f_U(u)\,du = \int_{-\infty}^{-1} f_U(u)\,du + \int_{-1}^{0} f_U(u)\,du + \int_{0}^{y} f_U(u)\,du = \int_{-\infty}^{-1} 0\,du + \int_{-1}^{0} 0.2\,du + \int_{0}^{y}(0.2 + 1.2u)\,du = 0 + 0.2[u]_{-1}^{0} + 0.2[u]_0^y + 1.2[u^2/2]_0^y = 0.2+0.2y+0.6y^2$
$F_Y(y) = \int_{-\infty}^{y} f_U(u)\,du = \int_{-\infty}^{-1} f_U(u)\,du + \int_{-1}^{0} f_U(u)\,du + \int_{0}^{1} f_U(u)\,du + \int_{1}^{y} f_U(u)\,du = \int_{-\infty}^{-1} 0\,du + \int_{-1}^{0} 0.2\,du + \int_{0}^{1}(0.2 + 1.2u)\,du + \int_{1}^{y} 1\,du = 0.2(0+1) + 0.2(1-0) + 1.2(1^2/2 - 0) = 0.2+0.2+0.6 = 1$
So the cumulative distribution function of Y is:

$$F_Y(y) = \begin{cases} 0 & \text{for } -\infty < y \leq -1 \\ 0.2 + 0.2y & \text{for } -1 < y \leq 0 \\ 0.2 + 0.2y + 0.6y^2 & \text{for } 0 < y \leq 1 \\ 1 & \text{for y} > 1 \end{cases}$$

Part c:
Where the solid line is f(y) and the dashed line is F(y):



Part d:
F(-1) = P(Y≤1) = 0
F(0) = P(Y≤0) = 0.2+0.2*0 = 0.2
F(1) = P(Y≤1) = 0.2+0.2(1)+0.6(1)^2 = 0.2+0.2+0.6 = 1

Part e:
P($0 \leq Y \leq 0.5$) = (0.2+0.2*0.5+0.6*0.5^2)-(0.2+0.2*0) = 0.2 + 0.1 + 0.15 - 0.2 = 0.25

Part f:
P(Y>0.5|Y>0.1) = $\frac{1-(0.2+0.2(0.5)+0.6(0.5)^2)}{1-(0.2+0.2(0.1)+0.6(0.1)^2)}$ = 0.55/0.774 = 0.7106

Problem 4.22
E(Y) = $\int_{-1}^{0} 0.2y\,dy + \int_{0}^{1}(0.2y + 1.2y^2)\,dy = 0.4$
E(Y^2) = $\int_{-1}^{0} 0.2y^2\,dy + \int_{0}^{1}(0.2y^2 + 1.2y^3)\,dy = 1.3/3$ Var(Y) = 1.3/3 - 0.4^2 = 0.2733

Problem 4.27
E(Y) = $\int_{-\infty}^{\infty} yf(y)\,dy = \int_0^1 y[3/2y^2+y]\,dy = \int_0^1 y[3/2y^3+y^2]\,dy = [\frac{3}{2}*\frac{y^4}{4}+\frac{y^3}{3}]_0^1$ = (3/2)*(1/4)+(1/3)

= 17/24
E(Y^2) = $\int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2[3/2y^2 + y]\,dy = \int_0^1 (3y^4/2 + y^3)\,dy = [\frac{3}{2} * \frac{y^5}{5} + \frac{y^4}{4}]_0^1 = 11/20$
Var(Y) = 11/20 - $(17/24)^2$ = 0.0483
W = 5 - 0.5Y
E(W) = 5 - 0.5(17/24) = 4.6458
Var(W) = 0.25*0.0483 = 0.012075