Homework 7 Solution

5.3

$P(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}, 0 \le y_1, 0 \le y_2, 1 \le y_1 + y_2 \le 3$

5.7

(a)

$$\begin{aligned}
P(Y_1 < 1, Y_2 > 5) &= \int_5^\infty \int_0^1 e^{-(y_1+y_2)} dy_1 dy_2 \\
&= \int_5^\infty e^{-y_2} dy_2 \int_0^1 e^{-y_1} dy_1 \\
&= e^{-5}(1 - e^{-1}) \\
&= 0.00426
\end{aligned}$$

(b)

$$\begin{aligned}
P(Y_1 + Y_2 < 3) &= \int_0^3 \int_0^{3-y_1} e^{-(y_1+y_2)} dy_2 dy_1 \\
&= \int_0^3 e^{-y_1} \int_0^{3-y_1} e^{-y_2} dy_2 dy_1 \\
&= \int_0^3 e^{-y_1}(1 - e^{y_1-3}) dy_1 \\
&= 1 - 4e^{-3} \\
&= 0.8009
\end{aligned}$$

5.9

(a)

$$\begin{aligned}
1 &= \int_0^1 \int_0^{y_2} k(1-y_2)dy_1dy_2 \\
&= k\int_0^1 y_2(1-y_2)dy_2 \\
&= k/6 \\
k &= 6
\end{aligned}$$

(b) Notice that $0 \le y_1 \le y_2 \le 1$.

$$\begin{aligned}
P(Y_1 \le 3/4, Y_2 \ge 1/2) &= P(Y_1 < 1/2, Y_2 \ge 1/2) + P(1/2 \le Y_1 \le 3/4, Y_2 > 1/2) \\
&= \int_{1/2}^1 \int_0^{1/2} 6(1-y_2)dy_1dy_2 + \int_{1/2}^{3/4} \int_{y_1}^1 6(1-y_2)dy_2dy_1 \\
&= 3/8 + 7/64 \\
&= 31/64 = 0.484375
\end{aligned}$$

5.15

$Y_1 \ge Y_2$

(a)

$$\begin{aligned}
P(Y_1 < 2, Y_2 > 1) &= \int_1^2 \int_1^{y_1} e^{-y_1}dy_2dy_1 \\
&= \int_1^2 (y_1-1)e^{-y_1}dy_1 \\
&= e^{-1} - 2e^{-2}
\end{aligned}$$

(b)

$$\begin{aligned}
P(Y_1 \ge 2Y_2) &= \int_0^\infty \int_{2y_2}^\infty e^{-y_1}dy_1dy_2 \\
&= \int_0^\infty e^{-2y_2}dy_2 \\
&= 1/2
\end{aligned}$$

(c)

$$\begin{aligned} P(Y_1 - Y_2 \leq 1) &= \int_0^\infty \int_{y_2+1}^\infty e^{-y_1} dy_1 dy_2 \\ &= \int_0^\infty e^{-(y_2+1)} dy_2 \\ &= e^{-1} \end{aligned}$$

5.21

(a) Hypergeometric with $N = 9, n = 3, r = 4$, pdf:

$$P(Y_1 = y) = \frac{\binom{4}{y}\binom{5}{3-y}}{\binom{9}{3}}, 0 \leq y \leq 3$$

(b)

$$\begin{aligned} P(Y_1 = 1 | Y_2 = 2) &= \frac{P(Y_1 = 1, Y_2 = 2)}{P(Y_2 = 2)} \\ &= \frac{\binom{4}{1}\binom{3}{2}\binom{9-4-3}{0}}{\binom{9}{3}} / \frac{\binom{3}{2}\binom{6}{1}}{\binom{9}{3}} \\ &= 2/3 \end{aligned}$$

(c)

$$\begin{aligned} P(Y_3 = 1 | Y_2 = 1) &= P(Y_1 = 1 | Y_2 = 1) \\ &= \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_2 = 1)} \\ &= \frac{\binom{4}{1}\binom{3}{1}\binom{9-4-3}{1}}{\binom{9}{3}} / \frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} \\ &= 8/15 \end{aligned}$$

(d) They are the same

5.23

$Y_2 < Y_1$

(a) $f_2(y_2) = \int_{y_2}^1 3y_1 dy_1 = \frac{3}{2}(1 - y_2^2), 0 \leq y_2 \leq 1$

(b) $y_2 \in [0, y_1]$

(c)

$$
\begin{aligned}
f_1(y_1) &= \int_0^{y_1} 3y_1 dy_2 \\
&= 3y_1^2, 0 \le y_1 \le 1 \\
f_{2|1}(y_2|y_1) &= \frac{f(y_1, y_2)}{f_1(y_1)} \\
&= \frac{1}{y_1}, 0 \le y_2 \le 1 \\
P(Y_2 > 1/2 | Y_1 = 3/4) &= \int_{1/2}^{3/4} \frac{1}{3/4} dy_2 \\
&= \frac{1}{3}
\end{aligned}
$$

5.25

(a) Both $Exp(1)$ distribution

$$
\begin{aligned}
f_1(y_1) &= e^{-y_1}, y_1 > 0 \\
f_2(y_2) &= e^{-y_2}, y_2 > 0
\end{aligned}
$$

(b) $\int_1^{2.5} e^{-x} dx = e^{-1} - e^{-2.5} = 0.2858$

(c) $y_2 \in (0, \infty)$

(d) $f_{1|2}(y_1|y_2) = f_1(y_1) = e^{-y_1}, y_1 > 0$

(e) $f_{2|1}(y_2|y_1) = f_2(y_2) = e^{-y_2}, y_2 > 0$

(f) They are the same

(g) They are the same

5.27

$y_1 \le y_2$

(a)

$$
\begin{aligned}
f_1(y_1) &= \int_{y_1}^{1} 6(1 - y_2) dy_2 = 3(1 - y_1)^2, 0 \le y_1 \le 1 \\
f_2(y_2) &= \int_0^{y_2} 6(1 - y_2) dy_1 = 6y_2(1 - y_2), 0 \le y_2 \le 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(Y_2 \leq 1/2 | Y_1 \leq 3/4) &= \frac{P(Y_2 \leq 1/2, Y_1 \leq 3/4)}{P(Y_1 \leq 3/4)} \\
&= \frac{\int_0^{1/2} \int_0^{y_2} 6(1-y_2) dy_1 dy_2}{\int_0^{3/4} 3(1-y_1)^2} \\
&= \frac{1/2}{63/64} = \frac{32}{63}
\end{aligned}
$$

(c) $f_{1|2}(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{1}{y_2}, 0 \leq y_1 \leq y_2 \leq 1$

(d) $f_{2|1}(y_2|y_1) = \frac{f(y_1,y_2)}{f_1(y_1)} = \frac{2(1-y_2)}{(1-y_1)^2}, 0 \leq y_1 \leq y_2 \leq 1$

(e) $P(Y_2 \geq 3/4 | Y_1 = 1/2) = \int_{3/4}^{1} f_{2|1}(y_2|y_1 = 1/2) dy_2 = \int_{3/4}^{1} 8(1-y_2) dy_2 = \frac{1}{4}$