Homework 9 Solution

6.1

CDF is $F_Y(y) = 2y - y^2, 0 \le y \le 1$

(a) $F_{U_1}(u) = P(U_1 \le u) = P(Y \le \frac{u+1}{2}) = F_Y(\frac{u+1}{2}) = 2(\frac{u+1}{2}) - (\frac{u+1}{2})^2$
$f_{U_1}(u) = \frac{1-u}{2}, -1 \le u \le 1$

(b) $F_{U_2}(u) = P(U_2 \le u) = P(Y \ge \frac{1-u}{2}) = 1 - F_Y(\frac{1-u}{2}) = 1 - 2(\frac{1-u}{2}) + (\frac{1-u}{2})^2 = (\frac{u+1}{2})^2$
$f_{U_2}(u) = \frac{u+1}{2}, -1 \le u \le 1$

(c) $F_{U_3}(u) = P(U_3 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u}) = P(0 \le Y \le \sqrt{u}) = F_Y(\sqrt{u}) = 2\sqrt{u} - u$
$f_{U_3}(u) = \frac{1}{\sqrt{u}} - 1, 0 \le u \le 1$

(d) $E(U_1) = \int_{-1}^{1} u f_{U_1}(u) du = -\frac{1}{3}$
$E(U_2) = \int_{-1}^{1} u f_{U_2}(u) du = \frac{1}{3}$
$E(U_3) = \int_{0}^{1} u f_{U_3}(u) du = \frac{1}{6}$

(e) $E(2Y - 1) = 2E(Y) - 1 = -\frac{1}{3}$
$E(1 - 2Y) = 1 - 2E(Y) = \frac{1}{3}$
$E(Y^2) = \frac{1}{6}$

6.4

$f_Y(y) = \frac{1}{4}\exp(-\frac{1}{4}y), F_Y(y) = 1 - \exp(-\frac{1}{4}y), y \ge 0$

(a) $F_Y(u) = F_Y(\frac{u-1}{3}) = 1 - \exp(-\frac{u-1}{12}), u \ge 1$
$f_U(u) = \frac{1}{12}\exp(-\frac{u-1}{12}), u > 1$

(b) $E(U) = 1 + 12 = 13$

6.5

$$
\begin{aligned}
F_Y(y) &= \frac{y-1}{4}, 1 \le y \le 5 \\
F_U(u) &= F_Y(\sqrt{\frac{u-3}{2}}) = [\sqrt{\frac{u-3}{2}} - 1]/4, 5 \le u \le 53 \\
f_U(u) &= \frac{1}{8\sqrt{2}} \frac{1}{\sqrt{u-3}}, 5 \le u \le 53
\end{aligned}
$$

6.23

(a) $Y = \frac{U+1}{2}, \frac{dy}{du} = \frac{1}{2}$
$f_U(u) = \frac{1}{2}2(1 - \frac{u+1}{2}) = \frac{1-u}{2}, -1 \le u \le 1$

(b) $Y = \frac{1-U}{2}, \frac{dy}{du} = -\frac{1}{2}$
$f_U(u) = -\frac{1}{2}2(1 - \frac{1-u}{2}) = \frac{u+1}{2}, -1 \le u \le 1$

(c) $Y = \sqrt{U}, \frac{dy}{du} = \frac{1}{2\sqrt{u}}$
$f_U(u) = \frac{1}{2\sqrt{u}}2(1 - \sqrt{u}) = \frac{1}{\sqrt{u}} - 1, 0 \le u \le 1$

6.28

$F_U(u) = P(-2\log Y \le u) = 1 - F_Y(\exp(-\frac{u}{2})) = 1 - e^{-\frac{u}{2}}; f_U(u) = \frac{1}{2}e^{-\frac{u}{2}}, u > 1$

6.30

$$
\begin{aligned}
f_I(i) &= \frac{1}{2}, 9 \le i \le 11 \\
I &= \sqrt{P/2} \\
\frac{di}{dp} &= (\frac{1}{2})^{\frac{3}{2}} p^{-\frac{1}{2}} \\
f_P(p) &= (\frac{1}{2})^{\frac{3}{2}} p^{-\frac{1}{2}} \frac{1}{2} = [4\sqrt{2p}]^{-1}, 162 \le p \le 242
\end{aligned}
$$

6.37

$p(y) = p^y(1-p)^{1-y}, y \in \{0, 1\}$

(a) $m_{Y_1}(t) = E(e^{tY_1}) = \sum_{y=0}^{1} e^{ty}p(y) = 1 - p + pe^t$

(b) $m_W(t) = E(e^{tW}) = \prod_{i=1}^{n} m_{y_i}(t) = (m_{Y_1}(t))^n = (1 - p + pe^t)^n$

(c) Binomial distribution with n trials and success probability p

6.60

$m_W(t) = m_{Y_1}(t) * m_{Y_2}(t)$:

$$\begin{aligned} m_{Y_2}(t) &= \frac{m_W(t)}{m_{Y_1}(t)} \\ &= \frac{(1-2t)^{-v/2}}{(1-2t)^{-v_1/2}} \\ &= (1-2t)^{-(v-v_1)/2} \end{aligned}$$

Thus Y_2 is a χ^2 random variable with $(v - v_1)$ degrees of freedom.