Module 2 – Section 6

Matched Pair Designs – Tests for Marginal Homogeneity

Overview

- 2 x 2 Tables
- Larger Tables

Variables

- Variable 1
  - $I = 2$ categories
- Variable 2
  - $J = 2$ categories
- Similar variables with same categories
  - Ex. Success/Failure, Yes/No, Correct/Incorrect, etc.

Data

Variable 1	Variable 2
Success	Failure
Success	Failure
Failure	Success
⋮	⋮
⋮	⋮
Failure	Success
Success	Success

Summary Data

- Form contingency table by cross classifying observations

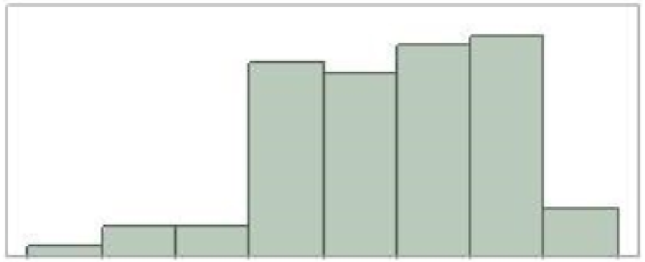
	Variable 2		
Variable 1	Success	Failure	Total
Success	Y_{11}	Y_{12}	$Y_{1.}$
Failure	Y_{21}	Y_{22}	$Y_{2.}$
Total	$Y_{.1}$	$Y_{.2}$	n

Ex. STAT 101 Questions

- Two questions presented to all students in an introductory statistics course
- Score both questions as either Correct or Incorrect

Ex. STAT 101 Questions

A histogram for the sale price of 100 cars is pictured below.



Question A3: Choose the best description of the horizontal axis for the histogram.	Question A4: Choose the best description of the vertical axis for the histogram.
a. The number of cars *b. The sale prices* c. The mean sales price for each bin d. The median sales price for each bin	*a. The number of cars* b. The sale prices c. The mean sales price for each bin d. The median sales price for each bin

Ex. Data

Variable 1	Variable 2
Correct	Correct
Correct	Correct
Correct	Correct
⋮	⋮
⋮	⋮
Incorrect	Incorrect
Incorrect	Incorrect

Ex. Contingency Table

	Question A4		
Question A3	Correct	Incorrect	Total
Correct	469	12	481
Incorrect	30	78	108
Total	499	90	589

Marginal Homogeneity

- Proportion of Success on each Variable
  - Variable 1: $p_{1.}$
  - Variable 2: $p_{.1}$
- Is the proportion of success on each variable the same?

	Variable 2		
Variable 1	Correct	Incorrect	Total
Correct	p_{11}	p_{12}	$p_{1.}$
Incorrect	p_{21}	p_{22}	$p_{2.}$
Total	$p_{.1}$	$p_{.2}$	1

Marginal Homogeneity

- Two marginal proportions:
  - $p_{1.} = p_{11} + p_{12}$
  - $p_{.1} = p_{11} + p_{21}$

- If $p_{1.} = p_{.1}$, then

$$p_{12} = p_{21}$$

	Variable 2		
Variable 1	Correct	Incorrect	Total
Correct	p_{11}	p_{12}	$p_{1.}$
Incorrect	p_{21}	p_{22}	$p_{2.}$
Total	$p_{.1}$	$p_{.2}$	1

McNemar’s Test of Marginal Homogeneity

- Proportion of success on Variable 1 is same as proportion of success on Variable 2.
  - $H_0: p_{1.} = p_{.1}$
- Proportion of success on Variable 1 is different from proportion of success on Variable 2.
  - $H_a: p_{1.} \neq p_{.1}$

Data

- Compare Y_{12} to Y_{21}
- Large differences indicate $p_{12} \neq p_{21}$ which implies $p_{1.} \neq p_{.1}$

	Variable 2		
Variable 1	Success	Failure	Total
Success	Y_{11}	Y_{12}	$Y_{1.}$
Failure	Y_{21}	Y_{22}	$Y_{2.}$
Total	$Y_{.1}$	$Y_{.2}$	n

Test Statistic

$$z^2 = \frac{(Y_{12} - Y_{21})^2}{Y_{12} + Y_{21}}$$

- Large values of z^2 indicate $p_{12} \neq p_{21}$ which implies $p_{1.} \neq p_{.1}$

P-value

- As long as $Y_{12} + Y_{21} \geq 20$, the distribution of z^2 can be well approximated by χ_1^2

$$P(\chi_1^2 > z^2)$$

Ex. STAT 101 Questions

	Question A4		
Question A3	Correct	Incorrect	Total
Correct	469	12	481
Incorrect	30	78	108
Total	499	90	589

Ex. Null and Alternative Hypotheses

- The proportion of correct responses on Question 1 is the same as the proportion of correct responses on Question 2.
  - $H_0 \colon p_{1.} = p_{.1}$
- The proportion of correct responses on Question 1 is different from the proportion of correct responses on Question 2.
  - $H_a \colon p_{1.} \neq p_{.1}$

Ex. Test Statistic

$$z^2 = \frac{(Y_{12} - Y_{21})^2}{Y_{12} + Y_{21}} = \frac{(12 - 30)^2}{12 + 30} = 7.714$$

Ex. P-value and Conclusion

$$P(\chi_1^2 > 7.714) = 0.0055$$

- Conclusion: We have very strong evidence of a significant difference in the proportion of students correctly answering both questions.

Confidence Interval

- Estimate $p_{1.} - p_{.1}$

$$\hat{p}_{1.} - \hat{p}_{.1} = \frac{Y_{1.}}{n} - \frac{Y_{.1}}{n}$$

$$SE(\hat{p}_{1.} - \hat{p}_{.1}) = \frac{\sqrt{(Y_{12} + Y_{21}) - (Y_{12} - Y_{21})^2/n}}{n}$$

Confidence Interval

$$(\hat{p}_{1.} - \hat{p}_{.1}) \pm z_{1-\frac{\alpha}{2}} SE(\hat{p}_{1.} - \hat{p}_{.1})$$

Ex. STAT 101 Questions

$$SE(\hat{p}_{1.} - \hat{p}_{.1}) = \frac{\sqrt{(12+30) - (12-30)^2/589}}{589} = 0.0109$$

$$\hat{p}_{1.} - \hat{p}_{.1} = \frac{481}{589} - \frac{499}{589} = \frac{-18}{589} = -0.0306$$

Ex. STAT 101 Questions

$$(\hat{p}_{1.} - \hat{p}_{.1}) \pm z_{1-\frac{\alpha}{2}} SE(\hat{p}_{1.} - \hat{p}_{.1})$$

$$= -0.0306 \pm 1.96(0.0109) = (-0.0520, -0.0092)$$

Ex. STAT 101 Questions

- We are 95% confident the proportion of all STAT 101 students who answer correctly when asked to identify the horizontal axis in a histogram is between 0.0092 and 0.0520 lower than the proportion of all STAT 101 students who answer correctly when asked to identify the vertical axis in a histogram.

Extension to Larger Tables

- Variable 1
  - Question with $I = J$ categories
- Variable 2
  - Question with $I = J$ categories
- Similar variables with the same categories.

Population Proportions

	Variable 2				
Variable 1	Cat 1	Cat 2	Cat 3	Cat 4	Total
Cat 1	p_{11}	p_{12}	p_{13}	p_{14}	$p_{1.}$
Cat 2	p_{21}	p_{22}	p_{23}	p_{24}	$p_{2.}$
Cat 3	p_{31}	p_{32}	p_{33}	p_{34}	$p_{3.}$
Cat 4	p_{41}	p_{42}	p_{43}	p_{44}	$p_{4.}$
Total	$p_{.1}$	$p_{.2}$	$p_{.3}$	$p_{.4}$	1

Population Proportions

- Proportion of responses on each Variable
  - Variable 1: $p_{1.}, p_{2.}, p_{3.}, p_{4.}$
  - Variable 2: $p_{.1}, p_{.2}, p_{.3}, p_{.4}$
- Is the proportion of responses the same for each variable?

Extension of McNemar's Test

- Distribution of Variable 1 is same as distribution of Variable 2.
  - $H_0 : p_{j.} = p_{.j}$ for all $j = 1, 2, \dots, J$
- Distribution of Variable 1 is different than distribution of Variable 2.
  - $H_a : p_{j.} \neq p_{.j}$ for at least one $j = 1, 2, \dots, J$

Test Statistic

- $\hat{d}_j = \frac{Y_{j.} - Y_{.j}}{n}$ for $j = 1, 2, \dots, J-1$

- $\hat{v}_{jj} = \frac{Y_{j.} + Y_{.j} - 2Y_{jj}}{n}$ for $j = 1, 2, \dots, J-1$

- $\hat{v}_{ij} = \frac{-(Y_{ij} + Y_{ji})}{n}$ for $i \neq j$,

  i and $j = 1, 2, \dots, J-1$

Test Statistic

$$W = n\hat{d}'\hat{V}^{-1}\hat{d}$$

where $\hat{d} = (\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_{J-1})$ and
$\hat{V}$ is a symmetric $J - 1$ by $J - 1$ matrix
with components $\hat{v}_{jj}$ and $\hat{v}_{ij}$

P-value

- P-value
  - $P(\chi^2_{J-1} > W)$

Ex. Baby Pictures

- Correct Baby – Baby B
- Baby B is also wearing a hat
- Are respondents choosing baby B due to the hat instead of resemblance with father?

Ex. Variables

- Question 1 – Pick baby when father is not pictured.
  - Baby A, B, C, D
- Question 2 – Pick baby when father is pictured.
  - Baby A, B, C, D
- Do respondents choose the babies with the same probabilities for both questions?

Ex. Data

Question 1	Question 2
A	A
A	A
A	A
⋮	⋮
⋮	⋮
D	D
D	D

Ex. Contingency Table

	Father Pictured				
Father Not Pictured	Baby A	Baby B	Baby C	Baby D	Total
Baby A	11	6	2	11	30
Baby B	14	14	15	23	66
Baby C	5	11	14	24	54
Baby D	10	3	11	21	45
Total	40	34	42	79	195

Ex. Null and Alternative Hypotheses

- H_0: $p_{A.} = p_{.A}$; $p_{B.} = p_{.B}$; $p_{C.} = p_{.C}$; $p_{D.} = p_{.D}$
- H_A: at least one $p_{j.} \neq p_{.j}$ for $j = A, B, C, D$

Ex. Test Statistic and P-value

- Test Statistic
  - $W = 24.2329$
- P-value
  - $P(\chi_3^2 > 24.2329) = 0.00002$

Ex. Conclusion

- The distribution of responses on the two questions is not the same.
  - Without father pictured, respondents were drawn to some babies over others. Baby B was one of them.
  - Once father was pictured, respondents were not drawn to baby B.