Homework 11

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Problem 8.4
Part a:
They are equal.

Part b:
$MSE(\hat{\theta}) > V(\hat{\theta})$

Problem 8.6
Part a:
$E(\hat{\theta_3}) = E(a\hat{\theta_1} + (1-a)\hat{\theta_2})$
$E(\hat{\theta_3}) = aE(\theta_1) + (1-a)E(\hat{\theta_2})$
$E(\hat{\theta_3}) = a\theta + (1-a)\theta = \theta$

Part b:
If $\hat{\theta_1}$ and $\hat{\theta_2}$ are independent.
$V(\hat{\theta_3}) = V(a\hat{\theta_1} + (1-a)\hat{\theta_2}) = a^2V(\hat{\theta_1}) + (1-a)^2V(\hat{\theta_2})$
To minimize the variance of $\hat{\theta_3}$, then constant a can be determined like so: $V(\hat{\theta_3}) = a^2{\sigma^2}_1 + (1-a)^2{\sigma^2}_2$
$\frac{d}{da}(V(\hat{\theta_3})) = 0 => 2a{\sigma^2}_1 + 2(1-a)(-1){\sigma^2}_2 = 0$
$a({\sigma^2}_1 + {\sigma^2}_2) = {\sigma^2}_2$
So $a = \frac{{\sigma^2}_2}{{\sigma^2}_1+{\sigma^2}_2}$

Problem 8.8
Part a:
$\hat{\sigma_1}, \hat{\sigma_2}, \hat{\sigma_3}, \hat{\sigma_5}$ are simple linear combinations of $Y_1, Y_2 and Y_3$. So it can be shown that these four estimators are unbiased. $\hat{\sigma_4}$ has an exponential distribution with mean $\theta/3$ so this estimator is biased.

Part b:
$V(\hat{\sigma_1}) = \theta^2$
$V(\hat{\sigma_2}) = \theta^2/2$
$V(\hat{\sigma_3}) = 5\theta^2/9$
$V(\hat{\sigma_5}) = \theta^2/9$
So the estimator $V(\hat{\sigma_5})$ is unbiased and also has the smallest variance.

Problem 8.12
Part a:
There is uniform distribution. So $E(Y_i) = \frac{\theta+\theta+1}{2} = \theta + 0.5$. So $E(\bar{Y}) = \theta + 0.5$ and $B(\bar{Y}) = 0.5$

Part b:
Based on $\bar{Y}$, the unbiased estimator is $\bar{Y} - 0.5$.

Part c:
$V(\bar{Y}) = \frac{1}{12n}$. So $MSE(\bar{Y}) = \frac{1}{12n} + 0.25$

Problem 8.18
The density function for $Y_{(1)}$ is like so:
$g_{(1)}(y) = \frac{n}{\theta}(1 - \frac{y}{\theta})^{n-1}, 0 \leq y \leq \theta$.
Then the $E(Y_{(1)}) = \theta/(n+1)$ and the unbiased estimator for θ is $(n+1)Y_{(1)}$

Problem 9.2
Part a:
$E(\hat{\mu_1}) = 1/2(E(Y_1) + E(Y_2)) = 1/2(\mu + \mu) = \mu$
$E(\hat{\mu_2}) = \frac{\mu}{4} + \frac{(n-2)\mu}{2(n-2)} + \frac{\mu}{4} = \mu$
$E(\hat{\mu_3}) = E(\bar{Y}) = \mu$
So all three estimators are unbiased.

Part b:
First, find the variances of the three estimators.
$V(\hat{\mu_1}) = \frac{1}{4}(\sigma^2 + \sigma^2) = 1/2\sigma^2$
$V(\hat{\mu_2}) = \frac{\sigma^2}{16} + \frac{(n-2)\sigma^2}{4(n-2)^2} + \frac{\sigma^2}{16} = \frac{\sigma^2}{8} + \frac{\sigma^2}{4n-8}$
$V(\hat{\mu_3}) = \frac{\sigma^2}{n}$
So the efficiency$(\hat{\mu_3}, \hat{\mu_2}) = \frac{n^2}{8(n-2)}$ and efficiency$(\hat{\mu_3}, \hat{\mu_1}) = \frac{n}{2}$

Problem 9.3
Part a:
The density function of $\hat{\theta_2} = Y_{(n)} : g_n(y) = n(y - \theta)^{n-1}, \theta \leq y \leq \theta + 1$.
So $E(\hat{\theta_1}) = E(\bar{Y}) - 1/2 = \theta + 1/2 - 1/2 = \theta$. and $E(\hat{\theta_2}) = E(Y_{(n)}) - \frac{n}{n+1} = \theta$

Part b:
Find the variances:
$V(\hat{\theta_1}) = V(\bar{Y}) = \frac{\sigma^2}{n} = 1/12n$
$V(\hat{\theta_2}) = V(Y_{(n)}) = \frac{n}{(n+2)(n+1)^2}$
So efficiency$(\hat{\theta_1}, \hat{\theta_2}) = \frac{12n^2}{(n+2)(n+1)^2}$

Problem 9.7
We know that the estimator $\hat{\theta_1}$ is an unbiased estimator so $MSE(\hat{\theta_1}) = V(\hat{\theta_1}) = \theta^2$. The $\hat{\theta_2} = \bar{Y}$ is unbiased for θ and $V(\hat{\theta_2}) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$. So the efficiency$(\hat{\theta_1}, \hat{\theta_2}) = \frac{1}{n}$