STAT 330 (Spring 2021) - Exam 1

March 5, 2021

1. Counting (3 random questions)

   (a) How many different passwords are there that are 8 digits long and only contain lowercase letters?
   Answer: $26^8 = 208,827,064,576$

   (b) How many debit card pins are there if you can only use unique numbers (0-9) and the pin is 6 numbers long?
   Answer: n = 10, k = 6 $\frac{n!}{(n-k)!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151,200$

   (c) One coin is flipped and one 6-sided die is rolled. How many different combinations of coin flips and die rolls are there?
   Answer: Two possible coin flips and 6 possible rolls, so 2*6 = 12.

   (d) A combination lock has 3 dials each containing the numbers 0 to 9. How many different combinations are there?
   Answer: There are $n = 10$ digits and $k = 3$ choices. The digits can be repeated and order matters and thus this is a permutation with replacement. The number of combinations are $10^3 = 1000$.

   (e) A game of pool contains 15 balls numbered from 1 to 15 and the goal is to knock these balls into the pockets of the pool table. How many different orders are there for knocking all 15 balls into the pockets?
   Answer: There are $n = 15$ balls and $k = 15$ choices. The balls cannot be repeated and order matters and thus this is a permutation without replacement. The number of different orders is 15!.

2. Consider rolling two 4-sided dice. (3 random questions)

   (a) What is the sample space for this experiment?
   Answer:
   $\{11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44\}$

   (b) What is the probability of rolling a sum of 3?
   Answer: A sum of 3 occurs in two ways: $\{12, 21\}$. $2/16 = 0.125$

   (c) What is the probability that exactly one die has a 4?
   Answer: This occurs 6 times $\{14, 24, 34, 41, 42, 43\}$ and thus the probability is $6/16 = 0.375$.

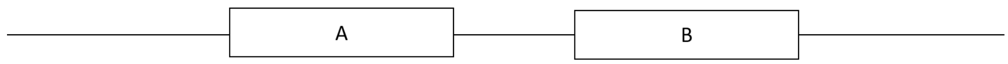
   (d) What is the probability at least one of the dice is an even number?
   Answer: Calculate one minus the probability that neither dice is an even number, i.e. both dice are odd numbers. This occurs 4 times $\{11, 13, 31, 33\}$ and thus the probability is $1 - 4/16 = 12/16 = 0.75$.

   (e) What is the probability the sum of the two dice is less than or equal to 5?
   Answer: The sum of the two dice are less than or equal to 5 when the following results are observed $\{11, 12, 13, 14, 21, 22, 23, 31, 32, 41\}$. Thus this occurs with probability $10/16 = 0.625$.

3. Reliability. For the following questions assume the reliability of component A is 0.9, component B is 0.8, and component C is 0.7.

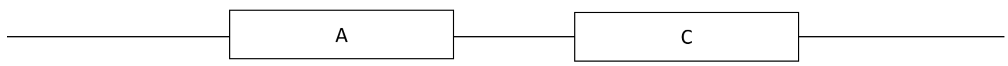
   (a) Reliability serial (1 random question)

      i. What is the reliability of a system that has components A and B in serial?



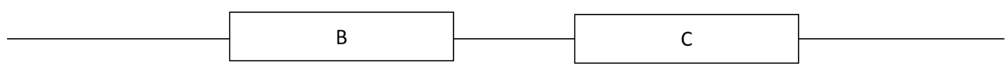
Answer: Both components need to work, so $P(A \text{ and } B) = P(A)P(B) = 0.9 * 0.8 = 0.72$.

      ii. What is the reliability of a system that has components A and C in serial?



Answer: Both components need to work, so $P(A \text{ and } C) = P(A)P(C) = 0.9 * 0.7 = 0.63$.

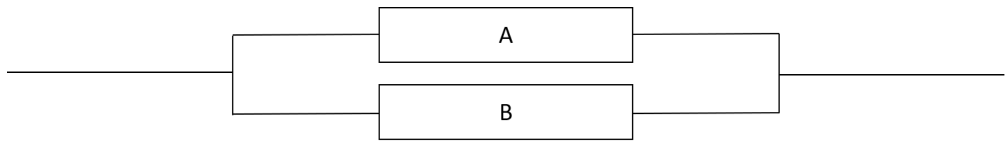
      iii. What is the reliability of a system that has components B and C in serial?



Answer: Both components need to work, so $P(B \text{ and } C) = P(B)P(C) = 0.8 * 0.7 = 0.56$.

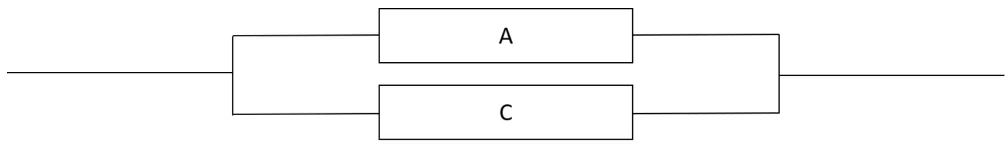
(b) Reliability parallel (1 random question)

i. What is the reliability of a system that has components A and B in parallel?



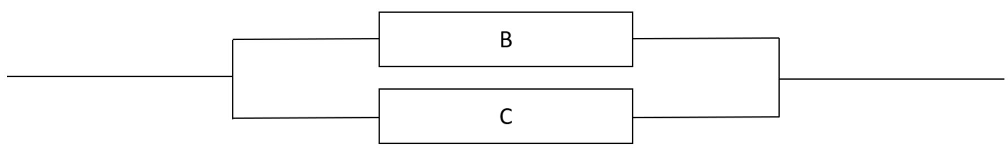
Answer: At least one component needs to work, so $P(A \text{ or } B) = 1 - P(\overline{A} \text{ and } \overline{B}) = 1 - (1 - 0.9)(1 - 0.8) = 0.98$.

ii. What is the reliability of a system that has components A and C in parallel?



Answer: At least one component needs to work, so $P(A \text{ or } C) = 1 - P(\overline{A} \text{ and } \overline{C}) = 1 - (1 - 0.9)(1 - 0.7) = 0.97$.

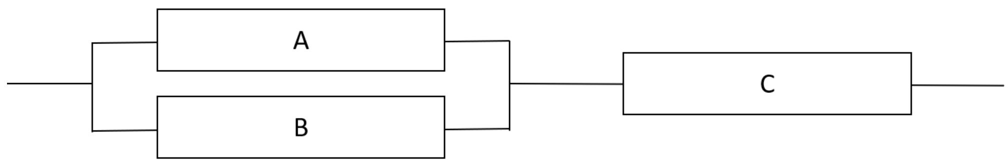
iii. What is the reliability of a system that has components B and C in parallel?



Answer: At least one component needs to work, so $P(B \text{ or } C) = 1 - P(\overline{B} \text{ and } \overline{C}) = 1 - (1 - 0.8)(1 - 0.7) = 0.94$.

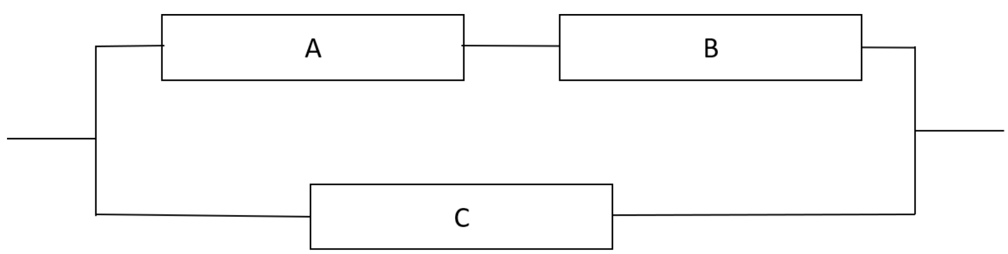
(c) Reliability both (1 random question)

i. What is the reliability of a system that has components A and B parallel and then serial with component C?



Answer: First determine the probability the A and B parallel sub-system works. At least one component needs to work, so $P(A \text{ or } B) = 1 - P(\overline{A} \text{ and } \overline{B}) = 1 - (1 - 0.9)(1 - 0.8) = 0.98$.
Then both this subsystem and component C need to work, so $P((A \text{ or } B) \text{ and } C) = P(A \text{ or } B)P(C) = 0.98 * 0.7 = 0.686$.

ii. What is the reliability of a system that has components A and B serial and then parallel with component C?



Answer: First determine the probability the A and B serial sub-system works. At least one component needs to work, so $P(A \text{ and } B) = 0.9 * 0.8 = 0.72$.
Then either this subsystem or component C need to work which is 1 minus the probability that they both don't work. so $P((A \text{ and } B) \text{ or } C) = 1 - P(\overline{A \text{ and } B} \text{ and } \overline{C}) = 1 - P(\overline{A \text{ and } B})P(\overline{C}) = 1 - (1 - 0.72)(1 - 0.7) = 0.916$.

4. In Candy Crush, players can use boosters to help defeat levels. For a particular level, the following table provides the joint probability of players beating the level and number of boosters used.

	Number of boosters		
Beat level	0	1	2
Yes	0.00	0.20	0.50
No	??	0.10	0.05

Answer the following questions based on this table. (3 random questions)

(a) What is the probability that a user uses 2 boosters and beats the level?
Answer: 0.50

(b) What is the probability that a user uses no boosters and does not beat the level?
Answer: Probabilities must sum to 1, so $1 - 0.2 - 0.5 - 0.1 - 0.05 = 0.15$.

(c) What is the marginal probability a user beats the level?
Answer: $0.2 + 0.5 = 0.7$

(d) What is the conditional probability that a user beats the level given that they use 2 boosters?
Answer: $0.5/(0.5 + 0.05) = 10/11 = 0.909$

(e) Are the number of boosters and whether or not the user beats the level independent?
Answer: No since $0.5 = P(W, B = 2) \neq P(W)P(B = 2) = 0.7 * 0.55 = 0.385$.

5. Diagnostic testing (1 random question)

(a) Suppose a diagnostic test has a sensitivity of 0.99 and a specificity of 0.95. The disease being tested for has an overall prevalence of 0.1. If a test result comes back positive, what is the probability of having the disease?

Answer:

$$\frac{0.99 * 0.1}{0.99 * 0.1 + (1 - 0.95)(1 - 0.1)} = 0.6875$$

(b) Suppose a diagnostic test has a sensitivity of 0.95 and a specificity of 0.99. The disease being tested for has an overall prevalence of 0.1. If a test results comes back positive, what is the probability the individual has the disease?

Answer:

$$\frac{0.95 * 0.1}{0.95 * 0.1 + (1 - 0.99)(1 - 0.1)} = 0.9134615$$

6. Let $X \sim Bern(1/3)$. (all questions)

(a) $E[X]$
Answer: $1/3 = 0.3333333$

(b) $Var[X]$
Answer:

$$\frac{1}{3}\left(1-\frac{1}{3}\right)=\frac{2}{9}=0.2222222$$

(c) $P(X = 0)$
Answer: $2/3 = 0.6666667$

7. Let $Y \sim Bin(16, 0.4)$. (3 random questions)

   (a) $E[Y]$
   Answer: 16*0.4 = 6.4

   (b) $Var[Y]$
   Answer: 16*0.4(1-0.4) = 3.84

   (c) $P(Y = 4)$
   Answer: $\binom{16}{4}0.4^4(1-0.4)^{16-4} = 0.1014206$

   (d) $P(Y \leq 10)$
   Answer: $\sum_{y=0}^{10}\binom{16}{y}0.4^y(1-0.4)^{16-y} = 0.9808581$

8. Let $X \sim Geo(0.8)$. (3 random questions)

   (a) $E[X]$
   Answer: $1/0.8 = 1.25$

   (b) $Var[X]$
   Answer: $\frac{1-0.8}{0.8^2} = 0.3125$

   (c) $P(X = 1)$
   Answer: 0.8

   (d) $P(X > 3)$
   Answer: $(1 - 0.8)^3 = 0.008$

9. Let $Y \sim Po(5)$. (3 random questions)

(a) $E[Y]$
Answer: 5

(b) $Var[Y]$
Answer: 5

(c) $P(Y = 3)$
Answer: $\frac{5^3 e^{-5}}{3!} = 0.1403739$

(d) $P(Y < 5)$
Answer: $P(Y < 5) = P(Y \leq 4) = \sum_{y=0}^{4} \frac{5^y e^{-5}}{y!} = 0.4404933.$

10. Suppose you roll a fair 20-sided die 2 times and you win if you get a 20 in either roll. (3 random questions)

    Answer: Let Y be the number of 20s you get and assume $Y \sim Bin(n, p)$ with $n = 2$ and $p = 1/20$.

    (a) What is the expected number of 20s?
    Answer: $E[Y] = 2 * (1/20) = 1/10 = 0.1$

    (b) What is the variance in the expected number of 20s?
    Answer: $Var[Y] = 2 * (1/20)(1 - 1/20) = 0.095$

    (c) What is the probability you win?
    Answer: You win if Y is greater than or equal to 1, so $P(W) = P(Y >= 1) = 1 - P(Y = 0) = 1 - 0.9025 = 0.0975$

11. A particular website has 15 visitors per hour. Assume each hour is independent of all other hours. (3 random questions)

Answer: Let X be the number of visitors in the next hour.

(a) What is the expected number of visitors in the next hour?

Answer: $E[X] = 15$

(b) What is the variance of the number of visitors in the next hour?

Answer: $Var[X] = 15$

(c) What is the probability there will be exactly 13 visitors in the next hour?

Answer: $P(X = 13) = \frac{15^{13}e^{-15}}{13!} = 0.0956068$

(d) What is the probability there will be more than 10 visitors in the next hour?

Answer: $P(X > 10) = 1 - P(X \leq 10) = \sum_{y=0}^{4} \frac{15^y e^{-15}}{y!} = 0.8815356$

12. In Minecraft, you can trade with Piglins to obtain Ender Pearls. You continue trading with a Piglin until you obtain one Ender Pearl. For the following questions, assume the probability of obtaining an Ender Pearl is 5%. (3 random questions)

    Answer: Let Y be the number of trades when you receive your first Ender Pearl and assume $Y \sim Geo(0.05)$.

    (a) What is the expected number of trades when you receive your first Ender Pearl?
    Answer: $E[Y] = 1/0.05 = 20$.

    (b) What is the variance in the number of trades when you receive your first Ender Pearl?
    Answer: $Var[Y] = (1 - 0.05)/0.05^2 = 380$

    (c) What is the probability you get the Ender Pearl on your first trade?
    Answer: 0.05

    (d) What is the probability you will have to trade more than 30 times before you get your first Ender Pearl?
    Answer: $P(Y > 30) = (1 - 0.05)^{30} = 0.2146388$.