# Stat 330: Module 4 Homework Solution

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1. Classify each of the following stochastic processes as discrete-time or continuous-time, and discrete-space or continuous-space.

   **Answer:**

   (a) continuous time and continuous state

   (b) discrete time and continuous state

   (c) continuous time and discrete state

   (d) discrete time and discrete state

   (e) discrete time and discrete state

2. A certain machine used in a manufacturing process can be in one of three states: Fully operational ("full"), partially operational ("part"), or broken ("broken"). If the machine is fully operational today, there is a .7 probability it will be fully operational again tomorrow, a .2 chance it will be partially operational tomorrow, and otherwise tomorrow it will be broken. If the machine is partially operational today, there is a .6 probability it will continue to be partially operational tomorrow and otherwise it will be broken (because the machine is never repaired in its partially operational state). Finally, if the machine is broken today, there is a .8 probability it will be repaired to fully operational status tomorrow; otherwise, it remains broken. Let X = the state of the machine on day n.

   **Answer:**

   (a) {full, part, broken}

   (b)

   $$P = \begin{array}{c} \\ \text{full} \\ \text{part} \\ \text{broken} \end{array} \begin{array}{c} \begin{array}{ccc} \text{full} & \text{part} & \text{broken} \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.8 & 0 & 0.2 \end{pmatrix} \end{array}$$

   (c)

   $$P^2 = P \times P = \begin{pmatrix} 0.57 & 0.26 & 0.17 \\ 0.32 & 0.36 & 0.32 \\ 0.72 & 0.16 & 0.12 \end{pmatrix}$$

   Thus $P^2[\text{broken}, \text{broken}] = 0.12$.

3. Information bits (0s and 1s) in a binary communication system travel through a long series of relays. At each relay, a "bit-switching" error might occur. Suppose that at each relay, there is a 4% chance of a 0 bit being switched to a 1 bit and a 5% chance of a 1 becoming a 0. Let X0 = a bit's initial parity (0 or 1), and let Xn = the bit's parity after traversing the nth relay.

   **Answer:**

   (a)

   $$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.96 & 0.04 \\ 0.05 & 0.95 \end{pmatrix} \end{array}$$

   (b) Let $P_0 = (0.8, 0.2)$, then

   $$P_0 \times P = (0.778, 0.222)$$

   and thus we see 77.8% 0s and 22.2% 1s exiting the first relay.

(c)
$$P_0 \times P^3 = P_0 \times P \times P \times P = (0.7398, 0.2602)$$
and thus we see 74% 0s and 26% 1s exiting the third relay.

4. A hamster is placed into the three-chambered circular habitat shown in the figure below. Sitting in any chamber, the hamster is equally likely to next visit either of the two adjacent chambers. Let Xn = the nth chamber visited by the hamster.

**Answer:**

(a)
$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \end{array}$$

(b)
$$P^2 = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$
all elements $> 0$ so this Markov chain is regular.

(c) $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

(d) $\frac{1}{2}$

5. A Markov chain has 3 possible states: A, B, and C. Every hour, it makes a transition to a different state. From state A, transitions to states B and C are equally likely. From state B, transitions to states A and C are equally likely. From state C, it always makes a transition to state A.

**Answer:**

(a)
$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) Let $P_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then
$$P_0 \times P \times P = (0.4167, 0.25, 0.333).$$

(c) $P^4$ has all entries >0 thus this Markov chain is regular.

(d) $(\frac{4}{9}, \frac{2}{9}, \frac{1}{3})$

6. Radio blackouts are among the most common space weather events to affect Earth. Minor radio blackouts occur, on average, twice per year. Use a Poisson process to model the phenomenon.

**Answer:**

(a) $X \sim \text{Pois}(1)$, the $P(X = 3) = 0.0613$.

(b) $X \sim \text{Pois}(4)$, the $E(X) = 4$.

(c) $T \sim \text{Exp}(2)$ then $1 - e^{-2t} = 0.5$ then $t = 0.3466$ years.

(d) $T$ is time till 4th blockout the $T \sim \text{Gamma}(4, 2)$, $P(T < 2) = P(X \geq 4)$ when $X \sim \text{Pois}(4)$ then $P(X \geq 4) = 0.5665$,

7. Suppose that minor errors occur on a computer in a space station, which will require re-calculation. Assume the occurrence of errors follows a Poisson process with a rate of 1/2 per hour.

**Answer:**

(a) $P(X_{24} = 0) = 6.1442 \times 10^{-6}$ when $X_{24} \sim \text{Pois}(12)$

(b) $P(X_{24} > 25) = 1 - P(X_{24} \leq 25) = 0.000308$

(c) $P(X_2 \leq 5) = P(X_2 - X_0 \leq 4) = 0.9963$ where $X_2 - X_0 \sim \text{Pois}(1)$.

8. During the daily lunch rush, arrivals at the drive-thru at a nearby McDonald follow a Poisson process with a rate of 1 customers per minute.

**Answer:**

(a) $X_{60} \sim \text{Pois}(60)$, then $E(X_{60}) = 60$ and $SD(X_{60}) = \sqrt{(60)} = 7.746$

(b) If $X$ be the number of customer in 5 mins, then $X \sim \text{Pois}(5)$ and $P(X > 10) = 1 - P(X \leq 10) = 0.01369$.

(c) If $T$ be the time until next customer, then $T \sim \text{Exp}(1)$ and $P(T < 0.5) = 0.3935$.

(d) If $T$ be the time until 100th cunstomer, then $T \sim \text{Gamma}(100, 1)$ and $E(T) = 100$, $Var(T) = 100$, and $SD(T) = 10$.

9. Extra Credit (2 points) A variation of a poisson process is a Birth and Death process. In a Birth and Death process, the state of a system can change if something enters the system (a birth) or leaves the system (a death). Suppose that we model the time till a birth as B $\sim Exp(\lambda)$ and the time till a death as D $\sim Exp(\mu)$, where B and D are independent of each other. Say we are interested in the time till the system changes state. Define a random variable as: Y = time till state changes. We can write Y as: Y = Min(B, D). Give the distribution of Y and its parameter(s).

**Answer:**

Assume $B \sim \text{Exp}(\lambda)$ and $D \sim \text{Exp}(\mu)$ and B and D are independent of each other. If $Y \sim \text{Min}(B, D)$, then

$$P(Y > t) = P(B > t, D > t) = P(B > t)P(D > t) = e^{-t\lambda}e^{-t\mu} = e^{-t(\lambda+\mu)}$$

and the cdf of Y is

$$F_Y(t) = P(Y \leq t) = 1 - e^{-t(\lambda+\mu)}.$$

This is the cdf of an exponential distribution with rate $\lambda + \mu$, i.e. $Y \sim \text{Exp}(\lambda + \mu)$.