Method of Moments & Maximum Likelihood Estimation

STAT 330 - Iowa State University

Outline

In this lecture students will learn about two general methods of finding estimators for parameters.

Estimating Parameters

2 General Methods for estimating parameters:

1. Method of moments estimation (MoM)
2. Maximum likelihood estimation (MLE)

1st Attempt at finding estimators

$$\mu = \mathbb{E}(X) \longrightarrow \hat{\mu} = \frac{1}{n}\sum X_i = \bar{X}$$

$$\sigma^2 = Var(X) \longrightarrow \hat{\sigma}^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2 = S^2$$

"Plug in Estimators"

Method of Moments (MoM)

Method of Moments (MoM)

Definition:

- The *k^{th} moment* of a R.V X is defined as $\mu_k = E(X^k)$
- The *k^{th} sample moment* is defined as $m_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$

The method of moments (MoM) estimators for parameters are found by equating (known) sample moments to (unknown) population moments, and then solving for the parameters in terms of the data.

- If our model has more than one unknown parameter, we need to make equations with more than one moment.
- In general, need k equations to derive MoM estimators for k parameters.

MOM Cont.

To obtain MoM estimators for k parameters: Set the sample moments (m_k) equal to population moments (μ_k), and solve.

- $m_1 = \mu_1 \rightarrow \frac{1}{n}\sum x_i = E(X)$
- $m_2 = \mu_2 \rightarrow \frac{1}{n}\sum x_i^2 = E(X^2)$

  $\vdots$
- $m_k = \mu_k \rightarrow \frac{1}{n}\sum x_i^k = E(X^k)$

Note:

- MoM estimators may be biased
- Sometimes you can get estimates outside of parameter space

MoM Examples

MoM Example

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Example 1: Let $X_1, \ldots, X_n \overset{iid}{\sim} Geo(p)$

Estimate one parameter $p \rightarrow$ need the first moment.

- 1^{st} (population) moment: $\mu_1 = E(X) = \frac{1}{p}$.
- 1^{st} sample moment is $m_1 = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$

Set 1^{st} moment equal 1^{st} sample moment, and solve for p.

$$\frac{1}{p} = \bar{X} \rightarrow \boxed{\hat{p}_{MoM} = \frac{1}{\bar{X}}}$$

MoM Examples Cont.

?

Example 2: Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Estimate two parameters $\rightarrow$ need first two moments

Set the first two moments equal to the first two sample moments.

1. $\frac{1}{n}\sum_{i=1}^{n} X_i = E(X)$
2. $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X^2)$

For our random variables, $E(X) = \mu$ and $Var(X) = \sigma^2$

From Eq 1, we have $\frac{1}{n}\sum X_i = E(X) = \mu$

$\rightarrow \hat{\mu}_{MoM} = \frac{1}{n}\sum X_i$

$\rightarrow \boxed{\hat{\mu}_{MoM} = \bar{X}}$

MoM Examples Cont.

$Var(X) = E(X^2) - [E(X)]^2 \to E(X^2) = Var(X) + [E(X)]^2 = \sigma^2 + \mu^2$

From Eq 2. we have:

$$\underline{\frac{1}{n}\sum_{i=1}^{n} X_i^2} = E(X^2) = \underline{\sigma^2 + \mu^2}$$

$$\to \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$$

$$\to \underline{\hat{\sigma}^2_{MoM}} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \hat{\mu}^2_{MoM}$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \quad *$$

"Biased version" $\Longrightarrow$

$$\boxed{= \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

Maximum Likelihood Estimation (MLE)

Likelihood Function

We have $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$, where $f_X(x)$ has (unknown) parameter θ.

The model for our data is the *joint distribution* of $X_1, \ldots, X_n$

$$f_X(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

When the joint distribution is viewed as a function of the unknown parameter, it is referred to as the *likelihood function*

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f_X(x_i)$$

Likelihood Example

Example 3: $X_1 \ldots, X_n \overset{iid}{\sim} Pois(\lambda)$

The marginal distribution of each X_i is

$$f_X(x) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$\frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \cdot \ldots$$

The joint distribution/likelihood function is

$$\frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\begin{aligned}
\mathcal{L}(\lambda) = f(x_1, \ldots, x_n) &= \prod_{i=1}^{n} f_X(x_i) \\
&= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\
&= \boxed{\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}} \Rightarrow \text{function of } \lambda
\end{aligned}$$

Maximum Likelihood Estimation (MLE)

Definition

A *maximum likelihood estimator* $\hat{\theta}_{MLE}$ of θ is the function that "maximizes the likelihood (probability) of the data"

Thus, the MLE maximizes the joint distribution model or likelihood function:

$$\hat{\theta}_{MLe} = \underset{\theta \in \Theta}{\operatorname{argmax}} \mathcal{L}(\theta) = \underset{\theta \in \Theta}{\operatorname{argmax}} \prod_{i=1}^{n} f(x_i)$$

MLE Examples

Example 4: Flip a coin 10 times. Let X be the # of heads obtained. A reasonable model for X is $Bin(n = 10, p)$ where p is our unknown parameter that we would like to estimate.

Suppose we observe the value $x = 3$. (only 1 data value).

Since there's only 1 data value, the likelihood/joint distribution is just the marginal distribution $f(x)$:

$$\begin{aligned}\mathcal{L}(p) = f(x) &= \binom{10}{x} p^x (1-p)^{10-x} \\ &= \binom{10}{3} p^x (1-p)^{10-3} \\ &= \boxed{120p^3(1-p)^7}\end{aligned}$$

MLE Examples Cont.

What value of p maximizes the likelihood?

$p = .2 \Rightarrow L(.2) = .201$

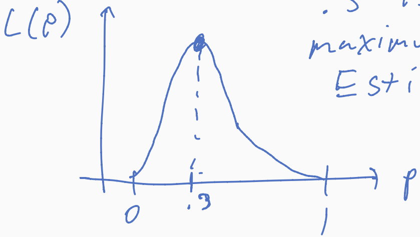
$p = .9 \Rightarrow L(.9) = .000008$

$\boxed{p = .3 \Rightarrow L(.3) = .267}$

$p = .4 \Rightarrow L(.4) = .215$

what we observed

$\mathbb{P}(X = 3 \mid p = p^*)$



.3 is the maximum likelihood Estimate of p

General Calculation of MLE

- Maximizing the likelihood from $L(\theta)$ when there are multiple observed values becomes difficult.
- The common trick is to use the *log-likelihood function* instead:

$$\ell(\theta) = log\,\mathcal{L}(\theta)$$

log = ln

where $\ell(\cdot)$ is the natural-log

$\rightarrow$ Since $\ell(\cdot)$ is increasing, the same θ that maximizes log-likelihood $\ell(\cdot)$ also maximizes the likelihood $\mathcal{L}(\theta)$

- Use calculus to find the maximum of $\ell(\theta)$

General Calculation of MLE cont.

Finding MLE:

1. Find the likelihood function: $\boxed{\mathcal{L}(\theta) = \prod_{i=1}^{n} f(x_i)}$
2. Find the log-likelihood function: $\ell(\theta) = log\mathcal{L}(\theta)$
3. Take the first derivative: $\ell^{'}(\theta) = \frac{d}{d\theta}\ell(\theta)$
4. Set $\ell^{'}(\theta) = 0$ and solve for θ

   $\rightarrow$ this is your $\boxed{\hat{\theta}_{MLE}}$

* 5. Check if second derivative $\ell^{''}(\theta) < 0$ to make sure $\hat{\theta}_{MLE}$ is maximum

MLE Examples

MLE Examples

Example 5: Roll a (6-sided) die until you get a 6, and record the number of rolls. Repeat for 100 trials. For $i = 1, \dots 100$,

$$X_i = \# \text{ of rolls until you obtain a 6 in the } i^{th} trial$$

$$X_i \overset{iid}{\sim} Geo(p) \text{ and } \boxed{f(x_i) = p(1-p)^{x_i - 1}}$$

? marginal PMF

Data:

x	1	2	3	4	5	6	7	8	9
#	18	20	8	9	9	5	8	3	5
x	11	14	15	16	17	20	21	27	29
#	3	3	3	1	1	1	1	1	1

MLE Examples Cont.

1. Find the likelihood function $\mathcal{L}(p)$:

$p(1-p)^{x_1-1} \cdot p(1-p)^{x_2-1} \cdot \dots$

$$\mathcal{L}(p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p(1-p)^{x_i-1} = \boxed{p^n(1-p)^{\sum_{i=1}^{n} x_i - n}}$$

2. Find the log-likelihood function $\ell(p) = log\mathcal{L}(p)$:

$$\ell(p) = log\mathcal{L}(p)$$

$\ell(p) = n\log(p) + \left(\sum x_i - n\right)\log(1-p)$

MLE Examples Cont.

3. Take the 1^{st} derivative w.r.t p: $\ell^{'}(p)$:

$$\ell^{'}(p) = \frac{d}{dp}\ell(p) = \frac{d}{dp}\left[nlog(p) + (\sum_{i=1}^{n} x_i - n)log(1-p)\right]$$

$$\ell^{'}(p) = \frac{n}{p} - \frac{\sum x_i - n}{1-p}$$

MLE Example

4. Set $\ell'(p) = 0$ and solve for p:

$$\frac{d}{dp}\ell(p) \overset{set}{=} 0$$

$$\frac{n}{p} = \frac{\sum x_i - n}{1-p}$$

$$\Rightarrow \frac{1-p}{p} = \frac{\sum x_i - n}{n}$$

$$\Rightarrow \frac{1}{p} - \cancel{1} = \frac{\sum x_i}{n} - \cancel{1} \Rightarrow \hat{p}_{mle} = \frac{n}{\sum x_i}$$

$$\Rightarrow \boxed{\hat{p}_{mle} = \frac{1}{\bar{X}}}$$

MLE Examples Cont.

5. 2nd derivative test to confirm we have maximum:

   $\frac{d^2}{dp^2}\ell(p)$

$$-\frac{n}{p^2} - \frac{(\sum x_i - n)}{(1-p)^2} < 0$$

$$\hat{p}_{mle} = \frac{1}{\bar{X}}$$

MLE Example Cont.

from data

$n = 100$ $\quad \bar{x} = 5.68$

$\sum x_i = 568$

$$\boxed{\hat{p}_{mle} = \frac{1}{5.68} = .176}$$

Plug in the data into our MLE:

Recap

Students should now be familiar with Method of Moments and Maximum Likelihood estimation. They should be able to find estimators for parameters using each method and give numerical estimates based on data.