Homework 11 Solution

8.4

(a) $MSE = V(\hat{\theta})$

(b) $MSE > V(\hat{\theta})$

8.6

(a) $E(\hat{\theta}_3) = aE(\hat{\theta}_1) + (1-a)E(\hat{\theta}_2) = \theta(a+1-a) = \theta$

(b) $V(\hat{\theta}_3) = V(a\hat{\theta}_1 + (1-a)\hat{\theta}_2) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2 = (\sigma_1^2+\sigma_2^2)a^2 - 2\sigma_2^2 a + \sigma_2^2$

Minimize: $a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

8.8

(a) $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_5$ are unbiased.

(b) $V(\hat{\theta}_1) = \theta^2, V(\hat{\theta}_2) = \theta^2/2, V(\hat{\theta}_3) = 5\theta^2/9, V(\hat{\theta}_5) = \theta^2/3$

It's $\hat{\theta}_5$

8.12

(a) $E(\bar{Y}) = \theta + 0.5$: $B(\bar{Y}) = E(\bar{Y}) - \theta = 0.5$

(b) Unbiased estimator is $\bar{Y} - 0.5$

(c) $V(\bar{Y}) = V(Y_1)/n = \frac{1}{12n}$: $MSE = \frac{1}{12n} + 0.25$

8.18

$$
\begin{aligned}
F_{(1)}(y) &= P(Y_{(1)} < y) = 1 - P(Y_{(1)} > y) \\
&= 1 - \prod_{1}^{n} P(Y_i > y) \\
&= 1 - (1 - F(y))^n \\
&= 1 - (1 - y/\theta)^n \\
f_{(1)}(y) &= \frac{n}{\theta}(1 - y/\theta)^{n-1}, y \in [0, \theta] \\
E(Y_{(1)}) &= \int_0^\theta y f_{(1)}(y) dy \\
&= \frac{1}{n+1}\theta
\end{aligned}
$$

So $(n+1)Y_{(1)}$ is an unbiased estimator of θ.

9.2

(a) $E(Y_i) = \mu, \forall i = 1, 2, \cdots, n$
Thus $E(\hat{\mu}_1) = E(\hat{\mu}_2) = E(\hat{\mu}_1) = \mu$

(b)

$$
\begin{aligned}
V(\hat{\mu}_1) &= \sigma^2/2 \\
V(\hat{\mu}_2) &= (1/16 + 1/4(n-2) + 1/16)\sigma^2 \\
V(\hat{\mu}_3) &= \sigma^2/n \\
eff(\hat{\mu}_3, \hat{\mu}_1) &= \frac{n}{2} \\
eff(\hat{\mu}_3, \hat{\mu}_2) &= \frac{n^2}{8(n-2)}
\end{aligned}
$$

9.3

(a) $E(Y_i) = \theta + 0.5, E(Y_{(n)}) = \theta + \frac{1}{n+1}$: $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$

(b)

$$\begin{aligned}
F_{(n)}(y) &= (y-\theta)^n \\
f_{(n)}(y) &= n(y-\theta)^{n-1} \\
V(\hat{\theta}_1) &= \frac{1}{12n} \\
V(\hat{\theta}_2) &= E(Y_{(n)}^2) - E(Y_{(n)})^2 \\
&= \frac{n}{(n+2)(n+1)^2} \\
eff(\hat{\theta}_1, \hat{\theta}_2) &= \frac{12n^2}{(n+2)(n+1)^2}
\end{aligned}$$

9.7

$$\begin{aligned}
MSE(\hat{\theta}_1) &= V(\hat{\theta}_1) = \theta^2 \\
MSE(\hat{\theta}_2) &= V(\hat{\theta}_2) = \theta^2/n \\
eff(\hat{\theta}_1, \hat{\theta}_2) &= \frac{1}{n}
\end{aligned}$$