Descriptive Statistics

STAT 330 - Iowa State University

Outline

In this lecture students will be introduced to descriptive statistics. We begin with the definition of a statistic, and describe various numerical summaries of data such as:

1. the sample mean
2. the sample variance
3. the sample median
4. sample quantiles

Statistics

Statistics

capital X's

Definition: Statistics

A *statistic*, $T(X_1, \ldots, X_n)$ is a function of random variables.

- Start with taking a *simple random sample (SRS)* of size n from some population/distribution.

$$\boxed{X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)}$$

- We can then obtain *statistics* based on $X_1, \ldots, X_n$
- Since a statistic is a function $T(\cdot)$ of random variables, the statistic is also a random variable. *
- Thus, the statistic will have its own distribution called the *sampling distribution of the statistic* (more on this later!)

Statistics Cont.

Definition: Observed Statistics

The *observed statistics*, $T(x_1, \ldots, x_n)$ is the statistic function with observed values plugged in.

- *Descriptive statistics:* Describing what our sample data looks like (graphically or numerically)
- *Inferential statistics:* Use the statistic to infer/learn about the "true" distribution, $f_X(x)$, that generated the data.

Note:

R.V world

- Use capital letters (X, $\bar{X}$, S^2, etc) to represent random variables.
- Use small letters (x, $\bar{x}$, s^2, etc) to represent observations and observed statistics.

Numbers

$\mathbb{E}(X)$

$\mathrm{Var}(X)$

Mean and Variance

Sample Mean and Variance

$E(X)=\mu$, $Var(X)=\sigma^2$

Let $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$ where $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$

- *Sample mean* is defined as $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$
  - $\rightarrow$ estimates the population mean μ.
- *Sample variance* is defined as $\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$
  - $\rightarrow$ estimates the population variance σ^2
  - $\rightarrow$ an estimate of the $Var(X) = E[(X - E(X))^2]$ can be found as $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$
  - $\rightarrow$ typically, n in the above denominator is replaced with $n-1$ to get S^2 (more on this later)
- *Sample standard deviation* is $S = \sqrt{S^2}$

Note: The quantities above are R.V's since they are functions of R.V's $X_1, \ldots, X_n$.

Observed Sample Mean and Variance

- To obtain the *observed sample mean* and *observed sample variance*, plug in observed data values $(x_1, \ldots, x_n)$ into sample mean and variance formulas

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$$

$$s = \sqrt{s^2}$$

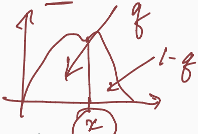
Note: The quantities above are not random variables since you have plugged in data values. They are values such as $2.4, 100,$ *etc*.

Quantiles

Quantiles

Definition: Quantiles (population)

The q^{th} *quantile* of a distribution, $f_X(x)$, is a value x such that $P(X < x) \leq q$ and $P(X > x) \leq 1 - q$.



This is also called the $100 \cdot q^{th}$ *percentile*.

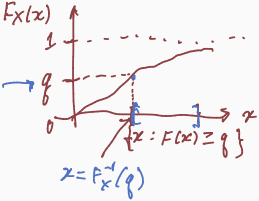
$Q_1 = 0.25^{th}$ quantile, $Q_2 = 0.5^{th}$ quantile (median), and $Q_3 = 0.75^{th}$ quantile

Definition: Quantile Function

The *quantile function* is defined as:

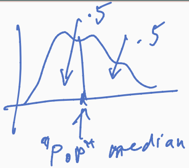
$$F_X^{-1}(q) = min\{x : F_X(x) \geq q\}$$

[0,1]



Median

The *median* is the 0.5^{th} quantile (or 50^{th} percentile)
$\rightarrow$ can be written as $F_X^{-1}(0.5)$



The *sample median* is calculated by:

1. Order sampled values in increasing order: : $X_{(1)}, \ldots, X_{(n)}$

   smallest value — largest value

   $1, 3, \boxed{5}, 7, 9$

   - If n is odd, take the middle value
     $\rightarrow$ median $= X_{\lceil \frac{n}{2} \rceil}$ $\quad X_{\lceil \frac{5}{2} \rceil} \rightarrow X_{(3)}$
   - If n is even, average the two middle values
     $\rightarrow$ median $= \frac{X_{\frac{n}{2}} + X_{\frac{n}{2}+1}}{2}$ $\quad 1, 3, \boxed{5, 7}, 8, 10$ $\quad 6$

Note: Since the above values are functions of R.V's, they are R.Vs.
Obtain the *observed sample median* by plugging in the observed values $(x_1, \ldots, x_n)$ from data.

Q_1 and Q_3

Other sample quantiles we are typically interested in are

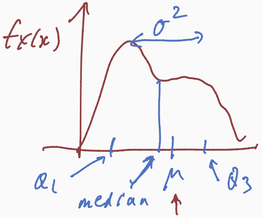
- $Q_1 = 0.25^{th}$ quantile
- $Q_3 = 0.75^{th}$ quantile

Many ways to calculate quantiles. Our method for a general q^{th} sample quantile is ...

1. Compute $(n+1) \cdot q$
    - If this value is an integer, use $(n+1)q^{th}$ ordered value
    - Else, use the average of the 2 surrounding values

Example

Example 1: A sample $\boxed{X_1, \dots, X_n \overset{iid}{\sim} f_X(x)}$ was taken where $X_i =$ CPU time for a randomly chosen task. The ordered observed values are $15, 34, 35, 36, 43, 48, 49, 62, 70, 82$ (secs)



"Population" value	sample statistic	
μ	$\bar{X}_n$	$\to \frac{1}{n}\sum X_i$
σ^2	S^2	
median	$\widehat{med}$	$X_{[\frac{n}{2}]}$
Q_1	$\hat{Q}_1$	⋮
Q_3	$\hat{Q}_3$	

Sample mean

$$\bar{x}_{10} = \frac{1}{10}\left[15 + 34 + \dots + 82\right]$$

$$= \boxed{47.4 \text{ secs}}$$

Sample variance

$$s^2 = \frac{1}{9}\left[(15 - 47.4)^2 + \dots + (82 - 47.4)^2\right]$$

$$= \boxed{384.04} \to s = \sqrt{384.04}$$

Example Cont.

Sample median

$$\widehat{med} = \frac{x_{(5)} + x_{(6)}}{2} = \frac{43+48}{2} = 45.5$$

$$\hat{Q}_1 = (n+1)q = (11)(.25) = 2.75 = \frac{x_{(2)} + x_{(3)}}{2} = 34.5$$

$$\hat{Q}_3 = \frac{x_{(8)} + x_{(9)}}{2} = \frac{62+70}{2} = 66$$

Right now, we're only using these statistics to describe the sample of CPU speeds.

- sample mean and median (Q_2) tell us "typical" values
- sample variance tells us how "spread out" / how variable the data are
- Q_1 and Q_3 "rank" where values fall in our sample

Mode, Range, IQR

Mode, Range, and IQR

Other common descriptive statistics to describe the data:

- *Mode:* The most frequent value in our sample. Can have multiple modes in data set
- *Range:* Max - Min $= X_{(n)} - X_{(1)}$

  $\rightarrow$ describes the "total" variability of the data
- *Interquartile Range (IQR):* $Q_3 - Q_1$

  $\rightarrow$ describes the variability of the middle 50% of data

Robust Statistics

- With all the different options for statistics, how do we choose which ones to use?

  $\rightarrow$ It depends on your data set

- Statistics that are not affected by extreme values are called *robust statistics* mean to median (Center)

Example 2:

Stats	pre-Bezos	post-Bezos	Robust?
mean	\$60k	way-Bigger	NO
median	\$60K	Slightly Bigger	Yes
Std. Dev	\$10K	way Bigger	NO
IQR	\$25K	Slighty Bigger	Yes

Recap

Students should now be familiar with the concept of a statistic. They should be able to distinguish between random statistics and observed statistics. They should be able to calculate some observed statistics such as the sample mean, sample variance, and others.