Confidence Intervals

STAT 330 - Iowa State University

Outline

In this lecture students will learn about Confidence Intervals. We define what a Confidence Interval is, and derive intervals for means and proportions. They will also see how we can choose the sample size to obtain an interval with a desired width and confidence level.

Confidence Intervals

Confidence Intervals

- MLE gives us a "point estimate" of the unknown parameter.
- But $\hat{\theta}$ probably won't *exactly* equal θ due to sampling error.

  $\rightarrow P(\theta = \hat{\theta}) = 0$
- Create a confidence interval to give range of reasonable values for the unknown parameter θ.

Point Estimator

?

p = proportion of voters who favor some Bill

Example 1: Polling

Today's poll shows 58% of people favor the new bill. The margin of error is $\pm 3\%$.

The confidence interval for the proportion of people that favor the bill is $[0.55, 0.61]$.

Confidence Interval

Definition

A random interval $[a, b]$ is a $(1-\alpha)100\%$ *confidence interval* for the parameter θ if it contains θ with probability $(1-\alpha)$

$$P(a \leq \theta \leq b) = 1 - \alpha$$

- $(1-\alpha)$ is called the confidence level
- When you estimate an unknown parameter θ, it should be accompanied by a confidence interval

- Interpretation: We are $[(1-\alpha)\%]$ confident that the [insert population parameter + context] is between [insert interval + units].

Constructing Confidence Intervals

In this class, we will construct normal distribution based intervals.

Suppose we have an estimator $\hat{\theta}$ for unknown parameter θ.

1. $\hat{\theta}$ is unbiased: $E(\hat{\theta}) = \theta$
2. $\hat{\theta}$ follows a normal distribution.

We can standardize $\hat{\theta}$ to get

$$Z = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \sim N(0, 1)$$

where $SE(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$ = standard deviation of $\hat{\theta}$

Constructing Confidence Intervals

$Z \sim N(0,1)$

$1-\alpha$

$\mathbb{P}\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1-\alpha$

Let $z_{\alpha/2}$ be the $1-\frac{\alpha}{2}$ quantile of the standard normal distribution.

$$P\left(-z_{\alpha/2} \leq \boxed{\frac{\hat{\theta}-\theta}{SE(\hat{\theta})}} \leq z_{\alpha/2}\right) = 1-\alpha$$

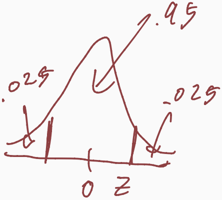
Constructing Confidence Intervals

Isolating θ in the middle, we get

$$P\left(\underbrace{\hat{\theta} - z_{\alpha/2}SE(\hat{\theta})}_{a} \leq \theta \leq \underbrace{\hat{\theta} + z_{\alpha/2}SE(\hat{\theta})}_{b}\right) = 1 - \alpha$$

Thus, a $(1-\alpha)100\%$ confidence interval for θ is

$$\boxed{\hat{\theta} \pm z_{\alpha/2}SE(\hat{\theta})}$$



Common choices for α are $0.01, 0.05,$ and 0.1

$(1-\alpha)100\%$	80	90	95	98	99
$z_{\alpha/2}$	1.282	1.645	1.96	2.326	2.576

Constructing Confidence Intervals Cont.

We will make confidence intervals for four cases:

1. μ (population mean)
2. p (population proportion)
3. $\mu_1 - \mu_2$ (difference in population means)
4. $p_1 - p_2$ (difference in population proportions)

Confidence intervals for all 4 of the above cases can be constructed using normal distribution based inference.

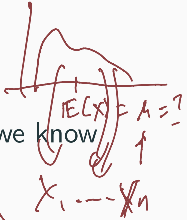
Follow the same general procedure to construct these intervals.

Confidence Interval for Mean

Confidence Interval for μ

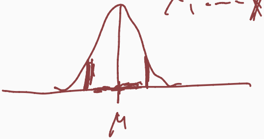
Confidence interval for the population mean

$X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$ with $E(X_i) = \mu$ and $Var(X_i) = \sigma$



First, we estimate μ using the *statistic* $\bar{X}$. From CLT, we know

- $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$
- $SE(\bar{X}) = \sqrt{Var(\bar{X})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$



A $(1-\alpha)100\%$ confidence interval for μ is

$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

?

$$\left[\hat{\theta} \pm z_{\alpha/2}\, se(\hat{\theta})\right]$$

In most cases, the population standard deviation σ will be unknown. Replace σ with the sample standard deviation s.

$$\boxed{\bar{X} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}}$$

Confidence Interval for μ Cont.

If we want a 95% confidence interval, then

$$1 - \alpha = 0.95$$
$$\rightarrow \alpha = 0.05$$
$$\rightarrow \alpha/2 = 0.025$$

$z_{\alpha/2} = z_{0.025}$ is the 0.975^{th} quantile of the $N(0,1)$ distribution.
$\rightarrow$ Using the $z - table$, we get $z_{0.025} = 1.96$

The 95% confidence interval for μ is

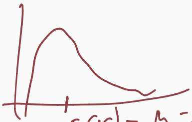
$$\begin{cases} \bar{X} \pm 1.96\dfrac{\sigma}{\sqrt{n}} & \text{when } \sigma \text{ is known} \\ \boxed{\bar{X} \pm 1.96\dfrac{s}{\sqrt{n}}} & \text{when } \sigma \text{ is unknown} \end{cases}$$

Example

Example 2: A random sample of 50 batteries were taken for a particular brand. For the sample, the mean lifetime is 72.5 hours and variance is 19.3 hours^2. Find a 95% confidence interval for the true mean lifetime of batteries from that particular brand.

X_i = lifetime of Battery i

$X_1 \dots X_n \underset{\sim}{iid} f_X(x)$



$E(X) = \mu = ?$

Estimator = $\bar{X}$

Estimate = $\bar{x} = 72.5$

S^2 Estimates σ^2

$s^2 = 19.3$

$\bar{X} \pm \underbrace{z_{\alpha/2}} s/\sqrt{n}$

$z_{\alpha/2} = 1.96$

Example Cont.

$$\underbrace{72.5}_{\text{"Guess"}} \pm \underbrace{1.96 \frac{\sqrt{14.3}}{\sqrt{50}}}_{\text{"MOE"}}$$

$$72.5 \pm 1.22$$

$$\boxed{(71.28, 73.72)}$$

I am 95% confident the mean lifetime of these batteries is between 71.28 hours and 73.72

Selecting Sample Size for Means

How to decide sample size?

- Can choose the sample size n to obtain a desired level of confidence & width for our confidence interval.
- Margin or error (Δ) is half the width of the confidence interval
  margin of error $= \Delta = z_{\alpha/2} SE(\bar{X}) = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$
- The bigger the sample size, the smaller the standard error of the estimator, and smaller the size of our interval

To attain a particular margin of error Δ, we need a sample size

$$n \geq \left(\frac{z_{\alpha/2} \sigma}{\Delta} \right)^2$$

Confidence Interval for Proportion

Confidence Interval for p

Confidence interval for the population proportion

- In this scenario, we want to estimate the proportion of population belonging to a particular category.
- Any individual in the population either belongs to the category of interest ("1"), or they don't ("0").
- Thus, we can think of each random variable X as a Bernoulli distribution with unknown parameter p
- We ultimately want to estimate and find a confidence interval for p.

All voters

$\boxed{1, 1, 0, 0 \quad 1, 0 \ldots} \longrightarrow X_1 = \begin{cases} 1 & \text{Yes} \\ 0 & \text{No} \end{cases}$

$X \sim \text{Bern}(p)$

Confidence Interval for p Cont.

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$

First, estimate p using the *statistic* $\hat{p} = \frac{\sum X_i}{n}$ = sample proportion.

- $E(\hat{p}) = E\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n} E(\sum_{i=1}^n X_i) = \frac{1}{n} np = p$ (unbiased)
- $Var(\hat{p}) = Var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} Var(\sum X_i) = \frac{np(1-p)}{n^2} = \boxed{\frac{p(1-p)}{n}}$

  $\rightarrow SE(\hat{p}) = \sqrt{Var(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}$

Since $\hat{p}$ is the mean of the Bernoulli $X's$, CLT for means applies

$$\hat{p} \approx N\left(p, \frac{p(1-p)}{n}\right)$$

Thus a $(1-\alpha)100\%$ confidence interval for p is

$* \ \hat{se}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$$\boxed{\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

Example

Example 3: In a random sample of 1000 U.S. adults, 38.8% stated they believed in the existence of ghosts. Find a 90% confidence interval for the population proportion of all U.S. adults who believe in the existence of ghosts.

$$X_i = \begin{cases} 1 & \text{ghosts} \\ 0 & \text{no ghosts} \end{cases}$$

$X_1 \dots X_n \overset{iid}{\sim} \text{Bern}(p)$?

$\hat{p} = .388$

90% conf: 1.65

$n = 1000$

$$.388 \pm 1.65\sqrt{\frac{(.388)(.612)}{1000}}$$

$\Rightarrow \quad .388 \pm .025$

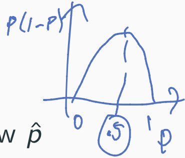
$$\boxed{(.363, .413)}$$

Selecting Sample Size for Proportions

How to decide sample size?

- Just as before, we can select the sample size based on how large we want our margin or error to be

  margin or error $= \Delta = z_{\alpha/2} SE(\hat{p}) = \boxed{z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$



- Issue: We haven't taken the sample, so we don't know $\hat{p}$
- Solution: Use $\hat{p} = 0.5$ for most conservative sample size

  $\Delta = z_{\alpha/2}\sqrt{\frac{0.5 \cdot 0.5}{n}} = z_{\alpha/2}\sqrt{\frac{0.5^2}{n}}$

To attain a particular margin of error Δ, we need a sample size

$$\boxed{n \geq \left(\frac{z_{\alpha/2} \cdot 0.5}{\Delta}\right)^2}$$

Sample Size Calculation Cont.

Example 4: Political polls typically use 95% confidence and report margin of errors of 3%: $\boxed{\hat{p} \pm 0.03.}$

What sample size do we need to for such a poll?

$z = 1.96$

$\Delta = .03$

$$n \geq \left(\frac{(.5)(1.96)}{.03} \right)^2$$

$= 1067. \ldots$

$\Rightarrow$ Sample of size $\boxed{1068}$

FOX NEWS POLL

All results are for release after 6:00PM/ET Thursday, May 21, 2020.

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Methodology

Interviews were conducted May 17-20, 2020 among a random national sample of 1,207 registered voters (RV). Landline (261) and cellphone (946) telephone numbers were randomly selected for inclusion in the survey using a probability proportionate to size method, which means phone numbers for each state are proportional to the number of voters in each state.

Results based on the full sample have a margin of sampling error of ± 3 percentage points.

The Fox News Poll is conducted under the joint direction of Beacon Research (D) (formerly known as Anderson Robbins Research) and Shaw & Company Research (R).

Fieldwork conducted by Braun Research, Inc. of Princeton, NJ.
Fox News Polls before 2011 were conducted by Opinion Dynamics Corporation.

Results are of registered voters, unless otherwise noted.
An asterisk (*) is used for percentages of less than one-half percent.
A dash (-) represents a value of zero.
Some percentages may not add to 100% due to rounding. In the same way, percentages in "total" columns may be one point more or less than the sum of their parts due to rounding.

Recap

Students should now with comfortable with a Confidence Interval in statistics. They should be able to calculate intervals for a mean or proportion and interpret what the interval tells us. They should also be able to calculate the sample size for a certain level of confidence and margin of error.