Homework 12 Solution

9.19

$$
\begin{aligned}
E(\bar{Y}) &= E(Y_1) \\
&= \int_0^1 y * \theta y^{\theta-1} dy \\
&= \frac{\theta}{\theta+1} \\
E(Y_1^2) &= \frac{\theta}{\theta+2} \\
Var(\bar{Y}) &= Var(Y_1)/n \\
&= [E(Y_1^2) - E(Y_1)^2]/n \\
&= \frac{\theta}{n(\theta+1)^2(\theta+2)}
\end{aligned}
$$

Since (1) $E(\bar{Y}) = \frac{\theta}{\theta+1}$ and (2) $Var(\bar{Y}) \to 0$ as $n \to \infty$, $\bar{Y}$ is a consistent estimator of $\frac{\theta}{\theta+1}$.

9.37

Likelihood $l(p) = p^{\sum X_i}(1-p)^{n-\sum X_i} = g(\sum X_i, p) * h(\underset{\sim}{X})$, where $g(\sum X_i, p) = p^{\sum X_i}(1-p)^{n-\sum X_i}, h(\underset{\sim}{X}) = 1$.

Thus $\sum X_i$ is sufficient for p.

9.39

By additivity of independent Poisson random variables: $\sum Y_i \sim Poisson(n\lambda)$

Thus the joint conditional distribution of $(Y_1, \cdots, Y_n)$ given $\sum Y_i = x$ is:

$$
\begin{aligned}
P(Y_1 = y_1, \cdots, Y_n = y_n | \sum Y_i = x) &= \frac{P(Y_1 = y_1, \cdots, Y_n = y_n, \sum Y_i = x)}{P(\sum Y_i = x)} \\
&= \frac{P(Y_1 = y_1, \cdots, Y_n = y_n)}{P(\sum Y_i = x)} \\
&= \frac{\prod_{i=1}^n [\lambda^{y_i} e^{-\lambda}/y_i!]}{(n\lambda)^x e^{-n\lambda}/x!} \\
&= \frac{x!}{n^x \prod y_i!}
\end{aligned}
$$

Free of parameter λ. Thus $\sum Y_i$ is sufficient for λ.

9.62

$(Y_{(1)} - \frac{1}{n})$ is unbiased and sufficient for θ: $(Y_{(1)} - \frac{1}{n})$ is an MVUE for θ.

9.63

(a)

$$
\begin{aligned}
F(y|\theta) &= (y/\theta)^3, 0 \le y \le \theta \\
F_{(n)}(y|\theta) &= (\frac{y}{\theta})^{3n} \\
f_{(n)}(y|\theta) &= \frac{3ny^{3n-1}}{\theta^{3n}}, 0 \le y \le \theta
\end{aligned}
$$

(b)

$$
\begin{aligned}
E(Y_{(n)}) &= \frac{3n}{3n+1}\theta \\
E(\frac{3n+1}{3n} Y_{(n)}) &= \theta
\end{aligned}
$$

$\frac{3n+1}{3n} Y_{(n)}$ is unbiased and sufficient of θ: MVUE.

9.71

$E(Y) = 0, E(Y^2) = \sigma^2$: $\hat{\sigma}^2 = m_2' = \frac{1}{n} \sum_1^n Y_i^2$

9.77

$E(Y) = 1.5\theta$: $\hat{\theta} = m'1 * \frac{2}{3} = \frac{2}{3}\bar{Y}$

9.81

The log-likelihood:

$$\begin{aligned} l(\theta) &= -n\log\theta - \frac{1}{\theta}\sum y_i \\ l'(\hat\theta) &= 0 \\ \hat\theta &= \bar{Y} \end{aligned}$$

$\bar{Y}$ is the MLE for θ. By invariance property of MLE, $\bar{Y}^2$ is the MLE for θ^2.

9.97

(a) $E(Y) = 1/p$: $\hat{p} = 1/m_1' = \frac{1}{\bar{Y}}$

(b)

$$\begin{aligned} l(p) &= p^n(1-p)^{\sum y_i - n} \\ l'(p) &= np^{n-1}(1-p)^{\sum y_i - n} - (\sum y_i - n)p^n(1-p)^{\sum y_i - n - 1} \\ &= p^{n-1}(1-p)^{\sum y_i - n - 1}[n(1-p) - (\sum y_i - n)p] \\ &= 0 \\ \hat{p} &= \frac{1}{\bar{Y}} \end{aligned}$$

8.40

(a) $Y \sim N(\mu, 1), Y - \mu \sim N(0,1)$: $P(-1.96 < Y - \mu < 1.96) = P(Y - 1.96 < \mu < Y + 1.96) = 0.95$

The 95% confidence interval for μ is $[Y - 1.96, Y + 1.96]$

(b) Similarly, $P(-1.645 < Y - \mu) = P(\mu < Y + 1.645) = 0.95$

The 95% upper confidence interval for μ is $Y + 1.645$

(c) Similarly, The 95% lower confidence interval for μ is $Y - 1.645$

8.56

(a) iid binomial random variable with $\hat{p} = 0.45$: the 98% CI is

$$\hat{p} \pm \Phi(0.99)\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.45 \pm 0.041$$

(b) 0.5 not included in the CI: there is not compelling evidence that a majority of adults feel that movies are getting better.

8.60

(a) iid normal random variable with $\mu = 98.25, \sigma = 0.73$: the 99% CI is

$$\mu \pm \Phi(0.995)\frac{\sigma}{\sqrt{n}} = 98.25 \pm 0.165$$

(b) 98.6 not included in the CI: It is possible that the standard for "normal body temperature" is no longer valid.