Module 1 – Section 5

Confidence Intervals for a Population Proportion

Outline

- Confidence Interval Methods
- Confidence Levels and Coverage Rates
- Sample Size Calculations

Population Proportion p

- $p =$ proportion of population belonging to a particular category (category of interest or success)
- Goal = estimate value of p
- Take simple random sample of size n from population.

Estimating p

- Denote $Y =$ number of successes in the sample of size n.
- Estimate p with $\hat{p} = \frac{Y}{n}$
- $\hat{p} =$ sample proportion of observations in category of interest

Sampling Distribution of $\hat{p}$

- If $np \geq 10$ and $n(1-p) \geq 10$:

$$\hat{p} \approx N\left(p, \sqrt{\frac{p(1-p)}{n}}\right) \text{ or } \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \approx N(0,1)$$

Sampling Distribution of $\hat{p}$

$$1-\alpha \quad = \quad P\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}\right)$$

$$\cong \quad P\left(-z_{1-\frac{\alpha}{2}} < \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} < z_{1-\frac{\alpha}{2}}\right)$$

Normal Approximation Method

- Solve equation on slide 6 for p.
- Replace p with $\hat{p}$ in formula for $SE(\hat{p})$.
- Approximate $(1 - \alpha)\%$ CI for p:

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Ex. Ingots

- $n = 400$; $\hat{p} = 0.16$
- Approximate 95% CI for p is

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.16 \pm 1.96\sqrt{\frac{0.16(0.84)}{400}}$$

$$= (0.1241, 0.1959)$$

Ex. Ingots

- We are approximately 95% confident the proportion of ingots that will crack using the new manufacturing method is between 0.1241 and 0.1959.

Wilson's Score Method

- Solve equation on slide 6 for p.
- No substitution for p in formula $SE(\hat{p})$.
- Correspondence to Score Method Hypothesis Test: $(1-\alpha)\%$ CI consists of values of p_0 where H_0 is not rejected for the given level of α in hypothesis test using the score method with hypotheses:

$$H_0: p = p_0 \text{ vs. } H_a: p \neq p_0$$

Wilson's Score Method

- Approximate $(1-\alpha)\%$ CI for p:

$$\frac{\hat{p}+\frac{1}{2n}z_{1-\alpha/2}^{2}\pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}+\frac{z_{1-\alpha/2}^{2}}{4n^{2}}}}{1+\frac{1}{n}z_{1-\alpha/2}^{2}}$$

Ex. Ingots

$$\frac{\hat{p} + \frac{1}{2n} z_{1-\alpha/2}^2 \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{1-\alpha/2}^2}{4n^2}}}{1 + \frac{1}{n} z_{1-\alpha/2}^2} = \frac{0.16 + \frac{1}{800}(1.96)^2 \pm 1.96\sqrt{\frac{0.16(0.84)}{400} + \frac{1.96^2}{4(400)^2}}}{1 + \frac{1}{400}(1.96)^2}$$

$$= (0.1273, 0.1991)$$

Ex. Ingots

- We are approximately 95% confident that the proportion of ingots that will crack using the new manufacturing method is between 0.1273 and 0.1991.

Summary of CIs

Method	Lower CI Bound	Upper CI Bound
Normal Approximation	0.1241	0.1959
Wilson's Score	0.1273	0.1991

Confidence Level

- An individual CI either does or does not contain the population parameter.
- However, $(1-\alpha)$% of all possible confidence intervals should contain population parameter.

Coverage Rate

- The percentage of confidence intervals containing the population parameter when drawing many samples from a given population.
  - Draw Sample – Calculate CI – Does CI contain parameter?
  - Draw Sample – Calculate CI – Does CI contain parameter?
  - Repeat many, many times.

Coverage Rate and Confidence Level

- If the sampling distribution and all other assumptions are correct, the coverage rate should be approximately equal to the confidence level for the CI.

Coverage Rates for CI for p

- Problems obtaining this coverage rate for CI for p
  - Using normal distribution as approximation to discrete binomial distribution.
  - $SE(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$ contains parameter value

Why have different methods?

- In practice, methods for CI for p produce different coverage rates.
  - Coverage rate for Normal Approximation depends heavily on assumption: $np \geq 10$ and $n(1-p) \geq 10$.
  - Coverage rate for Wilson's Score Method is less affected by the values of n and p.

Which one to use?

- Normal Approximation Method
  - Easier to calculate (by hand) and explain (at first)
  - Method taught in most intro courses
- Wilson's Score Method
  - Harder to calculate (by hand) and explain
  - Better coverage rates
  - Method used by most computer packages

Is that it?

- No!
- Other methods are available.
- These seem to perform best and/or are the most popular.

Sample Size Calculations

- Use Normal Approx. Method Margin of Error

$$M = z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- What sample size is required for a given margin of error and confidence level?

Sample Size Calculations

$$n \geq \left(\frac{Z_{1-\frac{\alpha}{2}}}{M}\right)^2 \hat{p}(1-\hat{p})$$

- Answer to question requires value of $\hat{p}$.
- Want to obtain required sample size before taking sample.

Solution #1

- Use worse-case scenario ($\hat{p} = 0.5$)

$$n \geq \left(\frac{0.5 z_{1-\alpha/2}}{M}\right)^2$$

  - Calculated CI's margin of error will be no worse, but usually better than targeted value M.
  - If $\hat{p}$ is much less than 0.4 or much greater than 0.6, solution gives you much larger sample size than really needed to obtain targeted margin of error M.

Solution #2

- Use best-guess estimate ($\hat{\hat{p}}$ = estimate of $\hat{p}$)

$$n \geq \left(\frac{z_{1-\frac{\alpha}{2}}}{M}\right)^2 \hat{\hat{p}}\,(1-\hat{\hat{p}})$$

  - Generally gives more accurate estimate of required sample size.
  - Calculated CI could have larger margin of error than targeted value M.

Ex. Ingots

- $M = 0.02$ or 2%
  - Width of CI will be 0.04 or 4%
- Confidence Level = 95%
  - $z_{1-\alpha/2} = 1.96$

Solution #1

$$n \geq \left(\frac{0.5 z_{1-\frac{\alpha}{2}}}{M}\right)^2 = \left(\frac{0.5(1.96)}{0.02}\right)^2 = 2401$$

Solution #2

- Use $\hat{\hat{p}} = 0.2$

$$n \geq \left(\frac{Z_{1-\frac{\alpha}{2}}}{M}\right)^2 \hat{\hat{p}}\,(1 - \hat{\hat{p}}) = \left(\frac{1.96}{0.02}\right)^2 0.2(0.8) = 1536.64$$

- Sample size $n = 1537$

Ex. Ingots

- Comparison
  - Solution #1 – 2401 ingots needed
  - Solution #2 – 1537 ingots needed
- Large difference between two methods

Diminishing Returns

- Not 1-1 correspondence between M and n
- Ex. Ingots – Values of M and n for 95% CI (Using $\hat{\hat{p}} = 0.2$)

Margin of Error (M)	Sample Size Needed (n)
0.08	97
0.04	384
0.02	1,537
0.01	6,147
0.005	24,586

Diminishing Returns

- Decreasing M by 50% (one-half) required 4 times the sample size n.
- Many national polls use $M = 0.03$ or 3%
  - Requires approximately 1,000 respondents