Homework 9

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Problem 6.1
Part a:
Define $U_1 = 2Y - 1$
$U_1 + 1 = 2Y$ so $Y = \frac{U_1+1}{2}$
$|\frac{dY}{dU_1}| = 1/2$
The pdf of U_1 is $g(U_1) = f(y)|\frac{dY}{dU_1}| = 2(1 - \frac{U_1+1}{2}) * 1/2 = 1 - \frac{U_1+1}{2}$ for $-1 < U_1 > 1$, 0 otherwise

Part b:
Define $U_2 = 1 - 2Y$
$U_2 - 1 = -2Y$ so $Y = \frac{U_2-1}{2}$
$|\frac{dY}{dU_2}| = 1/2$
Then the pdf of U_2 is $g(U_2) = f(y)|\frac{dY}{dU_2}| = 2(1-(-\frac{U_2-1}{2}))*1/2 = \frac{U_2+1}{2}$ for $-1 < U_2 < 1$, 0 otherwise

Part c:
Define $U_3 = Y^2$ so $Y = \sqrt{U_3}$
$|\frac{dY}{dU_3}| = \frac{1}{2\sqrt{U_3}}$
Then the pdf of U_3 is $g(U_3) = f(y)|\frac{dY}{dU_3}| = 2(1 - \sqrt{U_2}) * \frac{1}{2\sqrt{U_3}} = \frac{1-\sqrt{U_3}}{\sqrt{U_3}}$ for $0 < U_3 < 1$, 0 otherwise
E(U_3) = $\int_0^1 U_3 g(U_3)\, dU_3 = \int_0^1 U_3 * \frac{1-\sqrt{U_3}}{\sqrt{U_3}}\, dU_3 = 1/6$

Part d:
E(U_1) = $\int_{-1}^1 U_1 g(U_1)\, dU_1 = \int_{-1}^1 U_1(1 - \frac{U_1+1}{2})\, dU_1 =$ -1/3
E(U_2) = $\int_{-1}^1 U_2 g(U_2)\, dU_2 = \int_{-1}^1 U_2 * \frac{U_2+1}{2})\, dU_2 = 1/3$
E(U_3) = $\int_0^1 U_3 g(U_3)\, dU_3 = \int_0^1 U_3 * \frac{1-\sqrt{U_3}}{\sqrt{U_3}}\, dU_3 = 1/6$

Part e:
E(y) = $\int_0^1 y f(y)\, dy = \int_0^1 y * 2(1-y)\, dy = 1/3$
E(y^2) = $\int_0^1 y^2 f(y)\, dy = \int_0^1 y^2 * 2(1-y)\, dy = 1/6$
E(U_1) = E(2Y-1) = 2E(Y)-1 = -1/3
E(U_2) = E(1-2Y) = E(1) - E(2Y) = 1-2E(Y) = 1/3
E(U_3) = E(Y^2) = 1/6

Problem 6.4
Part a:
probability function for U is P($U \le u$) = P(3Y+1≤u) = P($Y \le (u-1)/3$)
Y follows an exponential distribution with mean of 4 tons, the probability function for U is $P(U \le u) = P(Y \le (u-1)/3) = 1 - e^{-(u-1)/12}$ for u≥1

So the density function for U is $f_U(u) = \frac{dP(U \leq u)}{du} = \frac{d(1-e^{-(u-1)/12})}{du} = \frac{e^{-(u-1)/12}}{12}$ for u ≥ 1

Part b:
Given the density function from part a, the E(U) is:
E(U) = $\int_1^\infty ue^{-(u-1)/12}/12\, du$
use integration by parts where t=u-1: $-13e^{t/12}|_0^\infty = 13$
E(U) = 13

Problem 6.5
Let Y be the waiting time until delivery of a new component, Y U(1,5). So the distribution function of Y would be $F_Y(y) = \frac{y-1}{4}, 1 \leq y \leq 5$
Calculate the distribution function of U: $F_U(u) = P(U \leq u) = P(2Y^2 + 3 \leq u) = P(Y \leq \sqrt{(u-3)/2}) = F_Y(\sqrt{(u-3)/2} = 1/4(\sqrt{(u-3)/2} - 1)$
Calculate the probability density function of U: $f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\frac{1}{4}(\sqrt{(u-3)/2} - 1) = \frac{1}{16}(\frac{u-3}{2})^{-1/2}$
The probability density function for U is:
$$f_U(u) = \begin{cases} \frac{1}{16}(\frac{u-3}{2})^{-1/2} & 5 \leq u \leq 53 \\ 0 & \text{otherwise} \end{cases}$$

Problem 6.23
Part a:
$U_1 = 2Y - 1$. Assume that $h(y) = 2y - 1$. So $h^{-1}(u_1) = \frac{u_1+1}{2} => \frac{dh^{-1}}{du_1} = 1/2$. Then the probability density function of U1 is
$f_{U_1}(u_1) = f_Y(h^{-1}(u_1))|\frac{dh^{-1}}{du_1}| = \frac{1-u_1}{2} for 0 \leq \frac{u_1+1}{2} \leq 1$
So, $f_{U_1}(u_1) = \begin{cases} \frac{1-u_1}{2} & -1 \leq u_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Part b:
$U_2 = 1 - 2Y$. Assume that $y = \frac{1-u_2}{2}$. So $h^{-1}(u_2) = \frac{1-u_2}{2} => \frac{dh^{-1}}{du_2} = -1/2$. Then the probability density function of U2 is
$f_{U_2}(u_2) = f_Y(h^{-1}(u_2))|\frac{dh^{-1}}{du_2}| = \frac{1+u_2}{2} for 0 \leq \frac{1-u_2}{2} \leq 1$
So, $f_{U_2}(u_2) = \begin{cases} \frac{1+u_2}{2} & -1 \leq u_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Part c:
$U_3 = Y^2$. Assume that $h(y) = y^2$. So $h^{-1}(u_3) = \sqrt{u_3} => \frac{dh^{-1}}{du_3} = \frac{1}{2\sqrt{u_3}}$. Then the probability density function of U3 is
$f_{U_3}(u_3) = f_Y(h^{-1}(u_3))|\frac{dh^{-1}}{du_3}| = \frac{1}{\sqrt{u_3}} - 1 for 0 \leq u_3 \leq 1$
So, $f_{U_3}(u_3) = \begin{cases} \frac{1}{\sqrt{u_3}} - 1 & 0 \leq u_3 \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Problem 6.28
Given that Y has a uniform (0,1) distribution, the probability density function of Y is $f_Y(y) = 1, 0 \leq y \leq 1$
The probability density function of U is given by $f_U(u) = f_Y(y)|\frac{dy}{du}|$ when Y-0, U=∞ and when Y=1, U=0

-u/2=ln(y) or y = $e^{-u/2}$ => $\frac{dU}{dY} = -1/2e^{-u/2}$
The probability distribution of U is $= 1/2e^{-u/2}, 0 \leq u < \infty$ which is the probability density function of an exponential random variable with mean 2. If Y Uniform(0,1), $U = -2ln(y)$ has an exponential distribution with mean 2.

Problem 6.30
Let a fluctuating electric current be a uniform distribution on interval (9,11).
$p_I(i) = \begin{cases} \frac{1}{2} & 9 \leq i \leq 11 \\ 0 & \text{otherwise} \end{cases}$
Define $p = 2I^2 => I = (p/2)^{1/2} => h^{-1}(p) = (p/2)^{1/2}$
$\frac{dh^{-1}}{dp} = \frac{d(h^{-1}(p))}{dp} = \frac{d(p/2)^{1/2}}{dp} = \frac{1}{4}(\frac{2}{p})^{1/2}$
$f_P(p) = f_I(h^{-1}(p))|\frac{d(h^{-1}(p))}{dp}| = \frac{1}{2}\frac{1}{4}(\frac{2}{p})^{1/2}$
so $f_P(p) = \begin{cases} \frac{1}{8}(\frac{2}{p})^{1/2} & 162 \leq p \leq 242 \\ 0 & \text{otherwise} \end{cases}$

Problem 6.37
Part a:
For Bernoulli distribution: $m_{y_1}(t) = E(e^{ty_1}) = \sum_{y1=0or1} e^{ty_1}p^{y_1}(1-p)^{1-y_1} = (1-p) + pe^t$

Part b:
Since $Y_1..., Y_n$ are i.i.d, $m_W(t) = [m_{y_1}(t)]^n = [(1-p) + pe^t]^n$

Part c:
The distribution of W is Binomial(n,p)

Problem 6.60
Let $W = Y_1 + Y_2$ where Y1 and Y2 are independent. If W has χ^2 distribution with ν degrees of freedom, $f_Y(y) = \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)}e^{-w/2}y^{(\nu/2)-1}, w > 0$.
The moment generating function of W is $M_W(t) = E[e^{tw}] = \int_0^\infty \frac{(\frac{1}{2})^{\nu/2}}{\Gamma(\nu/2)}e^{-w/2}e^{tw}w^{(\nu/2)-1}\,dw = (1-2t)^{-\nu/2}, t < 1/2$
Similarly, Y1 has a χ^2 distribution with $\nu_1 < \nu$ degrees of freedom. The moment generating function of Y1 is $M_{Y_1}(t) = (1-2t)^{-\nu_1/2}, t < 1/2$. We already know that W=Y1+Y2. So then moment generating function of W is $M_W(t)v = M_{Y_1}(t)M_{Y_2}(t) = (1-2t)^{-1/2(\nu-\nu_1)}, t < 2$.
So Y2 has a χ^2 distribution with $\nu - \nu_1$ degrees of freedom.