Cavendish Balance

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I. INTRODUCTION

The main purpose of this lab was to experimentally determine the universal gravitational constant, G, using the Cavendish Balance. In short, the Cavendish Balance is a torsion balance in which two large masses cause a gravitational attraction of two smaller masses such that torsion pendulum imparts an equilibrating force on the system, causing an oscillation of the system. A simple diagram of the Cavendish Balance may be seen in Fig. 1. Please note, the Cavendish Balance is an extremely sensitive piece of equipment where a simple bump of the apparatus will cause an oscillation of upwards of a full two hours. Furthermore, alignment of the instrument proves to take many hours and will be discussed in-depth later in this report.

From one data set along with the dimensions of the Cavendish Balance given within the lab handout, three values of G are able to be determined with varying methods. Method I determines the value of G via the final deflection of the light spot from the first equilibrium point ,S_1, to the second equilibrium point, S_2. Method II determines the value of G based on the offset of equilibrium positions inherent the data set through a curve fit. Finally, Method III determines the value of G via the acceleration of the light spot over the first two minutes of data. Each analysis method will be discussed in detail later in this report.

II. THEORETICAL BACKGROUND

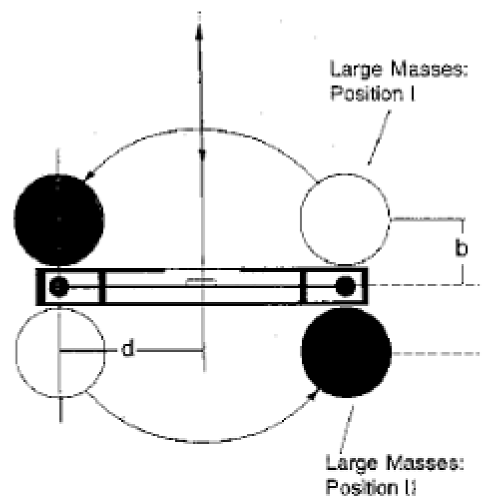
A. Cavendish Balance

In essence, the Cavendish Balance is quite a simple apparatus. However, the Cavendish Balance contains some nuances when aligning that will be discussed later in the report. As seen in Fig. 1, the apparatus consists of two lead balls (Q) placed onto a swivel support, two tungsten balls (q), and a wire connected to the two tungsten balls, acting as a torsion pendulum. Also, attached to the torsion pendulum is a mirror, with which a light beam is reflected onto the chalkboard.

B. Method I: Measurement by Final Deflection

When the large masses are in Position I of Fig.2 the gravitational attraction F is given by the law of universal gravitation:

FIG. 1. TOP VIEW DESCRIPTION



$$F = \frac{Gm_1m_2}{b^2}, \tag{1}$$

where m_1 is the larger mass, m_2 is the smaller mass, and b is distance between the center of the large mass touching the case and the center of the small mass. The gravitational attraction between the masses produces a new torque τ_{grav} on the system given by:

$$\tau_{\text{grav}} = 2Fd, \tag{2}$$

where d is the length of the lever arm of the pendulum's cross beam. Since the system is in equilibrium the torsion band is supplying an equal and opposite torque. This torque τ_{band} is the torsion constant of the band ,κ, times the angle with which it is twisted ,θ:

$$\tau_{\text{band}} = -\kappa\theta \tag{3}$$

Combining Eq. (1), (2), and (3) and that $\tau_{\text{band}} = -\tau_{\text{grav}}$, the following is obtained:

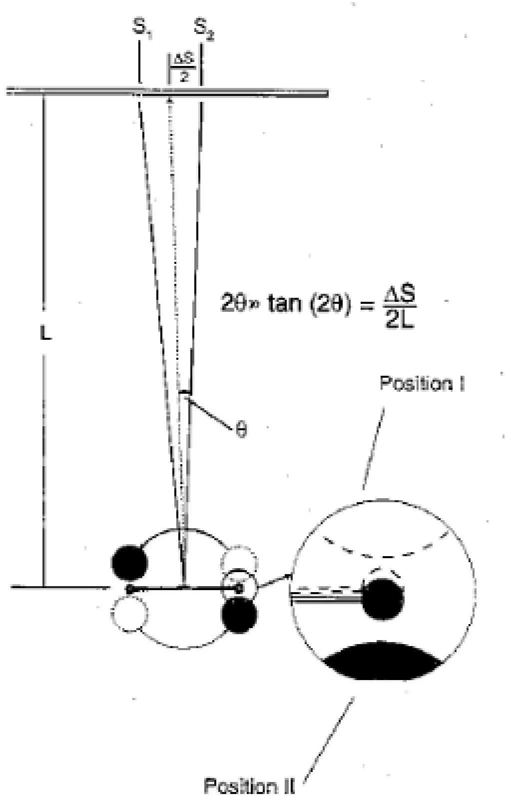
$$\kappa\theta = \frac{2dGm_1m_2}{b^2}, \tag{4}$$

and rearranging the equation for G produces:

$$G = \frac{\kappa\theta^2}{2dm_1m_2}. \tag{5}$$

To determine the values of θ and κ oscillations of the pendulum must be observed. When the large masses are placed (and after some time), the initial equilibrium is noted as S_1. Then, when the masses are swiveled to their other position it will cause the pendulum to oscillate until it comes to equilibrium at a new position S_2. This can bee seen in Fig. 3:

FIG. 2. ANGLE VIEW DESCRIPTION



At the new equilibrium position the torsion wire will still be twisted through and angle θ but in the opposite direction of its initial twist. The total change of angle is 2θ. Hence the doubling of the angle as shown below:

$$\Delta S = S_2 - S_1, \tag{6}$$
$$4\theta = \Delta S/L, \tag{7}$$
$$\theta = \Delta S/4L \tag{8}$$

The torsion constant can be determined by observing the period T of the oscillations and then using the equation:

$$T^2 = \frac{4\pi^2 I}{\kappa} \tag{9}$$

Where I is the moment of inertia of the small mass system. The moment of inertial of the mirror and support system for the small masses is negligibly small compared to the masses themselves, so the total moment of inertia is:

$$I = 2m_2\left(d^2 + \frac{2r^2}{5}\right). \tag{10}$$

Then, it can be seen that:

$$\kappa = 8\pi^2 m_2 \frac{d^2 + \frac{2r^2}{5}}{T^2} \tag{11}$$

Substituting Eq. (8), and Eq. (11) into Eq. (5) generates the equation for G as displayed below:

$$G = \pi^2 \Delta S b^2 \frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L d}. \tag{12}$$

All of the values on the right hand side of this equation are either measurable or given in the lab handout. So, by measuring the total deflection of the light spot and the period of oscillation, the value of G can be calculated:

$$G = \pi^2 \Delta S b^2 \frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L d} \tag{13}$$

where ΔS is the change in equilibrium position one (S_1) and equilibrium position two (S_2), b is the distance from the center of the lead ball touching the casing to the center of the tungsten ball, d is the distance from the center of the tungsten ball to the torsion axis, L is the distance from the mirror to the chalkboard, m_1 is the mass of the one lead ball, T is the period of the oscillations, and r is the ...

Furthermore, the error propagation of such a calculation is not trivial and may be viewed below in Eq. (22) through Eq. (29).

C. Method II: Measurement by Equilibrium Positions

When the large masses (lead balls) are moved from Position I to Position II, the torsion balance oscillates for a time, typically more than an hour, and then comes to rest at a new equilibrium position. This oscillation can be described as a damped sine wave with an offset.

It's important to note that this offset is note visible to the eye. The offset displays the value for the equilibrium point of the balance. ΔS can be found by determining the difference between the equilibrium points of Position I and Position II. The rest of the theory is identical to that in "Method 1: Measurement by Final Deflection" (Sec. II B).

$$C_{fit} = A_1 e^{(A_2 t)} \cos\left(\frac{2\pi}{A_3} t + A_4\right) + A_5 \tag{14}$$

where A_1 describes the amplitude of the oscillation, A_3 is the period of the oscillations, and A_5 is the equilibrium position for the simple harmonic oscillation. It is important to note, instead of using Mathematica, the "curve_fit" function from scipy in Python was used in order to generate the curve fits on as shown below.

D. Method III: Measurement by Acceleration

With the large masses in Position I, the gravitational attraction F of the small masses, m_2, to their neighboring large masses, m_1, is determined by the law of universal gravitation, as shown below:

$$F = \frac{G m_1 m_2}{b^2} \tag{15}$$

where b is the distance from the center of the large mass when it is touching the case to the center of the small mass, as mentioned above.

The force caused by the movement of the masses is balanced by the torque impeded onto the twisted torsion ribbon, which acts to place the system in equilibrium. The angle of this twist, θ, is determined by noting the position of the light spot where the reflected beam shines off the chalkboard. After moving the masses to Position II, the system begins to oscillate until it reaches a new equilibrium point.

Due to the oscillation of the small masses being rather long (around eight to nine minutes), it's important to note that the small masses do not move a significant amount with the re-positioning of the large masses from Position I to Position II. Also, due to the symmetry of the setup, the large masses exert the same amount of gravitational force on the small masses, but in the opposite direction. Furthermore, the equilibrating force from the torsion pendulum has not changed. Thus, the total force (F_{net}) acting on the system to accelerate the small masses is equivalent to twice the original gravitational force as in Eq. (15):

$$F_{\text{total}} = 2F = 2\frac{G m_1 m_2}{b^2} \tag{16}$$

Each small mass, therefore, accelerates towards its neighboring large mass with an initial acceleration (a_0) as described below:

$$m_2 a_0 = \frac{2 G m_1 m_2}{b^2} \tag{17}$$

Over time, the friction on the system begins to slow the movement of the small masses and the torsion ribbon becomes more relaxed. This coincides with a decrease in the force on the small masses as well as a decrease in the acceleration of the small masses. Observing the acceleration of the small masses over a relatively small time period (two minutes in this case), Eq. (17) may be utilized to determine G. Given the nature of the damped harmonic oscillation of the system the initial acceleration is constant within about 5% in the first tenth of the an oscillation (Note: roughly 50 seconds for this lab report). So, relatively good results may be obtained if the acceleration is measured within the first minute of the large masses being re-positioned, and the following relationship may be used to determine G:

$$G = \frac{b^2 a_0}{2 m_1} \tag{18}$$

where b is the distance from the center of the lead ball touching the casing to the center of he tungsten ball, a_0 is the acceleration of the light spot across the oscillation, and m_1 is the mass of a lead ball.

The acceleration of the small masses may be measured by observing the displacement of the light spot on the screen. Taking into the account the doubling of the angle on the reflection:

$$\Delta S = \Delta s \left(\frac{2L}{d}\right) \tag{19}$$

where Δs is the linear displacement of the small masses, d is the distance from the center of mass of the small masses to the axis of rotation of the torsion balance, L is the distance of the scale from the mirror of the balance, and ΔS is the displacement of the light spot on the screen.

Then, using the equation of motion for an object with a constant acceleration ($x = l/ta^2$) the acceleration may be calculated as shown below:

$$a_0 = \frac{2\Delta s}{t^2} = \frac{\Delta S d}{T^2 L} \tag{20}$$

Thus, by observing the motion of the light spot over a small amount of time, the acceleration of the small masses may be determined using Eq. (20), and the universal gravitational constant can be determined using Eq. (18).

$$G = \frac{b^2 \Delta S d}{2 m_1 T^2 L} \tag{21}$$

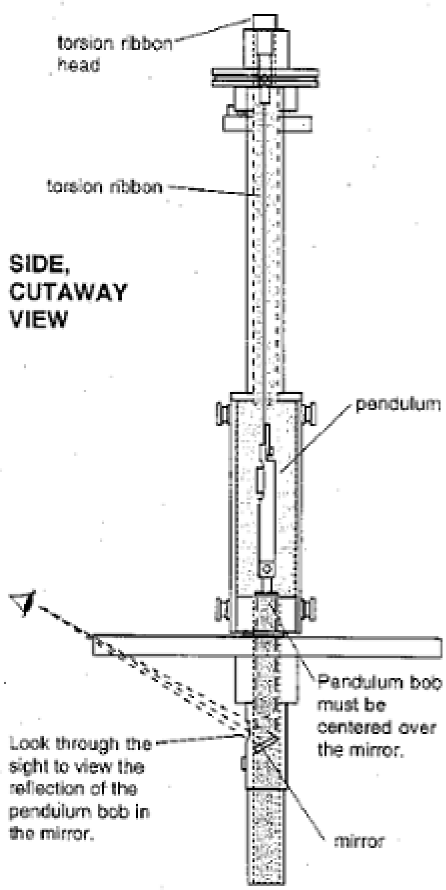
Furthermore, the above equation simply displays the combination of Eq. (20) and Eq. (18) in an attempt to limit calculation time.

III. EXPERIMENTAL DESIGN AND PROCEDURE

A. Cavendish Balance Alignment

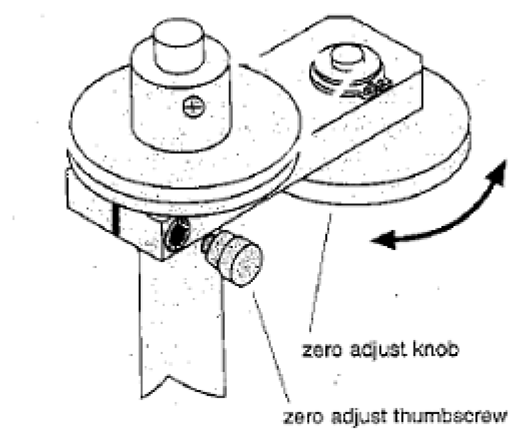
Initially, the most the important adjustment lies in the feet of the Cavendish Balance. The level of the feet determine the perpendicularity of the torsion pendulum, which, in turn, causes the light spot to move across the screen in a perfectly horizontal motion. In order to view whether the pendulum is perfectly perpendicular, the mirror in Fig. 4 was used to ensure that a small ring of light lay around the round bottom of the torsion pendulum. If no light was viewed from the bottom of the torsion pendulum, then the light spot would not move horizontally across the chalk board. Furthermore, to measure the level motion of the light spot, a ruler was placed on the board with a level and a line was drawn across the board using a meter stick.

FIG. 3. Perpendicular Alignment of Torsion Pendulum



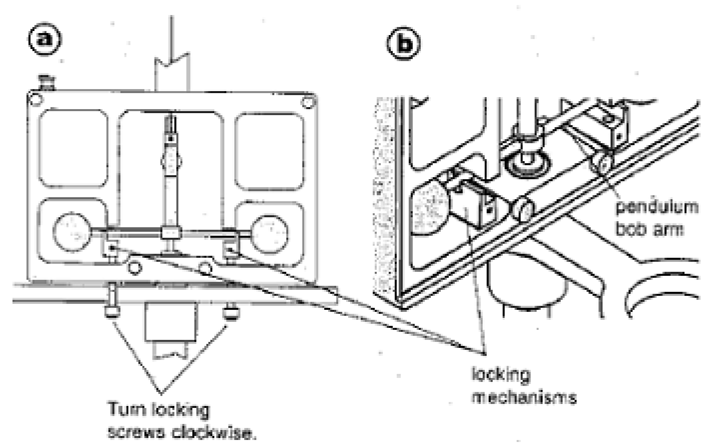
With the apparatus aligned perfectly perpendicular, the rotational alignment of the bob must be corrected. As shown in Fig. 5, the zero adjustment thumb screw was loosened and the zero adjust nob was turned either counterclockwise or clockwise, for motion of the light spot leftwards or rightwards respectively. It is important to note that this step required the most amount of time as a simple adjustment of the zero adjust nob needed at least one hour to settle sufficiently. After twisting the zero adjust nob, several moments had to pass before tightening the zero adjust thumbscrew. Ultimately, the light spot on the chalkboard should align vertically with the reflection of the laser off the glass casing. Unfortunately, the alignment process prevented a third trial of data from being completed. If one more day was granted to the lab, then a third trial would be able to be executed correctly.

FIG. 4. Rotational refinement of pendulum bob



In an attempt to dampen the severity of the oscillations, the locking mechanisms in Fig. 6 were pushed up to stop the oscillation of the small masses. This increased the speed with which adjustments could be made initially. However, as the light spot neared the reflection of the light from the glass casing, the use of the locking mechanisms ceased.

FIG. 5. Locking Mechanisms



IV. CALCULATIONS AND ANALYSIS

A. Method I: Measurement by Final Deflection

B. Method II: Measurement by Equilibrium Positions

C. Method III: Measurement by Acceleration

V. CONCLUSION

TABLE I. Displays the probabilities of all the three dice totals for the first group of data.

Probability		3 Dice Total
1/216	for	3 and 8
3/216	for	4 and 17
6/216	for	5 and 16
10/216	for	6 and 15
15/216	for	7 and 14
21/216	for	8 and 13
25/216	for	9 and 12
27/216	for	10 and 11

$$\delta G = \sqrt{\left(\frac{\partial G}{\partial \Delta S}\delta \Delta S\right)^2 + \left(\frac{\partial G}{\partial b}\delta b\right)^2 + \left(\frac{\partial G}{\partial d}\delta d\right)^2 + \left(\frac{\partial G}{\partial r}\delta r\right)^2 + \left(\frac{\partial G}{\partial T}\delta T\right)^2 + \left(\frac{\partial G}{\partial m_1}\delta m_1\right)^2 + \left(\frac{\partial G}{\partial L}\delta L\right)^2} \tag{22}$$

$$\frac{\partial G}{\partial \Delta S} = \pi^2 b^2 \left(\frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L d}\right) \tag{23}$$

$$\frac{\partial G}{\partial b} = 2\pi^2 b \Delta S \left(\frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L d}\right) \tag{24}$$

$$\frac{\partial G}{\partial d} = \pi^2 b^2 \Delta S \left(\frac{5d^2 - 2r^2}{5T^2 m_1 L d}\right) \tag{25}$$

$$\frac{\partial G}{\partial r} = \pi^2 b \Delta S \left(\frac{\frac{4}{5}r^2}{T^2 m_1 L d}\right) \tag{26}$$

$$\frac{\partial G}{\partial T} = -2\pi^2 b \Delta S \left(\frac{d^2 + \frac{2}{5}r^2}{T^3 m_1 L d}\right) \tag{27}$$

$$\frac{\partial G}{\partial m_1} = -\pi^2 b \Delta S \left(\frac{d^2 + \frac{2}{5}r^2}{T^2 {m_1}^2 L d}\right) \tag{28}$$

$$\frac{\partial G}{\partial L} = -\pi^2 b \Delta S \left(\frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L^2 d}\right) \tag{29}$$

TABLE II. Depicts the χ^2 Table for the individual rolls of the dice and the sum of the rolls of the three die.'

Number of Rolls	White	Green	Red	Sum
216	8.444444	2.444444	5.833333	13.498307
432	9.305556	3.805556	5.055556	13.716005
648	7.537037	2.462963	2.314815	15.511675
864	5.833333	1.305556	3.027778	19.474881

TABLE III. Depicts the P-value for the degree of freedom from a χ^2 table.

	Degrees of Freedom	P-values
Individual	5	1.145
Sum of 3 Die	15	7.261