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1 a

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{Gm}{r^2} \to \frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) = \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} = -\frac{2}{\rho r}\frac{dp}{dr} - 4\pi G\rho$$

By multiplying a r^2 to the relation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + \frac{2}{\rho r}\frac{dp}{dr} = \left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G r^2 \rho$$

Thus, if we define $\rho \equiv \rho_c \theta^n$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K \rho_c^{\frac{1}{n}}(n+1)\frac{d\theta}{dr}\right) = -4\pi G \rho_c \theta^n \tag{1}$$

Now, if

$$r \equiv \alpha\xi$$

In which

$$\xi = r\sqrt{\frac{4\pi G}{(n+1)K\rho_c^{\frac{1}{n}-1}}} \tag{2}$$

Thus,

$$\frac{d}{d\xi} = \frac{d}{dr}\left(\frac{4\pi G}{(n+1)K\rho_c^{\frac{1}{n}-1}}\right)^{-\frac{1}{2}}$$

Therefore, the eq.(1) will be

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{3}$$

2 b

It is impossible to solve this equation analytically in a general way. However, it is possible to solve for some specific powers. Now, we are looking at a few values of n.

2.1 $n = 0$

The general equation will be reduced to

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2 \rightarrow \int d\left(\xi^2\frac{d\theta}{d\xi}\right) = -\int \xi^2 d\xi \rightarrow \xi^2\frac{d\theta}{d\xi} = c_1 - \frac{\xi^3}{3}$$

Thus,

$$\theta(\xi) = \int \xi \frac{C - 1 - \frac{\xi^3}{3}}{\xi^2} = \theta_0 - c_1\xi^{-1} - \frac{1}{6}\xi^2$$

Applying BCs, $(\theta(0) = 1, \theta'(0) = 0)$

$$\theta_1(\xi) = 1 - \frac{1}{6}\xi^2 \tag{4}$$

2.2 $n = 1$

In this case, the equation will be

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta\xi \tag{5}$$

Which is a spherical Bessel differential equation.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + [k^2r^2 - n(n+1)]R = 0$$

One of the answers for $k = 1$ and $n = 0$ is that

$$\theta(\xi) = AJ_0(\xi) + Bn_0(\xi)$$

By applying BCs ($\theta(0) = 1, \theta'(0) = 0$) B will vanish and only J will be remained.

$$\theta_2(\xi) = \frac{sin\xi}{\xi} \approx 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} + ... \tag{6}$$

2.3 $n = 5$

First, by making a Emden's transformation

$$\theta = Ax^{\omega}z, \omega = \frac{2}{n-1}$$

The main equation will be

$$\frac{d^2z}{dt^2} + (2\omega - 1)\frac{dz}{dt} + \omega(\omega - 1)z + A^{n-1}z^n = 0$$

It is possible to show

$$\frac{d^2z}{dt^2} = \frac{z}{4}(1 - z^4)$$

That the solution is

$$\theta_5(\xi) = (1 + \frac{\xi^2}{3})^{-\frac{1}{2}} = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{24} + ... \quad (7)$$

It is allowed to write the function $\theta^n(\xi)$ in terms of power series, $\theta^n(\xi) = \sum_i \theta_i$, around the center. So, the final general solution around the center should be something like what is derived in the above:

$$\theta_n(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + ... \quad (8)$$

2.4 Mathematica

There are mathematica codes to solve the equation analytically

2.4.1 $n = 0$

In[161]:= Clear[y] DSolve[y''[t] + (2/t)*y'[t] == -1, y[0.0001] == 1, y'[0.0001] == 0, y[t], t]

Out[162]= y[t] -> (0.166667 (-2.*10$^-12 + 6.t - 1.t^3$))$/t$

I used a small number close to zero, because the syntax "DSolve" does not work when t is exactly zero.

2.4.2 $n = 1$

In[163]:= Clear[y] DSolve[y''[t] + (2/t)*y'[t] == -t, y[0.0001] == 1, y'[0.0001] == 0, y[t], t]

Out[164]= y[t] -> (0.0833333 (-3.*10$^-16 + 12.t - 1.t^4$))$/t$

2.4.3 General Code

There is also a general code with a numerical method that is attached.

2.5 Total Mass

It is clear that

$$M = \int_0^R dr 4\pi r^2 \rho = \int_0^R dr 4\pi r^2 \rho_c \theta^n$$

From the eq.(2), $r = \alpha\xi$. Thus,

$$M = 4\pi\alpha^3 \rho_c \int_0^R dr r^2 \theta^n$$

By using the main equation, the eq. (3), and substituting θ^n

$$M = -4\pi\alpha^3 \rho_c \int_0^R d\xi \xi^2 \frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -4\pi\alpha^3 \rho_c \int_0^{\xi_n} d\xi \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)$$

Therefore, $(R = \alpha\xi_n)$

$$M = -4\pi\alpha^3 \rho_c \xi_n^2 \frac{d\theta_n}{d\xi} = -4\pi\rho_c R^3 \frac{\theta_n'}{\xi_n} \tag{9}$$

From the eq. (2)

$$\rho_c = \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{\frac{n}{n-1}}$$

So,

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{\frac{n}{n-1}} R^3 \frac{\theta'_n}{\xi_n} = -\left(4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \theta'_n\right) \left(\frac{\xi_n}{R}\right)^{\frac{2n}{n-1}} \frac{R^3}{\xi_n}$$

Finally,

$$M = -\left(4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \theta'_n \xi_n^{\frac{n+1}{n-1}}\right) R^{\frac{3-n}{1-n}} \tag{10}$$

2.6 c

In this part, everything is the same as before, except details in the prefactor. SO, we can still use the eq. (10). In the code, the equation below is considered.

$$R = yM^x$$

By making an equivalence to the eq. (10), we will know

$$y = -\left(4\pi \left[\frac{(n_* + 1)K_*}{4\pi G}\right]^{\frac{n_*}{n_*-1}} \theta'_n \xi_n^{\frac{n_*+1}{n_*-1}}\right)^{-\frac{1-n_*}{3-n_*}}$$

And

$$x = \frac{1-n_*}{3-n_*}$$

According to the code,

$$x \approx 1.8 \rightarrow n_* \approx 5.4 \rightarrow y \approx 3.7 \times 10^7 - 0.43 K_*^{\frac{n_*}{n_*-3}} {\theta'_n}^{\frac{1-n_*}{n_*-3}} \xi_n^{\frac{n_*+1}{n_*-3}}$$