Exercise: Derivation of Hotelling's Rule

The following is the Wikipedia definition of Hotelling's rule:[1]

> Hotelling's rule states that the most socially and economically profitable extraction path of a non-renewable resource is one along which the price of the resource, determined by the marginal net revenue from the sale of the resource, increases at the rate of interest. It describes the time path of natural resource extraction which maximizes the value of the resource stock.

If some non-renewable resource is sold at price $p(t)$ and its marginal extraction cost is $c(t)$, Hotelling's rule states that the growth rate of the marginal net revenue $p(t) - c(t)$ is determined by

$$\frac{\frac{d}{dt}\left(p(t) - c(t)\right)}{p(t) - c(t)} = r(t),$$

where $r(t)$ is the interest rate. The price less marginal cost must increase at the same rate as the interest rate. Let's derive this equation.

To simplify the calculation, assume that $r(t) = r$ is constant. Let $R(t)$ denote the stock of the non-renewable resource such as fossil fuel at time t and $E(t)$ the amount of extraction. The dynamics of the resource stock is determined by

$$\dot{R}(t) = -E(t).$$

Assume that there is a single owner of the resource, who maximizes the sum of discounted future profits from sales of the resource. The benefit of extraction, which is identical to the revenue, is $p(t)E(t)$ and the cost of extraction is assumed to be $c(t)E(t)$. Let $\gamma(t) = p(t) - c(t)$. He therefore maximizes

$$\int_0^\infty e^{-rt}\gamma(t)E(t)dt \qquad \text{subject to } \dot{R}(t) = -E(t).$$

Problem.

1. The problem has two important variables $R(t)$ and $E(t)$. Which is the stock variable? Which is the flow variable?
2. Set up the Hamiltonian function $\mathcal{H}(R, E, \lambda)$.
3. Hamiltonian differentiated by the flow variable must be zero. Obtain this first-order condition.
4. Hamiltonian differentiated by the stock variable must be equal to $-\dot{\lambda}$. Obtain this first-order condition.
5. Use equations obtained in Problems 3 and 4 to get the desired equation:

$$\frac{\dot{\gamma}}{\gamma} = r.$$

The price of the resource increases as the resource becomes scarecer. The profit maximization implies that the rate of this increase is identical to the interest rate.

[1] *Wikipedia: The free encyclopedia*, s.v. "Hotelling's rule." [https://en.wikipedia.org/w/index.php?title=Hotelling%27s_rule&oldid=690259933]