Problem Set (not for grading)

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[1] Cobb–Douglas function

Show that a Cobb–Douglas function

$$F(K, AL) = K^{\alpha}(AL)^{\beta}$$

is of constant returns if and only if $\alpha + \beta = 1$.

Remark 1. Proof for an "if and only if" statement typically consists of two parts. To show that "A holds if and only if B holds", (1) assume A and then prove B and (2) assume B and then prove A. There is no general rule for the order of (1) and (2).

[2] CES function

In macroeconomics, we use CES functions very often. CES functions are a family of functions of the form:

$$F(K, L) = \left[\alpha K^{\theta} + (1 - \alpha)L^{\theta}\right]^{\frac{1}{\theta}}, \quad 0 < \alpha < 1, \quad \theta < 1.$$

(a) Show that F has constant marginal returns.

(b) Show that the margianl rate of technical substitution,[1] defined by

$$MRTS_{KL} := \frac{\left(\frac{\partial F}{\partial L}\right)}{\left(\frac{\partial F}{\partial K}\right)},$$

satisfies

$$MRTS_{KL} = \frac{\alpha}{1 - \alpha}\left(\frac{K}{L}\right)^{1-\theta}.$$

(c) Represent $\ln\left(\frac{K}{L}\right)$ as a function of $\ln MRTS_{KL}$ to show that the elasticity of substitution, defined by,

$$\sigma = \frac{d \ln (K/L)}{d \ln MRTS_{KL}},$$

[1] Marginal rate of technical substitution is the amount of reduction of one factor in reaction to an increase in another factor by one extra unit, necessary to make output remain constant.

satisfies

$$\sigma = \frac{1}{1-\theta}.$$

(d) The Cobb–Douglas family of production functions is a subset of CES with $\theta \to 0$. To see this, use l'Hospital's rule to compute:

$$\lim_{\theta \to 0} \ln F(K, L) = \lim_{\theta \to 0} \frac{\alpha K^\theta + (1-\alpha) L^\theta}{\theta}.$$

[3] Solow model

We have derived that in the Solow model capital per unit of effective labor $k(t) = K(t)/[A(t)L(t)]$ satisfies

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t),$$

where $0 < s < 1$ is the constant saving rate, $f(k) = F(k, 1)$ the production function per effective labor, δ the depreciation rate, g the technical growth rate, n the population growth rate.

(a) Derive the steady state growth rate for aggregate production $Y(t) = F(K(t), A(t)L(t))$.

(b) Derive the steady state growth rates for aggregate consumption $C(t) = (1-s)Y(t)$ and per-capita consumption $C(t)/L(t)$.