## Discrete-time Ramsey model

In discrete-time, the flow budget constraint is much easier to handle. Let $S_t$ be the bank balance of the representative household at the begining of period $t$, where $t = 0, 1, \ldots$ The budget constraint for the household in period $t$ is

$$S_{t+1} - S_t + c_t \left(\frac{L_t}{H}\right) \leq w_t \left(\frac{L_t}{H}\right) + r_t S_t,$$

where $S_0 \geq 0$. The interpretations doesn't change of course. Expenditure of the household consists of the increment of bank balance and their consumption. Income consists of wage and interest. The flow budget constraints requires that the expenditure doesn't exceed income in each period. Rearranging terms and defining $C_t = c_t L_t / H$ and $W_t = w_t L_t / H$, we have the following simpler inequality:

$$S_{t+1} + C_t \leq W_t + (1 + r_t) S_t.$$

Let's arrange flow budget constraints chronologically:

$$\begin{aligned} S_1 + C_0 &\leq W_0 + (1 + r_0) S_0 \\ S_2 + C_1 &\leq W_1 + (1 + r_1) S_1 \\ S_3 + C_2 &\leq W_2 + (1 + r_2) S_2 \\ &\vdots \end{aligned}$$

These inequalities are written in terms of current values. Let's rewrite them in present values. To do this, we multiply the discount factors:

$$\begin{aligned} S_1 + C_0 &\leq W_0 + (1 + r_0) S_0 && \times \frac{1}{1 + r_0} \\ S_2 + C_1 &\leq W_1 + (1 + r_1) S_1 && \times \frac{1}{(1 + r_0)(1 + r_1)} \\ S_3 + C_2 &\leq W_2 + (1 + r_2) S_2 && \times \frac{1}{(1 + r_0)(1 + r_1)(1 + r_2)} \\ &\vdots && \vdots \end{aligned}$$

Define for notational simplicity

$$R_t = (1 + r_0) \times \cdots \times (1 + r_t),$$

which is the gross interest rate between period 0 and period $t$. We obtain the "present-value flow budget constraints"

$$\begin{aligned}
R_0^{-1}S_1 + R_0^{-1}C_0 &\le R_0^{-1}W_0 + S_0 \\
R_1^{-1}S_2 + R_1^{-1}C_1 &\le R_1^{-1}W_1 + R_0^{-1}S_1 \\
R_2^{-1}S_3 + R_2^{-1}C_2 &\le R_2^{-1}W_2 + R_1^{-1}S_2 \\
&\vdots
\end{aligned}$$

Notice that the coefficient for the last term in each inequality is $R_{t-1}^{-1}$ because $(1+r_t)R_t^{-1} = R_{t-1}^{-1}$. Sum up each sides of the above inequalities up to period $T$ to get

$$R_T^{-1}S_{T+1} + \sum_{t=0}^{T} R_t^{-1}C_t \le S_0 + \sum_{t=0}^{T} R_t^{-1}W_t.$$

By taking the limit of $T \to \infty$, we have

$$\sum_{t=0}^{\infty} R_t^{-1}C_t \le S_0 + \sum_{t=0}^{\infty} R_t^{-1}W_t,$$

which means that the lifetime consumption doesn't exceed initial wealth plus lifetime earning. Here, we assumed that

$$\lim_{T\to\infty} R_T^{-1}S_{T+1} \ge 0.$$

This is called the No-Ponzi game condition. A Ponzi game or Ponzi scheme is to roll over debt forever. If you borrow to repay, your debt (negative $S$) will explode and $R_T^{-1}S_{T+1} < 0$ will hold. This condition rules out such case.

**Euler equation** We formulate the household's optimiation problem as

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t)\frac{L_t}{H}$$

subject to

$$\sum_{t=0}^{\infty} R_t^{-1}c_t\left(\frac{L_t}{H}\right) \le S_0 + \sum_{t=0}^{\infty} R_t^{-1}w_t\left(\frac{L_t}{H}\right),$$

$$S_0: \text{ given.}$$

The Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t)\frac{L_t}{H} + \lambda\left[S_0 + \sum_{t=0}^{\infty} R_t^{-1}w_t\left(\frac{L_t}{H}\right) - \sum_{t=0}^{\infty} R_t^{-1}c_t\left(\frac{L_t}{H}\right)\right].$$

The first-order condition,

$$\frac{\partial\mathcal{L}}{\partial c_t} = \left(\frac{1}{1+\rho}\right)^t u'(c_t)\frac{L_t}{H} - \lambda R_t^{-1}\frac{L_t}{H} = 0,$$

implies

$$u'(c_t) = \lambda R_t^{-1}\left(\frac{1}{1+\rho}\right)^{-t}, \quad t = 0, 1, \ldots$$

Similarly, $\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0$ implies

$$u'(c_{t+1}) = \lambda R_{t+1}^{-1} \left( \frac{1}{1+\rho} \right)^{-t-1}.$$

We thus have the Euler equation:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{R_{t+1}}{R_t} \frac{(1+\rho)^t}{(1+\rho)^{t+1}} = \frac{1+r_{t+1}}{1+\rho}.$$

If we assume that

$$u(c) = \frac{c^{1-\theta}}{1-\theta},$$

then

$$\frac{c_{t+1}}{c_t} = \left( \frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}}.$$

Taking log, we have

$$\ln c_{t+1} - \ln c_t = \frac{\ln(1+r_{t+1}) - \ln(1+\rho)}{\theta}. \tag{1}$$

Compare this equation to the continuous-time Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta},$$

or

$$\frac{d}{dt} (\ln c(t)) = \frac{r(t)-\rho}{\theta}. \tag{2}$$

You observe similarity in (1) and (2).