Units of measurement in continuous- and discrete-time models

mail@**kenjisato**.jp

June 17, 2016

In the last session (June 15), we considered a model economy with three capital stocks, K (capital; seed potatoes), L (labor force) and A (labor-augmenting knowledge). L and A appear in the multiplicative form, AL, which is called the effective labor. A is somewhat abstract and so keep the following example in mind: Scientific knowledge on agriculture is much deeper today than in 100 years ago and thus we require less labor to produce the same quantity of output. In our simple model, the stock variables assume the following properties.

- L grows (or shrinks) at a constant rate, n, without any particular reason.
- A grows (or shrinks) at a constant rate, g, without any particular reason.
- K grows (or shrinks) according to saving (investment) decisions of the people.

Notice that the first and second items in the above list are mere simplifying assumptions. You can and will consider a model in which these assumptions are partially dropped.

According to our notation for time, we divide the flow of time into periods.

- periods $t = 0, 1, 2, \dots$ in a discrete-time model;
- periods $[t, t + \Delta t)$, $t \geq 0$, in a continuous-time setting, with small Δt,

In each period, several economic activities take place:

- production,
- consumption,
- saving, and
- investment,

which correspond to flow variables Y, C, S and I, respectively.

<u>Notice the following fact, which I didn't make clear in the last session.</u> Flow variables are measured by

- the unit of quantity in discrete-time models, and

- **the unit of [quantity/year] in continuous-time models.**

To see this, consider a situation in which the people consume $C = 365$ potatoes per year. Under a discrete-time setting, the quantity of consumed potatoes are measured/counted by kilograms, pounds or pieces. There still is freedom of choice but we can intuitively understand whatever choice we make. When we translate this into continuous time, a difficulty comes up. We cannot measure the flow variables by a unit of quantity. Consider a small Δt and divide one year into $N = 1/\Delta t$ subperiods. If $\Delta t = 1/365$ or $N = 365$, (average) daily consumption is $C/N = 1$. What if we naively take the limit of $\Delta t \to 0$, or $N \to \infty$? The consumption per hour is $1/24$, and $1/(24 \times 60)$ per minute, $1/(24 \times 60 \times 60)$ per second and so on. **The quantity consumed in an infinitessimally short period $[t, t + \Delta t]$ becomes nil. It is therefore meaningless to measure $C(t)$, $S(t)$, $I(t)$ and $Y(t)$ by the unit for quantity**.

We observed a similar issue when we considered infinitely frequent compounding. In that case we used an annual interest rate as a definition of the instantaneous interest (or growth) rate. Although the real (effective) interest rate becomes different from the annual rate, the latter works very well as a normalizer. When defining flow variables, we need to use a similar concept. A flow variable in a continuous-time model represents the instantaneous speed of economic activity. **The statement that the consumption at time t is $C(t)$ means the following fact: If the speed of consumption, $C(t)$, maintains for one year, the total consumption in one year from now will become $C(t)$.** In the last session, we saw such expressions as

$$Y(t)\Delta t = C(t)\Delta t + S(t)\Delta t. \tag{1}$$

This means that the quantity of production in period $[t, t + \Delta t]$ is equal to the sum of quantities for consumption and saving in the same period. The identity is written in terms of the unit of quantity. If you rewrite it into the identity in terms of speed, we get

$$Y(t) = C(t) + S(t). \tag{2}$$

(2) is more accurate than (1) because we implicitly assume in (1) that the speeds of the activities stay constant during the period of $[t, t + \Delta t]$, which is a good approximation if Δt is sufficiently small. To sum up, equations such as

$$Y_t = C_t + S_t \tag{3}$$

in discrete-time models are equations in terms of quantity [kg., lbs., pieces, etc]; in contrast, in continuous-time models, such equations as

$$Y(t) = C(t) + S(t) \tag{4}$$

are measured in terms of speed [pieces/year, lbs./year, pieces/year, etc.].

Let's review all the relationship among activities.

Production The quantity of produce (or speed of production, in continuous-time) is determined by a production function

$$Y = F(K, AL). \tag{5}$$

It should be natural to assume that you get more Y (output; new potatoes) with more K (capital; seed potatoes) with fixed AL (effective labor; combination of knowledge and labor) or with more AL with fixed K.

Consumption and Saving The people decide how much of the produce to eat and how much to save for production in the future. The quantity consumed is denoted by $C = C(Y)$ and the quantity saved by $S = S(Y)$ as functions of Y. It always holds that

$$Y = C + S, \tag{6}$$

under the assumptions of no government or trade. Since saving (potatoes they decide not to eat) automatically becomes investment (potatoes they decide to use for next planting), we have

$$S = I, \tag{7}$$

where $I = I(Y)$ is the quantity of saving.

Analysts (like you) have freedom to model the behavior (or the function forms) of people to some extent. Influential models are the following two.

- $C(Y) = (1 - s)Y$, where s is a constant rate (Solow-type model)
- $C(Y)$ is determined as the solution of consumers' optimal decition (Ramsey-type model)

Another possible assumption when you model a short-run economic behavior, which lies out of our scope in this course, is the following:

- $C(Y) = a + bY$, (Keynes-type consumption function)

See Chapter 8 of [Romer 4e] for a detailed discussion on the term, a.

Investment and Capital Accumulation The invested (or saved) I for next production. As the result of the above-described activites during the period, the have the-end-of-the-period stock of size

$$K + I - \delta K \tag{8}$$

in a discrete-time setting, while in continuous time, it is

$$K + I\Delta t - (\delta\Delta t)K. \tag{9}$$

The first-term, K, represents the beginning-of-the-period stock of capital (seed potatoes), the second, I (or $I\Delta t$), newly added capital (gross capital formation; newly added seed potatoes), and the third, $-\delta K$ (or $-(\delta\Delta t)K$), the capital depriciation. The quantity δK (or $(\delta\Delta t)K$) out of K becomes too old to produce in the next period and so they have no choice but dumping them. As a result, they need to invest at least as much as δK (or $(\delta\Delta t)K$) to keep the same amount of capital stock. $I - \delta K$ (or $I\Delta t - (\delta\Delta t)K$), is called the net investment.

Summary We can now make the following summary table.

Table 1: Basic model of economic growth in discrete time, approximately discrete time and continuous time

	Dicrete time, in period t	Continuous time, in period $[t, t + \Delta t]$	Continuous, in terms of speed
Production	$Y_t = F(K_t, A_t L_t)$	$Y(t)\Delta t = F(K(t), A(t)L(t))\Delta t$	$Y(t) = F(K(t), A(t)L(t))$
Consumption and Saving	$Y_t = C_t + S_t$	$Y(t)\Delta t = C(t)\Delta t + S(t)\Delta t$	$Y(t) = C(t) + S(t)$
Investment	$I_t = S_t$	$I(t)\Delta t = S(t)\Delta t$	$I(t) = S(t)$
Capital Accumulation	$K_{t+1} = K_t + I_t - \delta K_t$	$K(t + \Delta t) = K(t) + I(t)\Delta t - (\delta \Delta t)K(t)$	$\dot{K}(t) = I(t) - \delta K(t)$
Knowledge growth	$A_{t+1} = (1 + g)A_t$	$A(t + \Delta t) = (1 + g\Delta t)A(t)$	$\dot{A}(t) = gA(t)$
Labor growth	$L_{t+1} = (1 + n)L_t$	$L(t + \Delta t) = (1 + n\Delta t)L(t)$	$\dot{L}(t) = nL(t)$

The discrete-time model (first column) and the continuous-time model (third column) have different appearances but they are different representations for the same reality. The gap between them is easily filled by considering the discrete-time approximation of the continuou-time model.[1]

The approximated system is useful for computation. When you perform a simulation analysis of continuous-time model, you need to derive a discre-time approximation (second column) because computers cannot solve complex differential equations analytically. Difference equations like the first and second columns can be computed in a step-by-step basis, at which computers are very good.

[1] As I noted elsewhere, it is more natrural to consider a continuous-time model to be an approximation for discrete-time nature of available data. But it is mathematically valid to say that the second column is a discrete-time approximation for the continuous-time model (third column).