Solution to Quiz [16MA2G]

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Problem

Let f be of a Cobb-Douglas form $f(k) = k^{\alpha}$ with $0 < \alpha < 1$. Use the Lagrangian method to solve the following optimization problem:

$$\begin{aligned}
&\max\left[\ln c_0 + \beta \ln c_1\right] \\
&\text{subject to} \\
&\qquad c_0 = f(k_0) - k_1, \\
&\qquad c_1 = f(k_1) - k_2, \\
&\qquad k_0 : \text{ given.}
\end{aligned}$$

Solution

Note that (c_0, c_1, k_1, k_2) cannot be optimal with $k_2 > 0$. The optimal solution satisfies $k_2 = 0$; this condition is the simplest version of the so-called transversality condition. Now set up the Lagrangian under the condition of $k_2 = 0$:

$$\mathcal{L} := \ln c_0 + \beta \ln c_1 + \lambda_0 \left(k_0^{\alpha} - k_1 - c_0\right) + \lambda_1 \left(k_1^{\alpha} - c_1\right).$$

The first-order condition for optimality consists of the following five equations.

$$\frac{\partial \mathcal{L}}{\partial c_0} = \frac{1}{c_0} - \lambda_0 = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{\beta}{c_1} - \lambda_1 = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial k_1} = -\lambda_0 + \alpha \lambda_1 k_1^{\alpha - 1} = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_0} = k_0^{\alpha} - k_1 - c_0 = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = k_1^{\alpha} - c_1 = 0. \tag{5}$$

By (1), (2) and (3), we have

$$\frac{c_1}{c_0} = \beta \frac{\lambda_0}{\lambda_1} = \beta \alpha k_1^{\alpha-1}, \tag{6}$$

which is referred to as the Euler condition (Note that $\alpha k_1^{\alpha-1} = f'(k_1) = 1 + r$, under $\delta = 1$, where r is the rental rate. We have $c_1/c_0 = \beta(1+r)$). Since (5) gets us $c_1 = k_1^\alpha$, we obtain

$$\frac{k_1}{c_0} = \alpha\beta.$$

By (4),

$$(1+\alpha\beta)c_0 = k_0^\alpha$$

and

$$c_0 = \frac{1}{1+\alpha\beta} k_0^\alpha.$$

We therefore have

$$k_1 = \frac{\alpha\beta}{1+\alpha\beta} k_0^\alpha$$

and

$$c_1 = k_1^\alpha = \left(\frac{\alpha\beta}{1+\alpha\beta} k_0^\alpha \right)^\alpha.$$

Since the utility is concave and the constraint is convex, the above necessary conditions are also sufficient for optimality. In sumary, the following values are the optimal solution:

$$\begin{aligned}
c_0^* &= \frac{1}{1+\alpha\beta} k_0^\alpha, \\
k_1^* &= \frac{\alpha\beta}{1+\alpha\beta} k_0^\alpha, \\
c_1^* &= \left(\frac{\alpha\beta}{1+\alpha\beta} k_0^\alpha \right)^\alpha, \\
k_2^* &= 0.
\end{aligned}$$