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Summary of the Solow Model

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1 Framework

Let's first review our basic model economy without the set of Solow assumptions:

- There is only one good in the economy which is consumed and used for production.
- Production requires capital and effective labor (labor *multiplied by* knowledge; Harrod neutral).
- No government, no international trade.
- All saving goes to investment.
- Technology and labor force grow exogenously.

Table 1 is the mathematical formulation of the above list.

Table 1: Basic model of economic growth in discrete time, approximately discrete time and continuous time

	Dicrete time, in period t	Continuous time, in period $[t, t+\Delta t]$	Continuous, in terms of speed
Production	$Y_t = F(K_t, A_t L_t)$	$Y(t)\Delta t = F(K(t), A(t)L(t))\Delta t$	$Y(t) = F(K(t), A(t)L(t))$
Consumption and Saving	$Y_t = C_t + S_t$	$Y(t)\Delta t = C(t)\Delta t + S(t)\Delta t$	$Y(t) = C(t) + S(t)$
Investment	$I_t = S_t$	$I(t)\Delta t = S(t)\Delta t$	$I(t) = S(t)$
Capital Accumulation	$K_{t+1} = K_t + I_t - \delta K_t$	$K(t+\Delta t) = K(t) + I(t)\Delta t - (\delta\Delta t)K(t)$	$\dot{K}(t) = I(t) - \delta K(t)$
Knowledge growth	$A_{t+1} = (1+g)A_t$	$A(t+\Delta t) = (1+g\Delta t)A(t)$	$\dot{A}(t) = gA(t)$
Labor growth	$L_{t+1} = (1+n)L_t$	$L(t+\Delta t) = (1+n\Delta t)L(t)$	$\dot{L}(t) = nL(t)$

This framework does not determine the dynamics of the economy by itself. Look at the identity:

$$Y_t = C_t + S_t \quad \text{and} \quad Y(t) = C(t) + S(t).$$

The rule how consumers make their decisions about consumption and saving is unclear. We need to model the decision making. The Solow model and the Romer model differs in this point.

2 Solow model

The model assumes that consumers make consumption–saving decision according to the following criterion.

- Fixed fraction, $0 < s < 1$, of output is saved; that is,
  - $S = sY$,
  - $C = (1 - s)Y$,
  - In a sense, the Solow model ignores consumption side of the economy and focuses on the production side.

In the Solow model, we add the following assumptions for analytical reason. The dynamics of the model is determined without the following assumptions but adding them does us good in two ways. Firstly, the analysis becomes dramatically simple. Analytical tractability helps us interpret a result.[1]

- F has constant marginal returns.
  - Analysis in terms of $k = K/AL$, effective labor.
  - We defined the intensive-form production function $f(k) = F(k, 1)$.
- f satisfies the Inada-type conditions.
  - $f(0) = 0$,
  - $f'(k) > 0$,
  - $\lim_{k \to 0} f'(k) = +\infty$,
  - $\lim_{k \to \infty} f'(k) = 0$.

Figure 1 depicts the flow of goods in the Solow model, during an infinitessimally short period.

The fundamental dynamic relation in the Solow model is the following:

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t). \tag{1}$$

Since

$$\begin{aligned} \ln k(t) &= \ln \frac{K(t)}{A(t)L(t)} \\ &= \ln K(t) - \ln A(t) - \ln L(t), \end{aligned}$$

[1] Macroeconomics in different fields use different tools. Today, many macroeconomists, especially those who work on such short-run phenomena as business cycles and monetary problems, build a large scale model to have a truer description of the real economy and employ computer simulation and statistical tests to verify their hypotheses. Macroeconomics focusing on a more long-run economic behavior tend to prefer a simple model easier to understand the underlying mechanism of the phenomenon. Solow, Ramsey and Diamond OLG models are considered fundamental to both methodologies.

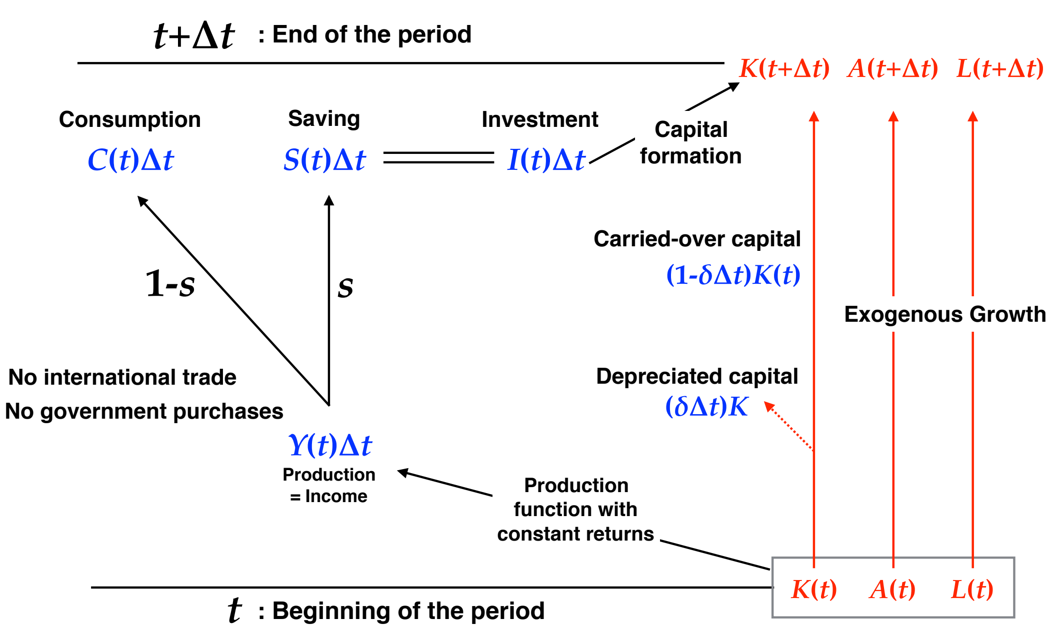

Figure 1: Solow model: Diagram for economic activities during a very short period $[t, t + \Delta t]$

we have

$$
\begin{aligned}
\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} && \left(\tfrac{d}{dt}\ln x(t) = \tfrac{\dot{x}(t)}{x(t)}\right) \\
&= \frac{sF(K(t), A(t)L(t)) - \delta K(t)}{K(t)} - g - n && \text{(Capital accumulation equation)} \\
&= \frac{s \cdot \frac{F(K(t),A(t)L(t))}{A(t)L(t)}}{\frac{K(t)}{A(t)L(t)}} - \delta - g - n \\
&= \frac{sf(k(t))}{k(t)} - (\delta + g + n). && \text{(Definition of } k(t) \text{ and } f(k)\text{)}
\end{aligned}
$$

Multipling both sides $k(t)$, we get

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t).$$

- $sf(k)$ is investment per unit of effective labor and
- $(\delta + g + n)k$ is the break-even investment per unit of effective labor.
  - Since the capital per effective labor depreciates at the rate of δ, even if AL is fixed, investment of δk is required to keep the beginning-of-period level of capital per effective labor.
  - Since AL grows at the rate of $g + n$, K/AL shrinks at the rate of $g + n$ with K fixed.
  - In fact, K grows and AL shrinks simultaneously and thus the overall variation of K/AL is a shrinking of rate $\delta + g + n$.

Note the following facts:

- If $\dot{k}(t) > 0$, k is growing at moment t;
- if $\dot{k}(t) < 0$, k is shrinking at moment t; and
- if $\dot{k}(t) = 0$, k is not moving at moment t.[2]

These facts are better understood with approximation. Since

$$\dot{k}(t) \simeq \frac{k(t + \Delta t) - k(t)}{\Delta t},$$

if $\Delta t > 0$, $k(t + \Delta t)$ denotes the capital per effective labor at an instant after time t. For a sufficiently small $\Delta t > 0$,

- $\dot{k}(t) > 0$ is equivalent to $k(t + \Delta t) > k(t)$, i.e. k is growing;
- $\dot{k}(t) < 0$ is equivalent to $k(t + \Delta t) < k(t)$, i.e., k is shrinking;
- $\dot{k}(t) = 0$ is implied by $k(t + \Delta t) = k(t)$, i.e., k stays steadily.

[2]Strictly speaking, this is a false statement.

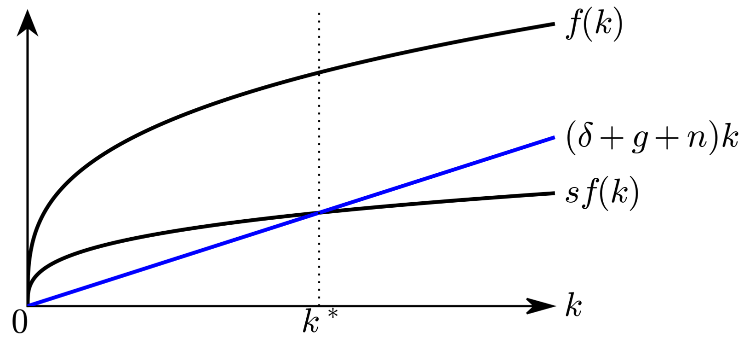

Figure 2: Fundamental equation

3 Dynamics of k in the Solow model

The only components in (1) are $sf(k)$ and $(\delta+g+n)k$. Scaling down f to the fraction s gets us the former. The latter is a linear function, whose graph is a straight line that starts from the origin. If the graph of $sf(k)$ lies above $(\delta+g+n)k$, then $\dot{k}>0$ holds and thereby k growing. If the graph of $sf(k)$ lies below $(\delta+g+n)k$ on the other hand, then $\dot{k}<0$ holds and thus k decreasing. At the intersections of the two graphs, $\dot{k}=0$ holds and the value of k stays steadily at the original level. Under the Inada-type conditions, the Solow model has two such steady states, $k=0$ and $k^*>0$. Since $k=0$ is not interesting economically, we focus on k^*, which we call *the* steady state.

See Figure 2. The capital per unit of effective labor, k, grows if k is smaller than k^* and it shrinks if it is greater than k^*. In drawing the figure, we utilize the Inada-type assumptions made above. With those properties, $sf(k)$ and $(\delta+g+n)k$ crosses only once for $k>0$ and at the intersection, $sf(k)$ crosses $(\delta+g+n)k$ from the North-West to the South-East. As is seen in the figure, k moves toward k^* from wherever it starts off (except for $k(0)=0$). This property is called the stability of the steady state. Since the economy eventually settles down to the steady state, k^* may be considered to be a long-run level of capital per unit of effective labor.

Although k stays constant at k^*, we observe dynamic phenomena in the steady state. For the capital per capita in the steady state, we have

$$\frac{K(t)}{L(t)}=A(t)k(t)=A(t)k^*.$$

The growth of K/L is thus driven solely by the growth of A. Similarly, from

$$\frac{Y(t)}{L(t)}=A(t)f(k(t))=A(t)f(k^*)$$

we observe that the output per capita, Y/L, is also driven solely by the growth of A. Since the consumption is a fixed fraction of output,

$$\frac{C(t)}{L(t)}=\frac{sY(t)}{L(t)}=sA(t)f(k(t))=A(t)\cdot sf(k^*)$$

grows at the rate, g. For the aggregate capital, output, and consumption, we have

$$K(t) = A(t)L(t)k^*, \quad Y(t) = A(t)L(t)f(k^*) \quad \text{and} \quad C(t) = A(t)L(t) \cdot sf(k^*).$$

The common growth rate of these variables are $g + n$, the sum of that for technical change and that for population growth.

4 Kaldor's stylized facts

Kaldor (1961) observed the following facts, which are referred to as the stylized facts. [See Jones and Romer (2010)]

1. Labor productivity has grown at a sustained rate.
2. Capital per worker has also grown at a sustained rate.
3. The real interest rate or return on capital has been stable.
4. The ratio of capital to output has been stable.
5. Capital and labor have captured stable shares of national income.
6. Among the fast growting countries of the world, there is an appreciable variation in the rate of growth.

These facts are so stable that a model of growth is expected to satisfy them. The balanced growth path of the Solow model can explain 1 to 5. There have been many attempts to include 6 by considering the transition path of the Solow model and by building other models. If we borrow the mathematical notation defined before, 1 – 5 of the stylized facts can be translated into the following list:

1. Y/L grows at a sustained rate.
2. K/L grows at a sustained rate.
3. $r = \frac{\partial F}{\partial K} - \delta$ is constant.
4. K/Y is constant.
5. rK/Y and wL/Y are constant (w is the wage rate).

Exercise. Show that the balanced growth path in the Solow model satisfies 1 to 5 of the stylized facts.