Solution to [16MA2B]

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[1] Basic properties of growth rates.

Romer 4e, Problem 1.1. The growth rate of a variable equals the time derivative of its log, i.e. $\dot{X}(t)/X(t) = \frac{d}{dt}[\ln X(t)]$, where $\dot{X}(t) = \frac{dX}{dt}(t)$. Use this fact to show:

(a) The growth rate of the product of two variables equals the sum of their growth rates. That is, if $Z(t) = X(t)Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] + [\dot{Y}(t)/Y(t)]$.

(b) The growth rate of the ratio of two variables equals the difference of their growth rates. That is, if $Z(t) = X(t)/Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] - [\dot{Y}(t)/Y(t)]$.

(c) If $Z(t) = X(t)^\alpha$, then $\dot{Z}(t)/Z(t) = \alpha\dot{X}(t)/X(t)$.

Sketch of Proof. Note the following facts:

$$\begin{aligned}
\ln[X(t)Y(t)] &= \ln X(t) + \ln Y(t) \\
\ln\frac{X(t)}{Y(t)} &= \ln X(t) - \ln Y(t) \\
\ln X(t)^\alpha &= \alpha \ln X(t).
\end{aligned}$$

We can compute, for example, $\dot{Z}(t)$ for $Z(t) = X(t)^\alpha$ as follows. By differentiating both sides of

$$\ln Z(t) = \alpha \ln X(t),$$

we obtain

$$\frac{\dot{Z}(t)}{Z(t)} = \alpha\frac{\dot{X}(t)}{X(t)}.$$

□

Remark 1. There are a few things you should remember. Function $\ln x$ (or simply $\log x$) is the inverse function of e^x. That is,

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x,$$

where e is Napier's constant defined by

$$e := \lim_{N\to\infty} \left(1+\frac{1}{N}\right)^N. \tag{1}$$

For any $t>0$,

$$\begin{aligned} e^t &= \left[\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N\right]^t \\ &= \lim_{Nt\to\infty}\left(1+\frac{t}{Nt}\right)^{Nt}. \end{aligned}$$

In some math textbooks, e^x (x is any real number) is simply defined by

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2}$$

thereby obtaining

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{3}$$

Roughly speaking, differentiating both sides of (2) gets us

$$\frac{d}{dx}e^x = \frac{d}{dx}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = e^x.$$

Since $y=\ln x$ is the inverse function of e^x, it holds that $x=e^y$. Thus, we get the formula for the derivative of ln:

$$\frac{d}{dx}(\ln x) = \frac{1}{\frac{d}{dy}e^y} = \frac{1}{e^y} = \frac{1}{x}.$$

The formula for the growth rate,

$$\frac{d}{dt}[\ln X(t)] = \frac{\dot{X}(t)}{X(t)},$$

can be obtained by using the formula for derivatives of compound function: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.

[2] Effective interest rate.

Assume that a bank offers an annual interest rate of 6% **compounded monthly** and that you make a deposit of one thousand dollars ($1,000) at the bank today.

(a) How much do you expect to have in the bank account in one year from now? (There is no other engagement with the bank before and after that deposit.)

[Solution] The monthly interest rate is $\frac{0.06}{12}$. So the formula for the bank balance in one year is

$$1,000 \times \left(1+\frac{0.06}{12}\right)^{12}.$$

(b) Compute the effective rate of interest.

[Solution] The effective rate is

$$\left(1+\frac{0.06}{12}\right)^{12}-1.$$

(c) How do the above results change if the interest is compounded daily?

[Solution] The bank balance in one year, with daily compounding, is

$$1,000\times\left(1+\frac{0.06}{365}\right)^{365}.$$

The effective rate is

$$\left(1+\frac{0.06}{365}\right)^{365}-1.$$

Remark 2. If you divide one year into N subperiods and the interest is compouned N times per year, you bank balance will be

$$1,000\times\left(1+\frac{0.06}{N}\right)^{N}.$$

Take the limit of $N\to\infty$ and we get

$$\lim_{N\to\infty}1,000\times\left(1+\frac{0.06}{N}\right)^{N}=1,000\times e^{0.06}.$$

Tha it, **infinitely often compounding** gives rise to e^r. This is a basic fact for continuous-time models.

If you divide one year into N subperiods, each subperiod's length is $1/N$ year. Let $\Delta t=1/N$. The annual effective rate for Δt-compounding is

$$(1+0.06\Delta t)^{\frac{1}{\Delta t}}.$$

Since taking limit of $\Delta t\to 0$ is equivalent to $N\to\infty$, the above rate is also $e^{0.06}$. We will use this type of manipulation very often.