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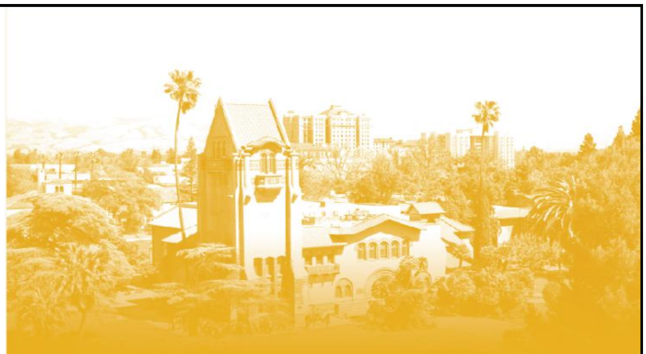

Basic Probability Concepts

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Agenda

- Principles of Counting
- Basic Probability Concepts

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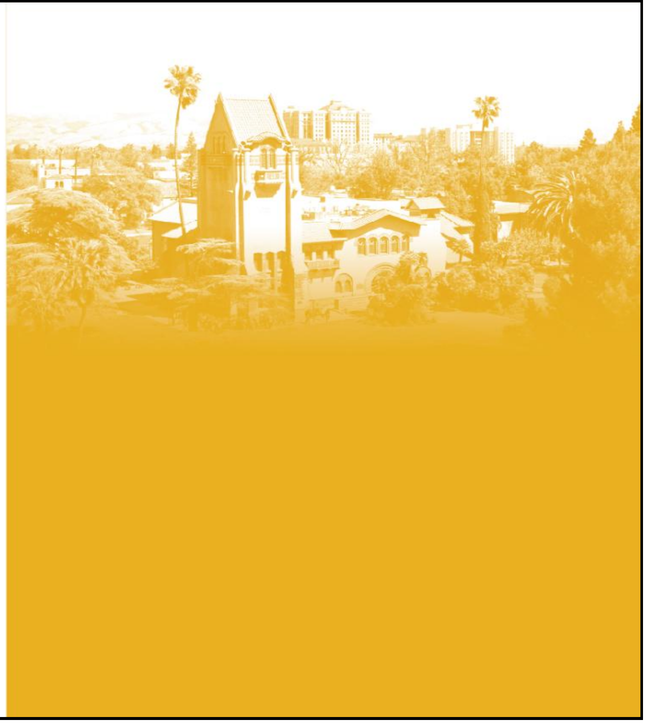
Principles of Counting



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Why Counting?

- To perform probability and statistics calculations, it is important to know how to count the size of a population or the size of a sample drawn from the population correctly.
- Knowing how to count in various situations is very crucial.

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Examples

Suppose your video streaming service has these types of movies:

- You want to watch three movies tonight, one of each type. How many different combinations of movies you can watch?

Type	Movies
action	674
romance	913
comedy	84

- In this case, order doesn't matter: Your choice of movie of any one type does not depend on your choice of any other type.

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Examples

There are 26 letters in the alphabet. How many 1-letter sequences can you make?

- How many 2-letter sequences are there?
- How many 5-letter sequences are there?
- In this case, order does matter: AB is different from BA

k	Number of sequences (N)
1 letter	$26^1 = 26$
2 letters	$26 \times 26 = 26^2 = 676$
3 letters	$26 \times 26 \times 26 = 26^3 = 17{,}576$
4 letters	$26 \times 26 \times 26 \times 26 = 26^4 = 456{,}976$
5 letters	$26 \times 26 \times 26 \times 26 \times 26 = 26^5 = 11{,}881{,}376$
6 letters	$26 \times 26 \times 26 \times 26 \times 26 \times 26 = 26^6 = 308{,}915{,}776$

$N^k = 26^k$

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Examples

Now, for the same 26 letters in the alphabet. What if there can be no repeated letters in the sequences?

k	Number of sequences (N)
1 letter	26
2 letters	26 x 25 = 650
3 letters	26 x 25 x 24 = 15,600
4 letters	26 x 25 x 24 x 23 = 358,800
5 letters	26 x 25 x 24 x 23 x 22 = 7,893,600
6 letters	26 x 25 x 24 x 23 x 22 x 21 = 165,765,600

- The # of ways for a sequence of k objects chosen without repetition from a collection of n objects:

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \dots \ \cdot (n-k+1)$$

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Factorial Notation

- The product $n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \dots \ \cdot 3 \cdot 2 \cdot 1$ is n factorial, denoted as $n!$
- Note that $n!$ grows very rapidly:
- Factorial notation can simplify formulas.

n	n!	
0	1	by definition
1	1	= 1
2	2	= 2 · 1
3	6	= 3 · 2 · 1
4	24	= 4 · 3 · 2 · 1
5	120	= 5 · 4 · 3 · 2 · 1
6	720	
7	5,040	
8	40,320	
9	362,880	
10	3,628,800	
11	39,916,800	
12	479,001,600	
13	6,227,020,800	
14	87,178,291,200	

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Sequence without Repetition Revisited

- Recall that the # of ways for a sequence of k objects chosen without repetition from a collection of n objects is

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \dots \ \cdot (n-k+1)$$

- Using factorial notation, we can simplify this as:

$${}_nP_k = \boxed{\frac{n!}{(n-k)!}}$$

Permutation

- What about the # of permutations of n objects in a circle?

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Example: Counting Numbers

- How many 3-digit numbers are there using the digits 1 through 9 that have no repeated digits?

- How many of these 3-digit numbers from above are odd? Here are some hints:
  - How many have 1, 3, 5, 7, or 9 as their third digit?
  - How many choices are there for the third digit?
  - It depends on what we chose for the first two digits!

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Example: Counting Numbers

- Start by counting digits from left to right.
- There are 4 even digits: 2, 4, 6, and 8. There are 5 odd digits: 1, 3, 5, 7, and 9.

Possible digit sequences	Number in each sequence
even even odd	4 x 3 x 5 = 60
odd even odd	5 x 4 x 4 = 80
even odd odd	4 x 5 x 4 = 80
odd odd odd	5 x 4 x 3 = 60
TOTAL	60 + 80 + 80 + 60 = 280

- Is there a better way to compute this?

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Example: Forming Baseball Teams

Suppose you need to choose a baseball team of 9 children out of 15 children. How many possible teams are there?

Questions to ask:

- Does order matter?
- Is there repetition?

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Counting When Order Doesn't Matter

How many different committees of 4 students can you make from a group of 15 students?

- In this case, order doesn't matter.
- We can start with the permutation where order does matter: $\frac{n!}{(n-k)!} = \frac{15!}{(15-4)!} = \frac{15!}{11!}$
- Then, we determine the # of possible committees if order matters: $\frac{n!}{(n-k)!} = \frac{4!}{(4-4)!} = 4!$
- Thus, the # of possible committees of 4 students from 15 students: $\frac{15!}{11!} \cdot \frac{1}{4!} = \frac{15!}{4! \cdot 11!}$
- In general, the # of ways to make a combination of k objects without repetition among n objects, and order doesn't matter is:

$${}_nC_k = \boxed{\frac{n!}{k!\,(n-k)!}}$$ Combination

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Binomial Coefficients

- The combination formula can also be written as:

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- $\binom{n}{k}$ is also known as a binomial coefficient.
- One important property of $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

e.g. Counting the number of ways to make a committee of 4 students out of 15 students is the same as counting the number of ways to leave 11 students off the committee.

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More Counting Situations

- For the case when you have n objects and would like to r different partitions with n_1 elements in the first partition, n_2 in the second, and so forth.
- The number of ways to partition is:

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad n = n_1 + n_2 + \cdots + n_r$$

Example: In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

$$\binom{7}{3, 2, 2} = \frac{7!}{3!\, 2!\, 2!} = 210$$

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Counting Scenarios So Far

	Repetitions allowed (with replacement)	**No Repetitions** (without replacement)
Sequences (order matters)	n^k	${}_nP_k = \dfrac{n!}{(n-k)!}$
Combinations (order doesn't matter)	?	${}_nC_k = \dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

What about combinations or collections with repetitions or replacements?

- Permutations: Arrangements of items in a sequence where order does matter.
- Combinations: Arrangements of items in a collection where order does not matter.

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Combinations with Replacements or Repititions

Problem: Determine the # of combinations for choosing r items from n distinct items.

- Imagine you have:
  - r stars (⭐), representing the items you choose.
  - $n-1$ separators (|), representing the separations between different types of items.
- For simplicity, let's consider 3 types of fruits (A, B, C) and choose 2 fruits. Here are the different combinations with repetitions:

AA
AB
AC
BB
BC
CC

Choose A and A (AA) - ⭐ ⭐||
Choose A and B (AB) - ⭐|⭐|
Choose A and C (AC) - ⭐||⭐
Choose B and B (BB) - |⭐ ⭐|
Choose B and C (BC) - |⭐|⭐
Choose C and C (CC) - ||⭐ ⭐

Similar to the problem of:
- Total items = $n-1+k$
- Select k from them

$$\binom{n+k-1}{k}$$

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{k!\,(n-1)!}$$

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Summary of Different Counting Scenarios

	Repetitions allowed (with replacement)	**No Repetitions** (without replacement)
Sequences (order matters)	n^k	${}_nP_k = \frac{n!}{(n-k)!}$
Combinations (order doesn't matter)	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$	${}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

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Basic Probability Concepts

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What is Probability?

- Probability is a branch of mathematics that deals with the likelihood or chance of different outcomes.
- It measures how likely it is for an event to occur, using a scale from 0 to 1 (0 means the event cannot happen and 1 means it will certainly happen).
- It creates mathematical models to study chance or randomness.
- Originally arose from studying games of chance.
  - The probability that a flipped fair coin will land heads is 1/2.
  - The probability that a card drawn from a shuffled deck of 52 cards is an ace is 4/52.

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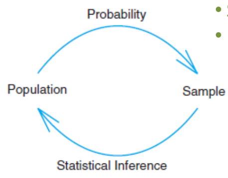
Role of Probability

- What's the relation and distinction of probability and statistics?



- Probability allow us to draw conclusions about characteristics of (hypothetical) data taken from the population, based on known features of the population.
- Statistics allows us to draw conclusions about the population, with inferential statistics making clear use of elements of probability.

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Some Probability Definitions

- Experiment: A process or activity that generates measurable results (a set of data).
  - E.g. Flipping a coin or rolling a dice.
- Outcome: Each distinct result of an experiment. Aka sample point.
  - E.g. The outcome of flipping a coin is H or T.
- Sample Space (S): The set of all possible outcomes of an experiment.
  - E.g. Sample space of rolling a six-sided die is $S = \{1, 2, 3, 4, 5, 6\}$.
- Event (E): A collection of outcomes; a specific subset of outcomes from S.
  - E.g. The event of rolling an even number of a six-sided die: $E = \{2, 4, 6\}$.
- Complementary Event ($\bar{E}$): The event that represents all outcomes in the sample space that are not in E.
  - E.g. If E is rolling a 3 on a die, $\bar{E}$ is rolling anything other than 3, i.e., {1, 2, 4, 5, 6}.

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Some Probability Definitions (cont)

Here are definitions based on two events A and B:

- Intersection of A and B ($A \cap B$): The event containing all elements that are common to A and B.
  - E.g. $A = \{2, 4, 6\}$ and $B = \{4, 5, 6\}$, then $A \cap B = \{2, 4, 5, 6\}$.
- Union of A and B ($A \cup B$): The event containing all elements that belong to A or B or both.
  - E.g. Let $A = \{a, b, c\}$ and $B = \{d, e\}$, then $A \cup B = \{a, b, c, d, e\}$.

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Some Probability Definitions (cont)

Here are definitions based on two events A and B:

- Mutually Exclusive Events: Two events that cannot occur at the same time.
  - E.g. When flipping a coin, the events “heads” and “tails” are mutually exclusive.
  - Can be written as $A \cap B = \phi$
- Independent Events: Two events are independent if the occurrence of one does not affect the occurrence of the other.
  - E.g. Rolling a die and flipping a coin are independent events.

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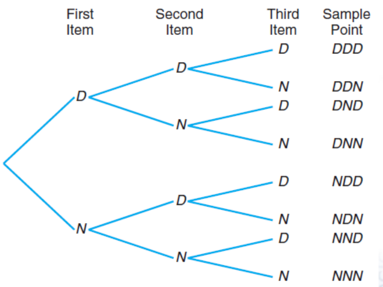
Example: Selection of Three Items

- Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N.
- Here's a list of all the elements of the sample space:

S = {DDD, DDN, DND, DNN, NDD, NDN, NND, NNN}

- A tree diagram can also be constructed.



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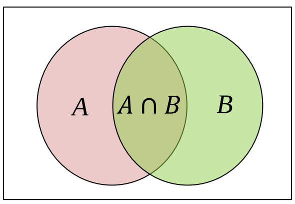
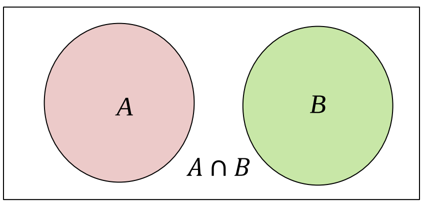
Venn Diagrams

- We can represent the relationships of different sets using Venn diagrams.





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Example

Construct a Venn diagram that would capture the following events:

- A: the card is red,
- B: the card is the jack, queen, or king of diamonds,
- C: the card is an ace.

What is the region for $A \cap C$?

What would these expressions equal to?

- $A \cap \bar{A}$
- $A \cap \bar{A}$
- $\overline{(A \cap B)}$
- $\overline{(A \cup B)}$

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Statement of a Sample Space

Sample spaces with a large or infinite number of sample points are best described by a **statement** or **rule method**.

- If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written as:

$$S = \{x \mid x \text{ is a city with a population over 1 million}\}$$

- If S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, we write the rule as:

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

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Some Probability Definitions

- The probability of an event E is the ratio of the number of outcomes N_E favorable to event E to the total number of possible outcomes N.

$$P(E) = \frac{N_E}{N}$$

- E.g: Let $N_E = 4$ ace cards and $N = 52$ total cards. Then, $P(drawing\ an\ ace\ card) = 4/52$.

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Some Probability Definitions

An empirical approach that uses experiments to count the occurrences of event E.

- Repeat an experiment a number of times. If event E occurs 30% of the time, then 0.3 can be a good approximation to the probability of event E.
- If n is the number of trials of the experiment and event E occurs on N_E of those trials, then

$$P(E) \approx \frac{N_E}{N}$$

- The approximation improves as value of n increases.

Some Basic Probability Laws

- For any event A:, the probability $P(A)$ is:

$$0 \leq P(A) \leq 1$$

  - The probability of event A occurring ranges from never (probability 0) to always (probability 1).

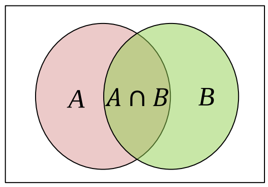
- The **complement** $\bar{A}$ of an event A is the event that A does not occur:

$$P(\bar{A}) = 1 - P(A)$$

Probabilities of Two Events

Let A be the set of outcomes favorable to some event E_A and B be the set of outcomes favorable to some event E_B. There may be some common outcomes in both A and B.

- $A \cup B$ is "A union B": event A or event B occurs
- $A \cap B$ is "A intersect B": event B and event B occurs



$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- What if event A and event B are mutually exclusive events? $P(A \text{ and } B) = P(A \cap B) = 0$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

- If event A and event B are **independent**, then:

$$P(A \cap B) = P(A) \cdot P(B)$$

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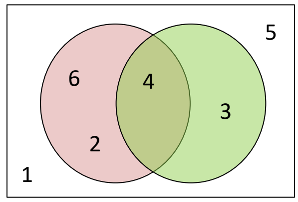
Example: Probabilities of Two Events

What is the probability that a roll of a die produces an even number <u>or</u> a number between 2 and 5?

- Let A = the set of even number outcomes
  Let B = the set of between 2 and 5 outcomes



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(\{2,4,6\}) + P(\{3,4\}) - P(\{4\})$$
$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6}$$

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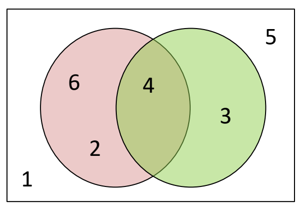
Example: Probabilities of Two Events

What is the probability that a roll of a die produces an even number <u>and</u> a number between 2 and 5?

- Let A = the set of even number outcomes
  Let B = the set of between 2 and 5 outcomes



$$P(A \cup B) = P(A) \cdot P(B)$$
$$= P(\{2,4,6\}) \cdot P(\{3,4\})$$
$$= \frac{3}{6} \cdot \frac{2}{6} = \frac{6}{36} = \frac{1}{6}$$