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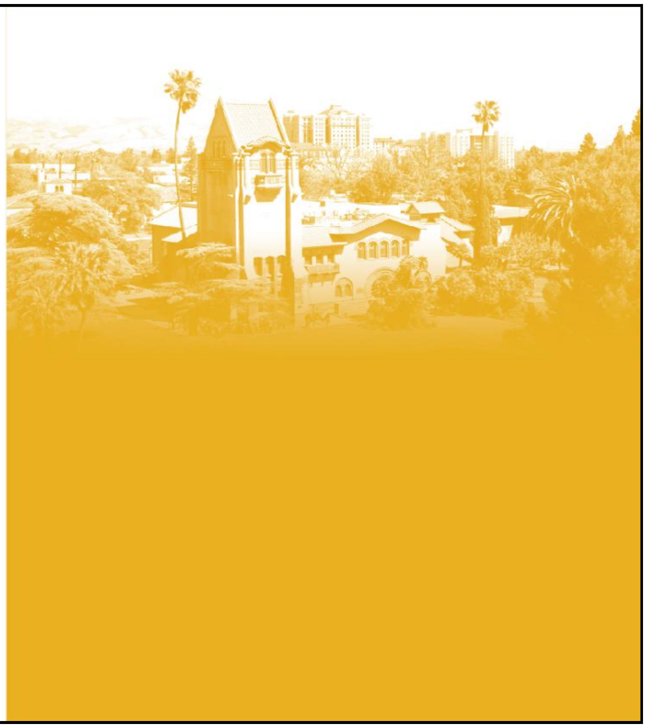
Constrained Optimization



1

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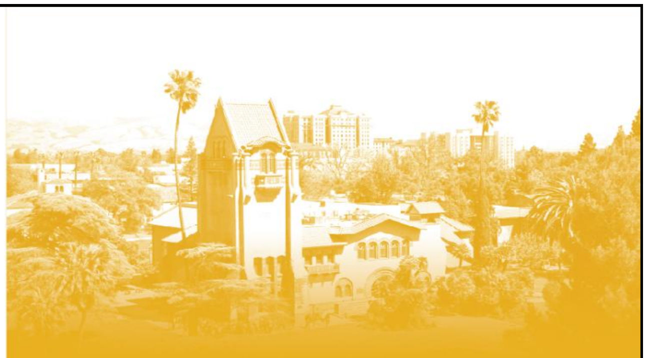
Agenda

- Multivariate Calculus Review
- Lagrange Multipliers & Constrained Optimization

2



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Multivariate Calculus Review

3

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Partial Derivatives

- For functions with 2 or more variables, we get partial derivatives:

$$f(x,y)$$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x} \qquad \frac{\partial f}{\partial y^2}$$

4

4

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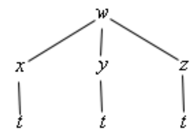
Chain Rule

- For multivariate functions, we can have intermediate variables between the dependent and independent variables.

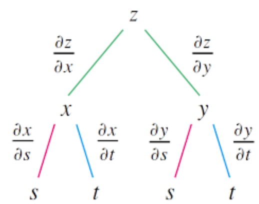
$$w(x,y,z) = 2xyz$$
where $x = t, y = 2t, z = t^2$



- We can use chain rule to find the partial derivatives:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
one intermediate variable

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
two intermediate variables



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Example

Use chain rule to find the partial derivatives of the multivariable function:

$$w(x,y) = x^2y + x$$
$$x = 1 + t$$
$$y = 2 + t^2$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

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Example

Use chain rule to find the partial derivatives of the multivariable function:

$$z(r,\theta) = \ln(r) + r^2 \sin\theta$$
$$r = 3s^2 - t$$
$$\theta = 2t^2 - \frac{4}{s^2}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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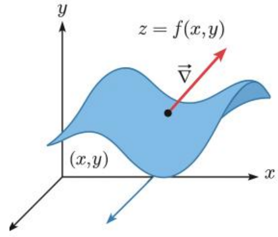
Gradient

The gradient of a scalar function $f(x_1, x_2, \ldots, x_n)$ is a vector that points in the direction of the greatest rate of increase of the function:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right)$$



Example: Find the gradient vector of $f(x,y) = x^2 + y^2$

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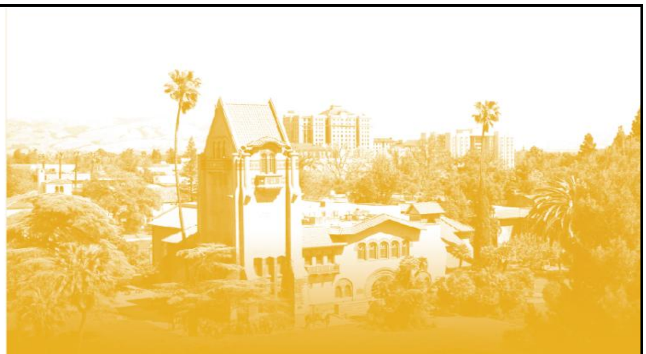
Extrema & Optimization



9

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Extrema

- In functions of several variables $f(x, y, \ldots)$, **extrema** (or critical points) are points where the function reaches local or global maximum or minimum values.
- Types of extrema:
  - Local Maximum: Function value is greater than all nearby values.
  - Local Minimum: Function value is less than all nearby values.
  - Global Maximum/Minimum: Highest/lowest value over the entire domain.
- To find extrema,
  - Compute the gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots\right)$
  - Set gradient to zero: $\nabla f = 0$
  - Solve for the critical points.

10

10

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Extrema of 2D Functions

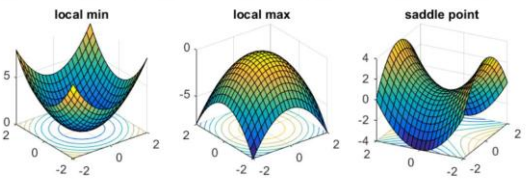
- For functions of 2 variables, we have the Second Derivative Test:
- The Hessian matrix contains all the second partial derivatives of $f(x, y)$:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

- The determinant of is called the discriminant

$$D = f_{xx}f_{yy} - {f_{xy}}^2$$

  - $D > 0$ ➔ extrema
    - $f_{xx} > 0$ local minimum
    - $f_{xx} < 0$ local maximum
  - $D < 0$ ➔ saddle points
  - $D = 0$ ➔ inconclusive



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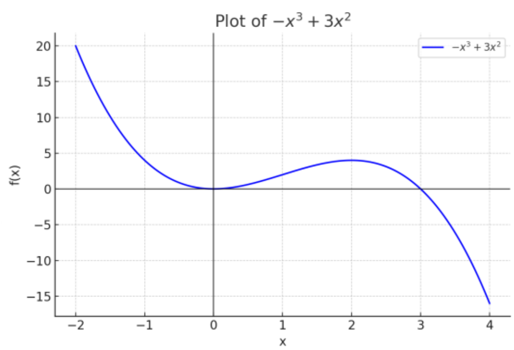
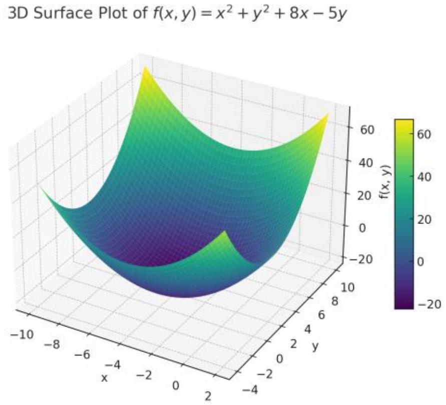
Example: 1D Function

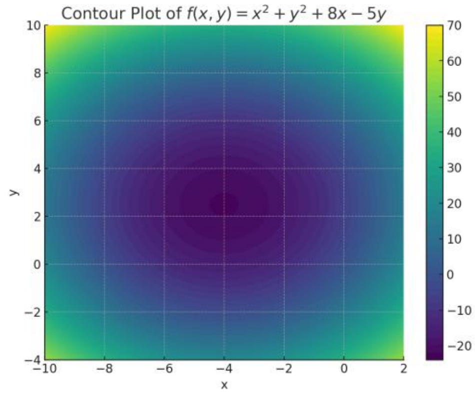
Find the critical values of the function $f(x) = -x^3 + 3x^2$



Example: 2D Function

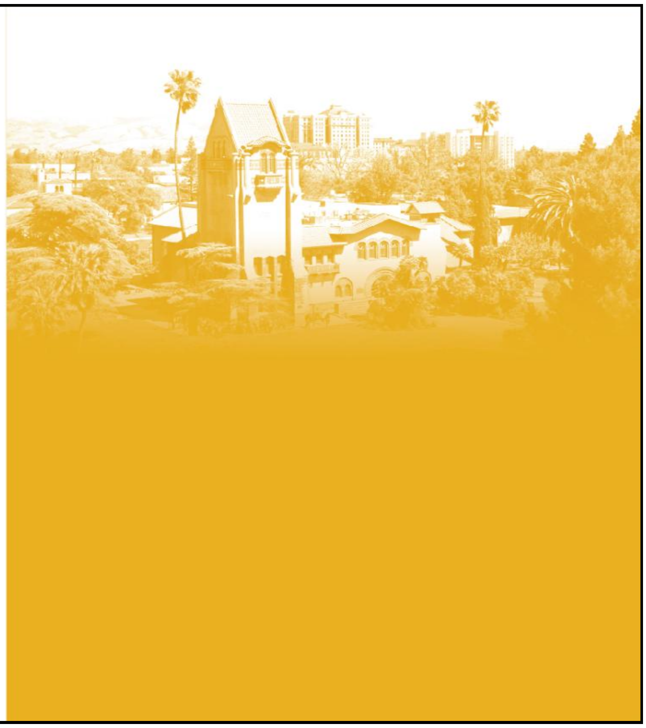
Find the critical values of the function $f(x) = x^2 + y^2 + 8x - 5y$





13

13



Lagrange Multipliers
&
Constrained Optimization

14

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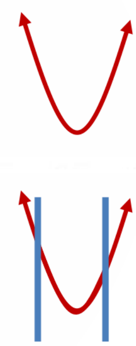
Constrained Optimization

- Constrained optimization is about maximizing or minimizing an objective function subject to one or more constraints.
- Constraints can be anything that limits the feasible region of the optimization problem, such as inequalities, equalities, or bounds



Example: a company wants to maximize its profits subject to constraints on its production capacity and resources.

- In this case, the objective function is the profit function, and the constraints are the production capacity and resource availability

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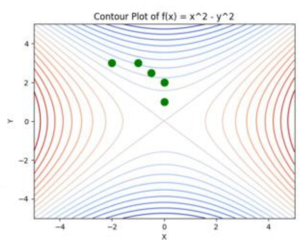
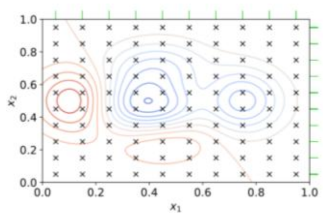
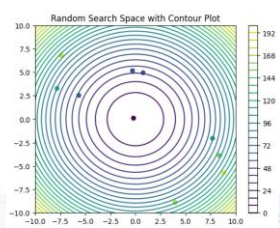
Constrained Optimization Techniques

Constrained optimization problems can be solved using various techniques:

- Lagrange Multipliers
- Penalty Methods
- Gradient Descent
- Grid Search
- Random Search







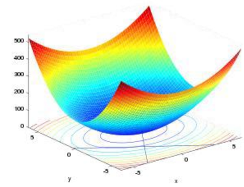
Lagrange Multipliers

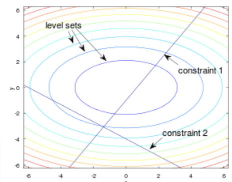
Lagrange multipliers is a method to find extrema of function $f(x, y, \ldots)$ subject to equality or inequality constraints $g(x, y, \ldots) = 0$

- To solve this constrained optimization problem, we define the Lagrange multiplier as:

$$\nabla f = \lambda \nabla g$$

- Then, we solve for $\lambda, x, y, \ldots$
- Evaluate $f(x, y, \ldots)$ with these values of x, y will give the extrema values





17

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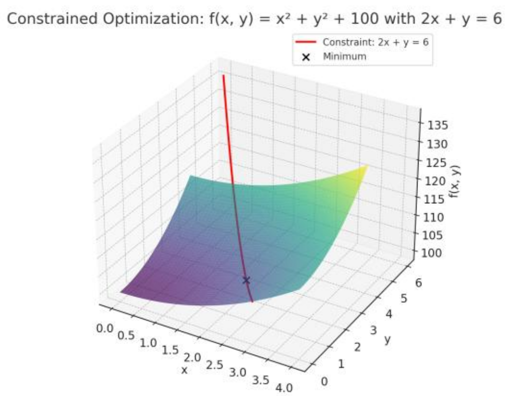
Example: Constrained Optimization

Find the extrema of the function $f(x) = x^2 + y^2 + 100$ subject to the constraint $2x + y = 6$



18

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Challenges with Constrained Optimization

- Local minima
- Vanishing gradient
- Exploding gradient
- Saddle point