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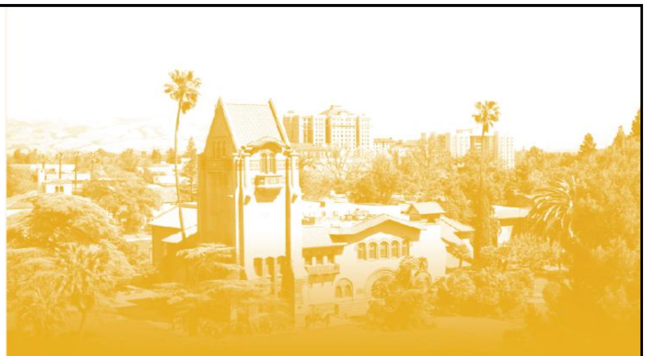
Probability Concepts (Part 2)



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Agenda

- Review of Basic Probability Concepts
- Marginal, Joint & Conditional Probabilities
- Independent Events
- Bayes' Theorem
- Law of Large Numbers

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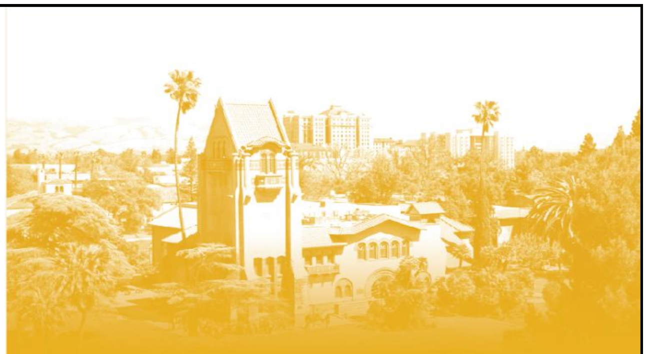

Review of Basic Probability Concepts

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Many Sided Dices

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Summary of Different Counting Scenarios

	Repetitions allowed (with replacement)	**No Repetitions** (without replacement)
Sequences (order matters)	n^k	${}_nP_k = \dfrac{n!}{(n-k)!}$
Combinations (order doesn't matter)	$\binom{n+k-1}{k} = \dfrac{(n+k-1)!}{k!\,(n-1)!}$	${}_nC_k = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

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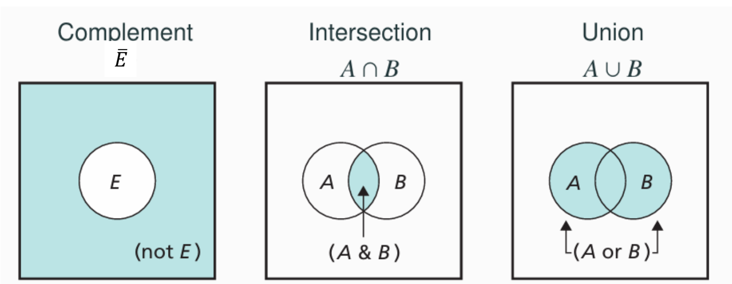
Intersections, Unions, and Complements

- $\bar{E}$ is "the complement of E": event that A does not occur
- $A \cup B$ is "A union B": either event A or event B occurs
- $A \cap B$ is "A intersect B": both event B and event B occurs



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Probabilities of Two Events

- Complement Rule:

$$P(A) + P(\bar{A}) = 1$$

- General Addition Rule:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If event A and event B are mutually exclusive or disjoint events: $P(A \text{ and } B) = P(A \cap B) = 0$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

- If event A and event B are independent:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$$

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Review Questions

DNA is made of sequences of nucleotides: A, C, G, T. How many DNA sequences of length 3 are there?

(i) 12 (ii) 24 (iii) 64 (iv) 81

How many DNA sequences of length 3 are there with no repeats?

(i) 12 (ii) 24 (iii) 64 (iv) 81

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Review Questions

Given $P(A) = 0.32$, $P(B) = 0.12$, $P(C) = 0.15$

- If the $P(A \text{ or } B)$ is 0.44, are A and B mutually exclusive? Why or why not?
- If the $P(B \text{ or } C)$ is 0.2, are B and C mutually exclusive? Why or why not?

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Review Questions

A one-pair hand consists of two cards having one rank and the remaining three cards having three other ranks. The probability of a one-pair hand is:

1) less than 5%
2) between 5% and 10%
3) between 10% and 20%
4) between 20% and 40%
5) greater than 40%

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Marginal, Joint & Conditional Probabilities

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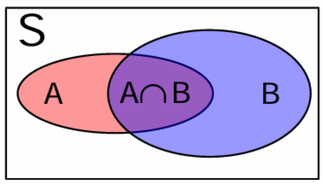
Conditional Probability

- Given two events A and B. The probability of event A occurs given that event B is known to occur is called the conditional probability, denoted by $P(A \mid B)$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{if } P(B) > 0$$



- Can rewrite this as:

$$\boxed{P(A \cap B) = P(A \mid B) \cdot P(B)}$$ General Multiplication Rule

- Also,

$$P(B \cap A) = P(B \mid A) \cdot P(A)$$

$$\Rightarrow \boxed{P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)}$$ This is VERY IMPORTANT!

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Example: General Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table.

- What is the probability that both cards are Kings?
- If the card drawn is known to be a face card (J, Q, K), what is the probability that it is a K?
- What is the probability of that the card is a face card given that it is NOT a K?
- What is the probability of that the card is a face card given that it is a K?

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Example: More Conditional Probability Cases

The probability that a regularly scheduled flight departs on time is P(D) = 0.83; the probability that it arrives on time is P(A) = 0.82; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$.

Find the probability that a plane:

a) arrives on time, given that it departed on time,

b) departed on time, given that it has arrived on time, and

c) arrives on time despite late departure.

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Marginal Probabilities of Contingency Tables

If a faculty member is selected at random, what is the probability that he/she is a full professor?

Age (year)	Full professor	Associate professor	Assistant professor	Lecturer	**Total**
Under 40	54	173	220	23	470
40-49	156	125	61	6	348
50-59	145	68	36	4	253
60+	75	15	3	0	93
Total	430	381	320	33	1164

- Probabilities that involve only one of the categorical variables in a contingency table are called marginal probabilities.

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Joint Probabilities of Contingency Tables

What is the probability that he/she is a full professor and under 40?

Age (year)	Full professor	Associate professor	Assistant professor	Lecturer	**Total**
Under 40	54	173	220	23	470
40-49	156	125	61	6	348
50-59	145	68	36	4	253
60+	75	15	3	0	93
Total	430	381	320	33	1164

- Probabilities that involve combination of categories of both categorical variables in a contingency table are called joint probabilities.
- This is the same as the overall proportions.

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Conditional Probabilities of Contingency Tables

If we know that the selected faculty member is under 40, what is the probability that he/she is a full professor?

Age (year)	Full professor	Associate professor	Assistant professor	Lecturer	**Total**
Under 40	54	173	220	23	470
40-49	156	125	61	6	348
50-59	145	68	36	4	253
60+	75	15	3	0	93
Total	430	381	320	33	1164

- The conditional probabilities of a column variable given row variable are simply the row proportions for contingency tables.

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Conditional Probabilities of Contingency Tables

If we know that the selected faculty member is a full professor, what is the probability that he/she is under 40?

Age (year)	Full professor	Associate professor	Assistant professor	Lecturer	**Total**
Under 40	54	173	220	23	470
40-49	156	125	61	6	348
50-59	145	68	36	4	253
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Total	430	381	320	33	1164

- The conditional probabilities of a row variable given column variable are simply the column proportions for contingency tables.

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Probabilities of Contingency Tables

Suppose the contingency table is given in relative frequencies:

Age (year)	Full prof.	Assoc. prof.	Assist. prof.	Lect.	**Total**
Under 40	0.046	0.149	0.189	0.020	0.404
40-49	0.134	0.107	0.052	0.005	0.299
50-59	0.125	0.058	0.031	0.003	0.217
60+	0.064	0.013	0.003	0.000	0.080
Total	0.369	0.327	0.275	0.028	1.000

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- These are simply the joint probabilities.
- Both the marginal & conditional probabilities can be computed from the joint probabilities.
- e.g. $$P(\text{full prof} \mid \text{under 40}) = \frac{P(\text{full prof \& under 40})}{P(\text{under 40})} = \frac{0.046}{0.404}$$

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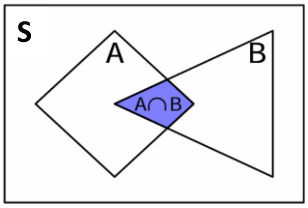
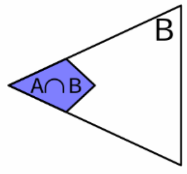
$P(A \mid B)$ vs $P(A \cap B)$

- $P(A \cap B)$ is the probability that A and B both occur (we are unsure whether B will occur)
- $P(A \mid B)$ is the probability that A occurs given that B has occurred
- $P(A \cap B) = \dfrac{P(A \cap B)}{P(\Omega)}$ ➔ The sample space is S
- $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ➔ The sample space is B





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General Multiplication Rule for Multiple Events

- The General Multiplication Rule for 2 events is:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B \mid A)$$

- For multiple events, we have:

$$P(A\ B\ C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A\ B)$$

$$P(A\ B\ C\ D) = P(A) \cdot P(B \mid A) \cdot P(C \mid A\ B) \cdot P(D \mid A\ B\ C)$$

$$P(A\ B\ C\ D\ E) = P(A) \cdot P(B \mid A) \cdot P(C \mid A\ B) \cdot P(D \mid A\ B\ C) \cdot P(E \mid A\ B\ C\ D)$$

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Example: General Multiplication Rule for Several Events

Five cards are dealt from a deck of well-shuffled card. What is the chance that none of them are hearts?

What about at least one heart among the five cards?

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Law of Total Probability

Law of Total Probability

- The total probability of an event expressed via the probabilities of different mutually exclusive scenarios that cover all possibilities
- It's a fundamental rule that relates marginal probabilities to conditional probabilities.

- Consider an event A with distinct outcomes $(A_1, A_2, \cdots, A_n)$, then for any event B

$$P(B) = \Sigma_i^n P(B \cap A_i) = \Sigma_i^n P(B \mid A_i) \cdot P(A_i) = P(B \mid A_1) \cdot P(A_1) + \cdots + P(B \mid A_n) \cdot P(A_n)$$

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Example: Law of Total Probability

Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available.

What is the chance that a purchased bulb will work for longer than 5000 hours?

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Independent Events

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Independence and Conditional Probabilities

- This means that Giving B doesn't tell us anything about A
- If A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

B happens doesn't affect how likely A happens

- Any of these can be true too if A and B are independent :

$$P(B \mid A) = P(B)$$

A happens doesn't affect how likely B happens

$$P(A \mid B) = P(A \mid \bar{B})$$

How likely A happens isn't affected by B happens or not

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Independent Events vs Disjoint Events

- If A and B are independent,

$$P(A \cap B) = P(A) \cdot P(B)$$

- If A and B are disjoint:

$$A \cap B = \emptyset \;\; \Rightarrow \;\; P(A \cap B) = 0$$

- If $P(A) > 0$ and $P(B) > 0$,
  - Independent events cannot be disjoint.
  - Disjoint events cannot be independent.
- Conceptually, A and B are disjoint means that one happens prevents the other from happening, so one's occurrence definitely affects the other's.

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Multiplication Law for Independent Events

- When A and B are independent

$$P(A \cap B) = P(A) \cdot P(B)$$

- This is simply the Multiplication Law:

$$P(A \cap B) = P(A) \cdot P(B \mid A) \text{ in which } P(B \mid A) \text{ reduces to } P(B)$$

- More generally, for k independent events:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1) \cdot P(A_2) \cdots P(A_k) \quad \text{if } A_1, A_2, \cdots, A_k \text{ are independent}$$

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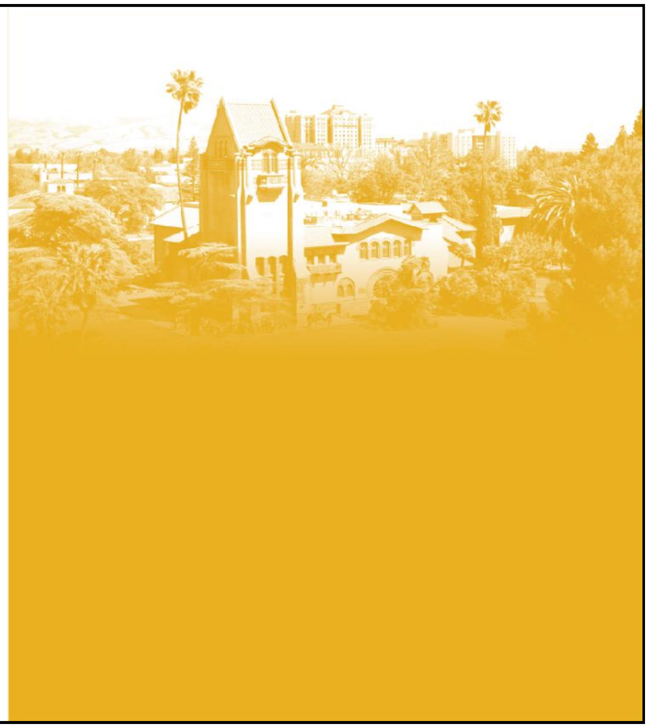

Bayes' Theorem

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Example: Bad Loan Risks

A bank's loan officer knows that:

- 5% of all loan applicants are bad risks.
- 92% of all loan applicants who are bad risks are also rated bad risks by a credit advisory service.
- 2% of all loan applicants are actually good risks but were rated bad risks by the credit advisory service.

What is the probability that a loan applicant who was <u>rated a bad risk</u> by the credit service is <u>truly a bad risk</u>?

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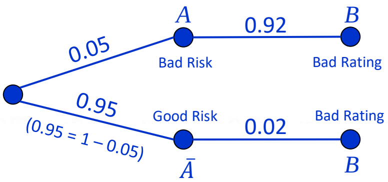
Example: Bad Loan Risks (cont)

- Let A be the event that the loan applicant is truly a **bad risk**. So, $\bar{A}$ represents good risk.
- Let B be the event that the credit advisory service **rates** a loan applicant a **bad** risk.

$P(A) = 0.05$ $P(B \mid A) = 0.92$ $P(B \mid \bar{A}) = 0.02$	• 5% of all loan applicants are bad risks. • 92% of all loan applicants who are bad risks are also rated bad risks by a credit advisory service. • 2% of all loan applicants are actually good risks but are rated bad risks by the credit advisory service.



$P(B \mid A) \cdot P(A) = (0.92)(0.05) = 0.046$

$P(B \mid \bar{A}) \cdot P(\bar{A}) = (0.02)(0.95) = 0.019$

- What is $P(A \mid B)$? (Who is truly a bad loan risk given a bad credit rating?)

$$P(B) = P(B \mid A) \cdot P(A) + P(B \mid \bar{A}) \cdot P(\bar{A}) = 0.046 + 0.019 = 0.065 \qquad P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} = \frac{0.046}{0.065} = 0.71$$

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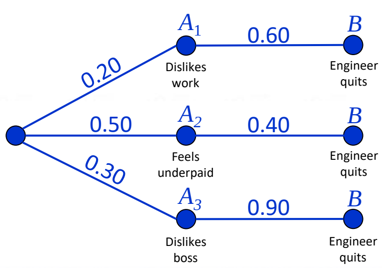
Example: Disgruntled Engineers

A company does a survey to understand why its engineers are quitting. The survey indicates:

- 20% dislike their work ➔ probability of quitting = 0.60
- 50% feel underpaid ➔ probability of quitting = 0.40
- 30% dislike their boss ➔ probability of quitting = 0.90

What is the most likely reason that an engineer quit?



$P(B \mid A_1) \cdot P(A_1) = (0.60)(0.20) = 0.12$

$P(B \mid A_2) \cdot P(A_2) = (0.40)(0.50) = 0.20$

$P(B \mid A_3) \cdot P(A_3) = (0.90)(0.30) = 0.27$

$\left.\right\} P(B) = 0.59$

$$P(B) = P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2) + P(B \mid A_3) \cdot P(A_3) = 0.12 + 0.20 + 0.27 = 0.59$$

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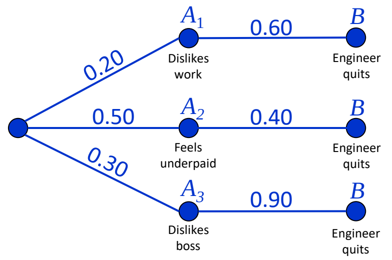
Example: Disgruntled Engineers (cont)



$P(B \mid A_1) \cdot P(A_1) = (0.60)(0.20) = 0.12$

$P(B \mid A_2) \cdot P(A_2) = (0.40)(0.50) = 0.20$

$P(B \mid A_3) \cdot P(A_3) = (0.90)(0.30) = 0.27$

$P(B) = 0.59$

What is the most likely reason that an engineer quit?

$$P(A_k \mid B) = \frac{P(B \mid A_k) \cdot P(A_k)}{P(B)}$$

$$P(A_1 \mid B) = \frac{P(B \mid A_1) \cdot P(A_1)}{P(B)} = \frac{(0.60)(0.20)}{0.59} = 0.20$$

$$P(A_2 \mid B) = \frac{P(B \mid A_2) \cdot P(A_2)}{P(B)} = \frac{(0.40)(0.50)}{0.59} = 0.34$$

$$P(A_3 \mid B) = \frac{P(B \mid A_3) \cdot P(A_3)}{P(B)} = \frac{(0.90)(0.30)}{0.59} = 0.46$$

most likely reason that an engineer quit

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Bayes' Theorem

- The conditional probability formula we have seen so far is a special case of the Bayes' Theorem (for 2 events).

$$P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A) \quad \Rightarrow \quad \boxed{P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}}$$

- For multiple events, the Bayes' Theorem states that:

$A_1, \cdots, A_n$ all other possible mutually exclusive outcomes of A besides A_k

$$P(A_k \mid B) = \frac{P(B \mid A_k) \cdot P(A_k)}{P(B)}$$

$$= \frac{P(B \mid A_k) \cdot P(A_k)}{P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2) + \cdots + P(B \mid A_n) \cdot P(A_n)}$$

$$\boxed{P(A_k \mid B) = \frac{P(B \mid A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)}}$$

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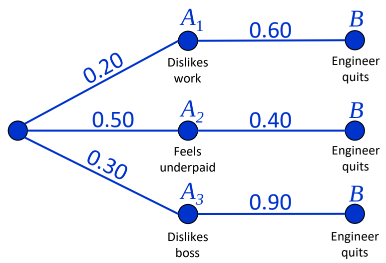
Bayes' Theorem

$$P(A_k \mid B) = \frac{P(B \mid A_k) \cdot P(A_k)}{P(B)} = \frac{P(B \mid A_k) \cdot P(A_k)}{\Sigma_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)}$$



- B is an observable event ➔ An engineer quits.
- The probabilities $P(A_i)$ are prior probabilities where i = 1, 2, ..., n
  - P(dislikes work) etc before observing the quit event B.
- The problem is to find each posterior probability $P(A_k \mid B)$ for k = 1, 2, ..., n.
  - P(dislikes work | quit) etc after observing the quit event B.

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Bayesian vs Frequentist Statistics

- Bayesian statistics is based on the Bayesian interpretation of probability, which incorporates one's degree of belief in an event. The degree of belief may be based on prior knowledge of the event, previous experimental results, or personal beliefs.
- Bayesian statisticians' interpretation of probability is different from the frequentists' interpretation. Recall that the latter views probability as the limit of the relative frequency of an event after many experimental trials.
- Many frequentist statisticians viewed Bayesian statistics unfavorably due to philosophical and practical considerations. But with modern computational power, the use of Bayesian statistics is increasing.

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The Monty Hall Problem

- Monty Hall hosted a popular TV game show.
- You are a contestant on the show.
  - Monty shows you three closed doors #1, #2, and #3.
  - Behind one door is a new car, but behind each of the other doors is a goat.
  - You pick one of the doors.
  - Monty opens one of the other two doors and reveals a goat.
  - He gives you the option to switch to the third door.
  - You want to win that car! Should you stay with your original pick, or should you switch?

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The Monty Hall Problem (cont)

- After seeing a goat behind the door that Monty opened:
  - Should you stay with the door you originally picked?
  - Should you switch to the other unopened door?
  - Does it make any difference whether you stay or switch?

- This is a conditional probability problem!

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The Monty Hall Problem – An Intuitive Explanation

- If you stay with your original door:
  - Your likelihood of winning remains the same: one in three.
- If you switch to the other closed door:
  - One time in three, your original door was right, so you lose by switching.
  - Two times in three, your original door was wrong, so you win by switching.

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