https://github.com/nmansard/wan24

Optimal control for walking robots

Theory and practice with Crocoddyl

Nicolas Mansard

Gepetto, LAAS-CNRS & ANITI

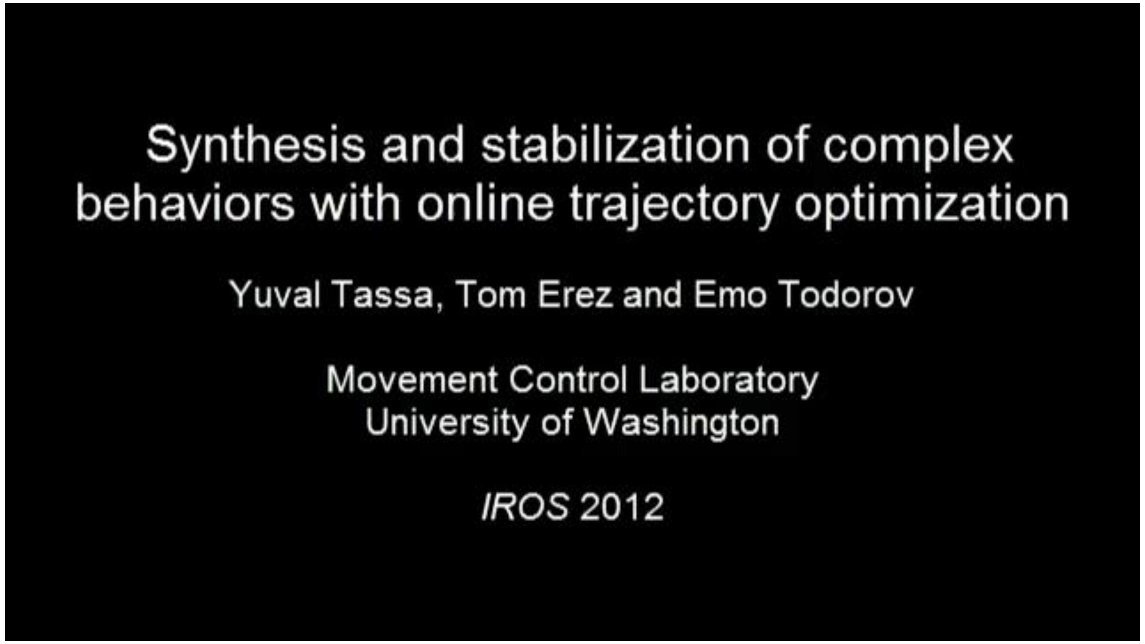
#02: Optimal control and its optimality principles

Autonomous Driving

Information Theoretic Model Predictive Control [Williams et al. 2018]

OC with Linear Inverted Pendulum Model
[Herdt et al. 2010]

OC with Centroidal Momentum Dynamics and Full Body Kinematics
[Ponton et al. 2018], [Carpentier et al. 2018], [Dai et al. 2014], [Herzog et al. 2015]



[Tassa et al. 2010]
DDP with Full-Body Dynamics
(realtime control)

[Mordatch et al. 2012]
Nonlinear Optimization for Multi-Contact Tasks

Problem definition

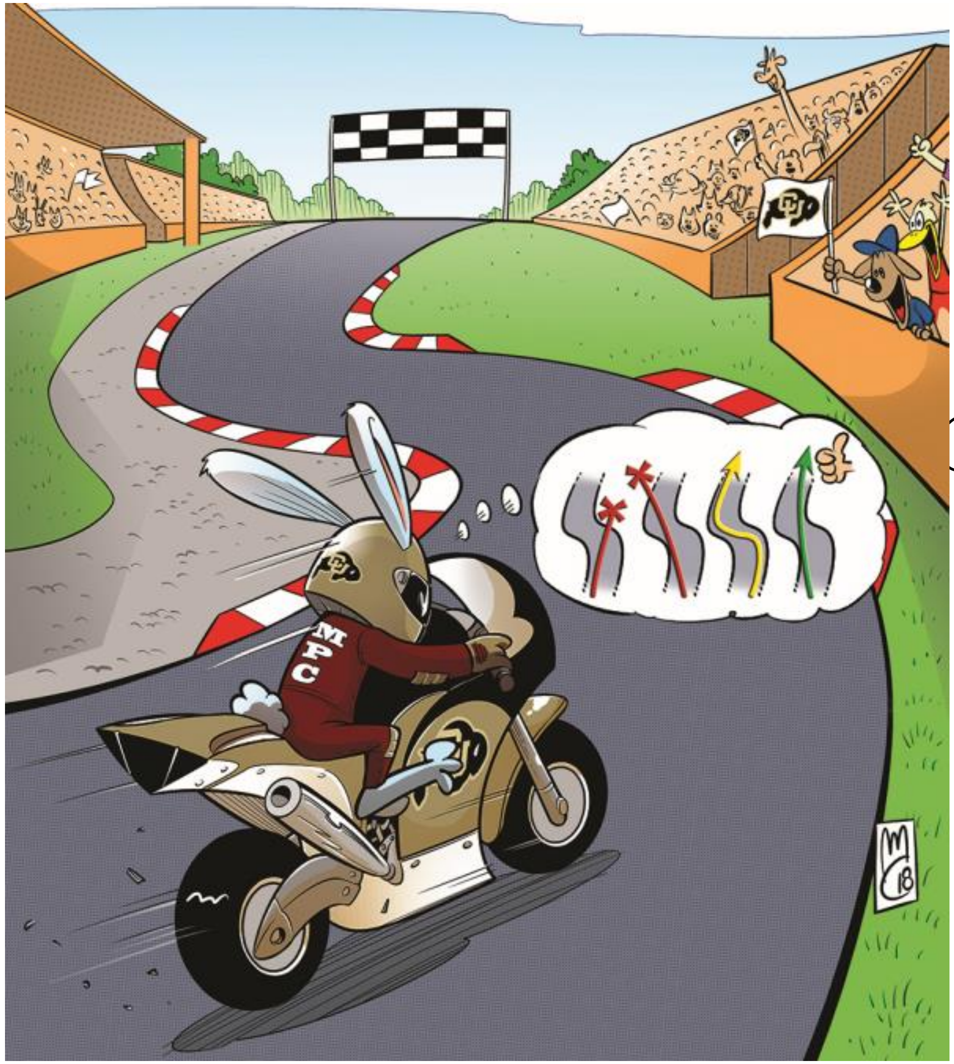
$$\min_{\{x\},\{u\}} \int_0^T l\big(x(t), u(t)\big)dt + l_T\big(x(T)\big)$$

so that $\quad x_0 = \hat{x}$

$$\forall t,\ \dot{x}(t) = f\big(x(t), u(t)\big)$$

- $\{x\}$ and $\{u\}$ are functions of t:
  $\{x\}: t \in \Re \rightarrow x(t) \in \Re^{nx}$
  $\{u\}: t \in \Re \rightarrow u(t) \in \Re^{nu}$
- The terminal time T is fixed (for now)

Starting example



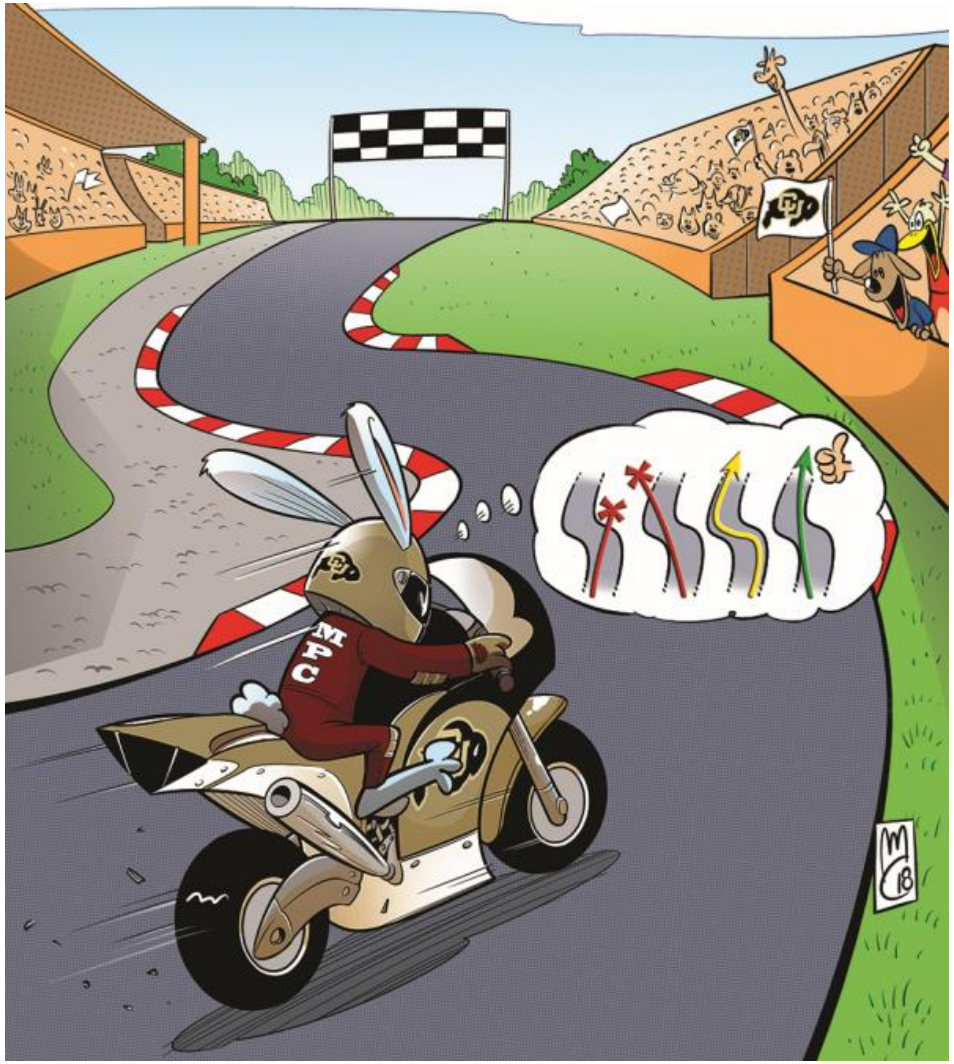
Original artwork by Michele Carminati, commissioned by Marco M. Nicotra (U. Colorado Boulder)

$$\min_{\substack{\{x\}=(Q,\dot{Q}),\\ \{u\}=\tau}} \int_0^T \sum_l l(x_t, u_t)\, dt$$

so that $x_0 = \hat{x}$

$\forall$t, $\dot{x}(t) = f(x(t), u(t))$

Starting example



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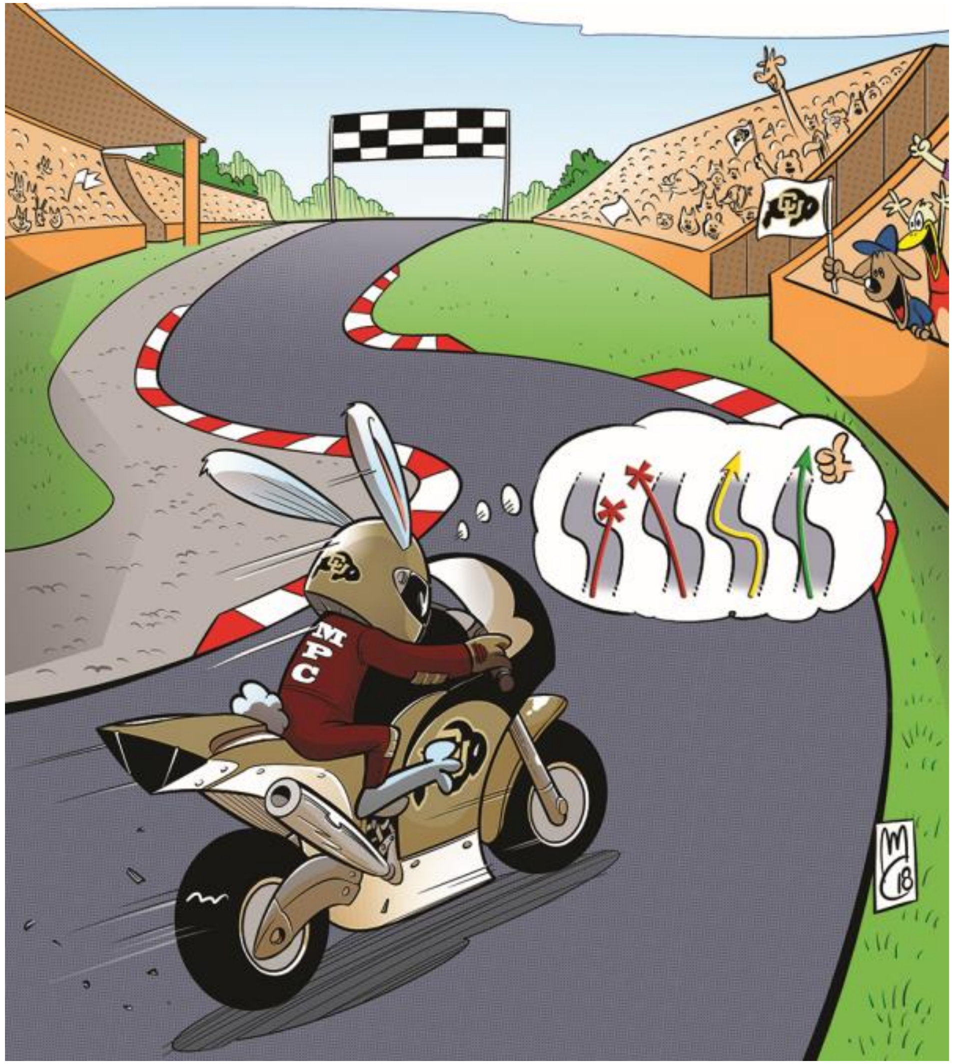
so that $\quad \forall t,\ \dot{x}(t) = f(x(t), u(t))$

$u_0 = \pi(x_0)$

$\{x\},\{u\}$

$\approx +\infty$

Starting example



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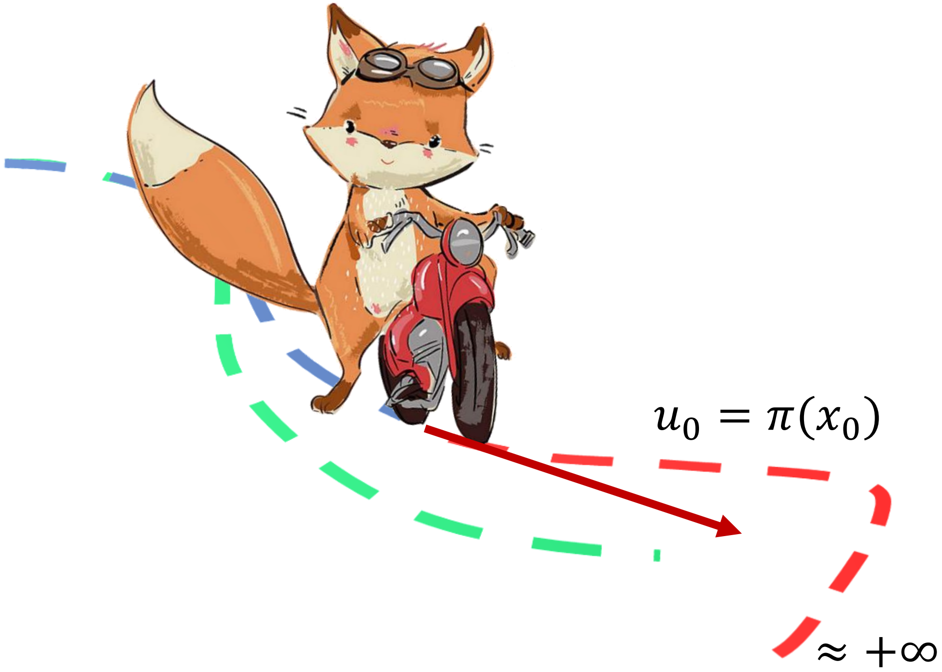


Optimal control problem

$$\underbrace{\min_{\{x\},\{u\}}}_{\text{Find control inputs to minimize cost}} \int_0^T \underbrace{l\big(x(t),u(t)\big)}_{\text{stage costs}} dt + \underbrace{l_T\big(x(T)\big)}_{\text{terminal cost}}$$

Find control inputs to minimize cost

stage costs

terminal cost

$$x_0 = \hat{x}$$ initial dynamics

$$\dot{x}(t) = f\big(x(t),u(t)\big)$$ deterministic dynamics

$$\mathrm{g}\big(x(t),u(t)\big) \geq 0$$ state and control constraints

Optimal control problem (discretized)

$$\min_{\{x\},\{u\}} \sum_{t=0}^{T-1} l(x_t, u_t) + l_T(x_T)$$

Find control inputs to minimize cost

stage costs

terminal cost

$$x_0 = \hat{x}$$ initial dynamics

$$x_t = f(x_t, u_t)$$ deterministic dynamics

$$\mathrm{g}\,(x_t, u_t) \geq 0$$ state and control constraints

Transcribing: *"representing" the reality*

See you later!

$$\min_{\substack{\underline{x}:t\longrightarrow x(t)\\ \underline{u}:t\longrightarrow u(t)}} \int_0^T l\big(x(t),u(t)\big)dt + l_T\left(x(T)\right)$$

$$\text{s.t.} \quad \forall t, \dot{x}(t) = f(x(t),u(t))$$

Optimal control problem (OCP)
with continuous variables
(infinite-dimension)

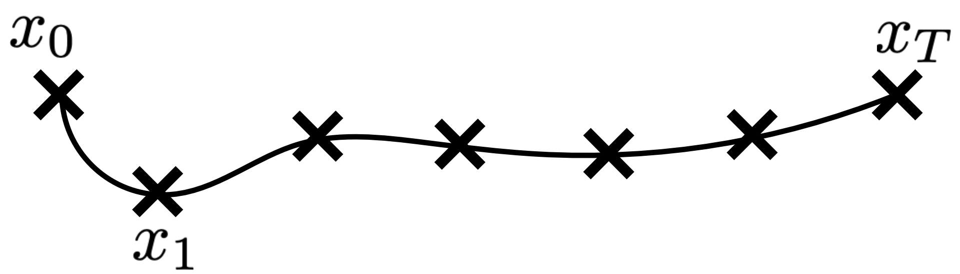
$$\min_{\substack{\underline{x}=\theta_{x1}\dots\theta_{xn}\\ \underline{u}=\theta_{u1}\dots\theta_{un}}} \sum_t l\big(x(t|\theta),u(t|\theta)\big) + l_T\left(x(T|\theta)\right)$$

$$\text{s.t.} \quad at\ some\ t, \dot{x}(t|\theta) = f(t|\theta_x,\theta_u)$$

Nonlinear optimization problem (NLP)
with static variables
(finite dimension)

$\theta_x\ \theta_u$ represents the continuous $\underline{x}$,$\underline{u}$ trajectories

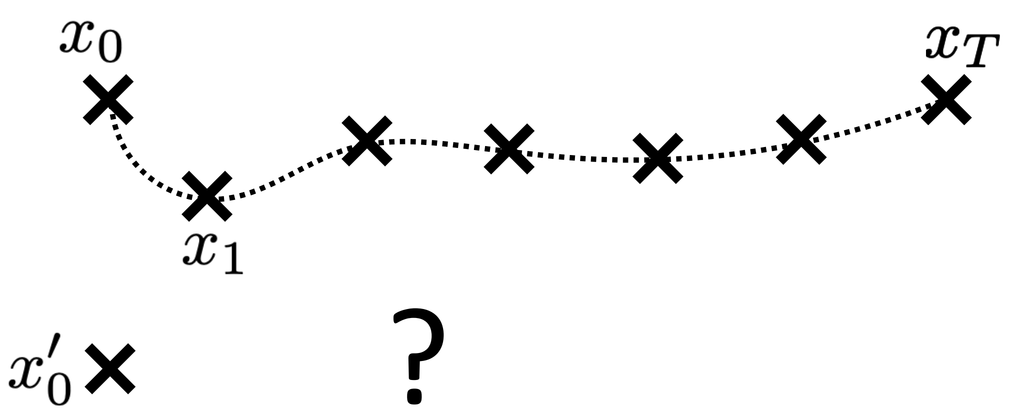
Optimal control problem



$$\{x\} = x_0, \cdots, x_T$$

$$\{u\} = u_0, \cdots, u_{T-1}$$

Optimal control problem



$$\{x\} = x_0, \cdots, x_T$$

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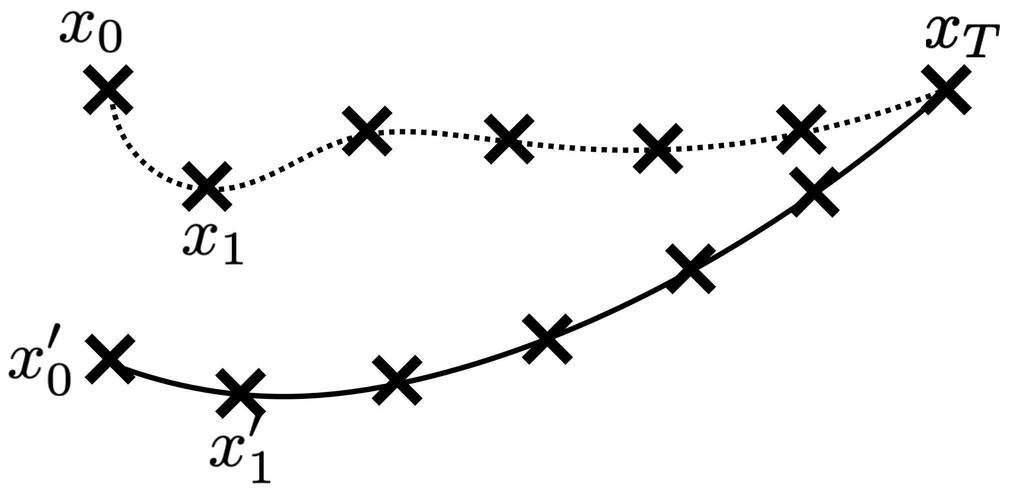
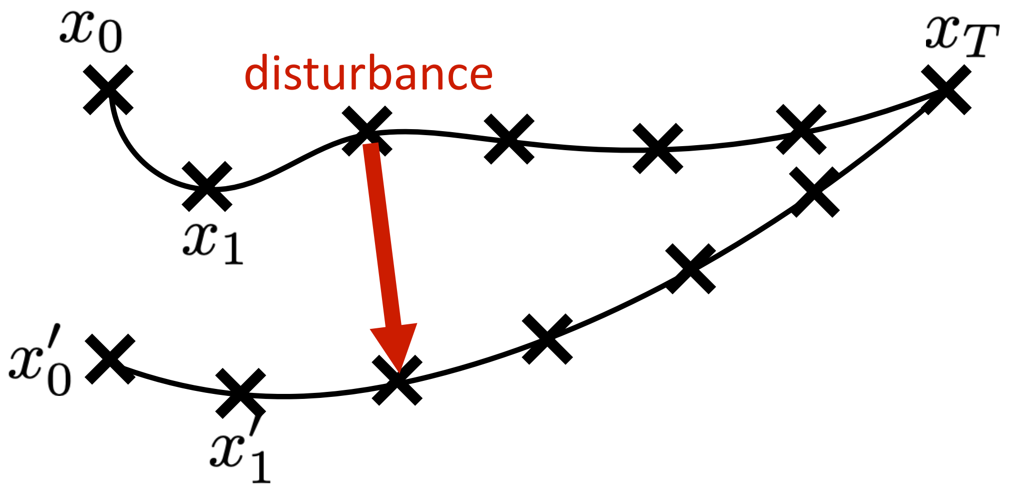
Optimal control problem



$$\{x'\} = x_0{\prime}, \cdots, x'_T$$
$$\{u'\} = u'_0, \cdots, u'_{T-1}$$

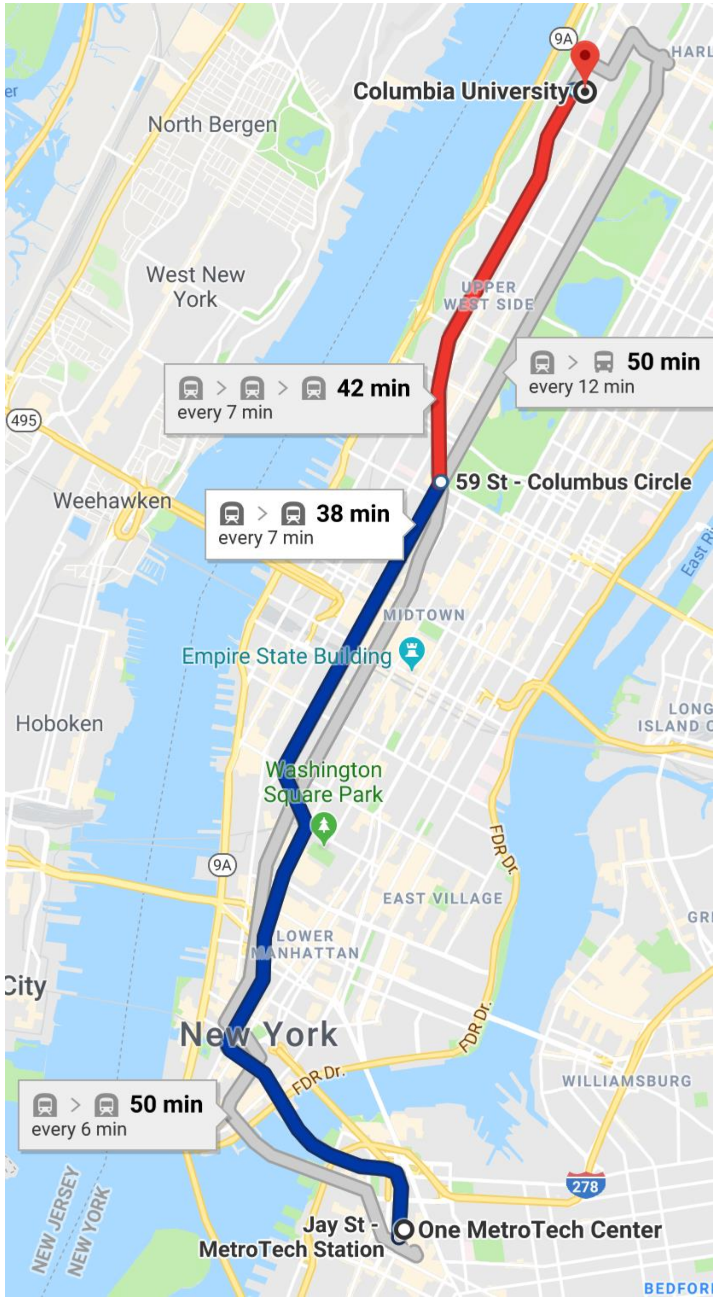
Optimal control problem



$\pi(x)$ => control policy

$\{u\}^*$ the optimal control trajectory

$\pi^*(x)$ the optimal control policy



How can we find the optimal control?

- The Principle of Optimality breaks down the problem
  - *Subpath of optimal paths are also optimal for then own subproblem*

How can we find the optimal control?

- The Principle of Optimality breaks down the problem

Optimal Cost to Go or Value Function

$$V_t(x_t) = \min_{u_t,\cdots,u_{N-1}} \sum_{k=t}^{T-1} l_k(x_k, u_k) + l_T(x_T)$$

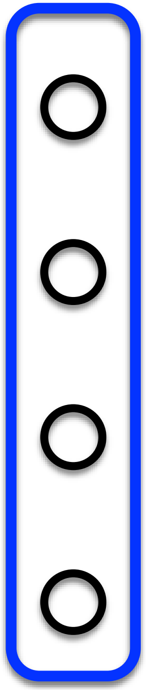
Bellman's Principle of Optimality

$$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$

$$x_{t+1} = f_t(x_t, u_t)$$

Dynamic programming

$$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$



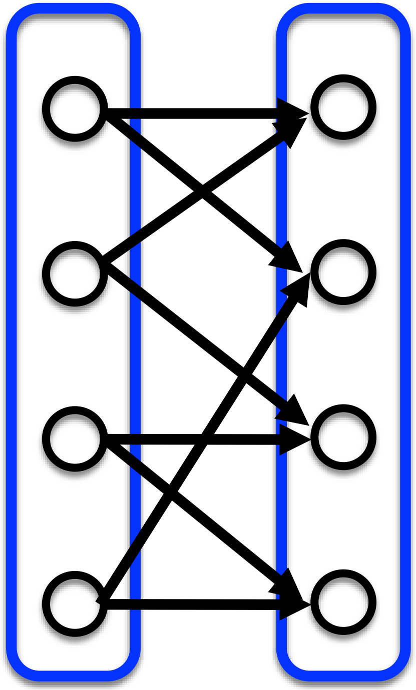
Final States

Stage T

$V_T(x_T)$

Dynamic programming

$$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$



Stage T-1

$V_{T-1}(x_{T-1})$

$\pi_{T-1}(x_{T-1})$

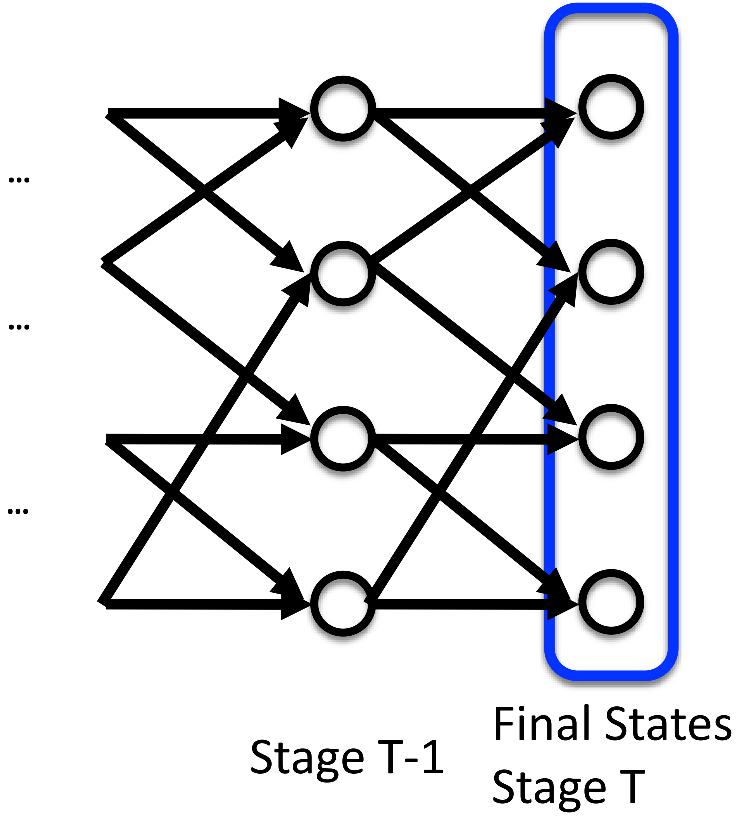
Final States

Stage T

$V_T(x_T)$

Dynamic programming

$$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$

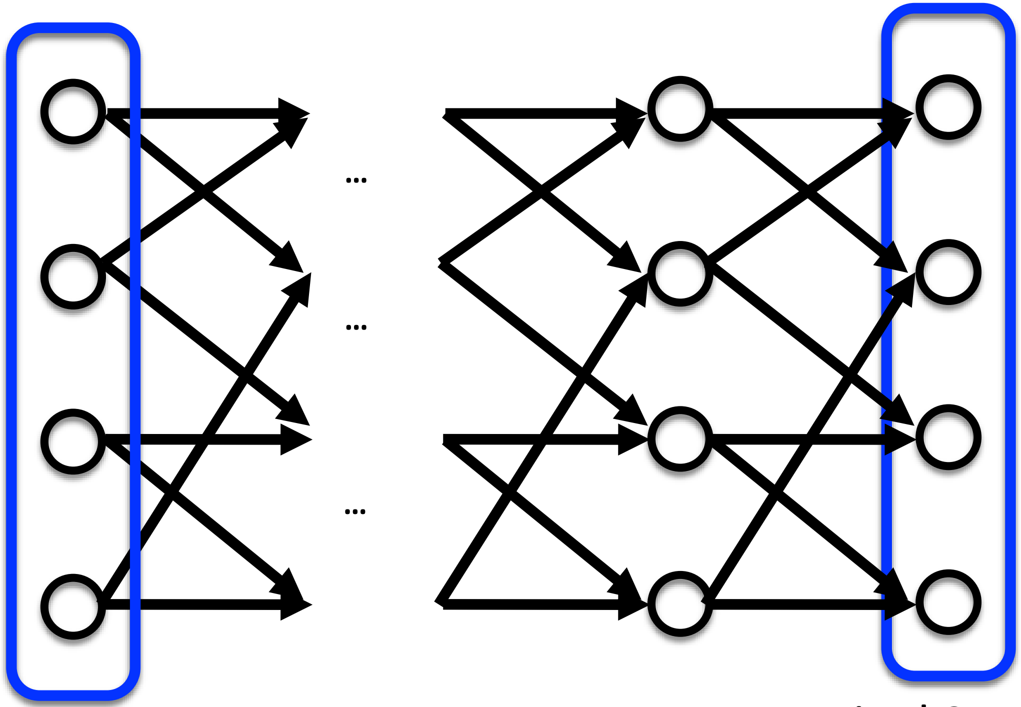


$V_{T-1}(x_{T-1})$ $V_T(x_T)$

$\pi_{T-1}(x_{T-1})$

Dynamic programming

$$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$



Stage 0

$V_0(x_0)$

$\pi_0(x_0)$

Stage T-1

$V_{T-1}(x_{T-1})$

$\pi_{T-1}(x_{T-1})$

Final States

Stage T

$V_T(x_T)$

Dynamic programming

Bellman Equation $$V_t(x_t) = \min_{u_t} l_t(x_t, u_t) + V_{t+1}(x_{t+1})$$

Problems:
- Curse of dimensionality
- minimization in Bellman equation

$\Rightarrow$ Approximate solution to Bellman equation
(DDP, trajectory optimization, reinforcement learning, etc)

Solving Bellman's Equations

[1] Bonnali'19 ArX:1903.00155
[2] Mordach'14 DOI:2185520.2185539
[3] Posa'14 DOI:0278364913506757
[4] Winkler'18 IEEE:2798285
[5] Rajamaki'17 DOI:3099564.3099579

Bellman's Equation $V_t = \min_{u_t} l(x_t, u_t) + V_{t+1}(f(x_t, u_t))$

Resolution Method: Stochastic - Deterministic

LQR
(exact solution)

Non LQR
(approximate solution)

Indirect Methods
Pontryagin's Maximum Principle
Rockets, Cars (small dimensions)

GuSTO [1]

Direct Methods
(Most popular in robotics)

"local"
Trajectory optimization

"global"
Value/Policy optimization

Collocation
CIO [2] TOWR [4]
TrajOpt
"Direct" trajectory optim [3]

Shooting
DDP
Multiple shooting
CMAES, PI^2

Guided policy
search

Explicit MPC
Q learning
Actor Critic
DDPG, TRPO, PPO

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Find control inputs to minimize cost

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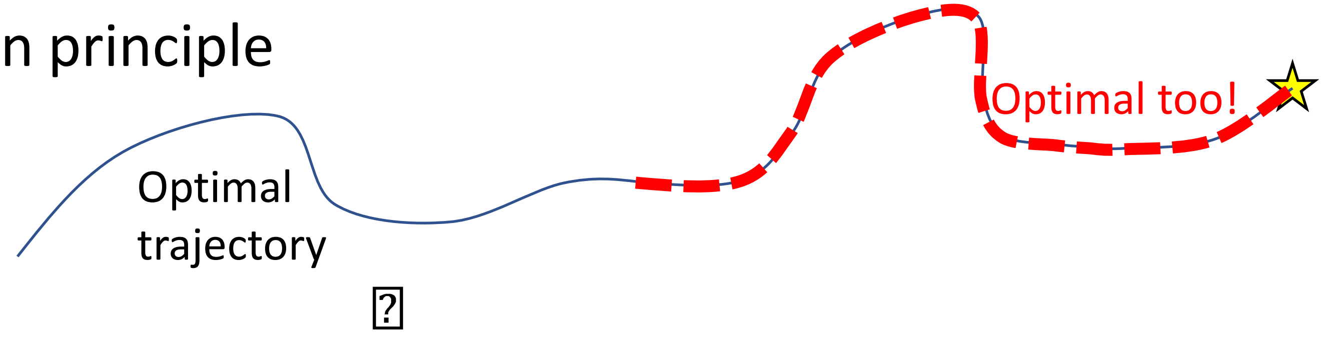
Optimality principles

- Belman principle



- Hamiltonien $H(x,u,\ ) = l(x,u) + \langle \lambda | f(x,u) \rangle$
- Hamilton Jacobi Belman equation

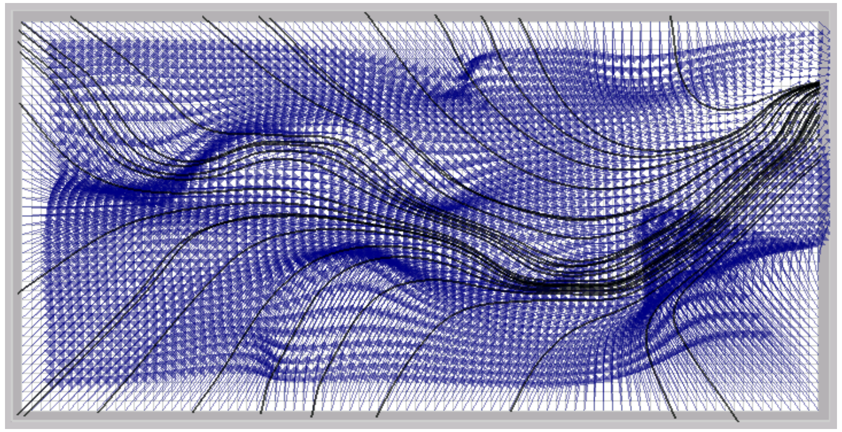
$$u^*(x) = \min_u H\left(x\ \ ,u\ \ ,-\frac{\partial V}{\partial x}(x)\right)$$

- Pontryagin Maximum principle

$$u^*(t) = \min_u H\left(x(t),u\ \ ,\lambda(t)\ \ \right)$$

Optimality principles

- Curse of dimensionality



- Solved in particular cases
  - Nonholonomic car-like robots
  - Reachability sets