227-0689-00L System Identification **3 October 2018**

Solution 1: Spectra and periodograms

Problem 1:

Consider the sinusoidal correlation function

$$I_S(N) = \frac{1}{N} \sum_{k=0}^{N-1} y(k) \sin(\omega_u k)$$

applied to a linear system, $y = Gu + v$, with sinusoidal input $u(k) = \alpha \cos(\omega_u k)$ and zero-mean Gaussian noise v.

- Show that

$$\lim_{N \longrightarrow \infty} E\left\{I_S(N)\right\} = \frac{-\alpha}{2} \left|G(e^{j\omega_u})\right| \sin\left(\arg\left(G(e^{j\omega_u})\right)\right).$$

- What can you say about the variance of $I_S(N)$ as N goes to infinity, given that v is a white noise, i.e., $\{v(k)\}$ is a sequence of independent Gaussian random variables with zero mean and variance one?

Hint: For any non-zero $\omega \in (-\pi, \pi)$ and any ϕ, we have

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sin(\omega k + \phi) = 0.$$

Solution

The response of the linear system to the sinusoidal input is

$$y(k) = \alpha \left|G(e^{j\omega_u})\right| \cos\left(\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) + v(k) + t(k),$$

where $t(k)$ is the transient part and we know that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} |t(k)| = 0$. Note that

$$\begin{aligned}
I_S(N) &= \frac{1}{N}\sum_{k=0}^{N-1} y(k)\sin(\omega_u k) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \alpha \left|G(e^{j\omega_u})\right| \cos\left(\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right)\sin(\omega_u k) + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\sin(\omega_u k) \\
&\quad + \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k)
\end{aligned}$$

So, one has

$$\begin{aligned}
I_S(N) &= \frac{1}{N}\sum_{k=0}^{N-1} \alpha \left|G(e^{j\omega_u})\right| \frac{1}{2}\left(\sin\left(2\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) - \sin\left(\arg\left(G(e^{j\omega_u})\right)\right)\right) \\
&\quad + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\sin(\omega_u k) + \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k) \\
&= \frac{\alpha}{2N}\left|G(e^{j\omega_u})\right| \sum_{k=0}^{N-1} \sin\left(2\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) - \frac{\alpha}{2N}\left|G(e^{j\omega_u})\right| \sum_{k=0}^{N-1} \sin\left(\arg\left(G(e^{j\omega_u})\right)\right) \\
&\quad + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\sin(\omega_u k) + \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k) \\
&= -\frac{\alpha}{2}\left|G(e^{j\omega_u})\right| \sin\left(\arg\left(G(e^{j\omega_u})\right)\right) \\
&\quad + \frac{\alpha}{2N}\left|G(e^{j\omega_u})\right| \sum_{k=0}^{N-1} \sin\left(2\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) + \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k) \\
&\quad + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\sin(\omega_u k)
\end{aligned}$$

Since $E\{v(k)\} = 0$, it follows

$$\begin{aligned}
E\left\{I_S(N)\right\} = &-\frac{\alpha}{2}\left|G(e^{j\omega_u})\right| \sin\left(\arg\left(G(e^{j\omega_u})\right)\right) \\
&+ \frac{\alpha}{2N}\left|G(e^{j\omega_u})\right| \sum_{k=0}^{N-1} \sin\left(2\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) + \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k).
\end{aligned}$$

According to following inequality,

$$|\frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k)| \le \frac{1}{N}\sum_{k=0}^{N-1} |t(k)\sin(\omega_u k)| \le \frac{1}{N}\sum_{k=0}^{N-1} |t(k)|$$

we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} t(k)\sin(\omega_u k) = 0.$$

Also, from given hint one has

$$\lim_{N\to\infty} \frac{\alpha}{2N} \left|G(e^{j\omega_u})\right| \sum_{k=0}^{N-1} \sin\left(2\omega_u k + \arg\left(G(e^{j\omega_u})\right)\right) = 0.$$

So, it concludes that

$$\lim_{N\to\infty} E\left\{I_S(N)\right\} = -\frac{\alpha}{2} \left|G(e^{j\omega_u})\right| \sin\left(\arg\left(G(e^{j\omega_u})\right)\right).$$

Regarding the variance of $I_S(N)$, we have

$$\begin{aligned}
\operatorname{var}\left\{I_S(N)\right\} &= E\left\{(\frac{1}{N}\sum_{k=0}^{N-1} v(k)\sin(\omega_u k))^2\right\} \\
&= \frac{1}{N^2} E\left\{(\sum_{k=0}^{N-1} v(k)\sin(\omega_u k))^2\right\} \\
&= \frac{1}{N^2}(\sum_{k=0}^{N-1} E\left\{v(k)\right\}^2 \sin(\omega_u k)^2 + \sum_{k,l=0,\ k\neq l}^{N-1} E\left\{v(k)v(l)\right\}\sin(\omega_u k)\sin(\omega_u l))
\end{aligned}$$

Since, $E\left\{v(k)v(l)\right\} = 0$, when $k \neq l$, and $E\left\{v(k)^2\right\} = 1$, we have

$$\begin{aligned}
\operatorname{var}\left\{I_S(N)\right\} &= \frac{1}{N^2}\sum_{k=0}^{N-1} \sin(\omega_u k)^2 \\
&\leq \frac{1}{N^2}\sum_{k=0}^{N-1} 1 \\
&\leq \frac{1}{N}
\end{aligned}$$

Therefore, since $\operatorname{var}\left\{I_S(N)\right\} \geq 0$, it follows that

$$\lim_{N\to\infty} \operatorname{var}\left\{I_S(N)\right\} = 0.$$

Problem 2:

Let $\{u(k)\}$ be a stationary stochastic process with $R_u(\tau) = E\{u(k)u(k-\tau)\}$. Also, let $\phi_u(\omega)$ be its power spectral density (spectrum). Assume that

$$\sum_{k=1}^{\infty} |kR_u(k)| < \infty.$$

Let $U_N(\omega)$ be defined as $U_N(\omega) = \sum_{k=1}^{N} u(k)e^{-j\omega k}$. Prove that $\lim_{N\longrightarrow\infty} E\{\frac{1}{N}|U_N(\omega)|^2\} = \phi_u(\omega)$.

Solution

According to the definition of $U_N(\omega)$, we have

$$\begin{aligned}
E\{\frac{1}{N}|U_N(\omega)|^2\} &= \frac{1}{N}E\{U_N(\omega)U_N(\omega)^*\} \\
&= \frac{1}{N}E\{(\sum_{k=1}^{N} u(k)e^{-j\omega k})(\sum_{k=1}^{N} u(k)e^{-j\omega k})^*\} \\
&= \frac{1}{N}E\{(\sum_{k_1=1}^{N} u(k_1)e^{-j\omega k_1})(\sum_{k_2=1}^{N} u(k_2)^* e^{j\omega k_2})\} \\
&= \frac{1}{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} E\{u(k_1)u(k_2)^*\}e^{-j\omega k_1}e^{j\omega k_2} \\
&= \frac{1}{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} R_u(k_1-k_2)e^{-j\omega(k_1-k_2)} && \text{(from definition of } R_u\text{)} \\
&= \sum_{k=-N+1}^{N-1}(1-\frac{|k|}{N})R_u(k)e^{-j\omega k} && \text{(setting } k \text{ as } k_1-k_2\text{)} \\
&= \sum_{k=-N+1}^{N-1} R_u(k)e^{-j\omega k} - \frac{1}{N}\sum_{k=-N+1}^{N-1}|k|R_u(k)e^{-j\omega k}.
\end{aligned}$$

From this equality and definition of power spectral density, $\phi_u(\omega) = \sum_{k=-\infty}^{\infty} R_u(k)e^{-j\omega k}$, one has

$$\begin{aligned}
|\phi_u(\omega) - \frac{E\{|U_N(\omega)|^2\}}{N}| &= |\sum_{|k|\geq N} R_u(k)e^{-j\omega k} + \frac{1}{N}\sum_{k=-N+1}^{N-1}|k|R_u(k)e^{-j\omega k}| \\
&\leq |\sum_{|k|\geq N} R_u(k)e^{-j\omega k}| + \frac{1}{N}\sum_{k=-N+1}^{N-1}|\,|k|R_u(k)e^{-j\omega k}| && \text{(triangle inequality)} \\
&\leq |\sum_{|k|\geq N} R_u(k)e^{-j\omega k}| + \frac{1}{N}\sum_{k=-N+1}^{N-1}|kR_u(k)| && \text{(since } |e^{-j\omega k}|=1\text{)} \\
&\leq |\sum_{|k|\geq N} R_u(k)e^{-j\omega k}| + \frac{1}{N}\sum_{k=-\infty}^{\infty}|kR_u(k)|
\end{aligned}$$

As N goes to infinity, since $\sum_{k=-N+1}^{N-1} R_u(k)e^{-j\omega k}$ converges to $\phi_u(\omega)$, we know that

$$\lim_{N\longrightarrow\infty} |\sum_{|k|\geq N} R_u(k)e^{-j\omega k}| = 0.$$

Therefore, from the assumption $\sum_{k=1}^{\infty}|kR_u(k)| < \infty$ and the inequality derived above, it follows

$$\lim_{N\longrightarrow\infty} |\phi_u(\omega) - \frac{E\{|U_N(\omega)|^2\}}{N}| = 0,$$

or equivalently, $\lim_{N\longrightarrow\infty} E\{\frac{1}{N}|U_N(\omega)|^2\} = \phi_u(\omega)$.

MATLAB exercises:

1. Write a MATLAB function that takes an input vector, $u(k)$, $k = 0, \ldots, N-1$ and returns the vector, $U_N(e^{j\omega_n})$, for

$$\omega_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N-1.$$

   The functional relationship between $u(k)$ and $U_N(e^{j\omega})$ is given by,

$$U_N(e^{j\omega}) = \sum_{k=0}^{N-1} u(k)\, \mathrm{e}^{-j\omega k}, \quad (j = \sqrt{-1}).$$

   This looks like an FFT and you may use an `fft` call in your function. See the MATLAB notes section at the end of these exercises for caveats.

2. Write a MATLAB script file (.m file) which performs the following calculations:

   a) Generate $e(k)$, a 1024 point $\mathcal{N}(0,1)$ distributed random sequence.

   b) Calculate and plot (on a log-log scale) the periodogram of $e(k)$. The periodogram is defined as $\frac{1}{N}|E_N(e^{j\omega})|^2$.

   c) Given a discrete-time plant,

$$P(z) = \frac{1}{z^2 - 0.9z + 0.5},$$

   calculate $w(k)$, the response of $P(z)$ to the input signal, $e(k)$. Assume that the sample time of $P(z)$ is specified as $T = 1$ second.

   d) Calculate the periodogram of $w(k)$.

   e) How is the periodogram of $w(k)$, $\frac{1}{N}|W_N(e^{j\omega})|^2$, related to $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|$ (asymptotically) ? Here $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|$ is the magnitude of the Bode plot of $P(z)$. Provide a plot comparing these quantities. To do this, plot both $\frac{1}{N}\left|W_N(e^{j\omega})\right|^2$ and $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|^2$ in the same figure. You should also plot $\frac{1}{N}\left|W_N(e^{j\omega})\right|^2 - \left|P\left(\mathrm{e}^{j\omega_n}\right)\right|^2$ to examine the error between the two quantities.

   f) Repeat the above for 2048 and 4096 length sequences. Plot all of your comparisons and see if your assertion in part e) is correct.

Solution hints

1. The definition of the DFT given here coincides with the MATLABfunction `fft`. Different definitions can be found in various literature including [Ljung, 1999]. It is important to be aware of the frequency interval the DFT is being evaluated on.

2. a) Use the MATLAB function `randn` to create a normally distributed random sequence.

   b) Use the function you wrote in 1. and the MATLAB function `abs` to compute the periodogram. Figure 1.1 shows the periodogram of e plotted with the MATLAB function `loglog` for the non-negative frequencies $\omega = 0, \frac{2\pi}{1024}, 2\frac{2\pi}{1024}, \ldots, \pi$.

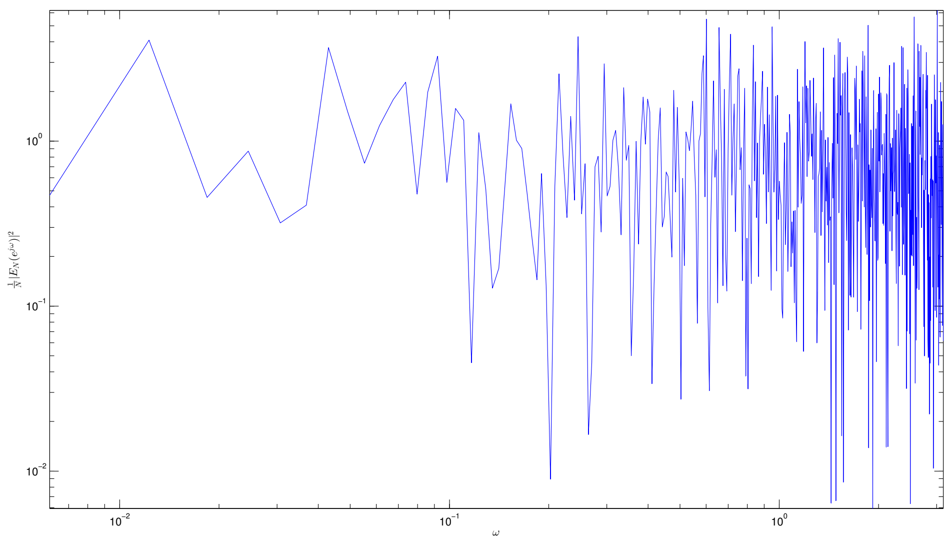

Figure 1.1: Periodogram of e

c) Define the discrete-time linear time-invariant (LTI) plant in MATLAB with the function `tf`. Define a time vector containing the discrete time steps kT_s, $k = 0, 1, \ldots N-1$ and compute the system response to the input signal with the MATLAB function `lsim`.

d) Calculate the periodogram of the output sequence as done for the input sequence in b).

e) The periodogram of w is an asymptotically unbiased estimate of the spectrum of w. As the input signal e is a white noise sequence the expected value of periodogram of w converges to the mangitude of the frequency response squared as $N \longrightarrow \infty$, $\lim_{N\longrightarrow\infty} E\left\{\frac{1}{N}|W_N(e^{j\omega})|^2\right\} = |P(e^{j\omega})|^2$. Figure 1.2 compares the periodogram of w with the squared magnitude of the plant transfer function.

f) Figures 1.3 and 1.4 show the results of the above procedure repeated for data lengths 2028 and 4096 respectively. Note that the periodogram still appears jagged for large data lengths as it only converges to the spectrum in expectation.

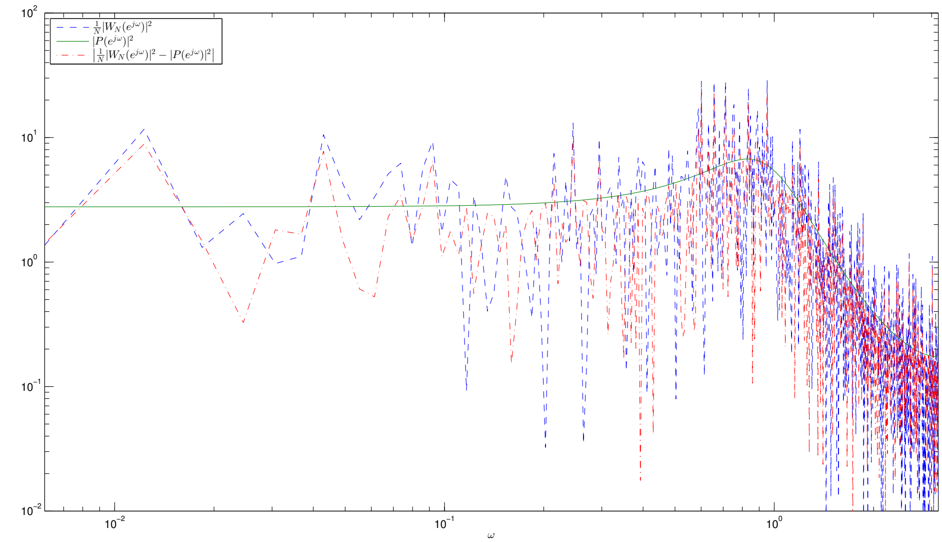

Figure 1.2: Experiment with data length N=1024

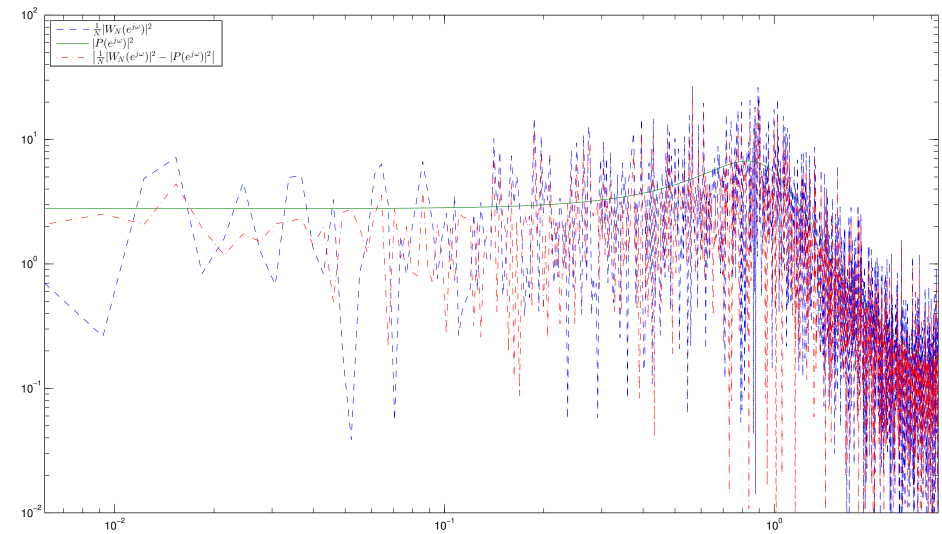

Figure 1.3: Experiment with data length N=2048

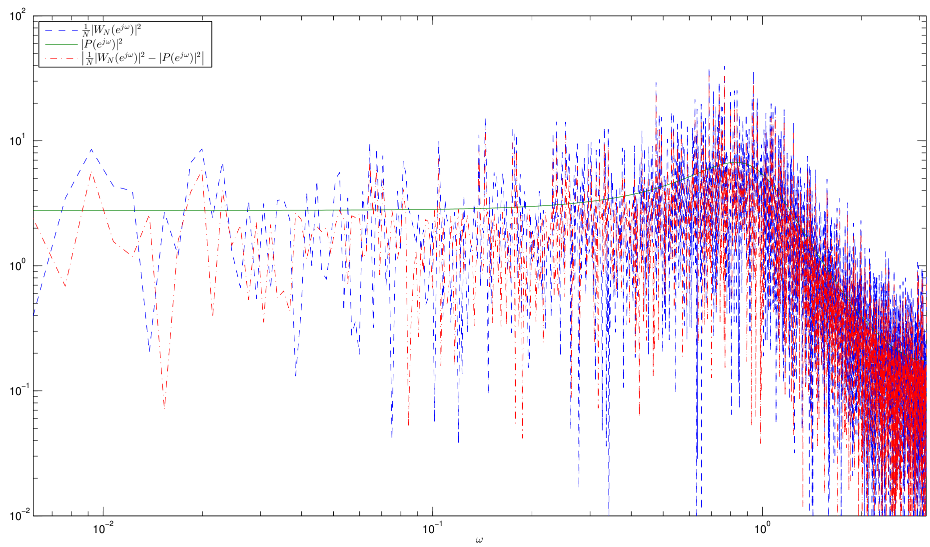

Figure 1.4: Experiment with data length N=4096