

Exercise 6: Frequency Domain Subspace and Closed-loop identification

Background reading

The background material for this exercise is class lectures 7 and 8.

Problem 1:

Consider the state space description of a discrete-time system:

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k)\end{aligned}$$

Here we examine several steps and assumptions in arriving at the results of Lecture 7.

- (a) Consider the impulse response of the system. In the lecture notes, this was given as

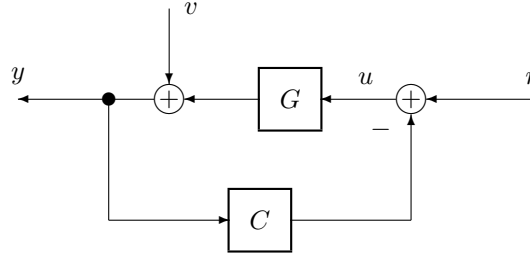
$$g(k) = \begin{cases} 0, & \text{if } k < 0 \\ D, & \text{if } k = 0 \\ CA^{k-1}B & \text{if } k > 0 \end{cases}$$

Explain why $g(k)$ has the above form. Additionally, derive why $G(e^{j\omega}) = \sum_{k=0}^{\infty} g(k)e^{-j\omega k}$ can be written as $G(e^{j\omega}) = C(e^{j\omega}I - A)^{-1}B + D$.

- (b) Suppose $A \in \mathbb{R}^{n \times n}$ is nilpotent. How does this affect the impulse response, Hankel matrix, and singular value decomposition, compared to a full rank A ?
- (c) One of the potential pitfalls listed for the subspace identification method is that it can result in an unstable model for a stable system. Give a numerical example of a stable linear discrete-time system which, when the system order is truncated, results in an unstable system. Explain why this happened.
- (d) Another disadvantage listed for the subspace identification method is that the truncated SVD of a Hankel matrix might no longer be Hankel. Show that the truncated SVD for any real-valued 2x2 Hankel matrix is still Hankel. Give an example of a 3x3 Hankel matrix which has a truncated SVD which is no longer Hankel. Why is having a non-Hankel truncated SVD a disadvantage?

Problem 2

Consider a plant, G , operating in closed-loop with a controller, C , as illustrated in the following diagram.



We can choose the external excitation signal, r , but of course we cannot choose the noise signal v . Assume that these two signals have the spectra, $\Phi_r(\omega)$ and $\Phi_v(\omega)$ respectively. We can also assume that these two signals are independent and that v is zero mean.

Now suppose we run an experiment and measure y and u . We use a frequency domain window, $W_\gamma(\omega)$, with width parameter γ , to estimate the spectral densities $\hat{\Phi}_y$ and $\hat{\Phi}_u$ of the output, y , and input, u , signals respectively. Show that the filtered estimate of G ,

$$\hat{G}_N(e^{j\omega}) := \frac{\hat{\Phi}_{yu}^N(\omega)}{\hat{\Phi}_u^N(\omega)}$$

satisfies,

$$\lim_{N, \gamma \rightarrow \infty} \hat{G}_N(e^{j\omega}) = \frac{G(e^{j\omega})\hat{\Phi}_r(\omega) - C(e^{-j\omega})\hat{\Phi}_v(\omega)}{\hat{\Phi}_r(\omega) + |C(e^{j\omega})|^2\hat{\Phi}_v(\omega)}.$$

Problem 3

Consider two independent stochastic variables V and W , uniformly distributed on $[0, 1]$. That is,

$$f_V(v) = \begin{cases} 1 & 0 < v < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_W(w) = \begin{cases} 1 & 0 < w < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Prove that the ratio of the two variables, $Z = V/W$, is a stochastic variable with probability density function

$$f_Z(z) = \begin{cases} 1/2 & 0 < z < 1 \\ \frac{1}{2z^2} & z \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of this distribution?

MATLAB exercise:

This exercise will look at identification in closed-loop.

Consider a continuous-time plant, $G(s)$, defined by,

$$G(s) = \frac{(s^2 + 2\zeta_z\omega_z s + \omega_z^2)}{(s^2 + 2\zeta_p\omega_p s + \omega_p^2)} \frac{5000}{(s + 50)(s + 200)},$$

where the pole and zero damping and modal frequency coefficients are,

$$\zeta_z = 0.1, \quad \omega_z = 3.0, \quad \zeta_p = 0.1, \quad \text{and} \quad \omega_p = 3.5.$$

At the input of this plant is a zero-order hold with a sample period of,

$$T_s = 0.02 \quad \text{seconds.}$$

Denote the discrete-time, zero-order hold equivalent of this plant by $G_d(z)$. This can be calculated in MATLAB via the command:

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>> Gdz = c2d(Gs,Ts,'zoh');
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The input to $G_d(z)$ is $u(k)$. Unfortunately our measurement of the output has noise, $v(k)$, giving the measured output as,

$$y(k) = G_d(z)u(k) + v(k).$$

In the following we will simulate the noise as a normally distributed signal with variance, $\lambda = 0.1$.

Now implement a discrete-time PI controller for this system. The controller is,

$$C_d(z) = \frac{1.25z - 0.75}{z - 1}.$$

We will implement this in a standard unity gain feedback configuration. The input to $C_d(z)$ is $r(k) - y(k)$, where $r(k)$ is a reference input to be tracked. The output of $C_d(z)$ is the control actuation, $u(k)$.

1. Calculate the discrete-time sensitivity function,

$$S_d(z) = \frac{1}{1 + G_d(z)C_d(z)},$$

and check that all of its poles are inside the unit circle. The complementary sensitivity function, $T_d(z)$ is the transfer function from the reference, $r(k)$, to the output, $y(k)$, and can be calculated by,

$$T_d(z) = 1 - S_d(z).$$

It has the same poles as $S_d(z)$.

2. Run the following series of identification experiments:
 - a) Excite $r(k)$ with a PRBS signal of maximum amplitude ± 0.1 . Use a periodic signal and discard the first period. Measure $y(k)$ and $u(k)$. Use time-domain averaging to improve the signal to noise ratio.

b) Estimate the ETFE in the usual way (i.e. $Y_N(\omega)/U_N(\omega)$).

Compare (on a Bode plot), your estimate of $G_d(z)$ with its true frequency response.

- Run the above experiment, again using a periodic excitation for $r(k)$ and measuring the “error signal”,

$$e(k) = r(k) - y(k).$$

Discarding the first period and using time-domain averaging will improve the signal to noise ratio. Now use these measurements to estimate the sensitivity function, $S_d(z)$. Compare (on a Bode plot and an error magnitude plot) your estimate of $S_d(z)$ with its true frequency response.

- For the next experiment we set $r(k) = 0$, but introduce another excitation signal, $w(k)$, at the output of the controller. In other words, the actuation signal going into the plant is now,

$$u(k) = w(k) + C_d(z)(r(k) - y(k)), \quad \text{with } r(k) = 0.$$

Now run an experiment, this time specifying $w(k)$ as an excitation signal with the same properties as $r(k)$ in the previous experiment. Measure $w(k)$ and $y(k)$. Note that the relationship between these two signals (with $r(k) = 0$) is,

$$y(k) = \frac{G_d(z)}{1 + G_d(z)C_d(z)} w(k) + \frac{1}{1 + G_d(z)C_d(z)} v(k),$$

and so you will be able to estimate $G_d(z)S_d(z)$ from this experiment.

- From your estimates of $S_d(z)$ and $G_d(z)S_d(z)$, divide one by the other to get an estimate for $G_d(z)$. Compare this (on a Bode plot and an error magnitude plot) with the true frequency response and your previous estimate of $G_d(z)$.