227-0689-00L System Identification **22 October 2018**

Solution 4: Smoothing ETFE Estimates and PRBS

We will not provide solutions for the MATLAB parts of the exercises. These will instead be discussed in the exercise session.

Problem 1:

Averaging the estimates $\hat{G}_r(e^{j\omega_n})$ from several experiments $r = 1, \ldots, R$ can improve the estimate $\hat{G}(e^{j\omega_n}) = \sum_{r=1}^{R} \alpha_r(e^{j\omega_n})\hat{G}_r(e^{j\omega_n})$ at frequency ω_n (cf. Eq. (6.41) in [Ljung, 1999] and Lecture Slide 4.2). The optimal weighting $\alpha_r(e^{j\omega_n})$ at each frequency ω_n is proportional to the inverse variance of $\hat{G}_r(e^{j\omega_n})$.

To demonstrate the optimality in a minimum variance sense, let p_r with $r = 1, \ldots, R$, be independent random variables with identical mean $E\{p_r\} = m$ and individual variance $E\{(p_r - m)^2\} = \lambda_r$. For $p = \sum_{r=1}^{R} \alpha_r p_r$ determine α_r, $r = 1, \ldots, R$, such that

(1.a) $E\{p\} = m$,

(1.b) $E\{(p - m)^2\}$ is minimised.

Hint: Use Lagrange multipliers to solve the minimisation problem in (1.b).

Solution

Since all variables have a mean value of m, i.e. $E\{p_r\} = m$ for $r = 1, \ldots, R$, it follows from (1.a) that

$$m \;=\; E\{p\} \;=\; E\left\{ \sum_{r=1}^{R} \alpha_r p_r \right\} \;=\; \sum_{r=1}^{R} \alpha_r E\{p_r\} \;=\; \sum_{l=r}^{R} \alpha_r m\,, \qquad \Rightarrow \qquad \sum_{l=r}^{R} \alpha_r \;=\; 1\,. \tag{4.1}$$

Hence, the weighting α_r can be used to emphasise a certain experiment r.

To determine the optimal weighting, we start with the variance of the weighted sum,

$$
\begin{aligned}
E\{(p-m)^2\} &= E\left\{\left(\sum_{r=1}^{R}\alpha_r p_r - m\right)^2\right\} \\
&= E\left\{\left(\sum_{r=1}^{R}\alpha_r p_r - \sum_{r=1}^{R}\alpha_r m\right)^2\right\} \\
&= E\left\{\sum_{r=1}^{R}\alpha_r^2 (p_r-m)^2 + \sum_{j\neq k}\alpha_j\alpha_k \underbrace{(p_j-m)(p_k-m)}_{=0}\right\} \\
&= E\left\{\sum_{r=1}^{R}\alpha_r^2 (p_r-m)^2\right\} \\
&= \sum_{r=1}^{R}\alpha_r^2\, E\left\{(p_r-m)^2\right\} \\
&= \sum_{r=1}^{R}\alpha_r^2\, \lambda_r .
\end{aligned}
\tag{4.2}
$$

where we used the square-of-sum rule, $\left(\sum_i a_i\right)^2 = \sum_i a_i^2 + \sum_{i\neq j} a_i a_j$, to expand the brackets in (4.2), and the assumption that p_r are assumed to be independent.

Hence to minimize the variance, $E\{(p-m)^2\}$, we can equivalently minimize $\sum_{r=1}^{R}\alpha_r^2\lambda_r$ subject to $\left(\sum_{r=1}^{R}\alpha_r = 1\right)$. We can solve this optimization problem using the Lagrange multiplier $\nu \in \mathbb{R}$ with the Lagrangian $\mathcal{L}$ defined as (see Wikipedia for examples)

$$
\mathcal{L}(\alpha_1,\ldots,\alpha_R,\nu) \;=\; \sum_{r=1}^{R}\alpha_r^2\lambda_r + \nu\left(\left(\sum_{r=1}^{R}\alpha_r\right) \;-\; 1\right) . \tag{4.3}
$$

Setting the gradient $\nabla_{\alpha,\nu}\mathcal{L} = 0$ yields the system of equations for this constrained optimization problem,

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\alpha_r} &= 2\alpha_r\lambda_r + \nu \;=\; 0\,, \\
\frac{\partial\mathcal{L}}{\partial\nu} &= \sum_{r=1}^{R}\alpha_r - 1 \;=\; 0\,,
\end{aligned}
\tag{4.4}
$$

where the second equation is exactly the original constraint. The first set of R equations yields the solution to the problem

$$
\alpha_r \;=\; -\frac{\nu}{2}\frac{1}{\lambda_r}\,, \qquad \text{for all } r = 1,\ldots,R \tag{4.5}
$$

where ν can be obtained from the second equation as,

$$
\sum_{r=1}^{R}\alpha_r \;=\; \sum_{r=1}^{R}\left(-\frac{\nu}{2}\frac{1}{\lambda_r}\right) \;=\; -\frac{\nu}{2}\sum_{r=1}^{R}\frac{1}{\lambda_r} \;=\; 1 \qquad \Leftrightarrow \qquad \nu \;=\; -2\left(\sum_{r=1}^{R}\frac{1}{\lambda_r}\right)^{-1} . \tag{4.6}
$$

Hence, the optimal weighting α_r is proportional to its respective inverse variance, and is given by,

$$\alpha_r = \frac{\frac{1}{\lambda_r}}{\left(\sum_{k=1}^{R} \frac{1}{\lambda_k}\right)}. \tag{4.7}$$

Problem 2:

It is not straightforward to use PRBS for multi-input systems. Consider the following: Let $s^M = s(t)$, $t = 1, \ldots, M$, be a maximum length M PRBS signal. Consider an identification experiment over a multiple of $2M$ samples by letting $u_1 = u_2 = s^M$ for the first M samples, and $u_1 = -u_2 = s^M$ for the next M samples. Show that this gives the same input covariance matrix as exciting one input at a time with $\sqrt{2}s^M$, i.e., $\tilde{u}_1 = \sqrt{2}s^M, \tilde{u}_2 = 0$ for the first M samples and then $\tilde{u}_1 = 0, \tilde{u}_2 = \sqrt{2}s^M$ for the next M samples.

Solution

For the first case, with $u_1 = u_2 = s^M$ for the first M samples, and $u_1 = -u_2 = s^M$ for the next M samples, we have

$$R_{u_i u_i} = \frac{1}{2M}\sum_{t=1}^{2M} u_i(t)u_i(t) = \frac{1}{M}\sum_{t=1}^{M} u_i(t)u_i(t) \quad i = 1, 2.$$

$$R_{u_1 u_2} = \frac{1}{2M}\sum_{t=1}^{2M} u_1(t)u_2(t) = \frac{1}{2M}\left[\sum_{t=1}^{M} u_1(t)u_2(t) - \sum_{t=1}^{M} u_1(t)u_2(t)\right] = 0.$$

For the second case, with $\tilde{u}_1 = \sqrt{2}s^M, \tilde{u}_2 = 0$ for the first M samples and then $\tilde{u}_1 = 0, \tilde{u}_2 = \sqrt{2}s^M$ for the next M samples, we have

$$R_{\tilde{u}_i \tilde{u}_i} = \frac{1}{2M}\sum_{t=1}^{2M} \tilde{u}_i(t)\tilde{u}_i(t) = \frac{1}{2M}\sum_{t=1}^{M} \sqrt{2}u_i(t)\sqrt{2}u_i(t) = \frac{1}{M}\sum_{t=1}^{M} u_i(t)u_i(t), \quad i = 1, 2.$$

$$R_{\tilde{u}_1 \tilde{u}_2} = \frac{1}{2M}\sum_{t=1}^{2M} \tilde{u}_1(t)\tilde{u}_2(t) = \frac{1}{2M}\left[\sum_{t=1}^{M} \tilde{u}_1(t)\cdot 0 + \sum_{t=M+1}^{2M} 0 \cdot \tilde{u}_2(t)\right] = 0.$$

MATLAB Exercise 1:

Plot periodograms for maximum length PRBS of order 5 to 8 and show how these spectral estimates change as the number of periods increases. What are advantages and disadvantages of PRBS signals?

Solution hints

Generate a PRBS of different orders using the Matlab command idinput. Check Matlab help to understand the different inputs to the function to define the order and periodicity of the PRBS.

Then compute the fft of the generated signal for each order and compute the periodogram. The results should look like Fig. 4.1.

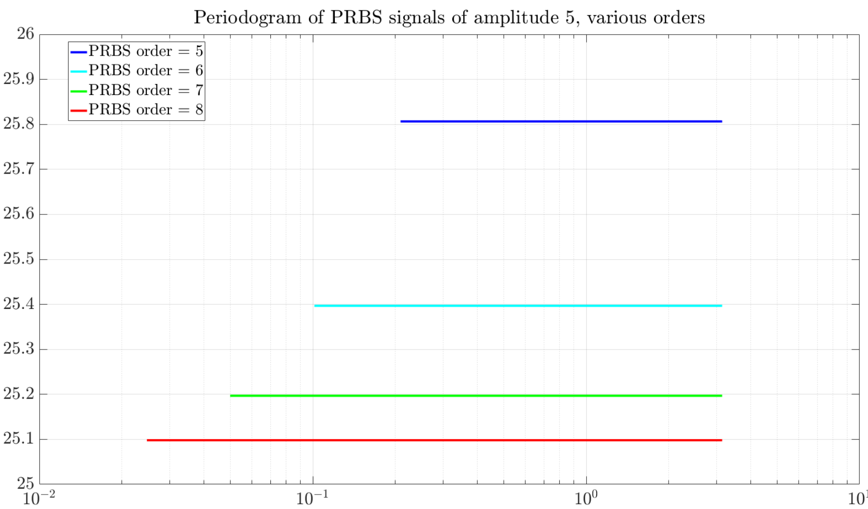

Figure 4.1: Periodogram of PRBS of different orders.

The advantages and disadvantages of the PRBS signal are shown in CH. 13.3 of Ljung on the pages 421-422.

Matlab **Exercise 2:**

Complete Exercise 3 from last week using the time domain Hann window.

Solution hints

Similar to the solution to Problem Set 3 in the frequency domain, form the vector of time domain points (symmetric and centered at zero) at which the window should be calculated (this vector should have the length of 1024) and then use *WtHann* in order to calculate the window for the different values of γ. Obtained windows in time and frequency domain are shown in Fig. 4.2. As can be seen, a bigger value of γ results in a wider window in the time domain.

For the time domain smoothing, one needs first to calculate the input signal autocorrelation and the input-output crosscorrelation. These vectors should then be multiplied with the time domain windows and their DFTs should be calculated in order to obtain the input spectrum and the cross spectrum. By point-wise dividing the two vectors, we obtain the smoothed ETFE. Fig. 4.3 shows the obtained smoothed ETFEs. As can be seen the estimates for $\gamma = 10$ and $\gamma = 5$ over-smooths the ETFE and introduces a large bias. The estimates with $\gamma = 50$ and $\gamma = 100$ provide a reasonable

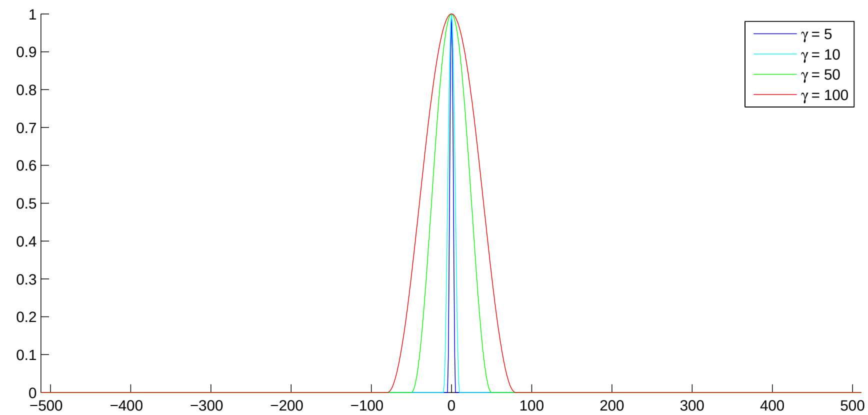

Figure 4.2: Hann windows in time domain.

smoothing/bias trade-off and therefore the search for the best value of γ should be done in this range.

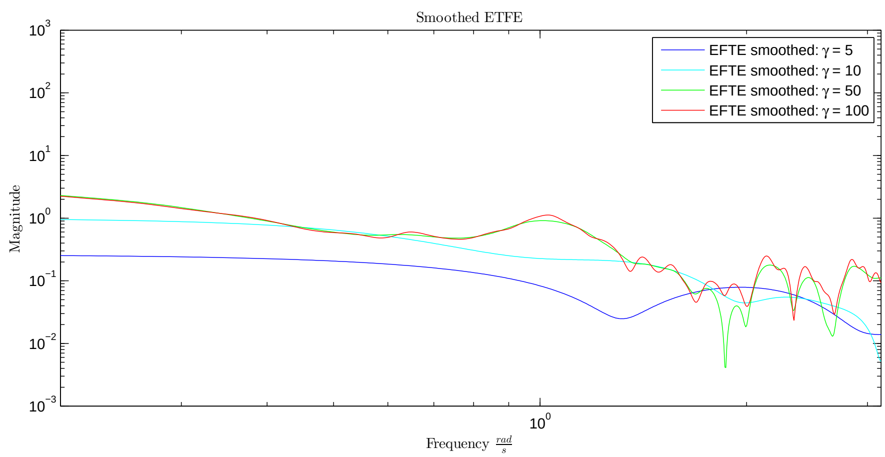

Figure 4.3: Time-domain windowed ETFE (for different γ).