227-0689-00L System Identification **17 October 2018**

Solution 03: Smoothing ETFE Estimates

We will not provide solutions for the MATLAB parts of the exercises. These will instead be discussed in the exercise session.

Problem 1:

Prove that if $r(k)$ is a fixed signal (for example a transient) and $u(k)$ is a periodic signal (with period $= M$), then

$$\lim_{N\to\infty} \frac{|R_N(e^{jw_n})|}{|U_N(e^{jw_n})|} = 0$$

and converges with a rate of $1/N$.

Solution

The DFT for a single length M period of a signal is

$$U_M(e^{j\omega_n}) = \sum_{k=0}^{M-1} u(k)e^{-jk\omega_n}, \quad m = n, ..., M.$$

Now consider increasing the experiment length to $N = mM$ where m is an integer. This adds $m-1$ more periods to the experiment. Then,

$$\begin{aligned} U_N(e^{j\omega_n}) =& U_{mM}(e^{j\omega_n}) = \sum_{k=0}^{mM-1} u(k)e^{-jk\omega_n} = \sum_{r=0}^{m}\sum_{s=0}^{M-1} u(s+rM)e^{-j(s+rM)\omega_n} \\ &= \sum_{r=0}^{m}\sum_{s=0}^{M-1} u(s)e^{-js\omega_n}e^{-jrM\omega_n} = m\sum_{s=0}^{M-1} u(s)e^{-js\omega_n} = mU_M(e^{j\omega_n}) \end{aligned}$$

since $e^{-jrM\omega_n} = 1$ as $M\omega_n = 2\pi n$. So the magnitude of the DFT frequencies are scaled by m. Only M frequencies can be calculated; energy at other frequencies is zero. This implies that the limit converges with rate $1/m$ and therefore also $1/N$ since $N = mM$.

Problem 2:

We are given data $\{y(k), u(k)\}$, $k = 1, \ldots, N$, from a noise corrupted identification experiment:

$$y(k) \;=\; G(z)\, u(k) \;+\; v(k),$$

with $v(k)$ a zero-mean stochastic signal.

Suppose that we filter our experimental data with stable LTI filters (or transfer functions) for the input and output signals giving,

$$\begin{aligned} y^F(k) &= L_y(z)\, y(k), \\ u^F(k) &= L_u(z)\, u(k). \end{aligned}$$

The DFTs of the filtered signals are $Y_N^F(\omega)$ and $U_N^F(\omega)$. Now calculate an ETFE estimate using the filtered signals,

$$\hat{G}^F(\mathrm{e}^{j\omega}) \;:=\; \frac{Y_N^F(\omega)}{U_N^F(\omega)}.$$

Hint: Start with:

$$Y_N(\omega) = G(e^{j\omega})U_N(\omega) + R_N^y(\omega) + V_N(\omega)$$

and use the Fourier Transforms resulting from Theorem 2.1 in the textbook of Ljung:

$$U_N^F(\omega) = L_u(e^{j\omega})U_N(\omega) + R_N^{uf}(\omega)$$
$$Y_N^F(\omega) = L_y(e^{j\omega})Y_N(\omega) + R_N^{yf}(\omega)$$

Questions:

a) Calculate the expected value of the filtered estimate,

$$E\{\hat{G}^F(e^{j\omega})\},$$

and its mean-square error at frequency ω,

$$E\left\{\left|\hat{G}^F(e^{j\omega}) - G(e^{j\omega})\}\right|^2\right\}.$$

b) What are the asymptotic properties of these quantities as $N \to \infty$? You may use $\lim_{N\to\infty} E\{\frac{1}{N}|X(\omega)|^2\} = \Phi_x(\omega)$.

c) Can you think of reasonable choices for $L_y(z)$ and $L_u(z)$?

Solution

a)+b)+c) Let

$$U_N(\omega) := \sum_{k=1}^{N} u(k)e^{-j\omega k}$$

be the Fourier transform of u and let U_N^F, Y_N, Y_N^F and V_N be defined analogously as the Fourier transforms of u^F, y, y^F and v respectively. By Theorem 2.1 in Ljung it holds:

$$Y_N(\omega) = G(e^{j\omega})U_N(\omega) + R_N^y(\omega) + V_N(\omega)$$

Also by Theorem 2.1, the Fourier transforms of the filtered signals u^F and y^F satisfy:

$$\begin{aligned} U_N^F(\omega) &= L_u(e^{j\omega})U_N(\omega) + R_N^{uf}(\omega) \\ Y_N^F(\omega) &= L_y(e^{j\omega})Y_N(\omega) + R_N^{yf}(\omega) \end{aligned}$$

Combining the equalities above we have

$$Y_N^F(\omega) = L_y(e^{j\omega})G(e^{j\omega})U_N(\omega) + L_y(e^{j\omega})R_N^y(\omega) + L_y(e^{j\omega})V_N(\omega) + R_N^{yf}(\omega)$$

and the ETFE based on the filtered signals becomes

$$\begin{aligned} \hat{G}_N^F(\omega) &:= \frac{Y_N^F(\omega)}{U_N^F(\omega)} \\ &= \frac{L_y(e^{j\omega})G(e^{j\omega})U_N(\omega) + L_y(e^{j\omega})R_N^y(\omega) + L_y(e^{j\omega})V_N(\omega) + R_N^{yf}(\omega)}{L_u(e^{j\omega})U_N(\omega) + R_N^{uf}(\omega)}. \end{aligned}$$

Noting that $E\{V_N\} = 0$ we get

$$E\{\hat{G}_N^F(\omega)\} = \frac{L_y(e^{j\omega})G(e^{j\omega})U_N(\omega) + L_y(e^{j\omega})R_N^y(\omega) + R_N^{yf}(\omega)}{L_u(e^{j\omega})U_N(\omega) + R_N^{uf}(\omega)}.$$

As $N \to \infty$ we asymptotically have

$$\lim_{N\to\infty} \left| \frac{R_N^{uf}(\omega)}{U_N(\omega)} \right| = 0, \lim_{N\to\infty} \left| \frac{R_N^{y}(\omega)}{U_N(\omega)} \right| = 0 \text{ and } \lim_{N\to\infty} \left| \frac{R_N^{yf}(\omega)}{U_N(\omega)} \right| = 0,$$

and thus

$$\lim_{N\to\infty} E\{\hat{G}_N^F(\omega)\} = \frac{L_y(e^{j\omega})}{L_u(e^{j\omega})}G(e^{j\omega}).$$

Note that if we select $L_y(\omega) = L_u(\omega)$ we get an unbiased estimate of the transfer function. Any other choice will result in bias.

To compute the mean-square error of the ETFE we first note that (arguments of the transfer functions are omitted for brevity)

$$\hat{G}_N^F - G = \frac{1}{L_uU_N + R_N^{uf}} \Big((L_y - L_u)GU_N + L_yR_N^y + L_yV_N + R_N^{yf} - R_N^{uf}G \Big)$$

This means that $|\hat{G}^F - G|^2$ can be written as

$$\begin{aligned}
|\hat{G}_N^F - G|^2 = \frac{1}{|L_u U_N + R_N^{uf}|^2}\Big(& |(L_y - L_u) G U_N|^2 \\
& + 2\Re\{(L_y - L_u) G U_N \big(L_y R_N^y + L_y V_N + R_N^{yf} - R_N^{uf} G\big)\} \\
& + |L_y V_N|^2 \\
& + 2\Re\{L_y V_N \big(L_y R_N^y + R_N^{yf} - R_N^{uf} G\big)\} \\
& + |L_y R_N^y + R_N^{yf} - R_N^{uf} G|^2 \Big)
\end{aligned}$$

Again noting that $E\{V_N\} = 0$ we have

$$\begin{aligned}
E\{|\hat{G}_N^F - G|^2\} = \frac{1}{|L_u U_N + R_N^{uf}|^2}\Big(& |(L_y - L_u) G U_N|^2 \\
& + 2\Re\{(L_y - L_u) G U_N \big(L_y R_N^y + R_N^{yf} - R_N^{uf} G\big)\} \\
& + |L_y|^2 E\{|V_N|^2\} \\
& + |L_y R_N^y + R_N^{yf} - R_N^{uf} G|^2 \Big)
\end{aligned}$$

Asymptotically as $N \to \infty$, we note that $\lim_{N\to\infty} E\{\frac{1}{N}|V_N|^2\} = \Phi_v$ (see solution to exercise 2) and we have

$$\begin{aligned}
\lim_{N\to\infty} E\{|\hat{G}_N^F - G|^2\} = \lim_{N\to\infty} \frac{1}{|L_u U_N + R_N^{uf}|^2}\Big(& |(L_y - L_u) G U_N|^2 \\
& + 2\Re\{(L_y - L_u) G U_N \big(L_y R_N^y + R_N^{yf} - R_N^{uf} G\big)\} \\
& + |L_y|^2 E\{|V_N|^2\} \\
& + |L_y R_N^y + R_N^{yf} - R_N^{uf} G|^2 \\
& = \frac{1}{|L_u U_N|^2}\Big(|(L_y - L_u) G U_N|^2 \\
& + |L_y|^2 \Phi_v \\
& = \frac{|L_y - L_u|^2}{|L_u|^2}|G|^2 + \frac{|L_y|^2}{|L_u|^2}\frac{\Phi_v}{|U_N|^2}
\end{aligned}$$

Note that the choice of $L_y(\omega) = L_u(\omega)$ gives the asymptotic variance as the noise/signal power spectrum, which is the same as the unfiltered case. If we have a very high noise level at a particular frequency then there may be an advantage to choosing $L_y(\omega)$ to filter out this noise. This is the case if $|L_y - L_u|^2|G|^2 < \Phi_v$. In this case we will get a bias error but the overall mean-square error may still be lower.

MATLAB exercise:

Consider the discrete time system,

$$G(z) = \frac{0.1z}{(z^2 - 1.7z + 0.72)(z^2 - 0.98z + 0.9)},$$

and noise model,

$$H(z) = \frac{0.5(z - 0.9)}{(z - 0.25)}.$$

The sample time can be taken as 1 second. The noise system, $H(z)$, is driven by a $N(0, 1)$ white noise, and the measured output is the sum of the outputs of $G(z)$ and $H(z)$.

a) Generate a 1024 point MATLAB simulation for an identification "experiment."

b) Estimate $G(z)$ via the unsmoothed ETFE and compare it to the true system; particularly in the frequency range around the resonant peak. Plot both the transfer functions and the magnitude of the errors.

c) Split the data record in 4 and calculate a "smoothed" ETFE by averaging the 4 ETFEs. Compare this to the true system as well as the original 1024 point ETFE. To compare these, calculate the Mean Square Error (MSE) in each case (making sure to normalize for the different data lengths). Again plot the transfer functions and the magnitude of the errors.

We will also look at the effect of windowing the data by smoothing in both the time and frequency domain. To achieve this write two MATLAB functions to calculate a Hann window response in both the time and frequency domains:

`WHfdom(gamma,omega)`	frequency domain Hann window
`WHtdom(gamma,tau)`	time domain Hann window

where `gamma` is the width parameter.

d) Plot the frequency response of the Hann window for $\gamma = 5$, 10, 50 and 100, on a frequency grid of 1024 points.

e) Plot the time domain weight for the same values of γ.

f) Calculate the smoothed ETFE for each of these values of γ. Do this in both the time and frequency domains to check that your functions are working correctly.

g) Experiment and find a value of γ that gives the best resolution of the resonant peak.

Solution hints

In this section we give some hints on how to solve the problem and provide the plots that are generated by a correct solution of the problem.

a) You should first generate the noise (e) and input (u) signals of appropriate length as randomly distributed vectors by using MATLAB command *randn*. Transfer functions $G(z)$ and $H(z)$ can be defined by using function *tf*. The noise free signal y' and the colored noise v can then be obtained by using the *lsim* function with the transfer functions $G(z)$ and $H(z)$ and the vectors u and e as appropriate inputs. Finally the measured output can be generated by adding together y' and v.

b) You can calculate the DFT of the input and output signals ($U_N(\omega)$ and $Y_N(\omega)$) by applying MATLAB *fft* function to the vectors u and y. The ETFE is then obtained by point-wise division of $Y_N(\omega)$ by $U_N(\omega)$ (./ in MATLAB). Note that due to the periodicity of the DFT, you actually need to take care of just half of the points in $U_N(\omega)$ and $Y_N(\omega)$. You can calculate the frequency response of the true systems $G(z)$ and $H(z)$ for the same frequencies at which the DFT of u and y is calculated by using the function *freqresp*. By using the *loglog* command you can plot the amplitude of the estimate and the actual frequency responses in logarithmic scale for both axes. The result should be similar as the one shown in Fig. 03.1. As can be seen the estimate is quite noisy around the resonant peak, which is why additional smoothing is needed in order to improve the estimate for this frequency range which is of particular interest.

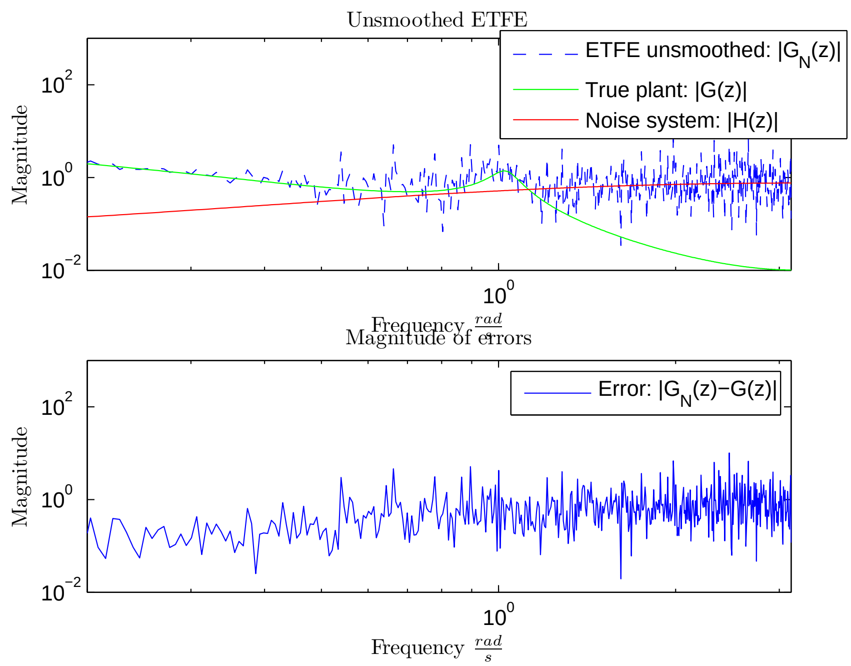

Figure 03.1: Comparison of the unsmoothed ETFE with the actual frequency response.

c) You can form 4 new input and output signals (that we denote u_i and y_i, $i = 1, 2, 3, 4$) by splitting the initial vectors u and y into 4 vectors of same length. For each i you can then repeat the steps from part b) in order to calculate the ETFE $\hat{G}_i, i = 1, 2, 3, 4$. The smoothed estimate is calculated by taking the mean over these 4 estimates that can be done by using the *mean* function. Note that the ETFE is in this case calculated for different frequency values than in b) since different data lengths are used. By using the same procedure for plotting the results as in b) one should obtain a result similar to the one shown in Fig. 03.2. As can be seen the estimate is less noisy this time and the estimate of the resonant peak is improved.

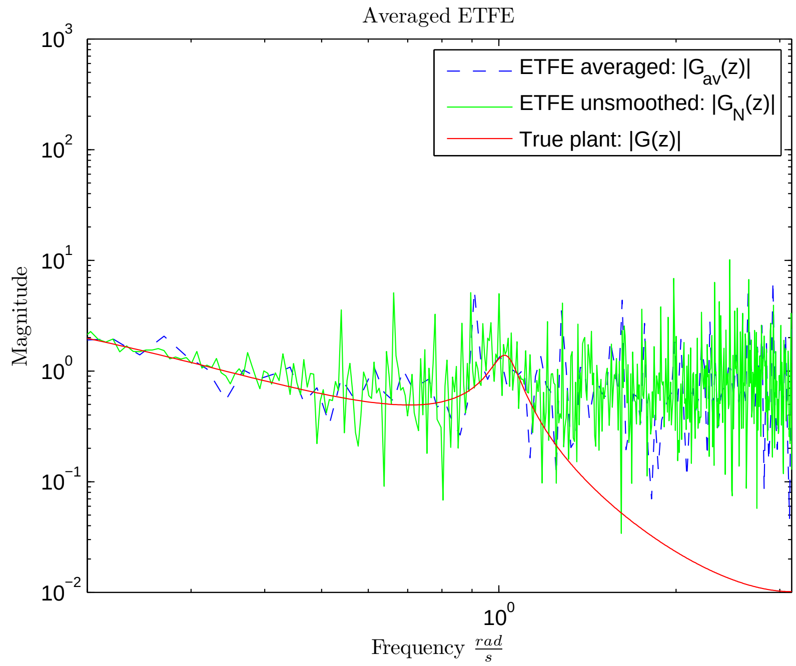

Figure 03.2: Comparison of the averaged and the unsmoothed ETFE with the actual frequency response.

d) First form the vector of desired frequencies (symmetric and centered at zero) at which the Hann window should be calculated (this vector should have the length of 1024) then use the function *WHfdom* in order to calculate the window for the given values of γ.

e) Similar as for d) form the vector of time domain points (symmetric and centered at zero) at which the window should be calculated (this vector should have the length of 1024) and then use *WHtdom* in order to calculate the window for the different values of γ. Obtained windows in time and frequency domain are shown in Fig. 03.3. As can be seen bigger value of γ results in a narrower window in frequency domain and a wider window in time domain.

f) For smoothing in the frequency domain, one can start from the unsmoothed ETFE calculated using the full data length as in b). Then this estimate can be smoothed by calculating the appropriate integrals (see slides for the exact formulas). These integrals can be calculated in MATLAB by using the finite sum approximation. For the time domain smoothing, one needs first to calculate the input signal autocorrelation and the input-output crosscorrelation. These vectors should then be multiplied with the time domain windows and their DFTs should be calculated in order to obtain the input spectrum and the cross spectrum. By point-wise dividing the two vectors, we obtain the smoothed ETFE. Fig. 03.4 shows the obtained smoothed ETFEs. As can be seen, the estimates for $\gamma = 10$ and $\gamma = 5$ overly smooth the ETFE and

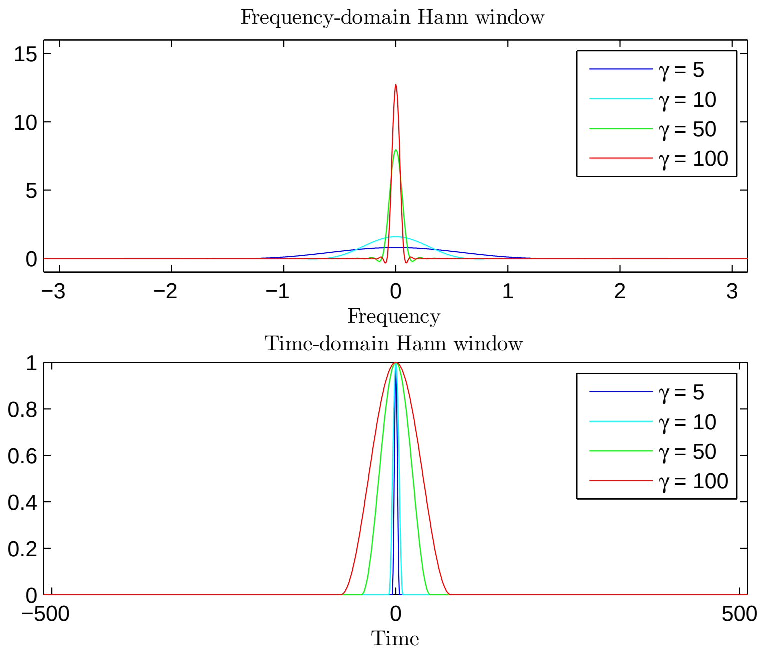

Figure 03.3: Hann windows in frequency and time domain.

introduce a large bias. The estimates with $\gamma = 50$ and $\gamma = 100$ provide a reasonable smoothing/bias trade-off and therefore the search for the best value of γ (see point g)) should be done in this range.

g) By repeating the procedure described in g) for different values of γ between 50 and 100, we find that the best estimate of the resonance peak is obtained for $\gamma = 80$. Comparison of the actual resonant peak and its smoothed estimate is shown in Fig. 03.5.

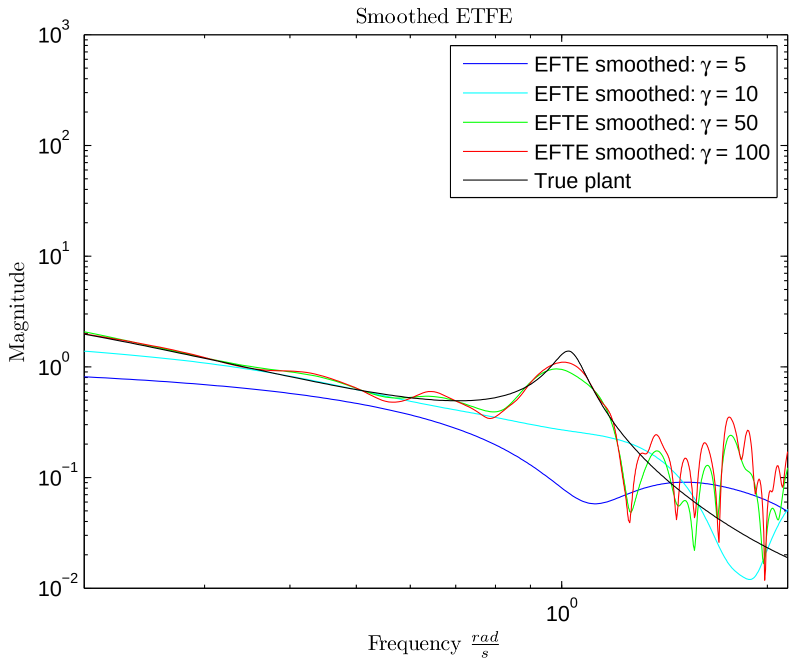

Figure 03.4: Comparison of the windowed ETFE (for different γ) with the actual frequency response.

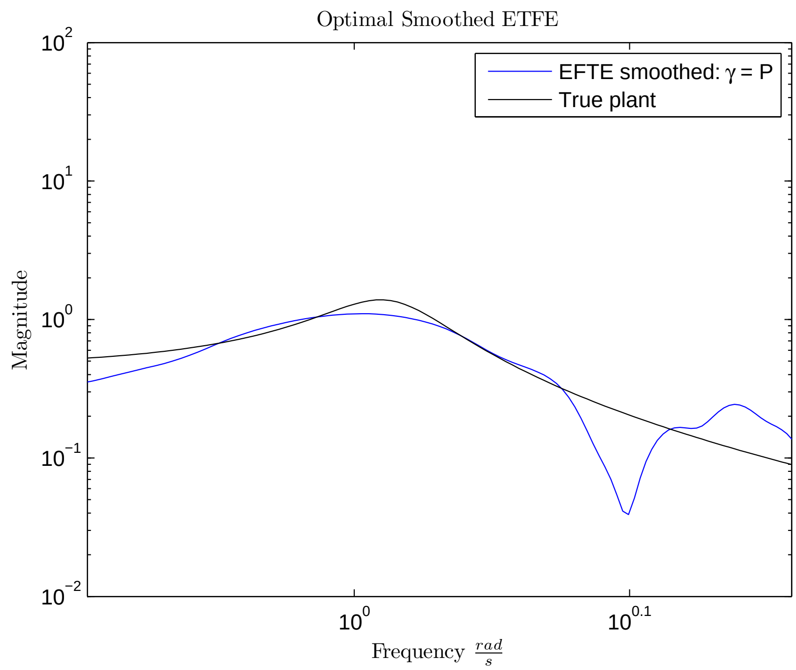

Figure 03.5: Optimal smoothed estimate of the resonant peak compared with the actual frequency response.