227-0689-00L System Identification **26 November 2018**

Exercise 9: Instrumental variable methods

Background reading

The background material for this exercise is Sections 7.5 and 7.6 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Suppose that a true description of a certain system is given by

$$y(k) + a_1 y(k-1) + ... + a_{n_a} y(k-n_a) = b_1 u(k-1) + ... + b_{n_b} u(k-n_b) + e(k)$$

where $\{e(k)\}$ is white noise independent of the input. Let the regressor $\phi(k)$ be defined as usual as

$$\phi(k) = [-y(k-1) \ldots - y(k-n_a) \quad u(k-1) \ldots u(k-n_b)]^T .$$

Moreover, let $\tilde{\phi}(k)$ be given by

$$\tilde{\phi}(k) = [-y_0(k-1) \ldots - y_0(k-n_a) \quad u(k-1) \ldots u(k-n_b)]^T ,$$

where

$$y_0(k) + a_1 y_0(k-1) + ... + a_{n_a} y_0(k-n_a) = b_1 u(k-1) + ... + b_{n_b} u(k-n_b) .$$

Note that y_0 is the noise-free response of the true system.

Prove that for the vector of instrumental variables $z(k) = [u(k-1)\, u(k-2)]^T$ we have

$$\mathbb{E}[z(k)\phi(k)^T] = \mathbb{E}[z(k)\tilde{\phi}(k)^T] .$$

Problem 2:

Consider the following system:

$$y(k) + ay(k-1) = b_1 u(k-1) + b_2 u(k-2) + v(k).$$

Parameters of the system should be estimated by using the instrumental variables method. To this end, it has been decided to use delayed inputs as instruments:

$$z(k) = [u(k-1)\, u(k-2)\, u(k-3)]^T.$$

Assuming that $u(k)$ is white noise with zero mean and unit variance and that it is uncorrelated with $v(k)$, find for which values of parameters a, b_1 and b_2, the instrumental variables are correlated with the regression variables (i.e. $E\{z(k)\varphi^T(k)\}$ is nonsingular).

Matlab exercise:

The dataset 'Data_ex9.mat' needed for this exercise can be downloaded from the resources page on Piazza.

Consider data, $y(k)$ and $u(k)$, collected from the system,

$$y(k) = \frac{B(z)}{A(z)}u(k) + C(z)e(k), \quad e(k) \sim \mathcal{N}(0, \lambda)\,.$$

The polynomials are of the form:

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2}\,, \\ B(z) &= b_1 z^{-1}\,, \\ C(z) &= 1 + c_1 z^{-1}\,. \end{aligned}$$

For the questions below, the following experimental data is provided:

- `ex9_u`: the input signal, $u(k)$, for $k = 1, 2, \ldots, K$,
- `ex9_y`: the corresponding output signal, $y(k)$, for $k = 1, 2, \ldots, K$.

The system is at rest and that there is no noise for $k \leq 0$,

$$u(k) = y(k) = e(k) = 0 \text{ for all } k \leq 0\,.$$

1. Using pseudo-linear regression (PLR) over the entire data, estimate the problem parameters $\hat{\theta}_{\text{PLR}} = [\hat{a}_1\ \hat{a}_2\ \hat{b}_1\ \hat{c}_1]^T$.
   (a) Formulate a pseudo-linear regression that, when solved, estimates the parameters $\hat{\theta}_{\text{PLR}}$. Express the one-step-ahead prediction estimator in terms of the solution to the pseudo-linear regression.
   (b) For the given input and output sequence, `ex9_u` and `ex9_y`, solve the above problem for the estimated parameters $\hat{\theta}_{\text{PLR}}$. Generate the resulting vector of prediction errors, $\epsilon(k) = y(k) - \hat{y}(k|\hat{\theta}_{\text{PLR}})$, $k = 1, \ldots, K$.
2. Use an instrumental variable (IV) method to estimate $\hat{\theta}_{\text{IV}} = [\hat{a}_1\ \hat{a}_2\ \hat{b}_1]^T$. Starting from a least-squares initialization, compute suitable instruments $\zeta(k)$.
   (a) Describe your choice of instrumental variables, $\zeta(k)$.
   (b) Estimate $\hat{\theta}_{\text{IV}}$ using your chosen $\zeta(k)$.