227-0689-00L System Identification	**26 September 2018**

Exercise 1: Spectra and periodograms

Background reading

The background material for this exercise is Sections 2.1, 2.2, 2.3, and 6.2 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999). You should also read Chapter 1 for a general overview of the topic.

Problem 1:

Consider the sinusoidal correlation function

$$I_S(N) = \frac{1}{N} \sum_{k=0}^{N-1} y(k) \sin(\omega_u k)$$

applied to a linear system, $y = Gu + v$, with sinusoidal input $u(k) = \alpha \cos(\omega_u k)$ and zero-mean Gaussian noise v.

- Show that

$$\lim_{N \longrightarrow \infty} E\left\{I_S(N)\right\} = \frac{-\alpha}{2} \left|G(e^{j\omega_u})\right| \sin\left(\arg\left(G(e^{j\omega_u})\right)\right).$$

- What can you say about the variance of $I_S(N)$ as N goes to infinity, given that v is a white noise, i.e., $\{v(k)\}$ is a sequence of independent Gaussian random variables with zero mean and variance one?

Hint: For any non-zero $\omega \in (-\pi, \pi)$ and any ϕ, we have

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sin(\omega k + \phi) = 0.$$

Problem 2:

Let $\{u(k)\}$ be a stationary stochastic process with $R_u(\tau) = E\{u(k)u(k-\tau)\}$. Also, let $\phi_u(\omega)$ its spectrum. Assume that

$$\sum_{\tau=1}^{\infty} |\tau R_u(\tau)| < \infty.$$

Let $U_N(\omega)$ be defined as $U_N(\omega) = \sum_{k=1}^{N} u(k)e^{-j\omega k}$. Prove that

$$\lim_{N \longrightarrow \infty} E\{\frac{1}{N}|U_N(\omega)|^2\} = \phi_u(\omega).$$

MATLAB **exercises:**

1. Write a MATLAB function that takes an input vector, $u(k)$, $k = 0, \ldots, N-1$ and returns the vector, $U_N(e^{j\omega_n})$, for
$$\omega_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N-1.$$
The functional relationship between $u(k)$ and $U_N(e^{j\omega})$ is given by,
$$U_N(e^{j\omega}) = \sum_{k=0}^{N-1} u(k)\, \mathrm{e}^{-j\omega k}, \quad (j = \sqrt{-1}).$$
This looks like an FFT and you may use an `fft` call in your function. See the MATLAB notes section at the end of these exercises for caveats.

2. Write a MATLAB script file (.m file) which performs the following calculations:
   a) Generate $e(k)$, a 1024 point $\mathcal{N}(0, 1)$ distributed random sequence.
   b) Calculate and plot (on a log-log scale) the periodogram of $e(k)$. The periodogram is defined as $\frac{1}{N}|E_N(e^{j\omega})|^2$.
   c) Given a discrete-time plant,
   $$P(z) = \frac{1}{z^2 - 0.9z + 0.5},$$
   calculate $w(k)$, the response of $P(z)$ to the input signal, $e(k)$. Assume that the sample time of $P(z)$ is specified as $T = 1$ second.
   d) Calculate the periodogram of $w(k)$.
   e) How is the periodogram of $w(k)$, $\frac{1}{N}|W_N(e^{j\omega})|^2$, related to $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|$ (asymptotically) ? Here $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|$ is the magnitude of the Bode plot of $P(z)$. Provide a plot comparing these quantities. To do this, plot both $\frac{1}{N}\left|W_N(e^{j\omega})\right|^2$ and $\left|P\left(\mathrm{e}^{j\omega_n}\right)\right|^2$ in the same figure. You should also plot $\frac{1}{N}\left|W_N(e^{j\omega})\right|^2 - \left|P\left(\mathrm{e}^{j\omega_n}\right)\right|^2$ to examine the error between the two quantities.
   f) Repeat the above for 2048 and 4096 length sequences. Plot all of your comparisons and see if your assertion in part e) is correct.

MATLAB **notes**

Use the commands `help fft` and `help periodogram` to get more information about these functions. Note that the MATLAB calcuation of the DFT (via the FFT algorithm) differs significantly from Ljung's definition. The scale factor, indexing and sign of the exponent are all different. You can also use Ljung's definition but you will have to adjust the answer.

The `periodogram` function in the signal processing toolbox actually estimates the periodogram. It zero pads the data and introduces window functions in its calculation of the estimate. It does not do the calculation given in Ljung for the periodogram.

The functions `rand` and `randn` give different distributions. Make sure that you use the correct one.