227-0689-00L System Identification **12 October 2018**

Solution 2: Empirical Transfer Function Estimation

We will not provide solutions for the MATLAB parts of the exercises. These will instead be discussed in the exercises sessions.

Problem 1:

Consider an experiment on the LTI system $G\left(e^{j\omega}\right)$ given by

$$y(k) = G\left(e^{j\omega}\right) u(k) + v(k),$$

where v is a noise signal. The ETFE $\hat{G}_N\left(e^{j\omega}\right)$ is unbiased if transients are neglected (cf. Lecture 3 and Lemma 6.1 in Ljung (1999)). This does not imply that $|\hat{G}_N\left(e^{j\omega}\right)|$ is an unbiased estimate of $|G\left(e^{j\omega}\right)|$. Show that

$$E\left\{\left|\hat{G}_N\left(e^{j\omega}\right)\right|^2\right\} = \left|G\left(e^{j\omega}\right)\right|^2 + \frac{\phi_v\left(e^{j\omega}\right)}{\frac{1}{N}|U_N\left(e^{j\omega}\right)|^2}$$

asymptotically for large N with $\phi_v\left(e^{j\omega}\right)$ defined as the noise spectrum.

Solution

Neglecting transients we can expand $\hat{G}_N\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) + \frac{V_N\left(e^{j\omega}\right)}{U_N\left(e^{j\omega}\right)}$, such that

$$\begin{aligned}
E\left\{\left|\hat{G}_N\left(e^{j\omega}\right)\right|^2\right\} =& E\left\{\hat{G}_N\left(e^{j\omega}\right)\overline{\hat{G}_N\left(e^{j\omega}\right)}\right\}\\
=& E\left\{\hat{G}_N\left(e^{j\omega}\right)\hat{G}_N\left(e^{-j\omega}\right)\right\}\\
=& E\left\{\left(G\left(e^{j\omega}\right) + \frac{V_N\left(e^{j\omega}\right)}{U_N\left(e^{j\omega}\right)}\right)\left(G\left(e^{-j\omega}\right) + \frac{V_N\left(e^{-j\omega}\right)}{U_N\left(e^{-j\omega}\right)}\right)\right\}\\
=& \left|G\left(e^{j\omega}\right)\right|^2 + \frac{G\left(e^{j\omega}\right)}{U_N\left(e^{-j\omega}\right)} E\left\{V_N\left(e^{-j\omega}\right)\right\} +\\
& \frac{G\left(e^{-j\omega}\right)}{U_N\left(e^{j\omega}\right)} E\left\{V_N\left(e^{j\omega}\right)\right\} + \frac{E\left\{\left|V_N\left(e^{j\omega}\right)\right|^2\right\}}{\left|U_N\left(e^{j\omega}\right)\right|^2}
\end{aligned}$$

If we assume that $v(k)$ is zero-mean, then it follows that

$$E\left\{V_N\left(e^{-j\omega}\right)\right\} = E\left\{V_N\left(e^{j\omega}\right)\right\} = 0.$$

Since the periodogram of $v(k)$ is an asymptotically unbiased estimator of the spectrum $\phi_v\left(e^{j\omega}\right)$, such that

$$E\left\{\left|V_N\left(e^{j\omega}\right)\right|^2\right\} = N\phi_v\left(e^{j\omega}\right),$$

it follows that

$$E\left\{\left|\hat{G}_N\left(e^{j\omega}\right)\right|^2\right\} = \left|G\left(e^{j\omega}\right)\right|^2 + \frac{\phi_v\left(e^{j\omega}\right)}{\frac{1}{N}|U_N\left(e^{j\omega}\right)|^2}$$

asymptotically for large N. Note that this also holds for transients with $\hat{G}_N\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) + \frac{V_N\left(e^{j\omega}\right)}{U_N\left(e^{j\omega}\right)} + \frac{R_N\left(e^{j\omega}\right)}{U_N\left(e^{j\omega}\right)}$.

Problem 2:

a) With a fixed frequency $\omega_u = \frac{2\pi r}{N}$, the input $u(k) = \alpha\cos(\omega_u k)$ is applied to a linear time-invariant system and the output $y(k),\ k = 0, 1, \ldots, N-1$ is measured. In Problem Set 1 the sinsuoidal correlation function was defined as

$$I_S(N) := \frac{1}{N}\sum_{k=0}^{N-1} y(k)\sin(\omega_u k) \tag{2.1}$$

and it was shown that

$$I_S(N) \xrightarrow[N\to\infty]{} \frac{-\alpha}{2}|G(e^{j\omega_u})|\sin(\arg(G(e^{j\omega_u}))), \tag{2.2}$$

where the expectation operator was omitted. In the same way, it is possible to define and show the following:

$$I_C(N) := \frac{1}{N}\sum_{k=0}^{N-1} y(k)\cos(\omega_u k) \tag{2.3}$$

$$I_C(N) \xrightarrow[N\to\infty]{} \frac{\alpha}{2}|G(e^{j\omega_u})|\cos(\arg(G(e^{j\omega_u}))). \tag{2.4}$$

Using (2.2) and (2.4), suggest an estimate for $G(e^{j\omega})$ at frequency $\omega = \omega_u$, $\bar{G}(e^{j\omega_u})$.

b) Making use of the definition of the discrete Fourier transform of y

$$Y_N(e^{j\omega}) = \sum_{k=0}^{N-1} y(k)e^{-j\omega k} \tag{2.5}$$

show that

$$\bar{G}(e^{j\omega_u}) = \frac{2Y_N(e^{j\omega_u})}{N\alpha}. \tag{2.6}$$

c) Evaluate the discrete Fourier transform $U(e^{j\omega})$ of the input $u(k) = \alpha\cos(\omega_u k)$ at the frequency $\omega = \omega_u$ in order to show that equation (2.6) is a particular case of the empirical transfer function estimate.

Hint: you may make use of $\sum_{k=0}^{N-1} e^{j\frac{2\pi rk}{N}} = \begin{cases} N, & \text{if } r = 0, \\ 0, & \text{if } 1 \le r < N \end{cases}$

Solution

a) From equation (2.4) we notice that $I_C(N)$ tends to the real part of $\alpha\, G(e^{j\omega_u})/2$, when N goes to infinity. In the same way, from (2.2) we see that $I_S(N)$ tends to the imaginary part of $-\alpha\, G(e^{j\omega_u})/2$. Then it is natural to estimate

$$\bar{G}(e^{j\omega_u}) = \frac{I_C(N) - jI_S(N)}{\alpha/2}\,. \tag{2.7}$$

b) By manipulating the Fourier transform (2.5) evaluated at $\omega = \omega_u$ we obtain

$$\begin{aligned} Y_N(e^{j\omega_u}) &= \sum_{k=0}^{N-1} y(k)e^{-j\omega_u k} = \sum_{k=0}^{N-1} y(k)\cos(\omega_u k) - j\sum_{k=0}^{N-1} y(k)\sin(\omega_u k) \\ &= N(I_C(N) - jI_S(N)) = \frac{\alpha N\bar{G}(e^{j\omega_u})}{2} \end{aligned} \tag{2.8}$$

where in the third equality we inserted the definitions (2.3) and (2.1) and the last equality follows from the estimate (2.7). Then (2.8) can be read as (2.6).

c) Let us evaluate the discrete Fourier transform $U_N(e^{j\omega})$ of the input $u(k) = \alpha\cos(\omega_u k)$ at the frequency $\omega = \omega_u$:

$$\begin{aligned} U_N(e^{j\omega_u}) &= \alpha\sum_{k=0}^{N-1}\cos(\omega_u k)e^{-j\omega_u k} = \alpha\sum_{k=0}^{N-1}\frac{e^{j\omega_u k} + e^{-j\omega_u k}}{2}e^{-j\omega_u k} \\ &= \frac{\alpha}{2}\left(\sum_{k=0}^{N-1} e^{j\omega_u k}e^{-j\omega_u k} + \sum_{k=0}^{N-1} e^{-j\omega_u k}e^{-j\omega_u k}\right) = \frac{\alpha}{2}\left(\sum_{k=0}^{N-1} 1 + \sum_{k=0}^{N-1} e^{-j2\omega_u k}\right) \\ &= \frac{\alpha N}{2} + \frac{\alpha}{2}\underbrace{\sum_{k=0}^{N-1} e^{-j\frac{2\cdot 2\pi rk}{N}}}_{=0 \text{ (see hint)}} = \frac{\alpha N}{2} \end{aligned}$$

By using $U_N(e^{j\omega_u}) = \frac{\alpha N}{2}$ we can rewrite (2.6) as

$$\bar{G}(e^{j\omega_u}) = \frac{Y_N(e^{j\omega_u})}{U_N(e^{j\omega_u})}\,, \tag{2.9}$$

which corresponds to the empirical transfer function estimate evaluated at the frequency $\omega = \omega_u$.

MATLABexercise:

Consider the discrete time system, $G(z)$, and noise model, $H(z)$,

$$G(z) = \frac{0.1z}{(z^2 - 1.7z + 0.72)}, \quad H(z) = 1.5\,\frac{z - 0.92}{z - 0.5}.$$

The measured system output, $y(k)$, defined as the sum of the outputs of $G(z)$ and $H(z)$, such that

$$y(k) = Gu(k) + He(k),$$

where the noise signal $v(k) = He(k)$ is driven by Gaussian white noise $e \sim \mathcal{N}(0, 0.01)$. The sample time can be taken as 1 s.

a) Generate a random periodic input signal u of length $L = r \cdot M$ where r is the number of periods and M the number of samples per period. The input over each period can be taken as Gaussian white noise $\mathcal{N}(0, 4)$. In a first "experiment", generate output data y for a periodic input u of $r = 5$ periods with $M = 1024$.

b) Compute the autocorrelation of the input u. Plot the autocorrelation over an appropriate number of lags to confirm the number of periods r that u contains.

c) Next, we want to construct the ETFE $\hat{G}_N(e^{j\omega})$ from the output data of the "experiment". To do so, average the system input and output over $r - 1$ periods where the data from the first period should be discarded to reduce transient effects in the identification. Calculate the ETFE $\hat{G}_N\left(e^{j\omega}\right)$ from the averaged data. Plot $|\hat{G}_N(e^{j\omega})|$ and $|G(e^{j\omega})|$ in the frequency range $0 < \omega \leq \pi$ and compute the RMS error of the estimate $|\hat{G}_N(e^{j\omega})|$ over the same frequency range. Comment on the effect of averaging.

d) Repeat steps a) and c) for different input signals of periods r={2,5,10,20} and $M = 1024$. Comment on the accuracy of the ETFE $\hat{G}_N(e^{j\omega})$ over the frequency range $0 < \omega \leq \pi$ for increasing r. To explore the variation of the RMS error, repeat the "experiments" for each r several times.

Solution hints

General hint: Check Matlab **help** if not sure about any Matlab commands introduced in the tutorial exercises.

a) Generate the discrete time systems $G(z)$ and $(H(z)$ in Matlab as discrete time transfer functions using the command **tf**. Generate the Gaussian white noise vector e of length $L = r \cdot M$ using **randn**. To generate the periodic input vector also of Gaussian white noise, generate the first period of length M using **randn** and copy this sequence appropriately to obtain the periodic input vector e of length L. Define a time vector of length L and run the first experiment using **lsim** where you need to first compute the noise signal $v(k) = He(k)$ with **lsim** and add the resulting noise to the uncorrupted solution. The resulting input and output vectors for our case are shown in Fig. 2.1.

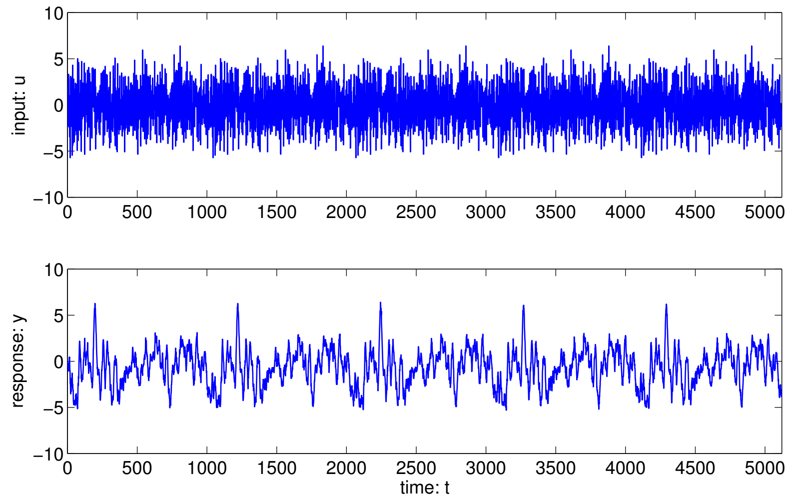

Figure 2.1: Experiment inputs u and output y for $r = 5$ and $M = 1024$.

b) In this exercise you should notice that the input signal is periodic and hence use the definition of the autocorrelation for periodic signals as given in slide 2.19. The the autocorrelation for periodic signals should be calculated over the lags $\tau = \{-L/2+1, ..., L/2\}$. You should try to implement both definitions yourself and then compare the results to what Matlab does in **xcorr** for the different options. Fig. 2.2 shows the autocorrelation using both definitions and we only show the autocorrelation for finite signals over lags $\tau = \{-L/2+1, ..., L/2\}$ for comparison.

c) Next, we average the input and output data. For the input signal this simply corresponds to the first period of u leading to the average input vector $\bar{u}$ of length M. The output signal is averaged over $r-1$ periods to arrive at an average output vector $\bar{y}$ also of length M, such that $\bar{y}(k) = \dfrac{1}{r-1} \sum_{i=2}^{r} y\left(k + M(i-1)\right)$. Note that we have neglected the first period ($r = 1$) to reduce effects of transients. As a result of averaging we have reduced the numbers of samples in the experiment to $N = M$.

   To calculate the ETFE $\hat{G}_N\left(e^{j\omega}\right)$ from the averaged data, follow the example in Slide 3.13 where **u** and **y** correspond to $\bar{u}$ and $\bar{y}$, respectively, and $N = M$. The resulting ETFE $\hat{G}_N\left(e^{j\omega}\right)$ is compared against the true plant $G_N\left(e^{j\omega}\right)$ in Fig. 2.3.

d) Next, we repeat the experiment for increasing number if periods $r = \{2, 5, 10, 20\}$. Figure 2.4 shows an improvement of the ETFE $\hat{G}_N\left(e^{j\omega}\right)$ especially at higher frequencies with increasing r. Increasing the periods r cancels out the added noise v as we have generated the added noise as (zero-mean) Gaussian white noise and the noise is uncorrelated with the input u.

   The results in Fig. 2.4 are limited to one realisation of noise and input signals only. To demonstrate the convergence of the ETFE with statistically more relevant data, we have repeated the above experiments for $2 \geq r \geq 20$ 100 times for each r and presented the root-mean square (rms) error of $|\hat{G}_N\left(e^{j\omega}\right)|$ in Fig. 2.5 using Matlab **boxplot**. Check Matlab **help boxplot** to understand how to read the plot.

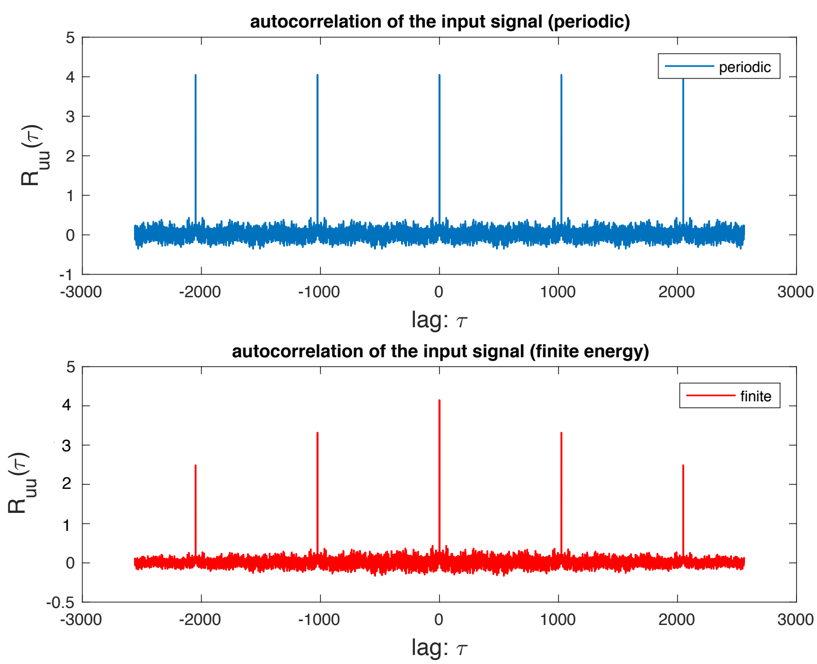

Figure 2.2: Autocorrelation of input signal u for periodic and finite signals.

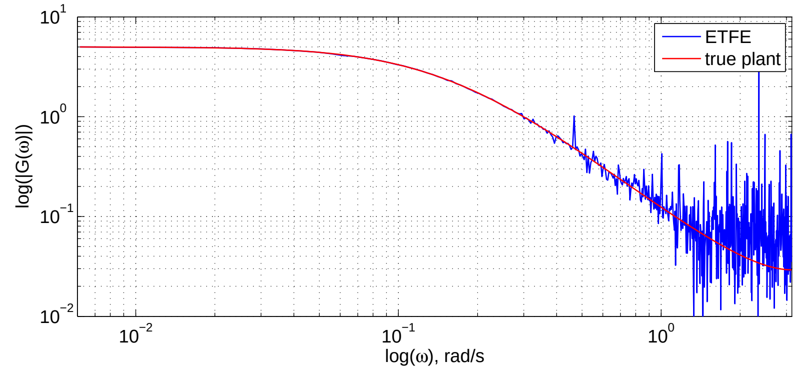

Figure 2.3: ETFE $\hat{G}_N\left(e^{j\omega}\right)$ compared against the true plant $G_N\left(e^{j\omega}\right)$ for averaged $\bar{u}$ and $\bar{y}$.

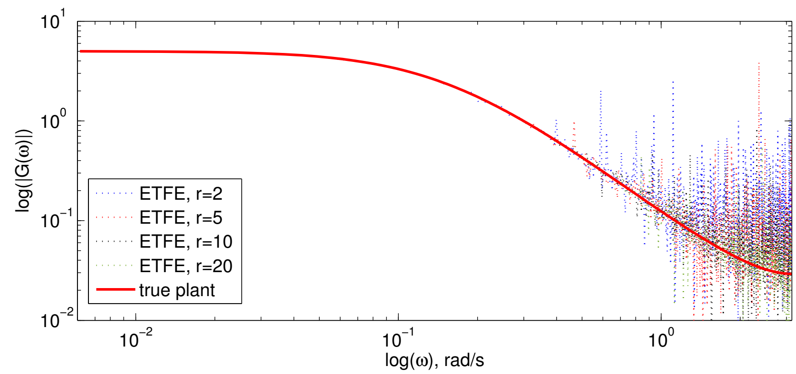

Figure 2.4: ETFE $\hat{G}_N\left(e^{j\omega}\right)$ compared against the true plant $G_N\left(e^{j\omega}\right)$ for averaged $\bar{u}$ and $\bar{y}$ for different input signals of periods $r = \{2, 5, 10, 20\}$.

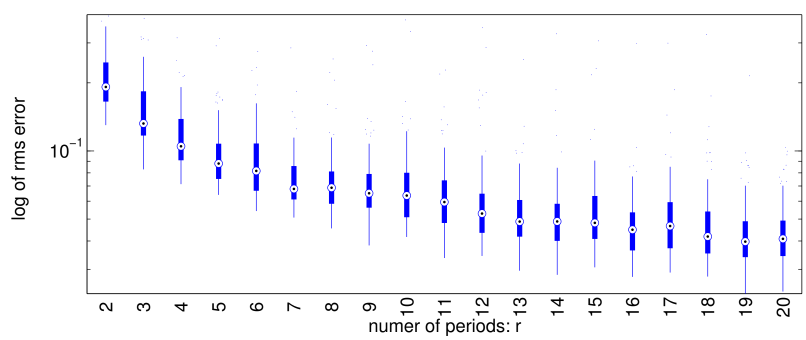

Figure 2.5: Statistical information of the rms error of $|\hat{G}_N\left(e^{j\omega}\right)|$ against $|G_N\left(e^{j\omega}\right)|$ for $2 \geq r \geq 20$ over 100 experiments for each r.