Exercise 8: Prediction error methods

Background reading

The background material for this exercise is Sections 3.2, 4.2, and 7.2 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1: Euler Approximation

Consider the continuous-time model given by

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-1}{\tau} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\beta}{\tau} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

where τ, $\beta > 0$.

Apply the bilinear transform defined for a general continuous signal $h(t)$ as

$$\frac{dh}{dt} \approx \frac{h(k+T) - h(k)}{T} = \frac{2}{T} \frac{z-1}{z+1} h(k),$$

where $z \in \mathbb{C}$ and $T > 0$ is the sampling period to obtain an approximation of the discrete-time transfer function $G(e^{j\omega})$ in dependency of $\theta = \begin{bmatrix} \tau & \beta \end{bmatrix}^\top$. Assume that a constant input $u(t)$ is applied at times $kT \leq t \leq kT + T$.

Problem 2: Optimal predictor for a first-order ARMAX model

Consider the model

$$y(k) + ay(k-1) = bu(k-1) + e(k) + ce(k-1),$$

where $|c| < 1$ and $e(k) \sim \mathcal{N}(0, \lambda^2)$. The parameter vector is given by $\theta = \begin{bmatrix} a & b & c \end{bmatrix}^\top$.

1. Show that an optimal predictor $\hat{y}(t|t-1,\theta)$ which minimizes the prediction error variance σ is given by
$$\hat{y}(k|k-1,\theta) = -ay(k-1) + bu(k-1) + ce(k-1).$$
2. What difficulties can be encountered when implementing the optimal predictor? How can this be avoided?
3. Find a practical recursive implementation as a function of the innovation of the optimal predictor.

Matlab exercises

Consider the ARMAX model

$$A(z)y(k) = B(z)u(k) + C(z)e(k),\ k = 1, \ldots, N,$$

where $N = 10^4$ with the matrices defined by

$$\begin{aligned} A(z) &= 1 - 1.5z^{-1} + 0.7z^{-2} \\ B(z) &= 1.0z^{-1} + 0.5z^{-2} \\ C(z) &= 1 - 1.0z^{-1} + 0.2z^{-2}, \end{aligned}$$

where $e(k) \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1)$. Fix an input sequence $u(k)$ from the following ARMAX process

$$u(k) = 0.1u(k-1) + 0.12\,u(k-2) + e_u(k-1) + 0.2\,e_u(k-2).$$

where $e_u \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1)$.

Assuming that the transfer function $C(z)$ is exactly known:

1. Obtain least-squares (LS) estimates $\hat{A}_{LS}(z)$ and $\hat{B}_{LS}(z)$ for $A(z)$ and $B(z)$ respectively.
2. Plot the predicted values along with the true response $y(k)$ for validation set.
3. Repeat (1) for a different set of realizations of $e(k)$ and plot the histograms of the parameters.