227-0689-00L System Identification **01 October 2018**

Exercise 2: Empirical Transfer Function Estimation

Background reading

The background material for this exercise is Sections 6.1, 6.2 and 6.3 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

Problem 1:

Consider an experiment on the LTI system $G\left(e^{j\omega}\right)$ given by

$$y(k) = G\left(e^{j\omega}\right) u(k) + v(k),$$

where v is a noise signal. The ETFE $\hat{G}_N\left(e^{j\omega}\right)$ is unbiased if transients are neglected (cf. Lecture 3 and Lemma 6.1 in Ljung (1999)). This does not imply that $|\hat{G}_N\left(e^{j\omega}\right)|$ is an unbiased estimate of $|G\left(e^{j\omega}\right)|$. Show that

$$E\left\{\left|\hat{G}_N\left(e^{j\omega}\right)\right|^2\right\} = \left|G\left(e^{j\omega}\right)\right|^2 + \frac{\phi_v\left(e^{j\omega}\right)}{\frac{1}{N}|U_N\left(e^{j\omega}\right)|^2}$$

asymptotically for large N with $\phi_v\left(e^{j\omega}\right)$ defined as the noise spectrum.

Problem 2:

a) With a fixed frequency $\omega_u = \frac{2\pi r}{N}$, the input $u(k) = \alpha\cos(\omega_u k)$ is applied to a linear time-invariant system and the output $y(k),\ k = 0, 1, \ldots, N-1$ is measured. In Problem Set 1 the sinsuoidal correlation function was defined as as

$$I_S(N) := \frac{1}{N}\sum_{k=0}^{N-1} y(k)\sin(\omega_u k) \tag{2.1}$$

and it was shown that

$$I_S(N) \xrightarrow[N\to\infty]{} \frac{-\alpha}{2}|G(e^{j\omega_u})|\sin(\arg(G(e^{j\omega_u}))), \tag{2.2}$$

where the expectation operator was omitted. In the same way, it is possible to define and show the following:

$$I_C(N) := \frac{1}{N}\sum_{k=0}^{N-1} y(k)\cos(\omega_u k) \tag{2.3}$$

$$I_C(N) \xrightarrow[N\to\infty]{} \frac{\alpha}{2}|G(e^{j\omega_u})|\cos(\arg(G(e^{j\omega_u}))). \tag{2.4}$$

Using (2.2) and (2.4), suggest an estimate for $G(e^{j\omega})$ at frequency $\omega = \omega_u$, $\bar{G}(e^{j\omega_u})$.

b) Making use of the definition of the discrete Fourier transform of y

$$Y_N(e^{j\omega}) = \sum_{k=0}^{N-1} y(k)e^{-j\omega k} \tag{2.5}$$

show that

$$\bar{G}(e^{j\omega_u}) = \frac{2Y_N(e^{j\omega_u})}{N\alpha}\,. \tag{2.6}$$

c) Evaluate the discrete Fourier transform $U(e^{j\omega})$ of the input $u(k) = \alpha\cos(\omega_u k)$ at the frequency $\omega = \omega_u$ in order to show that equation (2.6) is a particular case of the empirical transfer function estimate.

Hint: you may make use of $\sum_{k=0}^{N-1} e^{j\frac{2\pi rk}{N}} = \begin{cases} N, & \text{if } r = 0, \\ 0, & \text{if } 1 \leq r < N \end{cases}$

Matlab exercises:

Consider the discrete time system, $G(z)$, and noise model, $H(z)$,

$$G(z) = \frac{0.1z}{(z^2 - 1.7z + 0.72)}, \quad H(z) = 1.5\,\frac{z - 0.92}{z - 0.5}.$$

The measured system output, $y(k)$, defined as the sum of the outputs of $G(z)$ and $H(z)$, such that

$$y(k) = Gu(k) + He(k),$$

where the noise signal $v(k) = He(k)$ is driven by Gaussian white noise $e \sim \mathcal{N}(0, 0.01)$. The sample time can be taken as 1 s.

a) Generate a random periodic input signal u of length $L = r \cdot M$ where r is the number of periods and M the number of samples per period. The input over each period can be taken as Gaussian white noise $\mathcal{N}(0, 4)$. In a first "experiment", generate output data y for a periodic input u of $r = 5$ periods with $M = 1024$.

b) Compute the autocorrelation of the input u. Plot the autocorrelation over an appropriate number of lags to confirm the number of periods r that u contains.

c) Next, we want to construct the ETFE $\hat{G}_N(e^{j\omega})$ from the output data of the "experiment". To do so, average the system input and output over $r-1$ periods where the data from the first period should be discarded to reduce transient effects in the identification. Calculate the ETFE $\hat{G}_N\left(e^{j\omega}\right)$ from the averaged data. Plot $|\hat{G}_N(e^{j\omega})|$ and $|G(e^{j\omega})|$ in the frequency range $0 < \omega \leq \pi$ and compute the RMS error of the estimate $|\hat{G}_N(e^{j\omega})|$ over the same frequency range. Comment on the effect of averaging.

d) Repeat steps a) and c) for different input signals of periods r={2,5,10,20} and $M = 1024$. Comment on the accuracy of the ETFE $\hat{G}_N(e^{j\omega})$ over the frequency range $0 < \omega \leq \pi$ for increasing r. To explore the variation of the RMS error, repeat the "experiments" for each r several times.