227-0689-00L System Identification **22 October 2018**

Exercise 5: Experiment design, drifts and offsets

Background reading

The background material for this exercise is Sections 6.4, 3.2 and 13.3 of Ljung (*System Identification; Theory for the User*, 2nd Ed., Prentice-Hall, 1999).

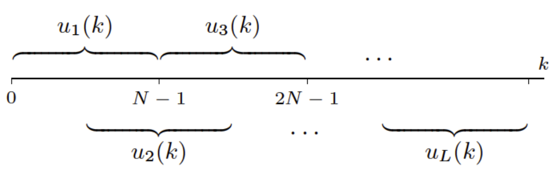
Problem 1:

Derive the spectrum of a PRBS signal with amplitude U and the maximal period length M.

Problem 2:

Consider an input signal $u(k)$ of length K. Using the Welch's method, apply the following time-domain windowing to $u(k)$.

First, split $u(k)$ into L overlapping segments of length N (with N being an odd number) $\{u_1(k), \ldots, u_L(k)\}$ as illustrated below. The segments overlap at 50%. Assume N is chosen so that all the data are split equally. That is, $\frac{N-1}{2}(L+1) = K$.



Then, window each segment $u_l(\cdot)$ with a symmetric *Bartlett window* $w_l(\cdot)$ centred at $k = \frac{(N-1)}{2}l$. Note that each $w_l(\cdot)$ is zero everywhere except for $k \in \left(\frac{N-1}{2}(l-1), \frac{N-1}{2}l\right)$. Hence, consider the windowed signal $u_w(\cdot)$ defined by

$$u_w(k) := \sum_{l=1}^{L} w_l(k)u(k) \qquad k = 0, \ldots, K-1\,.$$

Answer to the following questions.

(i) Derive an explicit expression for $u_w(k)$ as a function of k, $u(k)$, N, and L.

(ii) Assume $u(k) \geq 0 \, \forall k$. How does the difference $\sum_{k=0}^{K-1} u(k) - \sum_{k=0}^{K-1} u_w(k)$ depend on L and K?

(iii) Show that the energy signal $\sum_{l=1}^{L} |w_l(k)u(k)|^2$ satisfies

$$\sum_{k=\frac{N-1}{2}}^{\frac{N-1}{2}L} |u(k)|^2 \geq \sum_{k=\frac{N-1}{2}}^{\frac{N-1}{2}L} \sum_{l=1}^{L} |w_l(k)u(k)|^2 \geq \frac{1}{2} \sum_{k=\frac{N-1}{2}}^{\frac{N-1}{2}L} |u(k)|^2$$

Hint: you may use the fact that $\min_{x \in \mathbb{R}} x^2 + (1-x)^2 = 1/2$ and $\max_{x \in [0,1]} x^2 + (1-x)^2 = 1$.

Matlab **Exercise 1:**

Define the state-space matrices,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2\gamma \end{bmatrix}, \quad \text{with } \gamma > 0.$$

Now consider the signal, $y(k)$, calculated via the following iteration. We first initialize $x(0)$ with entries of 0 or 1, with at least one non-zero entry. Then, for a given number of iterations calculate,

$$\begin{aligned} z &= A\,x(k) \\ x(k+1) &= \mathrm{mod}_2(z) \qquad \text{where } \mathrm{mod}_2() \text{ is the remainder after division by 2.} \\ y(k) &= C\,x(k) - \gamma. \end{aligned}$$

The signal, $y(k)$, is periodic. What is its maximum possible period? (Hint: every component of $x(k)$ is either 0 or 1. If you get back to the $x(0)$ state after some period of time M, (i.e. $x(M) = x(0)$) then the sequence will repeat.)

Simulate this for at least twice the maximum period. Verify the smallest period, M, in your simulated data. Use $\gamma = 5$ for your simulations and plot $y(k)$.

Calculate an estimate of the autocorrelation function, $R_y(\tau)$ for $0 \leq \tau \leq M-1$. Use the knowledge that $y(k)$ is periodic in your calculation and don't smooth your estimate. Plot $\hat{R}_y(\tau)$ for $0 \leq \tau \leq M-1$.

We now want to compare this to other periodic signals in terms of the spectra. Generate 10 examples of the following signals:

1. $y(k)$ as above with 10 different, non-zero, initial conditions.
2. $e(k)$, a normally distributed signal saturated such that $\max_k |e(k)| = \gamma$ and having period M.
3. $w(k)$, a uniformly distributed signal on the interval $[-\gamma, \gamma]$, also having period M.

For each of the above estimate the spectra (via a non-smoothed estimate of $R_y(\tau)$, $R_e(\tau)$, and $R_w(\tau)$) and average each signal over all 10 instances to get the spectral estimates: $\hat{\Phi}_y(\omega)$, $\hat{\Phi}_e(\omega)$, $\hat{\Phi}_w(\omega)$. Plot these on the same graph and note how much energy each has.

Matlab **Exercise 2:**

Consider a normally distributed random signal, $u(k)$, with period 2^8. Suppose we measure $u(k)$ but our measurement is corrupted in two ways:

1. Offset error: $u_o(k) = \texttt{randn}(2^8) + \beta_o$.
2. Drift error: $u_d(k) = \texttt{randn}(2^8) + \alpha_d\,(k - 2^7)$.

We want to examine these effects in the spectral domain.

Generate 10 random instances of $u(k)$, $u_o(k)$, and $u_d(k)$ of length 2^8. Use $\beta_o = 0.1$ and $\alpha_d = 1/2^9$. For each instance calculate the DFTs, $U_N(\omega)$, $U_{oN}(\omega)$, and $U_{dN}(\omega)$. Average each over the 10 instances and compare each on the same plot. What is the effect of the offset and drift?

Note that both of these effects commonly occur in sensors.