Non isomorphic p-adic local fields with isomorphic additive and multiplicative groups

This short note is the fruit of some joint reflexion with Béranger Seguin from the following question asked by Bruno Deschamps during the *Journées Galoisiennes d'hiver 2024*. The initial question was the following

Does there exist two (possibly non-commutative) fields whose additive and multiplicative groups are isomorphic but which are not isomorphic as fields ?

While asking this question, Bruno had a non-commutative example in mind. Nonetheless, we began to think about this question to find a commutative example of such phenomenon. Our result is the following

Proposition 0.1. *For each prime p, there exists an explicit pair of p-adic fields (i.e. finite extension of $\mathbb{Q}_p$) whose additive groups are isomorphic, whose multiplicative groups are isomorphic, but which are not isomorphic as fields.*

The strategy is the following : for K and L two p-adic fields, we first notice that their additive groups and multiplicative groups are isomorphic if and only if $[K:\mathbb{Q}_p] = [L:\mathbb{Q}_p]$ and the cardinal of the roots of unity $\mu_\infty(K)$ and $\mu_\infty(L)$ are equal. We follow by finding an example where these equalities are satisfied without being isomorphic $\mathbb{Q}_p$-extensions. We conclude by proving that any isomorphism of fields between p-adic fields is an isomophism of $\mathbb{Q}_p$-extensions.

Remark 0.2. If both K and L contain a p-th root of unity, the two equalities characterising the multiplicative groups are also equivalent to an isomorphism between the absolute Galois group $\mathcal{G}_K \cong \mathcal{G}_L$. This is the main theorem of [JR79].

Lemma 0.3. *For K and L two p-adic fields, the groups $(K,+)$ and $(L,+)$ are isomorphic as soon as $[K:\mathbb{Q}_p] = [L:\mathbb{Q}_p]$.*

Proof. Both of them are $\mathbb{Q}_p$-vector spaces of the same dimension. □

Lemma 0.4. *For K and L two p-adic fields, the groups $(K^\times,\times)$ and $(L^\times,\times)$ are isomorphic if and only if*

$$[K:\mathbb{Q}_p] = [L:\mathbb{Q}_p] \text{ and } |\mu_\infty(K)| = |\mu_\infty(L)| \qquad (\star).$$

Proof. We use [Neu99, Chapter II, Proposition 5.7] stating that

$$K^\times \cong \mathbb{Z} \times G \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

where G is a finite group. Hence, G identifies with the groupe of finite order element in $K^\times$, i.e. $\mu_\infty(K)$. Hence we obtain the isomorphism provided that the two equalities are verified.

On the other direction, suppose that $K^\times \cong L^\times$ as groups. We can identify μ_∞ on both side by taking the torsion part. Moreover, after quotienting by the torsion part and taking reduction modulo p, we obtain an $\mathbb{F}_p$-vector space of dimension $[K:\mathbb{Q}_p]+1$ (resp. $[L:\mathbb{Q}_p]+1$) giving the equalities between the degrees. □

Lemma 0.5 (See part 2 in [JR79]). *The fields $L_1 = \mathbb{Q}_p(\zeta_p, \sqrt[p]{p})$ and $L_3 = \mathbb{Q}_p(\zeta_p, \sqrt[p]{\zeta_p - 1})$ verify $(\star)$ but are not isomorphic as $\mathbb{Q}_p$-extensions.*

Proof. Their degree are both $(p-1)p$.

The extension $L_1|\mathbb{Q}_p$ is normal with Galois group isomorphic to $\mathrm{Aff}(\mathbb{F}_p)$ whose abelianisation is $\mathbb{F}_p$. Hence $\mu_\infty(L_1) = \mu_p$.

The extension $L_3|\mathbb{Q}_p$ is not normal (see [JR79, §2, example (1)]). Hence $\mu_\infty(L_3) = \mu_p$.

The fact that L_1 is normal but not L_3 also shows that they are non isomorphic $\mathbb{Q}_p$-extensions. □

Theorem 0.6. *Any isomorphism of fields between p-adic fields is continuous.*

Proof. As they are henselian discretely valued, we can use [Rib99, Chapter 3, X].

A more elementary proof in the case of local fields can be found in [Mar24]. □

Corollary 0.7. *Any field isomorphism of p-adic fields is an isomorphism of $\mathbb{Q}_p$-extensions.*

Proof. Thanks to the previous theorem, such isomorphism is continuous and it fixes $\mathbb{Q}$. Henceforth, it fixes $\mathbb{Q}_p$. □

Another commutative example was suggested to me a few days later by Sylvain Rideau-Kikuchi.

Proposition 0.8. *The fields $\mathbb{Q}^{\mathrm{alg}}$ and $\mathbb{Q}(t)^{\mathrm{alg}}$ have isomorphic additive groups, isomorphic multiplicative groups, but are not isomorphic as fields.*

Proof. Their additive groups are isomorphic as they both are countable, infinite dimensionnal $\mathbb{Q}$-vector spaces.

For K being one of these fields, we have an exact sequence of abelian groups

$$1 \to \mu_\infty \to K^\times \to \mathbb{Q}^{\mathbb{N}} \to 1$$

where $\mu_\infty \cong \mathbb{Q}/\mathbb{Z}$ thanks to the field being algebraically. To identify the quotient, first remark that it is uniquely (because we quotiented by the roots of unity) divisible (because the field is algebraically closed) and countable hence a $\mathbb{Q}$-vector space of finite or countable dimension. It is not of finite dimension cause the classes of the primes numbers are a free family (use unicity of prime decomposition). Finally, the exact sequence is split as a section is given by chosing a compatible families of roots of lifts of any $\mathbb{Q}$-basis of the quotient. Hence the multiplicative groups are isomorphic.

They are not isomorphic as fields cause one of them is algebraic over $\mathbb{Q}$ and the other is not. □

References

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