Math 578 Lab A1

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This lab looks at matrix methods. The first section investigates the growth factor of Gaussian Elimination with Partial Pivoting defined by

$g(A) = \frac{max_{i,j}|u_{ij}|}{max_{i,j}|a_{ij}|}$

Where $PA = LU$

```python
import numpy as np
import matplotlib.pyplot as plt
import time

# Q2 Answer
def genMat(n,mu = 0,sigma = 1):
    return sigma*np.random.randn(n,n) + mu


# Q3 Function
def gelim(A):
    n = np.shape(A)[0]
    P = np.eye(n)
    L = np.eye(n).astype('float')
    U = A.astype('float')

    for i in range(n):

        # If all values are 0 there is no solution
        if all(U[i:n,i] == 0):
            print('No Solution')
            return None

        # Find the row with maximum subcolumn index
        p = np.argmax(abs(U[i:n,i]))
        p = p + i # Max row index

        # Swap rows, track permutations
        U[[i,p]] = U[[p,i]]
        P[[i,p]] = P[[p,i]]
        L[[i,p],0:i] = L[[p,i],0:i]


        # Reduce, track reduction
        for j in range(i+1,n):
            m = U[j,i]/U[i,i]
            L[j,i] = m
            U[j] = U[j,:] - m*U[i,:]


    return P,L,U


# Test if PLU Decomp worked
A = np.array([[1,-1,3],
             [4,-2,1],
             [-3,-1,4]
             ])

P,L,U = gelim(A)  # A matrix I found online
print('LU')
print(L.dot(U))
print('PA')
print(P.dot(A))


A = genMat(4) # works for some dense matrix that is likely full rank
P,L,U = gelim(A)
print('LU')
print(L.dot(U))
print('PA')
print(P.dot(A))


A = (np.eye(5)) # Should get the identity (ie nothing happens)
P,L,U = gelim(A)
print('LU')
print(L.dot(U))
print('PA')
print(P.dot(A))
```

```
LU
[[ 4. -2.  1.]
 [-3. -1.  4.]
 [ 1. -1.  3.]]
PA
[[ 4. -2.  1.]
 [-3. -1.  4.]
 [ 1. -1.  3.]]
LU
[[ 1.29715638  0.21443491  1.30809353  0.09120403]
 [ 0.41804249  2.65492307 -0.34584008  0.83629517]
 [ 0.54017645 -1.14161729 -0.42793014  0.5206604 ]
 [-0.14198159 -0.67538617 -1.11255036 -0.77442825]]
PA
[[ 1.29715638  0.21443491  1.30809353  0.09120403]
 [ 0.41804249  2.65492307 -0.34584008  0.83629517]
 [ 0.54017645 -1.14161729 -0.42793014  0.5206604 ]
 [-0.14198159 -0.67538617 -1.11255036 -0.77442825]]
LU
[[ 1.  0.  0.  0.  0.]
 [ 0.  1.  0.  0.  0.]
 [ 0.  0.  1.  0.  0.]
 [ 0.  0.  0.  1.  0.]
 [ 0.  0.  0.  0.  1.]]
PA
[[ 1.  0.  0.  0.  0.]
 [ 0.  1.  0.  0.  0.]
 [ 0.  0.  1.  0.  0.]
 [ 0.  0.  0.  1.  0.]
 [ 0.  0.  0.  0.  1.]]
```

Ok, we have now implemented PLU decomposition and verified it worked on a number of matricies.

Now let's do Question 4 and plot the growth factor as a function of matrix size. For each value of n we will average the growth factor for a number of matricies. Computing power is limiting here so we will generate 7 random matricies for each value of n. This is about how long it took me to go for a 8k run and make lunch, which seems a reasonably generous number of matricies.

First, implement a function that returns the growth factor.

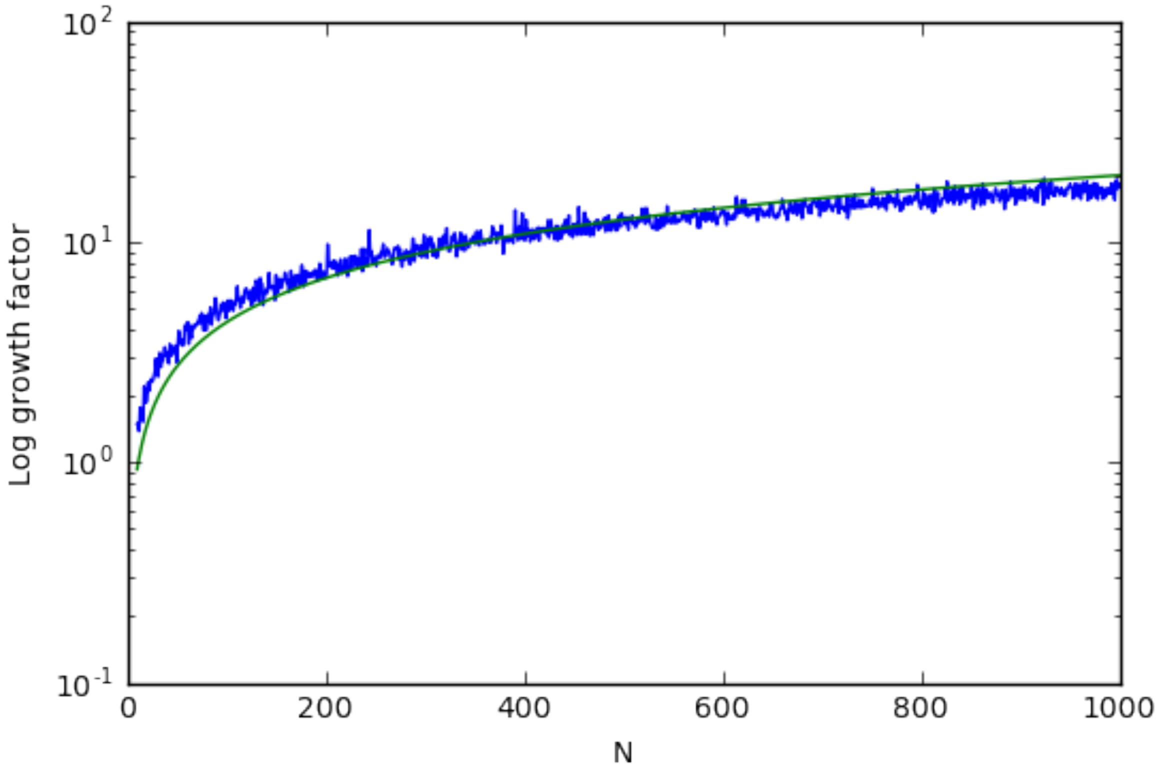
```python
def growth(A,U):
    maxA = np.max(abs(A))
    maxU = np.max(abs(U))

    return maxU / maxA

growthN = np.zeros(1001)
```

```
start = time.time()
for n in range(1000,10,-1):
    print('{}/{}'.format(n,990))
    ga = []
    for k in range(7):
        A = genMat(n)
        P,L,U = gelim(A)
        ga.append(growth(A,U))

    growthN[n] = (np.mean(ga))

t = time.time() - start
np.save('growthFile_2',growthN)
print('Time taken (m): ' + str(t/60))
```

```
Time taken (m): 72.37368076642355
```

Running the above code takes a while. So we limit our number of samples to 7. We plot seperately to avoid re-running the code too often.

```
growthN = growthN[growthN != 0] # Get rid of the zeros from initialization

a = 2/3
c = 0.2

fit = [c*n**a for n in range(10,1000)]

plt.plot(range(10,1000),growthN)
plt.plot(range(10,1000),fit)
plt.xlabel('N')
plt.ylabel('Log growth factor')
plt.yscale('log')
plt.show()
```



When assuming $g \approx cn^{\alpha}$ this experiment suggests that when doing Gaussian elimination with partial pivoting on matricies with values iid sampled from random standard normal distributions that the growth factor grows roughly as a polynomial of degree

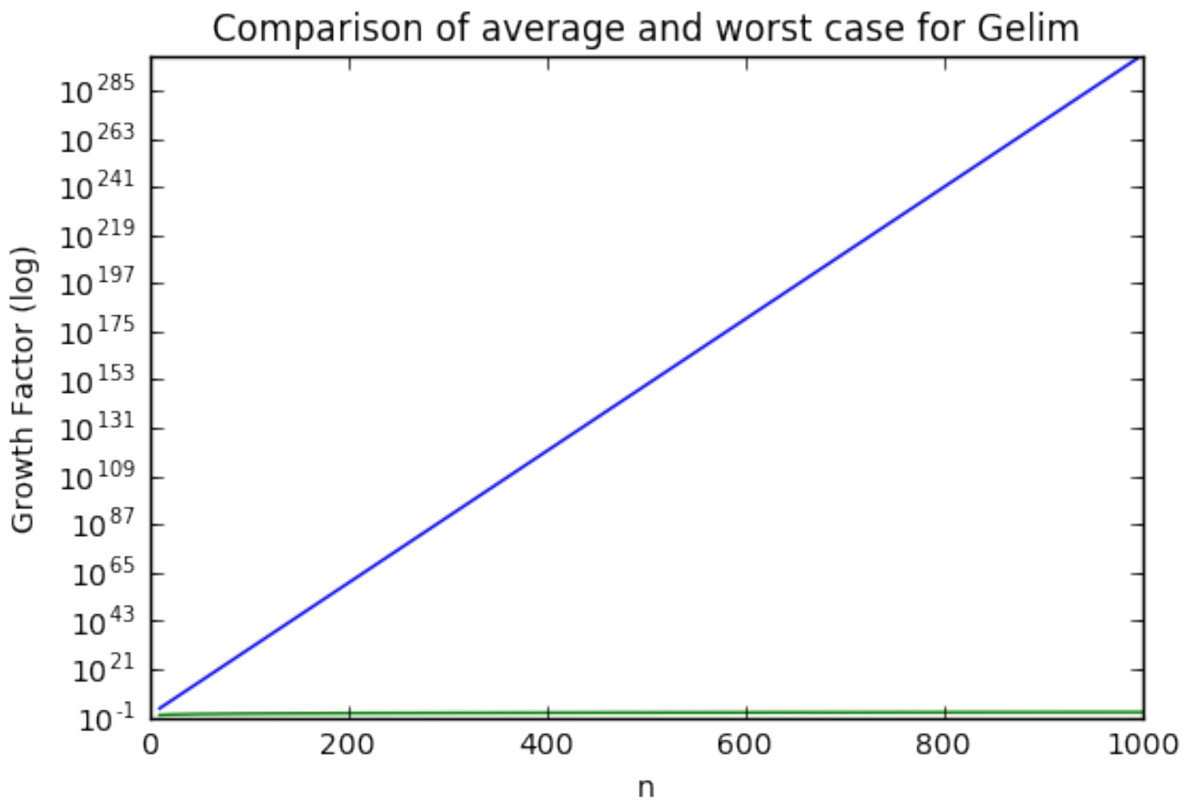
$\alpha = 2/3$

Further experiments would be needed to be more certain of this value. Although this does not suggest this relationship holds outside of $[10, 1000]$ it is certainly promising for most use cases of gaussian elimination with partial pivoting.

This is far far better than the worst case scenario of growing like an base 2 exponential. For comparison I have plotted below the curves cn^{α} where $c = 1/5, \alpha = 2/3$ and 2^n on log scale. See just how much quicker the worst case scenario grows.

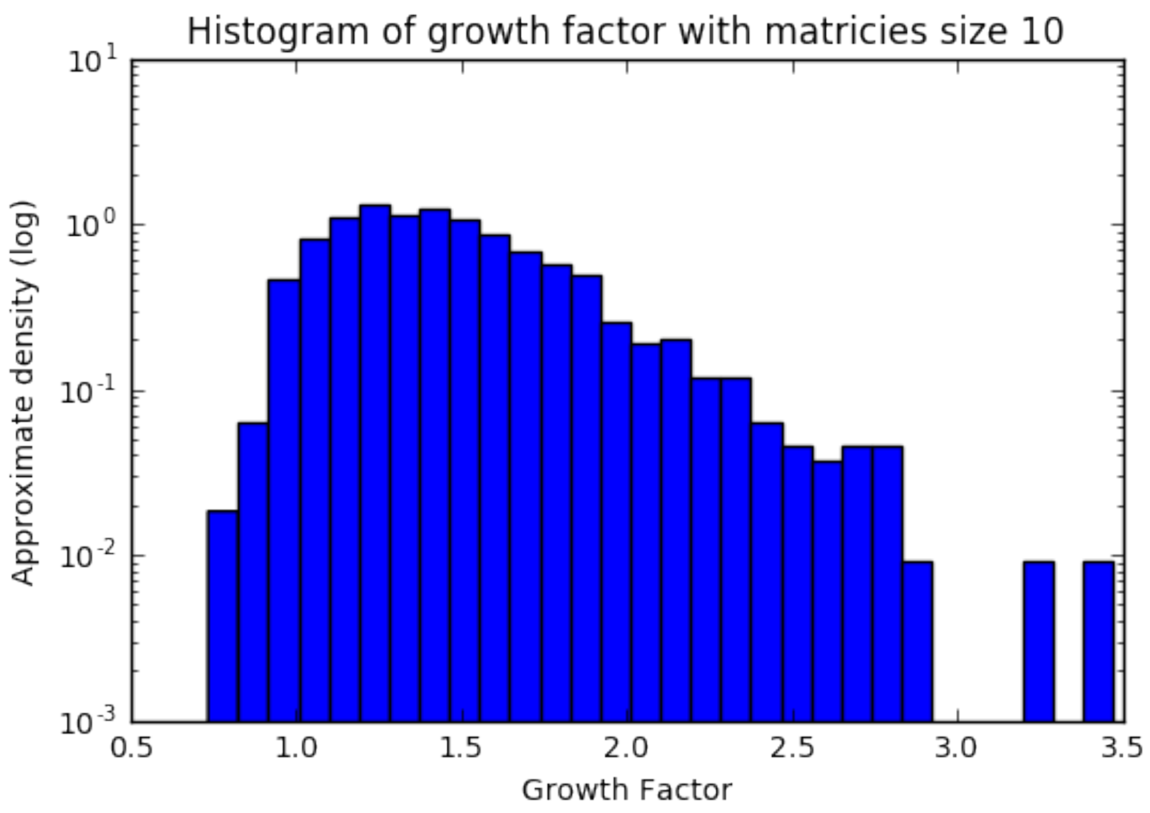
```
worst = [2**n for n in range(10,1000)] # so fast it overflows :(
plt.plot(range(10,1000),worst)
plt.plot(range(10,1000),fit)
plt.title('Comparison of average and worst case for Gelim')
plt.xlabel('n')
plt.ylabel('Growth Factor (log)')
plt.yscale('log')
plt.show()
```

```
//anaconda/lib/python3.5/site-packages/matplotlib/ticker.py:1597: RuntimeWarning:
overflow encountered in power
  ticklocs = b ** decades
```



Now, let's get a feel for the probability distribution. We need enough samples to get a reasonably good histogram so we choose smaller values of n for efficiency. I've experimented a little and found a good trade-off between accuracy and not burning my computer to be ≈ 1000.

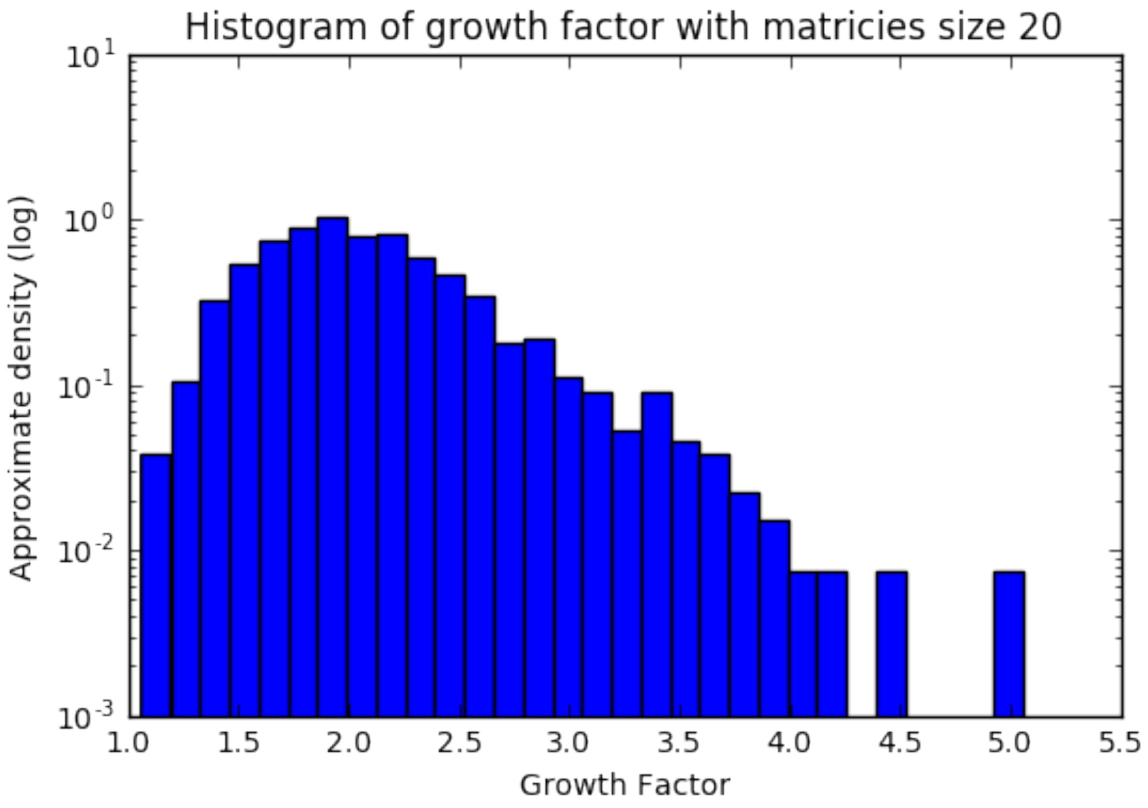
```
n_values = [10]
n_samples = 1200
for n in n_values:
    ga = []
    for k in range(n_samples): # takes a while already with 10 per n
        A = genMat(n)
        P,L,U = gelim(A)
        ga.append(growth(A,U))

    plt.figure() # Create new figure for each n
    plt.hist(ga,30, normed=True) # I'll let python choose the number and size of
bins
    plt.title('Histogram of growth factor with matricies size {}'.format(n))
    plt.ylabel('Approximate density (log)')
    plt.xlabel('Growth Factor')
    plt.yscale('log', nonposy='clip')
    plt.show()
```

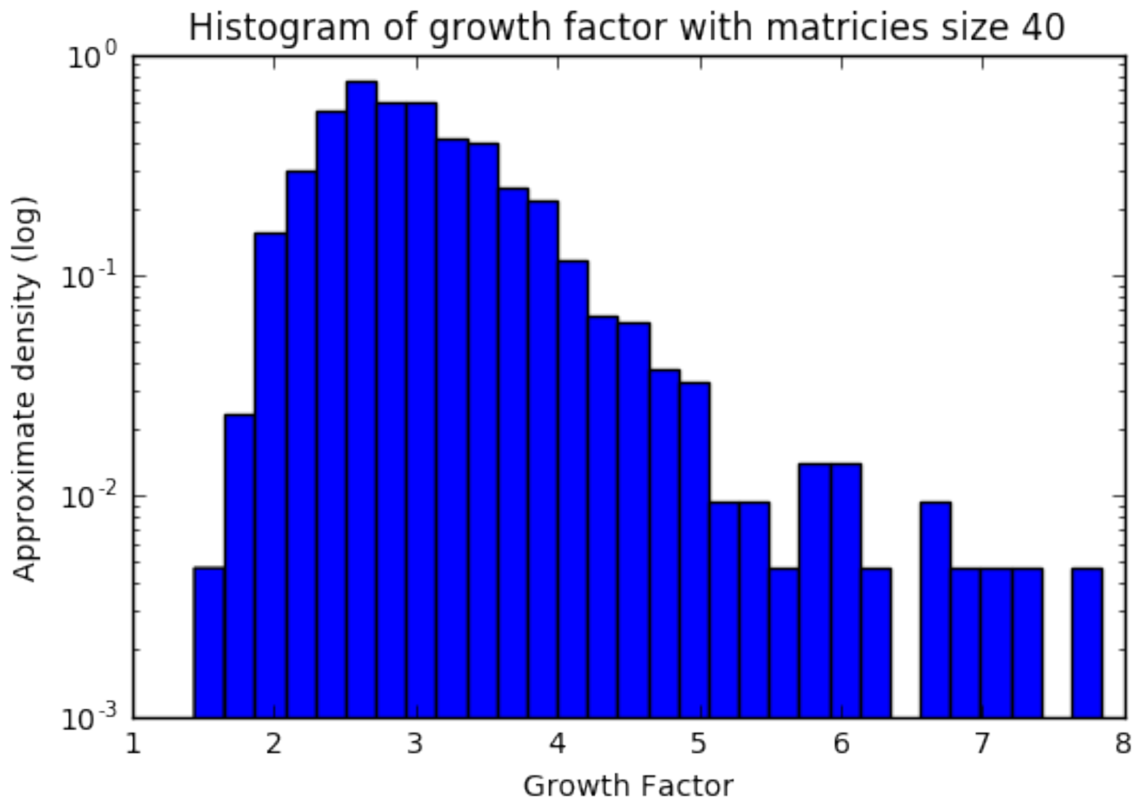


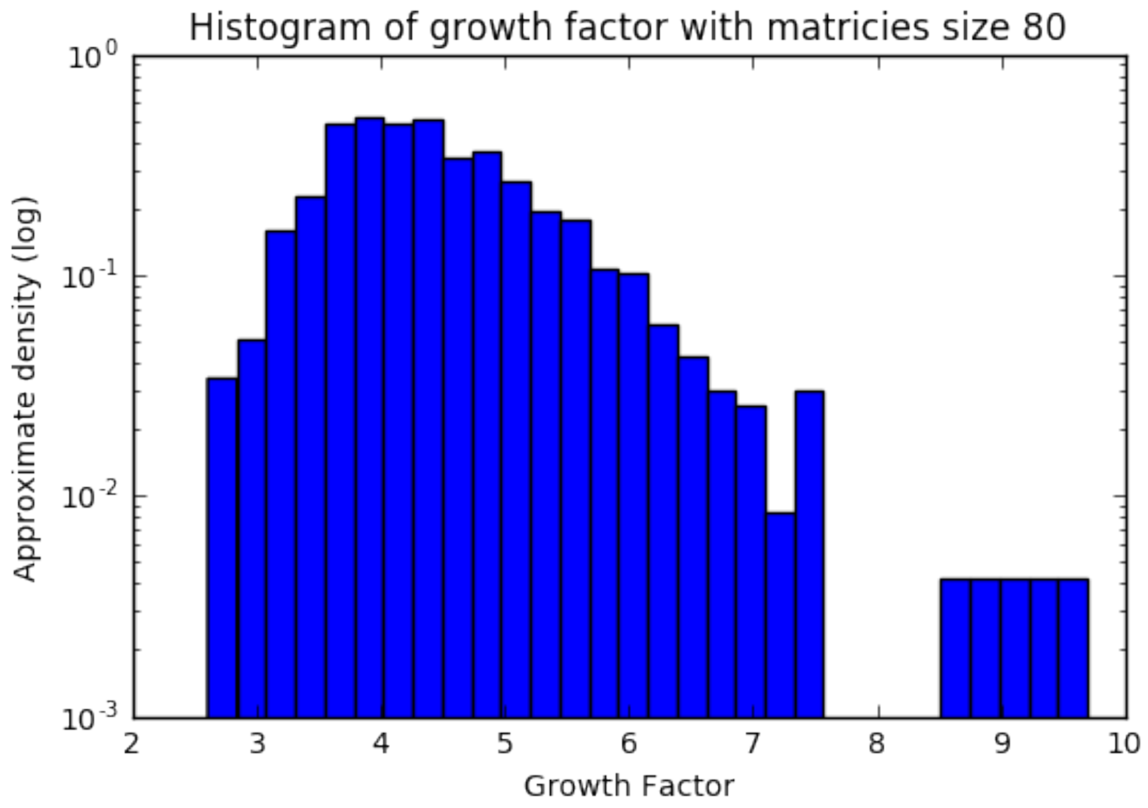
It appears that the density decays exponentially as growth factor increases. This is easy to tell as the y-axis is on a log scale and it looks to be decaying linearly in the plot. Of course, it's only a conjecture right now.

We can repeat this experiment now with different values of n.

```
n_values = [20,40,80]
n_samples = 1000
for n in n_values:
    ga = []
    for k in range(n_samples): # takes a while already with 10 per n
        A = genMat(n)
        P,L,U = gelim(A)
        ga.append(growth(A,U))

    plt.figure() # Create new figure for each n
    plt.hist(ga,30, normed=True) # I'll let python choose the number and size of
bins
    plt.title('Histogram of growth factor with matricies size {}'.format(n))
    plt.ylabel('Approximate density (log)')
    plt.xlabel('Growth Factor')
    plt.yscale('log', nonposy='clip')
    plt.show()
```




Histogram of growth factor with matricies size 40
Approximate density (log)
10^0
10^{-1}
10^{-2}
10^{-3}
1
2
3
4
5
6
7
8
Growth Factor


Histogram of growth factor with matricies size 80
Approximate density (log)
10^0
10^{-1}
10^{-2}
10^{-3}
2
3
4
5
6
7
8
9
10
Growth Factor

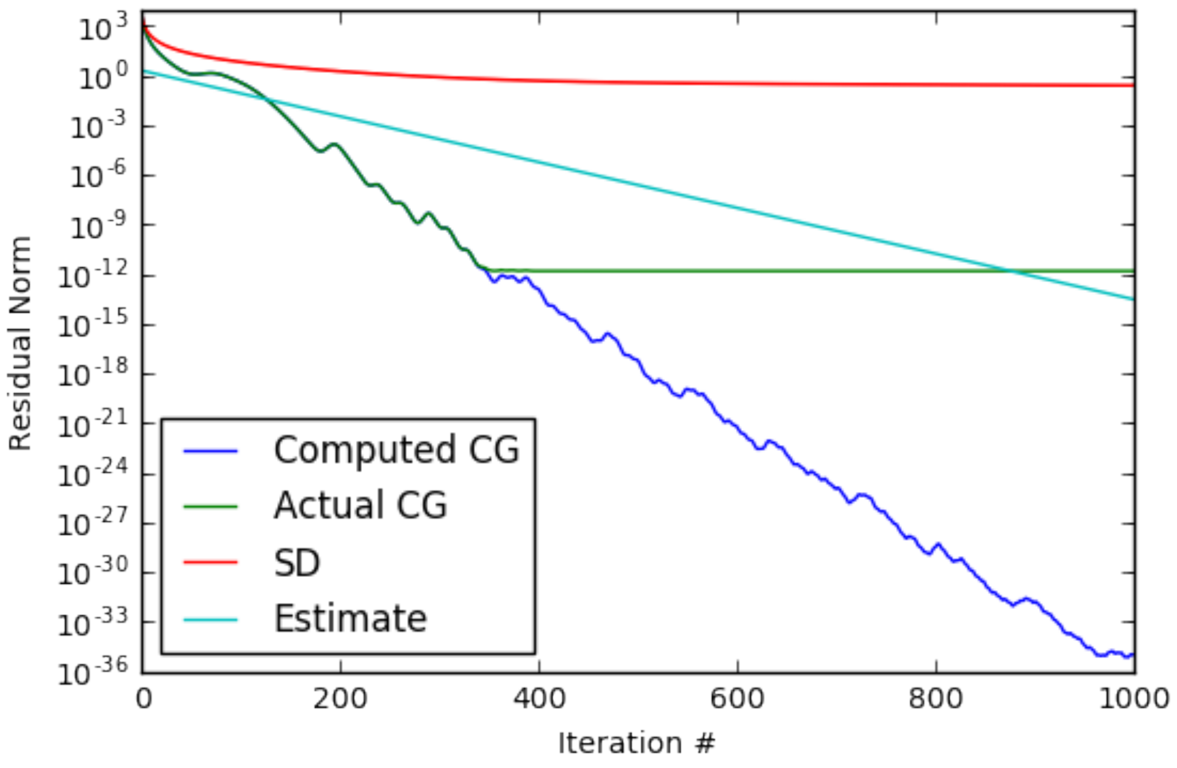
After repeating the experiment with $n = 20, 40, 80$ I am more sure of the conjecture that the density decays exponentially as the growth factor increases. Clearly, we start to lack samples for very high growth factors, but in the high density region it is clear that the density decays exponentially.

Part 2

Now we will be solving $Ax = b$ with steepest descent and CG.

```
# First let's create a function to make A and b as described in the question

def makeAb(): # we don't do this much so we can be inefficient
    A = np.eye(1000)
    for i in range(999):
        A[i,i] = i+1
        A[i-1,i] = 1
        A[i+1,i] = 1

    A[998,999] = 1 # The above code misses two spots, fill them in
    A[999,999] = 1000
    return A,np.ones(1000)



# Question 1
def steepestDescent(A,b,x,iters):
    resNorm = np.zeros(iters)
    r = b - A.dot(x)
    for i in range(iters):
        Ar = A.dot(r) # for efficiency
        a = (np.transpose(r).dot(r)) / (np.transpose(r).dot(Ar))
        x = x + a*r
        r = r - a*Ar
        resNorm[i] = np.linalg.norm(r) # Compute residual norms as well



    return x,resNorm

# Question 2
def cg(A,b,x,iters):
    compResNorm = np.zeros(iters)
    actualNorm = np.zeros(iters)
    r = b - A.dot(x)
    p = r
    for i in range(iters):
        Ap = A.dot(p) # for efficiency
        a = (np.transpose(p).dot(r)) / (np.transpose(p).dot(Ap))
        x = x + a*p
        r = r - a*Ap
        beta = np.transpose(Ap).dot(r) / np.transpose(Ap).dot(p)
        p = r - beta*p

        resNorm[i] = np.linalg.norm(r) # Compute residual norms as well
        actualNorm[i] = np.linalg.norm(b - A.dot(x))

    return x,resNorm,actualNorm


# Setup for question 3
iters= 1000
A,b = makeAb()
condition = np.linalg.cond(A)
print('Condition number: ' + str(condition))
sqrtk = np.sqrt(condition) # for cleaner algebra

x,compResNorm_cg,actualResNorm_cg = cg(A,b,b,iters) # Use b as an initial guess
x,resNorm_sd = steepestDescent(A,b,b,iters)
estimate = [2 * ((sqrtk - 1) / (sqrtk + 1))**i for i in range(1,1001)]



plt.plot(compResNorm_cg)
plt.plot(actualResNorm_cg)
plt.plot(resNorm_sd)
plt.plot(estimate)
plt.legend(['Computed CG','Actual CG','SD','Estimate'],loc = 'lower left')
plt.xlabel('Iteration #')
plt.ylabel('Residual Norm')
plt.yscale('log')
plt.show()
```

```
Condition number: 3942.9624167
```



Comments:

A is a reasonable conditioned matrix with $\kappa = 3942.9624167$. From what I've found searching this value is neither particulary good or bad.

The steepest descent method works well at first and then slows down drastically. It clearly performed much worse that CG did. Both methods are easy to implement and similiarly efficient. There really seems to be no real situation in which SD would be preferable to CG.

The conjugate gradient method converges extremely fast. It is interesting to note how the actual CG residuals and the computed CG residuals diverge drastically around 390 iterations. At this point the CG method stops converging at all despite the computed CG residuals shrinkings quickly. This well known problem of CG is caused by cumulative rounding errors that eventually prevent the search directions from being A-conjugate. This could be remedied by occasionally replacing the actual residuals with the computed residuals.

We see that CG converges quicker than the rate of the estimate. This is expected as the estimate is only an inequality. Again it's interesting to note that once CG breaks down this estimate no longer holds. One would have to be cautious to ensure CG does not break down if a good bound on your solution error is required.