CS161 Homework 3 Problem 3-6

(a)

$$
\begin{aligned}
X_i &= \text{the number of probes required for the ith insertion} \\
A_j &= \begin{cases} 1 & \text{if } j\text{th slot is occupied} \\ 0 & \text{otherwise} \end{cases} \\
\Pr(A_j = 1) &= \frac{i-1}{m} \\
\Pr(X_i = k) &= \left(\frac{i-1}{m}\right)^{k-1}\left(1 - \frac{i-1}{m}\right) \\
\Pr(X_i > k) &= 1 - \Pr(X_i \leq k) \\
&= 1 - \sum_{j=1}^{k}\left(\frac{i-1}{m}\right)^{j-1}\left(1 - \frac{i-1}{m}\right) \\
&= 1 - \left(1 - \left(\frac{i-1}{m}\right)^{k}\right) \\
&= \left(\frac{i-1}{m}\right)^{k} \\
&\leq \left(\frac{n-1}{m}\right)^{k} \\
&\leq \left(\frac{(m/2)-1}{m}\right)^{k} \\
&= \left(\frac{m-2}{2m}\right)^{k} \\
&\leq \left(\frac{m}{2m}\right)^{k} \\
&= \left(\frac{1}{2}\right)^{k}
\end{aligned}
$$

Thus, $\Pr(X_i > k) \leq \left(\frac{1}{2}\right)^k$.

(b) From part a, we have $P(X_i > k) \leq \left(\frac{1}{2}\right)^k$. Plug in $k = 2\log n$ and we get:

$$\begin{aligned}
\Pr(X_i > k) &\leq \left(\frac{1}{2}\right)^k \\
\Pr(X_i > 2\log n) &\leq \left(\frac{1}{2}\right)^{2\log n} \\
&= 2^{-2\log n} \\
&= 2^{\log n^{-2}} \\
&= n^{-2}
\end{aligned}$$

Thus, we have that $\frac{1}{n^2}$ is an upper bound for $P(X_i > 2\log n)$, so $P(X_i > 2\log n) = O\left(\frac{1}{n^2}\right)$

(c)

$$\begin{aligned}
\Pr(X > k) &= \Pr(X_1 > k) \text{or} \Pr(X_2 > k) \text{or} \Pr(X_3 > k) \text{or...or} \Pr(X_n > k) \\
&= \Pr(X_1 > k) + \Pr(X_2 > k) + \Pr(X_3 > k) + ... + \Pr(X_n > k) \\
&= \sum_{i=1}^{n} \Pr(X_i > k)
\end{aligned}$$

In this case, $k = 2\log n$, so we have:

$$\begin{aligned}
\Pr(X > 2\log n) &= \sum_{i=1}^{n} \Pr(X_i > 2\log n) \\
&\leq \sum_{i=1}^{n} \frac{1}{n^2} \\
&= n\left(\frac{1}{n^2}\right) \\
&= \frac{1}{n}
\end{aligned}$$

Thus, $\frac{1}{n}$ is an upper bound for $\Pr(X > 2\log n)$, so $\Pr(X > 2\log n) = O\left(\frac{1}{n}\right)$.

(d) We know $\Pr(X > 2\log n) = \frac{c}{n}$, where c is some constant. That means $\Pr(X \leq$

$2 \log n) = 1 - \frac{c}{n}$.

$$
\begin{aligned}
E[X] &= \sum_{x} x \Pr(X = x) \\
&\leq \left(\max_{x > 2 \log n} x \right) \frac{c}{n} + \sum_{x \leq 2 \log n} x \Pr(X = x) \\
&\leq \left(\max_{x > 2 \log n} x \right) \frac{c}{n} + \left(\max_{x \leq 2 \log n} x \right) \left(1 - \frac{c}{n} \right) \\
&= n \frac{c}{n} + (2 \log n) \left(1 - \frac{c}{n} \right) \\
&= n \frac{c}{n} + (2 \log n) \left(\frac{n - c}{n} \right) \\
&= c + (2 \log n) \left(\frac{n - c}{n} \right) \\
&\leq c + (2 \log n)
\end{aligned}
$$

Thus, $E[X] = O(\log n)$.