

CS161 Homework 3 Problem 3-6

(a)

 X_i = the number of probes required for the i th insertion

$$A_j = \begin{cases} 1 & \text{if } j\text{th slot is occupied} \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr(A_j = 1) = \frac{i-1}{m}$$

$$\Pr(X_i = k) = \left(\frac{i-1}{m}\right)^{k-1} \left(1 - \frac{i-1}{m}\right)$$

$$\begin{aligned} \Pr(X_i > k) &= 1 - \Pr(X_i \leq k) \\ &= 1 - \sum_{j=1}^k \left(\frac{i-1}{m}\right)^{j-1} \left(1 - \frac{i-1}{m}\right) \\ &= 1 - \left(1 - \left(\frac{i-1}{m}\right)^k\right) \\ &= \left(\frac{i-1}{m}\right)^k \\ &\leq \left(\frac{n-1}{m}\right)^k \\ &\leq \left(\frac{(m/2)-1}{m}\right)^k \\ &= \left(\frac{m-2}{2m}\right)^k \\ &\leq \left(\frac{m}{2m}\right)^k \\ &= \left(\frac{1}{2}\right)^k \end{aligned}$$

Thus, $\Pr(X_i > k) \leq \left(\frac{1}{2}\right)^k$.

(b) From part a, we have $P(X_i > k) \leq \left(\frac{1}{2}\right)^k$. Plug in $k = 2 \log n$ and we get:

$$\begin{aligned} \Pr(X_i > k) &\leq \left(\frac{1}{2}\right)^k \\ \Pr(X_i > 2 \log n) &\leq \left(\frac{1}{2}\right)^{2 \log n} \\ &= 2^{-2 \log n} \\ &= 2^{\log n^{-2}} \\ &= n^{-2} \end{aligned}$$

Thus, we have that $\frac{1}{n^2}$ is an upper bound for $P(X_i > 2 \log n)$, so $P(X_i > 2 \log n) = O\left(\frac{1}{n^2}\right)$

(c)

$$\begin{aligned} \Pr(X > k) &= \Pr(X_1 > k) \text{ or } \Pr(X_2 > k) \text{ or } \Pr(X_3 > k) \text{ or } \dots \text{ or } \Pr(X_n > k) \\ &= \Pr(X_1 > k) + \Pr(X_2 > k) + \Pr(X_3 > k) + \dots + \Pr(X_n > k) \\ &= \sum_{i=1}^n \Pr(X_i > k) \end{aligned}$$

In this case, $k = 2 \log n$, so we have:

$$\begin{aligned} \Pr(X > 2 \log n) &= \sum_{i=1}^n \Pr(X_i > 2 \log n) \\ &\leq \sum_{i=1}^n \frac{1}{n^2} \\ &= n \left(\frac{1}{n^2} \right) \\ &= \frac{1}{n} \end{aligned}$$

Thus, $\frac{1}{n}$ is an upper bound for $\Pr(X > 2 \log n)$, so $\Pr(X > 2 \log n) = O\left(\frac{1}{n}\right)$.

(d) We know $\Pr(X > 2 \log n) = \frac{c}{n}$, where c is some constant. That means $\Pr(X \leq 2 \log n) = 1 - \frac{c}{n}$.

$$2 \log n) = 1 - \frac{c}{n}.$$

$$\begin{aligned}
 E[X] &= \sum_x x \Pr(X = x) \\
 &\leq \left(\max_{x > 2 \log n} x \right) \frac{c}{n} + \sum_{x \leq 2 \log n} x \Pr(X = x) \\
 &\leq \left(\max_{x > 2 \log n} x \right) \frac{c}{n} + \left(\max_{x \leq 2 \log n} x \right) \left(1 - \frac{c}{n} \right) \\
 &= n \frac{c}{n} + (2 \log n) \left(1 - \frac{c}{n} \right) \\
 &= n \frac{c}{n} + (2 \log n) \left(\frac{n - c}{n} \right) \\
 &= c + (2 \log n) \left(\frac{n - c}{n} \right) \\
 &\leq c + (2 \log n)
 \end{aligned}$$

Thus, $E[X] = O(\log n)$.