RESEARCH STATEMENT: ALGEBRA AND GEOMETRY OF CATEGORICAL SPECTRA

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1. Overview

My general interest lies in the interaction between the *objects* of topology, algebraic/arithmetic geometry, and theoretical physics and the *methods* of homotopical or higher-categorical[1] algebra. In these fields, the notion of *spectra* has been effectively used to exploit the structures that were previously hidden. Spectra originated in stable homotopy theory and spread to the world of mathematics as the homotopical analog of abelian groups. Roughly speaking, they are like chain complexes, and are built up from abelian groups in layers, but typically in a much more interesting way. The most basic spectrum, the *sphere* $\mathbb{S}$, is the group completion of the symmetric monoidal *category of finite sets and automorphisms* $\mathsf{Fin}^{\simeq}$, whereas the classical counterpart $\mathbb{Z}$ is the group completion of the *set of isomorphism classes* $\mathbb{N}$ of finite sets. In other words, the sphere sees the homological complexity of finite groups and some part of the "$\mathbb{F}_1$-geometric" information. Many important examples, such as algebraic K-theory, topological cyclic homology, and Thom spectra, do not exist as chain complexes.

My current dissertation project focuses on the foundation of *categorical spectra*, which is a novel higher categorical generalization of spectra (§2). The goal is to build a theory that is robust enough for some sort of algebraic geometry. The primary use of this notion is in functorial field theories; as such, it sees the rich interplay between higher algebra and mathematical physics (§4, 5.1). Aside from that, it sees deeper structures in noncommutative algebraic geometry (§5.1, 5.2). My original inspiration [Mas21] comes from the papers by Connes and Consani [CC20] in search of absolute/$\mathbb{F}_1$-algebra, even though the current direction has diverted from their work.

As we saw, Waldhausen's *brave new algebra* philosophy of taking spectra seriously as building blocks of algebra was hugely successful. I propose to take categorical spectra seriously as an algebraic object as well. Abelian groups and spectra both have *tensor products*. They are the unique symmetric monoidal structures promoting the respective free functors from sets and spaces to symmetric monoidal functors. Since all of algebraic geometry relies on tensor products, the following question is fundamental:

Question 1.1. *Can we equip the $(\infty,1)$-category CatSp of categorical spectra with a natural presentably (symmetric) monoidal structure?*

It turns out to be a much subtler question than in spectra (§3). Based on the lax Gray tensor product on the $(\infty,1)$-category $\infty\mathsf{Cat}$ of (∞,∞)-categories, I proved the following:

Theorem 1.2. *There exists a unique (noncommutative) tensor product on CatSp promoting $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ to a monoidal functor with respect to the lax Gray tensor product. It acts additively on the categorical levels, which takes $\mathbb{Z} \cup \{\pm\infty\}$-values. Moreover, the tensor product localizes to the full subcategory of categorical spectra with duals $\mathsf{CatSp}^{\mathrm{dual}}$, i.e., those with stable cells admitting adjoints.*

The tensor unit of $\mathsf{CatSp}^{\mathrm{dual}}$, denoted by $\mathsf{Bord}^{\mathrm{fr}}$, is the delooping of free symmetric monoidal (∞,∞)-category on a single fully dualizable object. The notation suggests the interpretation as the categorical Thom spectrum for framing and the expected geometric description as the stably framed cobordism category.

Corollary 1.3. *$\mathsf{Bord}^{\mathrm{fr}}$ is the initial algebra of $\mathsf{CatSp}^{\mathrm{dual}}$, which passes to the sphere $\mathbb{S}$ in Sp. If we use the internal hom of categorical spectra, we have $[\mathsf{Bord}^{\mathrm{fr}}, X] \xrightarrow{\sim} X$.*

[1] I distinguish these two terms: I use *homotopical* to mean something belongs to the second column of the table, and *higher categorical* to mean something related to the third column. The term *higher* can ambiguously mean both.

Note that the last statement is stronger than a typical cobordism hypothesis: if we use the category of symmetric monoidal functors, we would only get the underlying groupoid of X. Modulo the (stable) cobordism hypothesis, the algebra structure on $\mathsf{Bord}^{\mathrm{fr}}$ is given by the cartesian product of manifolds. This should be regarded as a typical example where "lax" consideration recovers noninvertible information, indicating that the tensor product is something useful.

The following table of analogy is to give an idea about the context where categorical spectra fit in:

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\mathsf{Set} = \mathsf{Cat}_{(0,0)}$	spaces/∞-groupoids $\mathsf{S} = 0\mathsf{Cat}$	(∞,∞)-categories $\infty\mathsf{Cat}$
—	homotopy n-type	(∞,n)-category
$(1,1)$-categories $\mathsf{Cat}_{(1,1)}$	$(\infty,1)$-categories Cat	(∞,∞)-categories $\infty\mathsf{Cat}$
Cartesian product $\times$	Cartesian product $\times$	lax Gray tensor product $\otimes$
abelian groups Ab	spectra Sp	categorical spectra CatSp
	grouplike $\mathbb{E}_\infty$-spaces $\simeq \mathsf{Sp}^{\mathrm{cn}}$	symmetric monoidal (∞,∞)-categories
abelian categories	(pre)stable categories	(stable $\infty\mathsf{Cat}^{\otimes}$-bimodules?)
—	loop $\Omega(X,x) = \mathrm{Aut}_X(x)$	$\Omega(X,x) = \mathsf{End}_X(x)$
—	suspension $\Sigma = (-) \wedge \mathrm{B}\mathbb{Z}$	$\Sigma = \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1} = (-) \oslash \mathrm{B}\mathbb{N}$
free functor $\mathsf{Set} \to \mathsf{Ab}$	suspension spectra $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$	$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$	finite set categorical spectrum $\mathbb{F}$
tensor product $\otimes_{\mathbb{Z}}$	tensor (smash) product $\otimes_{\mathbb{S}}$	tensor product $\otimes_{\mathbb{F}}$

A New feature, which is also a big challenge, in the categorical context is the *noninvertibilty* of cells and the resulting *asymmetry.* Higher algebra comes with some hierarchy (the third row of the table). In homotopical theory, operations that let objects leak out of their homotopical hierarchy are the *colimits.* Now the categorical hierarchies are closed under limits and colimits, and *lax colimits* plays a similar role to colimits in homotopy theory. The non-invertibility of higher cells in the constructed object causes asymmetry and makes them harder to control. For this reason, lax constructions are dreaded; to say the least, it is a challenging task to construct a robust theory, even though it seems unavoidable if we try to retain noninvertible information. However, the algebra in this world is not a barren wasteland. On the contrary, there is emerging evidence that interesting algebra can be done with categorical spectra. For instance, Johnson-Freyd and Reutter have recently developed a version of Galois theory and announced in [Joh23a] that the Galois-closedness (in characteristic 0) characterizes the "universal physical targets of TQFT," capturing non-classical algebraic extensions starting with the category of super vector spaces, along the lines of Freed and Hopkins' paper [FH21] on reflection positivity.

2. Categorical spectra

First, we go straight to the definition of categorical spectra. Let $\infty\mathsf{Cat}$ be the $(\infty,1)$-category of (∞,∞)-categories[2]. A *pointed* (∞,∞)*-category* (X,x) is an (∞,∞)-category X with a distinguished object (basepoint) x. We often suppress the basepoint from notation. The following definition, which is analogous to the definition of spectra, was independently introduced by at least a few groups of people:[3].

Definition 2.1. The *loop* functor $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ sends a pointed (∞,∞)-category (X,x) to $\Omega(X,x) \coloneqq (\mathsf{End}_X(x), \mathrm{id}_x)$, the (monoidal) (∞,∞)-category of endomorphisms. It admits a left adjoint called the *suspension* Σ. A *categorical spectrum* is a sequence $X = (X_n)_{n\in\mathbb{N}}$ of pointed

[2]We exclusively work with the colimit of the inclusions $n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat}$ in Pr^{L}, not in Pr^{R}. It is characterized as being initial among the homotopy fixed points of enrichment endofunctor $(-)\text{-}\mathsf{Cat} : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$[Gol23].

[3]I originally called them ∞-spectra in an approach to absolute algebras [Mas21] until I came across Stefanich's thesis [Ste21], which gave them more descriptive name of *categorical spectra.* The paper spends a chapter on its formal foundation and utilizes it to package powerful functorialities of higher quasicoherent sheaves. He claims to have learned the notion from Teleman, who called them *anticategories.* Horiuchi [Hor18] essentially speculates about categorical spectra in the last section. Johnson-Freyd and Reutter use the term *towers* in [Reu23][Joh23b], who attribute it to Scheimbauer.

(∞,∞)-categories with equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. More precisely, the $(\infty,1)$-category of categorical spectra is the limit of the right adjoints

$$\mathsf{CatSp} \coloneqq \lim(\cdots \to \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*) \quad \in \mathsf{Pr}^{\mathsf{R}}.$$

We denote the functor $X \mapsto X_0$ by $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$, which has the left adjoint Σ^∞. We also have *shift* autofunctors $(-)[n]$ with $\Omega = [-1]$ and $\Sigma = [1]$.

While it is a *natural* definition, it is a priori unclear how *useful* the notion is. Part of the goal of this statement is to persuade the reader that it is not a meaningless generalization but an interesting object to study. We first note that this is a common generalization of symmetric monoidal (∞,∞)-categories and spectra:

Remark 2.2. The symmetric monoidal (∞,∞)-categories embed into categorical spectra by the *infinite delooping* $\mathrm{B}^\infty : \infty\mathsf{SMCat} \hookrightarrow \mathsf{CatSp}$, whose image should be regarded as *connective categorical spectra* $\mathsf{CatSp}^{\mathrm{cn}}$; as a consequence of Baez-Dolan delooping hypothesis, commutative monoid objects in $\infty\mathsf{Cat}$ are precisely the infinite loop objects, so the limit tower in the above definition factors through the forgetful functor $\infty\mathsf{SMCat} \to \infty\mathsf{Cat}_*$ to give the limit diagram of right adjoints

$$\mathsf{CatSp} \xrightarrow{\sim} \lim(\cdots \to \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat}),$$

with the left adjoint B^∞. We have $\Sigma^\infty = \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty}$. It also follows that CatSp is semiadditive (i.e., has biproducts $\oplus$).

Remark 2.3. Spectra is an instance of categorical spectra $(X_n)_n$ whose components X_n are all ∞-groupoids. The inclusion $\mathsf{Sp} \hookrightarrow \mathsf{CatSp}$ has both left and right adjoints: the localization (left adjoint) $(-)^{\mathrm{gp}}$ is the group completion functor, which level-wise inverts cells and group completes, whereas the colocalization (right adjoint) $\mathbb{G}_m$[4] takes levelwise the maximal Picard subgroupoid.

Remark 2.4. Notice $\mathsf{Sp} \cap \infty\mathsf{SMCat} = \mathsf{Sp}^{\mathrm{cn}} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S})$. The above fact that CatSp is the limit of $\infty\mathsf{SMCat}$ corresponds to that Sp is the stabilization of $\mathsf{Sp}^{\mathrm{cn}}$. While Ω^∞ restricts to the one for spectra, Σ^∞ does not: the relation is $\Sigma^\infty_{\mathsf{Sp}} \simeq (\Sigma^\infty)^{\mathrm{gp}}$. The free object on a point in categorical spectra is the symmetric monoidal groupoid of finite sets $\mathbb{F} \coloneqq \mathrm{B}^\infty\mathsf{Fin}^{\simeq} = \Sigma^\infty_+(*)$, while in spectra it is the sphere $\mathbb{S}$. The fact $\mathbb{S} = \mathbb{F}^{\mathrm{gp}}$ is known as the Barratt-Priddy-Quillen theorem[5][BP72]. Baez-Dolan delooping hypothesis itself is a categorical version of May's recognition principle for n-fold loop spaces[May72], whose $n = \infty$ case is often used to motivate the notion of spectra. Note that we work *without* group completion, which is somewhat difficult to analyze and sometimes quite destructive. For instance, May's recognition principle can be separated into the delooping hypothesis and a less formal fact about group completion.

Sp is the $n = -\infty$ case of the following categorical hierarchy:

Definition 2.5 ([Ste21, Notation 13.2.21])**.** Let $-\infty \le n \le \infty$. The full subcategory $n\mathsf{CatSp} \subset \mathsf{CatSp}$ of *n-categorical spectra* consists of objects $X = (X_k)_{\in\mathbb{N}}$ such that X_k is an $(\infty, \max\{0, n+k\})$-category. In particular, $\mathsf{Sp} = -\infty\mathsf{CatSp}$, $\mathsf{CatSp} = \infty\mathsf{CatSp}$. For $n \in \mathbb{Z}$, $n\mathsf{CatSp}$ is a shift of $0\mathsf{CatSp}$.

Before closing this section, we sketch a few more typical examples.

Example 1. ([Hau18][Ste21, §13.3.10]) Let $\mathcal{C}$ be an $(\infty,1)$-category with finite limits. $n\mathsf{Span}(\mathcal{C})$ is an (∞,n)-category with the same objects as $\mathcal{C}$, whose morphism from x to y is a span $x \leftarrow z \to y$, 2-morphisms are spans of spans, and so on, up through n-morphisms. A symmetric monoidal structure is given by the objectwise Cartesian product, whose unit is terminal. Then $\{n\mathsf{Span}(\mathcal{C})\}$ forms a categorical spectrum.

Example 2. ([Ste21, §13.3.6]) n-modules and presentable categorical spectrum: Let $\mathsf{V} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$. One hopes to define a large 1-categorical spectrum $\underline{\mathsf{V}} = \{n\mathsf{Mod}_{\mathsf{V}}\}$ by iterating the construction $\mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$ and enhancing them to n-categories. It does not work naively as $\mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$ is not presentable in the original universe, but it can be fixed by working in the very large $(\infty,1)$-category

[4]This is taken from [Joh23a]. I used Pic until recently, but now I adopt the 0-th level notation for consistency.

[5]It is sometimes rephrased as $K(\mathbb{F}_1) = \mathbb{S}$ because the finite sets can be considered as perfect modules over the mythical absolute base field $\mathbb{F}_1$, which seems to suggest that spectral algebraic geometry sees some $\mathbb{F}_1$-geometric information.

$\mathsf{CAT}^{\mathrm{ccpl}}$ of cocomplete $(\infty,1)$-categories and appropriately pushing the resulting object back into the original universe. In particular, one can define the categorical spectra $\underline{\mathsf{S}} = \{n\mathsf{Pr}\}$ and $\underline{\mathsf{Sp}} = \{n\mathsf{Pr}_{\mathrm{st}}\}$ of presentable (stable) n-categories. If A is an $\mathbb{E}_\infty$-ring, we define $\underline{A} := \Omega\underline{\mathsf{Mod}_A(\mathsf{Sp})}$.

Example 3. ([Hau17][JS17][Ste21, §13.3.12]) Morita categorical spectrum: Let $\mathcal{C}$ be a symmetric monoidal n-category (with good relative tensor products). Then one can construct a symmetric monoidal $(n+k)$-category $\mathsf{Morita}_k(\mathcal{C})$, whose objects are $\mathbb{E}_k$-algebras in $\mathcal{C}$, a morphism $A \to B$ is an $\mathbb{E}_{k-1}$-algebra object in (A,B)-bimodules, and so on. Then $\{\mathsf{Morita}_k(\mathcal{C})\}$ forms a n-categorical spectrum $\mathsf{Morita}(\mathcal{C})$.

Example 4. For an (∞,n)-category $\mathcal{C}$ (for finite n), [Lur09b, §3.2] outlines the definition of $\mathsf{Fam}_n^k(\mathcal{C})$. When $\mathcal{C} = *$, $\mathsf{Fam}_n^k(*) = n\mathsf{Span}(\mathsf{S}_{k\text{-fin}})$ is the n-category of spans in k-finite (i.e., k-truncated π-finite) spaces. Roughly speaking, the functor $\mathsf{Fam}_n^k(\mathcal{C}) \to \mathsf{Fam}_n^k(*)$ exhibits $\mathsf{Fam}_n^k(\mathcal{C})$ as the category of spans of k-finite spaces decorated by cells of $\mathcal{C}$ in a coherent way. There is a natural morphism $\mathcal{C} \to \mathsf{Fam}_n^k(\mathcal{C})$ exhibiting $\mathsf{Fam}_n^k(\mathcal{C})$ as the universal k-semiadditive n-category mapping out of $\mathcal{C}$, as proven by [Har20] in the $n=1$ case and the general case (including the definition of k-semiadditive n-category) is announced by [Sch23]. If $\mathcal{C}$ itself is k-semiadditive, it gives the *finite path integral* functor[6] $\int : \mathsf{Fam}_n^k(\mathcal{C}) \to \mathcal{C}$. If $X = (X_n)$ is a 0-categorical spectrum, it is immediate from the definition that $\mathsf{Fam}^k(X) := \{\mathsf{Fam}_n^k(X_n)\}_{n\geq 0}$ again forms a categorical spectrum. One can characterize it as the universal k-semiadditive categorical spectrum mapping out from X.**check**

3. Tensor product

In this section, we address the Question 1.1. We start with a bit of history. The tensor product of abelian groups is characterized as the unique presentably symmetric monoidal structure promoting Free : $\mathsf{Set} \to \mathsf{Ab}$ to a symmetric monoidal functor. The tensor product (aka. smash product) of spectra is similar, but it demands more sophisticated groundwork. While Boardman[Boa65] provided arguably the best definition $(1,1)$-categorically possible that time, even humble desiderata were shown to be incompatible with the point-set approach[Lew91], and several eclectic point-set constructions followed, e.g., [Elm+07], but it had to wait until Lurie's solid foundation of $(\infty,1)$-categories[Lur09a] for a truly canonical construction. As an example of microcosm principle[7], Lurie first constructed a symmetric monoidal structure $\otimes$ on Pr^{L}, the large category of presentable categories, characterized by the following properties: presheaf functor $\mathcal{P} : \mathsf{Cat} \to \mathsf{Pr}^{\mathsf{L}}$ is symmetric monoidal, and $\otimes$ distributes over colimits. A (commutative) algebra object in Pr^{L} is precisely a presentable (symmetric) monoidal category whose monoidal product is distributive over colimits.

Remark 3.1 ([Lur17])**.** $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ is an idempotent $\mathbb{E}_0$-algebra in Pr^{L}, i.e., $\Sigma^\infty_+ \otimes \mathrm{id} : \mathsf{Sp} \simeq \mathsf{S} \otimes \mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$ is an equivalence. Since the forgetful functor $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_\infty}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ is an equivalence, Sp uniquely promotes to a object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ so that Σ^∞_+ is symmetric monoidal. It is the unit object of the monoidal subcategory $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}} \subset \mathsf{Pr}^{\mathsf{L}}$ of stable presentable categories.

This robust implementation (together with the whole ∞-categorical setup) unlocked the explosive development of spectral algebraic geometry and algebraic K-theory in the past 15 years or so.

Now, it is natural to ask if we can do the same for CatSp:

Question 3.2 (=1.1)**.** *Can we equip the category* CatSp *of categorical spectra with a natural presentably (symmetric) monoidal structure?*

The answer turns out to be tricky. We would like a characterization similar to Remark 3.1, but we cannot expect $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \xrightarrow{\Sigma^\infty_+} \mathsf{CatSp}$ to be idempotent as an $\mathbb{E}_0$-algebra in Pr^{L}. In fact, the category $\infty\mathsf{Cat}$ is already not idempotent over S, as objects of $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ have only one compatible $\mathbb{E}_1$-structure, which uniquely promotes to an $\mathbb{E}_\infty$-structure, but as we will see, $\infty\mathsf{Cat}$ admits an asymmetric monoidal structure with the terminal unit. A more reasonable question is whether

[6]The importance of this functor is explained in [Fre+09]. $\mathsf{Fam}_n^k(\mathcal{C})$ classifies classical field theories, and the composition with $\int$ gives the *quantization*. An important example is the Dijgraaf-Witten theory. **write the definition?**

[7]Microcosm principle tells you that, to talk about an object with a certain structure (e.g. a commutative monoid), you must first equip the ambient category with the corresponding structure (e.g. a symmetric monoidal structure).

$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is idempotent, but to make sense of it, we must choose an algebra structure on $\infty\mathsf{Cat}$. The obvious first choice is the Cartesian monoidal structure, but the suspension fails to be a module homomorphism over it for a simple reason:[8]: if X, Y are m, n-categories respectively, then $X \wedge \Sigma Y$ is a $\max\{m, n+1\}$-category, while $\Sigma(X \wedge Y)$ is a $\max\{m,n\}+1$-category, so we have $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$ in general. In other words, the suspension is not given by smashing with $\vec{S}^1 := \mathrm{B}\mathbb{N} = \Sigma S^0$[9].

To fix this problem, we adopt a monoidal structure, the *(lax Gray) tensor product*, that acts additively on the category levels. Recall the full subcategory $\boxed{\square} = \{\square^n \mid n \geq 0\} \subset \infty\mathsf{Cat}$ of *the cubes*. The picture below describes the first few examples:

(insert pictures here)

The category $\boxed{\square}$ admits a natural monoidal structure so that $\square^n$ is the n-th power of $\square^1$. The *tensor product* $\otimes$ on $\infty\mathsf{Cat}$ is a presentably monoidal structure extending the monoidal structure on $\boxed{\square}$. The uniqueness follows from the density of $\boxed{\square} \subset \infty\mathsf{Cat}$ [Cam22], and the existence follows from Loubaton's thesis [Lou23], which builds on previous works [Ver08][VRO23][10]. The internal hom of the tensor product is the (∞,∞)-category of functors and lax natural transformations: $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \mathsf{Fun}^{\mathrm{lax}}(Y, Z))$.

Remark 3.3. We denote the pointed version ("lax smash" product) by $\oslash$. The suspension functor can be identified with $(-) \oslash \vec{S}^1$, which has the obvious structure of left $\infty\mathsf{Cat}^{\otimes}$-module morphism.

However, to mimic Lurie's strategy, we must promote it to a bimodule homomorphism. The left modules over a noncommutative algebra do not inherit a monoidal structure. It is, in fact, a key technical point: giving the structure of a bimodule homomorphism to $(-) \oslash S : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ is equivalent to lifting S to the *center* (aka. Hochschild cohomology) of $\infty\mathsf{Cat}_*^{\oslash}$, and in general higher coherence can be difficult to spell out. In our case $\vec{S}^1$ turns out to be *half-central* in the following sense: let $D : A \to A$ be a monoidal functor with $D^2 \simeq \mathrm{id}$. As an algebra morphism is a (pro)functor between the deloopings, D can be seen as a A-bimodule; explicitly, it is the identity bimodule A except that the left action is twisted by D. We define the *half-center* of A with respect to D as $\mathrm{Hom}_{\mathsf{BMod}_A}(A, D) \simeq \mathrm{Hom}_{\mathsf{BMod}_A}(D, A)$. The following theorem is relatively formal after showing (1):

Theorem 3.4.
1. *$\vec{S}^1 = \mathrm{B}\mathbb{N} \in \infty\mathsf{Cat}_*$ canonically lifts to the half-center with respect to the total dual (which flips the domain and codomain of all cells). In particular, Σ^2 canonically promotes to a $\infty\mathsf{Cat}^{\otimes}$-bimodule morphism.*
2. *With the induced bimodule structure from above, $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is an idempotent $\mathbb{E}_0$-algebra in $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}})$. In particular, it uniquely promotes to an $\mathbb{E}_1$-algebra object.*
3. *The presentably monoidal structure on CatSp given by forgetting along the lax monoidal functor $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Pr}^{\mathsf{L}}$ satisfies the universal property of $\infty\mathsf{Cat}^{\otimes}[(\vec{S}^1)^{-1}]$.*

Remark 3.5. Categorical filtration makes CatSp into a filtered monoidal category, i.e., The tensor product of an n-categorical spectrum and an m-categorical spectrum is a $(m+n)$-categorical spectrum. In particular, $0\mathsf{CatSp}$ is a monoidal subcategory. Here we take $-\infty + \infty = -\infty$. In other words, $\mathsf{Sp} \subset \mathsf{CatSp}$ is a tensor-ideal. The localization is smashing by the sphere spectrum $\mathbb{S}$. Since $\mathsf{Sp} \subset \mathsf{CatSp}$ is a monoidal subcategory, we have the inclusion $\mathsf{Alg}(\mathsf{Sp}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.

Remark 3.6. $\infty\mathsf{SMCat} = \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ is a monoidal subcategory. It follows that $\infty\mathsf{SMCat}$ admits a unique $\mathbb{E}_1$-monoidal structure $\otimes$ that makes $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat}^{\otimes} \to \infty\mathsf{SMCat}^{\otimes}$ monoidal. From footnote

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[8] However, the connective part $\mathsf{CatSp}^{\mathrm{cn}}$ can be easily given a symmetric monoidal structure; as in [GGN16], for any $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$, one has $\mathsf{CMon}(\mathcal{C}) \simeq \mathsf{CMon}(\mathsf{S}) \otimes \mathcal{C}$ and $\mathsf{CMon}(\mathsf{S}) \in \mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^{\mathsf{L}})$, so a unique symmetric monoidal structure $\circledast$ making $\mathrm{Free}_{\mathbb{E}_\infty} : \mathcal{C}^{\times} \to \mathsf{CMon}(\mathcal{C})^{\circledast}$ symmetric monoidal. This product $\circledast$ does not commute with delooping.

[9] $\mathrm{B}\mathbb{N}$ is a 1-category freely generated by a single object and a single endomorphism of the object.

[10] Loubaton proved the equivalence of $\infty\mathsf{Cat}$ and a combinatorial model called complicial sets, where the Gray tensor product was constructed by Verity. It is not a priori clear if the transferred tensor product satisfies this characterization, but it follows from the fact that the 0-truncation commutes with the Verity's Gray tensor product and [Lou23, Theorem 4.3.3.26] that the gaunt categories are closed under Gray cylinders in $\infty\mathsf{Cat}$, as communicated by Loubaton. A less model-dependent approach is taken by [CM23] for $(\infty,2)$-categories, but the extension to (∞,∞)-category is not straightforward (Campion very recently told me that he managed to work through it). We will only use the model-independent characterization.

8, $\infty\mathsf{SMCat}$ also has a symmetric monoidal structure $\circledast$. The identity functor $\infty\mathsf{SMCat}^{\circledast} \to \infty\mathsf{SMCat}^{\otimes}$ is lax monoidal and the two monoidal structures agree on $\mathsf{CMon}(\mathsf{S})$. In particular, we have the inclusion $\mathsf{Rig}_{\mathbb{E}_1}(\mathsf{S}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.

Remark 3.7. With an appropriate cocomplete variant of tensor product of categorical spectra in $\mathsf{CATSP}^{\mathrm{ccpl}}$, we expect that $\mathsf{CAlg}(\mathsf{S}) \to \mathsf{CATSP}^{\mathrm{ccpl}}; R \mapsto \underline{R}$ of Example 2 has a lax monoidal structure. **multiplicative delooping as opposed to additive delooping?**

4. Categorical spectra with duals

A context where categorical spectra naturally appear is the study of functorial field theories. Various sorts of cobordism categories appear as their domains, and the typical targets are some extensions of the higher module categories. The important feature of cobordism categories is that they are universal among the symmetric monoidal higher categories with duals (with some extra structure). In this section, we discuss them from the viewpoint of categorical spectra.

We say an (∞, n)-category has adjoints if, for $k < n$, any k-morphism has both left and right adjoints. Note that it is not reasonable to require the existence of adjoints for n-morphisms; for the top dimensional cells, adjoints are the same as inverses, as the unit and counit must be invertible. In particular, the statement "$\mathcal{C}$ is an (∞, n)-category with adjoints" depends on n and implies that either X is an (∞, n)-cateogry but not an $(\infty, n-1)$-category, or X is an ∞-groupoid. This leads to the following category-level dependent definition of *categorcial spectra with duals*:

Definition 4.1. An *n-categorical spectrum with duals*[11] is a categorical spectrum $X = (X_k)_{k\geq 0} \in n\mathsf{CatSp}$ where X_k is an $(n+k)$-category with adjoints.

Previous consideration implies that $n\mathsf{CatSp}^{\mathrm{dual}} \cap m\mathsf{CatSp}^{\mathrm{dual}} = \mathsf{Sp} = -\infty\mathsf{CatSp}^{\mathrm{dual}}$ for any $m \neq n$. Since one can write the condition of having adjoints as being local with respect to a certain set of morphisms corepresenting adjunctions, $n\mathsf{CatSp}^{\mathrm{dual}} \hookrightarrow \mathsf{CatSp}$ has the localization L_n. With some computation of the tensor product $\square^1 \otimes \mathsf{Adj} \in 3\mathsf{Cat}$ of the interval and the generic adjunction, we obtain the following:

Theorem 4.2. *The localizations $L_n : n\mathsf{CatSp} \to n\mathsf{CatSp}^{\mathrm{dual}}$ for $\infty \leq n \leq \infty$ are compatible with the graded monoidal structure on $\{n\mathsf{CatSp}\}$, i.e., for an L_n-equivalence f and an m-categorical spectrum X, the morphisms $f \otimes X$ and $X \otimes f$ are L_{m+n}-equivalences. In particular, there exist unique $\mathbb{E}_1$-monoidal structures on $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$ promoting $L = L_\infty$ and L_0 to monoidal functors.*

Part of the reasons to be interested in categorical spectra with duals is the potential to restore some commutativity of tensor products; adding adjoints is a milder version of adding inverses. More concretely, passage to adjoints and mates gives an equivalence $X \to X^{\mathrm{op}}$ so we get $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$[12]. I am currently working to upgrade this into a braiding of $(0)\mathsf{CatSp}^{\mathrm{dual}}$:

Conjecture 1. The graded $\mathbb{E}_1$-monoidal structure on $\{n\mathsf{CatSp}^{\mathrm{dual}}\}$ promotes to an $\mathbb{E}_2$-structure.

Ideally, we would like an $\mathbb{E}_\infty$ structure, but it is possible that the choice of identifications $X \to X^{\mathrm{op}}$, for example, whether I use left or right adjoint, affect the canonicity of braiding, so $\mathbb{E}_2$ seems to be a sensible conjecture for now.

The framed version of the cobordism hypothesis says that, for $0 \leq n < \infty$, the category $\mathrm{B}^\infty\mathsf{Bord}_n^{\mathrm{fr}}$ is the free n-categorical spectra with a single fully dualizable object, or equivalently, $\mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ is the free 0-categorical spectra on a single fully dualizable $(-n)$-cell, i.e., $L_0(\mathbb{F}[-n])$. The latter makes it clear that the "point" in $\mathsf{Bord}_n^{\mathrm{fr}}$ is secretly given an n-framing. Using tensor algebra, it can even be combined into a single equation $\mathsf{Bord}_\bullet^{\mathrm{fr}}[-\bullet] \coloneqq \bigoplus_{n\geq 0} B^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}} = L_0\,\mathrm{Tens}(\mathbb{F}[-1])$, which gives the graded $\mathbb{E}_1$-rig structure on bordism categories given by cartesian product of manifolds; one can think this as encoding various compactifications of field theories at once.

I do not know the similar geometric identification of the tensor unit $L\mathbb{F}$ of $\mathsf{CatSp}^{\mathrm{dual}}$, but since $\pi_*((L\mathbb{F})^{\mathrm{gp}}) = \pi_*\mathbb{S} = \Omega_*^{\mathrm{fr}}$, the following is a reasonable guess:

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[11] this terminology reflects the fact that the existence of adjoints in X_{n+1} forces the existence of duals in the symmetric monoidal ∞-category $X_n \simeq \Omega X_{n+1}$.

[12] $(-)^{\mathrm{op}} : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ is an antimonoidal involution that flips all odd dimensional cells. It induces a similar functor on CatSp.

Conjecture 2. $L\mathbb{F} = \mathsf{Bord}^{\mathrm{fr}}$, where $\mathsf{Bord}^{\mathrm{fr}}$ is the bordism (∞,∞)-category of stably framed manifolds. In the usual notatiton for cobordism hypothesis, it says $\mathrm{ev}_* : \mathsf{Fun}^{\otimes}(\mathsf{Bord}^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{\simeq}$ for any $\mathcal{C} \in \infty\mathsf{SMCat}^{\mathrm{dual}}$.

The following corollary of the theorem formally enhances the conjecture using the internal hom of categorical spectra:

Corollary 4.3. *If $X \in \mathsf{CatSp}^{\mathrm{dual}}$, we have $[L\mathbb{F}, X] \xrightarrow{\sim} X$.*

In other words, by considering lax natural transformations, we can recover the whole object X, not just the underlying groupoid of it. Assuming Conjecture 2, the algebra structure of the unit $L\mathbb{F}$ is given by the cartesian product on manifolds in $\mathsf{Bord}^{\mathrm{fr}}_n$. This category-level incarnation of the ring structure of the sphere was speculated in [Yua], but it requires our language to correctly formulate. Similarly to the finite-dimensional version, Conjecture 2 implies that the infinite piecewise-linear group PL acts on $\mathsf{Bord}^{\mathrm{fr}}$ by change of stable framings. This allows us to define the cobordism categories with various stable tangential structures as *categorical Thom spectra*. The Corollary 4.3 also implies that PL acts on any categorical spectra with duals.

We would like a similar enhancement of the cobordism hypothesis for $\mathsf{Bord}^{\mathrm{fr}}_n$, but the obvious analog with the internal hom of 0-categorical spectra fails, and some of the consequences, analogous to above, are too strong to be true. It is an interesting question to salvage this, which is also related to the question of constructing the cobordism categories with arbitrary tangential structures as the *categorical Madsen-Tillmann spectra*. Another related ongoing work is to interpret stronger variants of the cobordism hypothesis, including the cobordism hypothesis with singularities and tangle hypothesis, in terms of corepresenting categorical spectra, and hopefully to make reductions from one to another formal. It will be necessary to work with parametrized categorical spectra at some point; Thom categorical spectra construction can be seen as an instance of it (parametrized over BPL), and it is essential in incorporating non-tangential geometric structures, as in [GP23].

5. Open Directions

Because of the novel nature of this project, there are a plethora of interesting problems to be asked.

5.1. **Deeper Brauer spectrum and categorical etale topology.** Recall from Example 2 that if R is an $\mathbb{E}_\infty$-ring, we can associate a categorical spectrum $\underline{R}$ by considering higher module categories. $\mathbb{G}_m(\underline{R})$ is a classical (but typically not bounded below) spectrum and contains valuable information about R: for instance, $\Omega^{\infty-2}\mathbb{G}_m(\underline{R}) = \mathrm{Br}(R)$ is the Brauer space of R. The structure of $\mathrm{Br}(R)$ is well-understood (when R is connective) by [AG14] using etale cohomology. The key input here was the etale local triviality of Brauer groups. So far, nothing analogous has been known for the "deeper Brauer spectrum" $\mathbb{G}_m(\underline{R})$. The twist here is that the theory of categorical spectra thinks that the usual etale topology is too coarse. For instance, the super vector spaces $\underline{\mathsf{sVect}_{\mathbb{C}}}$ gives an extension of $\underline{\mathsf{Vect}_{\mathbb{C}}}$, even though $\mathbb{C}$ is classically an algebraically closed field. It was announced in [Joh23a](where I took the word "deeper" from) that, at least when X is (a small version of) "presentable stable" categorical spectra and X_0 contains a ring of characteristic 0, one can define the *etale homotopy type* $\mathrm{et}(X) \in \mathrm{Pro}(\mathsf{S}_{\mathrm{fin}})$, and if X is *Galois-closed*, i.e., when the etale homotopy type is trivial, then one has $\mathbb{G}_m(X) = I_{\mathbb{Q}/\mathbb{Z}}$. This gives some hope for structural understanding of $\mathbb{G}_m(\underline{R})$, once the following fundamental problem is answered:

Problem 1. *Give the "categorically-etale" topology on (suitably presentable stable) categorical spectra in a way that $R \mapsto \mathbb{G}_m(\underline{R})$ forms a sheaf. Do we need to generalize the notion of topos and sheaves?*

A (1-)categorical version of etaleness has received attention also in tensor triangulated geometry (see e.g. [Ram23][NP23]), and I speculate that these fields will eventually meet.

5.2. **Algebraic K-theory.** It is an interesting problem to define the K-theory of suitably compactly generated or dualizable presentable stable categorical spectra, including the class of $\underline{\mathsf{Mod}_R}$, in a way that $K(\underline{\mathsf{Mod}_R})$ enhances the classical $K(R)$, and is also related to secondary K-theory of [MS21] and higher. It should admit a trace map to higher Hochschild homology; this should appear as the value of the tori of the TQFT defined using the theory of higher quasicoherent sheaves of [Ste21].

Optimistically, one hopes that the redshift phenomena get some illuminating description here, in light of the "category level vs chromatic level" picture, along the lines of [TV09][BDR03][BS23]. Another hope is that the appearance of lax pullbacks in the elegant description of the K-theory of pullbacks of $\mathbb{E}_1$-rings in [LT19] gets an illuminating explanation.

5.3. **Reduction principles in higher category theory.** One major difficulty working with categorical spectra is the lack of a robust reduction principle, corresponding to the t-structure in stable categories and spectral sequences. The root of the problem seems to be that a categorical hierarchy is something more closed than the homotopical hierarchy: it is easy to "leak out" from one homotopical level to higher by colimits, so things can be decomposed into simpler pieces by colimits. Since categorical levels are closed under limits and colimits, the role of decomposing objects must be replaced by lax colimits (unstraightening is a particular example). They are in general harder to control, but we still hope for usable reduction principles in restricted settings, for instance, for suitably stable presentable categorical spectra or when stable cells are fully dualizable.

5.4. **stability of higher categories.** Theorem 3.4 implies that being a CatSp-module is a property of $\infty\mathsf{Cat}^{\otimes}$-bimodules. This corresponds to the fact that being an Sp-module is a property (stability) of presentable $(\infty, 1)$-categories. In $(\infty, 1)$-category theory, the stability has intrinsic characterizations. It would be useful to have ones for CatSp-modules. The difficulty here is in understanding the $\infty\mathsf{Cat}^{\otimes}$-bimodule structure itself. The data should amount to the *Gray cylinder* endofunctors $(-) \otimes \square^1$ and $\square^1 \otimes (-)$ together with infinitely many coherence data (which lives in $\mathbf{❑}$) and conditions (forcing the action of $\mathcal{P}(\mathbf{❑})$ to factor through the localiation $\infty\mathsf{Cat}$). Alternatively, one can start from an (∞, ∞)-category $\mathcal{C}$ and contemplate if it has a natural $\infty\mathsf{Cat}^{\otimes}$-bimodule structure that is stable. Because $\mathcal{C}$ itself is enriched in $\infty\mathsf{Cat}$, it makes sense to think about $\infty\mathsf{Cat}$-weighted (co)limits, and in particular about (partially) (op)lax (co)limits (in a similar fashion to [Ber20]). Understanding stability in this setting seems to be a meaningful question.

5.5. **Directed homotopy theory, Boolean and natural cohomology.** Grothendieck's homotopy hypothesis identifies ∞-groupoids with weak homotopy types of topological spaces. In the same spirit, ∞-categories can be thought of as homotopy types of some sorts of *directed* spaces. The categorical spectrum can be seen as a (co)homology theory on such. In this viewpoint, we have the following conceptual problem:

Problem 2. *Give an excisive-functor-style description of categorical spectra.*

This is closely related to a characterization of stability we speculated in the last section. It is an unavoidable question if we hope to develop (at least the first-order) Goodwillie calculus and apply it to deformation theory. In any way, it gives an invariant of directed spaces. Directed topological spaces have found many applications, including practical ones, so computing the cohomology of those spaces is a problem of interest. Cohomology theory given by ordinary spectra does not see any information of directedness: if X is a directed space (i.e., ∞-category) and E is a spectrum, $E^*(X) = [\Sigma^\infty_+ X, E] \simeq [\Sigma^\infty_+ X \otimes \mathbb{S}, E] \simeq E^*(|X|)$, where $|X|$ is a groupoidification of X. The next basic example would be the Eilenberg-Mac Lane spectra of semifields. Finite semifields are either a finite field $\mathbb{F}_p$ or the Boolean semifield $\mathbb{B}$, whose addition is idempotent; $1 + 1 = 1$. This "characteristic one" linear over $\mathbb{B}$ is developed in [CC19]. It is also proven that in [Gus+23]that a projective $\mathbb{B}$-module valued 1-dimensional TQFT corresponds to nondeterministic finite state automaton. Analogous to [Mil58], the following problem is of fundamental computational interest:

Problem 3. *Compute the Steenrod algebra $[H\mathbb{B}, H\mathbb{B}]$ and the dual Steenrod algebra $H\mathbb{B} \otimes H\mathbb{B}$. What about the natural cohomology $H\mathbb{N}$?*

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