SAG 1.4.9 – 1.4.11

② Postnikov tower

$X \in \mathrm{SpDM}$ **Recall**
$\begin{array}{c} \shortparallel \\ (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{array}$

$$\begin{array}{ccc} \mathrm{SpDM}^{\le n} & \underset{\tau_{\le n}}{\overset{\perp}{\rightleftarrows}} & \mathrm{SpDM} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\mathfrak{X}, \tau_{\le n}\mathcal{O}_{\mathfrak{X}}) & \longleftarrow\!\shortmid & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{array}$$

Prop $\mathrm{SpDM} \to \cdots \to \mathrm{SpDM}^{\le 2} \xrightarrow[\tau_{\le 0}]{} \mathrm{SpDM}^{\le 1} \xrightarrow[\tau_{\le 0}]{} \mathrm{SpDM}^{\le 0}$

is a limit diagram in CAT_∞

proof Let $G : \mathrm{SpDM} \longrightarrow \varprojlim_n \mathrm{SpDM}^{\le n}$

fully faithful

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{SpDM}}(X, Y) & \longrightarrow & \varprojlim \mathrm{Map}_{\mathrm{SpDM}^{\le n}}(\tau_{\le} X, \cancel{\tau_{\le}} Y) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{\infty\mathrm{Top}}(\mathfrak{X}, \mathfrak{Y}) & \\ & \rotatebox{90}{$\in$} & \\ & f^* : \mathfrak{Y} \to \mathfrak{X} & \end{array}$$

need to show equiv. on fibers

fib over f^* : $\mathrm{Map}_{\mathrm{Shv}^{\mathrm{loc}}_{\mathrm{CAlg}}(\mathfrak{X})}(f^*\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{X}})$

$\longrightarrow \varprojlim \mathrm{Map}_{\mathrm{Shv}^{\mathrm{loc}}_{\mathrm{CAlg}}(\mathfrak{X})}(f^*\mathcal{O}_{\mathfrak{Y}}, \tau_{\le n}\mathcal{O}_{\mathfrak{X}})$

without "loc" follows from

$\underset{\text{hypercopleteness} + \pi_n \text{ equiv.}}{\underline{1.4.8.1}} \to \mathcal{O}_{\mathfrak{X}} \xrightarrow{\sim} \varprojlim_n \tau_{\leq n} \mathcal{O}_{\mathfrak{X}}$

hypercopleteness + π_n equiv.

checked on affines

add in "loc": "map is local" is a π_0 condition

direct summand corr. to local maps

corresponds in both sides.

ess. surj $\mathrm{ob}(\varprojlim \mathrm{SpDM}^{\leq n}) \ni \left(\begin{array}{c} (\mathfrak{X}_n, \mathcal{O}_n) = X_n \\ + \\ \tau_{\leq n} X_{n+1} \simeq X_n \end{array} \right)$

may assume $\mathfrak{X}_n = \mathfrak{X}_0 =: \mathfrak{X}$

$(\mathfrak{X}, \mathcal{O}_n)$ with $\mathcal{O}_n \simeq \tau_{\leq n} \mathcal{O}_{n+1}$

$\rightsquigarrow$ want:

$$\begin{array}{ccc} \mathrm{SpDM} & \longrightarrow & \varprojlim_n \mathrm{SpDM}^{\leq n} \\ ①\ \Psi & & \Psi \\ (\mathfrak{X}, \varprojlim \mathcal{O}_n) & \xleftrightarrow{②} & \{(\mathfrak{X}, \mathcal{O}_n)\}_n \end{array}$$

① ② : both local

passing to an affine cover of X_0

may assume X_0 : affine.

$\rightsquigarrow$ X_n is affine (affineness can be checked on π_0),

$$X_n = \mathrm{Sp\acute{e}t}\, A_n \overset{\lim}{\rightsquigarrow} A$$

$(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_n)$
$=$
$(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_0)$: affine
by X_0 : affine.

can check: $\mathrm{Sp\acute{e}t}\, A$ is $(\mathcal{X}, \varprojlim \mathcal{O}_{\mathrm{Sp\acute{e}t} A_n})$

① is clear

② by CAlg^{cn} : Postnikov complete. $\square$

③ étale morphisms of ∞-topoi (HTT 6.3.5)

$U \in \mathcal{X}$: ∞-topos

fix

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{X}_{/U} \arrow[r, shift left=3, "\pi_!"] \arrow[r, phantom, "\perp"] \arrow[r, shift right=3, "\pi_*"'] & \mathcal{X} \arrow[l, "\pi^*"'] \end{tikzcd} \qquad (\mathcal{X}_{/U}, \pi^*, \Delta)$$

$\pi_!$ ← forget

π^* ← $- \times U$

π_* ← "sections over U"

$$\begin{tikzcd} & U = 1_{\mathcal{X}/U} \arrow[dl, "id"'] \arrow[d, "\Delta"] \\ U & U \times U = \pi^* U \arrow[l, "pr"] \end{tikzcd}$$

universal among

$$\begin{tikzcd} \mathcal{X}_{/U} \arrow[d, shift right=2, "g^*"'] \\ \mathcal{Y} \arrow[u, shift right=2, "g_*"'] \arrow[r, shift right=2, "f_*"'] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift right=2, "f^*"'] \end{tikzcd} \qquad (\mathcal{Y}, f^*, \phi)$$

$$1_{\mathcal{Y}} \xrightarrow{\phi} f^* U$$

g^*
$\cap$

$g^* \circ \pi^*$
"

i.e.

$$\begin{tikzcd}
\mathrm{Fun}^*(\mathcal{X}/_U, \mathcal{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \arrow[d, "ev_U"] & \ni f^* \arrow[d, mapsto] \\
\mathcal{Y}_* \arrow[r, "\text{forget}"'] & \mathcal{Y} & f^*U
\end{tikzcd}$$

$\mathcal{Y}_* \subset \mathrm{Fun}(\Delta^1, \mathcal{Y})$

$(1_\mathcal{Y} \to f^*U) \mapsfrom f^*U$

$\quad\quad \shortparallel$

$g^*(1_{\mathcal{X}/U} \xrightarrow{\Delta} \pi^*U)$

$\cancel{F(f^*U, \mathcal{Y})}$

<u>Cor</u>

$$\begin{tikzcd}
\mathrm{Map}_\mathcal{Y}(1_\mathcal{Y}, f^*U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}^*(\mathcal{X}/_U, \mathcal{Y}) \arrow[r] \arrow[d, "\text{left fib}"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}_* \arrow[d, "\text{left fib}"] & 1_\mathcal{Y} \to f^*U \arrow[d, mapsto] \\
\{f^*\} \arrow[r] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \arrow[r, "ev_U"] & \mathcal{Y} & f^*U
\end{tikzcd}$$

$\{f^*\} \mapsto f^*U$

fib seq

$$\rightsquigarrow \quad \begin{tikzcd}
\mathcal{Y}/_{f^*U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}/_U \arrow[d, "\pi_*"] \\
\mathcal{Y} \arrow[r, "f_*"'] & \mathcal{X}
\end{tikzcd} \quad \text{in } \infty\mathrm{Top}$$

<u>again étale</u>

<u>proof</u> $\mathrm{Map}_{\infty\mathrm{Top}}(\mathcal{Z}, -)$

$$\begin{tikzcd}
P(g^* f^* U, Z) \arrow[r] \arrow[d, equal] & \mathrm{Fun}^*(\mathcal{Y}_{/f^*U}, Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & \mathrm{Fun}^*(\mathcal{Y}, Z) \arrow[d] & g^* \arrow[l, Rightarrow] \\
P((f\circ g)^* U, Z) \arrow[r] & \mathrm{Fun}^*(\mathcal{X}_{/U}, Z) \arrow[r] & \mathrm{Fun}^*(\mathcal{X}, Z)
\end{tikzcd}$$

Cor plug $\mathcal{Y} = \mathcal{X}_{/V} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{X}$ étale

$$\begin{tikzcd}
\mathrm{Fun}^*_{\mathcal{X}}(\mathcal{X}_{/V}, \mathcal{X}_{/V}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^*(\mathcal{X}_{/V}, \mathcal{X}_{/V}) \arrow[d] \\
\{f^*\} \arrow[r] \arrow[u, bend left, "\cong"] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{X}_{/V})
\end{tikzcd}$$

$$\mathrm{Map}_{\infty\mathrm{Top}/\mathcal{X}}(\mathcal{X}_{/V}, \mathcal{X}_{/U}) \simeq \mathrm{Map}_{\mathcal{X}_{/V}}(1_{\mathcal{X}_{/V}}, f^*U)$$

by fib seq

$$\cong \mathrm{Map}_{\mathcal{X}}(\underbrace{f_! 1_{\mathcal{X}_{/V}}}_{\overset{\|}{V}}, U)$$

$$\begin{tikzcd}
V \arrow[r] & U \\
\mathcal{X}_{/V} \arrow[r, dashed] \arrow[dr] & \mathcal{X}_{/U} \arrow[d] \\
& \mathcal{X}
\end{tikzcd}$$

$\mathcal{X}_{/V} = (\mathcal{X}_{/U})_{/V}$

Cor.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f_*"] \arrow[dr, "h_*"'] & \mathcal{Y} \arrow[d, "g_*:\ \text{étale}"] \\
& Z
\end{tikzcd}$$

f_* : étale $\Leftrightarrow$ h_* : étale

Cor

$$\begin{tikzcd}
\mathcal{X}^{\Delta^1} \arrow[d, "\mathrm{cod}"] \\
\mathcal{X} \arrow[r] & \mathrm{CAT}_\infty^{op} \supset \mathrm{Pr}^R
\end{tikzcd}$$

$U \mapsto \mathcal{X}_{/U}$

$$\text{fully faithful} \longrightarrow \infty\mathrm{Top}^{\text{ét}}$$

$$\text{ess im} = \left(\infty\mathrm{Top}^{\text{ét}}\right)_{/\mathfrak{X}}$$

colimit along étale mor

$$\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}} \hookrightarrow \infty\mathrm{Top}_{/\mathfrak{X}} \to \infty\mathrm{Top} \hookrightarrow \mathrm{Pr}^R \to \mathrm{CAT}_\infty^{op}$$

proof

$\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}} \simeq \mathfrak{X}$: admits small colim

$\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}} \hookrightarrow \infty\mathrm{Top}_{/\mathfrak{X}}$: preserves them

$\infty\mathrm{Top} \to \mathrm{CAT}_\infty^{op}$: creates colimits

$\mathfrak{X} \to \mathrm{CAT}_\infty^{op}$: colim-pres, by descent

$$\operatorname{colim} \mathfrak{X}_{/U_\alpha} \simeq \mathfrak{X}_{/\operatorname{colim} U_\alpha}$$

More generally

$\boxed{\text{Thm}}$

$$K \ni \alpha \mapsto \mathfrak{X}_\alpha$$

$$\infty\mathrm{Top}^{\text{ét}} \hookrightarrow \infty\mathrm{Top}$$

$\infty\mathrm{Top}^{\text{ét}}$: admits small colim

$\hookrightarrow$: preserves them

By thm $\operatorname{colim}_{\text{along ét}} \mathfrak{X}_\alpha =: \mathfrak{X} \rightsquigarrow \mathfrak{X}_\alpha \xrightarrow{\text{ét}} \mathfrak{X}$, $\quad \mathfrak{X}_{/U_\alpha} \cong \mathfrak{X}_\alpha$

$\rightsquigarrow \operatorname{colim} \mathfrak{X}_{/U_\alpha} \simeq \mathfrak{X}$

$$\coprod U_\alpha \twoheadrightarrow 1$$

ref: 21.4.7 or DAG V

② Étale mor of locally ringed ∞-topoi

Def $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\infty\mathrm{Top}_{\mathrm{CAlg}}$ is étale

if $\simeq$ $(\mathcal{Y}_{/V}, \mathcal{O}_{\mathcal{Y}}|_V)$ $\exists V \in \mathcal{Y}$.

$$\begin{tikzcd} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[rr] \arrow[dr, "\simeq"'] & & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \\ & (\mathcal{Y}_{/V}, \mathcal{O}_{\mathcal{Y}}|_V) \arrow[ur] & \end{tikzcd}$$

underlying topos étale
+
$\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}|_V$ condition

$\Longleftrightarrow$ f : cartesian edge of $\infty\mathrm{Top}_{\mathrm{CAlg}} \to \infty\mathrm{Top}$

étale surjection if $V \twoheadrightarrow 1_{\mathcal{Y}}$

Remark (1)

$$\begin{tikzcd} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \arrow[dr, "g\,:\,\text{étale}"] & \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[ur, "f"] \arrow[rr, "h"'] & & (\mathcal{X}'', \mathcal{O}_{\mathcal{X}''}) \end{tikzcd}$$

$\Rightarrow$ f : étale iff h : étale

or sHen loc, ét

(2) $\infty\mathrm{Top}_{\mathrm{CAlg}}$ obj : locally ring ∞-topoi
mor : Étale mor
($\Rightarrow$ local)

admits small colim

preserved by

$\infty\mathrm{Top}_{\mathrm{CAlg}}$ $\quad$ $\infty\mathrm{Top}$ $\quad$ $\infty\mathrm{Top}^{\mathrm{loc}}$ or sHen

$\mathrm{colim}\,(\mathcal{X}_\alpha, \mathcal{O}_\alpha)$ $\qquad$ Let $\mathcal{X} := \mathrm{colim}\, \mathcal{X}_\alpha$

||

$= \left(\mathcal{X}, \lim_{\alpha} (f_\alpha)_* \mathcal{O}_\alpha\right)$

$\mathcal{X}/U_\alpha$

$\mathcal{X}_\alpha \xrightarrow{(f_\alpha)_*} \mathcal{X}$

$\mathcal{O}_\alpha \qquad \bullet$

<u>Rmk</u> $X \to Y$

f: étale is local on the source :

If $\exists \coprod_\alpha U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$ s.t.

$f_\alpha : (\mathcal{X}/U_\alpha, \mathcal{O}_{\mathcal{X}}|_{U_\alpha}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$
étale

$\Rightarrow f$: étale

i.e. $\left(\begin{array}{c} \xrightarrow{\text{ét.surj}} \\ \searrow \; \leftrightarrows \; \swarrow \\ \text{étale} \end{array} \right)$

<u>proof</u> $\mathcal{X}_0 \underset{\text{full}}{\subset} \mathcal{X}$

$U \in \mathcal{X}_0 :\Longleftrightarrow X|_U \to X \xrightarrow{f} Y$: étale

① $\left\{ \begin{array}{l} \bullet\ \mathcal{X}_0 \text{ is a sieve} \qquad V \to U \in \mathcal{X}_0 \\ \qquad\qquad\qquad\qquad \Rightarrow V \in \mathcal{X}_0 \\ \bullet\ \mathcal{X}_0 \text{ closed under small colim} \end{array} \right.$

(by the prop above)

$U_\alpha \in \mathcal{X}_0$. $U = \coprod U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$

$\Longrightarrow 1_{\mathcal{X}} \in \mathcal{X}_0$. i.e. f: étale.

$U \times U \times U \in \mathcal{X}_0$

$\downarrow$ ①

U

(4) étale mor in SpDM^{nc}

$$K \longrightarrow (\mathrm{SpDM}^{nc})^{\text{ét}} \subset (\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}})^{\text{ét}}$$ closed under colim

$\alpha \longmapsto X_\alpha^{\in}$

$\mathrm{SpDM}^{nc} \subset_{\text{full sub}} \infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}$ preserves them.

<u>proof</u> $(\mathcal{X}_\alpha, \mathcal{O}_\alpha) \in \mathrm{SpDM}^{nc}$

$\mathcal{X}_\alpha = \mathcal{X}/U_\alpha$

$\downarrow$ colim

$(\mathcal{X}, \mathcal{O})$ in $\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}$

$\underline{\underline{\coprod U_\alpha \twoheadrightarrow 1}}$ $\in \mathrm{SpDM}^{nc}$

<u>e.g.</u> G : discre group

$BG \longrightarrow \mathrm{SpDM}^{nc}$ $\quad \operatorname{colim}_{BG_{\parallel}} X$

$* \circlearrowleft G \qquad X \circlearrowleft G \qquad X/G$

act by equiv

$\Downarrow$

étale

$\left(\begin{matrix} X /\!/ G \\ [X/G] \end{matrix} \; ? \right)$

(5) relation to étale ring maps ? (1.4.10)

<u>Thm</u> $\phi^* : A \longrightarrow B$ in CAlg is étale

iff $\mathrm{Spét}\, B \xrightarrow[\phi_*]{} \mathrm{Spét}\, A$ is étale.

<u>Cor</u> $f : X \longrightarrow Y$ in SpDM^{nc} is étale

$\underline{\text{iff}}$ $\forall$ comm square

$$\left[\begin{tikzcd} \mathrm{Spét}\, B \arrow[r, "\text{ét}"] \arrow[d, "\phi_*"] & X \arrow[d, "f"] \\ \mathrm{Spét}\, A \arrow[r, "\text{ét}"'] & Y \end{tikzcd}\right]$$

alg map $A \xrightarrow[\phi^*]{} B$ is étale

$(\Rightarrow)$
$$\begin{tikzcd} \mathrm{Spét}\, B \arrow[rr, "\phi^*"] \arrow[dr, "\text{ét}"'] & & \mathrm{Spét}\, A \arrow[dl, "\text{ét}"] \\ & Y & \end{tikzcd}$$
$\Rightarrow \phi_* : \text{ét}$
$\Updownarrow$ Thm
$\phi^* : A \to B$ etale.

$(\Leftarrow)$ Cover Y by affines V_α

$\rightsquigarrow$ cover X by affines $U_{\alpha\beta}$

$$\begin{tikzcd} \text{aff} : \underbrace{\coprod} U_{\alpha\beta} \arrow[r, dashed, "\text{cover}"] \arrow[d, "\text{étale}"] & X \arrow[d] \\ \text{aff} : V_\alpha \arrow[r, "\text{ét}"'] & Y \end{tikzcd} \quad \Rightarrow \quad f : \text{étale}.$$

$\boxed{\natural}$

Proof of Thm

<u>Thm</u> $\phi^* : A \longrightarrow B$ in CAlg is étale

iff $\mathrm{Spét}\, B \xrightarrow[\phi_*]{} \mathrm{Spét}\, A$ is étale

$(\Rightarrow)$ by construction

$$\mathrm{Shv}^{\text{ét}}_B \simeq (\mathrm{Shv}^{\text{ét}}_A)_{/\underbrace{h^B}_{\text{corep. sheaf by } B}}$$

($\Leftarrow$)

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathrm{Sp\acute{e}t}\, A \\ \text{\rotatebox{90}{$\cong$}} & & \| \\ (\mathfrak{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U) & \longrightarrow & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{array}$$

$$\begin{tikzcd} & & U \arrow[u, phantom, "\in"] \\ \coprod_{j \in J} W_j \arrow[rru] \arrow[rr, "\text{refine}"'] & & \coprod_{i \in I} V_i \arrow[u, two heads] \end{tikzcd}$$

finite

by W_j : corep by B_j : étale B-alg.

$\left[V_i := \text{sheaf in } \mathrm{Shv}^{\text{ét}}_A \text{ corep. by } A_i : \text{étale } A\text{-alg} \right.$

$\left(\text{refinement} : \begin{array}{ccc} J & \xrightarrow[\leftarrow]{\text{id}} & \cancel{I}\, J \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ j & \longmapsto & r(j) \end{array} \qquad \begin{tikzcd} W_j \arrow[r] \arrow[dr] & V_{r(j)} \arrow[d] \\ & U \end{tikzcd} \right)$

$\coprod W_j$ $=$ corep by $\prod B_j =: B'$ sheaf in $\mathrm{Shv}^{\text{ét}}_B$

$\coprod V_j$ $=$ corep by $\prod A_j = A'$ in $\mathrm{Shv}^{\text{ét}}_A$

$$\begin{array}{ccccccc} (X_{/U})_{/\amalg W_j} & \longrightarrow & X_{/\amalg V_j} & \longrightarrow & X_{/U} & \longrightarrow & X \\ \| & & \| & & \| & & \| \\ \mathrm{Sp\acute{e}t}\, B' & \longrightarrow & \mathrm{Sp\acute{e}t}\, A' & \longrightarrow & \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathrm{Sp\acute{e}t}\, A \end{array}$$

seq of DM stk

in CAlg

$$
\begin{tikzcd}
B' & A' \arrow[l] & B \arrow[l] \arrow[ll, bend left, "\text{ét}"] & A \arrow[l] \arrow[ll, bend right, "\text{ét}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & B \arrow[r, "\text{ét}"] \arrow[d] & B' \arrow[d] \arrow[ddr, bend left, "id"] & \\
A \arrow[r, "\text{ét}"'] & A' \arrow[r, "\text{ét}"'] \arrow[drr, bend right] \arrow[ur, phantom, "r"] & A' \otimes_B B' \arrow[dr] & \\
 & & & B'
\end{tikzcd}
$$

$\rightsquigarrow$ B': retract of $A' \otimes_B B'$ in $CAlg_A$

$$A \xrightarrow[\text{ét}]{} A' \otimes_B B'$$

$\rightsquigarrow$ B': Étale $/A$

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr, "\text{étale}"'] & B \arrow[d, shift left, "\text{étale faithfully flat}"] \\
 & B' \arrow[u, shift left]
\end{tikzcd}
$$

étale !!

∎

⑫ Limits in SpDM

<u>Prop.</u> (1) SpDM^{nc} admits finite limits, preserved by $\mathrm{SpDM}^{nc} \hookrightarrow \infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{CAlg}}$

(2)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\phi'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\phi"] \\ Y' \arrow[r] & Y \end{tikzcd}$$
pullback of ncDM stacks

ϕ : étale $\Rightarrow$ ϕ' : étale

(3) $\mathrm{Spét}(A \otimes_R B) \simeq \mathrm{Spét}\, A \times_{\mathrm{Spét}\, R} \mathrm{Spét}\, B$

(4)
$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$
in SpDM^{nc}

if X, Y, Y' : connective $\Rightarrow$ X' : connective.

<u>Proof</u> (3) $\mathrm{Spét} : \mathrm{CAlg} \underset{\Gamma}{\overset{\perp}{\rightleftarrows}} \infty\mathrm{Top}^{\mathrm{sHen}\ op}_{\mathrm{CAlg}}$

(2)
$$\begin{tikzcd} (\mathcal{X}_{/f^*U}, \mathcal{O}_\mathcal{X}|_{f^*U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{Y}_{/U}, \mathcal{O}_\mathcal{Y}|_U) \arrow[d] \\ (\mathcal{X}, \mathcal{O}_\mathcal{X}) \arrow[r] & (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \end{tikzcd}$$
pullback in $\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}$ SpDM^{nc}.

Étale = Étale on underlying ∞-topos + cartesian edge of $\infty\mathrm{Top}_{\mathrm{CAlg}}$
$$\begin{tikzcd} \mathrm{SpDM}^{nc} \arrow[r, hook] & \infty\mathrm{Top}_{\mathrm{CAlg}} \arrow[d] \\ & \infty\mathrm{Top} \end{tikzcd}$$

$(\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathrm{SpDM}^{nc} \Rightarrow (\mathcal{X}_{/f^*U}, \mathcal{O}_\mathcal{X}|_{f^*U})$

(1) DAG V, Prop 2.3.21

(4) cover Y by affines

pullback
$$\begin{tikzcd} X'_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_\alpha \arrow[d] \\ Y'_\alpha \arrow[r] & \mathrm{Sp\acute{e}t}\, A_\alpha \end{tikzcd}$$

cover X_α, Y'_α by affines

$\rightsquigarrow$ reduced to affine case

follows from A, B, R: connective

$\Longrightarrow A \underset{R}{\otimes} B$ connective $\boxed{\text{S}}$

cf 1.4.8.1

<u>Cor</u> SpDM^{nc} is <u>idempotent</u> complete

$\uparrow$

in 1-cat $\mathcal{C}$

$$\left\{ \text{retract } A \underset{i}{\overset{r}{\leftrightarrows}} X \right\} \overset{\text{injective}}{\rightsquigarrow} \left\{ X \circlearrowleft e = ir \ \text{idempotent} \ e^2 = e \right\}$$

$ri = id$

$\leftarrow - - -$

$$A \xrightarrow{i} X \underset{id_X}{\overset{e}{\rightrightarrows}} X \xrightarrow{r} A \qquad\qquad X \xrightarrow{e} X$$

i | eq

r | coeq

$\mathcal{C}$ is idempotent complete if this is surjective as well.

($\Uparrow$ $\mathcal{C}$ admits coeq or eq) in ∞-cat $\quad e^2 \cancel{=} e \quad e^3 \simeq e^2$

$\simeq$ $\quad$ $\begin{matrix} e^3 \simeq e^2 \\ \wr\!\!|_{S_2} \quad \wr\!\!| \\ e^2 \simeq e \end{matrix}$ $\sim \sim'$

$$\mathrm{Idem} = N\big(\overset{\mathrm{id}}{\circlearrowright} * \underset{e}{\circlearrowleft} \ , \ e = e^2 \big)$$

retracts can be recovered by colim of this diag.

Def $\mathcal{C}$: idem. compl. if

$\forall\ \mathrm{Idem} \longrightarrow \mathcal{C}$ admits colim.

Remark • Idem : infinite,

• when $\mathcal{C}$: n-category for $n < \infty$,

$\mathcal{C}$ admits finite lim or colim

$\Downarrow$

idem complete

• Idem weakly contractible

• $\mathrm{Idem} \underset{\text{retract}}{\overset{\longleftarrow}{\hookrightarrow}} \mathrm{Idem}^{\triangleright} \rightsquigarrow$ K-filtered for $\forall K \geq \omega$

If $\mathcal{C}$ admits filtered colim or lim

$\Rightarrow$ idempotent complete

Proof $\infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ admits filtered limits

full. $\cup$ $\quad$ (21.4.3.1. (3))

$\mathrm{SpDM}^{\mathrm{nc}}$ $\quad$ $\Rightarrow$ idempotent complete.

enough to show: closure under retracts in $\infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{CAlg}}$

$$\begin{tikzcd}
 & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \mathrm{SpDM}^{nc} \arrow[dr, "r"] & \\
(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \arrow[ur, "i"] \arrow[rr, equal] & & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})
\end{tikzcd}$$

verify conditions in 1.4.8.1

(1) $(\mathfrak{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathfrak{X}}) \in \mathrm{DM}$

(2) $\mathfrak{X} \longrightarrow \mathrm{Shv}_{\mathcal{E}}(\mathfrak{X}^{\heartsuit})$ 1-loc refl. is étale

(3) $\pi_n\mathcal{O}_{\mathfrak{X}}$: qcoh on $(\mathfrak{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathfrak{X}})$

(4) $\mathcal{O}_{\mathfrak{X}}$: hypercomplete

(1) $(\mathfrak{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathfrak{X}})$: retract of $(\mathcal{Y}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{Y}}) \in \mathrm{DM}$ in $\underbrace{1\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{CAlg}^{\heartsuit}}}_{\text{2-cat}}$

$\cap$ DM.

finite complete

$\underbrace{}$ $\Downarrow$ idem compl.

(2)
$$\begin{tikzcd}
\mathfrak{X} \arrow[r] \arrow[dr, "i"] \arrow[d] \arrow[rr, bend left, "r"] & \mathcal{Y} \arrow[r] \arrow[d, "\text{ét}"] & \mathfrak{X} \arrow[d] \\
\mathrm{Shv}(\mathfrak{X}^{\heartsuit}) \arrow[r] & \mathrm{Shv}(\mathcal{Y}^{\heartsuit}) \arrow[r] & \mathrm{Shv}(\mathfrak{X}^{\heartsuit})
\end{tikzcd}$$

$\mathcal{Z} \to \mathcal{Y}$, $\mathcal{Z} \to \mathrm{Shv}(\mathfrak{X}^{\heartsuit})$ ét

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, shift right] \arrow[loop below, "\mathrm{id}"] & \mathcal{Z} \arrow[l, shift right] \arrow[loop below, "e = ir"]
\end{tikzcd}$$

in $\infty\mathrm{Top}_{/\mathrm{Shv}(\mathfrak{X}^{\heartsuit})}$ closed under retracts

$\cup$ — closed under colim.

$\sim \in \infty\mathrm{Top}^{\text{ét}}_{/\mathrm{Shv}(\mathfrak{X}^{\heartsuit})}$

$$\left(\ e : \text{étale} \qquad \begin{tikzcd} Z \arrow[rr, "e"] \arrow[dr, "\text{ét}"'] & & Z \arrow[dl, "\text{ét}"] \\ & \mathrm{Shv}(\mathfrak{X}^{\heartsuit}) & \end{tikzcd} \right),$$

(3) point: $\mathrm{QCoh}^{\heartsuit}_{\pi_0 \mathfrak{X}} \subset \mathrm{Mod}^{\heartsuit}_{\pi_0 \mathfrak{X}}$ closed under retracts

by def
$$\begin{tikzcd} \mathcal{F} \arrow[loop left, "\mathrm{id}"] \arrow[r, shift right, "i"'] & \mathcal{G} \arrow[l, shift right, "r"'] \end{tikzcd} \qquad \mathcal{G} \text{ qcoh.}$$

$$\begin{tikzcd} \mathcal{O}_{\mathfrak{X}}(U) \otimes_{\mathcal{O}_{\mathfrak{X}}(U)} \mathcal{F}(U) \arrow[r] \arrow[d, shift right, "i"'] \arrow[dr, phantom, "\Uparrow \cong"] & \mathcal{F}(U) \arrow[d, shift right, "i"'] \\ \mathcal{O}_{\mathfrak{X}}(U) \otimes_{\mathcal{O}_{\mathfrak{X}}(U)} \mathcal{G}(U) \arrow[r, "\cong"] \arrow[u, shift right, "r"'] & \mathcal{G}(U) \arrow[u, shift right, "r"'] \end{tikzcd}$$

(4) $\mathcal{O}_{\mathfrak{X}}$: hypercomplete

$$\begin{tikzcd} & r_* \mathcal{O}_{\mathcal{Y}} \arrow[dr] & \\ \mathcal{O}_{\mathfrak{X}} \arrow[ur] \arrow[rr, "\cong"'] & & r_* i_* \mathcal{O}_{\mathfrak{X}} \end{tikzcd} \qquad \text{hypercomplete}$$

closed under retracts $\Leftarrow$ hypercomplete objects are closed under limits ($+\ \Omega^{\infty}$ preserves limits) ∎

Addendum

$$
\begin{tikzcd}
\mathrm{Idem} = \text{Nerve of } \Bigg(* \arrow[loop above, "\mathrm{id}"] \arrow[loop below, "e"] \;\Bigg|\; e^2 = e \Bigg) \arrow[d, "\text{cofinal}"] \\
\mathrm{Idem}^+ = \text{Nerve of } \Bigg(* \arrow[loop above, "\mathrm{id}"] \arrow[loop below, "e"] \arrow[r, shift right, "r"'] & * \arrow[l, shift right, "i"'] \arrow[loop below, "\mathrm{id}"] \;\Bigg|\; e = ir \Bigg) \\
\mathrm{Ret} \arrow[u, "\simeq"', "\text{Inner anodyne}"]
\end{tikzcd}
$$

$$
\mathrm{Ret} = \left(
\begin{tikzcd}
 & * \arrow[dr, "r"] \arrow[d, phantom, "\text{is}"] & \\
* \arrow[ur, "i"] \arrow[rr, equal] & & *
\end{tikzcd}
\right) = \Delta^2 / \Delta^{\{0,2\}}
$$

$$
\begin{tikzcd}
\{\mathrm{Ret} \to \mathcal{C}\} \arrow[r, phantom, "\simeq"] & \{\mathrm{Idem}^+ \to \mathcal{C}\} \arrow[r, shift right, "\text{restriction}"'] & \{\mathrm{Idem} \to \mathcal{C}\} \arrow[l, dashed, shift right, "\text{left/right Kan ext}"']
\end{tikzcd}
$$

"honest" retract diagram

$\simeq$ if ↑