Deformation Theory §7.4.1

Review $\mathcal{A}$: presentable ("cat of algebras")

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, bend right, "L"'] \arrow[d, shift right, "-1"'] \\
T_{\mathcal{A}} \arrow[ur, bend left, "sq\text{-}0"] \arrow[r, shift right, "\Omega^\infty"'] \arrow[dr] & \mathcal{A}^1 \arrow[l, shift right, "\Sigma^\infty_+"'] \arrow[u, shift right, "ev_0"'] \arrow[d, "ev_1"] \\
 & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
CAlg \arrow[d, shift right, "L"'] \\
Mod \arrow[u, shift right]
\end{tikzcd}
\qquad \left(\text{or} \quad
\begin{tikzcd}
\overbrace{Alg}^{Alg_{\mathbb{E}_1}(\mathcal{C})} \arrow[d, shift right] \\
{}_A\text{"}BMod\text{"}_A \arrow[u, shift right]
\end{tikzcd} \right)$$

$${}_A\text{"}BMod\text{"}_A \simeq Mod^{\mathbb{E}_1}(\mathcal{C})$$

- Denote $(A, M) \in T_{\mathcal{A}}$ the object $M \in Sp(\mathcal{A}^{/A})$
- $ev_0 \circ \Omega^\infty(A, M) =: A \oplus M$ triv. sq-zero ext

(· $L_A = (A, L_A)$)

$$\left(\begin{array}{ccc}
Mod_A \ni M & \longmapsto & \text{"}A\oplus M\text{"} \in CAlg \\
\simeq & & \\
S^0 \quad \cancel{CAlg^{aug} \simeq} \quad * & & \\
Sp(CAlg^{A/}) & \rightleftarrows & CAlg_{A/}
\end{array} \right)$$

§7.3.6

Packaged in $\mathcal{M}^T(\mathcal{A})$ the tangent corr. to

- $\mathcal{M}^T(\mathcal{A}) \xrightarrow[P=(p_1, p_2)]{} \Delta^1 \times \mathcal{A}$ categorical fib

• $p_1 : \mathcal{M}^T(\mathcal{A}) \longrightarrow \Delta^1$ cart & cocart

classifying
$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & T_{\mathcal{A}} \arrow[l, shift left, "sq\,0"] \end{tikzcd}$$
$p^{-1}(0)$ $\qquad$ $p^{-1}(1)$

i.e. mor in $\mathcal{M}^T(\mathcal{A})$ are:

- mor of alg $A \to B$
- ~ modules $(A,M) \to (B,N)$

  $f : A \to B$, $f_! M \to N$
  $\qquad\qquad\qquad\;\; B \otimes_A M$

- $$\frac{\dfrac{A \longrightarrow (B,N)}{f_! L_A \longrightarrow N \text{ in } \mathrm{Mod}_B}}{A \longrightarrow B \oplus N \text{ in } \mathrm{CAlg}_{/B}}$$

• $p_2 : \mathcal{M}^T(\mathcal{A}) \longrightarrow \mathcal{A}$

$$\begin{tikzcd} \mathcal{M}^T(\mathcal{A}) \arrow[r] & \mathcal{A} \\ \mathcal{A} \sqcup T_{\mathcal{A}} \arrow[u, hook] \arrow[ur, "\text{underlying objects}"'] & \end{tikzcd}$$

$(\mathrm{id}_{\mathcal{A}} \sqcup ev_1 \circ \Omega^\infty \quad)$

Classically (discrete case) ← in $\mathrm{CAlg}^{\heartsuit}$, $\mathrm{Mod}_R^{\heartsuit}$,

$$0 \to M \to \tilde{R} \to R \to 0 \quad \text{sq-zero ext}$$

sq-zero ext is trivial ⟺ ∃ section ⟺ ($\tilde{R}$ mod str. on M descends to R mod)

⟺ $M^2 = 0$

- $\mathrm{Aut}_R(R \oplus M)$
  $\simeq \mathrm{Alg}_R(R, R\oplus M)$
  $\simeq \mathrm{Der}(R, M)$
  $\simeq \mathrm{Ext}^0_R(\Omega_R, M) \simeq \pi_1 \mathrm{Hom}_R(\Omega_R, M[1])$

↑ mapping spectrum

$$\left[\begin{array}{l} \pi_i \mathrm{Hom}_R(M,N) = \mathrm{Ext}^{-i}_R(M,N) \\ \quad \cong \\ \mathrm{Hom}_R(M[i], N) \\ \qquad\qquad N[i] \end{array} \right]$$

- $\pi_0 \mathrm{Hom}_R(\Omega_R, M[1]) \simeq \mathrm{Ext}^1_R(\Omega_R, M) \to \{\text{sq zero ext of } R \text{ by } M\}$

L_R

$$\varphi \quad \left[\eta : 0 \to M \to \tilde{M} \xrightarrow{f} \Omega_R \to 0 \right]$$

fib

$$\begin{array}{ccccccc} & & P_2 & & & & \\ \cdots \to & & P_1 & \to & P_0 & \to & \Omega_R \to 0 \\ & \searrow^{0} & \downarrow & & \downarrow & \swarrow & \\ 0 \to & & M & \to & \tilde{M} & & \end{array}$$

$$\Omega_R \to M$$

$$\begin{array}{ccccc} (r, \tilde{m}) \in \tilde{R} & \to & R & & x \\ \downarrow & \lrcorner & \downarrow d & & \downarrow \\ \tilde{M} & \xrightarrow{f} & \Omega_R & & dx \end{array}$$

$dr = f(\tilde{m})$

$$(r_1, \tilde{m}_1) \cdot (r_2, \tilde{m}_2)$$

$$= (r_1 r_2,\ r_2 \tilde{m}_1 + r_1 \tilde{m}_2)$$

Def $\mathcal{D}er(\mathcal{A})$: ∞-cat of derivations

$$\begin{tikzcd}
\mathcal{D}er(\mathcal{A}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathcal{M}^T(\mathcal{A})) \arrow[d, "p_*"] \\
\mathcal{A} \arrow[r, hook] & \mathrm{Fun}(\Delta^1, \Delta^1 \times \mathcal{A}) \\[-1.5em]
A \arrow[r, mapsto] & \left((0,A) \xrightarrow{id_A} (1,A) \right)
\end{tikzcd}$$

objects $A \longrightarrow (A, M)$ in $\mathcal{M}^T(\mathcal{A})$

$$\left(\begin{tikzcd} A \arrow[r] \arrow[d, mapsto] & (A,M) \arrow[d, mapsto] \\ (0,A) \arrow[r] & (1,A) \end{tikzcd} \text{ in } \Delta^1 \times \mathcal{A} \right)$$

$L_A \longrightarrow M$ derivation

Def $\widetilde{\mathcal{D}er}(\mathcal{A})$: ∞-cat of extended derivations

$\cap$ full

$$\begin{tikzcd}
\square \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}^T(\mathcal{A})) \arrow[d, "p_*"] \\
\mathrm{Fun}(\Delta^1, \mathcal{A}) \arrow[r, hook] & \mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1 \times \mathcal{A})
\end{tikzcd}$$

$$\tilde{A} \xrightarrow{f} A \longmapsto \left[\begin{tikzcd} (0, \tilde{A}) \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "f"] & (0, A) \arrow[d, "id_A"] \\ (1, A) \arrow[r, "id_A"'] & (1, A) \end{tikzcd} \right]$$

objects of $\boxed{}$: $\sigma : (\Delta^1)^2 \to \mathcal{M}^T(\mathcal{A})$ of the form (0)

$$\begin{tikzcd}
\tilde{A} \arrow[r, "f"] \arrow[d] & A \arrow[d] \\
(A, N) \arrow[r] & (A, M)
\end{tikzcd}$$

— derivation $\boxed{L_A \to M}$

$\wr$

$N \to M$ in Mod_A

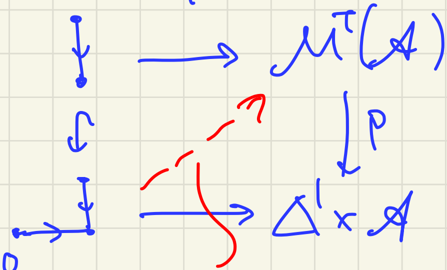
$\sigma \in \widetilde{\mathrm{Der}}(\mathcal{A})$ if further

(1) $N = 0$ (in $\mathrm{Sp}(\mathcal{A}^{/A})$ Mod_A)

(2) pullback in $\mathcal{M}^T(\mathcal{A})$

(1) $\Longleftrightarrow$ (A, N) : p-initial in $\mathcal{M}^T(\mathcal{A})$

$\Longleftrightarrow$



p-Kan ext

$$\widetilde{\mathrm{Der}}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Der}(\mathcal{A}) \quad \text{trivial Kan fib}$$

HTT 4.3.2.15

$\exists s$: section

Def $\Phi : \mathcal{D}er(\mathcal{A}) \xrightarrow{s} \widetilde{\mathcal{D}er}(\mathcal{A}) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{A})$

$$L_A \xrightarrow{\eta} M \qquad \begin{array}{ccc} A^\eta & \to & A \\ \downarrow & \lrcorner & \downarrow \\ (A,0) & \to & (A,M) \end{array} \qquad \boxed{} \qquad A^\eta \longrightarrow A$$

~~$\mathcal{A}[1]$~~

square-zero ext of A by $M[-1]$ associated to η

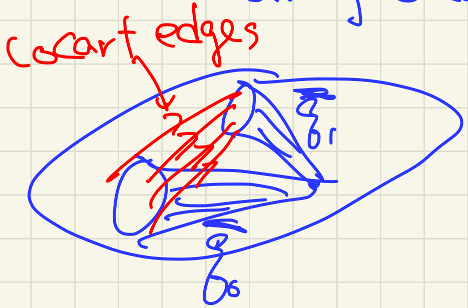
A^η

$\tilde{A} \xrightarrow{f} A$ is a sq-zero ext if it lies in the ess. im.

HTT 4.3.1.9 relative colimit can be "pushed" along cocart edges

$$\begin{array}{c} X \\ p \downarrow \\ S \end{array}$$

inn. fib of ∞-cats



$\bar{q}_1$: p-colim

$\Longleftrightarrow \bar{q}_0$: p-colim

Rmk 7.4.1.7
$$\begin{array}{ccc} A^\eta & \xrightarrow{f} & A \\ \downarrow & \lrcorner & \downarrow \eta \\ (A,0) & \xrightarrow{0} & (A,M) \end{array} \qquad \begin{array}{c} \mathcal{M}^T(\mathcal{A}) \\ \downarrow p_1 \\ \Delta^1 \\ \downarrow p_1' \\ \ast \end{array} \qquad \begin{array}{ccc} 0 & \to & 0 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \to & 1 \end{array}$$

pull back $\Longleftrightarrow$ p_1'-pull back

~~$(p_1' \circ p_1)$~~-pullback iff p_1-pullback

pullback $\Leftrightarrow$ p_1 - pullback

$$\Leftrightarrow \quad \begin{tikzcd} A^\eta \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "d_\eta"] \\ A \arrow[r, "d_0"'] & A \oplus M \end{tikzcd}$$

$$[L_A \xrightarrow[0]{} M]$$

$\cancel{p_1}$ - pullback

$$\left\{ \begin{tikzcd} \underline{\mathcal{A}} \arrow[r, phantom, "="] \arrow[d] & \mathcal{M}^T(\mathcal{A}) \arrow[d, "\text{cocart}"] \\ \{0\} \arrow[r, phantom, "\subset"] & \Delta^1 \end{tikzcd} \right.$$

HTT 4.3.1.16

ex 7.4.1.9 $M \in Sp(\mathcal{A}^{/A})$

$\eta : L_A \xrightarrow[0]{} M$

$$\rightsquigarrow \begin{tikzcd} (A, M[-1]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (A, 0) \arrow[d] \\ (A, 0) \arrow[r] & (A, M) \end{tikzcd}$$

lim in $Sp(\mathcal{A}^{/A})$

$$\downarrow \Omega^\infty$$

$\mathcal{A}^{/A}$

$\cap$

$\mathcal{A}$

$$\rightsquigarrow \begin{tikzcd} A \oplus M[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "d_0"] \\ A \arrow[r, "d_0"'] & A \oplus M \end{tikzcd}$$

(cf $M = 0$ $A^\eta \simeq A$)

Warning: Φ fails to be faithful (sq-zero ext $f : \tilde{A} \to A$ does not determine η & M)

- No obvious intrinsic characterization of sq zero ext

Lurie says: { A-mod str. can't be recovered } $\to$ fib f

Want to identify ~~as~~ classes of sq-zero ext with intrinsic characterization on which Φ is invertible.

<u>n-small ext</u>

Sp

$\mathcal{C}$: presentable symmetric monoidal closed stable ∞-cat

w/ t-str. compatible w/ $\otimes$ $\left(\mathbb{1} \in \mathcal{C}_{\geq 0},\ \mathcal{C}_{\geq 0} \text{ closed under } \otimes \right)$

$$\begin{cases} \mathrm{Sp}^{cn} \underset{\tau}{\rightleftarrows} \mathrm{Sp} \ \boxed{\text{sym monoidal}} & \mathrm{Alg}^{cn} \underset{\tau}{\rightleftarrows} \mathrm{Alg} \\ \mathcal{C}_{\geq 0} \underset{\tau}{\rightleftarrows} \mathcal{C} & \\ X, Y \qquad X, Y & \\ X \otimes Y \mapsto X \otimes Y \in \mathcal{C}_{\geq 0} & \end{cases}$$

$\mathcal{A}^{(1)} = \mathrm{Alg}(\mathcal{C})$ assoc alg

$f : A \longrightarrow B$ in $\mathcal{A}^{(1)}$, $I := \mathrm{fib}(f)$

Note I have the mult given by

$$I \otimes_A I \longrightarrow I \otimes_A A \simeq I$$

(In fact, I : nonunital alg in the category ${}_A\mathrm{BMod}_A(\mathcal{C})$)

0 : initial alg

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array}$$

limits are computed in the cat of nonunital alg $\Updownarrow$ by underlying objects

Def (Assoc case) $n \geq 0$, $f : A \to B$ in $\mathcal{A}^{(1)} := \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$

- f : <u>n-connective ext</u> $\iff$ $A \in \mathcal{C}_{\geq 0}$, $I = \mathrm{fib}(f) \in \mathcal{C}_{\geq n}$ [in ${}_A\mathrm{BMod}_A(\mathcal{C})$]
- f : <u>n-Small</u> $\iff$ n-conn. & $I \in \mathcal{C}_{\leq 2n}$, $I \otimes_A I \to I$: null.

$$\mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathcal{A}^{(1)}) \underset{\text{full}}{\subset} \mathrm{Fun}_{n\text{-con}}(\Delta^1, \mathcal{A}^{(1)}) \underset{\text{full}}{\subset} \mathrm{Fun}(\Delta^1, \mathcal{A}^{(1)})$$

$$\mathcal{D}er^{(1)} := \mathcal{D}er(\mathcal{A}^{(1)}) \ni (A, M, \eta : L_A^{(1)} \to M[1])$$

$A \in \mathcal{A}^{(1)}$, $M \in {}_A\mathrm{BMod}_A(\mathcal{C})$; η: map of A–A bimod.

$\cup$ full

$\mathcal{D}er^{(1)}_{n\text{-con}}$: $A \in \mathcal{C}_{\geq 0}$, $M \in \mathcal{C}_{\geq n}$

$\cup$ full

$\mathcal{D}er^{(1)}_{n\text{-sm}}$: $M \in \mathcal{C}_{\leq 2n}$

$$\left[L_{B/A} \to B \otimes_A B \xrightarrow[m]{} B \right]$$

$$\left(\Omega_{B/A} \to I \otimes_A I \xrightarrow[m]{} I \right)$$

For $\mathbb{E}_k$ $1 \leq k \leq \infty$

$$\mathcal{A}^{(k)} := \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) = \left\{ \begin{tikzcd} & \mathcal{C}^{\otimes} \arrow[d] \\ \mathbb{E}_k^{\otimes} \arrow[ur] \arrow[r] & \mathbb{E}_\infty^{\otimes} \end{tikzcd} \right\} \quad \text{(operad map)}$$

forget $\theta \downarrow$

$$\mathcal{A}^{(1)} := \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$$

$\mathbb{E}_\infty^{\otimes} \simeq \mathrm{Fin}_*$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\otimes} \arrow[d] \\ \mathbb{E}_k \arrow[r] \arrow[ur, dashed, red] \arrow[uu, dashed, red, bend left, "\mathrm{Fin}_*"] & \mathbb{E}_\infty \end{tikzcd}$$

$\mathcal{A} = \mathcal{A}^{(\infty)}$ only write this

- f: n-small in $\mathcal{A}$ $\overset{\text{def}}{\iff}$ $\theta(f)$: n-small
- same def of $\mathcal{D}er_{n\text{-sm}}$ w/ $L_A^{(1)}$ replaced by L_A $({}_A\mathrm{BMod}_A \rightsquigarrow \mathrm{Mod}_A)$

Thm (7.4.1.23) 26

$$\Phi : \mathrm{Der}(\mathcal{A}) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{A})$$
$$\cup \qquad\qquad \cup$$
$$\Phi_{n\text{-sm}} : \mathrm{Der}_{n\text{-sm}}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathcal{A})$$

[restricts to this because $I \to A^\eta \xrightarrow{f} A$

$I \underset{A^\eta}{\otimes} I \longrightarrow I$: null

7.4.1.12~18 ?]

[This is an equiv of categories]

Cor n-small $\Rightarrow$ square-zero

Cor $A \in \mathrm{CAlg}^{cn}$

[Sq-zero extension]

$$\to \cdots \to \tau_{\le 3} A \to \tau_{\le 2} A \to \tau_{\le 1} A \to \tau_{\le 0} A$$

$I = (\pi_n A)[n] \longrightarrow \tau_{\le n} A$

$\qquad\qquad\qquad\qquad\downarrow$

$\qquad\qquad\qquad\quad \tau_{\le n-1} A$

$\boxed{\text{n-small}}$

$\leftarrow (\pi_n A) \otimes (\pi_n A)$

$\longrightarrow \pi_{2n}(\pi_n A[n])$
$= 0$

Note $I \underset{A}{\otimes} I \xrightarrow{m} I$ is null iff $\pi_{2n}(m) = 0$

$\qquad\quad \cap \qquad\qquad \cap$

$\qquad \mathcal{C}_{\ge 2n} \qquad \mathcal{C}_{\le 2n}$

$\Leftrightarrow$ [$\underbrace{\pi_n I \otimes \pi_n I}_{\mathrm{Tor}_0} \longrightarrow \pi_{2n} I$ is 0]

$\tau_{\ge n+1} I \to I \to (\pi_n I)[n]$

I. Relative Cotangent Complex (first 1/3 of §7.3.3)

II. Connectivity estimate (1^{st} half of §7.4.3)

(III. Proof of Thm 7.4.1.26 $\mathbb{E}_\infty$-case (after ~~DAG~~ §3.2)

for connectivity est.

$\mathbb{E}_1$: $L^{(*)}_{B/A} \to B \otimes_A B \xrightarrow{m} B$

$\mathbb{E}_\infty$: $\otimes$ is coproduct of alg

avoid operadic difficulty

allows us to reuse §7.4.3

Throughout

$\mathcal{C}$: presentable sym monoidal closed stable

w/ $\underbrace{\text{t-str. compatible w/ } \otimes}$

$\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$

e.g. Sp 7.1.1.7

Convention

$\mathrm{CAlg}(\mathcal{C}) = \mathrm{CAlg}$

$\mathrm{Mod}(\mathcal{C}) = \mathrm{Mod}$

I. $A \xrightarrow{f} B$ in CAlg $\rightsquigarrow L_{B/A} \in \mathrm{Mod}_B$

Define $L_{(-)/(-)} : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \longrightarrow \mathrm{Mod}$

$(A \xrightarrow{f} B) \longmapsto L_{B/A} \in \mathrm{Mod}_B$

which fits into cofib. seq.

$B \otimes_A L_A \longrightarrow L_B \longrightarrow L_{B/A}$ in Mod_B

Setup

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \arrow[r, "L"] & \mathrm{Fun}(\Delta^1, \mathrm{Mod}) & \mathrm{Fun}^{p\text{-cofib}}(\Delta^1 \times \Delta^1, \mathrm{Mod}) \arrow[l, "\sim"'] \arrow[d, "ev_{1,1}"] \\ & & \mathrm{Mod} \end{tikzcd}$$

$\boxed{\mathcal{E}} \subset \mathrm{Fun}^{p\text{-cofib}}$, forget

$(A \xrightarrow{f} B) \mapsto (L_A \to L_B)$

HTT.4.3.2.5

$$\begin{tikzcd} (A, L_A) \arrow[r] \arrow[d] & (B, L_B) \arrow[d] \\ (A, 0) \arrow[r] & (B, L_{B/A}) \end{tikzcd} \longmapsto (B, L_{B/A})$$

$$\begin{tikzcd} \mathrm{Mod} \arrow[d, "p"] \\ \mathrm{CAlg} \end{tikzcd}$$

$\mathrm{Fun}^{p\text{-cofib}}((\Delta^1)^2, \mathrm{Mod})$

$\cap$ full

$$\begin{tikzcd} \boxed{\mathcal{E}} \arrow[r] \arrow[d, "\wr"] & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Mod}) \arrow[d, "p"] \\ \mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \arrow[r] & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{CAlg}) \end{tikzcd}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "f"] & B \end{tikzcd} \right)$$

obj of $\boxed{\mathcal{E}}$

$$\sigma : \begin{tikzcd} (A, M) \arrow[r] \arrow[d] & (B, N) \arrow[d] \\ (A, M') \arrow[r] & (B, N') \end{tikzcd} \quad \text{in } \mathrm{Mod}$$

$$\left[\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \mathrm{Mod} \\ \cdot \arrow[r] & \cdot \arrow[ur, dashed, "p\text{-Ran}"'] & \end{tikzcd} \right]$$

$\sigma \in \mathrm{Fun}^{p\text{-cof}} \iff$ ① $M' = 0$ in Mod_A

② σ : pushout in Mod

Since $\Delta^1\times\Delta^1 \xrightarrow{\sigma} \mathrm{Mod}$

$$\begin{tikzcd} \Delta^1\times\Delta^1 \arrow[r, "\sigma"] \arrow[dr, "p\circ\sigma"'] & \mathrm{Mod} \arrow[d, "p"] \\ & \mathrm{CAlg} \arrow[d, "q"] \\ & * \end{tikzcd}$$

$p\circ\sigma$: pushout

② $\Longleftrightarrow$ σ : p-pushout

$\Uparrow$ $p\circ\sigma$

$$\begin{tikzcd} \arrow[r, "f"] \arrow[dr, "p\circ f"'] & \arrow[d, "p"] \\ & \arrow[d, "q"] \\ & {} \end{tikzcd}$$

[HTT 4.3.1.?]

if $p\circ f$: q-colim

f : p-colim $\Longleftrightarrow$ f : $(q\circ p)$-colim

$\Longleftrightarrow$ (HTT.4.3.1.8 + 4.3.1.?)

$$\begin{tikzcd} f_!M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & N \arrow[d] \\ 0 \arrow[r] & N' \end{tikzcd}$$

in Mod_B

$0 \simeq f_!0$

$$\boxed{B\otimes_A L_A \to L_B \to L_{B/A} \text{ in } \mathrm{Mod}_B}$$

It is easy to prove :

- for $A\to B\to C$

we have

$$\begin{tikzcd} (B, L_{B/A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (C, L_{C/A}) \arrow[d] \\ (B, L_{B/B}) \arrow[r] & (C, L_{C/B}) \end{tikzcd}$$

in Mod

$\Longleftrightarrow$ <u>cofib seq</u>

$C\otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}$

-

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "f"] \\ A' \arrow[r] & B' \end{tikzcd}$$

in CAlg $\rightsquigarrow$ $(B, L_{B/A}) \to (B', L_{B'/A'})$: p-cocart in Mod

$(\Longleftrightarrow B'\otimes_B L_{B/A} \xrightarrow{\sim} L_{B'/A'})$

II If $A \xrightarrow{f} B$: equiv $\rightsquigarrow$ $L_{B/A} \simeq 0$
in $\mathrm{CAlg}(\mathcal{C})$ $\qquad \mathrm{cofib}(f) \simeq 0$

measure failure of f: equiv.

$\exists$ natural comparison map

$$\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \longrightarrow L_{B/A} \quad \text{in } \mathrm{Mod}_B$$

<u>Thm 7.4.3.12</u> If $f : A \to B$ in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$
(DAG IV Thm 2.4) $\quad \mathrm{cofib}(f) \in \mathcal{C}_{\geq n}$

$$\Longrightarrow \mathrm{fib}(\varepsilon_f) \in \mathcal{C}_{\geq 2n}$$

<u>Construction of ε_f</u>

$$\begin{tikzcd}
(A, L_A) \arrow[r] \arrow[d] & (B, L_B) \arrow[d] \\
(A, 0) \arrow[r] & (B, L_{B/A}) \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd} \quad \text{in } T_{\mathcal{C}} \simeq \mathrm{Mod}(\mathcal{C})$$

cocart

$$\Longleftrightarrow \quad \begin{tikzcd}
A \arrow[rr, "f"] \arrow[d] \arrow[r, dashed, "f'"] & B^{\eta} \arrow[r, "f''"] \arrow[d] & B \arrow[d, "\eta"] \\
(A, 0) \arrow[rr] \arrow[r] & (B, 0) \arrow[r] & (B, L_{B/A})
\end{tikzcd}$$

in $\underset{\sim}{\mathcal{M}^T(\mathcal{C})}$

$$\left(\begin{tikzcd}
\cdot \arrow[r] \arrow[rr, bend left, "\text{forgetful}"] \arrow[dr, "p"'] & \mathcal{C} \times \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^1 &
\end{tikzcd} \right)$$

classifies

$$\left(\mathcal{C} \underset{\perp}{\overset{L}{\rightleftarrows}} T_{\mathcal{C}} \right)$$

$\uparrow$ sq-zero ext $\uparrow$
$p^{-1}(0)$ $\qquad p^{-1}(1)$

$\mathcal{C} \quad T_{\mathcal{C}}$

$$\begin{tikzcd}
 & & \mathrm{cof}(f'') \simeq L_{B/A} \in \mathrm{Mod}_B \\
A \arrow[r, "f"] \arrow[dr] & B \arrow[ur] \arrow[r] & \mathrm{cof}(f) \arrow[u] \quad \text{in } \mathrm{Mod}_A \\
 & B'' \arrow[u, "f''"'] & \\
L_{B/A}[-1] \arrow[ur] & &
\end{tikzcd}$$

$$\rightsquigarrow \; B \otimes_A \mathrm{cof}(f) \xrightarrow[\varepsilon_f]{} L_{B/A} \quad \text{in } \mathrm{Mod}_B$$

Outline of proof of Thm

We say $A \xrightarrow{f} B$ in $\mathrm{CAlg}(\mathcal{C})$ is <u>n-good</u> if $\mathrm{fib}(\varepsilon_f) \in \mathcal{C}_{\geq 2n}$

<u>Step 1</u> Stability properties of n-good morphisms

1 <u>(a)</u>

$$\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "h"'] & & C
\end{tikzcd}
\quad \text{in } \mathrm{CAlg}(\mathcal{C}_{\geq 0}) \quad
\begin{cases}
\cdot\ f, g : n\text{-good} \\
\cdot\ \mathrm{cofib}(f), \mathrm{cofib}(g) \in \mathcal{C}_{\geq n}
\end{cases}$$

$$\Longrightarrow h : n\text{-good}$$

2 <u>(b)</u>

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}
\quad \text{in } \mathrm{CAlg}(\mathcal{C}) \quad
\begin{array}{l}
\cdot\ B, B' : \text{connective} \\
\cdot\ f : n\text{-good}
\end{array}$$

$$\Longrightarrow f' : n\text{-good}$$

<u>Step 2</u> by "killing homotopy" argument (Lem 7.4.3.15) 3

$$\begin{tikzcd}
A = A_n \arrow[r, "g_n"] \arrow[dr, "f = f_n"'] & A_{n+1} \arrow[r, "g_{n+1}"] \arrow[d] & A_{n+2} \arrow[r] \arrow[dl, "f_{n+2}"] & \cdots \arrow[r, "g_{2n}"] & A_{2n+1} \arrow[dlll, bend left, "f_{2n+1}"] \\
 & B & & &
\end{tikzcd}
\quad \text{in } \mathrm{CAlg}(\mathcal{C}_{\geq 0})$$

$\mathrm{Sym}^* : \mathcal{C} \rightleftarrows \mathrm{CAlg}(\mathcal{C})$

which satisfies :

(i) $\mathrm{cofib}(\underset{A_m \to B}{f_m}) \in \mathcal{C}_{\geq m} \subset \mathcal{C}_{\geq n}$

(ii) $\forall m \in \{n, \cdots, 2n\}$ $\exists M_m \in \mathcal{C}_{\geq m-1}$ g_m is given by

$$\begin{tikzcd} \mathrm{Sym}^* M_m \arrow[r, "\phi_m"'] \arrow[d] & \mathrm{Sym}^* 0 \simeq \mathbb{1} \arrow[d] \\ A_m \arrow[r, "g_m"] & A_{m+1} \end{tikzcd}$$

(initial alg) e.g. $\mathbb{S}$

in $\mathrm{CAlg}(\mathcal{C})$

$(\phi_m : \mathrm{Sym}^*(M_m \xrightarrow{0} 0))$

By step 1 we are reduced to proving

- ϕ_m $(n \leq m \leq 2n)$
- f_{2n+1}

are n-good

g_m : n-good $\overset{(b)}{\Leftarrow}$ ϕ_m

g_m, f_{2n+1} $\overset{(a)}{\Rightarrow}$ f : n-good

Step 3 ϕ_m : n-good reduces to :

④ Lem(d) $M \in \mathcal{C}_{\geq n-1}$ $\phi : \mathrm{Sym}^* M \longrightarrow \mathrm{Sym}^* 0 \simeq \mathbb{1}$

$\Rightarrow \phi$: n-good

(proven by explicit computation)

$\Omega_{k[x_1,\cdots,x_n]/k} \overset{?}{\simeq} k[x_1,\cdots,x_n](dx_1,\cdots,dx_n)$

$\sum f_i dx_i$

Step 4 f_{2n+1} : n-good

Note that $\tau_{\leq 2n} f_{2n+1}$: equivalence :

$$\begin{tikzcd} \tau_{\leq 2n} A_{2n+1} \arrow[r, "\tau_{\leq 2n}", "\sim"'] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & \tau_{\leq 2n} \mathrm{Cof}(f_{2n+1}) \simeq 0 \end{tikzcd}$$

$$L_{B/A}[-1] \longrightarrow \underbrace{\mathrm{fib}(\mathcal{E}_{f_{2n+1}})}_{\text{want to show}} \longrightarrow \underbrace{B \otimes_A \mathrm{cofib}(f_{2n+1})}_{\in\, \mathcal{C}_{\geq 2n+1}} \xrightarrow{\mathcal{E}_{f_{2n+1}}} L_{B/A}$$

want to show $\in \mathcal{C}_{\geq 2n}$

$B \in \mathcal{C}_{\geq 0}$, $\mathrm{cofib}(f_{2n+1}) \in \mathcal{C}_{\geq 2n+1}$

$L_{B/A}[-1] \in \mathcal{C}_{\geq 2n}$

$\Leftrightarrow L_{B/A} \in \mathcal{C}_{\geq 2n+1}$

$\Leftrightarrow \tau_{\leq 2n} L_{B/A} = 0$

$\rightsquigarrow$ reduced to proving

Lem 7.4.3.17 (5) For $A \xrightarrow{f} B$ in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$

$\tau_{\leq n} f$: equiv $\Rightarrow \tau_{\leq n} L_{B/A} \simeq 0$

Let's prove ① - ⑤ one by one

① : (a)

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[r] & \mathrm{cof}(f) \arrow[d] & B/A \arrow[d] \\ A \arrow[ur] \arrow[rr, "h"] & & C \arrow[r] \arrow[dr] & \mathrm{cof}(h) \arrow[d] & C/A \arrow[d] \\ & & & \mathrm{cof}(g) & C/B \end{tikzcd}$$

Mod_A

$\underbrace{C \otimes_B B \otimes_A}$

$$\begin{tikzcd} C \otimes_A \mathrm{cof}(f) \arrow[r] \arrow[dd, "C \otimes_B \varepsilon_f"'] \arrow[rdd, phantom, "\circlearrowright"] & C \otimes_A \mathrm{cof}(h) \arrow[r] \arrow[dd, "\varepsilon_h"] \arrow[rdd, phantom, "\circlearrowright"] & C \otimes_A \mathrm{cof}(g) \arrow[d, "\varepsilon'"] \arrow[dd, bend left, "\varepsilon''"] \\ & & C \otimes_B \mathrm{cof}(g) \arrow[d, "\varepsilon_g"] \\ C \otimes_B L_{B/A} \arrow[r] & L_{C/A} \arrow[r] & L_{C/B} \end{tikzcd}$$

cof. seq

cof_seq.

· $\mathrm{fib}(C \otimes_B \varepsilon_f) \simeq \underbrace{C}_{\text{connective}} \otimes_B \underbrace{\mathrm{fib}(\varepsilon_f)}_{2n\text{-conn.}} \in \mathcal{C}_{\geq 2n}$.

$\mathrm{Mod}_B \rightleftarrows \mathrm{Mod}_C$ ($\mathrm{Mod} \to \mathrm{CAlg}_g$, $B \to C$)

left adjoint

· $\exists$ fib seq

$$\mathrm{fib}\,\varepsilon' \to \mathrm{fib}\,\varepsilon'' \to \mathrm{fib}\,\varepsilon_g \in \mathcal{C}_{\geq 2n}$$

enough to show : $\mathrm{fib}\,\varepsilon' \in \mathcal{C}_{\geq 2n} \iff \mathrm{cof}\,\varepsilon' \in \mathcal{C}_{\geq 2n+1}$

· $\mathrm{cof}(g) \in \mathcal{C}_{\geq n}$

· $\mathrm{cof}(A \to B) \in \mathcal{C}_{\geq n}$

(next by Lem $\rightsquigarrow$ we get +1 connectivity

In general $\overset{\text{Lem}}{} \; A \xrightarrow{f} B$ in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$ with $\mathrm{cofib}(f) \in \mathcal{C}_{\geq n}$

$\rightsquigarrow \forall M, N \in \mathrm{Mod}_B^{cn}$

$$C = \mathrm{cofib}(M \otimes_A N \longrightarrow M \otimes_B N) \in \mathcal{C}_{\geq n+1}$$

$$\underset{\text{is}}{} \quad |M \otimes A^{\otimes \bullet} \otimes N| \longrightarrow |M \otimes B^{\otimes \bullet} \otimes N| \quad \underset{\text{is}}{}$$

$$M \otimes A^{\otimes n} \otimes N \longrightarrow M \otimes B^{\otimes n} \otimes N \longrightarrow C_n$$

$$C_\bullet : \Delta^{op} \to \mathcal{C} \quad \text{s.t.} \quad C = |C_\bullet| .$$

$\underline{n=0}$ $\quad M \otimes N \longrightarrow M \otimes N \longrightarrow \underline{C_0 = 0}$

$\underline{n>0}$ $\quad M \otimes A^{\otimes n} \otimes N \longrightarrow M \otimes B^{\otimes n} \otimes N \longrightarrow C_n \in \mathcal{C}_{\geq n}$

$$\begin{array}{c} M\otimes A \otimes A \otimes N \\ \downarrow \\ M \otimes A \otimes B \otimes N \\ \downarrow \\ M \otimes B \otimes B \otimes N \end{array}$$

<u>HA 1.2.4.5</u> SS for simplicial object

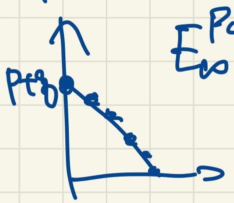
$X_\bullet : \Delta^{op} \to \mathcal{C} \qquad \mathrm{sk}_{n-1} X \to \mathrm{sk}_n X$

$\rightsquigarrow D(n) = |\mathrm{sk}_n X| \qquad D(0) \hookrightarrow D(1) \hookrightarrow \cdots$

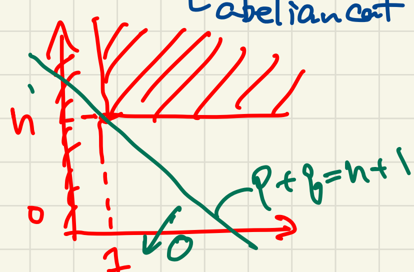
$E_1^{p,q}$: p th term of normalized chain cpx assoc. to $\pi_q X_\bullet : \Delta^{op} \to \mathcal{C}^\heartsuit$ ← abelian cat

at least under the current assumption $\Downarrow$

$\pi_{p+q}(|X_\bullet|)$



$E_\infty^{p,q}$



4

(d) First we prove:

classically $M = k^{\oplus n} \rightsquigarrow \underset{A}{\mathrm{Sym}} = k[x_1, \dots, x_n]$

$\Omega_A = A \otimes M \simeq A^{\oplus n}$

Prop 7.4.3.14

$M \in \mathcal{C}$, $A = \mathrm{Sym}^* M \in \mathrm{CAlg}(\mathcal{C})$

$L_A \simeq A \otimes M$ in Mod_A

proof $\mathrm{Map}_{\mathrm{Mod}_A}(L_A, N) \simeq \mathrm{Map}_{\mathrm{CAlg}/A}(A, A \oplus N)$, $A = \mathrm{Sym}^* M$

$$\begin{tikzcd}
& & N \\
M \arrow[urr] \arrow[rr] \arrow[d] & & A \oplus N \arrow[u] \arrow[d] \\
\mathrm{Sym}^* M \arrow[urr] \arrow[rr] & & A
\end{tikzcd}$$

$\simeq \mathrm{Map}_{\mathcal{C}/A}(M, A \oplus N)$

$\simeq \mathrm{Map}_{\mathcal{C}}(M, N)$

$\simeq \mathrm{Map}_{\mathrm{Mod}_A}(A \otimes M, N)$ $\square$

Now we prove (d): $M \in \mathcal{C}_{\geq n-1}$, $\phi : \mathrm{Sym}^* M \to \mathrm{Sym}^* 0 \simeq \mathbb{1}$

$\Longrightarrow \phi$: n-good.

($\mathrm{fib}(\varepsilon_\phi)$: $2n$-conn)

Claim $\varepsilon_\phi : \underset{\text{IS } ①}{\mathbb{1} \otimes_{\mathrm{Sym}^* M} \mathrm{cofib}(\phi)} \longrightarrow \underset{\text{IS } ②}{L_{\mathbb{1}/\mathrm{Sym}^* M}}$

$\mathrm{fib}(\varepsilon_\phi) \longrightarrow \bigoplus_{m \geq 1} \mathrm{Sym}^m(M[1]) \twoheadrightarrow M[1]$

$\mathrm{fib}(\varepsilon_\phi) = \underbrace{\bigoplus_{m \geq 2} \underbrace{\mathrm{Sym}^m(M[1])}_{\in \mathcal{C}_{\geq n}}}_{\in \mathcal{C}_{\geq 2n}}$

$\left(\mathrm{Sym}^m(M) = M^{\otimes m} / \Sigma_m \in \mathcal{C}_{\geq mn} \quad \forall M \in \mathcal{C}_{\geq n} \right)$

done

②

$$\underbrace{\mathbb{1} \otimes_{\mathrm{Sym}^* M} L_{\mathrm{Sym}^* M}}_{\substack{\cong \\ \underbrace{(\mathrm{Sym}^* M)\otimes M}_{\substack{\cong \\ M}}}} \longrightarrow L_{\mathbb{1}} \longrightarrow L_{\mathbb{1}/\mathrm{Sym}^* M} \quad \text{in } \mathrm{Mod}_{\mathbb{1}} \simeq \mathcal{C}$$

$$L_{\mathbb{1}} \simeq \mathrm{Sym}^*(0) \otimes 0 \simeq 0$$

$$L_{\mathbb{1}/\mathrm{Sym}^* M} \cong M[1]$$

①

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & M[1]
\end{tikzcd}
\quad \text{in } \mathcal{C}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathrm{Sym}^* M \arrow[r, "\phi"] \arrow[d] & \mathbb{1} \arrow[r] \arrow[d] & \mathrm{cofib}(\phi) \arrow[d] \\
\mathbb{1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Sym}^*(M[1]) \arrow[r] & \mathbb{1} \otimes_{\mathrm{Sym}^* M} \mathrm{cofib}(\phi)
\end{tikzcd}
: \text{exact}$$

in CAlg

$$\mathbb{1} \otimes_{\mathrm{Sym}^* M} \mathrm{cofib}(\phi) \cong \bigoplus_{m \geq 1} \mathrm{Sym}^m(M[1])$$

$$\mathbb{1} = \mathrm{Sym}^0(M[1]) \longrightarrow \mathrm{Sym}^*(M[1]) \longrightarrow \bigoplus_{m \geq 1} \mathrm{Sym}^m(M[1])$$

$$\begin{array}{ccc}
\cancel{\mathbb{1}} & \cancel{\longrightarrow} & \cancel{\mathbb{1}} \\
0 & \longrightarrow & M[1] \\
0 \otimes 0/\Sigma_2 & \longrightarrow & M[1]^{\otimes 2}/\Sigma_2 \\
\vdots & &
\end{array}$$

Aside

$$
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, equal] & \mathcal{M}^T(\mathcal{C}) \arrow[d] & T_{\mathcal{C}} \arrow[l] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] \\
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}^0 \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[l] \arrow[d] \arrow[u, shift right] \arrow[dl, phantom, "\llcorner", very near start] \\
\{0\} \arrow[r] & \Delta^1 & \{1\} \arrow[l]
\end{tikzcd}
$$

classifying

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[dr, shift left, "L"] \arrow[dr, phantom, "\perp"] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[l, shift left] \arrow[d, shift right, "\Sigma^\infty_+"'] \arrow[d, phantom, "\dashv"] \\
 & T_{\mathcal{C}} \arrow[u, shift right, "\Omega^\infty"'] \arrow[ul, shift left, "\text{sq zero}"]
\end{tikzcd}
$$

③ **Lem 7.3.4.15** $(n \geq 0)$

$A \xrightarrow{f} B$ in $\mathrm{CAlg}_{\geq 0}$, assume $\mathrm{cof}(f) \in \mathcal{C}_{\geq n}$

$\Rightarrow \exists M \in \mathcal{C}_{\geq n-1}$ $\exists M \xrightarrow{\phi} A$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd}
\mathrm{Sym}^* M \arrow[r] \arrow[d, "\phi \rightsquigarrow"'] & \mathrm{Sym}^* 0 \simeq \mathbb{1} \arrow[d] \arrow[dd, bend left] \\
A \arrow[r] \arrow[dr, bend right, "f"'] & A' \arrow[ul, phantom, "\ulcorner"] \arrow[d, dashed, "f'"] \\
& B
\end{tikzcd}$$

in CAlg

- $A' \in \mathrm{CAlg}_{\geq 0}$
- $\mathrm{cof}(f') \in \mathcal{C}_{\geq n+1}$

proof $M = \mathrm{fib}(f) \in \mathcal{C}_{\geq n-1}$

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
A \arrow[r, "f"'] & B \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}
\overset{\mathrm{Sym}^*}{\rightsquigarrow}
\begin{tikzcd}
\mathrm{Sym}^* M \arrow[r] \arrow[d] & \mathrm{Sym}^* 0 \arrow[d] \\
\mathrm{Sym}^* A \arrow[r] \arrow[d, "\mathrm{counit}"'] & \mathrm{Sym}^* B \arrow[ul, phantom, "\ulcorner"] \arrow[d] \arrow[dr, "\mathrm{counit}"] \\
A \arrow[r] \arrow[rr, bend right, "f"'] & A' \arrow[ul, phantom, "\ulcorner"] \arrow[r, dashed, "f'"] & B
\end{tikzcd}$$

in $\mathcal{C}$ in CAlg

$$\mathcal{C} \underset{U}{\overset{\mathrm{Free}}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{C}) \qquad \mathrm{Free} \dashv U$$

$\boxed{FU \to \mathrm{id}}$ counit

$A' \in \mathrm{CAlg}_{\geq 0}$

is

$A \otimes_{\mathrm{Sym}^* A} \mathrm{Sym}^* B \ (\simeq A \otimes_{\mathrm{Sym}^* M} \mathbb{1})$

It remains to prove $\mathrm{cof}(f') \in \mathcal{C}_{\geq n+1}$.

(i) $n=0$ $\quad$ $\underline{\mathrm{cofib}(f') \in \mathcal{C}_{\geq 1}}$

$$A' \longrightarrow B \longrightarrow \mathrm{cofib}(f') \quad \text{fib seq.}$$

$$\rightsquigarrow \pi_0 A' \longrightarrow \pi_0 B \longrightarrow \pi_0\, \mathrm{cofib}(f') \longrightarrow 0$$

$\uparrow$ surjective iff $\pi_0\,\mathrm{cofib}(f') \cong 0$ (in $\mathcal{C}^{\heartsuit}$)

$\therefore$ $\underline{\text{true}}$ because

$$\begin{tikzcd} \mathrm{Sym}^* B \arrow[rr] \arrow[dr] & & B \\ & A' \arrow[ur] & \end{tikzcd}$$

proj onto a summand $\mathrm{Sym}^1(B)$

(ii) $n \geq 1$

$$I := \mathrm{fib}(\underset{\S 3.1}{\mathrm{Sym}^* M} \longrightarrow \mathbb{1}) \simeq \bigoplus_{n\geq 1} \mathrm{Sym}^n(M)$$

$\mathrm{Sym}^n(M) = M^{\otimes n}/\!/\Sigma_n = (M^{\otimes n})_{h\Sigma_n}$

$$\begin{tikzcd} M \simeq \mathrm{Sym}^1 M \arrow[r, dashed, "s"] \arrow[dddr, bend right, "\mathrm{id}_M"'] & I \arrow[r] \arrow[d] & \mathrm{Sym}^* M \arrow[r] \arrow[d] \arrow[dr, phantom, "\otimes\ulcorner"] & \mathrm{Sym}^* 0 \arrow[d] & \text{fib seq.} \\ & I \otimes_{\mathrm{Sym}^* M} A \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & A' \arrow[d, "f'"] & \text{fib seq.} \\ & M \arrow[r] & A \arrow[r, "f"'] & B & \text{fib seq} \\ & \mathrm{fib}(f) \arrow[u, equal] & & & \end{tikzcd}$$

$- \otimes_{\mathrm{Sym}^* M} A$

cofib

$\rightsquigarrow \mathrm{cof}(g) \to 0 \to \mathrm{cof}(f')$: fib

$\rightsquigarrow \mathrm{cof}(f') \cong \mathrm{cof}(g)[1] \simeq \mathrm{fib}(g)[2]$

$$\mathrm{cof}(f') \in \mathcal{C}_{\geq n+1} \iff \mathrm{fib}(g) \in \mathcal{C}_{\geq n-1}$$

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\mathrm{id}_M"'] & I \otimes_{\mathrm{Sym}^* M} A \arrow[r, "g"] & M \end{tikzcd}$$

$I \otimes A \in \mathcal{C}_{\geq n-1}$

$\left(\{\mathcal{O}(n)\}_{n \in \mathbb{N}} \circlearrowleft \Sigma_n \qquad \mathrm{Set} \rightleftarrows \mathrm{Alg}_{\mathcal{O}} \quad X \mapsto \coprod_{n \geq 0} X^n \times_{\Sigma_n} \mathcal{O}(n) = \mathrm{Free}_{\mathcal{O}}(X) \right)$

---

⑤

<u>Lem 7.4.3.17</u> $f : A \to B \in \mathrm{CAlg}^{\mathrm{ch}}$. $\tau_{\leq n} f$: equiv

$$\Rightarrow \tau_{\leq n} L_{B/A} = 0$$

<u>proof</u> $B \otimes_A L_A \to L_B \to L_{B/A}$ fib seq.

$\tau_{\leq n}(B \otimes_A L_A) \to \tau_{\leq n} L_B \to \tau_{\leq n} L_{B/A}$: fib seq. in $(\mathrm{Mod}_B)_{\leq n}$

$\tau_{\leq n}(B \otimes_A L_A) \to \tau_{\leq n} L_B$ <u>equiv</u> $\Leftarrow$ $\tau_{\leq n} L_{B/A} = 0$

<u>take any $M \in (\mathrm{Mod}_B)_{\leq n}$</u>

$$\mathrm{Map}_{(\mathrm{Mod}_B)_{\leq n}}(\tau_{\leq n} L_B, M) \xrightarrow{\simeq} \mathrm{Map}_{(\mathrm{Mod}_B)_{\leq n}}(\tau_{\leq n}(B \otimes_A L_A), M)$$

$$\mathrm{Map}_{(\mathrm{Mod}_B)_{\leq n}}(\tau_{\leq n} L_B, M) \simeq \mathrm{Map}_{\mathrm{Mod}_B}(L_B, M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{/B}}(B, B \oplus M) \overset{\star}{\simeq} \mathrm{Map}_{\mathrm{CAlg}/\tau_{\leq n} B}(B, (\tau_{\leq n} B) \oplus M)$$

$$\mathrm{Map}_{(\mathrm{Mod}_B)_{\leq n}}(\tau_{\leq n}(B \otimes_A L_A), M) \simeq \mathrm{Map}_{\mathrm{Mod}_B}(B \otimes_A L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, M) \simeq \mathrm{Map}_{\mathrm{CAlg}/A}(A, A \oplus M) \overset{\star}{\simeq}$$

$$
\circledast
$$

$$
\begin{tikzcd}
 & & & & \mathrm{Map}_{\mathrm{CAlg}/\tau_{\leq n}B}(A, (\tau_{\leq n}B)\oplus M) \\
 & A\oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B\oplus M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\tau_{\leq n}B)\oplus M \arrow[d] & \\
A \arrow[ur, dashed, bend left] \arrow[urr, dashed, bend left=40] \arrow[r, equal] & A \arrow[r] & B \arrow[r] & (\tau_{\leq n}B) & \text{in CAlg.}
\end{tikzcd}
$$

$$
\left(
\begin{tikzcd}
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & \tau_{\leq n}B \arrow[d, equal] \\
A \arrow[r] & B \arrow[r] & \tau_{\leq n}B
\end{tikzcd}
\right)
\oplus
\left(
\begin{tikzcd}
M \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d] \\
0 \arrow[r] & 0 \arrow[r, equal] & 0
\end{tikzcd}
\right)
$$

$$
\mathrm{Map}^{\tau_{\geq 0}}_{\mathrm{CAlg}/\tau_{\leq n}B}(B, \tau_{\leq n}B\oplus M) \xrightarrow{\sim} \mathrm{Map}^{\tau_{\geq 0}}_{\mathrm{CAlg}/\tau_{\leq n}B}(A, \tau_{\leq n}B\oplus M)
$$

$\tau_{\geq 0}$: right adj $\rightsquigarrow$ we may replace M by $\tau_{\geq 0}M$

and assume $M \in \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n}$

$$
\begin{tikzcd}
\tau_{\leq n}A \arrow[r] \arrow[d, "\sim"'] & \boxed{(\tau_{\leq n}B)\oplus M} \in \mathcal{C}_{\geq 0}\cap\mathcal{C}_{\leq n} \arrow[d] \\
\tau_{\leq n}B \arrow[r] \arrow[ur, dashed] & \tau_{\leq n}B
\end{tikzcd}
$$

$$
\left(\mathrm{CAlg}^{cn}_{\leq n} \underset{\perp}{\rightleftarrows} \mathrm{CAlg}^{cn}\right)
$$

$$\mathrm{Sp}_{\le n}^{\otimes\text{cn}} \underset{\perp}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Sp}^{\otimes}$$

$\boxed{\text{HA}}$ 2.2.1.10

$\mathcal{C}^{\otimes,\text{cn}} \subset \mathcal{C}^{\otimes}$

stable$^{\otimes}$ w/ t-str. compatible $\otimes$

$$\mathcal{C}^{\text{cn}} \underset{\perp}{\overset{\tau_{\le n}}{\rightleftarrows}} \underbrace{(\mathcal{C}^{\text{cn}})_{\le n}}$$

Can be promoted to

($\tau_{\ge n}$: monoidal
incl: lax mon)

$\tau_{\le n}(-\otimes -) = - \overset{\le n}{\otimes} -$

Recall

$$\mathcal{D}er = \left(\frac{A \longrightarrow (A, M[1]) \text{ in } \mathcal{M}^T}{L_A \xrightarrow{\eta} M[1]} \;\searrow\; A = A \text{ in } \mathrm{CAlg} \right)$$

$$\begin{tikzcd}
\mathcal{D}er \arrow[d, dashed, shift right, "\simeq"'] \arrow[dd, bend right=60, "\Phi"'] \\
\widetilde{\mathcal{D}er} \arrow[u, shift right, "\sim"'] \arrow[d] \\
\mathrm{Fun}(\Delta^1, \mathrm{CAlg})
\end{tikzcd}$$

$$\widetilde{\mathcal{D}er} = \left\{ \begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "\eta"] \\ (A,0) \arrow[r] & (A, M[1]) \end{tikzcd} \text{ in } \mathcal{M}^T \right\} \longrightarrow \begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd}$$

$$\mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \ni \tilde{A} \longrightarrow A$$

Observation $\exists\, \Psi : \mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \longrightarrow \mathcal{D}er$ left adj to Φ

$$\tilde{A} \to A \longmapsto (A \to (A, L_{A/\tilde{A}}))$$

proof

$$\mathcal{D}er = \left\{ \begin{tikzcd} \cdot \arrow[r, dashed] \arrow[d, dashed] & A \arrow[d] \\ \cdot \arrow[r, dashed] & (A, M[1]) \end{tikzcd} \right\} \begin{tikzcd} \arrow[r, shift right, "\perp"'] & \arrow[l, shift right, "\sim"'] \end{tikzcd} \left\{ \begin{tikzcd} & A \arrow[d] \\ (A,0) \arrow[r] & (A, M[1]) \end{tikzcd} \right\}$$

in $\mathcal{M}^T$

$$\begin{tikzcd}
\left\{ \begin{tikzcd} & A \arrow[d] \\ (A,0) \arrow[r] & (A, M[1]) \end{tikzcd} \right\} \arrow[d, shift left, "\text{formation of pb.}"] \arrow[dr, leftrightarrow, "\simeq"] & \\
\left\{ \begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d] \\ (A,0) \arrow[r] & (A, M[1]) \end{tikzcd} \right\} \arrow[u, shift left, "\text{forget}"] \arrow[r, phantom, "\supset"] \arrow[d, shift left, "\text{forget}"] & \widetilde{\mathcal{D}er} \; \text{pullback} \\
\left\{ \begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \\ (A,0) & \end{tikzcd} \right\} \arrow[u, shift left, "\text{p.o.}"] \arrow[r, shift right, two heads, "\sim"'] & \left\{ \tilde{A} \to A \right\} = \mathrm{Fun}(\Delta^1, \mathrm{CAlg}) \arrow[l, shift right, "\sim"']
\end{tikzcd}$$

(forget $\uparrow \dashv \downarrow$ formation of pb.; p.o. $\uparrow \dashv \downarrow$ forget; $\perp$ between the bottom arrows)

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd} \text{ in } \mathrm{CAlg}$$

$$f_!(\tilde{A}, 0) = \left\{ \begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ (\tilde{A}, 0) \arrow[r] & (A, L_{\tilde{A}/A}) \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \\ (A,0) & \end{tikzcd} \right\}$$

def of $L_{\tilde{A}/A}$:

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ (\tilde{A}, 0) \arrow[r] & (A, L_{\tilde{A}/A}) \end{tikzcd}$$

p-p.o. for CMod $\downarrow$ CAlg

Φ, Ψ restrict to diagrams with A : connective

($\mathcal{D}er'$, Fun')

$$\rightsquigarrow \quad \mathcal{D}er' \underset{\Phi'}{\overset{\Psi'}{\rightleftarrows}} \mathrm{Fun}'(\Delta^1, \mathrm{CAlg})$$

$$\tau \downarrow \dashv \uparrow i \qquad \circlearrowleft \qquad F \dashv G \leftarrow \text{res of } \Phi$$

$\mathcal{D}er''$

τ $\begin{cases} A : \text{conn} \\ M : 2n\text{-truncated} \end{cases}$

$$\left[\begin{array}{ccc} L_A & \xrightarrow{\eta} & M[1] \\ & \searrow^{\tau(\eta)} & \downarrow \\ & & (\tau_{\le 2n} M)[1] \end{array}\right]$$

$\underline{n \geq 0}$

Now we have

A : conn

M : 2n-tr.

A : conn

$$\mathcal{D}er'' \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Fun}'(\Delta^1, \mathrm{CAlg})$$

$$\cup \qquad\qquad\qquad \cup$$

$$\mathcal{D}er_{n\text{-sm}} \leftarrow\!-\!- \rightarrow \mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathrm{CAlg})$$

M : n-conn

in $\mathrm{Mod}_{\tilde{A}}$

$I = \mathrm{fib}(\tilde{A} \to A)$

$\rightsquigarrow \cdot\, I \in \mathcal{C}_{\ge n} \cap \mathcal{C}_{\le 2n}$

$\cdot$
$$\begin{array}{ccc} I \otimes_{\tilde{A}} I & \longrightarrow & I \\ \downarrow & & \uparrow \text{null} \\ \tilde{A} \otimes_{\tilde{A}} I & & \end{array}$$

It remains to prove :

① $\eta : A \to M[1]$ n-sm. $\Rightarrow$ $G\eta : A^\eta \to A$: n-sm.

② — " — $\Rightarrow F \circ G(\eta) \xrightarrow[\text{counit}]{} \eta$ is an equiv.

③ $\tilde{A} \xrightarrow[f]{} A$: n-small $\Longrightarrow$ Ff : n-sm.

④ — " — $\Rightarrow f \xrightarrow[\text{unit}]{} GFf$: equiv.

① Let $I = \mathrm{fib}(A^\eta \to A)$ $M \simeq I$ as A^η-mod. $\rightsquigarrow I \in \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq n}$.

$$\begin{tikzcd}
I \otimes_{A^\eta} I \arrow[r] \arrow[drr, bend left, "0"] & A^\eta \otimes_{A^\eta} I \arrow[r, "\simeq"] \arrow[d] & I \\
& A \otimes_{A^\eta} I \arrow[r] \arrow[d, "\simeq"] & A \otimes_A M \arrow[u, "\simeq"'] \\
& A \otimes_{A^\eta} M \arrow[ur] &
\end{tikzcd}$$

(fib seq) $\otimes I$

$$\left[\begin{array}{ccc} M & \xrightarrow[\text{forget}]{} & I \\ A\text{-mod} & & A^\eta\text{-mod} \end{array}\right]$$

$$\left(\begin{tikzcd}
I \arrow[r] \arrow[d, "\simeq"] & A^\eta \arrow[r] \arrow[d] & A \arrow[d] \\
M \arrow[r] & A \arrow[r] & A \otimes M
\end{tikzcd}\right) \text{ in } \underline{A^\eta\text{-mod}}$$

$$\begin{tikzcd}
A^\eta \otimes_{A^\eta} I \arrow[r] \arrow[d] & A^\eta \otimes_{A^\eta} A^\eta \arrow[r] \arrow[d] & A^\eta \otimes A \arrow[d] \\
A \otimes_{A^\eta} M & A^\eta \otimes_{A^\eta} M \arrow[l] \arrow[r] & A^\eta \otimes_{A^\eta} A
\end{tikzcd}$$

by adjunction

② $GFG\eta \xrightarrow[\sim]{} G\eta$ it suffices to prove G: conservative.

$$\begin{tikzcd}
A^{\eta} \arrow[r] \arrow[d, "\simeq"'] & A \arrow[r, "\eta"] \arrow[d, "\simeq"'] & M[1] \arrow[d] \arrow[d, phantom, "\simeq"] \\
B^{\eta'} \arrow[r] & B \arrow[r, "\eta'"'] & N[1]
\end{tikzcd}$$

Gf: equiv.

$f \Downarrow$ in $\mathcal{D}\mathrm{er}$

fib seq.

$\rightsquigarrow \underline{\underline{\eta \simeq \eta'}}$

③④

F:
$$\begin{aligned}
&(\tilde{A} \to A) \\
&\quad \downarrow \Psi' \\
&(A \to (A, L_{A/\tilde{A}})) \\
&\quad \downarrow \tau \\
&(A \xrightarrow{\eta} (A, \underbrace{\tau_{\leq 2n}(L_{\tilde{A}/A}[-1])}_{M \,\in\, \mathcal{C}_{\leq 2n}}[1]))
\end{aligned}$$

$G \downarrow$

$(A^{\eta} \to A)$

$\rightsquigarrow$ $\underline{\underline{\text{unit}}}$

$$\begin{tikzcd}
\mathrm{fib} f \arrow[r] \arrow[d, "g"'] & \tilde{A} \arrow[r, "f"] \arrow[d, "u_f"'] & A \arrow[d, "\mathrm{id}"] \\
M \arrow[r] & A^{\eta} \arrow[r, "GFf"'] & A
\end{tikzcd}$$

$\mathcal{C}_{\geq 2n} \ni \mathrm{fib} f$

To prove
$\left[\begin{array}{l} ③\ M \in \mathcal{C}_{\geq 2n} \\ ④\ u_f : \text{equiv} \end{array}\right]$

It suffices to prove

g: equiv.

Since fib f, $M \in \mathcal{C}_{\leq 2n}$,

$$g = \tau_{\leq 2n}\, g$$

claim: g is the composition

$$\begin{tikzcd}
I \arrow[r, "1"] \arrow[d, phantom, "\shortparallel"] & I \otimes_{\tilde{A}} A \arrow[d, "\varepsilon_f[-1]"] & \\
I \otimes_{\tilde{A}} \tilde{A} \arrow[ur, "\mathrm{id}\otimes f"] & L_{A/\tilde{A}}[-1] \arrow[r] & \tau_{\leq 2n}(L_{\tilde{A}/A}[-1]) = M
\end{tikzcd}$$

First Assume the claim. $\rightsquigarrow$ done.

- the first map
- the second map by Thm 7.4.3.12

$$I \longrightarrow \tilde{A} \longrightarrow A$$

$$I \otimes_{\tilde{A}} - \downarrow$$

$$I \otimes_{\tilde{A}} I \longrightarrow I \otimes_{\tilde{A}} \tilde{A} \xrightarrow{2} I \otimes_{\tilde{A}} A$$

null

$$\rightsquigarrow \quad \pi_k(I \otimes_{\tilde{A}} I) \xrightarrow{0} \pi_k(I \otimes_{\tilde{A}} \tilde{A}) \underset{\cong}{\xrightarrow{1}} \pi_k(I \otimes_{\tilde{A}} A)$$

$$0 \to 0 \to$$

in $\mathcal{C}^\heartsuit$

$$0 = \pi_{k-1}(I \otimes_{\tilde{A}} I) \in \mathcal{C}_{\geq 2n}$$

$k \leq 2n$ 0

$$\begin{tikzcd}
I \arrow[r] \arrow[dr] & I \otimes_{\tilde{A}} A \arrow[d, "\varepsilon_f[-1]"] \\
 & L_{A/\tilde{A}}[-1]
\end{tikzcd}$$

Ψ'

$$\begin{tikzcd}
I \arrow[rr, "\mathrm{fib}"] \arrow[ddr] \arrow[ddddr, "g"'] & & \tilde{A} \arrow[rr] \arrow[dr] \arrow[dddr, red] & & A \arrow[dd, "\eta'"] \arrow[dddd, bend left, "\eta"] \\
 & & & A^{\eta'} \arrow[ur] \arrow[dd] & \\
 & L_{A/\tilde{A}}[-1] \arrow[urr, "\mathrm{fib}"] \arrow[dd] & & & L_{A/\tilde{A}} \arrow[dd] \\
 & & & A^{\eta} \arrow[uuur] & \\
 & \tau_{\le 2n}(L_{A/\tilde{A}}[-1]) \arrow[urr, "\mathrm{fib}"] & & & \tau_{\le 2n+1} L_{A/\tilde{A}}
\end{tikzcd}$$

$G = \tau\Psi'$

Other Applications of Thm 7.4.3.12

(Cor 7.4.3.2 ~ Prop 7.4.3.9)

(important but easy to follow)