

SAG 1.4.9 - 1.4.11

② Postnikov tower

$$\begin{array}{ccc} X \in \mathrm{SpDM} & \text{Recall} & \mathrm{SpDM}^{\leq n} \xrightleftharpoons[\tau_{\leq n}]{\iota} \mathrm{SpDM} \\ \parallel & & \cup \qquad \qquad \cup \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & & (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \hookrightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

Prop $\mathrm{SpDM} \rightarrow \dots \rightarrow \mathrm{SpDM}^{\leq 2} \xrightarrow[\tau_{\leq 0}]{\iota} \mathrm{SpDM}^{\leq 1} \xrightarrow[\tau_{\leq 0}]{\iota} \mathrm{SpDM}^{\leq 0}$
is a limit diagram in CAT_{∞}

proof Let $G : \mathrm{SpDM} \rightarrow \varprojlim_n \mathrm{SpDM}^{\leq n}$

fully faithful

$$\mathrm{Map}_{\mathrm{SpDM}}(X, Y) \longrightarrow \varprojlim_n \mathrm{Map}_{\mathrm{SpDM}^{\leq n}}(\tau_{\leq n} X, \tau_{\leq n} Y)$$

$$\searrow \qquad \swarrow$$

$$\mathrm{Map}_{\mathrm{Top}}(X, Y)$$

$$f^* : Y \rightarrow X$$

need to show equiv. on fibers

fib over f^* : $\mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}^{\mathrm{loc}}(X)}(f^* \mathcal{O}_Y, \mathcal{O}_X)$

$$\longrightarrow \varprojlim_n \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}^{\mathrm{loc}}(X)}(f^* \mathcal{O}_Y, \tau_{\leq n} \mathcal{O}_X)$$

without "loc" follows from

$$\begin{array}{c} 1.4.8.1 \rightarrow \mathcal{O}_X \xrightarrow{\sim} \varprojlim_n \tau_{\leq n} \mathcal{O}_X \\ \uparrow \\ \text{hypercplteness} + \pi_n \text{ equiv.} \\ \uparrow \\ \text{checked on affines} \end{array}$$

add in "loc" "map is local" is a π_0 condition

direct summand corr. to local maps

corresponds in both sides.

$$\text{ess. surj ob} \left(\varprojlim_n \text{SpDM}^{\leq n} \right) \ni \left(\begin{array}{c} (\mathcal{X}_n, \mathcal{O}_n) = \mathcal{X}_n \\ + \\ \tau_{\leq n} \mathcal{X}_{n+1} \simeq \mathcal{X}_n \end{array} \right)$$

may assume $\mathcal{X}_n = \mathcal{X}_0 \Rightarrow \mathcal{X}$ \uparrow

$(\mathcal{X}, \mathcal{O}_n)$ with $\mathcal{O}_n \simeq \tau_{\leq n} \mathcal{O}_{n+1}$ \swarrow

\leadsto want: $\text{SpDM} \longrightarrow \varprojlim_n \text{SpDM}^{\leq n}$

$$\begin{array}{ccc} \textcircled{1} \downarrow & & \downarrow \\ (\mathcal{X}, \varprojlim_n \mathcal{O}_n) & \xrightarrow{\textcircled{2}} & \{(\mathcal{X}, \mathcal{O}_n)\}_n \end{array}$$

$\textcircled{1} \textcircled{2}$: both local

passing to ^{an} affine cover of X_0

may assume X_0 : affine.

$\leadsto X_n$ is affine (affineness can be checked on π_0)
 \parallel
 $\mathrm{Spét} A_n \xrightarrow{\sim} A$

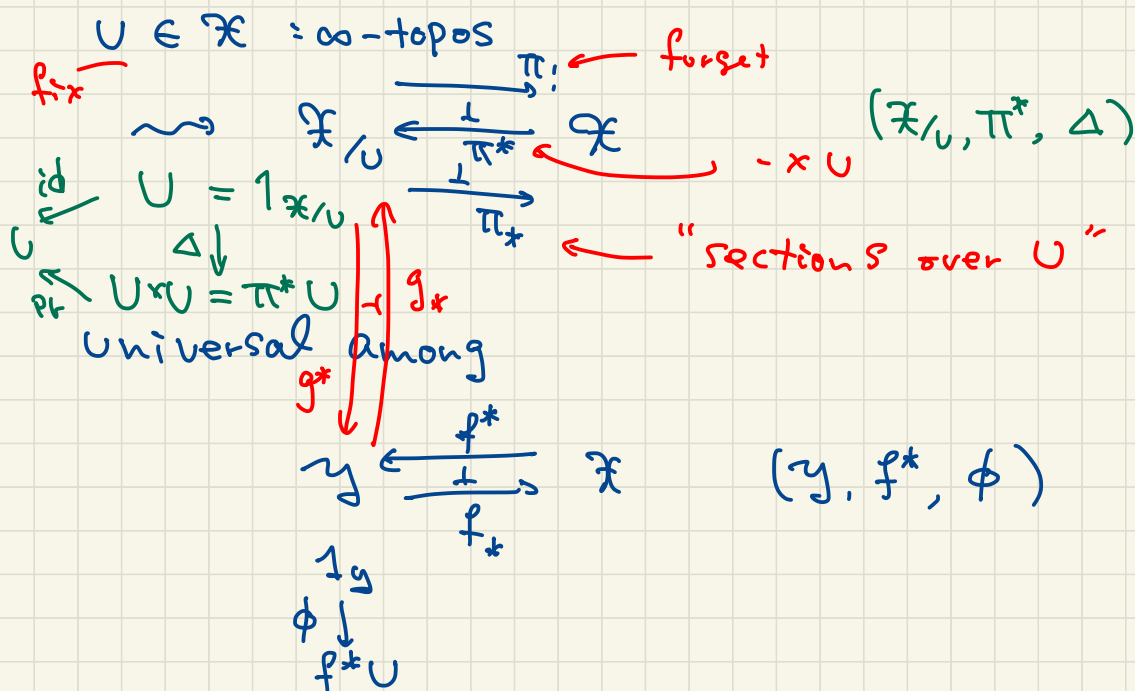
can check: $\mathrm{Spét} A$

$$(\mathcal{X}, \varprojlim \mathcal{O}_{\mathrm{Spét} A_n})$$

① is clear

② by $\mathrm{CAlg}^{\mathrm{cu}}$: Postnikov complete. \square

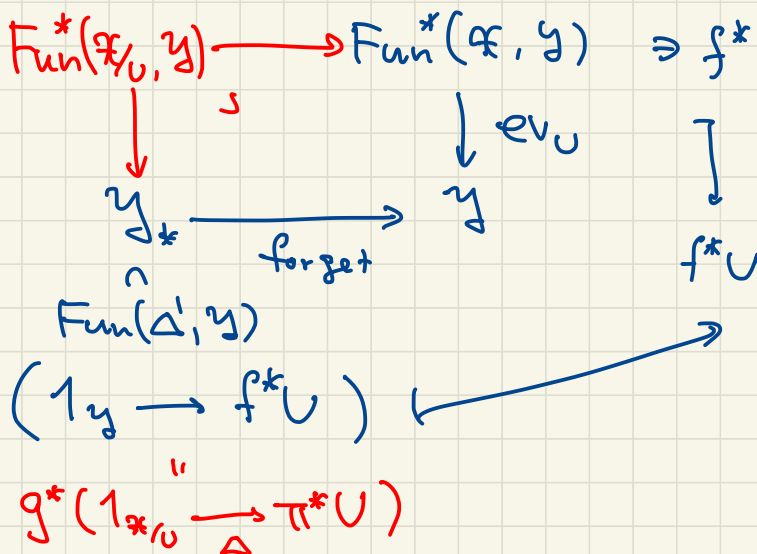
② Étale morphisms of ∞ -topoi (HTT 6.3.5)



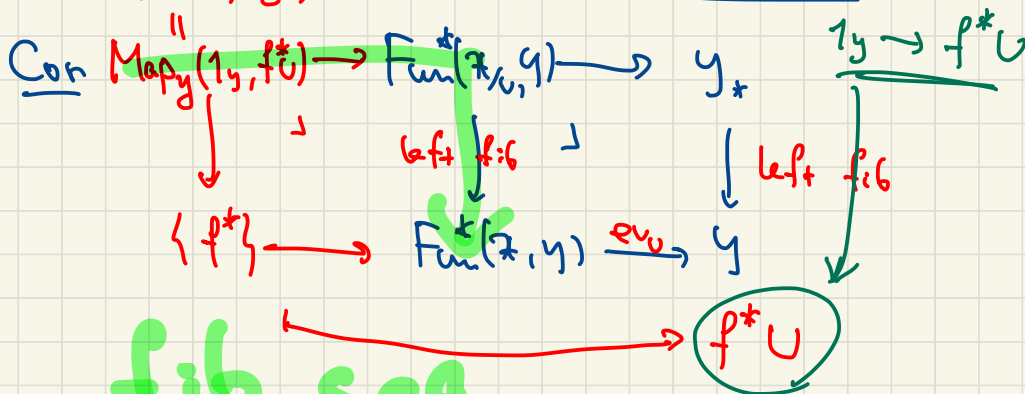
g^*

$g^* \circ \pi^*$

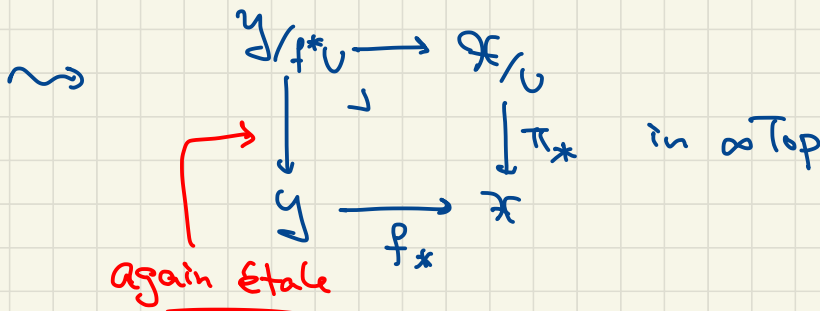
i.e.



$P(f^*U, y)$



fib seq



proof $\text{Map}_{\infty\text{Top}}(\mathbb{Z}, -)$

$$P(g^* f^* U, Z) \longrightarrow \text{Fun}^*(Y/f^* U, Z) \longrightarrow \text{Fun}^*(Y, Z) \Rightarrow g^*$$

$$\parallel \quad \downarrow \quad \downarrow \quad \downarrow$$

$$P((f \circ g)^* U, Z) \longrightarrow \text{Fun}^*(X/U, Z) \xrightarrow{f!} \text{Fun}^*(X, Z)$$

Cor plug $Y = X/U \xleftarrow{f^*} X$ étale

$$\text{Fun}_X^*(X/U, X/U) \longrightarrow \text{Fun}^*(X/U, X/U)$$

$$\uparrow \quad \downarrow \quad \downarrow$$

$$\text{is } \{f^*\} \longrightarrow \text{Fun}^*(X, X/U)$$

$$\text{Map}_{\infty \text{Top}/X}(X/U, X/U) \xrightarrow{\sim} \text{Map}_{X/U}(1_{X/U}, f^* U)$$

by fib seq

$$\begin{array}{ccc} V & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow \\ X/U & \xrightarrow{\quad} & X/U \\ \downarrow & & \downarrow \\ (X/U)_U & \xrightarrow{\quad} & X \end{array}$$

$$\text{Map}_X(f! 1_{X/U}, U)$$

Cor.

$$\begin{array}{ccc} X & \xrightarrow{f_*} & Y \\ h_* \searrow & & \swarrow g_* \\ Z & & \end{array} \quad g_* : \text{étale}$$

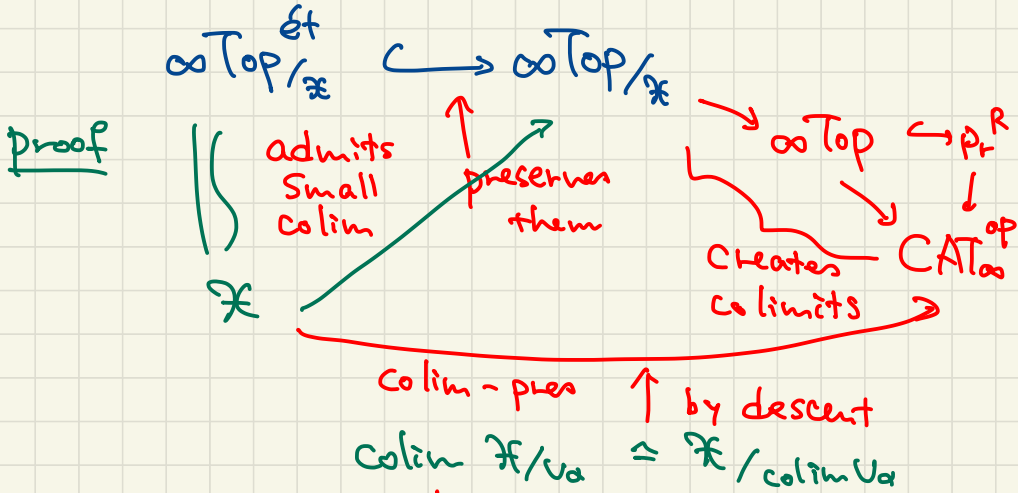
$$\begin{array}{c} f_* : \text{étale} \\ \Downarrow \\ h_* : \text{étale} \end{array}$$

Cor

$$\begin{array}{ccc} X^{\Delta'} & & \\ \downarrow \text{cod} & & \\ X & \xrightarrow{\quad} & \text{CAT}_{\infty}^{\text{op}} \supset \text{Pr}^R \\ \downarrow & \searrow & \\ U & \xrightarrow{\quad} & X/U \end{array}$$

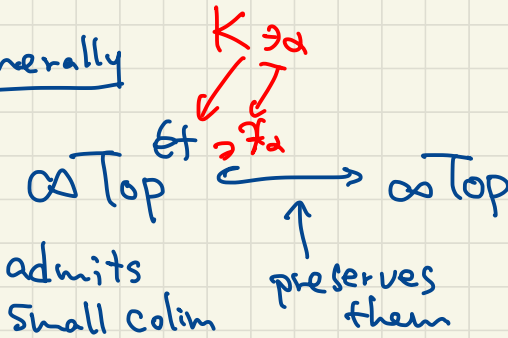
fully faithful $\rightarrow \infty \text{Top}^{\text{ét}}$
 $\text{ess im} = (\infty \text{Top}^{\text{ét}})_{/\mathbb{X}}$

colimit along étale mor



More generally

Thm



By thm $\text{colim}_{\text{along ét}} \mathbb{X}_{\alpha} =: \mathbb{X} \rightsquigarrow \mathbb{X}_{\alpha} \xrightarrow{\text{ét}} \mathbb{X}$

$\rightsquigarrow \text{colim } \mathbb{X}/U_{\alpha} \cong \mathbb{X}$

ref: 21.4.7
 or DAG V

$\coprod U_{\alpha} \rightarrow 1$

② Étale mor of locally ringed ∞ -topoi

Def $f: (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in ∞TopAlg is étale

if $\begin{matrix} & \searrow \cong & \nearrow \\ & (\mathcal{Y}_V, \mathcal{O}_{\mathcal{Y}|V}) & \end{matrix} \quad \exists V \in \mathcal{Y}.$

underlying topos étale

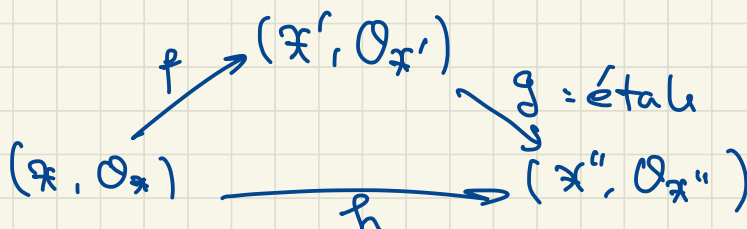
$\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}|V}$ condition

$\iff f: \text{cartesian edge of}$

$\infty\text{TopAlg} \downarrow \infty\text{Top}$

étale surjection if $V \twoheadrightarrow 1_{\mathcal{Y}}$

Remark (1)



$\Rightarrow f: \text{étale} \iff h: \text{étale}$
or $\text{stHen}_{\text{loc. ét}}$

(2) ∞TopAlg

admits small colim

(obj: locally ring ∞ -topoi
mor: étale mor
(\Rightarrow local)

preserved by

∞TopAlg

∞Top

$\infty\text{Top}^{\text{loc or stHen}}$

$\text{colim} (\mathcal{X}_{\alpha}, \mathcal{O}_{\alpha})$

let $\mathcal{X} := \text{colim } \mathcal{X}_{\alpha}$

||

$$= (\mathcal{X}, \lim_{\alpha} (f_{\alpha})_* \mathcal{O}_{\alpha})$$

$$\mathcal{X}_{\alpha} \xrightarrow{(f_{\alpha})_*} \mathcal{X}$$

$\mathcal{X}/\mathcal{U}_{\alpha}$

Rmk $X \rightarrow Y$

f : étale is local on the source :

$$\text{If } \exists \coprod_{\alpha} U_{\alpha} \rightarrow 1_X \text{ s.t.}$$

$$f_{\alpha}: (\mathcal{X}|_{U_{\alpha}}, \mathcal{O}_{\mathcal{X}}|_{U_{\alpha}}) \rightarrow (Y, \mathcal{O}_Y)$$

étale

$$\Rightarrow f: \text{étale}$$

proof

$$\mathcal{X}_0 \subset \mathcal{X}$$

full

$$U \hookrightarrow X|_U \rightarrow X \xrightarrow{f} Y$$

: étale

$$\left(\begin{array}{c} \text{i.e.} \\ \text{ét.surj} \\ \swarrow \quad \searrow \\ \text{étale} \end{array} \right)$$

- ①
- \mathcal{X}_0 is a sieve $V \rightarrow U \in \mathcal{X}_0 \Rightarrow V \in \mathcal{X}_0$
 - \mathcal{X}_0 closed under ^{small} colim (by the prop above)

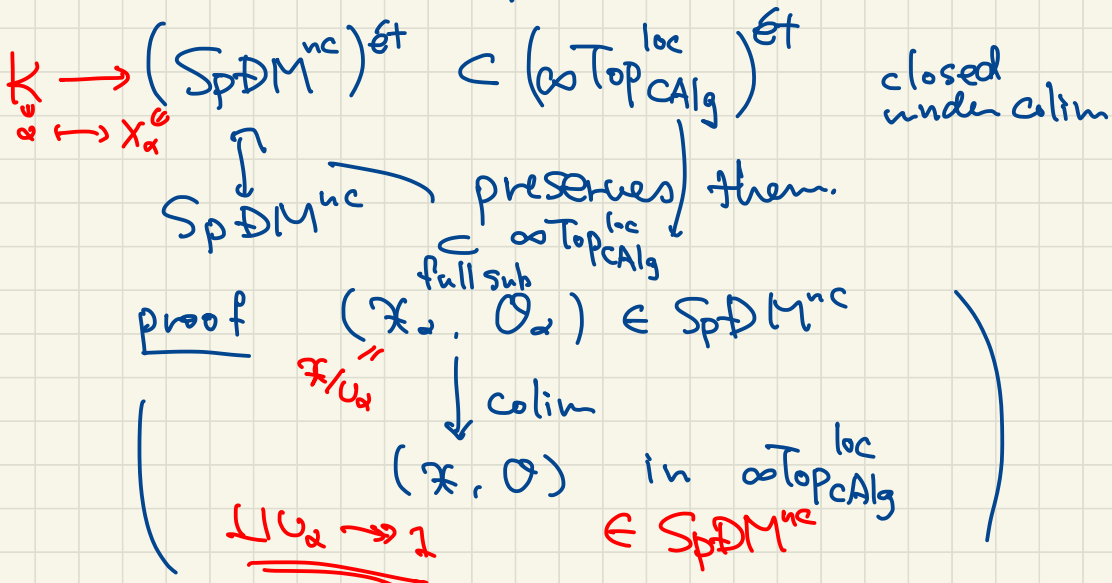
$$U_{\alpha} \in \mathcal{X}_0, U = \coprod U_{\alpha} \rightarrow 1_X$$

$$\Rightarrow 1_X \in \mathcal{X}_0, \text{ i.e. } f: \text{étale.}$$

$$U \times U \times U \in \mathcal{X}_0$$

$$\downarrow \textcircled{1}$$

② étale mor in $\mathrm{SpDM}^{\mathrm{nc}}$



e.g. G : discrete group

$$BG \longrightarrow \mathrm{SpDM}^{\mathrm{nc}}$$

$$*_{BG}$$

$$X \downarrow G$$

act by equiv
 \downarrow
 étale

$$\text{colim}_{BG} X$$

$$X/G$$

$$(X//G ?)$$

③ relation to étale ring maps ? (1.4.10)

Thm $\phi^* : A \longrightarrow B$ in CAlg is étale

iff $\mathrm{Spét} B \xrightarrow{\phi_*} \mathrm{Spét} A$ is étale.

Cor $f : X \longrightarrow Y$ in $\mathrm{SpDM}^{\mathrm{nc}}$ is étale

iff

\forall Comm square

$$\left[\begin{array}{ccc} \mathrm{Sp\acute{e}t} B & \xrightarrow{\acute{e}t} & X \\ \downarrow \phi_* & & \downarrow f \\ \mathrm{Sp\acute{e}t} A & \xrightarrow{\acute{e}t} & Y \end{array} \right]$$

alg map $A \xrightarrow{\phi^*} B$ is étale

$$\begin{aligned} (\Rightarrow) \quad \mathrm{Sp\acute{e}t} B \xrightarrow{\phi^*} \mathrm{Sp\acute{e}t} A &\Rightarrow \phi_* : \acute{e}t \\ &\quad \acute{e}t \searrow \quad \swarrow \acute{e}t \\ &\quad Y \end{aligned} \quad \begin{aligned} &\Downarrow \text{Thm} \\ &\phi^* : A \rightarrow B \\ &\text{étale.} \end{aligned}$$

(\Leftarrow) Cover Y by affines U_α

\leadsto cover X by affines $U_{\alpha\beta}$

$$\begin{aligned} \text{aff} : U_{\alpha\beta} &\overset{\text{cover}}{\dashrightarrow} X \\ \downarrow \text{étale} & \quad \downarrow \\ \text{aff} : U_\alpha &\xrightarrow{\acute{e}t} Y \end{aligned} \quad \Rightarrow \quad f : \text{étale.}$$



proof of Thm

Thm $\phi^* : A \rightarrow B$ in CAlg is étale

iff $\mathrm{Sp\acute{e}t} B \xrightarrow{\phi_*} \mathrm{Sp\acute{e}t} A$ is étale

(\Rightarrow) by construction

$$\mathrm{Shv}_B^{\text{ét}} \simeq (\mathrm{Shv}_A^{\text{ét}})_{/h^B}$$

corep. sheaf by B

$$(\Leftarrow) \quad \mathrm{Spet}_B \longrightarrow \mathrm{Spet}_A$$

$$(\mathcal{X}_U, \mathcal{O}_X|_U) \longrightarrow (\mathcal{X}, \mathcal{O}_X)$$

$$\coprod_{j \in J} W_j \xrightarrow{\text{refine}} \coprod_{i \in I} V_i$$

finite \rightarrow by W_j :
corep by B_j :
étale B-alg.

$V_i =$ sheaf in $\mathrm{Shv}_A^{\text{ét}}$
corep. by A_i : étale A-alg

$$\left(\begin{array}{ccc} \text{refinement : id} & & \\ J & \xrightarrow{\text{id}} & J \\ \downarrow & & \downarrow \\ j & \mapsto & r(j) \end{array} \quad \begin{array}{ccc} W_j & \longrightarrow & V_{r(j)} \\ & \searrow & \swarrow \\ & U & \end{array} \right)$$

$$\coprod W_j$$

corep by $\prod B_j =: B'$
sheaf in $\mathrm{Shv}_B^{\text{ét}}$

$$\coprod V_j$$

corep by $\prod A_j =: A'$
sheaf in $\mathrm{Shv}_A^{\text{ét}}$

$$(X_U)_{/h^W_j} \longrightarrow X_{/h^{V_j}} \longrightarrow X_U \longrightarrow X$$

$\text{Spet } B' \longrightarrow \text{Spet } A' \longrightarrow \text{Spet } B \longrightarrow \text{Spet } A$

seq of DM stk

in \mathcal{CAlg}

$$B' \xleftarrow{\quad} A' \xleftarrow{\quad} B \xleftarrow{\quad} A$$

$\xleftarrow{\quad}$
 $\xleftarrow{\quad}$

$$\begin{array}{c} B \xrightarrow{\text{ét}} B' \\ \downarrow \quad \downarrow \\ A \xrightarrow{\text{ét}} A' \xrightarrow{\text{ét}} A' \otimes_B B' \xrightarrow{\text{id}} B' \end{array}$$

$\leadsto B'$: retract of $A' \otimes_B B'$ in \mathcal{CAlg}_A

$$A \xrightarrow{\text{ét}} A' \otimes_B B'$$

$\leadsto B'$: étale / A

$$\begin{array}{c} A \xrightarrow{\quad} B \\ \text{étale} \searrow \quad \nearrow \text{étale} \\ B' \end{array}$$

étale !!
faithfully flat



