SAG 1.4.9 – 1.4.11

② Postnikov tower

$X \in \mathrm{SpDM}$
$\shortparallel$
$(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

Recall

$$\begin{tikzcd} \mathrm{SpDM}^{\le n} \arrow[r, shift left, hook] \arrow[r, phantom, "\perp"] & \mathrm{SpDM} \arrow[l, shift left, "\tau_{\le n}"] \end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{SpDM}^{\le n} & & \mathrm{SpDM} \\ \uplus & & \uplus \\ (\mathcal{X}, \tau_{\le n}\mathcal{O}_{\mathcal{X}}) & \leftarrow\!\shortmid & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

Prop $\mathrm{SpDM} \to \cdots \to \mathrm{SpDM}^{\le 2} \xrightarrow[\tau_{\le 0}]{} \mathrm{SpDM}^{\le 1} \xrightarrow[\tau_{\le 0}]{} \mathrm{SpDM}^{\le 0}$

is a limit diagram in CAT_∞

proof Let $G : \mathrm{SpDM} \longrightarrow \varprojlim_n \mathrm{SpDM}^{\le n}$

fully faithful

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{SpDM}}(X, Y) \arrow[rr] \arrow[dr] & & \varprojlim \mathrm{Map}_{\mathrm{SpDM}^{\le n}}(\tau_{\le}X, \cancel{\tau_{\le}}Y) \arrow[dl] \\ & \mathrm{Map}_{\infty\mathrm{Top}}(\mathcal{X}, \mathcal{Y}) & \end{tikzcd}$$

$\rotatebox{90}{$\in$}$
$f^* : \mathcal{Y} \to \mathcal{X}$

need to show equiv. on fibers

fib over f^* : $\mathrm{Map}_{\mathrm{Shv}^{loc}_{\mathrm{CAlg}}(\mathcal{X})}(f^*\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}})$

$\longrightarrow \varprojlim \mathrm{Map}_{\mathrm{Shv}^{loc}_{\mathrm{CAlg}}(\mathcal{X})}(f^*\mathcal{O}_{\mathcal{Y}}, \tau_{\le n}\mathcal{O}_{\mathcal{X}})$

without "loc" follows from

1.4.8.1 $\to \mathcal{O}_{\mathfrak{X}} \xrightarrow{\sim} \varprojlim_n \tau_{\leq n} \mathcal{O}_{\mathfrak{X}}$

hypercompleteness + π_n equiv.

checked on affines

add in "loc" "map is local" is a π_0 condition

direct summand corr. to local maps

corresponds in both sides.

ess. surj $ob(\varprojlim \mathrm{SpDM}^{\leq n}) \ni \left(\begin{array}{c} (\mathfrak{X}_n, \mathcal{O}_n) = X_n \\ + \\ \tau_{\leq n} X_{n+1} \simeq X_n \end{array} \right)$

may assume $\mathfrak{X}_n = \mathfrak{X}_0 =: \mathfrak{X}$

$(\mathfrak{X}, \mathcal{O}_n)$ with $\mathcal{O}_n \simeq \tau_{\leq n} \mathcal{O}_{n+1}$

$\rightsquigarrow$ want:

$$\begin{array}{ccc} \mathrm{SpDM} & \longrightarrow & \varprojlim_n \mathrm{SpDM}^{\leq n} \\ ① \Downarrow & & \Downarrow \\ (\mathfrak{X}, \varprojlim \mathcal{O}_n) & \overset{②}{\longleftrightarrow} & \{(\mathfrak{X}, \mathcal{O}_n)\}_n \end{array}$$

① ② : both local

passing to an affine cover of X_0

may assume X_0 : affine.

$\rightsquigarrow$ X_n is affine (affineness can be checked on π_0),

$$X_n = \mathrm{Sp\acute{e}t}\, A_n \xrightarrow{\lim} A$$

$(\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_n) = (\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_0)$: affine by X_0 : affine.

can check: $\mathrm{Sp\acute{e}t}\, A$ is $(\mathcal{X}, \varprojlim \mathcal{O}_{\mathrm{Sp\acute{e}t} A_n})$

① is clear

② by CAlg^{cn} : Postnikov complete. $\square$

② Étale morphisms of ∞-topoi (HTT 6.3.5)

$U \in \mathcal{X}$: ∞-topos

fix $\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{X}_{/U} \arrow[rr, bend left, "\pi_!"] & & \mathcal{X} \arrow[ll, "\pi^*"'] \arrow[ll, phantom, bend right, "\perp"] \\
\end{tikzcd} \qquad \mathcal{X}_{/U} \xrightarrow{\pi_*} \mathcal{X}$$

$\pi_!$: forget, $\pi^* = - \times U$, π_* : "sections over U"

$(\mathcal{X}_{/U}, \pi^*, \Delta)$

$U = 1_{\mathcal{X}/U}$, $U \xleftarrow{id} U$, $\Delta: U \to U \times U = \pi^* U$, $U \xleftarrow{pr} U \times U$

universal among

$$\mathcal{X}_{/U} \underset{g_*}{\overset{g^*}{\rightleftarrows}} \mathcal{Y}$$

$$\mathcal{Y} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{X} \qquad (\mathcal{Y}, f^*, \phi)$$

$$1_\mathcal{Y} \xrightarrow{\phi} f^* U$$

g^*
$\cap$

$g^* \circ \pi^*$
$=$

i.e.

$$\begin{tikzcd}
\mathrm{Fun}^*(\mathcal{X}/_U, \mathcal{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \arrow[d, "ev_U"] & \Rightarrow f^* \arrow[d, mapsto] \\
\mathcal{Y}_* \arrow[r, "forget"'] & \mathcal{Y} & f^*U
\end{tikzcd}$$

$$\mathcal{Y}_* \subset \mathrm{Fun}(\Delta^1, \mathcal{Y})$$

$$(1_{\mathcal{Y}} \to f^*U) \longmapsfrom f^*U$$

$$\parallel$$

$$g^*(1_{\mathcal{X}/U} \xrightarrow[\Delta]{} \pi^*U)$$

$$\cancel{F(f^*U, \mathcal{Y})}$$

$$\parallel$$

<u>Cor</u>

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{Y}}(1_{\mathcal{Y}}, f^*U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}^*(\mathcal{X}/_U, \mathcal{Y}) \arrow[r] \arrow[d, "\text{left fib}"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}_* \arrow[d, "\text{left fib}"] & 1_{\mathcal{Y}} \to f^*U \arrow[d, mapsto] \\
\{f^*\} \arrow[r] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \arrow[r, "ev_U"] & \mathcal{Y} & f^*U
\end{tikzcd}$$

$$\{f^*\} \longmapsto f^*U$$

fib seq

$$\rightsquigarrow \quad \begin{tikzcd}
\mathcal{Y}/f^*U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}/U \arrow[d, "\pi_*"] \\
\mathcal{Y} \arrow[r, "f_*"'] & \mathcal{X}
\end{tikzcd} \quad \text{in } \infty\mathrm{Top}$$

<u>again étale</u>

<u>proof</u> $\mathrm{Map}_{\infty\mathrm{Top}}(\mathcal{Z}, -)$

$$\begin{tikzcd}
P(g^*f^*U, Z) \arrow[r] \arrow[d, equal] & \mathrm{Fun}^*(\mathcal{Y}_{/f^*U}, Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & \mathrm{Fun}^*(\mathcal{Y}, Z) \arrow[d] & g^* \\
P((f\circ g)^*U, Z) \arrow[r] & \mathrm{Fun}^*(\mathcal{X}_{/U}, Z) \arrow[r] & \mathrm{Fun}^*(\mathcal{X}, Z)
\end{tikzcd}$$

$\longrightarrow f_!$

<u>Cor</u> plug $\mathcal{Y} = \mathcal{X}_{/V} \xleftarrow{f^*} \mathcal{X}$ étale

$$\begin{tikzcd}
\mathrm{Fun}^*_{\mathcal{X}}(\mathcal{X}_{/V}, \mathcal{X}_{/V}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^*(\mathcal{X}_{/V}, \mathcal{X}_{/V}) \arrow[d] \\
\{f^*\} \arrow[r] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{X}_{/V})
\end{tikzcd}$$

$$\mathrm{Fun}^*_{\mathcal{X}}(\mathcal{X}_{/V}, \mathcal{X}_{/V}) \overset{\sim}{\longleftarrow} \mathrm{Map}_{\infty\mathrm{Top}/\mathcal{X}}(\mathcal{X}_{/V}, \mathcal{X}_{/U}) \simeq \mathrm{Map}_{\mathcal{X}_{/V}}(1_{\mathcal{X}/V}, f^*U)$$

by fib seq

$$\mathrm{Map}_{\mathcal{X}_{/V}}(1_{\mathcal{X}/V}, f^*U) \cong \mathrm{Map}_{\mathcal{X}}(\underbrace{f_! 1_{\mathcal{X}/V}}_{\overset{\shortparallel}{V}}, U)$$

$$\begin{tikzcd}
V \arrow[r] & U \\
\mathcal{X}_{/V} \arrow[r, dashed] \arrow[dr] & \mathcal{X}_{/U} \arrow[d] \\
(\mathcal{X}_{/U})_{/V} \arrow[u, equal] & \mathcal{X}
\end{tikzcd}$$

<u>Cor.</u>

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f_*"] \arrow[dr, "h_*"'] & \mathcal{Y} \arrow[d, "g_*:\text{ étale}"] \\
& Z
\end{tikzcd}$$

f_* : étale
$\Updownarrow$
h_* : étale

<u>Cor</u>

$$\begin{tikzcd}
\mathcal{X}^{\Delta^1} \arrow[d, "\mathrm{cod}"] \\
\mathcal{X} \arrow[r] & \mathrm{CAT}_\infty^{op} \supset \mathrm{Pr}^R \\
U \arrow[r, mapsto] & \mathcal{X}_{/U}
\end{tikzcd}$$

($U \in \mathcal{X}$)

$\cup$

$\xrightarrow{\text{fully faithful}} \infty\mathrm{Top}^{\text{ét}}$

ess im $= (\infty\mathrm{Top}^{\text{ét}})_{/\mathfrak{X}}$

Colimit along étale mor

$$\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}} \hookrightarrow \infty\mathrm{Top}_{/\mathfrak{X}} \to \infty\mathrm{Top} \hookrightarrow \mathrm{Pr}^R \to \mathrm{CAT}_\infty^{op}$$

proof

$\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}}$ admits small colim; $\infty\mathrm{Top}^{\text{ét}}_{/\mathfrak{X}} \hookrightarrow \infty\mathrm{Top}_{/\mathfrak{X}}$ preserves them; creates colimits

$\mathfrak{X} \to \infty\mathrm{Top}_{/\mathfrak{X}}$

colim-pres

$\mathfrak{X} \to \mathrm{CAT}_\infty^{op}$

by descent

$$\operatorname{colim} \mathfrak{X}/U_\alpha \simeq \mathfrak{X}/\operatorname{colim} U_\alpha$$

More generally

$\boxed{\text{Thm}}$

$K \ni \alpha$

$$\infty\mathrm{Top}^{\text{ét}} \ni \mathfrak{X}_\alpha, \qquad \infty\mathrm{Top}^{\text{ét}} \hookrightarrow \infty\mathrm{Top}$$

$\infty\mathrm{Top}^{\text{ét}}$ admits small colim; $\infty\mathrm{Top}^{\text{ét}} \hookrightarrow \infty\mathrm{Top}$ preserves them

By thm $\operatorname{colim} \mathfrak{X}_\alpha =: \mathfrak{X}$ along ét $\rightsquigarrow$ $\mathfrak{X}/U_\alpha \cong \mathfrak{X}_\alpha \xrightarrow{\text{ét}} \mathfrak{X}$

$\rightsquigarrow \operatorname{colim} \mathfrak{X}/U_\alpha \simeq \mathfrak{X}$

$\coprod U_\alpha \twoheadrightarrow 1$

ref: 21.4.7
or DAG V

② Étale mor of locally ringed ∞-topoi

<u>Def</u> $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\infty\mathrm{Top}_{\mathrm{CAlg}}$ is étale

if $\simeq \searrow (\mathcal{Y}_{/V}, \mathcal{O}_{\mathcal{Y}}|_V) \nearrow$ $\exists V \in \mathcal{Y}$.

underlying topos étale
+
$\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}|_V$ condition

$\iff$ f : cartesian edge of $\infty\mathrm{Top}_{\mathrm{CAlg}} \to \infty\mathrm{Top}$

étale surjection if $V \twoheadrightarrow 1_{\mathcal{Y}}$

<u>Remark</u> (1)

$$\begin{tikzcd} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \arrow[dr, "g : \text{étale}"] & \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[ur, "f"] \arrow[rr, "h"'] & & (\mathcal{X}'', \mathcal{O}_{\mathcal{X}''}) \end{tikzcd}$$

$\Rightarrow$ f : étale iff h : étale

or strict

(2) $\infty\mathrm{Top}_{\mathrm{CAlg}}^{\text{loc, ét}}$ (obj : locally ring ∞-topoi
mor : étale mor
($\Rightarrow$ local))

admits small colim

preserved by

$\infty\mathrm{Top}_{\mathrm{CAlg}}$ $\quad$ $\infty\mathrm{Top}$ $\quad$ $\infty\mathrm{Top}^{\text{loc or strict}}$

$\mathrm{colim}\,(\mathcal{X}_\alpha, \mathcal{O}_\alpha)$ $\quad$ Let $\mathcal{X} := \mathrm{colim}\, \mathcal{X}_\alpha$
||

$$= \left(\mathcal{X}, \lim_{\alpha} (f_\alpha)_* \mathcal{O}_\alpha\right)$$

$\mathcal{X}_{/U_\alpha}$

$\mathcal{X}_\alpha \xrightarrow{(f_\alpha)_*} \mathcal{X}$

$\mathcal{O}_\alpha \quad \bullet$

<u>Rmk</u> $X \to Y$

f: étale is local on the source :

If $\exists \coprod_\alpha U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$ s.t.

$$f_\alpha : (\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \quad \text{étale}$$

$\Rightarrow f$: étale

i.e.
$$\left(\begin{tikzcd} \bullet \arrow[rr, "\text{ét.surj}"] \arrow[dr, "\text{étale}"'] & \arrow[d, phantom, "\Leftrightarrow"] & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}\right)$$

<u>proof</u> $\mathcal{X}_0 \underset{\text{full}}{\subset} \mathcal{X}$

$\mathcal{X}_0 \ni U \;:\Longleftrightarrow\; X|_U \to X \xrightarrow{f} Y$: étale

① • $\mathcal{X}_0$ is a sieve $\qquad V \to U \in \mathcal{X}_0 \Rightarrow V \in \mathcal{X}_0$

• $\mathcal{X}_0$ closed under small colim (by the prop above)

$U_\alpha \in \mathcal{X}_0$. $U = \coprod U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$

$\Rightarrow 1_{\mathcal{X}} \in \mathcal{X}_0$. i.e. f: étale.

$$\begin{tikzcd} U \times U \times U \in \mathcal{X}_0 \arrow[d, "①"] \\ U \end{tikzcd}$$

④ étale mor in SpDM^{nc}

$$K \longrightarrow (\mathrm{SpDM}^{nc})^{\acute{e}t} \subset (\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}})^{\acute{e}t}$$

closed under colim

$\alpha \mapsto X_\alpha$

SpDM^{nc} preserves them. $\subset_{\text{full sub}} \infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}$

<u>proof</u> $(\mathcal{X}_\alpha, \mathcal{O}_\alpha) \in \mathrm{SpDM}^{nc}$, $\mathcal{X}_\alpha = \mathcal{X}/U_\alpha$

$\downarrow$ colim

$(\mathcal{X}, \mathcal{O})$ in $\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}$ $\in \mathrm{SpDM}^{nc}$

$\underline{\underline{\coprod U_\alpha \twoheadrightarrow 1}}$

<u>e.g.</u> G : discre group

$BG \longrightarrow \mathrm{SpDM}^{nc}$ $\qquad \operatorname{colim}_{BG} X = X/G$

$* \circlearrowleft G \qquad X \circlearrowleft G$ act by equiv $\Downarrow$ étale

$\left(\begin{array}{l} X/\!/G \\ [X/G] \end{array} ? \right)$

⑤ relation to étale ring maps ? (1.4.10)

<u>Thm</u> $\phi^*: A \longrightarrow B$ in CAlg is étale iff $\mathrm{Sp\acute{e}t}\, B \xrightarrow[\phi_*]{} \mathrm{Sp\acute{e}t}\, A$ is étale.

<u>Cor</u> $f: X \longrightarrow Y$ in SpDM^{nc} is étale

iff $\forall$ comm square

$$\left[\begin{tikzcd} \mathrm{Sp\acute{e}t}\, B \arrow[r, "\acute{e}t"] \arrow[d, "\phi_*"] & X \arrow[d, "f"] \\ \mathrm{Sp\acute{e}t}\, A \arrow[r, "\acute{e}t"'] & Y \end{tikzcd}\right]$$

alg map $A \xrightarrow[\phi^*]{} B$ is étale

$(\Rightarrow)$

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, B \arrow[rr, "\phi^*"] \arrow[dr, "\acute{e}t"'] & & \mathrm{Sp\acute{e}t}\, A \arrow[dl, "\acute{e}t"] \\ & Y & \end{tikzcd}$$

$\Rightarrow$ ϕ_* : ét
$\Updownarrow$ Thm
$\phi^*: A \to B$ etale.

$(\Leftarrow)$ Cover Y by affines V_α

$\rightsquigarrow$ cover X by affines $U_{\alpha\beta}$

$$\begin{tikzcd} \text{aff} : \coprod U_{\alpha\beta} \arrow[r, dashed, "\text{cover}"] \arrow[d, "\text{étale}"] & X \arrow[d] \\ \text{aff} : V_\alpha \arrow[r, "\acute{e}t"'] & Y \end{tikzcd}$$

$\Rightarrow$ f : étale. ∎

Proof of Thm

Thm $\phi^* : A \longrightarrow B$ in CAlg is étale iff $\mathrm{Sp\acute{e}t}\, B \xrightarrow[\phi_*]{} \mathrm{Sp\acute{e}t}\, A$ is étale

$(\Rightarrow)$ by construction

$$\mathrm{Shv}^{\text{ét}}_B \simeq (\mathrm{Shv}^{\text{ét}}_A)_{/\underbrace{h^B}_{\text{corep. sheaf by } B}}$$

$(\Leftarrow)$

$$\begin{tikzcd}
\mathrm{Sp\acute{e}t}\, B \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Sp\acute{e}t}\, A \arrow[d, phantom, "="] \\
(\mathfrak{X}_{/U}, \mathcal{O}_{\mathfrak{X}}|_U) \arrow[r] & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})
\end{tikzcd}$$

$$U \in \mathfrak{X}$$

$$\begin{tikzcd}
 & U \\
\coprod_{j \in J} W_j \arrow[ur] \arrow[r, "\text{refine}"'] & \coprod_{i \in I} V_i \arrow[u, two heads]
\end{tikzcd}$$

finite $\rightarrow$ $j \in J$

by W_j : corep by B_j : étale B-alg.

[$V_i :=$ sheaf in $\mathrm{Shv}^{\text{ét}}_A$ corep. by A_i : étale A-alg]

(refinement : $J \xrightarrow{\mathrm{id}} \cancel{I}\ J$

$j \mapsto r(j)$

$$\begin{tikzcd}
W_j \arrow[r] \arrow[dr] & V_{r(j)} \arrow[d] \\
 & U
\end{tikzcd}$$
)

$\coprod W_j$ = corep by $\prod B_j =: B'$ sheaf in $\mathrm{Shv}^{\text{ét}}_B$

$\coprod V_j$ = corep by $\prod A_j = A'$ in $\mathrm{Shv}^{\text{ét}}_A$

$$\begin{array}{ccccccc}
(X_{/U})_{/\coprod W_j} & \longrightarrow & X_{/\coprod V_j} & \longrightarrow & X_{/U} & \longrightarrow & X \\
\| & & \| & & \| & & \| \\
\mathrm{Sp\acute{e}t}\, B' & \longrightarrow & \mathrm{Sp\acute{e}t}\, A' & \longrightarrow & \mathrm{Sp\acute{e}t}\, B & \longrightarrow & \mathrm{Sp\acute{e}t}\, A
\end{array}$$

seq of DM stk

in CAlg

$$\begin{tikzcd} B' & A' \arrow[l] & B \arrow[l] \arrow[ll, bend left, "\text{ét}"] & A \arrow[l] \arrow[ll, bend right, "\text{ét}"'] \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[r, "\text{ét}"] \arrow[d] & B' \arrow[d] \arrow[ddr, bend left, "id"] & \\ A \arrow[r, "\text{ét}"'] & A' \arrow[r, "\text{ét}"'] \arrow[drr, bend right] \arrow[ur, phantom, "r"] & A' \otimes_B B' \arrow[dr] & \\ & & & B' \end{tikzcd}$$

$\rightsquigarrow$ B': retract of $A' \otimes_B B'$ in $CAlg_A$

$$A \xrightarrow{\text{ét}} A' \otimes_B B'$$

$\rightsquigarrow$ B': Étale $/A$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\text{étale}"'] & & B \arrow[dl, shift left, "\text{étale, faithfully flat}"] \\ & B' \arrow[ur, shift left] & \end{tikzcd}$$

étale !!

$\boxed{\text{QED}}$