K vs TC

$(R, (p))$ henselian, (+ some conditions)

$\rightsquigarrow K(R)/p^r \longrightarrow TC(R)/p^r$

6.3 • equivalence if the domain is "Étale - Postnikov sheafified"

6.5 • $\pi_{\geq d}$-equiv, d depending on the "p-dim" of the local rings of R

6.6 K & Étale sheafified K agrees in deg $\geq d$.
("Lichtenbaum - Quillen type" statement)

Strategy

use descent $\begin{bmatrix} \text{étale} \\ \text{Nisnevich} \end{bmatrix}$, reduce to $\begin{bmatrix} \text{strictly henselian local} \\ \text{henselian local rings} \end{bmatrix}$

use rigidity to reduce to $\begin{bmatrix} \text{separably closed fields} \\ \text{fields } (/\mathbb{F}_p) \end{bmatrix}$
(Thm 4.36)

use 4.29 : explicit formula for K, TC.

main references [hyperdescent] Hyperdescent and étale K-thry

[Descent] Descent in Alg K-thry and a conjecture of Ausoni - Rognes

Topologies X : scheme,

$$X_{\text{ét}} \xrightarrow{\supset} X_{\text{Nis}} \xrightarrow{\supset} X_{\text{Zar}}$$ sites & "continuous" maps

finer ⟵⟶ coarser

- X_{Zar} : category of $U \to X$ open imm.

  + coverings = open cov.

- $X_{\text{ét}}$: category of $U \to X$ étale maps ($\subset \mathrm{Sch}/_X$)

  + coverings : contain $\{U_i \to X\}_{1\le i\le n}$

  s.t. $\coprod U_i \longrightarrow X$: surj

- X_{Nis} : category of $U \to X$ étale maps ($\subset \mathrm{Sch}/_X$)

  + coverings : contain $\{U_i \to X\}_{1\le i\le n}$

  s.t. $\coprod U_i \longrightarrow X$

  $\nwarrow$ $\uparrow x$ Spec k

  ($\forall x$, $\forall k$: field)

"Norms in motivic homotopy thry" Appendix A

$$\rightsquigarrow \underset{\mathrm{Ab}}{\mathrm{Shv}}(X_{\text{ét}}) \underset{\leftarrow}{\overset{\perp}{\rightarrow}} \underset{\mathrm{Ab}}{\mathrm{Shv}}(X_{\text{Nis}}) \underset{\rightarrow}{\overset{\leftarrow}{\perp}} \underset{\mathrm{Ab}}{\mathrm{Shv}}(X_{\text{Zar}})$$ geom mor

have enough points : $\{x^* : \mathrm{Shv}(X_\tau) \to \mathrm{Ab}\}$

$\begin{cases} \bullet \text{ jointly conservative} \\ \bullet \text{ reflects exact sequence} \end{cases}$

- X_{Zar} : a point of the underlying space

  stalk of $\mathcal{O}_X$: $\mathcal{O}_{X,x}$

- $X_{ét}$: a geometric point $\bar{x} : \operatorname{Spec} k \to X$
$$* \mapsto \bar{x}$$

k : separably closed field.

$$k(x) \hookrightarrow k(x)^{sep} \hookrightarrow k$$

$$\operatorname*{colim}_{\text{étale nbd}} \mathcal{O}_X(U)$$
$$\shortparallel$$
$$\mathcal{O}_{X,\bar{x}} \cong \mathcal{O}_{X,x}^{sh}$$

- X_{Nis} : $\operatorname{Spec} k \to X$, $* \mapsto x$ classifying a finite separable ext $k(x) \hookrightarrow k$

$$\left(\text{or } Y \in Et_{/X},\ y \in Y \quad \begin{array}{ccc} k = k(y) & \to & Y \\ \downarrow & & \downarrow \\ k(x) & \to & X \end{array}\right)$$

$$(\mathcal{O}_X)_{(Y,y)} \cong \mathcal{O}_{Y,y}^{h},$$
$$\shortparallel$$
$$\operatorname*{colim}_{\left(\substack{\text{ét nbd} \\ \text{w/ res. field} \\ k}\right)} \mathcal{O}_X(U)$$

<u>point</u> : TC satisfies étale descent

K satisfies Nisnevich descent) finite Galois descent

but not étale descent

<u>Thm 6.3</u> X: proper scheme $/R$ $(R,(p))$ henselian.

$$\hat{K}^{ét}(-) := \varprojlim_n \left(K(-)_{\leq n}\right)^{ét}$$

$$\left((-)^{ét} : PShv(X_{ét}) \longrightarrow Shv(X_{ét}) \right)$$

(Thm 1.3 [Hyperdescent] says under some mild conditions

$$\hat{K}^{ét}(-) = (K(-))^{ét}$$
)

$$\rightsquigarrow \quad \hat{K}^{ét}(X)/p^r \xrightarrow{\sim} TC(X)/p^r$$

<u>proof</u> TC/p is an étale Postnikov sheaf (Thm 5.16 of [hyperdescent])

$$\rightsquigarrow \quad \begin{array}{ccc} \hat{K}^{ét}(X) & \longrightarrow & TC/p \\ \downarrow & \nearrow & \\ \hat{K}^{ét}(X)/p & & \end{array}$$

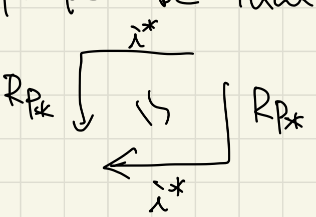
induced by cyc. trace.

- Gabber's "affine analog of the proper basechange thm"

(A, I) henselian $\quad H^j_{ét}(\mathrm{Spec}\,\underset{A}{R}, \mathcal{F}) \xrightarrow{\sim} H^j_{ét}(\mathrm{Spec}(\underset{A/I}{R/(p)}), i^*\mathcal{F})$

for any $\mathcal{F}$: torsion

- proper bc thm



$$\begin{array}{ccc} i^*\mathcal{F} & & \mathcal{F} \\ X_{\mathbb{F}_p} & \xrightarrow{i} & X \\ p \downarrow & \lrcorner & \downarrow p : \text{proper} \\ \mathrm{Spec}\,R/(p) & \xhookrightarrow{i} & \mathrm{Spec}\,R \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}\,\mathbb{F}_p & \xhookrightarrow{i} & \mathrm{Spec}\,\mathbb{Z} \end{array}$$

$$H^j_{ét}(X_{\mathbb{F}_p}, i^*\mathcal{F}) \xleftarrow{\simeq} H^j_{ét}(X; \mathcal{F})$$

$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

$$H^j_{ét}(\mathrm{Spec}\,R/pR, Rp_* i^*\mathcal{F}) \simeq H^j_{ét}(\mathrm{Spec}\,R, Rp_*\mathcal{F})$$

≀ ← proper bc

$H^s_{ét}(\mathrm{Spec}\, R/p\,,\ \boxed{i^* Rp_* \mathcal{F}})$ $\simeq$

Gabber's

$\left(E_2^{i,j} = R^i \Gamma_Y \circ R^j p_* \Rightarrow R^{i+j}\Gamma_X \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow g \\ & & * \end{array} \right)$

$X_{ét}$ has finite p-cohomological dimension

$\uparrow$ max n s.t. $\exists \mathcal{F}$ p-power tor

$H^n(X; \mathcal{F}) \neq 0$

$X = \bigcup_{i=1}^{n} U_i$ affine open $\quad (U_i \cap U_j$ affine$)$

induction on n $\quad H^{>n}(X, \mathcal{F}) = 0$

(p-power torsion)

$n=1$ case Artin-Schreier seq

- $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$ over $\mathbb{F}_p$ (of étale sheaves)

$x \longmapsto x - x^p$

- $H^{\bullet}_{ét}(X, \mathbb{G}_a) \simeq H^{\bullet}_{Zar}(X, \mathcal{O}_X) = 0$ if $\bullet \geq 1$

(Zariski vs étale qcoh sheaves) $\quad X$: affine

$H^{\bullet}_{ét}(X, \mathbb{Z}/p\mathbb{Z}) = 0$ if $\bullet \geq 2$

How to upgrade this ?? $\mathbb{Z}/p\mathbb{Z} \rightsquigarrow \mathcal{F}$

$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$

induction step: by MV seq.

⑥

$\mathrm{Ext}^p(h_x, \pi_q^{\text{ét}} \mathcal{F})$

(hyperdescent ex. 2.11 ??)

(Specializing Prop 2.13)

?

h_x

$$E^2_{p,q} = H^p_{\text{ét}}(X; \pi_q^{\text{ét}} \mathcal{F}) \Rightarrow \pi_{q-p} \hat{\mathcal{F}}^{\text{ét}}(X)$$

for $\mathcal{F} = K(-)/p$

$TC(-)/p$

$$\begin{tikzcd}
\pi_q^{\text{ét}} : \mathrm{Shv}^{\text{ét}}_{\mathrm{Sp}}(X) \arrow[r] \arrow[dr, "\pi_q"'] & (\mathrm{Shv}^{\text{ét}}(X))^{\heartsuit} \simeq \mathrm{Shv}^{\text{ét}}_{\mathrm{Ab}}(X) \\
& \mathrm{PShv}_{\mathrm{Ab}}(X) \arrow[u, "\text{sheafify}"']
\end{tikzcd}$$

- $\widehat{TC(-)/p}^{\text{ét}} = TC(-)/p$

$\rightsquigarrow$ to show $\hat{K}^{\text{ét}}/p \to TC/p$: equiv.

ETS

$$\begin{array}{ccc}
H^p_{\text{ét}}(X; \pi_q^{\text{ét}}(K(-)/p)) & \xrightarrow{\simeq} & H^p_{\text{ét}}(X; \pi_q^{\text{ét}}(TC(-)/p)) \\
\| & & \| \\
H^p_{\text{ét}}(X_{\mathbb{F}_p}; i^* \text{———}) & & H^p_{\text{ét}}(X_{\mathbb{F}_p}; i^* \text{———})
\end{array}$$

ETS $i^* \pi_q^{\text{ét}}(K(-)/p) \xrightarrow{\simeq} i^* \pi_q^{\text{ét}}(TC(-)/p)$

i_* fully faithful $\simeq$

$i^* i_* i^*(\pi_q^{\text{ét}} K(-)/p)$ $\cong$

ess. im $= \left(\begin{array}{l} \mathcal{F}_{\bar{x}} = 0 \\ \text{for } \bar{x} \in X_{\mathbb{F}_p} \end{array} \right)$

$$\boxed{i_* i^* \pi_q^{\text{ét}} K(-)/p \xrightarrow{\simeq} \pi_q^{\text{ét}}(TC(-)/p)}$$

of $\mathrm{Shv}^{\text{ét}}_{\mathrm{Ab}}(X)$

Taking stalks of $\bar{x} \in X$

- $\left(\pi_{g}\left(TC(-)/p\right)\right)_{\bar{x}} = 0$ for $\bar{x} \notin X_{\mathbb{F}_p}$ ($\rightsquigarrow$ p is invertible here)

  p invertible

- $\left(\pi_g K(-)/p\right)_{\bar{x}} \xrightarrow{\simeq} \left(\pi_g TC(-)/p\right)_{\bar{x}}$ for $x \in X_{\mathbb{F}_p}$

$(i_* g)_{\bar{x}} = g_{\bar{x}}$

$\mathcal{F}(U) \simeq \mathcal{F}(\mathcal{O}_{X,\bar{x}})$

$\operatorname{colim}_{(U,u)\text{ étale}}$

commute with filtered colim

reduced to proving

Thm 6.1 R strictly henselian local, residue char $= p$

then $K(R)/p \xrightarrow{\simeq} TC(R)/p$

- $k/\mathbb{F}_p$: ind-smooth
- 4.29 (1)(2)(3)

$\tilde{\nu}^{n+1}(R) \simeq \pi_{n-1}\left(K^{inv}(R)/p\right) = 0$

true for $n \geq 0$

8/17 **Thm 6.1** R : sHen^{loc}, residue char $= p$ (residue field k)

then $K(R)/p \xrightarrow{\sim} TC(R)/p$.

Proof by Thm 4.36

$$\begin{tikzcd}
K(R)/p \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{K(k)/p} \arrow[d, "\simeq"] \\
TC(R)/p \arrow[r] & \boxed{TC(k)/p}
\end{tikzcd}$$

Reduced to the case R : sep. closed field of char p.
$R = k$

(general)
Lemma If F : perfect field, E/F : field ext
$\Longrightarrow$ E : ind-smooth F-algebra.

Apply for $k/\mathbb{F}_p$

proof $E = \text{filt. colim}\,(\text{fin gen ext})$ [stacks 0300]

$\rightsquigarrow$ may assume E/F fin gen.

$\exists$ separating transcendence basis

$$\underbrace{E \,/\, F(x_1, \dots, x_n)}_{\text{finite separable}} \underbrace{/\, F}_{\text{purely trans.}}$$

finite separable $\Downarrow$ étale

purely trans. $\Downarrow$ smooth (check the lifting property)

$\operatorname{Spec} E \to \operatorname{Spec} F$ smooth

$\left(A \twoheadrightarrow A/I \text{ square-zero} \Rightarrow A^\times \to (A/I)^\times \text{ surj} \right)$

$\bigcup_{n \geq 0} k(x^{1/p^n}) = k(x^{1/p^\infty})/k$: separable but has no separating trans basis

k : ind-smooth $\mathbb{F}_p$-algebra. by Thm 4.29

[$\forall E/F$: separable $\Leftrightarrow$ $\forall E/F$ algebraic $\Rightarrow$ separable
(in a general sense)
($\Leftrightarrow$ Frobenius is isom)]

4.29

$$\begin{tikzcd}
 & & \pi_{n-1}(K^{inv}(R)/p) & & \\
0 \arrow[r] & \tilde{\nu}^{n+1}(R) \arrow[r] \arrow[ur, "(3)", "\cong"', "R:\text{local}"] & \pi_n(TC(R)/p) \arrow[r] \arrow[u] & \nu^n(R) \arrow[r] & 0 \\
 & & \pi_n(K(R)/p) \arrow[u] \arrow[ur, "(1)", "\cong \text{ if } R:\text{local}"'] & &
\end{tikzcd}$$

$n \geq 0$

(2) exact

(In particular 4.29 (3) in the paper :

ETS $\boxed{\begin{matrix}\tilde{\nu}^{n+2}(R) \\ \| \\ 0\end{matrix}}$ $\Longleftrightarrow$ $\cong$ $\begin{matrix}\pi_n(K^{inv}(R)/p) \\ \| \\ 0\end{matrix}$ is actually true for $n \geq -1$)

$$\tilde{\nu}^n(R) = \operatorname{coker}\left(\Omega^n_k \xrightarrow[1-C^{-1}]{} \Omega^n_k / d\Omega^{n-1}_k\right) \qquad R = k$$

recall $C^{-1} : \Omega^n_k \longrightarrow H^n(\Omega^*_k) \hookrightarrow \Omega^n_k / d\Omega^n_k$

characterized by
- graded alg hom
- $C^{-1}a = a^p$
- $C^{-1}(db) = b^{p-1}\,db$. " $= \dfrac{d(b^p)}{p}$ "

we prove $1 - C^{-1} : \Omega^n_k \to \Omega^n_k / d\Omega^{n-1}_k$: surj.

$$\omega \longmapsto [\omega]$$

$\omega = a \cdot dx_1 \cdots dx_n \quad (x_i \in k)$

If $u \in k$, then

$$(1 - C^{-1})(u\omega) = u\omega - (ua)^p x_1^{p-1} \cdots x_n^{p-1} dx_1 \cdots dx_n$$

$$= \underbrace{(u - u^p (ax_1 \cdots x_n)^{p-1})}_{=1} \underbrace{a\,dx_1 \cdots dx_n}_{=\omega}$$

$\exists$ solution to $0 = u - \boxed{\Sigma} u^p - 1$ in k ← sep. cl.

$(-)'$ ⇝ 1 ⇒ separable polynomial.

$\boxed{\Sigma}$

§6.2 Asymptotic comparison

- $(R, (p))$ henselian pair
- $R/(p)$ has finite Krull dim

Thm 6.5 $d := \max \left\{ 1, \sup_{x \in \operatorname{Spec} R/p} \log_p [k(x) : k(x)^p] \right\}$

the map $K(R)/p^r \to TC(R)/p^r$: $\pi_{\geq d}$-equiv $(\forall r)$

Proof · by rigidity (4.36) we may assume $R = R/(p)$

(i.e. R : $\mathbb{F}_p$-algebra)

- $K(R)/p^r \to \mathbb{K}(R)/p^r \to TC(R)/p^r$

$\uparrow$
equiv in deg ≥ 0

$\uparrow$ equiv in deg ≥ 1

$\rightsquigarrow$ we may replace K by $\mathbb{K}$.

- [Descent A.15] Any "weakly localizing" invariant satisfies Nisnevich descent

$$\mathrm{Cat}_\infty^{ex} \longrightarrow \mathcal{D}$$

$$[\text{Verdier quotient}] \longmapsto [\text{Cofiber seq}]$$

$\cdot\ \mathbb{K}(-)/p^r$, $\cdot\ TC(-)/p^r$ } weakly locali

- $\dim R < \infty \implies (\operatorname{Spec} R)_{Nis}$ has homotopy dim $< \infty$
  - [hyperdescent : 3.17]
  - When R: noeth : [SAG Chap 3]

S ho dim 0 $\quad S/S^n$ ho dim n

$\mathcal{X}$ has ho dim $\leq n$

$\iff \forall X \in \mathcal{X}_{\geq n} \quad \exists \begin{matrix} X \\ \downarrow \\ * \end{matrix}$

$\implies (\operatorname{Spec} R)_{Nis}$: loc. of ho. dim $< \infty$ (the topos is gen by affine étale A/R, $\dim A = \dim R$ $\rightsquigarrow (\operatorname{Spec} A)_{Nis}$: ho. dim $\leq n$ for some fixed n)

- $\{X_i\}$ generates $\mathcal{X}$ under colim s.t. $\mathcal{X}/X_i$ of ho dim $\leq n$
- locally of ho dim $< \infty \implies$ Postnikov complete.

Descent SS

$$H^s_{Nis}(X ; \pi_t^{Nis}(\mathbb{K}(-)/p^r)) \implies \pi_{t-s}\, \mathbb{K}(-)/p^r$$

$$H^s_{et}(X; \pi^{Nis}_t(TC(-)/p^r)) \Rightarrow \widehat{\pi}_{t-s}\, TC(-)/p^r$$

deg $\geq d$ equiv ?

enough to show equiv in $t-s \geq d$ ($\Rightarrow t \geq d$)

Taking stalks (@ $Y \in Et_{/X}$, $y \in Y$) ETS:

$$\pi_q(\mathbb{K}(-)/p^r)_y \longrightarrow \pi_q(TC(-)/p^r)_y \; : \cong \text{ when } q \geq d$$

$$X_{Nis} \xrightarrow{x^*} \in Set$$

$$\exists Y$$

$$\mathrm{Spec}\, k \xrightarrow[\text{fin sep}]{} \mathrm{Spec}\, k(x) \xrightarrow{x} X \xleftarrow{\text{fin ét}} Y, \quad \mathrm{Spec}\, k \xrightarrow{y} Y$$

$X = \mathrm{Spec}\, R$

$$\left[\mathcal{F}_{(Y,y)} = \operatorname*{colim}_{\substack{U \to X \text{ étale} \\ \Psi \quad \Psi \\ x' \to x \\ k(x') = k(y)}} \mathcal{F}(U) \right]$$

$$\Leftrightarrow \quad \pi_q(\mathbb{K}(\mathcal{O}^h_{Y,y})/p^r) \longrightarrow \pi_q(TC(\mathcal{O}^h_{Y,y})/p^r) \; : \; q \geq d (\geq 1)$$

$\pi_q(K(\mathcal{O}^h_{Y,y})/p^r)$

$\mathcal{O}^h_{Y,y} = A$: henselian local, residue field of A:

$k(\mathcal{O}^h_{Y,y}) \cong k(\mathcal{O}_{Y,y})$ fin sep ext of $k(x)$

$\uparrow$ fin sep

$k(x)$

$\exists x \in |\mathrm{Spec}\, R|$

A : henselian local with res field k

by rigidity we may assume $\underline{A = k}$

$$K(k)/p^r \longrightarrow TC(k)/p^r \quad \pi_{\geq d}\text{-equiv}$$

If char $k = p$, $d \geq \log_p[k : k^p]$

remark $k / k(x)$ fin sep. (étale) $\quad \log_p [k(x) : k(x)^p] \le d$

$$\log_p [k : k^p] = \dim_k \Omega^1_{k/\mathbb{F}_p} = \dim_{k(x)} \Omega^1_{k(x)/\mathbb{F}_p}$$

(theory of p-basis [07P2]

x_i : p-basis $\Leftrightarrow$ dx_i : basis of Ω^1)

$\Leftarrow \prod x_i^{(<p)}$: basis

by 4.29

$\Longleftrightarrow \tilde{\nu}^{>d}(k) = 0$

$$\log_p [k : k^p] = \dim \Omega^1_k \le d$$

$$\Downarrow$$

$$\Omega^n_k = 0 \quad \text{for } n > d.$$

$$1 - C^{-1} : 0 \longrightarrow 0 \quad \text{in deg } n > d.$$

$\hookrightarrow \tilde{\nu}^{>d}(k) = 0$. $\boxed{k}$