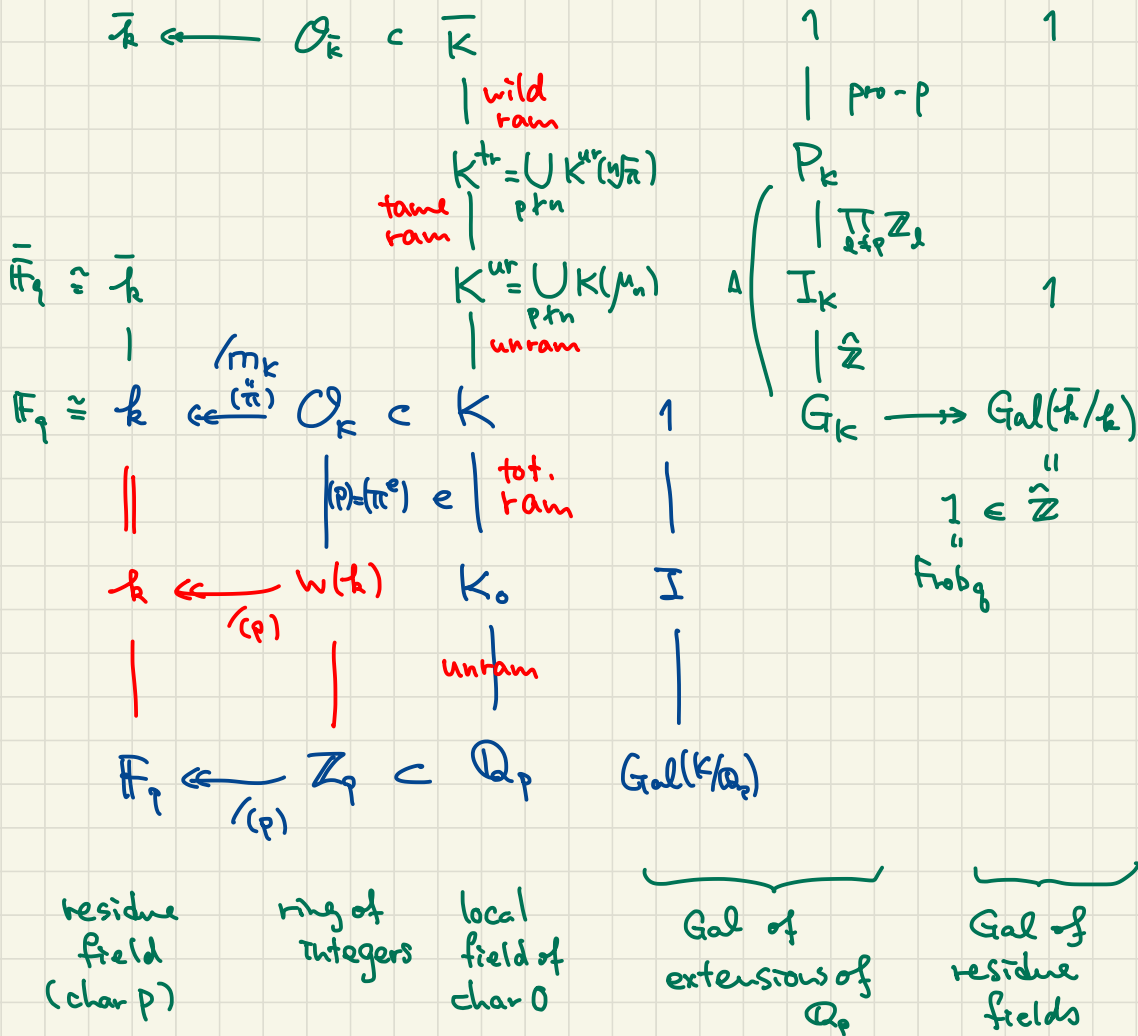
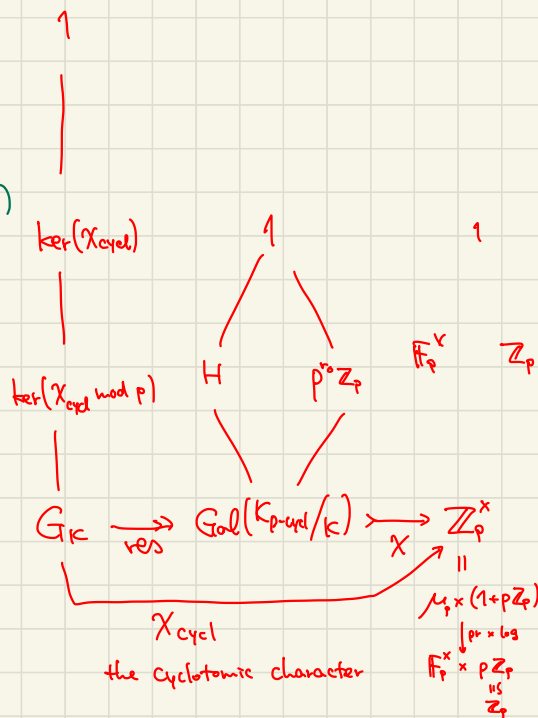
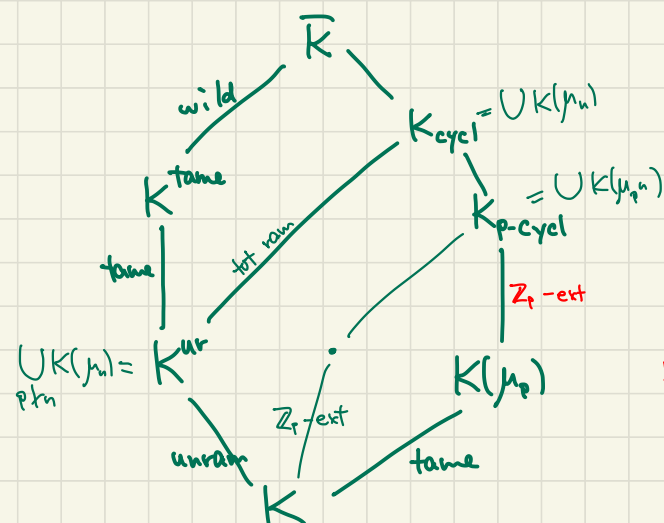


Cheat Sheet





$$\begin{array}{ccccccc}
 1 & \rightarrow & \mathbb{F}_p^{\times} & \xrightarrow[\text{pr}]{\text{pr}} & \mathbb{Z}_p^{\times} & \xrightarrow{\text{log}} & p\mathbb{Z}_p \rightarrow 1 \\
 & & \uparrow & & \uparrow \chi & & \uparrow \\
 1 & \rightarrow & H & \xrightarrow{\text{res}} & \text{Gal}(K_{p\text{-cycl}}/K) & \rightarrow & p^{\infty}\mathbb{Z}_p \rightarrow 1 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \mathbb{Z}_p & & \mathbb{Z}_p
 \end{array}$$

Hodge - Tate Representation

$$\begin{array}{ccc}
 B_{dR} & \longrightarrow & B_{HT} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_p(n) \\
 \text{filtration} & \downarrow & \uparrow \\
 & \text{gr.} & \text{as } G_K\text{-rep.}
 \end{array}$$

Hodge - Tate decomp

Thm (Faltings) X : smooth proper scheme / k

$\Rightarrow G_K$ -equiv isom

$$H_{\text{ét}}^k(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{a+b=k} H^b(X, \Omega_{X/K}^a) \otimes_K \mathbb{C}_p(-a)$$

(cf. analog for \mathbb{C} ,
 X : cpt Kähler : Hodge decomp)
Tate twist

Recall E : top field, G : top gp B : top E -alg

G acts on B , restricts to trivial on E

$$\text{Rep}_B(G) = \left\{ V : B\text{-mod} \left(\begin{array}{c} G \curvearrowright V : B\text{-semilin} \\ x \mapsto y \\ g(bx) = g(b)g(y) \end{array} \right) \right\}$$

$B \otimes_E - \uparrow$

$$\text{Rep}_E(G) = \left\{ \begin{array}{c} \text{E-linear rep of } G \\ \downarrow \text{triv.} \\ G \end{array} \right\}$$

$$\text{Rep}_B(G) : \text{abelian } \otimes_B, \text{Hom}_B(-, -)$$

non. closed

$$\text{Hom}_B(-, -)^G = \text{Hom}_{\text{Rep}_B(G)}(-, -) \left(\begin{array}{ccc} & g & \\ g \uparrow & \alpha & \uparrow g \\ & \varphi: V \rightarrow W & \end{array} \right)$$

B^n : free B -mod with componentwise G -action is called trivial

$$\text{Rep}_E^{B\text{-adm}}(G) \underset{\text{full}}{\subset} \text{Rep}_E(G) \quad \left\{ \begin{array}{l} B\text{-semilin rep} \end{array} \right.$$

||

$$\left\{ V : \text{f.in. dim / } E, B \otimes_E V \text{ is trivial} \right\} \quad \left\{ \begin{array}{l} \text{closed under} \\ \oplus, \otimes \\ (-)^\vee \end{array} \right.$$

Assumptions : (H1) B : domain

$$\left(\begin{array}{l} \text{FO's book} \\ \text{Def 2.8} \\ (F, G) \text{ regular} \end{array} \right) \Leftrightarrow \left\{ \begin{array}{l} \text{(H2) } (\text{Frac } B)^G = B^G \\ \text{(H3) } b \in B \setminus \{0\}, Eb \subset B \text{ is } G\text{-inv} \end{array} \right.$$

$\Rightarrow b \in B^\times$

follows:

B^G : field

$$b \in B^G \setminus \{0\}$$

$$\Rightarrow \exists b \quad G \text{ inv} \Rightarrow b \in B^{\times}$$

$$(b^{-1} \in B^G)$$

Prop Assume (H1) - (H3)

1.4.1 $V \in \text{Rep}_F(G)$ fin dim / F } TFAE:
 $W = B \otimes_F V$

(i) W : trivial (i.e. $V \in \text{Rep}_F^{B\text{-adm}}(G)$)

(ii) $\alpha_W: B \otimes_{B^G} W^G \longrightarrow W$ is an ISO
 (injective: automatic)

(iii) $\dim_{B^G} W^G = \dim_F V$

proof elementary

□

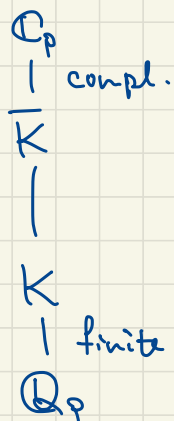
Cor From characterization (iii) it's easy to prove
 $\text{Rep}_F^{B\text{-adm}}(G) \subset \text{Rep}_F(G)$ closed under
 sub quotient.

$$\left(\begin{array}{l} 0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0 \\ V: B\text{-adm} \\ \Rightarrow V_1, V_2: B\text{-adm} \end{array} \right)$$

$$0 \rightarrow (B \otimes_F V_1)^G \rightarrow (B \otimes_F V)^G \rightarrow (B \otimes_F V_2)^G$$

④ Cyclotomic character, Tate twist

Setting



$$G = G_K = \text{Gal}(\bar{K}/K)$$

\mathbb{Z} triv.

$$E = \mathbb{Q}_p$$

conti \curvearrowright \mathbb{C}_p

$$\mu_{p^n}(K) \cong \begin{matrix} \mathbb{Z}/p^n\mathbb{Z} \\ \curvearrowright \end{matrix} \cong \begin{matrix} \mathbb{Z}/p^n\mathbb{Z} \\ \curvearrowright \end{matrix} \chi_n(\sigma) \text{ mult.}$$

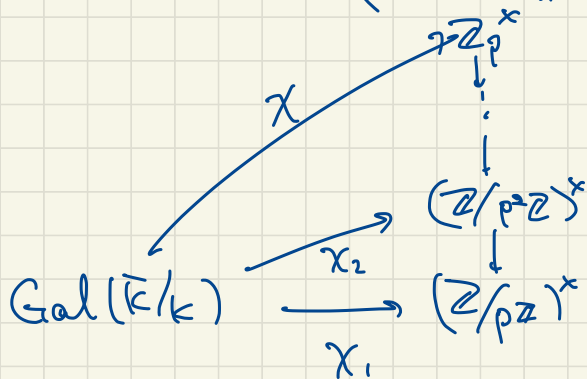
ζ_n prim. p^n th root of 1

$$\chi_n: \text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(K(\mu_{p^n})/K) \rightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times$$

σ

$$\mapsto (\zeta_n \mapsto \zeta_n^{\chi_n(\sigma)})$$

$\chi_n(\sigma)$



$$\leadsto \chi: G_K \longrightarrow \mathbb{Z}_p^\times \text{ cyclotomic character}$$

$$\parallel$$

$$GL_1(\mathbb{Z}_p) \hookrightarrow GL_1(\mathbb{Q}_p)$$

1-dim repn $\mathbb{Q}_p(1)$ of G_K / \mathbb{Q}_p

$$Q_p(n) = \begin{cases} Q_p(1)^{\otimes n} & (n \geq 0) \\ (Q_p(1)^\vee)^{\otimes (-n)} & (n < 0) \end{cases}$$

repn corr. char χ^n

↑
 Tate twist

$$V(n) = V \otimes_{Q_p} Q(n)$$

Aside: characters of G_{Q_p} are

§1.1.3

$$\left\{ \begin{array}{l} a \in \mathbb{Z}_p^\times \\ b \in \mathbb{F}_p^\times \end{array} \right. \left(G_K \rightarrow \mathbb{Z}_p^\times \xrightarrow{\text{mod } p} \mathbb{F}_p^\times \xrightarrow{[-]} \mathbb{Z}_p^\times \right)$$

$\chi^a \cdot \omega^b \cdot \lambda$
↑
unramified $\hat{\mathbb{Z}} \rightarrow \mathbb{Z}_p^\times$
(Frob) $\rightarrow \lambda$

$$1 \rightarrow 1+p\mathbb{Z}_p \rightarrow \mathbb{Z}_p^\times \rightarrow \mathbb{F}_p^\times \rightarrow 1$$

(1) \mathbb{Z}_p

$\mathbb{C}_p \cdot t^n$
" μ_p

$$B_{HT} := \mathbb{C}_p[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_p(n)$$

$$\underline{G_K\text{-mod}} \quad g \cdot (at^n) = \underbrace{ga}_{\text{semilinearity}} \cdot \chi(g)^n \cdot t^n$$

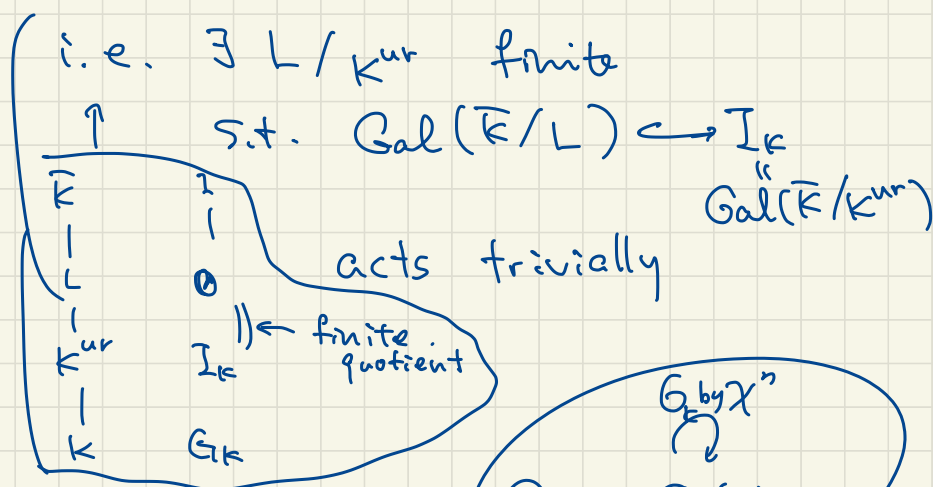
② Back to Fontaine's formalism.

② \mathbb{C}_p -admissibility

$$\left\{ \begin{array}{l} \cdot B = \mathbb{C}_p \\ \cdot E = \mathbb{Q}_p \\ \cdot G = G_K \end{array} \right.$$

Thm $V: \mathbb{Q}_p$ -linear, fin dim rep of G_K is \mathbb{Q}_p -admissible iff

$$\begin{array}{ccc} I_K & \hookrightarrow & G_K \longrightarrow \text{Aut}_{\mathbb{Q}_p}(V) \\ \downarrow & & \searrow \\ \text{(finite quotient)} & & \text{"potentially unramified"} \end{array}$$



proof hard \square

② B_{HT} -admissibility

Prop¹ V as above. is B_{HT} -adm. iff

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V = \mathbb{C}_p(n_1) \oplus \dots \oplus \mathbb{C}_p(n_d)$$

(such V is called a Hodge-Tate rep)

Prop. 2 n_1, \dots, n_d : Uniquely determined by V
called Hodge - Tate weights of V

Prop 3 B_{HT} satisfy (H1) - (H3)
 $B_{HT} = \mathbb{C}_p((t))$

Prop 2 : enough to check $\dim_K \operatorname{Hom}_{\operatorname{Rep}_{\mathbb{C}_p}(G_K)}(\mathbb{C}_p(n), \mathbb{C}_p(m))$

$$= \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

$$\operatorname{Hom}_{\operatorname{Rep}_{\mathbb{C}_p}(G_K)}(\mathbb{C}_p(n), \mathbb{C}_p(m))$$

$$= \operatorname{Hom}_{\mathbb{C}_p}(\mathbb{C}_p(n), \mathbb{C}_p(m))^{G_K}$$

$$= \mathbb{C}_p(m-n)^{G_K}$$

$$\dim_K (\mathbb{C}_p(m-n)^{G_K})$$

$$\underline{m \geq n} \quad \mathbb{C}_p^{G_K} = K \quad (\text{Ax-Sen-Tate})$$

$m \neq n$ enough to show $\mathbb{C}_p(i)$ is not

\mathbb{C}_p -admissible

biproduct: $\mathbb{C}_p(n)^{G_K} = 0$
 $\underline{n \neq 0}$

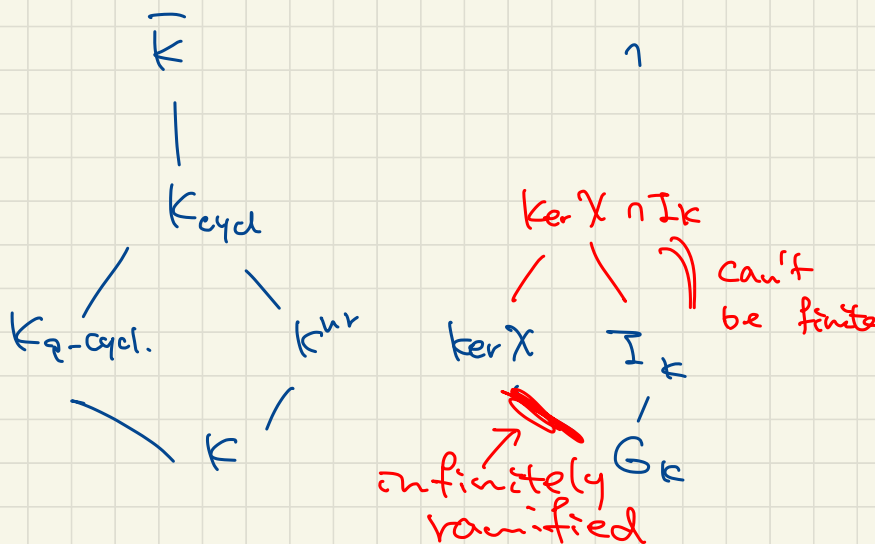
$$\left(\rightsquigarrow \dim_K \mathbb{C}_p(i)^{G_K} \overset{0}{=} < \overset{1}{=} \dim_{\mathbb{C}_p}(\mathbb{C}_p(i)) \right)$$

by the Thm this is equiv. to

$I_K \cap \mathcal{O}_p(\bar{\alpha})$ does not factor through a fin. quotient

χ_i

$$\begin{array}{ccc} I_K & \hookrightarrow & G_K \xrightarrow{\chi_i} \mathbb{Z}_p^\times \\ \downarrow & & \nearrow \\ I_K / I_K \cap \ker \chi_i & & \end{array}$$



Prop 1. Prop 3 : Exercise

follows easily from $(\mathcal{O}_p(\bar{\alpha}))^{G_K} = 0$