RESEARCH STATEMENT: ALGEBRA AND GEOMETRY OF CATEGORICAL SPECTRA

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1. Overview

My general interest lies in the interaction between the *objects* of algebraic topology, algebraic/arithmetic geometry, theoretical physics and the *methods* of homotopical or higher-categorical[1] algebra. In these fields, the notion of *spectra* has been effectively used to exploit the structures that was previously hidden. Spectra originated in stable homotopy theory and spread to the world of mathematics as the homotopical analog of abelian groups. Roughly speaking, they are like chain complexes, and is built up from abelian groups in layers, but typically in a much more interesting way. The most basic spectrum, the *sphere* $\mathbb{S}$, is the group completion of the symmetric monoidal *category of finte sets and automorphisms* $\mathsf{Fin}^{\simeq}$, whereas the classical counterpart $\mathbb{Z}$ is the group completion of the *set of isomorphism classes* $\mathbb{N}$ of finite sets. In other words, the sphere sees the homological complexity of finite groups and some part of "$\mathbb{F}_1$-geometric" information. Many important examples, such as algebraic K-theory, topological cyclic homology and Thom spectra, does not exist as a chain complex.

My current dissertation project focuses on the foundation of *categorical spectra*, which is a novel higher categorical generalization of spectra. The goal is to build a theory that is robust enough for some sort of algebraic geometry. Primary use of this notion is in functorial field theories; as such, it sees the rich interplay between higher algebra and mathematal physics. Aside from it, I expect it to see deeper structures in noncomutative algebraic geometry. My original inspiration[Mas21] comes from papers by Connes and Consani [CC20] in search of absolute ($\sim \mathbb{F}_1$-)algebra, even though the current direction has diverted from their work.

As we saw, Waldhausen's *brave new algebra* philosophy of taking spectra seriously as buliding blocks of algebra was hugely successful. I propose to take categorical spectra seriously as an algebraic object as well. Abelian groups and spectra both have *tensor products*. There are the unique symmetric monoidal structures promoting the respective free functors from sets and spaces to symmetric monoidal functors. Since all of algebraic geometry relies on tensor products, the following question is fundamental:

Question 1. *Can we equip the category* CatSp *of categorical spectra with a natural presentably (symmetric) monoidal structure?*

It turns out to be a much subtler question than in spectra. Based on the lax Gray tensor product on the $(\infty,1)$-category $\infty\mathsf{Cat}$ of (∞,∞)-categories, I proved the following:

Theorem 1.1. *There exists a unique (noncommutative) tensor product on* CatSp *promoting* $\Sigma^{\infty}_{+}: \infty\mathsf{Cat} \to \mathsf{CatSp}$ *to a monoidal functor with respect to the lax Gray tensor product. It acts additively on the categorical levels, which takes* $\mathbb{Z} \cup \{\pm\infty\}$*-values. Moreover, the tensor product localizes to the full subcategory of categorical spectra with duals* $\mathsf{CatSp}^{\mathrm{dual}}$*, i.e., those with the stable cells admitting adjoints.*

The tensor unit of $\mathsf{CatSp}^{\mathrm{dual}}$ is the delooping of free symmetric monoidal (∞,∞)-category on a single fully dualizable object. We denote it by $\mathsf{Bord}^{\mathrm{fr}}$ because it should be thought of as the categorical Thom spectrum for framing, and I expect it to have a geometric description as the stably framed cobordism category.

Corollary 1.2. $\mathsf{Bord}^{\mathrm{fr}}$ *is the initial algebra of* $\mathsf{CatSp}^{\mathrm{dual}}$*, which passes to the sphere* $\mathbb{S}$ *in* Sp*. If we use the internal hom of categorical spectra, we have* $[\mathsf{Bord}^{\mathrm{fr}}, X] \xrightarrow{\sim} X$.

[1] I distinguish these two terms: I use *homotopical* to mean something belongs to the second column of the table, and *higher categorical* to mean something related to the third column. The term *higher* can ambiguously mean the both.

Note that the last statement is stronger than a typical cobordism hypothesis: if we use the usual category of symmetric monoidal category, we would only get the underlying groupoid of X. Modulo the (stable) cobordism hypothesis, geometric description of algebra structure is given by the cartesian product of manifolds. This should be regarded as a typical example where "lax" consideration recovers noninvertible information, and as a good indication that the defined tensor product is something useful.

The following table of analogy is to give an idea about the context where categorical spectra fits in:

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\mathsf{Set} = \mathsf{Cat}_{(0,0)}$	spaces/groupoids $\mathsf{S} = \mathsf{Cat}_{(\infty,0)}$	∞-categories $\infty\mathsf{Cat} = \mathsf{Cat}_{(\infty,\infty)}$
—	homotopy n-type	n-category
$(1,1)$-categories $\mathsf{Cat}_{(1,1)}$	categories $\mathsf{Cat} = \mathsf{Cat}_{(\infty,1)}$	∞-categories $\infty\mathsf{Cat}$
Cartesian product $\times$	Cartesian product $\times$	lax Gray tensor product $\otimes$
abelian groups Ab	spectra Sp	categorical spectra CatSp
	grouplike $\mathbb{E}_\infty$-space $\simeq \mathsf{Sp}^{\mathrm{cn}}$	symmetric monoidal ∞-categories
abelian categories	(pre)stable categories	(stable $\infty\mathsf{Cat}^{\otimes}$-bimodules?)
—	loop $\Omega(X,x) = \mathrm{Aut}_X(x)$	$\Omega(X,x) = \mathsf{End}_X(x)$
—	suspension $\Sigma = (-) \wedge B\mathbb{Z}$	$\Sigma = \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1} = (-) \oslash B\mathbb{N}$
free functor $\mathsf{Set} \to \mathsf{Ab}$	suspension spectra $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$	$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$	finite set spectrum $\mathbb{F}$
tensor product $\otimes_\mathbb{Z}$	tensor (smash) product $\otimes_\mathbb{S}$	tensor product $\otimes_\mathbb{F}$

New feature, which is also a big challenge, in the categorical context is the *noninvertibilty* of cells and the resulting *asymmetry*. Higher algebra comes with hierarchy (the third row of the table). In homotopical theory, operations that let objects leak out of their homotopical hierarchy was the *colimits*. Now the categorical hierarchies are closed under limits and colimits, and *lax colimits* plays the similar role to colimits in homotopy theory. The noninvertibility of higher cells in the constructed object causes the asymmetry and makes them harder to control. For this reason, lax constructions are dreaded and often shunned; to say the least, it makes it a challenging task to construct a roboust theory, even though it seems unavoidable if we try to retain noninvertible information. However, the algebra in this world is not a barren wasteland. In fact, there are emerging evidence that interesting algebra can be done with categorical spectra. For instance, Johnson-Freyd and Reutter has recently developed a version of Galois theory and announced in [Joh23a] that the Galois-closedness (in characteristic 0) characterizes the "universal physical targets of TQFT," capturing non-classical algebraic extensions starting with the category of super vector spaces, along the lines of Freed and Hopkins' paper [FH21] on reflection positivity.

2. Categorical spectra

First, we go straight to the definition of categorical spectra. Let $\infty\mathsf{Cat}$ be the category of (∞,∞)-categories[2]. A *pointed ∞-category* (X,x) is an ∞-category X with a distinguished object (basepoint) x. We often surpress the basepoint from notation. The following definition, which is analogous to the definition of spectra, was independenly introduced by at least a few groups of people:[3].

Definition 2.1. The *loop* functor $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ sends a pointed ∞-category (X,x) to $\Omega(X,x) \coloneqq (\mathsf{End}_X(x), \mathrm{id}_x)$, the (monoidal) ∞-category of endomorphisms. It admits a left adjoint called the *suspension* Σ. A *categorical spectrum* is a sequence $X = (X_n)_{n\in\mathbb{N}}$ of pointed ∞-categories with equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. More precisely, the category of categorical spectra is the limit

$$\mathsf{CatSp} \coloneqq \lim(\cdots \to \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*) \quad \in \mathsf{Pr}^\mathsf{R}.$$

[2]We exclusively work with the colimit of the inclusions $n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat}$ in Pr^L, not in Pr^R. It is characterized as being initial among the homotopy fixed points of enrichment endofunctor $(-)$-$\mathsf{Cat} : \mathsf{Pr}^\mathsf{L} \to \mathsf{Pr}^\mathsf{L}$[Gol23].

[3]I called them ∞-spectra[Mas21] until I came across the Stefanich's thesis [Ste21], which called them *categorical spectra*. The paper spends a chapter on its formal foundation and utilizes it to package powerful functoriality of higher quasicoherent sheaves. He claims to have learned the notion from Teleman, who called them *anticategories*. Horiuchi [Hor18] essentially speculates about categorical spectra in the last section. Johnson-Freyd and Reutter use the term *towers* in [Reu23][Joh23b], who attribute it to Scheimbauer.

We denote the functor $X \mapsto X_0$ by $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$, which has the left adjoint Σ^∞. We also have *shift* autofunctors $(-)[n]$ with $\Omega = [-1]$ and $\Sigma = [1]$.

While it is a *natural* definition, it is a priori unclear how *useful* the notion is. Part of the goal of this statement is to persuade the reader that it is not a meaningless generalization but a interesting object to study. We first note that this is a common generalization of symmetric monoidal (∞,∞)-categories and spectra:

Remark 2.2. The symmetric monoidal (∞,∞)-categories embed into categorical spectra by the *infinite delooping* $\mathrm{B}^\infty : \infty\mathsf{SMCat} \hookrightarrow \mathsf{CatSp}$, whose image should be regarded as *connective categorical spectra* $\mathsf{CatSp}^{\mathrm{cn}}$; as a consequence of Baez-Dolan delooping hypothesis, commutative monoid objects in $\infty\mathsf{Cat}$ are precisely the infinite loop objects, so the limit tower in the above definition factors through the forgetful functor $\infty\mathsf{SMCat} \to \infty\mathsf{Cat}_*$ to give the limit diagram of right adjoints

$$\mathsf{CatSp} \xrightarrow{\sim} \lim(\cdots \to \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat}),$$

with the left adjoint B^∞. We have $\Sigma^\infty = \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty}$. It also follows that CatSp is semiadditive (i.e., has biproducts $\oplus$).

Remark 2.3. Spectra is an instance of categorical spectra $(X_n)_n$ whose components X_n are all ∞-groupoids. The inclusion $\mathsf{Sp} \hookrightarrow \mathsf{CatSp}$ has both left and right adjoints: the localization (left adjoint) $(-)^{\mathrm{gp}}$ is the group completion functor, which levelwise inverts cells and group completes, whereas the colocalization (right adjoint) $\mathbb{G}_m$[4] takes levelwise the maximal Picard subgroupoid.

Sp is the $n = -\infty$ case of the following categorical hierarchy:

Definition 2.4 ([Ste21, Notation 13.2.21])**.** Let $-\infty \le n \le \infty$. The category $n\mathsf{CatSp} \subset \mathsf{CatSp}$ of *n-categorical spectra* consists of objects $X = (X_k)_{\in\mathbb{N}}$ such that X_k is a $\max\{0, n+k\}$-category.

CatSp is the cases $n = \infty$. All intermediate cases are shifts of another. The inclusion from one to another admits both left and right adjoints, similarly to $m\mathsf{Cat} \hookrightarrow n\mathsf{Cat}$ for $m \le n$.

Remark 2.5. Notice $\mathsf{Sp} \cap \infty\mathsf{SMCat} = \mathsf{Sp}^{\mathrm{cn}} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S})$. The above fact that CatSp is the limit of $\infty\mathsf{SMCat}$ corresponds to that Sp is the stabilization of $\mathsf{Sp}^{\mathrm{cn}}$. While Ω^∞ restricts to the one for spectra, Σ^∞ does not: the relation is $\Sigma^\infty_{\mathsf{Sp}} \simeq (\Sigma^\infty)^{\mathrm{gp}}$. The free object on a point in categorical spectra is the symmetric monoidal groupoid of finite sets $\mathbb{F} := \mathrm{B}^\infty\mathsf{Fin}^{\simeq} = \Sigma^\infty_+(*)$, while in spectra it is the sphere $\mathbb{S}$. The fact $\mathbb{S} = \mathbb{F}^{\mathrm{gp}}$ is known as the Barratt-Priddy-Quillen theorem[5][BP72]. Baez-Dolan delooping hypothesis itself is a categorical version of May's recognition principle for n-fold loop spaces[May72], whose $n = \infty$ case is often used to motivate the notion of spectra. Note that we work *without* group completion, which is somewhat difficult to analyze and sometimes quite destructive. For instance, May's recognition principle can be separated into the delooping hypothesis and a less formal fact about group completion.

Before closing this section, we sketch a few more typical examples.

Example 1. ([Ste21, §13.3]) Let $\mathcal{C}$ be a category with finite limits. $n\mathsf{Span}(\mathcal{C})$ is the n-category with the same objects as $\mathcal{C}$, a morphism from x to y is a span $x \leftarrow z \to y$, a 2-morphisms are spans of spans, and so on, up thourgh n-morphisms. Symmetric monoidal structure is given by objectwise Cartesian product, so the unit is the terminal object. Then $\{n\mathsf{Span}(\mathcal{C})\}$ forms a categorical spectrum.

Example 2. n-modules and presentable categorical spectrum

Example 3. [Hau17][JS17] Morita categorical spectrum: Let $\mathcal{C}$ be a symmetric monoidal n-category (with good relative tensor products). Then one can construct a symmetric monoidal $(n+k)$-category $\mathsf{Morita}_k(\mathcal{C})$, whose objects are $\mathbb{E}_k$-algebras in $\mathcal{C}$, a morphism $A \to B$ is a $\mathbb{E}_{k-1}$-algebra object is (A,B)-bimodules, and so on. Then $\{\mathsf{Morita}_k(\mathcal{C})\}$ forms a n-categorical spectrum $\mathsf{Morita}(\mathcal{C})$.

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[4] This is taken from [Joh23a]. I used Pic until recently, but now adopting the 0-th level notation for consistency.

[5] It is sometimes rephrased as $K(\mathbb{F}_1) = \mathbb{S}$ because the finite sets can be considered as perfect modules over the mythical absolute base field $\mathbb{F}_1$, which seems to suggest that spectral algebraic geometry sees some $\mathbb{F}_1$-geometric information.

Example 4. For an n-category $\mathcal{C}$ (for finite n), [Lur09b, §3.2] outlines the definition of $\mathsf{Fam}_n(\mathcal{C})$. When $\mathcal{C} = *$, $\mathsf{Fam}_n(*) = n\mathsf{Span}(\mathsf{S}_{\mathrm{fin}})$ is the n-category of spans in π-finite spaces. Roughly speaking, the functor $\mathsf{Fam}_n(\mathcal{C}) \to \mathsf{Fam}_n(*)$ exhibits $\mathsf{Fam}_n(\mathcal{C})$ as the category of spans of π-finite spaces on whose n-cell is given a local system of n-cells of $\mathcal{C}$ in a compatible way. There is a natural morphism $\mathcal{C} \to \mathsf{Fam}_n(\mathcal{C})$.

In case each $\mathcal{C}$ is ∞-semiadditive, there exists a unique

The importance of this category is described in [Fre+09]: it classifies classical field theory valued in $\mathcal{C}$, and it provides a natural formalism for quantization. An example of classical field theory is the $\mathsf{Bord}_n \to n\mathsf{Span}(\mathsf{Mfld}^{\mathrm{op}}) \xrightarrow{\mathrm{Map}(-,X)} \mathsf{Fam}_n(*)$ for $X \in \mathsf{S}_{\mathrm{fin}}$. The quantization procedure in this language is just composing with the canonical "finite path integral functor" $\int : \mathsf{Fam}_n(\mathcal{C}) \to \mathcal{C}$. This morphism is characterized as the universal ∞-semiadditive category mapping from $\mathcal{C}$, as proven by [Har20] in the $n = 1$ case and the general case is announced by [Sch23].

Let $X = (X_n)$ be a categorical spectrum. It is immediate from the definition that $\mathsf{Fam}(X) := \{\mathsf{Fam}_n(X_n)\}$ again forms a categorical spectrum. In the same vein (or hopefully cleaner), we expect a univeral characterization of $\{\mathsf{Fam}_n(X_n)\}$.

3. Tensor product

In this section, we address the Question 1. We start with a bit of history. The tensor product of abelian groups is characterized as the unique presentably symmetric monoidal structure promoting Free : $\mathsf{Set} \to \mathsf{Ab}$ to a symmetric monoidal functor. The tensor product (aka. smash product) of spectra is similar, but it demanded more sophisticated groundwork. While Boardman[Boa65] provided arguably the best definition $(1,1)$-categorically possible that time, even humble desiderata were shown to be incompatible with the point-set approach[Lew91], and several eclectic point-set constructions followed, e.g., [Elm+07], but it had to wait until Lurie's solid foundation of $(\infty,1)$-categories[Lur09a] for a truely canonical construction. As an example of microcosm principle[6], Lurie first contstructed a symmetric monoidal structure $\otimes$ on Pr^{L}, the large category of presentable categories, characterized by the following properties: presheaf functor $\mathcal{P} : \mathsf{Cat} \to \mathsf{Pr}^{\mathsf{L}}$ is symmetric monoidal, and $\otimes$ distributes over colimits. A (commutative) algebra object in Pr^{L} is precisely a presentabe (symmetric) monoidal category whose monoidal product is distributive over colimits.

Remark 3.1 ([Lur17])**.** $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ is an idempotent $\mathbb{E}_0$-algebra in Pr^{L}, i.e., $\Sigma^\infty_+ \otimes \mathrm{id} : \mathsf{Sp} \simeq \mathsf{S} \otimes \mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$ is an equivalence. Since the forgetful functor $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_\infty}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ is an equivalence, Sp uniquely promotes to a object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ so that Σ^∞_+ is symmetric monoidal. It is the unit object of the monoidal subcategory $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}} \subset \mathsf{Pr}^{\mathsf{L}}$ of stable presentable categories.

This robust implementation (together with tho whole ∞-categorical setup) unlocked the explosive development of spectral algebraic geometry and algebraic K-theory in the past 15 years or so.

Now, as in Question 1, it is natural to ask if we can do the same for CatSp. The answer turns out to be tricky. We would like a characterization similar to Remark 3.1, but we cannot expect $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \xrightarrow{\Sigma^\infty_+} \mathsf{CatSp}$ to be idempotent as an $\mathbb{E}_0$-algebra in Pr^{L}. In fact, the category $\infty\mathsf{Cat}$ is already not idempotent over S, as objects of $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ have only one compatible $\mathbb{E}_1$-structure, which uniquely promotes to an $\mathbb{E}_\infty$-structure, but as we will see, $\infty\mathsf{Cat}$ admits an asymmetric monoidal structure with the terminal unit. More reasonable question is whether $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is idempotent, but to make sense of it, we must choose an algebra structure on $\infty\mathsf{Cat}$. The obvious first choice is the Cartesian monoidal structure, but the suspension fails to be a module homomorphism over it for a simple reason:[7]: if X, Y are m, n-categories respectively, then $X \wedge \Sigma Y$ is a $\max\{m, n+1\}$-category, while $\Sigma(X \wedge Y)$ is a $\max\{m, n\}+1$-category, so we have $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$ in general. In other words, suspension is not given by smashing with $\vec{S}^1 := \mathrm{B}\mathbb{N} = \Sigma S^0$.

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[6] Microcosm principle tells you that, to talk about an object with certain structure (e.g. a commutative monoid), you must first equip the ambient category with the corresponding structure (e.g. a symmetric monoidal structure).

[7] However, the connective part $\mathsf{CatSp}^{\mathrm{cn}}$ can be easily given a symmetric monoidal structure; as in [GGN16], for any $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$, one has $\mathsf{CMon}(\mathcal{C}) \simeq \mathsf{CMon}(\mathsf{S}) \otimes \mathcal{C}$ and $\mathsf{CMon}(\mathsf{S}) \in \mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^{\mathsf{L}})$, so a unique symmetric monoidal structure $\circledast$ making $\mathrm{Free}_{\mathbb{E}_\infty} : \mathcal{C}^\times \to \mathsf{CMon}(\mathcal{C})^\circledast$ symmetric monoidal. This product $\circledast$ does not commute with delooping.

To fix this problem, we adopt a monoidal structure, the *(lax Gray) tensor product*, that acts additively on the category levels. Recall the full subcategory $\square = \{\square^n \mid n \geq 0\} \subset \infty\mathsf{Cat}$ of *the cubes*. The picture below describes the first few examples:

(insert pictures here)

The category $\square$ admits a natural monoidal structure so that $\square^n$ is the n-th power of $\square^1$. The *tensor product* $\otimes$ on $\infty\mathsf{Cat}$ is a presentably monoidal structure extending the monoidal structure on $\square$. The uniqueness follows from the density of $\square \subset \infty\mathsf{Cat}$ [Cam22], and the existence follows from Loubaton's thesis [Lou23], which bulids on previous works [Ver08][VRO23][8]. The internal hom of the tensor product is the ∞-category of functors and lax natural transformations: $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \mathsf{Fun}^{\mathrm{lax}}(Y, Z))$. We denote the pointed version ("lax smash" product) by $\oslash$.

Remark 3.2. The suspension functor can be identified with $(-) \oslash \vec{S}^1$, which is clearly a left $\infty\mathsf{Cat}^{\otimes}$-module morphism.

However, to mimic Lurie's strategy, we must promote it to a bimodule homomorphism. The left modules over a noncommutative algebra do not inherit a monoidal structure. It is in fact a key technical point: giving the structure of a bimodule homomorphism to $(-) \oslash S : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ is equivaent to lifting S to the *center* (aka. Hochshild cohomology) of $\infty\mathsf{Cat}_*^{\oslash}$, and in general higher coherence can be difficult to spell out. In our case, one can show that $\vec{S}^1$ turns out to be *half-central* in the following sense: let $D : A \to A$ be a monoidal functor such that $D^2 \simeq \mathrm{id}$. As an algebra morphism is a (pro)functor between the deloopings, D can be seen as a A-bimodule; explicitly, it is the identity bimodule A except that the left action is twisted by D. We define the *half-center* of A with respect to D as $\mathrm{Hom}_{\mathsf{BMod}_A}(A, D) \simeq \mathrm{Hom}_{\mathsf{BMod}_A}(D, A)$. The following theorem is relatively formal after showing (1):

Theorem 3.3.
(1) $\vec{S}^1 = \mathrm{B}\mathbb{N} \in \infty\mathsf{Cat}_*$ *canonically lifts to the half-center with respect to the total dual (which flips the domain and codomain of all cells). In particular,* Σ^2 *canonically promotes to a* $\infty\mathsf{Cat}^{\otimes}$*-bimodule morphism.*
(2) *With the induced bimodule structure from above,* $\Sigma^{\infty}_{+} : \infty\mathsf{Cat} \to \mathsf{CatSp}$ *is an idempotent* $\mathbb{E}_0$*-algebra in* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}})$*. In particular, it uniquely promotes to an* $\mathbb{E}_1$*-algebra object.*
(3) *The presentably monoidal structure on* CatSp *given by forgetting along the lax monoidal functor* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Pr}^{\mathsf{L}}$ *satisfies the universal property of* $\infty\mathsf{Cat}^{\otimes}[(\vec{S}^1)^{-1}]$.

Remark 3.4. Categorical filtration makes CatSp into a filtered monoidal category, i.e., The tensor product of an n-categorical spectrum and an m-categorical spectrum is a $(m+n)$-categorical spectrum. In particular, $0\mathsf{CatSp}$ is a monoidal subcategory. Here we take $-\infty + \infty = -\infty$. In other words, $\mathsf{Sp} \subset \mathsf{CatSp}$ is a tensor-ideal. The localization is smashing by the sphere spectrum $\mathbb{S}$. Since $\mathsf{Sp} \subset \mathsf{CatSp}$ is a monoidal subcategory, we have the inclusion $\mathsf{Alg}(\mathsf{Sp}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.

Remark 3.5. $\infty\mathsf{SMCat} = \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ is a monoidal subcategory. It follows that $\infty\mathsf{SMCat}$ admits a unique $\mathbb{E}_1$-monoidal structure $\otimes$ that makes $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat}^{\otimes} \to \infty\mathsf{SMCat}^{\otimes}$ monoidal. From footnote 7, $\infty\mathsf{SMCat}$ also has a symmetric monoidal structure $\circledast$. The identity functor $\infty\mathsf{SMCat}^{\circledast} \to \infty\mathsf{SMCat}^{\otimes}$ is lax monoidal and the two monoidal structures agree on $\mathsf{CMon}(\mathsf{S})$. In particular, we have the inclusion $\mathsf{Rig}_{\mathbb{E}_1}(\mathsf{S}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.

Remark 3.6. With an appropriate cocomplete variant of tensor product of categorical spectra in $\mathsf{CATSP}^{\mathrm{ccpl}}$, we expect that $\mathsf{CAlg}(\mathsf{S}) \to \mathsf{CATSP}^{\mathrm{ccpl}}; R \mapsto \underline{R}$ of example Example 2 has a lax monoidal structure. **multiplicative delooping as opposed to additive delooping?**

[8] Loubaton proved the equivalence of $\infty\mathsf{Cat}$ and a combinatorial model called complicial sets, where the Gray tensor product was constructed by Verity. It is not a priori clear if the transferred tensor product satisfies this characterization, but it follows from the fact that the 0-truncation commutes with the Verity's Gray tensor product and [Lou23, Theorem 4.3.3.26] that the gaunt categories are closed under Gray cylinders in $\infty\mathsf{Cat}$, as communicated by Loubaton. Less model-dependent approach is taken by [CM23] for $(\infty, 2)$-categories, but extension to (∞, ∞)-category is not straightforward. We will only use the model-independent characterization.

4. Categorical spectra with duals

A context where categorical spectra naturally appears is the study of functorial field theories. Various sorts of cobordism categories appear as their domains, and the typical targets are some extensions of the higher module categories. The important feature of cobordism categories is that they are universal among the symmetric monoidal higher categories with duals (with some extra structure). In this section, we discuss them from the viewpoint of categorical spectra. i

We say an (∞, n)-category has adjoints if for $k < n$, any k-morphism has both left and right adjoints. Note that it is not reasonable to require the existence of adjoints for n-morphisms; for the top dimensional cells, adjoints are the same as inverses, as the unit and counit must be invertible. In particular, the statement "$\mathcal{C}$ is an n-category with adjoints" depends on n and implies that either X is an (∞, n)-cateogry but not an $(\infty, n-1)$-category, or X is an ∞-groupoid. This leads to the following category level dependent definition of *categorcial spectra with duals*:

Definition 4.1. An *n-categorical spectrum with duals*[9] is a categorical specrum $X = (X_k)_{k\geq 0} \in n\mathsf{CatSp}$ where X_k is an $(n+k)$-category with adjoints.

Previous consideration implies that $n\mathsf{CatSp}^{\mathrm{dual}} \cap m\mathsf{CatSp}^{\mathrm{dual}} = \mathsf{Sp} = -\infty\mathsf{CatSp}^{\mathrm{dual}}$ for any $m \neq n$. Since one can write the condition of having adjoints as being local with respect to a certain set of morphisms corepresenting adjunctions, $n\mathsf{CatSp}^{\mathrm{dual}} \hookrightarrow \mathsf{CatSp}$ has the localization L_n. With some computation of the tensor product $\square^1 \otimes \mathsf{Adj} \in 3\mathsf{Cat}$ of the interval and the generic adjunction, we obtain the following:

Theorem 4.2. *The localizations $L_n : n\mathsf{CatSp} \to n\mathsf{CatSp}^{\mathrm{dual}}$ for $\infty \leq n \leq \infty$ are compatible with the graded monoidal structure on $\{n\mathsf{CatSp}\}$, i.e., for an L_n-equivalence f and an m-categorical spectrum X, the morphism $f \otimes X$ is an L_{m+n}-equivalence. In patricular, there exist unique $\mathbb{E}_1$-monoidal structures on $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$ promoting $L = L_\infty$ and L_0 to monoidal functors.*

Part of the reasons to be interested in categorical spectra with duals is the potential to restore some commutativity of tensor products; adding adjoints is a milder version of adding inverses. More concretely, passage to adjoints and mates gives an equivalence $X \to X^{\mathrm{op}}$ so we get $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$[10]. I am currently working to upgrade this into a braiding of $(0)\mathsf{CatSp}^{\mathrm{dual}}$:

Conjecture 1. The $\mathbb{E}_1$-monoidal structure on $n\mathsf{CatSp}^{\mathrm{dual}}$ promotes to an $\mathbb{E}_2$-structure.

Ideally, we would like an $\mathbb{E}_\infty$ structure, but it is possible that the choice of identifications $X \to X^{\mathrm{op}}$, for example whether I use left or right adjoint, affect the canonicity of braiding, so $\mathbb{E}_2$ seems to be a sensible conjecture for now.

The framed version of cobordism hypothesis says that, for $0 \leq n < \infty$, the category $\mathrm{B}^\infty \mathsf{Bord}_n^{\mathrm{fr}}$ is the free n-categorical spectra with a single fully dualizable object, or equivalently, $\mathrm{B}^{\infty - n}\mathsf{Bord}_n^{\mathrm{fr}}$ is the free 0-categorical spectra on a single fully dualizable $(-n)$-cell, i.e., $L_0(\mathbb{F}[-n])$. The latter makes it clear that the "point" in $\mathsf{Bord}_n^{\mathrm{fr}}$ is secrectly given an n-framing. Using tensor algebra, it can even be combined into a single equation $\mathsf{Bord}_\bullet^{\mathrm{fr}}[-\bullet] := \bigoplus_{n\geq 0} B^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}} = L_0 \operatorname{Tens}(\mathbb{F}[-1])$, which gives the graded $\mathbb{E}_1$-rig structuree on bordism categories given by cartesian product of manifolds; one can think this as encoding various compactifications of field theories at once.

I do not know the similar geometric identification of the tensor unit $L\mathbb{F}$ of $\mathsf{CatSp}^{\mathrm{dual}}$, but since $\pi_*((L\mathbb{F})^{\mathrm{gp}}) = \pi_*\mathbb{S} = \Omega_*^{\mathrm{fr}}$, the following is a reasonable guess:

Conjecture 2. $L\mathbb{F} = \mathsf{Bord}^{\mathrm{fr}}$, where $\mathsf{Bord}^{\mathrm{fr}}$ is the bordism (∞,∞)-category of stably framed manifolds. In the usual notatiton for cobordism hypothesis, it says $\mathrm{ev}_* : \mathsf{Fun}^{\otimes}(\mathsf{Bord}^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{\simeq}$ for any $\mathcal{C} \in \infty\mathsf{SMCat}^{\mathrm{dual}}$.

The following corollary of the theorem formally enhances the conjectrue using the internal hom of categorical spectra:

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[9]this terminology reflects the fact that the existence of adjoints in X_{n+1} forces the existence of duals in the symmetric monoidal ∞-category $X_n \simeq \Omega X_{n+1}$.

[10]$(-)^{\mathrm{op}} : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ is an antimonoidal involution that flips all odd dimensional cells. It induces a similar functor on CatSp.

Corollary 4.3. *If $X \in \mathsf{CatSp}^{\mathrm{dual}}$, we have $[L\mathbb{F}, X] \xrightarrow{\sim} X$.*

In other words, by considering lax natural transformations, we can recover the whole object X, not just the underlying groupoid of it. Assuming Conjecture 2, the algebra structure of the unit $L\mathbb{F}$ is given by the cartesian product on manifolds in $\mathsf{Bord}_n^{\mathrm{fr}}$. This category level incarnation of the ring structure of the sphere was speculated in [Yua], but it requires our language to correctly formulate. Similarly to the finite-dimensional version, Conjecture 2 implies that the infinite piecewise-linear group PL acts on $\mathsf{Bord}^{\mathrm{fr}}$ by change of stable framings. This allows us to define the cobordism categories with various stable tangential structures as *categorical Thom spectra*. The Corollary 4.3 also implies that PL acts on any categorical spectra with duals.

We would like a similar enhancement of the cobordism hypothesis for $\mathsf{Bord}_n^{\mathrm{fr}}$, but the obvious analog with the internal hom of 0-categorical spectra fails, and some of the consequences, analogous to above, are too strong to be true. It is an interesting question to salvage this, which is also related to the question of constructing the cobordism categories with arbitrary tangential structures as the *categorical Madsen-Tillmann spectra*. Another related ongoing work is to interpret stronger variants of cobordism hypothesis, including cobordism hypothesis with singularities and tangle hypothesis, in terms of corepresenting categorical spectra, and hopefully to make reduction from another formal. It will be necessary to work with parametrized categorical spectra at some point; Thom categorical spectra construction can be seen as an instance of it (parametrised BPL), and it essential in incorporating non-tangential geometric structures, as in [GP23].

5. Open Directions

Because of the novel nature of this project, there are plethora of interesting problems to be asked.

5.1. **Brauer categorical spectrum and higher etale topology.** Reference: [GL21]

Recall from the example 2 that if A is an $\mathbb{E}_\infty$ ring, we can associate a large categorical spectrum $\underline{A}$. $\mathbb{G}_m(\underline{A})$ contains valuable informations about A: we have $\Omega^{\infty-2}\mathbb{G}_m(\underline{A}) = \mathrm{Br}(A)$ is the Brauer space of A. At least when A is connective, the structure is well-understood by [AG14]. The key was the etale local triviality of Br. But nothing has been done beyond degree 3.

Problem 2. *Give the "categorically-etale" topology on presentable stable categorical spectra in a way that $A \mapsto \underline{A}$ and $A \mapsto \mathbb{G}_m(\underline{A}$ are sheaves. Do we need to generalize the notion of topos and sheaves?*

Announced in [Joh23a]: if X contains $\mathbb{Q}$ and "Galois-closed," then $\mathbb{G}_m(X) = I_{\mathbb{Q}/\mathbb{Z}}$.

This seems to describe the categorical etale behaviour of this higher Brauer spectrum. Classical algebraically closed fields like $\mathbb{C}$ is not algebarically closed in this world: 0-categorical algebraic closedness is only the first level of infinite hierarchy...

5.2. **algebraic K-theory.** It is an interesting problem to define the K-theory of suitably compactly generated or dualizable presentable stable categorical spectra, including the class of $\underline{\mathsf{Mod}_R}$, in a way that $K(\underline{\mathsf{Mod}_R})$ either recovers or enhances the classical $K(R)$, and is also related to secondary K-theory and higher. It should admit a trace map to higher hochshild homology; this should appear as the value of the tori of the TQFT defined using the theory of higher quasicoherent sheaves of [Ste21] (the B-model). Optimistically, one hopes that the redshift phenomena gets some illuminating description here, in light of "category level vs chromatic level" picture.

Anish said something about etaleness in TTG and seemed related. Very optimistic hope is that redshift gets more illuminating proof here, in light of category level vs chromatic level picture.

Another point worth mentioning is that, in the proof of

This is the crucial result that motivates many new development in the field.

Another point: in the proof of Land-Tamme's excision, appearance of lax pullback seems to be considered kind of random and not very well understood. I want to understand it using the language of higher categorical algebra and in relation to the noncommutative tensor product.

5.3. **t-structures. categorical chain complex.** One major difficulty working with categorical spectra is the lack of robust reduction principle, corresponding to the t-structure in stable categories and spectral sequences. The root of the problem seems to be that a categorical hierarchy is something more closed than the homotopical hierarchy: it is easy to "leak out" from one homotopical level to higher, generically by colimit-type constructions. Descent works precisely this way. Since one categorical level is closed both under limit and colimit, similar ideas inevitably fail. It seems that the idea that replaces are lax colimits. I didn't have time to check the relation to categorical chain complexes, but seems related in the case with duals. Thinking of Dold-Kan, there should be some truncation structure induced by truncated oriental objects.

when there are enough duals Lurie's categorical chain complexes may be used but I need to check how it really works and it may be totally irrelevant... At least it is a kind of reduction principle that allows to slice a symmetric monoidal n-category into 1-categorical level.

5.4. **stability of higher categories.** Theorem 3.3 implies that being a CatSp-module is a property of $\infty\mathsf{Cat}^{\otimes}$-bimodule. This corresponds to the fact that a Sp-module structure is actually a property of being stable presentable. In $(\infty, 1)$-category theory, the stability have intrinsic characterizations. It would be useful to have ones for CatSp-modules. The difficulty here is understanding $\infty\mathsf{Cat}^{\otimes}$-bimodule structure itself. The most important data will be the Gray cylinder. The rest should be a sort of coherence data (morphisms in $\square$) and conditions (localization along $\mathcal{P}(\square) \to \infty\mathsf{Cat}$).

Let $\mathcal{C}$ be an ∞-category. We may be able to see this as a Gray-bimodule in the following way, if it has enough $\infty\mathsf{Cat}$-weighted limits and colimits: Because $\mathcal{C}$ itself can be viewed as a category enriched in $\infty\mathsf{Cat}$, it makes sense to think about $\infty\mathsf{Cat}$-weighted (co)limits, and in particular about (partially) (op)lax (co)limits (in a similar fashion to [Ber20]).Then characterizing stability there seems to be a meaningful question.

5.5. **Directed homotopy theory, Boolean and natural cohomology.** Grothendieck's homotopy hypothesis identifies groupoids with the weak homotopy types of topological spaces. In the same spirit, ∞-categories can be thought of as homotopy types of some sorts of *directed* spaces. Categorical spectrum can be seen as a (co)homology theory on such. In this viewpoint, we have the folloing conceptual problem:

Problem 3. *Give an excisive-functor style description of categorical spectra.*

This is closely related to a characterization of stability we speculated in the last section. It is an unavoidable question if we hope to develop (at least the first-order) Goodwillie calculus and apply it to deformation theory. In any way, it gives an invariant of directed spaces. Directed topological spaces have found many applications, including practical ones, so computing cohomology of those spaces is be a problem of interest. Cohomology theory given by ordinary spectra does not see any information of directedness: if X is a directed space (i.e., ∞-category) and E is a spectrum, $E^*(X) = [\Sigma^\infty_+ X, E] \simeq [\Sigma^\infty_+ X \otimes \mathbb{S}, E] \simeq E^*(|X|)$, where $|X|$ is a groupoidification of X. The next basic example would be the Eilenberg-Mac Lane spectra of semifields. Finite semifields are either a finite field $\mathbb{F}_p$ or the boolian semifield $\mathbb{B}$, whose addition is idempotent; $1 + 1 = 1$. This "characteristic one" linear over $\mathbb{B}$ is developed in [CC19]. It is also proven that in [Gus+23]that a projective $\mathbb{B}$-module valued 1-dimensional TQFT corresponds to nondeterministic finite state automaton. Analogous to [Mil58], the following problem is of fundamental computational interest:

Problem 4. *Compute the Steenrod algebra $[H\mathbb{B}, H\mathbb{B}]$ and the dual Steenrod algebra $H\mathbb{B} \otimes H\mathbb{B}$. What about the natural cohomology $H\mathbb{N}$?*

5.6. **deformation theory.** In spectral algebraic geometry, deformation-theoretic technique is overwhelmingly important. This is because it allows us to induct on the truncation level. This was possible because, heuristically, square-zero extension flows out from the homotopical level. Now categorical level looks more "closed" and the only way to get out of it is by some lax construction. We will likely to need some first order Goodwillie derivative type technology for this, and understanding excisiveness is the first thing to do.

References

[AG14] Benjamin Antieau and David Gepner. "Brauer Groups and Étale Cohomology in Derived Algebraic Geometry". In: *Geometry & Topology* 18.2 (Apr. 7, 2014), pp. 1149–1244. ISSN: 1364-0380. DOI: 10.2140/gt.2014.18.1149.

[Ber20] John D. Berman. *On Lax Limits in Infinity Categories*. June 18, 2020. DOI: 10.48550/arXiv.2006.10851. preprint.

[Boa65] J. M. Boardman. *Stable Homotopy Theory*. University of Warwick, Coventry, 1965.

[BP72] Michael Barratt and Stewart Priddy. "On the Homology of Non-Connected Monoids and Their Associated Groups". In: *Commentarii Mathematici Helvetici* 47.1 (Dec. 1, 1972), pp. 1–14. ISSN: 1420-8946. DOI: 10.1007/BF02566785.

[Cam22] Tim Campion. *Cubes Are Dense in* (∞, ∞)*-Categories*. Sept. 19, 2022. URL: http://arxiv.org/abs/2209.09376. preprint.

[CC19] Alain Connes and Caterina Consani. "Homological Algebra in Characteristic One". In: *High.Struct.* 3.1 (Mar. 26, 2019), pp. 155–247. DOI: 10.21136/HS.2019.05.

[CC20] Alain Connes and Caterina Consani. "$\overline{\operatorname{Spec}\mathbb{Z}}$ AND THE GROMOV NORM". In: *Theory and Applications of Categories* 35.6 (2020), pp. 155–178.

[CM23] Timothy Campion and Yuki Maehara. *A Model-Independent Gray Tensor Product for* $(\infty, 2)$*-Categories*. Apr. 12, 2023. URL: http://arxiv.org/abs/2304.05965. preprint.

[Elm+07] A. Elmendorf et al. *Rings, Modules, and Algebras in Stable Homotopy Theory*. Vol. 47. Mathematical Surveys and Monographs. American Mathematical Society, Apr. 10, 2007. ISBN: 978-0-8218-4303-1 978-1-4704-1278-4. DOI: 10.1090/surv/047.

[FH21] Daniel S. Freed and Michael J. Hopkins. "Reflection Positivity and Invertible Topological Phases". In: *Geom. Topol.* 25.3 (May 20, 2021), pp. 1165–1330. ISSN: 1364-0380, 1465-3060. DOI: 10.2140/gt.2021.25.1165. arXiv: 1604.06527 [cond-mat, physics:hep-th, physics:math-ph].

[Fre+09] Daniel S. Freed et al. *Topological Quantum Field Theories from Compact Lie Groups*. Version 2. June 19, 2009. DOI: 10.48550/arXiv.0905.0731. preprint.

[GGN16] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of Multiplicative Infinite Loop Space Machines". In: *Algebr. Geom. Topol.* 15.6 (Jan. 12, 2016), pp. 3107–3153. ISSN: 1472-2739, 1472-2747. DOI: 10.2140/agt.2015.15.3107.

[GL21] David Gepner and Tyler Lawson. "Brauer Groups and Galois Cohomology of Commutative Ring Spectra". In: *Compositio Mathematica* 157.6 (June 2021), pp. 1211–1264. ISSN: 0010-437X, 1570-5846. DOI: 10.1112/S0010437X21007065.

[Gol23] Zach Goldthorpe. *Homotopy Theories of* (∞, ∞)*-Categories as Universal Fixed Points with Respect to Enrichment*. Aug. 16, 2023. DOI: 10.1093/imrn/rnad196. preprint.

[GP23] Daniel Grady and Dmitri Pavlov. *Extended Field Theories Are Local and Have Classifying Spaces*. Sept. 16, 2023. DOI: 10.48550/arXiv.2011.01208. preprint.

[Gus+23] Paul Gustafson et al. "Automata and One-Dimensional TQFTs with Defects". In: *Lett Math Phys* 113.5 (Sept. 5, 2023), p. 93. ISSN: 1573-0530. DOI: 10.1007/s11005-023-01701-y.

[Har20] Yonatan Harpaz. "Ambidexterity and the Universality of Finite Spans". In: *Proceedings of the London Mathematical Society* 121.5 (2020), pp. 1121–1170. ISSN: 1460-244X. DOI: 10.1112/plms.12367.

[Hau17] Rune Haugseng. "The Higher Morita Category of $\mathbb{E}_n$-Algebras". In: *Geom. Topol.* 21.3 (May 10, 2017), pp. 1631–1730. ISSN: 1364-0380, 1465-3060. DOI: 10.2140/gt.2017.21.1631.

[Hor18] Ryo Horiuchi. "Observations on the Sphere Spectrum". Department of Mathematical Sciences, Faculty of Science, University of Copenhagen, 2018. URL: https://soeg.kb.dk/permalink/45KBDK_KGL/fbp0ps/alma99122355005405763.

[Joh23a] Theo Johnson-Freyd. "Deeper Kummer Theory". Sept. 2023. DOI: 10.48660/23090104.

[Joh23b] Theo Johnson-Freyd. "SUPER DUPER VECTOR SPACES II: THE HIGHER-CATEGORICAL GALOIS GROUP OF $\mathbb{R}$". Aug. 18, 2023. URL: http://categorified.net/SuperDuperVec2.pdf.

[JS17] Theo Johnson-Freyd and Claudia Scheimbauer. "(Op)Lax Natural Transformations, Twisted Quantum Field Theories, and "Even Higher" Morita Categories". In: *Advances in Mathematics* 307 (Feb. 5, 2017), pp. 147–223. ISSN: 0001-8708. DOI: 10.1016/j.aim.2016.11.014.

[Lew91] L. Gaunce Lewis. “Is There a Convenient Category of Spectra?” In: *Journal of Pure and Applied Algebra* 73.3 (Aug. 30, 1991), pp. 233–246. ISSN: 0022-4049. DOI: 10.1016/0022-4049(91)90030-6.

[Lou23] Félix Loubaton. *Theory and Models of* (∞, ω)*-Categories*. Version 1. July 21, 2023. DOI: 10.48550/arXiv.2307.11931. preprint.

[Lur09a] Jacob Lurie. *Higher Topos Theory (AM-170):* Princeton University Press, Dec. 31, 2009. ISBN: 978-1-4008-3055-8. DOI: 10.1515/9781400830558.

[Lur09b] Jacob Lurie. *On the Classification of Topological Field Theories*. May 4, 2009. URL: http://arxiv.org/abs/0905.0465. preprint.

[Lur17] Jacob Lurie. *Higher Algebra*. Sept. 2017.

[Mas21] Naruki Masuda. “Towards Derived Absolute Algebraic Geometry”. Ph.D. Preliminary Oral Exam (Johns Hopkins University). Mar. 2021. URL: https://nmasuda2.github.io/notes/Oral_exam.pdf.

[May72] J. P. May. *The Geometry of Iterated Loop Spaces*. Vol. 271. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1972. ISBN: 978-3-540-05904-2 978-3-540-37603-3. DOI: 10.1007/BFb0067491.

[Mil58] John Milnor. “The Steenrod Algebra and Its Dual”. In: *Annals of Mathematics* 67.1 (1958), pp. 150–171. ISSN: 0003486X. DOI: 10.2307/1969932. JSTOR: 1969932.

[Reu23] David Reutter. “SUPER DUPER VECTOR SPACES I: THE HIGHER-CATEGORICAL GALOIS GROUP OF $\mathbb{R}$”. Aug. 18, 2023. URL: https://homepages.uni-regensburg.de/~lum63364/ConferenceFFT/Reutter.pdf.

[Sch23] Claudia Scheimbauer. “A Universal Property of the Higher Category of Spans and Finite Gauge Theory as an Extended TFT”. Feza Gursey Center Higher Structures Seminars. May 9, 2023. URL: https://researchseminars.org/talk/FezaGurseyHigher/20/.

[Ste21] German Stefanich. “Higher Quasicoherent Sheaves”. UC Berkeley, 2021. URL: https://escholarship.org/uc/item/19h1f1tv.

[Ver08] D. R. B. Verity. “Weak Complicial Sets I. Basic Homotopy Theory”. In: *Advances in Mathematics* 219.4 (Nov. 10, 2008), pp. 1081–1149. ISSN: 0001-8708. DOI: 10.1016/j.aim.2008.06.003.

[VRO23] Dominic Verity, Martina Rovelli, and Viktoriya Ozornova. “Gray Tensor Product and Saturated N-Complicial Sets”. In: *High.Struct.* 7.1 (May 21, 2023), pp. 1–21. DOI: 10.21136/HS.2023.01.

[Yua] Qiaochu Yuan. *From the Perspective of Bordism Categories, Where Does the Ring Structure on Thom Spectra Come From?* URL: https://mathoverflow.net/q/186440.