THESIS SUMMARY: TENSOR PRODUCT OF CATEGORICAL SPECTRA

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Abstract. It is a technical summary of my thesis project. For more background, please read my research statement.

Let $\infty\mathsf{Cat}$ be the $(\infty,1)$-category of (∞,∞)-categories, which is the colimit of the inclusions $n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat}$ in Pr^L. Let $\infty\mathsf{Cat}_*$ denote the $(\infty,1)$-category of pointed (∞,∞)-categories. The following definition was independently introduced by at least a few groups of people:

Definition 1. The *loop* $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ takes (X,x) to $(\mathsf{End}_X(x), \mathrm{id}_x)$, the (∞,∞)-category of endomorphisms. The left adjoint is the *suspension* Σ. The $(\infty,1)$-category of *categorical spectrum* is

$$\mathsf{CatSp} \coloneqq \lim(\cdots \to \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*) \quad \in \mathsf{Pr}^\mathsf{R},$$

It comes equipped with $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ with the left adjoint Σ^∞.

The symmetric monoidal (∞,∞)-categories embed into categorical spectra by the *infinite delooping* $\mathrm{B}^\infty : \mathsf{CMon}(\infty\mathsf{Cat}) \hookrightarrow \mathsf{CatSp}$. We have $\Sigma^\infty = \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty}$. We have the formula $X = \mathrm{colim}_n\, \Sigma^{\infty-n} X_n$ for a categorical spectrum $X = (X_n)$. Following [Ste21, Notation 13.2.21], we say $X = (X_k)$ is an *n-categorical spectrum* if X_k is an $(\infty, \max\{0, n+k\})$-category. In particular, $\mathsf{Sp} = -\infty\mathsf{CatSp}$, $\mathsf{CatSp} = \infty\mathsf{CatSp}$. For $n \in \mathbb{Z}$, $n\mathsf{CatSp}$ is a shift of $0\mathsf{CatSp}$.

Question 2. *Can we equip the category* CatSp *of categorical spectra with a natural presentably (symmetric) monoidal structure, in an analogous way to* Sp*?*

Unlike spectra, $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \to \mathsf{CatSp}$ is not idempotent as an $\mathbb{E}_0$-algebra in Pr^L. One can ask if $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is idempotent, but to make sense of it, we must choose an algebra structure on $\infty\mathsf{Cat}$. The suspension fails to be a module homomorphism over cartesian monoidal structure on $\infty\mathsf{Cat}$; the category level of the left- and right-hand side of $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$ does not match in general. Instead, we adopt the *(lax Gray) tensor product*, which is a presentably monoidal structure on $\infty\mathsf{Cat}$ taking an (∞,n)-category X and (∞,m)-category Y to an $(\infty,n+m)$-category $X \otimes Y$. We denote the pointed version ("lax smash" product) by $\oslash$. The suspension functor can be identified with $(-) \oslash \vec{S}^1$, where $\vec{S}^1 = \mathrm{B}\mathbb{N}$, which has the obvious structure of left $\infty\mathsf{Cat}^\otimes$-module morphism. To talk about the idempotence of Σ^∞, we must promote it to a bimodule homomorphism. This is the technical key, and it is done by lifting $\vec{S}^2 = \vec{S}^1 \oslash \vec{S}^1$ the center of $\infty\mathsf{Cat}_*^\oslash$. Because $\vec{S}^1$ is gaunt but $\vec{S}^2$ is not, it is easier to solve the coherence issue at the level of $\vec{S}^1$, lifting it to the "half-center" of $\infty\mathsf{Cat}_*^\oslash$:

Theorem 3.
(1) *The suspension* Σ *canonically lifts to* $\mathrm{Hom}_{\mathsf{End}(\mathrm{B}\infty\mathsf{Cat}_*^\oslash)}(\mathrm{id}, (-)^\circ)$*. As a consequence,* Σ^2 *and* Σ^∞_+ *canonically promotes to an* $\infty\mathsf{Cat}^\otimes$*-bimodule morphism.*
(2) $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ *is an idempotent* $\mathbb{E}_0$*-algebra in* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^\mathsf{L})$*. In particular, it uniquely promotes to an* $\mathbb{E}_1$*-algebra object.*
(3) *The presentably monoidal structure on* CatSp *given by forgetting along the lax monoidal functor* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^\mathsf{L}) \to \mathsf{Pr}^\mathsf{L}$ *satisfies the universal property of* $\infty\mathsf{Cat}^\otimes[(\vec{S}^1)^{-1}]$.
(4) *The tensor product acts additively on category levels:* $n\mathsf{CatSp} \otimes m\mathsf{CatSp} \to (n+m)\mathsf{CatSp}$*. In particular,* $0\mathsf{CatSp}$ *is a monoidal subcategory. When* $n = -\infty$*, it means* $\mathsf{Sp} \subset \mathsf{CatSp}$ *is a smashing localization by* $\mathbb{S}$*. We have* $\mathsf{Alg}(\mathsf{Sp}) \simeq \mathsf{Alg}_\mathbb{S}(\mathsf{CatSp}) \subset \mathsf{Alg}(\mathsf{CatSp})$.
(5) *The tensor product restricts to an* $\mathbb{E}_1$*-monoidal structure on* $\mathsf{CMon}(\infty\mathsf{Cat})$*, characterized by the fact that* $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat}^\otimes \to \infty\mathsf{SMCat}^\otimes$ *promotes to a monoidal functor.*

Remark 4. $\infty\mathsf{SMCat}$ also has a canonical symmetric monoidal structure $\circledast$ that is not compatible with the delooping. The identity functor $\infty\mathsf{SMCat}^\circledast \to \infty\mathsf{SMCat}^\otimes$ has a lax monoidal structure and the two monoidal structures agree on $\mathsf{CMon}(\mathsf{S})$.

Remark 5. For $\mathsf{V} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$, [Ste21, §13.3.6] defines a "higher presentable modules spectrum" $\underline{\mathsf{V}} = \{n\mathsf{Mod}_{\mathsf{V}}\}$ by roughly iterating the construction $\mathsf{V} \mapsto \mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$. For $R \in \mathsf{CAlg}(\mathsf{Sp})$, we define $\underline{R} = \Omega\underline{\mathsf{Mod}_R(\mathsf{Sp})}$. I am currently working on a cocomplete variant of the tensor product, with which we expect that $\mathsf{CAlg}(\mathsf{S}) \to \mathsf{CATSP}^{\mathrm{ccpl}}; R \mapsto \underline{R}$ has a lax monoidal structure. This example will be important in applications to noncommutative algebraic geometry.

Next, we focus on *categorical spectra with duals.* Recall that we say an (∞, n)-category *has adjoints* if, for $k < n$, any k-morphism has both left and right adjoints. Note that the statement depends on n.

Definition 6. An *n-categorical spectrum with duals* is a categorical spectrum $X = (X_k)_{k\geq 0} \in n\mathsf{CatSp}$ where X_k is an $(n+k)$-category with adjoints.

Theorem 7. *The localizations $L_n : n\mathsf{CatSp} \to n\mathsf{CatSp}^{\mathrm{dual}}$ for $-\infty \leq n \leq \infty$ are compatible with the graded monoidal structure on $\{n\mathsf{CatSp}\}$, i.e., for an L_n-equivalence f and an m-categorical spectrum X, the morphisms $f \otimes X$ and $X \otimes f$ are L_{m+n}-equivalences. In particular, the tensor product descends to $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$, which we will denote by $\otimes^L$.*

We have an antimonoidal involution $(-)^{\mathrm{op}} : \mathsf{CatSp} \to \mathsf{CatSp}$ flipping odd-dimensional stable cells. Passage to adjoints and mates gives an equivalence $X \to X^{\mathrm{op}}$ and so $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$. I am currently working to upgrade this into a braiding:

Conjecture 1. The graded $\mathbb{E}_1$-monoidal structure on $\{n\mathsf{CatSp}^{\mathrm{dual}}\}$ promotes to an $\mathbb{E}_2$-structure.

The framed cobordism hypothesis translates to that the framed cobordism categories organize into a tensor algebra $\bigoplus_{n\geq 0} B^{\infty - n}\mathsf{Bord}_n^{\mathrm{fr}} = L_0 \operatorname{Tens}(\Omega\mathbb{F})$. The graded $\mathbb{E}_1$-rig structure on bordism categories is given by the cartesian product of manifolds; one can think of this as encoding various compactifications of field theories at once.

I do not know the similar geometric identification of the tensor unit $L\mathbb{F}$ of $\mathsf{CatSp}^{\mathrm{dual}}$, but since $\pi_*((L\mathbb{F})^{\mathrm{gp}}) = \pi_*\mathbb{S} = \Omega_*^{\mathrm{fr}}$, the following is a reasonable guess:

Conjecture 2. $L\mathbb{F} = \mathsf{Bord}^{\mathrm{fr}}$, where $\mathsf{Bord}^{\mathrm{fr}}$ is the bordism (∞, ∞)-category of stably framed manifolds. In the usual notatiton for cobordism hypothesis, it says $\mathrm{ev}_* : \mathsf{Fun}^{\otimes}(\mathsf{Bord}^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{\simeq}$ for any $\mathcal{C} \in \infty\mathsf{SMCat}^{\mathrm{dual}}$.

The following corollary of the theorem formally enhances the conjecture using the internal hom of categorical spectra:

Corollary 8. *If $X \in \mathsf{CatSp}^{\mathrm{dual}}$, we have $[L\mathbb{F}, X] \xrightarrow{\sim} X$.*

In other words, by considering lax natural transformations, we can recover the whole object X, not just the underlying groupoid of it. Assuming **??**, the algebra structure of the unit $L\mathbb{F}$ is given by the cartesian product on manifolds in $\mathsf{Bord}_n^{\mathrm{fr}}$. This category-level incarnation of the ring structure of the sphere was speculated in [Yua], but it requires our language to correctly formulate. Similarly to the finite-dimensional version, **??** implies that the infinite piecewise-linear group PL acts on $\mathsf{Bord}^{\mathrm{fr}}$ by change of stable framings. This allows us to define the cobordism categories with various stable tangential structures as *categorical Thom spectra*. The **??** also implies that PL acts on any categorical spectra with duals.

We would like a similar enhancement of the cobordism hypothesis for $\mathsf{Bord}_n^{\mathrm{fr}}$, but the obvious analog with the internal hom of 0-categorical spectra fails, and some of the consequences, analogous to above, are too strong to be true. I am working to salvage this, as well as to address the related question of constructing the cobordism categories with arbitrary tangential structures as the *categorical Madsen-Tillmann spectra*. Another related ongoing work is to interpret stronger variants of the cobordism hypothesis, including the cobordism hypothesis with singularities and tangle hypothesis, in terms of corepresenting categorical spectra and to make reductions from one to another formal.

References

[Ste21] German Stefanich. "Higher Quasicoherent Sheaves". UC Berkeley, 2021. URL: https://escholarship.org/uc/item/19h1f1tv.

[Yua] Qiaochu Yuan. *From the Perspective of Bordism Categories, Where Does the Ring Structure on Thom Spectra Come From?* URL: https://mathoverflow.net/q/186440.