RESEARCH STATEMENT:
ALGEBRA AND GEOMETRY OF CATEGORICAL SPECTRA

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1. Overview

My general interest lies in the interaction between the *objects* of topology, algebraic/arithmetic geometry, and theoretical physics and the *methods* of homotopical or higher-categorical[1] algebra. In these fields, the notion of *spectra* has been effectively used to exploit the structures that were previously hidden. They originated in stable homotopy theory and spread to the world of mathematics, following Waldhausen's *brave new algebra* philosophy to take the notion of spectra seriously as a homotopical version of abelian groups, and thus as building blocks of algebraic geometry.

My current dissertation project focuses on the foundation of *categorical spectra*, an (∞,∞)-categorical generalization of spectra. A categorical spectrum is a sequence $X = (X_n)_{n\in\mathbb{Z}}$ of pointed higher categories, one entry identified with the basepoint's *endomorphisms*, or the *loop*, of the next: $X_n \xrightarrow{\sim} \Omega X_{n+1}$. In particular, $X_0 \simeq \Omega^n X_n$ (and similarly any X_k) is a n-fold loop object; they come equipped with multiplications, increasingly symmetric as n grows (just like for iterated loop spaces). The upshot is that X_0 naturally lifts to a symmetric monoidal higher category, which in turn gets identified with *connective* categorical spectra, i.e., those X determined by X_0.

Symmetric monoidal categories are ubiquitous. For instance, functorial quantum field theories (FQFTs) formalize QFTs as functors between symmetric monoidal higher categories, triggering rich interplay between higher algebra and mathematical physics. Even if we are only interested in symmetric monoidal higher categories, categorical spectra are a natural place to talk about many constructions, succeeding the role of spectra (§3.6, §4). However, there are plenty of interesting nonconnective spectra, typically arising from *iterated categorification*. Targets of FQFTs, varied in dimensions, are an example. In algebraic geometry, the coconnective part carries cohomological information. Similarly, nonconnective categorical spectra capture deeper structures in noncommutative algebraic geometry (§5.1, 5.2). For instance, Johnson-Freyd and Reutter have recently announced [Joh23a] that the underlying spectrum of a *Galois-closed* categorical spectrum (of characteristic 0) has the (coconnective and unbounded below) underlying spectrum $I_{\mathbb{Q}/\mathbb{Z}}$, the Brown-Comenetz dualizing spectrum, characterizing the *universal target of physical TQFTs* in the sense of [FH21]. It is also worth mentioning that my original inspiration [Mas21] is from the papers by Connes and Consani [CC20] in search of absolute/$\mathbb{F}_1$-algebra, even though the current direction has diverted from their work.

I propose to take categorical spectra seriously as an algebraic object as well. If we hope for a robust enough algebraic theory that allows some algebraic geometry, we must extend or refine the existing tensor product on abelian groups and spectra.

Question 1.1. *Can we equip the $(\infty,1)$-category CatSp of categorical spectra with a natural presentably (symmetric) monoidal structure?*

It turns out to be a much subtler question than in spectra (§3). Based on the lax Gray tensor product on the $(\infty,1)$-category $\infty\mathsf{Cat}$ of (∞,∞)-categories, I proved the following:

Theorem 1.2. CatSp *admits a unique presentably $\mathbb{E}_1$-monoidal structure promoting $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ into a monoidal functor. It acts additively on the categorical levels. Moreover, the tensor product localizes to the full subcategory $\mathsf{CatSp}^{\mathrm{dual}}$ of categorical spectra with duals.*

[1] I distinguish these two terms: I use *homotopical* to mean something belongs to the second column of the table, and *higher categorical* to mean something related to the third column. The term *higher* can ambiguously mean both.

The tensor unit of $\mathsf{CatSp}^{\mathrm{dual}}$, denoted by $\mathsf{Bord}^{\mathrm{fr}}$, is the delooping of the free symmetric monoidal (∞,∞)-category on a single fully dualizable object. The notation is to suggest the interpretation as the categorical Thom spectrum for framing as well as the expected geometric description as the stably framed cobordism category.

Corollary 1.3. *$\mathsf{Bord}^{\mathrm{fr}}$ is the initial algebra of $\mathsf{CatSp}^{\mathrm{dual}}$, which passes to the sphere $\mathbb{S}$ in Sp. Using the internal hom of categorical spectra, we have $[\mathsf{Bord}^{\mathrm{fr}}, X] \xrightarrow{\sim} X$ for any $X \in \mathsf{CatSp}^{\mathrm{dual}}$.*

Note that the last statement is stronger than a typical cobordism hypothesis: if we use the category of symmetric monoidal functors, we would only get the underlying groupoid of X. It is an example where "lax" consideration recovers noninvertible information. It is a new feature in the categorical context; higher algebra comes with some hierarchy (the third row of the table). In homotopy theory, objects go up the hierarchy via colimits. In category theory, this role of colimits is taken over by (partially) lax colimits. This makes it unavoidable to confront the dreaded lax constructions, making it a challenging task to develop a robust theory.

The following table of analogy is to give an idea about the context where categorical spectra fit in:

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\mathsf{Set} = \mathsf{Cat}_{(0,0)}$	spaces/∞-groupoids $\mathsf{S} = 0\mathsf{Cat}$	(∞,∞)-categories $\infty\mathsf{Cat}$
—	homotopy n-type	(∞, n)-category
$(1,1)$-categories $\mathsf{Cat}_{(1,1)}$	$(\infty,1)$-categories Cat	(∞,∞)-categories $\infty\mathsf{Cat}$
Cartesian product $\times$	Cartesian product $\times$	lax Gray tensor product $\otimes$
abelian groups Ab	spectra Sp	categorical spectra CatSp
	grouplike $\mathbb{E}_\infty$-spaces $\simeq \mathsf{Sp}^{\mathrm{cn}}$	symmetric monoidal (∞,∞)-categories
abelian categories	(pre)stable categories	(stable $\infty\mathsf{Cat}^{\otimes}$-bimodules?)
—	loop $\Omega(X,x) = \mathrm{Aut}_X(x)$	$\Omega(X,x) = \mathsf{End}_X(x)$
—	suspension $\Sigma = (-) \wedge \mathrm{B}\mathbb{Z}$	$\Sigma = \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1} = (-) \oslash \mathrm{B}\mathbb{N}$
free functor $\mathsf{Set} \to \mathsf{Ab}$	suspension spectra $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$	$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$	finite set categorical spectrum $\mathbb{F}$
tensor product $\otimes_{\mathbb{Z}}$	tensor (smash) product $\otimes_{\mathbb{S}}$	tensor product $\otimes_{\mathbb{F}}$

2. Categorical spectra

We first define categorical spectra. Let $\infty\mathsf{Cat}$ be the $(\infty,1)$-category of (∞,∞)-categories[2]. A *pointed (∞,∞)-category* $(X,x) \in \infty\mathsf{Cat}_*$ is a pair $X \in \infty\mathsf{Cat}$ and an object $x \in X$ (which we often suppress from notation). The following definition was independently introduced by at least a few groups of people[3]:

Definition 2.1. The *loop* functor $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ sends (X,x) to $\Omega X \coloneqq (\mathsf{End}_X(x), \mathrm{id}_x)$, the (∞,∞)-category of endomorphisms. It admits a left adjoint called the *suspension* Σ. A *categorical spectrum* is a sequence $X = (X_n)_{n\in\mathbb{N}}$ of pointed (∞,∞)-categories with equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. More precisely, the $(\infty,1)$-category of categorical spectra is the limit of the right adjoints

$$\mathsf{CatSp} \coloneqq \lim(\cdots \to \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*) \quad \in \mathsf{Pr}^{\mathsf{R}}.$$

We denote the functor $X \mapsto X_0$ by $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$, which has the left adjoint Σ^∞.

[2]Let Pr^{L}, Pr^{R} denote the $(\infty,1)$-category of presentable ∞-categories. We exclusively work with the colimit of the inclusions $n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat}$ in Pr^{L}, not in Pr^{R}. It is characterized as being initial among the homotopy fixed points of enrichment endofunctor $(-)$-$\mathsf{Cat} : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$[Gol23].

[3]I originally called them ∞-spectra in an approach to absolute algebras [Mas21] until I came across Stefanich's thesis [Ste21], which gave them the more descriptive name of *categorical spectra*. It spends a chapter on their formal foundation and utilizes it to package powerful functorialities of higher quasicoherent sheaves. He claims to have learned the notion from Teleman, who named it *anticategories*. Horiuchi [Hor18] essentially speculates on categorical spectra in the last section. Johnson-Freyd and Reutter use the term *towers* in [Reu23][Joh23b], who attribute it to Scheimbauer.

While the definition is *natural*, it is a priori unclear how *useful* it is. Part of my goal is to show that this is a reasonable notion to study. Note that this is a common generalization of symmetric monoidal (∞,∞)-categories and spectra:

Example 1. The symmetric monoidal (∞,∞)-categories embed into categorical spectra by the *infinite delooping* $\mathrm{B}^\infty : \infty\mathsf{SMCat} \coloneqq \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{\sim} \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$, whose image consists of *connective categorical spectra* $\mathsf{CatSp}^{\mathrm{cn}}$: as a consequence of Baez-Dolan delooping hypothesis, commutative monoid objects in $\infty\mathsf{Cat}$ are precisely the infinite loop objects, so the limit tower in the above definition factors through the forgetful functor $\infty\mathsf{SMCat} \to \infty\mathsf{Cat}_*$ to give the limit diagram of right adjoints

$$\mathsf{CatSp} \xrightarrow{\simeq} \lim(\cdots \to \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat}) \quad \in \mathsf{Pr}^{\mathsf{R}},$$

with the fully faithful left adjoint B^∞. We have $\Sigma^\infty = \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty}$. It follows that CatSp is semiadditive (i.e., has biproducts $\oplus$).

Example 2. Spectra are instances of categorical spectra $(X_n)_n$ such that all components X_n are ∞-groupoids. The inclusion $\mathsf{Sp} \hookrightarrow \mathsf{CatSp}$ has both left and right adjoints: the localization (left adjoint) $(-)^{\mathrm{gp}}$ is the group completion functor, which level-wise inverts cells and group completes, whereas the colocalization (right adjoint) $\mathbb{G}_m$[4] takes levelwise the maximal Picard subgroupoid.

Remark 2.2. We have $\mathsf{Sp} \cap \mathsf{CatSp}^{\mathrm{cn}} = \mathsf{Sp}^{\mathrm{cn}} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S})$. While $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ restricts to Ω^∞ for spectra, Σ^∞ does not: the relation is $\Sigma^\infty_{\mathsf{Sp}} \simeq (\Sigma^\infty)^{\mathrm{gp}}$. The free object on a point in CatSp is the symmetric monoidal groupoid of finite sets $\mathbb{F} \coloneqq \mathrm{B}^\infty \mathsf{Fin}^{\simeq} = \Sigma^\infty_+(*)$, as opposed to the sphere $\mathbb{S} \in \mathsf{Sp}$. The fact $\mathbb{S} = \mathbb{F}^{\mathrm{gp}}$ is known as the Barratt-Priddy-Quillen theorem[5][BP72]. Baez-Dolan delooping hypothesis is a categorical version of May's recognition principle for n-fold loop spaces[May72], whose $n = \infty$ case is often used to motivate the notion of spectra. Group completion is somewhat difficult to analyze and sometimes destructive, so working without group completion can be enlightening. For instance, May's recognition principle can be separated into the delooping hypothesis and a less formal fact about group completion.

Sp is the $n = -\infty$ case of the following categorical hierarchy:

Definition 2.3 ([Ste21, Notation 13.2.21])**.** Let $-\infty \leq n \leq \infty$. The full subcategory $n\mathsf{CatSp} \subset \mathsf{CatSp}$ of *n-categorical spectra* consists of objects $X = (X_k)_{\in\mathbb{N}}$ such that X_k is an $(\infty, \max\{0, n+k\})$-category. In particular, $\mathsf{Sp} = -\infty\mathsf{CatSp}$, $\mathsf{CatSp} = \infty\mathsf{CatSp}$. For $n \in \mathbb{Z}$, $n\mathsf{CatSp}$ is a shift of $0\mathsf{CatSp}$.

Before closing this section, we sketch a few more typical examples.

Example 3. ([Hau18][Ste21, §13.3.10]) For $\mathcal{C}$ an $(\infty,1)$-category with finite limits, $n\mathsf{Span}(\mathcal{C})$ is the (∞,n)-category with the same objects as $\mathcal{C}$, whose 1-morphism from x to y is a span $x \leftarrow z \to y$, 2-morphisms are spans of spans, and so on, up through n-morphisms. We see $n\mathsf{Span}(\mathcal{C}) \in n\mathsf{SMCat}$ by objectwise cartesian product. Then $\{n\mathsf{Span}(\mathcal{C})\}$ forms a 0-categorical spectrum.

Example 4. ([Ste21, §13.3.6]) n-modules and presentable n-categories: Let $\mathsf{V} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$. One hopes to define a large 1-categorical spectrum $\underline{\mathsf{V}} = \{n\mathsf{Mod}_{\mathsf{V}}\}$ by iterating the construction $\mathsf{V} \mapsto \mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$ and enhancing them to n-categories. It does not work naively as $\mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$ is not presentable in the original universe, but it can be fixed by working in the very large $(\infty,1)$-category $\mathsf{CAT}^{\mathrm{ccpl}}$ of cocomplete $(\infty,1)$-categories and appropriately pushing the resulting object back into the original universe. In particular, we define the categorical spectra $\{n\mathsf{Pr}\} \coloneqq \underline{\mathsf{S}}$ and $\{n\mathsf{Pr}_{\mathsf{st}}\} \coloneqq \underline{\mathsf{Sp}}$ of presentable (stable) n-categories. If A is an $\mathbb{E}_\infty$-ring, we define $\underline{A} \coloneqq \Omega\underline{\mathsf{Mod}_A(\mathsf{Sp})}$.

Example 5. ([Hau17][JS17][Ste21, §13.3.12]) Morita categorical spectra: If $\mathcal{C}$ is a symmetric monoidal n-category with good relative tensor products, we can associate an n-categorical spectrum $\{\mathsf{Morita}_k(\mathcal{C})\}_k$. An object of $\mathsf{Morita}_k(\mathcal{C})$ is an $\mathbb{E}_k$-algebras in $\mathcal{C}$, a morphism $A \to B$ is an $\mathbb{E}_{k-1}$-algebra in (A,B)-bimodules, and so on.

[4]This is taken from [Joh23a]. I used Pic until recently, but now I adopt the 0-th level notation for consistency.

[5]It is sometimes rephrased as $K(\mathbb{F}_1) = \mathbb{S}$ because the finite sets can be considered as perfect modules over the mythical absolute base field $\mathbb{F}_1$, which seems to suggest that spectral algebraic geometry sees some $\mathbb{F}_1$-geometric information.

Example 6. For finite n and an (∞, n)-category $\mathcal{C}$, [Lur09b, §3.2] outlines the definition of the n-category $\mathsf{Fam}_n^k(\mathcal{C})$. Roughly speaking, it is the n-category of spans of k-truncated π-finite spaces coherently decorated by cells of $\mathcal{C}$. There is a morphism $\mathcal{C} \to \mathsf{Fam}_n^k(\mathcal{C})$ exhibiting $\mathsf{Fam}_n^k(\mathcal{C})$ as the universal k-semiadditive n-category under $\mathcal{C}$, as proven by [Har20] in the $n = 1$ case and the general case (including the definition of k-semiadditive n-category) is announced by [Sch23]. If $\mathcal{C}$ itself is k-semiadditive, it gives the *finite path integral* functor[6] $\int : \mathsf{Fam}_n^k(\mathcal{C}) \to \mathcal{C}$. If $X = (X_n)$ is a categorical spectrum, almost by definition $\mathsf{Fam}^k(X) \coloneqq \{\mathsf{Fam}_n^k(X_n)\}_{n\geq 0}$ forms a categorical spectrum. One can define k-semiadditivity of categorical spectra so that $\mathsf{Fam}^k(X)$ is the universal such under X.

3. Tensor product

The tensor product[7] of abelian groups is characterized by the fact that Free : $\mathsf{Set} \to \mathsf{Ab}$ promotes into a symmetric monoidal functor. The tensor product of spectra is similar but demands more sophisticated groundwork. While Boardman[Boa65] provided arguably the best definition $(1,1)$-categorically possible that time, even humble desiderata were shown to be incompatible with the point-set approach[Lew91]. Several eclectic point-set constructions followed, e.g., [Elm+07], but a canonical construction had to wait until Lurie's solid foundation of $(\infty,1)$-categories[Lur09a]. As an example of microcosm principle[8], Lurie first constructed a symmetric monoidal structure $\otimes$ on Pr^L promoting the presheaf functor $\mathcal{P} : \mathsf{Cat} \to \mathsf{Pr}^\mathsf{L}$ into symmetric monoidal. A (commutative) algebra object in Pr^L is precisely a presentably (symmetric) monoidal category.

Remark 3.1 ([Lur17])**.** $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ is an idempotent $\mathbb{E}_0$-algebra in Pr^L, i.e., $\Sigma^\infty_+ \otimes \mathrm{id} : \mathsf{Sp} \simeq \mathsf{S} \otimes \mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$ is an equivalence. Since the forgetful functor $\mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^\mathsf{L}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^\mathsf{L})$ is an equivalence, Sp uniquely promotes to an object of $\mathsf{CAlg}(\mathsf{Pr}^\mathsf{L})$ whose unit is $\mathbb{S}$.

This robust implementation (together with the whole ∞-categorical setup) unlocked the explosive development of spectral algebraic geometry and algebraic K-theory in the past 15 years or so.

Question 3.2 (=1.1)**.** *Can we similarly equip the category* CatSp *of categorical spectra with a canonical presentably (symmetric) monoidal structure?*

The answer turns out to be tricky. We cannot expect $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \xrightarrow{\Sigma^\infty_+} \mathsf{CatSp}$ to be an idempotent $\mathbb{E}_0$-algebra in Pr^L. In fact, the category $\infty\mathsf{Cat}$ is already not idempotent over S. One can more reasonably ask if $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is idempotent, but to make sense of it, we must choose a monoidal structure on $\infty\mathsf{Cat}$. The obvious choice would be the Cartesian monoidal structure, but the suspension fails to be a module map over it for a simple reason[9]: if X, Y are m, n-categories respectively, then $X \wedge \Sigma Y$ is a $\max\{m, n+1\}$-category, while $\Sigma(X \wedge Y)$ is a $\max\{m,n\}+1$-category, so we have $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$ in general. In other words, the suspension is not given by smashing $\vec{S}^1 \coloneqq \mathrm{B}\mathbb{N} = \Sigma S^0$, where $\mathrm{B}\mathbb{N}$ is the free 1-category on an object and an endomorphism.

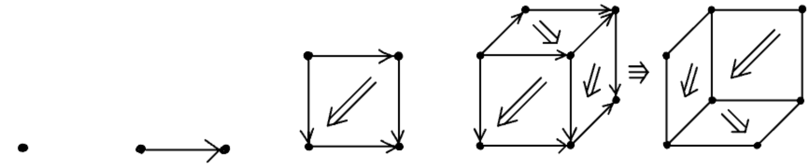
To fix this, we adopt a monoidal structure, the *(lax Gray) tensor product*, that acts additively on the category levels. Recall the full subcategory $\square = \{\square^n \mid n \geq 0\} \subset \infty\mathsf{Cat}$ of *the cubes*. The first few examples of the cubes are depicted below:

[6] The importance of this functor is explained in [Fre+09]. $\mathsf{Fam}_n^k(\mathcal{C})$ classifies classical field theories, and the composition with $\int$ gives the *quantization*. An important example is the Dijgraaf-Witten theory.

[7] Monoidal structures are assumed to distribute over colimits. We say "*presentably* symmetric monoidal" to emphasize.

[8] Microcosm principle tells you that, to talk about an object with a certain structure (e.g. a commutative monoid), you must first equip the ambient category with the corresponding structure (e.g. a symmetric monoidal structure).

[9] However, the connective part $\mathsf{CatSp}^{\mathrm{cn}}$ can be easily given a symmetric monoidal structure; as in [GGN16], for any $\mathcal{C} \in \mathsf{Pr}^\mathsf{L}$, one has $\mathsf{CMon}(\mathcal{C}) \simeq \mathsf{CMon}(\mathsf{S}) \otimes \mathcal{C}$ and $\mathsf{CMon}(\mathsf{S}) \in \mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^\mathsf{L})$, so a unique symmetric monoidal structure $\circledast$ making $\mathrm{Free}_{\mathbb{E}_\infty} : \mathcal{C}^\times \to \mathsf{CMon}(\mathcal{C})^\circledast$ symmetric monoidal. This product $\circledast$ does not commute with delooping.



$\square^0$ $\square^1$ $\square^2$ $\square^3$

The *tensor product* $\otimes$ on $\infty\mathsf{Cat}$ is a presentably monoidal structure characterized by $\square^n = (\square^1)^{\otimes n}$. The uniqueness follows from the density of $\square \subset \infty\mathsf{Cat}$ [Cam22], and the existence follows from Loubaton's thesis [Lou23], which builds on previous works [Ver08][VRO23][10]. The internal hom of the tensor product is the (∞,∞)-category of functors and lax natural transformations.

Remark 3.3. We denote the pointed version ("lax smash" product) by $\oslash$. The suspension functor can be identified with $(-) \oslash \vec{S}^1$, which has the obvious structure of left $\infty\mathsf{Cat}_*^{\oslash}$-module morphism.

This is still not sufficient to mimic Lurie's strategy; it does not make sense to ask if a left module over a noncommutative algebra is idempotent due to the lack of relative tensor products. The technical key is to promote Σ^∞ into a map in $\mathsf{BMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}^\mathsf{L})$. It turns out that Σ^2 can be promoted to a bimodule map by proving that $\vec{S}^1$ is "half-central" with respect to the total dual $(-)^\circ : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$[11]:

Theorem 3.4.
1. *The suspension Σ canonically lifts to $\mathrm{Hom}_{\mathsf{End}(\mathrm{B}\infty\mathsf{Cat}_*^{\oslash})}(\mathrm{id}, (-)^\circ)$. As a consequence, Σ^2 and Σ^∞_+ canonically promotes to an $\infty\mathsf{Cat}^{\otimes}$-bimodule morphism.*
2. *$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is an idempotent $\mathbb{E}_0$-algebra in $\mathsf{BMod}_{\infty\mathsf{Cat}^{\otimes}}(\mathsf{Pr}^\mathsf{L})$. In particular, it uniquely promotes to an $\mathbb{E}_1$-algebra structure.*
3. *The presentably monoidal structure on CatSp given by forgetting along the lax monoidal functor $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^\mathsf{L}) \to \mathsf{Pr}^\mathsf{L}$ satisfies the universal property of $\infty\mathsf{Cat}^{\otimes}[(\vec{S}^1)^{-1}]$.*

Remark 3.5. The tensor product acts additively on category levels: $n\mathsf{CatSp} \otimes m\mathsf{CatSp} \to (n+m)\mathsf{CatSp}$. In particular, $0\mathsf{CatSp}$ is a monoidal subcategory. When $n = -\infty$, it means $\mathsf{Sp} \subset \mathsf{CatSp}$ is a smashing localization by $\mathbb{S}$. We have $\mathsf{Alg}(\mathsf{Sp}) \simeq \mathsf{Alg}_{\mathbb{S}}(\mathsf{CatSp}) \subset \mathsf{Alg}(\mathsf{CatSp})$.

Remark 3.6. $\infty\mathsf{SMCat} \simeq \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ is a monoidal subcategory. It follows that $\infty\mathsf{SMCat}$ admits a unique presentably monoidal structure $\otimes$ that promotes $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat}^{\otimes} \to \infty\mathsf{SMCat}^{\otimes}$ to a monoidal functor. From footnote 9, $\infty\mathsf{SMCat}$ also has a symmetric monoidal structure $\circledast$. The identity functor $\infty\mathsf{SMCat}^{\circledast} \to \infty\mathsf{SMCat}^{\otimes}$ becomes lax monoidal and the two monoidal structures agree on $\mathsf{CMon}(\mathsf{S})$. In particular, we have the inclusion $\mathsf{Rig}_{\mathbb{E}_1}(\mathsf{S}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$, generalizing the previous remark.

Remark 3.7. With an appropriate cocomplete variant of tensor product of categorical spectra in $\mathsf{CATSP}^{\mathrm{ccpl}}$, we expect that $\mathsf{CAlg}(\mathsf{S}) \to \mathsf{CATSP}^{\mathrm{ccpl}}; R \mapsto \underline{R}$ of Example 4 has a lax monoidal structure.

4. Categorical spectra with duals

Categorical spectra naturally arise in the study of functorial field theories. The domains are some sort of cobordism categories and the typical targets are the higher module categories. An important feature of cobordism categories is the *cobordism hypothesis*; they have a universal characterization as the free symmetric monoidal higher categories with duals (with some extra structure). In this section, we discuss them from the viewpoint of categorical spectra.

We say an (∞,n)-category has adjoints if, for $k < n$, any k-morphism has both left and right adjoints. Note that it is not reasonable to require the existence of adjoints for n-morphisms because a

[10] Loubaton proves the equivalence of $\infty\mathsf{Cat}$ and a combinatorial model called complicial sets, where the Gray tensor product was constructed by Verity. It is not obvious if the transferred tensor product satisfies the characterization, but it follows from the fact that the 0-truncation commutes with the Verity's tensor product and that gaunt categories are closed under Gray cylinders in $\infty\mathsf{Cat}$[Lou23, Theorem 4.3.3.26], as communicated by Loubaton. A less model-dependent approach is taken in [CM23] for $(\infty,2)$-categories, but the extension to (∞,∞)-category is not straightforward (Campion recently communicated that he managed to work through it). We will only use the model-independent characterization.

[11] It is the monoidal involution that flips the source and the target of cells of all dimensions.

top-dimensional cell admits an adjoint iff it is invertible. In particular, the statement "$\mathcal{C}$ is an (∞, n)-category with adjoints" depends on n and implies either that X is an (∞, n)-cateogry but not an $(\infty, n-1)$-category or that X is an ∞-groupoid. This leads to the following category-level dependent definition of *categorcial spectra with duals*:

Definition 4.1. An *n-categorical spectrum with duals*[12] is a categorical spectrum $X = (X_k)_{k\geq 0} \in n\mathsf{CatSp}$ where X_k is an $(n+k)$-category with adjoints.

We have $n\mathsf{CatSp}^{\mathrm{dual}} \cap m\mathsf{CatSp}^{\mathrm{dual}} = \mathsf{Sp} = -\infty\mathsf{CatSp}^{\mathrm{dual}}$ for any $m \neq n$. The inclusion $n\mathsf{CatSp}^{\mathrm{dual}} \hookrightarrow \mathsf{CatSp}$ has a left adjoint L_n that freely adds adjoints to morphisms.

Theorem 4.2. *The localizations $L_n : n\mathsf{CatSp} \to n\mathsf{CatSp}^{\mathrm{dual}}$ for $-\infty \leq n \leq \infty$ are compatible with the graded monoidal structure on $\{n\mathsf{CatSp}\}$, i.e., for an L_n-equivalence f and an m-categorical spectrum X, the morphisms $f \otimes X$ and $X \otimes f$ are L_{m+n}-equivalences. In particular, there exist unique $\mathbb{E}_1$-monoidal structures on $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$ promoting $L = L_\infty$ and L_0 to monoidal functors.*

Part of the reasons to be interested in categorical spectra with duals is the potential to restore some commutativity of tensor products; adding adjoints is a milder version of adding inverses. More concretely, passage to adjoints and mates gives an equivalence $X \to X^{\mathrm{op}}$ so we get $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$[13]. I am currently working to upgrade this into a braiding of $(0)\mathsf{CatSp}^{\mathrm{dual}}$:

Conjecture 1. The graded $\mathbb{E}_1$-monoidal structure on $\{n\mathsf{CatSp}^{\mathrm{dual}}\}$ promotes to an $\mathbb{E}_2$-structure.

Ideally, we would like an $\mathbb{E}_\infty$ structure, but the choice of identifications $X \to X^{\mathrm{op}}$, e.g., if I use left or right adjoint, may affect the canonicity of braiding, so $\mathbb{E}_2$ seems to be a sensible conjecture for now.

Remark 4.3. The framed cobordism hypothesis says that, for $0 \leq n < \infty$, the category $\mathrm{B}^\infty\mathsf{Bord}_n^{\mathrm{fr}}$ is the free n-categorical spectra with a single fully dualizable object, or equivalently, $\mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ is the free 0-categorical spectra on a single fully dualizable $(-n)$-cell, i.e., $L_0(\mathbb{F}[-n])$. The latter makes it clear that the "point" in $\mathsf{Bord}_n^{\mathrm{fr}}$ is secretly given an n-framing. They can even be combined into a single tensor algebra $\mathsf{Bord}_\bullet^{\mathrm{fr}}[-\bullet] \coloneqq \bigoplus_{n\geq 0} \mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}} = L_0 \operatorname{Tens}(\mathbb{F}[-1])$, defining the graded $\mathbb{E}_1$-rig structure on bordism categories given by cartesian product of manifolds; one can think of this as encoding various compactifications of field theories at once.

I do not know the similar geometric identification of the tensor unit $L\mathbb{F}$ of $\mathsf{CatSp}^{\mathrm{dual}}$, but since $(L\mathbb{F})^{\mathrm{gp}} = \mathbb{S}$ is the Thom spectrum for stable framing, the following is a reasonable guess:

Conjecture 2. $L\mathbb{F}$ is $\mathsf{Bord}^{\mathrm{fr}}$, the bordism (∞,∞)-category of stably framed manifolds. In the usual notatiton for cobordism hypothesis, it says $\mathrm{ev}_* : \mathsf{Fun}^{\otimes}(\mathsf{Bord}^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{\simeq}$ for any $\mathcal{C} \in \infty\mathsf{SMCat}^{\mathrm{dual}}$.

The following corollary of the theorem formally enhances the conjecture:

Corollary 4.4. *If $X \in \mathsf{CatSp}^{\mathrm{dual}}$, we have $[L\mathbb{F}, X] \xrightarrow{\sim} X$, where $[-,-]$ is the internal hom of CatSp.*

In other words, by considering lax natural transformations, we recover the whole object X, not just the underlying groupoid of it. Assuming Conjecture 2, the algebra structure of the unit $L\mathbb{F}$ is given by the cartesian product on manifolds in $\mathsf{Bord}_n^{\mathrm{fr}}$. This category-level incarnation of the ring structure of the sphere was speculated in [Yua], but it requires our language to correctly formulate. Similarly to the finite-dimensional version, Conjecture 2 implies that the infinite piecewise-linear group PL acts on $\mathsf{Bord}^{\mathrm{fr}}$ by change of stable framings. This would allow us to define the cobordism categories with various stable tangential structures as *categorical Thom spectra*. Corollary 4.4 also implies that PL acts on any categorical spectra with duals.

We would like a similar enhancement of the cobordism hypothesis for $\mathsf{Bord}_n^{\mathrm{fr}}$, but the obvious analog with the internal hom of 0-categorical spectra fails, and some of the consequences, analogous to above, are too strong to be true. It is an interesting question to salvage this. Related to the question is to construct the cobordism categories with other tangential structures as the *categorical Madsen-Tillmann spectra*. Another related ongoing work is to interpret stronger variants of the cobordism hypothesis,

[12] The existence of adjoints in X_{n+1} forces the existence of duals in the symmetric monoidal ∞-category $X_n \simeq \Omega X_{n+1}$.

[13] $(-)^{\mathrm{op}} : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ is an antimonoidal involution that flips all odd dimensional cells. It induces a similar functor on CatSp.

including the cobordism hypothesis with singularities and tangle hypothesis, in terms of corepresenting categorical spectra, and to make reductions from one to another formal.

5. Open Directions

Because of the novel nature of this project, there are a plethora of interesting problems to be asked.

5.1. **Deeper Brauer spectrum and categorical etale topology.** Recall from Example 4 that if R is an $\mathbb{E}_\infty$-ring, we can associate a categorical spectrum $\underline{R}$ by considering higher module categories. $\mathbb{G}_m(\underline{R})$ is a classical (but typically not bounded below) spectrum and contains valuable information about R: for instance, $\Omega^{\infty-2}\mathbb{G}_m(\underline{R}) = \mathrm{Br}(R)$ is the Brauer space of R. The structure of $\mathrm{Br}(R)$ is well-understood (when R is connective) by [AG14] using etale cohomology. The key input here was the etale local triviality of Brauer groups. So far, nothing analogous is known for the "deeper Brauer spectrum" $\mathbb{G}_m(\underline{R})$. The twist here is that the theory of categorical spectra thinks that the usual etale topology is too coarse. For instance, even though $\mathbb{C}$ is classically algebraically closed field, the super vector spaces $\underline{\mathsf{sVect}_{\mathbb{C}}}$ gives an extension of $\underline{\mathsf{Vect}_{\mathbb{C}}}$. It was announced in [Joh23a](where I took the word "deeper" from) that, at least when X is (a small version of) "presentable stable" 0-categorical spectrum and X_0 contains $\mathbb{Q}$, one can define the *etale homotopy type* $\mathrm{et}(X) \in \mathrm{Pro}(\mathsf{S}_{\mathrm{fin}})$, and that the *Galois-closedness* ($\mathrm{et}(X) \simeq *$) of X implies $\mathbb{G}_m(X) = I_{\mathbb{Q}/\mathbb{Z}}$. This type of result, in conjunction with the following fundamental problem, should give a structural understanding of $\mathbb{G}_m(\underline{R})$:

Problem 1. *Give the "categorically-etale" topology on (suitably presentable stable) categorical spectra so that $R \mapsto \mathbb{G}_m(\underline{R})$ forms a sheaf. Do we need to generalize the notion of topos and sheaves?*

A (1-)categorical version of etaleness has received attention also in tensor triangulated geometry (see e.g. [Ram23][NP23]), and I speculate that these fields will eventually meet each other.

5.2. **Algebraic K-theory.** It is an interesting problem to define the K-theory of suitably compactly generated or dualizable presentable stable categorical spectra, including the class of $\underline{\mathsf{Mod}_R}$, in a way that $K(\underline{\mathsf{Mod}_R})$ enhances the classical $K(R)$, and is also related to secondary K-theory of [MS21] and higher. It should admit a trace map to higher Hochschild homology; this should appear as the value of the tori of the TQFT defined using the theory of higher quasicoherent sheaves of [Ste21]. Optimistically, one hopes that the redshift phenomena get some illuminating description here, in light of the "category level vs chromatic level" picture, along the lines of [TV09][BDR03][BS23]. Another hope is that the appearance of lax pullbacks in the elegant description of the K-theory of pullbacks of $\mathbb{E}_1$-rings in [LT19] gets an illuminating explanation.

5.3. **Reduction principles in higher category theory.** One major difficulty working with categorical spectra is the lack of a robust reduction principle, corresponding to the t-structure in stable categories and spectral sequences. The root of the problem seems to be that a categorical hierarchy is something more closed than the homotopical hierarchy: it is easy to "leak out" from one homotopical level to higher by colimits, so things can be decomposed into simpler pieces by colimits. Since categorical levels are closed under limits and colimits, the role of decomposing objects must be replaced by lax colimits (unstraightening is a particular example). They are in general harder to control, but we still hope for usable reduction principles in restricted settings, for instance, for suitably stable presentable categorical spectra or when stable cells are fully dualizable.

5.4. **Stability of higher categories.** Theorem 3.4 implies that being a CatSp-module is a property of $\infty\mathsf{Cat}^{\otimes}$-bimodules. This corresponds to the fact that being an Sp-module is a property (stability) of presentable $(\infty,1)$-categories. In $(\infty,1)$-category theory, the stability has intrinsic characterizations. It is desirable to have ones for CatSp-modules. The difficulty here is in understanding the $\infty\mathsf{Cat}^{\otimes}$-bimodule structure itself. The data amount to the *Gray cylinder* endofunctors $(-) \otimes \square^1$ and $\square^1 \otimes (-)$ together with infinitely many coherence data (coming from $\square$) and conditions (forcing the action of $\mathcal{P}(\square)$ to factor through $\infty\mathsf{Cat}$). Alternatively, one can start from an (∞,∞)-category $\mathcal{C}$ and contemplate if it has a natural $\infty\mathsf{Cat}^{\otimes}$-bimodule structure that is stable. Because $\mathcal{C}$ itself is enriched in $\infty\mathsf{Cat}$, it makes sense to talk about (partially) lax limits (in a similar fashion to [Ber20]). It is a meaningful question to characterize $\mathcal{C}$ that gives rise to a CatSp-module in this way.

5.5. **Directed homotopy theory, Boolean and natural cohomology.** Grothendieck's homotopy hypothesis identifies ∞-groupoids with weak homotopy types of topological spaces. In the same spirit, ∞-categories can be thought of as homotopy types of *directed* spaces. Categorical spectra are (co)homology theory on such. From this viewpoint, we have the following conceptual problem:

Problem 2. *Give an excisive-functor-style description of categorical spectra.*

This is closely related to a characterization of stability we speculated, and is inevitable if we are to develop (first-order) Goodwillie calculus or the cotangent complex formalism.

Cohomology theories given by ordinary spectra see no directed information: if X is a directed space and E is a spectrum, $E^*(X) = [\Sigma^\infty_+ X, E] \simeq [\Sigma^\infty_+ X \otimes \mathbb{S}, E] \simeq E^*(|X|)$, where $|X|$ is a groupoidification of X. The next basic example would be the Eilenberg-Mac Lane spectra of semifields. Finite semifields are either a finite field $\mathbb{F}_q$ or the Boolean semifield $\mathbb{B}$, whose addition is idempotent; $1 + 1 = 1$. This "characteristic one" linear algebra over $\mathbb{B}$ is developed in [CC19]. [Gus+23] shows that projective $\mathbb{B}$-module valued 1-dimensional TQFTs with defects correspond to nondeterministic finite state automata. Analogous to [Mil58], the following problem is of fundamental computational interest:

Problem 3. *Compute the Steenrod algebra* $[H\mathbb{B}, H\mathbb{B}]$ *and the dual Steenrod algebra* $H\mathbb{B} \otimes H\mathbb{B}$*. What about the natural cohomology* $H\mathbb{N}$*?*

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