Goal $\pi_* TC(H\mathbb{F}_p) = ?$

observation
- $H\mathbb{F}_p$ is p-complete
- If X : bdd below, $\underset{\text{II.4}}{\Longrightarrow}$ $TC(X)^\wedge_p = TC(X^\wedge_p) = TC(X^\wedge_p, p)$ ($\circlearrowleft C_{p^\infty}$)

  if moreover X : p-complete

  $\rightsquigarrow TC(X)^\wedge_p = TC(X) = TC(X, p)$

$$TC(H\mathbb{F}_p, p) \longrightarrow \underset{\underset{TC^-}{\parallel}}{THH(H\mathbb{F}_p)^{hT}} \xrightarrow[\text{can} - \varphi_p^{hT}]{} \underset{\underset{TP}{\cup}}{THH(H\mathbb{F}_p)^{tT}}$$

Thm
- $\pi_* THH(H\mathbb{F}_p) \simeq \mathbb{F}_p[u]$ $\quad |u| = 2$ $\quad$ ($\leftarrow$ Bökstedt)
- $\pi_* THH(H\mathbb{F}_p)^{hT} \simeq \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p)$ $\quad |\tilde{u}| = 2$, $|v| = -2$

  $\downarrow$ can $\quad v \mapsto v$, $\quad \tilde{u} \mapsto p v^{-1}$
- $\pi_* THH(H\mathbb{F}_p)^{tT} \simeq \mathbb{Z}_p[v^{\pm 1}]$
- $\pi_* TC(H\mathbb{F}_p) \simeq \begin{cases} \mathbb{Z}_p & i = 0, -1 \\ 0 & \text{else} \end{cases}$

④ HKR filtration on $HH(R)$

$$HH(R) := HH(HR/H\mathbb{Z}) = |\cdots \ R \otimes^L_{\mathbb{Z}} R \otimes^L_{\mathbb{Z}} R \ \cdots \ R \otimes^L_{\mathbb{Z}} R \rightrightarrows R|$$

$R \otimes^L_{\mathbb{Z}} R$ $\quad \updownarrow$ $\quad HR \otimes_{H\mathbb{Z}} HR$

$\widehat{\mathrm{Mod}_{H\mathbb{Z}}} \quad \simeq \quad D(\mathbb{Z})$

This is the (nonabelian) derived functor of more classical $HH(R/\mathrm{Mod}^\heartsuit_{\mathbb{Z}})$

- $HH(R/\mathbb{Z}) \simeq HH(HR/H\mathbb{Z})$ for flat $R/\mathbb{Z}$

- $\mathrm{CAlg}^{cn}_{H\mathbb{Z}} = \mathcal{P}_\Sigma$ (full-sub of polynomial algebras with finitely many generators $= \mathcal{C}$) $\quad$ HTT 5.5.8.25 ?

  freely adjoin sifted colim $\uparrow$

  $\mathbb{Z}[x]$ cpt proj gen

  $\mathcal{C} = \{$generated by $\mathbb{Z}[x]$ under finite coproducts $\}$, coproducts = tensor prodn

$\left(\begin{array}{c} \mathcal{P}_\Sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C}) \\ \text{finite product-preserving} \end{array} \right)$

"derived functor" means

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Mod}_{\mathbb{Z}} \\ \mathcal{P}_{\Sigma}(\mathcal{C}) \arrow[ur, "\mathbb{L}F"'] & \end{tikzcd}$$

left Kan ext

More explictly: $A \in \mathrm{CAlg}^{cn}_{H\mathbb{Z}}$ $\rightsquigarrow$ resolve A by free $\mathbb{Z}$-alg

i.e. $A_\bullet \xrightarrow{\mathrm{colim}} A$
simplicial obj, A_n : free.

$\rightsquigarrow |FA_\bullet| =: FA$

Another example: $\mathbb{L}\Omega^1_{(-)/\mathbb{Z}} =: L_{(-)/\mathbb{Z}}$ cotangent complex (Quillen)

• Let $R/\mathbb{Z}$ classical comm. alg. flat

$$\longrightarrow HH_1(R/\mathbb{Z}) = \frac{\mathrm{Ker}(R\otimes R \to R)}{\mathrm{Im}(R\otimes R\otimes R \to R\otimes R)}$$

$x\otimes y \mapsto xy - yx = 0$

$x\otimes y\otimes z \mapsto xy\otimes z - x\otimes yz + zx\otimes y$

$$\cong \Big\downarrow \qquad = \frac{R\otimes R}{x\otimes yz = xy\otimes z + xz\otimes y}$$

$$\Omega^1_{R/\mathbb{Z}} \qquad \begin{matrix} x\overset{\Psi}{\otimes} y \\ \downarrow \\ \ni x\,dy \end{matrix}$$

• $HH_*(R/\mathbb{Z}) = \pi_*(\text{simplicial comm. alg})$: graded comm ring with $x^2=0$ for $|x|$:odd.

$x\cdot x = (-1)^{|x||x|} x\cdot x$
$\ \ \| $
$\ -1$

$$\begin{tikzcd} HH_1(R/\mathbb{Z})[1] \arrow[r] \arrow[d, phantom, "\wr\|"] & HH_*(R/\mathbb{Z}) \\ \Omega^1_{R/\mathbb{Z}}[1] \arrow[d, hook] & \\ \Omega^\bullet_{R/\mathbb{Z}} \arrow[uur, dashed] & \end{tikzcd}$$

$\leftarrow$ graded alg map.

<u>HKR thm</u> if $R/\mathbb{Z}$ smooth $\Leftarrow$ fin.gen free
then this is an isom

The Whitehead tower

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\geq 2} HH(-/\mathbb{Z}) \arrow[r] \arrow[d] & \tau_{\geq 1} HH(-/\mathbb{Z}) \arrow[r] \arrow[d] & HH(-/\mathbb{Z}) \arrow[d] \\
\cdots & \Omega^2_{(-)/\mathbb{Z}}[2] & \Omega^1_{(-)/\mathbb{Z}}[1] & \Omega^0_{(-)/\mathbb{Z}}
\end{tikzcd}$$

$$\left(\begin{smallmatrix} \to & \to & \to & \to \\ \downarrow & \downarrow & \downarrow & \downarrow \end{smallmatrix}\right) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_{H\mathbb{Z}}) \xrightarrow{\ \mathrm{lan}\ } \mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{C}), \mathrm{Mod}_{H\mathbb{Z}})$$

$\mathcal{C} = \left(\begin{array}{l}\text{fin.-gen} \\ \text{free } \mathbb{Z}\text{-alg}\end{array}\right)$

$$\begin{tikzcd}
\rightsquigarrow \quad \cdots \arrow[r] & F_2 HH(HR/H\mathbb{Z}) \arrow[r] \arrow[d] & F_1 HH(HR/H\mathbb{Z}) \arrow[r] \arrow[d] & HH(HR/H\mathbb{Z}) \\
& L^2_{R/\mathbb{Z}}[2] & L^1_{R/\mathbb{Z}}[1] &
\end{tikzcd}$$

$L^n = (\mathbb{L}\Lambda^n)\, \underbrace{\mathbb{L}\Omega^1}_{= L}$

Prop
- $F_\bullet HH(HR/H\mathbb{Z})$: HKR filtration on $HH(HR/H\mathbb{Z})$
- $\mathrm{gr}_\bullet(F_\bullet HH) = L^\bullet$
- $\varprojlim F_n HH(HR/H\mathbb{Z}) = 0$

[From now on Notation: $HH(\cancel{H}R/\cancel{H}\mathbb{Z})$]

③ $HH_*(\mathbb{F}_p/\mathbb{Z})$

$L_{\mathbb{F}_p/\mathbb{Z}} \simeq \mathbb{F}_p[1]$

Proof ① $\underbrace{A \to B \to C}_{\text{any seq}} \Rightarrow C \overset{L}{\underset{B}{\otimes}} L_{B/A} \to L_{C/A} \to L_{C/B}$ fib seq

②
$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{po}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd} \Longrightarrow L_{B/A} \underset{B}{\otimes} D \xrightarrow{\sim} L_{D/C}$$

③ $L_{\mathbb{Z}[x]/\mathbb{Z}} = \Omega^1_{\mathbb{Z}[x]/\mathbb{Z}}$

Apply ③ to $\mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}$, $x \longmapsto 0$

$$\begin{tikzcd} L_{\mathbb{Z}[x]/\mathbb{Z}} \arrow[r] \arrow[d, equal, "③"'] & L_{\mathbb{Z}/\mathbb{Z}} \arrow[r] \arrow[d, equal] & L_{\mathbb{Z}/\mathbb{Z}[x]} \arrow[d, equal] \\ \Omega^1_{\mathbb{Z}[x]/\mathbb{Z}} = (x)/(x^2) & 0 & (x)/(x^2)[1] \end{tikzcd}$$

pushout sq

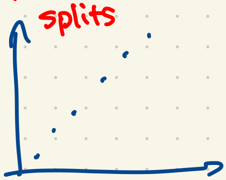
$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, "x \mapsto p"] \arrow[d, "x \mapsto 0"'] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$L_{\mathbb{F}_p/\mathbb{Z}} \simeq \mathbb{F}_p \underset{\mathbb{Z}}{\otimes} L_{\mathbb{Z}/\mathbb{Z}[x]} \simeq p\mathbb{Z}/p^2\mathbb{Z}[1] \simeq \mathbb{F}_p[1]$$

- $\mathbb{L}\Lambda^n(E[1]) \simeq \mathbb{L}\Gamma^n(E)[n]$ in $\mathcal{D}(E)$

$\pi_* HH(\mathbb{F}_p/\mathbb{Z}) \simeq \pi_*(\mathbb{L}\Lambda^\bullet L_{\mathbb{F}_p/\mathbb{Z}}) = \Lambda^\bullet(\mathbb{F}_p[1]) \overset{\pi_*}{\simeq} \mathbb{F}_p\langle u \rangle$ divided power alg, $|u| = 2$

HKR filtration splits



$\Lambda^\bullet(-) = \mathrm{Sym}(-[1])$

$$\begin{tikzcd} HH = F_0 \arrow[d] & \supset F_1 \arrow[d] & \supset \\ F_0/F_1 & F_1/F_2 \end{tikzcd}$$

ⓘ THH vs HH

$A \longrightarrow B$ in CAlg $\Longrightarrow$ $\mathrm{Mod}_B \rightleftarrows \mathrm{Mod}_A$ with $- \otimes_A B$: sym mon, $\mathrm{Hom}_A(B,-)$

$\Longrightarrow$ forgetful functor $\begin{cases} \text{pres lim \& colim} \\ \text{is lax sym mon} \end{cases}$

$R \in \mathrm{Alg}_B \rightsquigarrow$

$$\begin{tikzcd} {|\cdots \substack{\to\\\to\\\to\\\to} R \otimes_A R \otimes_A R \substack{\to\\\to\\\to} R \otimes_A R \rightrightarrows R|} \arrow[d] & =: HH(R/A) \arrow[d, "\simeq"] \\ {|\cdots \substack{\to\\\to\\\to} \otimes_B \quad \otimes_B|} & =: HH(R/B) \end{tikzcd}$$

$$\begin{matrix} A & & B \\ \text{\rotatebox{90}{$\simeq$}} & & \| \\ \mathbb{S} & \longrightarrow & H\mathbb{Z} \end{matrix} \qquad THH(R) \longrightarrow HH(R) = HH(HR/H\mathbb{Z})$$

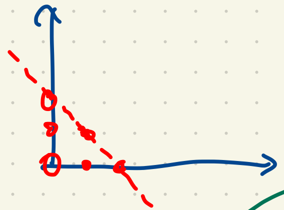
Prop. the fiber of this map is 2-connected.

$$I_n \longrightarrow \underbrace{R \otimes_A \cdots \otimes_A R}_{n} \longrightarrow \underbrace{R \otimes_B \cdots \otimes_B R}_{n}$$

$$\rightsquigarrow \mathrm{Fib} = |\cdots \, I_3 \rightrightarrows I_2 \rightrightarrows I_1|$$

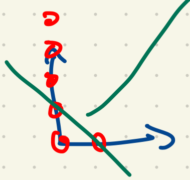
$\exists$ ss to compute the geom. realization of a (semi) simplicial objects induced by the sk. filtration (HA§1.2.4)

$$E_1^{p,q} = \pi_p I_{q+1} \Longrightarrow \pi_{p+q} \mathrm{Fib}$$



suffices to prove

- $\pi_{\leq 2} I_1 \overset{\neq 0}{=} 0$ ✓
- $\pi_{\leq 1} I_2 = 0$
- $\pi_{\leq 0} I_3 = 0$



$$\begin{array}{ccc} R \otimes_A R & \longrightarrow & R \otimes_B R \\ \uparrow & & \uparrow \\ \left| \begin{array}{c} R \otimes R \\ \uparrow\uparrow \\ R \otimes A \otimes R \\ \uparrow\uparrow\uparrow \end{array} \right| & \longrightarrow & \left| \begin{array}{c} R \otimes R \\ \uparrow\uparrow \\ R \otimes B \otimes R \end{array} \right| \end{array}$$

the same argument ($\mathbb{S} \to H\mathbb{Z}$ $\pi_{\leq 0}$-isom) $\boxed{\square}$

Cor $\tau_{\leq 2} \mathrm{THH}(H\mathbb{F}_p) \xrightarrow{\sim} \tau_{\leq 2} \mathrm{HH}(\mathbb{F}_p) \simeq \mathbb{F}_p \cdot u$

$$u \longmapsto u$$

Thm (Bökstedt)

$$\pi_* \mathrm{THH}(H\mathbb{F}_p) = \mathbb{F}_p[u] \quad (\longrightarrow \mathbb{F}_p\langle u \rangle)$$

(ii) $\mathrm{TC}^-(H\mathbb{F}_p) = \mathrm{THH}(H\mathbb{F}_p)^{hT}$

Homotopy fixed point ss (multiplicative)

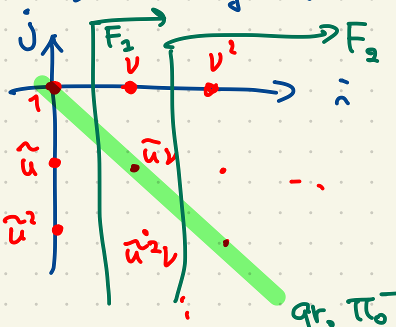
$$E_2^{i,j} = H^i(BT, \pi_{-j} \mathrm{THH}(H\mathbb{F}_p)) \Longrightarrow \pi_{-i-j}(\mathrm{THH}(H\mathbb{F}_p)^{hT})$$

$BT \simeq \mathbb{C}P^\infty$, trivial ?, $\pi_{-j}\mathrm{THH}(H\mathbb{F}_p) = \mathbb{F}_p \cdot u^{-j/2}$, $j \leq 0$ even

$E_2 \cong \mathbb{F}_p[\tilde{u}, v]$?

· $\tilde{u}$: lift of $u \in \pi_2 THH(H\mathbb{F}_p)$ in deg $(i,j) = (0,-2)$

· v in deg $(2,0)$



$gr_0\, \pi_0 THH(H\mathbb{F}_p)^{hT}$

$E_2 = E_\infty$ — only in even deg

$\boxed{\mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p) \cong \pi_* THH(H\mathbb{F}_p)^{hT}}$ is a $\mathbb{Z}_p$-alg generated by $\tilde{u}, v$, $\tilde{u}v = p$

$\pi_0 THH(H\mathbb{F}_p)^{hT} = \mathbb{Z}_p$

$p \in \pi_0 THH(H\mathbb{F}_p)^{hT}$ is in the first filtration F_1 $\Uparrow$ no p-power torsion

$$F_1 \hookrightarrow (\pi_0 THH \cdots) \longrightarrow \mathbb{F}_p$$
$$\in \qquad\quad p \longmapsto 0$$

$\tilde{u}^n v^n = p^n$ $\Rightarrow$

<u>Lemma</u> The image of p in $E_2^{2,-2} = F_1/F_2$ is $\tilde{u} \cdot v$

<u>proof</u>

Only depends on $\tau_{\le 2} THH(H\mathbb{F}_p) \simeq \tau_{\le 2} HH(\mathbb{F}_p) \simeq HH(\mathbb{F}_p)/F_2^{HKR}$

Recall : HKR filtration on $HH(\mathbb{F}_p) \supset F_1^{HKR} \supset F_2^{HKR} \supset \cdots$

$\mathbb{F}_p$ $\quad$ $\mathbb{L}_{\mathbb{F}_p/\mathbb{Z}}[1] = \mathbb{F}_p[1]$ $\quad$ $\mathbb{L}^2_{\mathbb{F}_p/\mathbb{Z}}[2] \ldots \longrightarrow$ above deg 2

$\underbrace{\qquad\qquad}_{\tau_{\le 2} HH(\mathbb{F}_p)}$

$\overline{HH(A)} := HH(A)/F_2^{HKR}(A)$

is an extension

$$\mathbb{L}_{A/\mathbb{Z}}[1] \longrightarrow \overline{HH(A)} \longrightarrow A$$

We may restrict attention to

$$H^{\cancel{*}\,2}(\mathbb{CP}^1, \pi_{\cancel{*}\,2} \overline{HH}(A))$$

$\mathbb{F}_p \ni u$

Need : $p = \tilde{u} \cdot v$

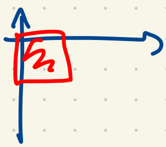
$$S^2 \ni * \longmapsto \overline{HH}(A)$$

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty = BT \longrightarrow \mathcal{D}(\mathbb{Z})$$

$\mathbb{CP}^1$ - structure (functor)

is given by

$$\begin{tikzcd} \mathrm{Aut}_{S^2}(*) \arrow[d, equal] \arrow[r] & \mathrm{Aut}(\overline{HH}(A)) \\ \Omega S^2 \arrow[d, equal] & \\ \mathrm{Free}_{\mathbb{E}_1} S^1 & \\ S^1 \arrow[u] \arrow[uuur] & \end{tikzcd}$$



$\Leftrightarrow$

$\overline{HH}(A) \longrightarrow \overline{HH}(A)[-1]$

$S^1 \longrightarrow \mathrm{Hom}(\overline{HH}(A), \overline{HH}(A))$

$\Sigma \overline{HH}(A) \longrightarrow \overline{HH}(A)$

$$\lim_{\mathbb{CP}^1} \overline{HH}(A) = \mathrm{fib}(\overline{HH}(A) \longrightarrow \overline{HH}(A)[-1]) \quad \longleftarrow \text{check later}$$

If A : smooth $/\mathbb{Z}$, then

$$\begin{tikzcd} \overline{HH}(A) \arrow[r] \arrow[d, "\tau_{\le 0}"'] & \overline{HH}(A)[-1] \arrow[r, squiggly, "\text{lan}"] & \overline{HH}(A) \arrow[r] \arrow[d] & \overline{HH}(A)[-1] \\ A \arrow[r] & \Omega^1_{A/\mathbb{Z}} \arrow[u, "\tau_{\ge 0}"'] & A \arrow[r] & \mathbb{L}^1_{A/\mathbb{Z}} \arrow[u] \end{tikzcd}$$

differential

conc. in deg 0

check

$\underline{\underline{A = \mathbb{F}_p}}$

$$\pi_0 \lim_{\mathbb{CP}^1} \overline{HH}(\mathbb{F}_p) = \pi_0\, \mathrm{fib}\Big(\overline{HH}(\mathbb{F}_p) \xrightarrow{\tau_{\le 0}} \mathbb{F}_p \longrightarrow \mathbb{L}^1_{\mathbb{F}_p/\mathbb{Z}} \longrightarrow \overline{HH}(\mathbb{F}_p)[-1]\Big)$$

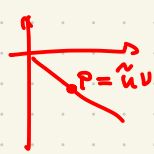
$$\mathbb{L}^1_{\mathbb{F}_p/\mathbb{Z}} = p\mathbb{Z}/p^2\mathbb{Z}[1]$$

$$= \pi_0\, \mathrm{fib}\big(\mathbb{F}_p \underbrace{\longrightarrow}_{\text{differential}} p\mathbb{Z}/p^2\mathbb{Z}[1]\big)$$

$$p\mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathrm{fib} \longrightarrow \mathbb{F}_p \longrightarrow p\mathbb{Z}/p^2\mathbb{Z}[1]$$

$$\mathrm{fib} = \mathbb{Z}/p^2\mathbb{Z}$$

$$\tilde{u} v \longmapsto p$$



③ $TP(H\mathbb{F}_p) = THH^{tT}$

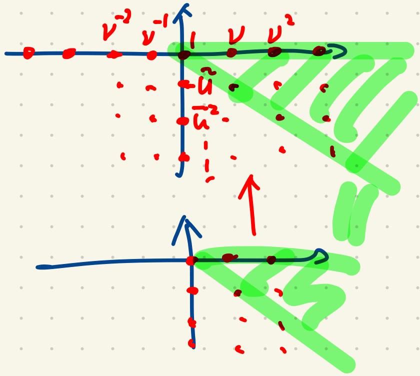
Tate SS $\quad E_\infty = E_2^{i,j} = \pi_{-i}\big((\pi_{-j} THH(H\mathbb{F}_p))^{tT}\big) \Rightarrow \pi_{-i-j}\big(\underbrace{THH(H\mathbb{F}_p)^{tT}}_{=TP}\big)$ multiplicative

$E_2^{i,j} \cong \mathbb{F}_p[u, v^{\pm 1}]$

$\uparrow$ can $\qquad\qquad \uparrow$

fixed pt $\quad E_\infty = E_2^{i,j} = \pi_{-i}\big((\pi_{-j} THH(H\mathbb{F}_p))^{hT}\big) \Rightarrow \pi_{i-j}\big(THH(H\mathbb{F}_p)^{hT}\big)$

$= \mathbb{F}_p[u, v]$



$\pi_{\leq 0} THH(H\mathbb{F}_p)^{tT}$

$= \pi_{\leq 0} TC^-(H\mathbb{F}_p)$

$v \cdot v^{-1} = 1$ in $E_2^{0,0}$

$\rightsquigarrow v \cdot v^{-1} - 1 \in p\mathbb{Z}_p \simeq p \cdot \pi_0 TP$

$\Rightarrow v \in (\pi_* TP)^\times$

$$\begin{array}{ccc} \boxed{\mathbb{Z}_p[v^{\pm 1}] \xrightarrow{\cong} \pi_* TP(H\mathbb{F}_p)} & & p \cdot v^{-1} \\ \uparrow \qquad\qquad\qquad \uparrow & & \uparrow \\ \mathbb{Z}_p[v] \xrightarrow[\cong]{} \pi_{\leq 0} \sim & & \tilde{u} \end{array}$$

$\tilde{u} v = p$

$$\mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p) \xrightarrow{\text{can}} \mathbb{Z}_p[v^{\pm 1}]$$

$$v \longmapsto v$$

$$\tilde{u} \longmapsto p \cdot v^{-1}$$

$\begin{cases} \cdot \text{ isom in } \pi_{\leq 0} \\ \cdot \text{ inj with image } p^j \text{ on } \pi_{2j} \quad (j \geq 0) \end{cases}$

$$\pi_* \varphi_p^{hT} : \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p) \longrightarrow \mathbb{Z}_p[v^{\pm 1}]$$

$\tilde{u} v = p \qquad \tilde{u} \longmapsto a \cdot v^{-1}$

$\qquad\qquad v \longmapsto b \cdot v \qquad\qquad \underline{ab = p}$

if b is a unit ($\rightsquigarrow$ isom in $\deg \leq 0$)

Consider

$$\begin{tikzcd}
\pi_{-2}\,THH(H\mathbb{F}_p)^{h\mathbb{T}} \arrow[r, "\pi_*\varphi_p^{h\mathbb{T}}", "\cong"'] \arrow[d] & \pi_2 THH(H\mathbb{F}_p)^{t\mathbb{T}} \arrow[r] \arrow[d, "0"] & \pi_2 H\mathbb{F}_p^{t\mathbb{T}} \arrow[d, "\cong"] & \mathbb{F}_p[v^{\pm 1}] \arrow[d] \\
\pi_{-2}\,THH(H\mathbb{F}_p) \arrow[r, "\pi_{-2}\varphi_p"'] & \pi_{-2}THH(H\mathbb{F}_p)^{tC_p} \arrow[r] & \pi_{-2}H\mathbb{F}_p^{tC_p} & \mathbb{F}_p[v^{\pm 1}] \\
0 & & 0 & \mathbb{F}_p
\end{tikzcd}$$

$v \longmapsto$ generator of $\mathbb{F}_p$

Contradiction

$\Downarrow$

a is a unit

$\Downarrow$

- isom in $\deg \geq 0$
- inj. with image $p^j\mathbb{Z}_p$ on π_{-2j} $(j \geq 0)$

$$S^1 \simeq \mathbb{T}/C_p \to BC_p \to B\mathbb{T}$$

Cor $\pi_* TC(H\mathbb{F}_p) = \mathbb{Z}_p[\varepsilon]/(\varepsilon^2)$

proof fiber seq

$$TC(H\mathbb{F}_p) \to \underbrace{THH(H\mathbb{F}_p)^{h\mathbb{T}}}_{TC^-} \xrightarrow{can - \varphi_p^{h\mathbb{T}}} \underbrace{THH(H\mathbb{F}_p)^{t\mathbb{T}}}_{TP}$$

π_* in even deg

$$0 \to \pi_{2i}TC \to \underset{\cong\, \mathbb{Z}_p}{\pi_{2i}TC^-} \xrightarrow{can - \varphi^{h\mathbb{T}}_{p,*}} \underset{\cong\, \mathbb{Z}_p}{\pi_{2i}TP} \to \pi_{2i-1}TC \to 0$$

if $i \neq 0$ exactly one of

can, $\varphi_p^{h\mathbb{T}}$ is an isom, another is divisible by p

$\Longrightarrow$ Isom

if $i = 0$ $id - id = 0$ $\pi_0 TC \cong \pi_{-1} TC \simeq \mathbb{Z}_p$ ⌟

$$H\mathbb{Z}_p = \tau_{\geq 0}TC(H\mathbb{F}_p) \to TC \to TC^- \xrightarrow{\varphi_p} TP$$

$$TC^- \to THH$$

$THH(H\mathbb{F}_p) \in CAlg(CycSp)$ $\left(\begin{array}{c} Sp \underset{TC}{\overset{(-)^{triv}}{\rightleftarrows}} CycSp \\ X \longmapsto X \circlearrowleft \mathbb{T} \text{ triv} \\ \varphi_p : X \to X^{h\mathbb{T}} \to X^{t\mathbb{T}} \end{array} \right)$

$$H\mathbb{Z}_p \to TC(\mathbb{F}_p) \rightsquigarrow H\mathbb{Z}_p^{triv} \to THH(\mathbb{F}_p)$$

$\qquad \| \qquad\qquad \wr\simeq$

$$sh_p H\mathbb{Z}_p^{triv} \xrightarrow[\simeq]{} sh_p THH(\mathbb{F}_p)$$

by previous computations + Lem IV.4.12

$X^{t\mathbb{T}}/n \simeq X^{tC_n}$

<u>fix p</u>

$$(-)_{sh} : CycSp^{cn} \longrightarrow CycSp^{cn}$$

$$X \longmapsto \tau_{\geq 0} X^{tC_p} \circlearrowleft \mathbb{T}/C_p \simeq \mathbb{T}$$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d, "\varphi_p"'] & \tau_{\geq 0} X^{tC_p} \arrow[d, "F\varphi_p"] \\
\tau_{\geq 0} X^{tC_p} \arrow[r] & X^{tC_p} & \underbrace{\tau_{\geq 0}}_{F} (\tau_{\geq 0} X^{tC_p})^{tC_p}
\end{tikzcd}$$