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SURVEY ARTICLE

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Some recent advances in topological Hochschild homology

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Abstract

We give an account of the construction of the Bhatt–Morrow–Scholze motivic filtration on topological cyclic homology and related invariants, focusing on the case of equal characteristic p and the connections to crystalline and de Rham–Witt theory.

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1 | INTRODUCTION

Let X be a smooth quasi-projective scheme over a field k. In this case, one has the algebraic K-theory spectrum $K(X)$ of X, defined by Quillen [106] using the exact category of vector bundles on X (in general, one should use perfect complexes as in [116]). One can think of $K(X)$ as a type of 'cohomology theory' for the scheme X, analogous to the topological K-theory of a compact topological space. With this in mind, the following fundamental result gives an analog of the classical Atiyah–Hirzebruch spectral sequence relating topological K-theory to singular cohomology.

Theorem 1.1 (The motivic filtration on algebraic K-theory, [51, 90]). *There is a functorial, convergent, decreasing multiplicative filtration* $\mathrm{Fil}^{\geqslant *}K(X)$ *and identifications* $\mathrm{gr}^i K(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i]$ *for* $i \geqslant 0$.

Here the $\mathbb{Z}(i)^{\mathrm{mot}}(X)$, called the *motivic cohomology* of X, are explicit cochain complexes introduced by Bloch [33] (see also [119]) in terms of algebraic cycles on $X \times \mathbb{A}^n_k$ for $n \geqslant 0$. In particular, we have that

$$H^{2i}(X, \mathbb{Z}(i)) \overset{\mathrm{def}}{=} H^{2i}(\mathbb{Z}(i)^{\mathrm{mot}}(X)) = \mathrm{CH}^i(X)$$

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is given by the Chow group $\mathrm{CH}^i(X)$ of codimension i cycles on X modulo rational equivalence.[†] The complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ (considered as objects of the derived category of abelian groups) can also be described as maps in the $\mathbb{A}^1$-motivic stable homotopy category from X into motivic Eilenberg–MacLane spectra.

Theorem 1.1 gives substantial information about algebraic K-theory, especially after profinite completion. After reducing modulo a prime l which is different from the characteristic, the Beilinson–Lichtenbaum conjecture proved by Voevodsky–Rost [120, 121] identifies mod l motivic cohomology as Zariski (or Nisnevich) sheaves,

$$\mathbb{Z}/l(i)^{\mathrm{mot}} \simeq \tau^{\leqslant i}(R\nu_* \mu_l^{\otimes i}), \tag{1}$$

for ν the pushforward from the étale to the Zariski topology. For example, Theorem 1.1 implies that in high degrees, we can compute mod l algebraic K-theory of a variety over $\mathbb{C}$ as the topological K-theory of the space of $\mathbb{C}$-points and the former is therefore finitely generated since the underlying homotopy type is that of a finite CW complex. (By contrast, K_0 with mod l coefficients can be enormous, cf., for instance, [110, 111, 117].)

After p-adic completion when the ground field k has characteristic p, the analog of the Beilinson–Lichtenbaum conjecture (1) is given by the theorems of Geisser–Levine [58] and Bloch–Kato–Gabber [34],

$$\mathbb{Z}/p(i)^{\mathrm{mot}} \simeq \Omega^i_{\log}[-i],$$

identifying the object $\mathbb{Z}/p(i)^{\mathrm{mot}}$ (which lives in the derived category of Zariski or Nisnevich sheaves on X) with the $-i$-shift of the subsheaf $\Omega^i_{\log} \subset \Omega^i$ of differential i-forms generated by $\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_i}{x_i}$, for the x_i local units. For example, this coupled with Theorem 1.1 implies that if X is a smooth variety over a perfect field of characteristic p, then the mod p K-theory $K_*(X;\mathbb{Z}/p)$ vanishes in degrees $> \dim(X)$.

However, the construction of Theorem 1.1 (and the definition of motivic cohomology) relies heavily on the smoothness assumption on X. The higher Chow groups of a singular variety X over a field k give an analog of Borel–Moore homology rather than cohomology; in particular, they are nilinvariant (while algebraic K-theory is far from nilinvariant) and lack a product structure. The existing constructions of the motivic filtration use the $\mathbb{A}^1$-invariance of K-theory (valid only in the regular case), and giving a general notion of the motivic filtration (or of motivic cohomology) on K-theory applicable to singular rings appears to be an open problem. Even the setting of regular rings in mixed characteristic is not fully understood (but see [89]).

Question 1.2. Is there a motivic filtration on $K(R)$ for any ring R which extends Theorem 1.1 when R is smooth over a field?[‡]

[†] One difference in this analogy is that algebraic K-theory historically preceded motivic cohomology, whereas singular cohomology preceded topological K-theory.

[‡] For affine schemes, one proposal is to left Kan extend motivic cohomology from smooth algebras using the result of Bhatt–Lurie (cf. [48, Appendix A] for an account) that the connective K-theory of rings is left Kan extended from smooth algebras. However, this construction will not satisfy Zariski descent and does not obviously globalize; moreover, it does not apply to nonconnective K-theory.

In the study of the algebraic K-theory of singular rings, the main new tool is the theory of trace methods. Trace methods provide maps from algebraic K-theory to more computable invariants built from Hochschild homology, and the basic tools are relative comparison results to the effect that the homotopy fiber of such maps satisfy excision and nil-invariance. The most general and powerful form of these results uses topological cyclic homology TC, introduced by Bökstedt–Hsiang–Madsen [37] in the p-complete case (see [47] for the integral version), which compares to K-theory via the cyclotomic trace map

$$K(R) \to \mathrm{TC}(R). \tag{2}$$

Theorem 1.3. *The homotopy fiber F of (2) has the following properties.*

(1) (Dundas–Goodwillie–McCarthy [47][†]) *F is nilinvariant.*
(2) (Land–Tamme [88]) *F satisfies excision, that is, given a pullback square of rings with the vertical arrows surjective, then F carries this to a pullback of spectra.*
(3) (Clausen–Mathew–Morrow [43]) *The profinite completion of the variant $F'(R) = \mathrm{fib}(K_{\geqslant 0}(R) \to \mathrm{TC}(R))$ is rigid for henselian pairs, that is, if (R, I) is a henselian pair, then $F'(R)/n \xrightarrow{\sim} F'(R/I)/n$ for any integer $n > 0$.*

When R is a $\mathbb{Q}$-algebra, then $\mathrm{TC}(R)$ agrees with the negative cyclic homology $\mathrm{HC}^-(R/\mathbb{Q})$, which is closely related to the de Rham cohomology of R, and parts (1) and (2) of the result are due, respectively, to Goodwillie [61] and to Cortiñas [44]. For example, compare [59] for applications of these results (at the time conjectural) to the calculation of the K-theory of singular curves over $\mathbb{Q}$ (for example, rings such as $\mathbb{Q}[x, y]/(xy)$). Part (3) in this case, or more generally when n is invertible on R, is the Gabber rigidity theorem [52].

In this survey, we concentrate on the situation after p-adic completion for p-adic rings, in which case parts (1) and (2) are due to McCarthy [101] and Geisser–Hesselholt [56]. In this case, it is known that the map (2) is not only useful for detecting 'infinitesimal' behavior, but is also an absolute approximation to p-adic K-theory R: specifically, it is p-adic étale (connective) K-theory; moreover, it is an equivalence in high enough degrees depending on the ring, under mild hypotheses. This follows from the work of Geisser–Levine [58] on the p-adic K-theory of smooth algebras in characteristic p, Geisser–Hesselholt [55] on TC of such rings, extended to the more general situation using the rigidity result of [43] (see also [42]).

Theorem 1.4.

(1) *For p-adic rings R, the trace (2) exhibits $\mathrm{TC}(R; \mathbb{Z}_p)$ as the p-completion of the étale K-theory of R.*[‡]
(2) *If R is p-complete, R/p has finite Krull dimension and $d = \sup_{x \in \mathrm{Spec}(R/p)} \log_p[k(x) : k(x)^p]$, then the map $K(R;\mathbb{Z}_p) \to \mathrm{TC}(R; \mathbb{Z}_p)$ is an equivalence in degrees $\geqslant \max(d, 1)$.*

The definition of TC is much more elaborate than that of K-theory and (in particular, in the p-adic setting) requires significant homotopical foundations, classically approached through equivariant stable homotopy theory [36], which were recently dramatically reworked (and simplified)

[†] See also [108] for an account using the approach to TC of [105].

[‡] Strictly speaking, this refers to the étale sheafification of the connective K-theory of R.

by Nikolaus–Scholze [105] (see also [17, 18] for ∞-categorical accounts of the theory of cyclotomic spectra). We refer to [73] for a modern survey of topological Hochschild and cyclic homology.

There is a sense, however, in which topological cyclic homology is a structurally simpler theory: while the building blocks of algebraic K-theory come from algebraic cycles, the building blocks for topological cyclic homology are the cotangent complex and its wedge powers. In practice, this means that the formal properties of TC are somewhat simpler (for example, TC has much better descent properties), and that TC is easier to compute. There is an extensive literature calculating various instances of the p-adic K-theory of p-adic rings using TC; see, for instance, [57, 69–72] for some examples.

The work of Bhatt–Morrow–Scholze [30] constructs analogs of the motivic filtration for topological Hochschild homology and its variants in great generality; the associated graded objects of this filtration are objects of deep interest in arithmetic geometry and especially p-adic Hodge theory, and have now been constructed purely algebraically using the prismatic theory [32]. Here we state first the analog for TC.

Theorem 1.5 (Bhatt–Morrow–Scholze [30]). *Let R be any p-complete ring.*[†] *Then there exists a natural multiplicative, $\mathbb{Z}_{\geqslant 0}^{op}$-indexed convergent filtration $\mathrm{Fil}^{\geqslant *}\mathrm{TC}(R;\mathbb{Z}_p)$ with associated graded terms $\mathrm{gr}^i\mathrm{TC}(R;\mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i]$, for the $\mathbb{Z}_p(i)(R)$ natural objects of the p-complete derived ∞-category. The constructions $\mathbb{Z}_p(i)$ (as functors to the derived ∞-category $\mathcal{D}(\mathbb{Z}_p)$) satisfy flat descent, and for regular $\mathbb{F}_p$-algebras reproduce the objects $R\Gamma_{\mathrm{proet}}(-, W\Omega^i_{\log})[-i]$.*

Theorem 1.5 is supposed to be an analog of Theorem 1.1 for TC. It is not entirely clear if this analogy can be made precise (that is, if both filtrations can be realized as instances of a common construction). However, it is at least known in the case where R is a smooth algebra over a field k of characteristic p (so that Theorem 1.1 is in effect), that the filtration $\mathrm{TC}(R;\mathbb{Z}_p)$ is the p-completion of the étale sheafification of the motivic filtration on $K(R)$ (indeed, both are Postnikov towers in the (pro-)Nisnevich and (pro-)étale topologies). The construction of Theorem 1.5 has the advantage of being very direct: the filtration is the Postnikov filtration when these invariants are considered as sheaves in the quasi-syntomic topology (Section 4).

The $\mathbb{Z}_p(i)$ in their broadest generality are supposed to be a general version of p-adic étale motivic cohomology for p-adic rings. They arise as a type of filtered Frobenius eigenspaces on *prismatic cohomology*, a new p-adic cohomology theory for p-adic formal schemes introduced by Bhatt–Scholze [32] of deep interest in integral p-adic Hodge theory and constructed in some cases in [29, 30]. The case of 'absolute' prismatic cohomology was originally constructed using topological Hochschild homology. For the formulation of the next result, we write THH for topological Hochschild homology equipped with its natural $\mathbb{T}$-action.

Theorem 1.6 [30]. *Let R be a formally smooth algebra over a perfectoid ring R_0. Then there is a complete, exhaustive $\mathbb{Z}$-indexed filtration on $\mathrm{TP}(R;\mathbb{Z}_p) \stackrel{\mathrm{def}}{=} \mathrm{THH}(R;\mathbb{Z}_p)^{t\mathbb{T}}$ such that $\mathrm{gr}^i\mathrm{TP}(R;\mathbb{Z}_p) \simeq \widehat{\mathbb{\Delta}}_R[2i]$.*

In fact, the above filtration is constructed using the similar descent techniques, which gives a construction of prismatic cohomology, independent of the prismatic site of [32]. When R_0 is

[†] The work [30] only treats the case where R is quasi-syntomic; it is shown in [8, Section 5] that the construction naturally extends (via left Kan extension) to all p-adic rings.

the ring of integers $\mathcal{O}_C$ in a complete, algebraically closed non-Archimedean field C, then the $\widehat{\Delta}_R$ recover the A_{inf}-cohomology of [29]: in particular, they specialize both to the de Rham cohomology of the formal scheme $\mathrm{Spf}(R)$ and the p-adic étale cohomology of the generic fiber.

We will not attempt to do justice to the new landscapes of integral p-adic Hodge theory. In this survey article, we will work through the characteristic p situation in some detail, in particular, constructing the filtration on $\mathrm{TP}(R; \mathbb{Z}_p)$ for R a smooth (or more generally quasi-syntomic, for example, lci) $\mathbb{F}_p$-algebra R and identifying the associated graded pieces in terms of crystalline cohomology. In equal characteristic p, absolute prismatic cohomology in this context reduces to crystalline cohomology, constructed using the (quasi-)syntomic site instead of the crystalline site (an approach that goes back to [50]). Finally, we will circle back to the motivation of algebraic K-theory, and explain how one can recover the calculations of the algebraic K-theory of the dual numbers over a perfect field [70, 113]. It would be interesting to revisit other such calculations.

Notation

We let $\mathbb{T}$ denote the circle group. We denote by Sp the ∞-category of spectra, with the smash product $\otimes$, and $\mathbb{S}$ the sphere spectrum. For a ring R, we let $\mathcal{D}(R)$ denote the derived ∞-category of R.

We will freely use the language of higher algebra [96], and in particular the theory of $\mathbb{E}_\infty$-ring spectra. We refer to [60] for a modern survey and introduction.

Throughout the paper, we fix a prime p. We will occasionally use the theory of δ-rings, but only in the p-torsionfree case; a p-torsionfree δ-ring consists of a commutative ring R equipped with an endomorphism $\varphi : R \to R$ which lifts the Frobenius modulo p. We refer to [32, Section 2] for an account of the theory of δ-rings in general.

We will often drop the notation of p-completions, since we will almost exclusively be working with p-complete objects.

2 | TOPOLOGICAL HOCHSCHILD HOMOLOGY

Let R be a commutative ring.

Definition 2.1. The *topological Hochschild homology* $\mathrm{THH}(R)$ is the universal $\mathbb{E}_\infty$-algebra equipped with a $\mathbb{T}$-action and a map $R \to \mathrm{THH}(R)$. As an $\mathbb{E}_\infty$-ring, there is an identification

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R} R,$$

and $\mathrm{THH}(R)$ can be obtained as the geometric realization of the cyclic bar construction (a simplicial $\mathbb{E}_\infty$-ring obtained from the tensor powers of R as a spectrum), [105, Section III.2].

Definition 2.1 (which works equally for an $\mathbb{E}_\infty$-ring R) is not the most flexible definition of THH, since THH is more generally defined for stable ∞-categories (it is a localizing invariant in the sense of [35], like algebraic K-theory); then THH of a commutative ring is defined as THH of its ∞-category of perfect complexes. For example, THH can be defined using factorization homology over the circle [16]. This perspective, while extremely important in the foundations of the theory (in particular, in producing the cyclotomic trace), plays less of a role in the work of [30], which focuses on commutative rings. The above formulation for $\mathbb{E}_\infty$-rings is due to [102].

Before describing some of the features of THH, we begin by reviewing the simpler algebraic analog of Hochschild homology.

Variant 2.2 (Classical Hochschild homology). Let R be a commutative k-algebra, for k a base ring. Then the *Hochschild homology* $\mathrm{HH}(R/k)$ is the universal $\mathbb{E}_\infty$-algebra[†] under k equipped with a $\mathbb{T}$-action and a map $R \to \mathrm{HH}(R/k)$ of $\mathbb{E}_\infty$-k-algebras. As an $\mathbb{E}_\infty$-k-algebra, one has

$$\mathrm{HH}(R/k) = R \otimes^L_{R\otimes^L_k R} R.$$

When $k = \mathbb{Z}$, we will sometimes drop the k in the above notation.

Hochschild homology over k is a very controllable construction, because of the classical Hochschild–Kostant–Rosenberg theorem. Since $\mathbb{T}$ acts on the $\mathbb{E}_\infty$-k-algebra $\mathrm{HH}(R/k)$, one obtains a commutative differential graded algebra structure on $\mathrm{HH}_*(R/k)$ (with the differential arising from the $\mathbb{T}$-action). The Hochschild–Kostant–Rosenberg theorem gives a natural isomorphism of commutative differential graded algebras for R smooth over k,

$$\mathrm{HH}_*(R/k) \simeq (\Omega^*_{R/k}, d), \tag{3}$$

where d is the de Rham differential.

Topological Hochschild homology is a much richer theory than classical Hochschild homology for p-adic rings (whereas for $\mathbb{Q}$-algebras, it reduces to Hochschild homology relative to $\mathbb{Q}$). Taking Hochschild homology over the base $\mathbb{S}$ leads to extra symmetries in the theory which are not available with an ordinary ring (for example, $\mathbb{F}_p$ or $\mathbb{Z}$) as the base; moreover, it leads to Bökstedt's computation of $\mathrm{THH}(\mathbb{F}_p)$. We begin by reviewing these aspects, following Nikolaus–Scholze [105]; see also the survey [73] for a more detailed overview.

Construction 2.3 (The cyclotomic Frobenius on THH, cf. [105, Section IV.2]). Given a commutative ring R, one has a natural $\mathbb{T}$-equivariant map

$$\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}, \tag{4}$$

using the natural embedding $C_p \subset \mathbb{T}$ and the $\mathbb{T} \simeq \mathbb{T}/C_p$-action on the right-hand side. To construct (4), we use the universal property of $\mathrm{THH}(R)$ to construct a map of $\mathbb{E}_\infty$-rings $R \to \mathrm{THH}(R)^{tC_p}$ and then extend it canonically to a $\mathbb{T}$-equivariant map as in (4). This in turn comes from the *Tate diagonal* [105, Section III.1]

$$R \to (R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R)^{tC_p} = (\otimes_{\mathbb{S}}^{C_p} R)^{tC_p},$$

(which exists for every spectrum) followed by the map $(\otimes_{\mathbb{S}}^{C_p} R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}$ obtained from the inclusion $C_p \subset \mathbb{T}$.

[†] One could also formulate the universal property in terms of animated (or simplicial) commutative k-algebras, without using the language of $\mathbb{E}_\infty$-rings.

The map φ is called the cyclotomic Frobenius and plays a central role in the theory. Its construction depends crucially on working over the sphere spectrum, and is thus a feature of THH that does not exist for ordinary Hochschild homology: by universal properties of the ∞-category of spectra, one shows [105, Section III.1] that there is a canonical, lax symmetric monoidal natural transformation, called the *Tate diagonal*,

$$X \to (X \otimes_{\mathbb{S}} X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p} \tag{5}$$

for any spectrum X. Indeed, the Tate diagonal is roughly analogous to (and refines) the map defined for every abelian group A,

$$A \to H^0_{\mathrm{Tate}}(C_p, A^{\otimes p}) = (A^{\otimes p})^{C_p}/\mathrm{norms},$$

given by the formula $a \mapsto a \otimes a \otimes \cdots \otimes a$. The analog of the map (5) does not exist in $\mathcal{D}(\mathbb{Z})$, the derived ∞-category of the integers $\mathbb{Z}$, which lacks the analogous universal property, and is a key reason why THH yields a richer theory.

In the work [105], it is shown that φ is enough to study the so-called (p-typical) 'cyclotomic structure' on the p-completion $\mathrm{THH}(R;\mathbb{Z}_p)$; in particular, it can be used to define the topological cyclic homology.

Construction 2.4 (Topological cyclic homology). Let R be a ring (or more generally a connective $\mathbb{E}_\infty$-ring). We define $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}}$ and $\mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}$ to be the $\mathbb{T}$-homotopy fixed points and Tate construction, respectively. We have two maps

$$\mathrm{can}, \varphi : \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p),$$

where can is the canonical map from $\mathbb{T}$-invariants to the $\mathbb{T}$-Tate construction, and φ is obtained by taking $\mathbb{T}$-invariants from (4) and using the identification $\mathrm{TP}(R;\mathbb{Z}_p) = (\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p})^{h\mathbb{T}/C_p}$, cf. [105, Lemma II.4.2]. In particular, since can identifies $\pi_0\mathrm{TC}^-(R;\mathbb{Z}_p)$ and $\pi_0\mathrm{TP}(R;\mathbb{Z}_p)$, we can regard φ as an endomorphism of the ring $\pi_0\mathrm{TC}^-(R;\mathbb{Z}_p)$. The spectrum $\mathrm{TC}(R;\mathbb{Z}_p)$ is the homotopy equalizer

$$\mathrm{TC}(R;\mathbb{Z}_p) = \mathrm{fib}(\varphi - \mathrm{can}) : \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p). \tag{6}$$

The expression (6) is very different from ones that appear in the more classical approach to THH. It plays an essential role in the work [30], and has many applications both structural and computational. For example, it implies the following basic structural feature of TC: for connective ring spectra, the construction TC/p commutes with filtered colimits [43, Theorem G].

By contrast, in the classical approach to THH and cyclotomic spectra via equivariant stable homotopy theory (cf. [98] for a survey), the objects TC^-, TP do not play a direct role. One constructs the structure of a genuine C_{p^n}-spectrum on $\mathrm{THH}(R)$, which enables one to form various fixed points $\mathrm{THH}(R)^{C_{p^n}}$, $n \geqslant 0$, together with maps

$$R, F : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n-1}}}, \quad V : \mathrm{THH}(R)^{C_{p^{n-1}}} \to \mathrm{THH}(R)^{C_{p^n}}.$$

The various fixed points are related to each other inductively using the cofiber sequences

$$\mathrm{THH}(R)_{hC_{p^n}} \to \mathrm{THH}(R)^{C_{p^n}} \xrightarrow{R} \mathrm{THH}(R)^{C_{p^{n-1}}}.$$

In particular, one forms the inverse limit

$$\mathrm{TR}(R) = \varprojlim_{R} \mathrm{THH}(R)^{C_{p^n}},$$

which is a connective $\mathbb{E}_\infty$-ring. The maps F, V act on $\mathrm{TR}(R)$, and $\mathrm{TC}(R;\mathbb{Z}_p)$ is the equalizer $\mathrm{fib}(F-1 : \mathrm{TR}(R;\mathbb{Z}_p) \to \mathrm{TR}(R;\mathbb{Z}_p))$. A major advance of the work [105] is the insight that much of the 'gluing' data that lead to the construction of the fixed points $\mathrm{THH}(R)^{C_{p^n}}$ is actually redundant (and is not needed to construct TC in particular). On the other hand, TR has recently been used to give an entirely new formulation (and 'decompletion') of the theory of cyclotomic spectra via the theory of topological Cartier modules due to Antieau–Nikolaus [10]; this has many advantages, including that it yields a natural t-structure on cyclotomic spectra.

The most fundamental calculation in topological Hochschild homology is that of $\mathbb{F}_p$.

Theorem 2.5 (Bökstedt). *We have* $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$ *with* $|\sigma| = 2$.

For a discussion of a proof of Theorem 2.5 (and in particular the multiplicative structure, which is crucial to everything that follows), cf. [73]. In particular, Theorem 2.5 is closely related to the result of Hopkins–Mahowald that the free $\mathbb{E}_2$-algebra over $\mathbb{S}$ with $p = 0$ is $H\mathbb{F}_p$. Moreover, using Bökstedt's theorem, one can in fact give a complete description of $\mathrm{THH}(\mathbb{F}_p)$ as a cyclotomic spectrum, cf. [105, Section IV-2].

It is instructive to compare Bökstedt's theorem with the calculation of $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})$. One sees easily (for example, using the Hochschild–Kostant–Rosenberg theorem, cf. [105, Proposition IV.4.3]) that $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z})$ is the divided power algebra $\Gamma^*[\sigma]$ (for the same class σ in degree two); in particular, replacing $\mathbb{Z}$ by $\mathbb{S}$ replaces the divided power algebra by the polynomial algebra.

For any $\mathbb{F}_p$-algebra R, one has the formula

$$\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{F}_p)} \mathbb{F}_p \simeq \mathrm{HH}(R/\mathbb{F}_p).$$

Given the above description of $\mathrm{THH}_*(\mathbb{F}_p)$, one may view $\mathrm{THH}(R)$ as a 'one-parameter deformation' of $\mathrm{HH}(R/\mathbb{F}_p)$ along σ. Upon taking circle-fixed points and passing to associated graded, this observation is ultimately connected to the fact that crystalline cohomology gives a one-parameter deformation of de Rham cohomology in characteristic p (along the parameter given by 'p').

Connections between THH and arithmetic have been explored in [57, 65, 71, 72], which relate the homotopy groups of the fixed points of THH to (various forms) of the de Rham–Witt complex. The first such result, in equal characteristic, gives a complete calculation of TR in the case of a regular $\mathbb{F}_p$-algebra:

Theorem 2.6 (Hesselholt [65]). *Let R be a regular $\mathbb{F}_p$-algebra.*[†] *Then there is an isomorphism* $\mathrm{TR}(R;\mathbb{Z}_p)_* \simeq W\Omega_R^*$, *where $W\Omega_R^*$ is the de Rham–Witt complex of Bloch–Deligne–Illusie [78]. This isomorphism carries the operators F, V on* $\mathrm{TR}(R;\mathbb{Z}_p)$ *to the similarly named operator F, V on* $W\Omega_R^*$.

[†] Recall that a regular $\mathbb{F}_p$-algebra is ind-smooth, by Néron–Popescu desingularization.

In particular, one obtains a complete calculation of TC using the fixed points of operator F on the de Rham–Witt forms. In mixed characteristic, [72] introduces analogs of the de Rham–Witt complex, and [57, 71, 72] discuss the connections between TR of smooth algebras in mixed characteristic and the de Rham–Witt complex. This has been used in [72] to verify the Lichtenbaum–Quillen conjecture for certain p-adic fields (prior to the general proof by Voevodsky–Rost). See also [93] for a new approach to this calculation inspired by the methods of [30].

3 | THE COTANGENT COMPLEX AND ITS WEDGE POWERS

The building blocks of all the constructions involved are the cotangent complex and its wedge powers; these are the 'animations' (for our purposes, left Kan extensions) of the usual differential forms functors. We begin with a brief review. Fix a base ring k.

Construction 3.1 (Left Kan extension). Let C be an ∞-category admitting sifted colimits. Let Poly_k be the category of finitely generated polynomial k-algebras, and let $F : \mathrm{Poly}_k \to C$ be a functor. Then one can construct the *left Kan extension* or *left derived functor* $LF : \mathrm{Ring}_k \to C$, extending the functor F to Ring_k. Explicitly, we define LF on all polynomial k-algebras (possibly on infinitely many variables) by forcing LF to commute with filtered colimits. Given an arbitrary k-algebra R, we can choose a simplicial resolution $P_\bullet \to R$ where each P_i is a polynomial k-algebra, and then $LF(R) = |F(P_\bullet)|$.

The above construction (in various forms classical, going back to Quillen) is a type of nonabelian left derived functor [95, Section 5.5.8]. More generally, we can express the above construction using the theory of animated rings. Let $\mathrm{Ani}(\mathrm{Ring}_k)$ be the ∞-category of animated k-algebras (also called simplicial commutative k-algebras; we refer to [41] for a discussion of this terminology). Then with C as above, one has an equivalence of ∞-categories $\mathrm{Fun}_\Sigma(\mathrm{Ani}(\mathrm{Ring}_k), C) \simeq \mathrm{Fun}(\mathrm{Poly}_k, C)$ between sifted-colimit preserving functors $\mathrm{Ani}(\mathrm{Ring}_k) \to C$ and functors $\mathrm{Poly}_k \to C$.

Definition 3.2 (The cotangent complex and its wedge powers). Let R be a k-algebra. Then the *cotangent complex* $L_{R/k} \in \mathcal{D}(R)$ is defined as the left derived functor of the functor $R \mapsto \Omega^1_{R/k}$ of Kähler differentials.[†] Similarly, the wedge power $\bigwedge^i L_{R/k} \in \mathcal{D}(R)$ (for $i \geqslant 0$) is defined as the derived functor of the functor $\bigwedge^i_R \Omega_{R/k}$ of differential i-forms; this can also be defined using the Dold–Puppe nonabelian derived exterior powers $\bigwedge^i : \mathcal{D}(R)^{\leqslant 0} \to \mathcal{D}(R)^{\leqslant 0}$ (cf. [97, Section 25.2.1] for a modern account) applied to the cotangent complex.

We refer to [114, Tag 08P5] for a comprehensive treatment of the cotangent complex. A basic fact about the cotangent complex is that it agrees with ordinary differential forms not only for polynomial k-algebras, but more generally for smooth k-algebras. The other fundamental tools are the transitivity sequence for a sequence of ring maps $A \to B \to C$, which yields a cofiber sequence

[†] In principle, the functor $L_{\cdot/k}$ takes values in $\mathcal{D}(k)$ as the argument varies, but with more effort (for example, using the ∞-category of pairs of an animated ring and a module over it) one can construct the functor as stated.

in $\mathcal{D}(C)$,

$$L_{B/A} \otimes_B^L C \to L_{C/A} \to L_{C/B}, \tag{7}$$

and the base-change property $L_{B/A} \otimes_A A' = L_{B\otimes_A A'/A'}$ for a map $A \to A'$ of k-algebras such that B, A' are Tor-independent over A (if they are not Tor-independent, one has to consider the derived tensor product $B \otimes_A^L A'$ as an animated ring itself).

Example 3.3. Using the cofiber sequence and base-change, we find that if $B = A/r$ for $r \in A$ a nonzerodivisor, then $L_{B/A} \simeq (r)/(r^2)[1]$.

Example 3.4. More generally, suppose $B = A/I$ for $I \subset A$ an ideal generated by a regular sequence; this (and its generalizations) will be one of the primary examples for us. Then $L_{B/A} = I/I^2[1]$, which is the suspension of a free B-module (cf. [114, Tag 08SH], [76, III.3.2]) A consequence is that $\bigwedge^i L_{B/A} = \Gamma^i(I/I^2)[i]$, for Γ^i the ith divided power functor on flat B-modules. This is a consequence of the décalage isomorphism of [76, Section I.4.3.2] between $\bigwedge^i(M[1]) = (\Gamma^i M)[i]$ for any $M \in \mathcal{D}(B)^{\leqslant 0}$.

Proposition 3.5. *Let R be a perfect $\mathbb{F}_p$-algebra. Then $L_{R/\mathbb{F}_p} = 0$.*

Proof. For every $\mathbb{F}_p$-algebra S, the Frobenius $S \to S$ induces zero on $L_{S/\mathbb{F}_p}$; this follows by inspection in the case of a polynomial $\mathbb{F}_p$-algebra and then follows in general by taking simplicial resolutions. The claim follows. □

The key structural result in [30] used in defining the motivic filtrations is the flat descent for the cotangent complex and its wedge powers. This was originally observed by Bhatt [23] and is treated further in [30, Section 3].

Theorem 3.6 [23, 30]. *Let k be a base ring. Then the construction $A \mapsto \bigwedge^i L_{A/k}$, as a functor from k-algebras to $\mathcal{D}(k)$, satisfies flat descent. More generally, for any k-module M, the construction $A \mapsto \bigwedge^i L_{A/k} \otimes_k M$ satisfies flat descent.*

This result is proved using the transitivity cofiber sequence for the cotangent complex, flat descent for modules, and an inductive argument on the degree i. As pointed out in [30], it remains open whether the above functors are flat hypersheaves.

4 | QUASI-SYNTOMIC RINGS

A key insight in [30] is that to understand invariants such as THH, it is extremely clarifying to work with very 'large' (for example, perfectoid) rings. In particular, one should make highly ramified extensions by adding lots of p-power roots. This strategy is expressed using the quasi-syntomic topology, a key construction of [30, Section 4]; all the filtrations of [30, Section 4] are defined on quasi-syntomic rings, and probably cannot be defined more generally (without sacrificing convergence properties). This class of rings is also extremely useful in other contexts, including in the prismatic Dieudonné theory of Anschütz–Le Bras [5].

Definition 4.1 (Quasi-syntomic rings). A ring R is *quasi-syntomic* if:

(1) R is p-complete, and the p-power torsion in R is bounded, that is, annihilated by p^N for $N \gg 0$;
(2) the cotangent complex $L_{R/\mathbb{Z}_p} \in \mathcal{D}(R)$ has the property that $L_{R/\mathbb{Z}_p} \otimes_R^L (R/p) \in \mathcal{D}(R/p)$ has Tor-amplitude in $[-1, 0]$.

We let QSyn denote the category of quasi-syntomic rings.

Example 4.2 (Complete intersections). Let $(R, \mathfrak{m})$ be a p-complete local Noetherian ring with $p \in \mathfrak{m}$. Then R is quasi-syntomic if R is a complete intersection, that is, if for any (or one) surjection $f : A \twoheadrightarrow \widehat{R}$ with A complete regular local, the kernel of f is generated by a regular sequence. Indeed, a result of Avramov [15] states that R is a complete intersection if and only if $L_{R/\mathbb{Z}_p}$ has Tor-amplitude in $[-1, 0]$. Therefore, if R is a complete intersection, then R is clearly quasi-syntomic.

The idea is that quasi-syntomic rings are those which behave like local complete intersections at the level of the cotangent complex (which is enough to control all Hochschild-type invariants). However, this class of rings includes many highly non-Noetherian examples.

Example 4.3 (Perfect rings). Any perfect $\mathbb{F}_p$-algebra R (that is, one where the Frobenius is an isomorphism) is quasi-syntomic. In fact, we have that $L_{R/\mathbb{F}_p} = 0$ as we saw in Proposition 3.5; the transitivity cofiber sequence applied to $\mathbb{Z}_p \to \mathbb{F}_p \to R$ thus implies that $L_{R/\mathbb{Z}_p}$ is the suspension of a rank 1 free R-module.

Example 4.4 (Witt vectors of perfect rings). If R is a perfect $\mathbb{F}_p$-algebra, then the ring of Witt vectors $W(R)$ is quasi-syntomic. In fact, $W(R)$ is p-torsionfree and $W(R)/p \simeq R$; one thus obtains that $L_{W(R)/\mathbb{Z}_p}$ vanishes p-adically, whence the claim.

The class of perfect $\mathbb{F}_p$-algebras admits a remarkable generalization to mixed characteristic, namely the class of integral perfectoid rings [29, Section 3.2] (based on the notion of perfectoid Tate ring introduced in [49, 82, 112]).

Definition 4.5 (Perfectoid rings). A p-adically complete ring R is called *perfectoid* if R can be expressed as the quotient $W(R')/\xi$, where R' is a perfect $\mathbb{F}_p$-algebra, and $\xi \in W(R')$ is an element of the form $[a] + pu$ where $u \in W(R')$ is a unit and $a \in R'$; here $[\cdot]$ denotes the Teichmüller lift.†

In the above, by replacing R' by its a-adic completion, which does not change the quotient $W(R')/([a] + pu)$, we may in fact assume that R' is a-adically complete. Note that this in particular implies that $R/[a]$ is an $\mathbb{F}_p$-algebra, and the Frobenius induces an isomorphism of $\mathbb{F}_p$-algebras, $R/[a]^{1/p} \xrightarrow{\sim} R/[a]$. This is in fact the essential feature of perfectoid rings:

Proposition 4.6 [29, Lemma 3.10]. *Let R be a ring such that there exists a nonzero divisor $\omega \in R$ such that:*

(1) *R is ω-adically complete;*
(2) *$\omega^p \mid p$;*

† Such elements ξ are also called *primitive* or *distinguished*; in the terminology of [32], a perfectoid ring R is the same data as the *perfect prism* $(W(R'), (\xi))$.

(3) *the Frobenius induces an isomorphism* $R/\omega \xrightarrow{\sim} R/\omega^p$.

Then R is perfectoid. Conversely, if R is perfectoid, then there exists an element ω such that $\omega^p \mid p$, and for any such element, the Frobenius map $R/\omega \to R/\omega^p$ is an isomorphism.

Remark 4.7. Let $R = W(R')/\xi$ be a perfectoid ring with $\xi = [a] + pu$. By a-adically completing R' if necessary, we may assume that R' is a-adically complete. In this case, R' is the *tilt* $R^\flat$ of R, namely, $R' = \varprojlim_{\varphi} R/p$. In fact, $R/p = R'/a$, and for any perfect $\mathbb{F}_p$-algebra S which is x-adically complete, we see that S agrees with the inverse limit perfection of S/x.

Remark 4.8. A perfectoid ring R is quasi-syntomic. Indeed, the p-complete cotangent complex $L_{R/\mathbb{Z}_p} = L_{R/W(R^\flat)}$ is the suspension of a free R-module of rank 1, since $W(R^\flat) \to R$ is the quotient by a nonzero divisor.

Example 4.9. The p-adic completion of the ring $\mathbb{Z}_p[p^{1/p^\infty}]$ is perfectoid. In fact, this ring can be written as the quotient of

$$W(\mathbb{F}_p[t^{1/p^\infty}])/([t] - p) = \left(\mathbb{Z}_p[u^{1/p^\infty}]\right)_{\hat{p}}/(u - p).$$

More generally, let R be a p-torsionfree, p-adically complete $\mathbb{Z}_p[p^{1/p^\infty}]$-algebra. Then R is perfectoid if and only if the Frobenius induces an isomorphism $\varphi : R/p^{1/p} \xrightarrow{\sim} R/p$.

Example 4.10. The ring $\mathbb{Z}_p[\zeta_{p^\infty}]_{\hat{p}}$ is perfectoid. In this case, we can form the ring $\mathbb{Z}_p[q^{1/p^\infty}]_{\widehat{(p,q-1)}} = W(\mathbb{F}_p[\epsilon^{1/p^\infty}]_{\widehat{(\epsilon-1)}})$ (via $q = [\epsilon]$) and the element $[p]_q := \frac{q^p-1}{q-1} = 1 + q + \cdots + q^{p-1}$. By considering the map $\mathbb{F}_p[\epsilon^{1/p^\infty}]_{\widehat{(\epsilon-1)}} \to \mathbb{F}_p, \epsilon \mapsto 1$, one checks that the coefficient of p in the Teichmüller expansion is a unit. Thus, we can take $R' = \mathbb{F}_p[\epsilon^{1/p^\infty}$ and $\xi = [p]_q$.

The original definition of a *perfectoid field* was given in [112]: a perfectoid field K is a complete nonarchimedean field K with ring of integers $\mathcal{O}_K \subset K$ such that the valuation of K is nondiscrete, p is topologically nilpotent, and the Frobenius on $\mathcal{O}_K/p$ is surjective. This implies that there exists a nonzero topologically nilpotent element $\omega \in \mathcal{O}_K$ with $\omega^p \mid p$ and such that the Frobenius induces an isomorphism $\mathcal{O}_K/\omega \xrightarrow{\sim} \mathcal{O}_K/\omega^p$. The datum of a perfectoid field is equivalent to the datum of a complete, rank 1 valuation ring which is perfectoid. The above two examples arise in this manner.

To obtain more examples of perfectoid rings, note that if R is a perfectoid ring, then the p-completion $R\langle t^{1/p^\infty}\rangle$ of $R[t^{1/p^\infty}]$ is perfectoid. More subtly, there is the construction of the 'perfectoidization' of a semiperfectoid ring. If R is a perfectoid ring and $I \subset R$ is a p-complete ideal, then there is a p-complete ideal $J \supset I$ such that R/J is perfectoid, and is the universal perfectoid ring to which R/I maps (cf. [32, Theorem 7.4]). This construction is not easy to describe explicitly in general. But if $I = (f)$ for $f \in R$ admitting a system of p-power roots $\{f^{1/p^n}\}_{n\geqslant 0}$, then J is the p-completion of $\bigcup_{n\geqslant 0}(f^{1/p^n})$.

We next review the quasi-syntomic topology on QSyn, cf. [30, Definition 4.1, Corollary 4.8]. This is a non-Noetherian version of the syntomic topology, cf. [50] or [114, Tag 0224], and in the p-complete context. Strictly speaking, it is QSyn^{op} that has the structure of a site.

Definition 4.11 (The quasi-syntomic site). A map $R \to R'$ in QSyn is a cover if:

(1) $R/p^n \to R'/p^n$ is faithfully flat for all $n \geqslant 0$;
(2) $L_{R'/R} \otimes^L_{R'} R'/p \in \mathcal{D}(R'/p)$ has Tor-amplitude in $[-1, 0]$.

The condition (1) is called *p-complete faithful flatness*, and is the appropriate replacement for faithful flatness in this (highly non-Noetherian) setup. Note that if R is Noetherian, then the condition (1) is simply faithful flatness due to [122].

Example 4.12 (Adding systems of p-power roots). Given a collection of elements $\{x_t \in R\}$, the ring R' obtained as the p-completion of $R[u_t^{1/p^\infty}, t \in T]/(u_t - x_t)$, that is, obtained by p-completely adding a system of p-power roots of the elements x_t, gives a cover of R in the quasi-syntomic topology. Iterating this construction transfinitely many times, one sees that every object of QSyn can be covered by an object where all elements admit compatible systems of p-power roots.

Example 4.13 (Covers of regular rings). Let R be a p-complete, regular noetherian ring. Then there is a quasisyntomic cover $R \to R_\infty$, with R_∞ perfectoid. Conversely, a p-complete Noetherian ring admitting such a cover is regular. This is proved in [27, Theorem 4.7].

Example 4.14 (p-complete valuation rings are quasisyntomic). This follows by a result of Gabber–Ramero [53, Theorem 6.5.8]. Moreover, if V is a valuation ring over $\mathbb{F}_p$, then $L_{V/\mathbb{F}_p}$ is a flat V-module.

An important general structural result for perfectoid rings, formulated in terms of the quasisyntomic site, is that locally one can add solutions to polynomial equations. This is highly non-trivial, since there is no obvious way to add such solutions while retaining the perfectoid property. For this result, compare [1, Section 2.5], [54, Theorem 16.9.17], [32, Theorem 7.12], and [41, Theorem 2.3.4].

Theorem 4.15 (André's lemma). *Let R be a perfectoid ring. Then there exists a map of perfectoid rings $R \to R_\infty$ such that:*

(1) *R_∞ is absolutely integrally closed, that is, every monic polynomial equation over R_∞ has a root in R_∞;*
(2) *$R \to R_\infty$ is a cover in* QSyn*. In fact, $R \to R_\infty$ can be taken to be the p-completion of an ind-syntomic map.*

Definition 4.16 (Quasi-regular semiperfectoid rings). A quasi-syntomic ring R is said to be *quasi-regular semiperfectoid* (or *quasi-regular semiperfect* if R is additionally an $\mathbb{F}_p$-algebra) if either of the following equivalent conditions hold.

(1) R receives a surjection from a perfectoid ring.
(2) R/p is semiperfect (that is, the Frobenius is surjective), and R receives a map from a perfectoid ring R_0.

To see that (2) implies (1), consider the map $\theta : W(R^\flat) \to R$ for $R^\flat$ the inverse limit perfection of R/p; this is surjective modulo p, hence surjective since R is p-complete. The extension

$R_0\hat{\otimes}_{\mathbb{Z}_p}W(R^\flat) \to R$ is also therefore surjective, and the source is perfectoid, whence (1). Note also that (1) implies (2) because the reduction mod p of a perfectoid ring is semiperfect.

We denote by QRSPerfd the category of quasi-regular semiperfectoid rings, equipped with the induced site structure. The subcategory QRSPerfd $\subset$ QSyn is a *basis* for the quasi-syntomic site: any object of QSyn admits a cover by an object of QRSPerfd. Moreover, the tensor product of two rings in QRSPerfd remains in QRSPerfd.

Remark 4.17. Suppose that R is a quasi-regular semiperfectoid ring. In this case, $L_{R/\mathbb{Z}_p}$ is the suspension of a p-completely flat R-module.

Heuristically quasi-syntomic rings are those which behave like lci rings, at least at the level of the cotangent complex (and after p-completion). We also discuss a class of $\mathbb{F}_p$-algebras which behave more like smooth algebras.

Definition 4.18 (Cartier smooth $\mathbb{F}_p$-algebras, [83]). Let R be an $\mathbb{F}_p$-algebra. We say that R is *Cartier smooth* if:

(1) the cotangent complex $L_{R/\mathbb{F}_p}$ is a flat R-module in degree zero;
(2) for each i, the inverse Cartier operator $C^{-1} : \Omega^i_{R/\mathbb{F}_p} \to H^i(\Omega^*_{R/\mathbb{F}_p})$ is an isomorphism. Here the inverse Cartier operator is the unique map of graded algebras $\Omega^*_{R/\mathbb{F}_p} \to H^*(\Omega^*_{R/\mathbb{F}_p})$ carrying $r \in R$ to the class of r^p and $ds, s \in R$ to the class of $s^{p-1}ds$. Compare [28, Proposition 3.3.4].

Example 4.19.

(1) Any smooth algebra over a perfect field (or more generally a perfect $\mathbb{F}_p$-algebra) is Cartier smooth, due to the classical Cartier isomorphism (cf. [81, Theorem 7.2] for an account).
(2) Any regular Noetherian $\mathbb{F}_p$-algebra is Cartier smooth. Indeed, this follows because Cartier smooth algebras are closed under filtered colimits and any regular Noetherian $\mathbb{F}_p$-algebra is ind-smooth by Néron–Popescu desingularization. However, one can also prove this claim directly, [28, Section 9.5].
(3) Any valuation ring over $\mathbb{F}_p$ is Cartier smooth. This follows from results of Gabber–Ramero [53, Theorem 6.5.8] and Gabber [84, Appendix A]. Conjecturally (by local uniformization, a weak form of resolution of singularities) valuation rings over $\mathbb{F}_p$ are ind-smooth, which would imply Cartier smoothness, but local uniformization is not known in general.
(4) A collection of elements $\{x_i\}_{i\in I}$ in an $\mathbb{F}_p$-algebra R is a *p-basis* if the elements $\prod_{i\in I} x_i^{a_i} \in R$, as $\{a_i\}_{i\in I}$ ranges over all finitely supported functions $I \to \{0, 1, \dots, p-1\}$, forms a basis for R as a module over itself via the Frobenius map. If R admits a p-basis, then R is Cartier smooth, cf. [28, Theorem 9.5.21] and its proof.

We refer to [83, 84] for some applications of the theory of Cartier smooth algebras. In particular, in [83, 84] it is shown that the calculation [55, 58] of the p-adic K-theory and topological cyclic homology of regular local $\mathbb{F}_p$-algebras also generalizes to local Cartier smooth $\mathbb{F}_p$-algebras (for example, valuation rings).

Remark 4.20. We do not know if the condition of Cartier smoothness guarantees that the Frobenius endomorphism is flat. By a classical theorem of Kunz (see [86] or [114, Tag 0EC0]), a

Noetherian $\mathbb{F}_p$-algebra is regular if and only if the Frobenius endomorphism is flat. All the above examples of Cartier smooth algebras have the property that the Frobenius is flat.

On the other hand, condition (2) in the definition of Cartier smoothness is definitely not implied by condition (1). For instance, there exist semiperfect $\mathbb{F}_p$-algebras R such that $L_{R/\mathbb{F}_p} = 0$ but such that R is not perfect; compare [22] for an example. In this case, the inverse Cartier operator reproduces the Frobenius $\varphi : R \to R$ in degree zero, which is not an isomorphism.

Question 4.21. Is there an analog of Cartier smoothness for arbitrary quasi-syntomic rings?

5 | SOME QUASI-SYNTOMIC SHEAVES

Throughout, we use the language of sheaves of spectra [97, Section 1.3] (this was implicitly used in the formulation of Theorem 3.6). Note that this is slightly more general than the theory introduced by Jardine [79], which corresponds to the subcategory of hypercomplete sheaves, cf. [46]. However, all the sheaves used in the constructions of [30] will be shown to be hypercomplete (this is a convenient feature of the quasi-syntomic site), so the distinction does not play a significant role in [30].

Definition 5.1 (Sheaves on QSyn). A spectrum-valued sheaf on QSyn is a functor F : QSyn → Sp such that:

(1) F preserves finite products;
(2) if $A \to B$ is a cover in QSyn, then the natural map $F(A) \to \varprojlim(F(B) \rightrightarrows F(B\hat{\otimes}_A B) \substack{\to\\\to\\\to} \dots)$ is an equivalence.

We let Shv(QSyn, Sp) denote the ∞-category of sheaves of spectra.

Definition 5.2 (Hypercomplete sheaves). We will say that a sheaf of spectra $F \in$ Shv(QSyn, Sp) is *hypercomplete* if F satisfies descent for hypercovers in the quasi-syntomic topology (rather than only for Čech covers as above). We let $\mathrm{Shv}_{\mathrm{hyp}}(\mathrm{QSyn}, \mathrm{Sp}) \subset \mathrm{Shv}(\mathrm{QSyn}, \mathrm{Sp})$ denote the subcategory of hypercomplete sheaves; this inclusion is the right adjoint of a Bousfield localization $(-)^h$: $\mathrm{Shv}(\mathrm{QSyn}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{hyp}}(\mathrm{QSyn}, \mathrm{Sp})$ called hypercompletion.

The presentable, stable ∞-category Shv(QSyn, Sp) admits a canonical t-structure (as sheaves on any site do). A Sp-valued sheaf $\mathcal{F}$ on QSyn is *connective* if for every $A \in$ QSyn and $x \in \pi_j(\mathcal{F}(A))$ for $j < 0$, there exists a quasi-syntomic cover $A \to B$ such that x is carried to zero in $\pi_j(\mathcal{F}(B))$. Similarly, $\mathcal{F}$ is *coconnective* if it takes values in coconnective spectra. The t-structure restricts to a t-structure on the hypercomplete sheaves, and every bounded-above sheaf is automatically hypercomplete.[†] With respect to this t-structure, the heart of Shv(QSyn, Sp) is the ordinary category of sheaves of abelian groups on QSyn.

Construction 5.3 (Postnikov towers). Given any $\mathcal{F} \in$ Shv(QSyn, Sp), we have its Postnikov tower $\{\mathcal{F}_{\leqslant n}\}_{n\in\mathbb{Z}}$ with respect to the above t-structure. The limit of this Postnikov tower is given by its hypercompletion $\mathcal{F}^h$. This is a consequence of the fact that the quasi-syntomic site is 'replete' in

[†] The hypercomplete sheaves are those sheaves which receive no maps from ∞-connected sheaves.

the sense of [31, Section 3]; compare [100, Proposition A.10]. In particular, if $\mathcal{F}$ is already hypercomplete, then $\mathcal{F}$ is the limit of its Postnikov tower.

A basic tool for working with sheaves on QSyn is restriction to the basis $\mathrm{QRSPerfd} \subset \mathrm{QSyn}$. In general, given a Grothendieck site, then it is a classical result [12, Exp. III, Theorem 4.1] that sheaves of sets or abelian groups are equivalent to sheaves on any basis of the site. The analog need not hold for sheaves of spaces or spectra, but it at least holds for hypercomplete sheaves in general, cf. [11, Appendix A] or [19, Proposition 3.12.11]. In the case of $\mathrm{QRSPerfd} \subset \mathrm{QSyn}$, it is actually true that arbitrary sheaves on QSyn identify with sheaves on the basis QRSPerfd, cf. [30, Proposition 4.31] or [75, Lemma C.3] (for a more general statement); the main point is that a pushout $B\widehat{\otimes}_A C$ in QSyn along quasi-syntomic covers with $B, C \in \mathrm{QRSPerfd}$ belongs to QRSPerfd. Note that this strategy of restricting to $\mathrm{QRSPerfd} \subset \mathrm{QSyn}$ is useful precisely because we are working with such 'infinitary' sites; it would be much less useful if we worked with more classical sites such as the syntomic or fppf site.

Now we discuss some examples of sheaves on QSyn.

Example 5.4. For each $i \geqslant 0$, the construction $R \mapsto \bigwedge^i L_{R/\mathbb{Z}_p}[-i]$ defines a sheaf of spectra on QSyn (due to Theorem 3.6, with a slight modification since we are working with p-completely faithful flatness). This sheaf belongs to the heart (so thus corresponds to a sheaf of ordinary abelian groups). In fact, this follows because it takes discrete values on the quasi-regular semiperfectoid rings.

Theorem 5.5. *The functors* $\mathrm{HH}(-;\mathbb{Z}_p), \mathrm{HH}(-;\mathbb{Z}_p)^{h\mathbb{T}}, \mathrm{HH}(-;\mathbb{Z}_p)^{t\mathbb{T}}$, $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, $\mathrm{TP}(-;\mathbb{Z}_p), \mathrm{TC}(-;\mathbb{Z}_p)$*, etc., all define hypercomplete sheaves on* QSyn.

In fact, all of these functors define (a priori not hypercomplete) sheaves on the (p-completely) flat topology on all rings, as in [30, Section 3]; for quasi-syntomic rings the argument shows that they are hypersheaves. One uses the Hochschild–Kostant–Rosenberg filtration [105, Proposition IV.4.1] to prove that $\mathrm{HH}(-;\mathbb{Z}_p)$ is a hypercomplete sheaf on QSyn starting from the fact that the p-complete cotangent complex and its wedge powers are sheaves on QSyn. Taking homotopy fixed points, we find that $\mathrm{HH}(-;\mathbb{Z}_p)^{h\mathbb{T}}$ is a hypercomplete sheaf. Similarly, using $\mathrm{THH}(-;\mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} = \mathrm{HH}(-;\mathbb{Z}_p)$ and taking the limit of the Postnikov tower of $\mathrm{THH}(\mathbb{Z})$, one bootstraps to $\mathrm{THH}(-;\mathbb{Z}_p)$ and the invariants defined from it.

6 | THE MOTIVIC FILTRATIONS OF [30]

To begin with, we describe the Hochschild–Kostant–Rosenberg filtration on Hochschild homology using the quasi-syntomic site.

Construction 6.1 (The Hochschild–Kostant–Rosenberg filtration). For a ring R and any R-algebra A, there is a functorial, complete multiplicative $\mathbb{Z}_{\geqslant 0}$-indexed descending filtration $\mathrm{Fil}^{\geqslant *}_{\mathrm{HKR}}\mathrm{HH}(A/R)$ on $\mathrm{HH}(A/R)$ with $\mathrm{gr}^i\mathrm{HH}(A/R) = \bigwedge^i L_{A/R}[i]$. This filtration is the Postnikov filtration when A is a polynomial algebra over R (using the Hochschild–Kostant–Rosenberg theorem to identify the graded pieces), and is more generally defined via left Kan extension, cf. [105, Proposition IV.4.1]. A universal property of this filtration has been given by Raksit [107].

The Hochschild–Kostant–Rosenberg filtration is the prototype of the motivic filtrations of [30]. However, the strategy is to define the filtration by descent from quasi-regular semiperfectoids, that is, by a right Kan extension process rather than a left Kan extension process. These filtrations will generally be more complicated to construct directly for polynomial algebras. To begin with, we show that the HKR filtration can be obtained for quasi-syntomic rings in such a fashion, after p-completion.

Construction 6.2 (The Hochschild–Kostant–Rosenberg filtration as a Postnikov filtration). Suppose R is a quasi-syntomic ring. On the category of quasi-syntomic R-algebras, we consider the functor $A \mapsto \mathrm{HH}(A/R;\mathbb{Z}_p)$, which defines a hypercomplete sheaf of spectra. We claim that the homotopy sheaves are concentrated in even degrees, and that $A \mapsto \mathrm{Fil}^{\geqslant *}_{\mathrm{HKR}}\mathrm{HH}(A/R;\mathbb{Z}_p)$ defines the double-speed Postnikov tower. In other words, $\mathrm{Fil}^{\geqslant i}_{\mathrm{HKR}}$ is the $2i$th connective cover in quasi-syntomic sheaves. This follows easily from the observation that if A/R is such that the p-completion $L_{A/R}$ is the suspension of a p-completely flat R-module, then $\mathrm{gr}^i\mathrm{HH}(A/R;\mathbb{Z}_p)$ is concentrated in degree $2i$ (for example, if A is a quasiregular semiperfectoid R-algebra), and the HKR filtration reduces to the double-speed Postnikov filtration on the individual spectrum $\mathrm{gr}^i\mathrm{HH}(A/R;\mathbb{Z}_p)$.

The starting point of the extension of the above strategy to invariants defined from THH is the following generalization of Bökstedt's theorem.

Theorem 6.3. *Let R be a perfectoid ring. Then one has an isomorphism $\mathrm{THH}_*(R;\mathbb{Z}_p) \simeq R[\sigma]$, for $|\sigma| = 2$. Moreover, one has $\pi_*(\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}) = R[u^{\pm 1}]$ for $|u| = 2$, and the Frobenius φ : $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$ exhibits the source as the connective cover of the target.*

Theorem 6.3 reduces to Bökstedt's theorem for $R = \mathbb{F}_p$, and is extended to an arbitrary perfectoid ring in [30, Section 6]. The case of $R = \mathcal{O}_{\mathbb{C}_p}$ had been previously proved in [67]. See also [73, Section 1.3] for an account of this result. The basic strategy is to bootstrap from the case $R = \mathbb{F}_p$, using the Hochschild–Kostant–Rosenberg theorem and that for any map of perfectoid rings $R \to R'$, the p-completed relative cotangent complex $L_{R'/R}$ vanishes.

In [73, Section 1.3], the constructions $\mathrm{TC}^-_*(R;\mathbb{Z}_p), \mathrm{TP}_*(R;\mathbb{Z}_p)$ are also identified. Let $A_{\mathrm{inf}} = A_{\mathrm{inf}}(R)$ be the Witt vectors of $R^\flat$, so one has a canonical surjection $\theta : A_{\mathrm{inf}} \to R$ with kernel generated by a nonzero divisor $\xi \in A_{\mathrm{inf}}$. Then one has isomorphisms:

$$\mathrm{TC}^-_*(R;\mathbb{Z}_p) = A_{\mathrm{inf}}[x,\sigma]/(x\sigma = \xi), \quad |x| = -2, |\sigma| = 2 \tag{8}$$

$$\mathrm{TP}_*(R;\mathbb{Z}_p) = A_{\mathrm{inf}}(R)[u^{\pm 1}], \quad |u| = 2. \tag{9}$$

Here $\sigma \in \pi_2\mathrm{TC}^-(R;\mathbb{Z}_p)$ is a lift of the generator in $\pi_2\mathrm{THH}(R;\mathbb{Z}_p)$. With respect to these isomorphisms, the canonical map $\mathrm{TC}^-_*(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p)$ carries x to u^{-1} and σ to ξu. The cyclotomic Frobenius $\varphi : \mathrm{TC}^-_*(R;\mathbb{Z}_p) \to \mathrm{TP}_*(R;\mathbb{Z}_p)$ carries $\sigma \mapsto u$ and $x \mapsto \varphi(\xi)u^{-1}$, and is the Witt vector Frobenius on π_0.

Identifying $\mathrm{TC}^-_*(R;\mathbb{Z}_p), \mathrm{TP}_*(R;\mathbb{Z}_p)$ for quasi-regular semiperfectoids is significantly more difficult (and the description in purely algebraic terms is a major result of [30, 32]). To begin with, we make the simple observation that these are concentrated in even degrees.

Corollary 6.4 (Evenness for quasi-regular semiperfectoids). *Let A be a quasi-regular semiperfectoid R-algebra. Then $\mathrm{THH}_*(A;\mathbb{Z}_p)$ is concentrated in even degrees. Consequently, $\mathrm{TC}^-_*(A;\mathbb{Z}_p), \mathrm{TP}_*(A;\mathbb{Z}_p)$ are concentrated in even degrees.*

Proof. This follows from the equivalence

$$\mathrm{THH}(A;\mathbb{Z}_p) \otimes_{\mathrm{THH}(R;\mathbb{Z}_p)} R \simeq \mathrm{HH}(A/R;\mathbb{Z}_p), \tag{10}$$

the Hochschild–Kostant–Rosenberg filtration (which shows that the latter is concentrated in even degrees). Then the $\mathbb{T}$-homotopy fixed point and Tate spectral sequences prove the remaining claims. □

Similarly from (10), one obtains:

Corollary 6.5. *Let A be a smooth algebra over the perfectoid ring R. Then $\mathrm{THH}_*(A;\mathbb{Z}_p) \simeq R[\sigma] \otimes_R \Omega^*_{R/\mathbb{Z}_p}$ with $|\sigma| = 2$.*

With respect to the above equivalence, the motivic filtration on $\mathrm{THH}(A;\mathbb{Z}_p)$ is such that σ belongs to filtration 1 and $\Omega^1_{A/R}$ belongs to filtration 1. This is not a Postnikov filtration, so it seems difficult to construct the filtration on $\mathrm{THH}(A;\mathbb{Z}_p)$ purely within the setting of smooth R-algebras. Thus, one needs to use instead the quasisyntomic site.

In particular, it follows from Corollary 6.4 that the constructions $\mathrm{THH}(-;\mathbb{Z}_p), \mathrm{TC}^-(-;\mathbb{Z}_p)$, $\mathrm{TP}(-;\mathbb{Z}_p)$, when considered as objects of $\mathrm{Shv}(\mathrm{QSyn}, \mathrm{Sp})$, have homotopy groups concentrated in even degrees. In fact, the same holds for $\mathrm{TC}(-;\mathbb{Z}_p)$.

Theorem 6.6 (The odd vanishing conjecture, [32, Section 14]). *The quasi-syntomic sheaf $\mathrm{TC}(-;\mathbb{Z}_p)$ has homotopy groups concentrated in even degrees.*

Theorem 6.6 (which was conjectured in [30]) is much more difficult than Corollary 6.4. In particular, the evenness of $\mathrm{TC}(R;\mathbb{Z}_p)$ does not hold for an arbitrary quasi-regular semiperfectoid ring, and the proof relies on André's lemma and the theory of prismatic cohomology.

Definition 6.7 (The motivic filtrations). The motivic filtration on $\mathrm{THH}(-;\mathbb{Z}_p)$ (respectively, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, $\mathrm{TP}(-;\mathbb{Z}_p), \mathrm{TC}(-;\mathbb{Z}_p)$) is given as the double speed Postnikov filtration in $\mathrm{Shv}(\mathrm{QSyn}, \mathrm{Sp})$; in other words, $\mathrm{Fil}^{\geqslant i}\mathrm{THH}(-;\mathbb{Z}_p)$ is the $2i$th connective cover of the quasi-syntomic sheaf $\mathrm{THH}(-;\mathbb{Z}_p)$.[†] We define the objects for $A \in \mathrm{QSyn}$,

$$\widehat{\mathbb{\Delta}}_A\{i\} = \mathrm{gr}^i\mathrm{TP}(A;\mathbb{Z}_p)[-2i], \tag{11}$$

$$\mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_A\{i\} = \mathrm{gr}^i\mathrm{TC}^-(A;\mathbb{Z}_p)[-2i], \tag{12}$$

$$\mathbb{Z}_p(i)(A) = \mathrm{gr}^i\mathrm{TC}(A;\mathbb{Z}_p)[-2i]. \tag{13}$$

All these define sheaves of p-complete, coconnective spectra on QSyn.

[†] In the definition of the motivic filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$, we want the formula (6) to work at the level of filtered spectra, which here follows from the odd vanishing conjecture. One could also directly define the motivic filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$ to ensure this, which is the approach taken in [30]. Then the fact that $\mathrm{Fil}^{\geqslant i}$ is the $2i$-connective cover (not simply the $(2i-1)$-connective cover) requires the odd vanishing conjecture.

Remark 6.8 (The motivic filtrations on QRSPerfd). A priori, the motivic filtrations are defined using the abstract theory of sheaves of spectra, and the t-structure there. However, if $A \in$ QRSPerfd, the motivic filtrations on $\mathrm{THH}(A;\mathbb{Z}_p), \mathrm{TC}^-(A;\mathbb{Z}_p), \mathrm{TP}(A;\mathbb{Z}_p)$ are very explicit: they are simply the double-speed Postnikov filtrations on these individual spectra. In other words, when restricted to quasi-regular semiperfectoid rings, the individual *homotopy groups* of $\mathrm{THH}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, $\mathrm{TP}(-;\mathbb{Z}_p)$ form sheaves of spectra, cf. [30, Section 7]. In particular, for a quasi-regular semiperfectoid ring A, we have $\widehat{\Delta}_A\{i\} = \pi_{2i}\mathrm{TP}(A;\mathbb{Z}_p)$; this is an invertible module over $\widehat{\Delta}_A = \pi_0\mathrm{TP}(A;\mathbb{Z}_p)$.

Indeed, the object $\widehat{\Delta}_A = \mathrm{gr}^0\mathrm{TP}(A;\mathbb{Z}_p)$ (for A quasi-syntomic) is perhaps the most fundamental of all the above structures and is closely related to prismatic cohomology [32]. Let us discuss some of the structure that it carries, which follows directly from its definition.

Let $A \in$ QRSPerfd. The *Nygaard filtration* on $\widehat{\Delta}_A = \pi_0\mathrm{TP}(A;\mathbb{Z}_p) = \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$ is the filtration that comes from the homotopy fixed point spectral sequence. In particular, we define $\mathcal{N}^{\geqslant i}\widehat{\Delta}_A = \pi_0(\tau_{\geqslant 2i}\mathrm{THH}(A;\mathbb{Z}_p))^{h\mathbb{T}} \subset \widehat{\Delta}_A$. This defines a descending, multiplicative, and complete filtration on $\widehat{\Delta}_A$ such that $\mathcal{N}^{\geqslant i}\widehat{\Delta}_A/\mathcal{N}^{\geqslant i+1}\widehat{\Delta}_A = \pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$. By descent, we obtain the Nygaard filtration on $\widehat{\Delta}_A$ for all quasi-syntomic A. For A quasi-regular semiperfectoid, $\widehat{\Delta}_A\{i\} = \pi_{2i}\mathrm{TP}(A;\mathbb{Z}_p)$ is an invertible $\widehat{\Delta}_A$-module (which can be trivialized, but not canonically in general) and the notation above $\mathcal{N}^{\geqslant i}\widehat{\Delta}_A\{i\} = \pi_{2i}\mathrm{TC}^-(A;\mathbb{Z}_p)$ is consistent. The *Frobenius* gives an endomorphism $\varphi : \widehat{\Delta}_A \to \widehat{\Delta}_A$ which for $A \in$ QRSPerfd comes from the cyclotomic Frobenius. The filtration and the Frobenius interact: we also have 'divided' Frobenii $\varphi_i : \mathcal{N}^{\geqslant i}\widehat{\Delta}_A\{i\} \to \widehat{\Delta}_A\{i\}$ for $i \geqslant 0$, which arise from the cyclotomic Frobenius on π_{2i}.

If A is a quasi-regular semiperfectoid algebra over the perfectoid ring R, then $\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$ has a finite filtration whose associated graded terms are (the p-completions of) $\bigwedge^j L_{A/R}[-j]$ for $0 \leqslant j \leqslant i$. Moreover, as A ranges over quasi-regular semiperfectoid R-algebras, we can trivialize the Breuil–Kisin twists $\widehat{\Delta}_A\{i\}$ for $i \in \mathbb{Z}$ using the description of $\mathrm{TC}^-_*(R;\mathbb{Z}_p)$, $\mathrm{TP}_*(R;\mathbb{Z}_p)$. In particular, we have that φ becomes divisible by $\varphi(\xi)^i$ (one typically writes $\tilde{\xi} = \varphi(\xi)$) on $\mathcal{N}^{\geqslant i}\widehat{\Delta}_A$ and we have a divided Frobenius $\varphi/\tilde{\xi}^i : \mathcal{N}^{\geqslant i}\widehat{\Delta}_A \to \widehat{\Delta}_A$. By descent, we obtain this structure for any quasi-syntomic R-algebra.

Example 6.9. In the base case of the perfectoid ring R, we have $\widehat{\Delta}_R = A_{\mathrm{inf}}$ and $\mathcal{N}^{\geqslant i}\widehat{\Delta}_R = \xi^i A_{\mathrm{inf}}$. Given a quasi-regular semiperfectoid R-algebra A, the ideal (ξ) is $\widehat{\Delta}_A$ is well-defined (and is contained in $\mathcal{N}^{\geqslant 1}\widehat{\Delta}_A$); however, it depends on the choice of perfectoid ring R. On the other hand, the ideal $(\tilde{\xi}) = (\varphi(\xi))$ is well-defined purely in terms of A without reference to R. In fact, it is the kernel of the map $\widehat{\Delta}_A = \pi_0(\mathrm{TP}(A;\mathbb{Z}_p)) \to \pi_0(\mathrm{THH}(A;\mathbb{Z}_p)^{tC_p})$.

In particular, by analyzing topological Hochschild homology and its homotopical structure, one obtains the above quasisyntomic sheaf of rings, equipped with the Frobenius and filtration. This is a structure of great interest to p-adic arithmetic geometry in mixed characteristic. For formally smooth algebras over a perfectoid ring, this agrees with the construction of A_{inf}-cohomology of [29] (and later [32]).

In the next couple of sections, we will discuss the situation in more detail in characteristic p, where one recovers the theory of crystalline cohomology.

7 | DERIVED DE RHAM COHOMOLOGY

In this section, we discuss some of the properties of p-adic derived de Rham cohomology, after [24]; see also [115]. Fix a base ring k.

Definition 7.1 (Derived de Rham cohomology [77, Section VIII.2]). Let R be a k-algebra. The *derived de Rham cohomology* $L\Omega_{R/k} \in \mathcal{D}(k)$ is the left derived functor of the functor $P \mapsto \Omega^\bullet_{P/k}$ sending a polynomial k-algebra to its de Rham complex considered as an $\mathbb{E}_\infty$-algebra over k. Moreover, $L\Omega_{R/k}$ is equipped with the descending, multiplicative derived Hodge filtration $\{L\Omega^{\geqslant *}_{R/k}\}$ obtained as the left Kan extension of the naive filtration on the de Rham complex of a polynomial k-algebra (that is, the ith filtration piece consists of j-forms for $j \geqslant i$).

Remark 7.2 (The Hodge completion). The Hodge completion of derived de Rham cohomology is often more tractable. For example, for a smooth k-algebra R, the Hodge completion of $L\Omega_{R/k}$ agrees with the usual de Rham complex; this follows by considering the map from derived to underived de Rham cohomology (with respective Hodge filtrations), and using that $\bigwedge^i L_{R/k} = \Omega^i_{R/k}$ for R/k smooth.

Example 7.3 (Derived de Rham cohomology in characteristic zero). Let $k = \mathbb{C}$, and let R be a finitely generated $\mathbb{C}$-algebra. On the one hand, derived de Rham cohomology of animated $\mathbb{C}$-algebras is easily seen to be the constant functor with value $\mathbb{C}$ in this case; indeed, this follows because the de Rham complex of a polynomial $\mathbb{C}$-algebra is acyclic in positive degrees. On the other hand, the Hodge completion of derived de Rham cohomology agrees with the singular cohomology (with $\mathbb{C}$-coefficients) of the associated complex points. This is a classical result of Grothendieck [62] for R smooth; compare [23] for a discussion in general.

In the sequel, we will only consider the p-adic version of derived de Rham cohomology, and we will simply drop the p-completion from the notation. We will also often drop the p-completion notation on the cotangent complex and its wedge powers.

Construction 7.4 (The derived conjugate filtration). Let $A \to B$ be a map of animated $\mathbb{F}_p$-algebras. By left Kan extension of the Postnikov filtration (and using the Cartier isomorphism), we see that $L\Omega_{B/A}$ admits a natural $B^{(1)} := B \otimes_{A,\varphi} A$-structure and an increasing, multiplicative, and exhaustive filtration $\mathrm{Fil}^*_{\mathrm{conj}} L\Omega_{B/A}$ in $\mathcal{D}(B^{(1)})$; the associated graded pieces are given by $\mathrm{gr}^i = \bigwedge^i L^{(1)}_{B/A}[-i]$.

A key consequence of the derived conjugate filtration, the fact that differential forms and the cotangent complex agree for smooth algebras, and the Cartier isomorphism for smooth algebras, is the following result. Note that it shows that derived de Rham cohomology behaves entirely differently in characteristic p than in characteristic zero.

Theorem 7.5 (Bhatt [24, Corollary 3.10]). *Given a smooth map $A \to B$ of rings, the p-complete derived de Rham cohomology $L\Omega_{B/A}$ agrees with the p-complete de Rham cohomology $\Omega^\bullet_{B/A}$ (with derived and classical Hodge filtrations matching).*

Remark 7.6. Let R be a Cartier smooth $\mathbb{F}_p$-algebra. Then the natural map $L\Omega_{R/\mathbb{F}_p} \to \Omega^\bullet_{R/\mathbb{F}_p}$ is an equivalence respecting Hodge filtrations. This also follows from the conjugate filtration. In fact, for each i, the map $L(\tau^{\leqslant i}\Omega_{R/\mathbb{F}_p}) \to \tau^{\leqslant i}\Omega^\bullet_{R/\mathbb{F}_p}$ is an equivalence; one sees this on associated graded terms, whence it follows from the assumptions.

A further aspect of the p-adic theory is the appearance of certain p-adic period rings when one applies p-adic derived de Rham cohomology to certain large rings, shown in [20] in the Hodge-completed case and explored further in [24]. This phenomenon arises from the natural appearance of divided powers, cf. [115, Proposition 3.16] for a detailed account.

Example 7.7 (Divided powers via derived de Rham cohomology). Consider the map $\mathbb{Z}_p[x] \to \mathbb{Z}_p$. Then the p-adic derived de Rham cohomology is given by the p-complete divided power algebra $(\mathbb{Z}_p[\frac{x^i}{i!}])_{\hat{p}}$: more precisely, the natural map $\mathbb{Z}_p[x] \to L\Omega_{\mathbb{Z}/\mathbb{Z}_p[x]}$ exhibits the target as the p-adic divided power completion of (x) in the source.

To see this, we observe that everything involved has a grading. Formally, we work in the ∞-category of nonnegatively graded animated rings $R_\star$ with $R_0 = \mathbb{Z}_p$. For any map $A \to B$ of such nonnegatively graded animated rings, the construction $L\Omega_{B/A}$ carries through in this ∞-category, and it is not difficult to see that the Hodge filtration converges for grading reasons (indeed, $L\Omega^{\geqslant i}_{B/A}$ is concentrated in internal degrees $\geqslant i$ whence $\varprojlim_i L\Omega^{\geqslant i}_{B/A} = 0$ in the graded derived ∞-category). In the graded ∞-category, the isomorphism $L\Omega_{\mathbb{Z}_p/\mathbb{Z}_p[x]} = (\mathbb{Z}_p[\frac{x^i}{i!}])_{\hat{p}}$ follows by passage to the associated graded of the Hodge filtration $\mathrm{gr}^*(L\Omega_{B/A}) = \bigwedge^* L_{B/A}[-*]$, using $L_{\mathbb{Z}_p/\mathbb{Z}_p[x]} = \mathbb{Z}_p[1]$ and the décalage isomorphism $\bigwedge^i(M[1]) = \Gamma^i(M)[i]$.[†] By forgetting the grading, we conclude the desired isomorphism.

In particular, if A is a p-complete ring and $x \in A$ is a nonzero divisor, then the p-adic derived de Rham cohomology of $A \to A/x$ is simply the p-complete divided power envelope of (x); this follows from the above by base-change.

Construction 7.8 (Derived de Rham cohomology as a quasi-syntomic sheaf). Let R be a quasi-syntomic ring; for simplicity, we assume R is p-torsionfree or an $\mathbb{F}_p$-algebra. On the category of quasi-syntomic R-algebras, the construction $A \mapsto L\Omega_{A/R}$ defines a sheaf of spectra, which belongs to the heart of the t-structure (in fact, it takes discrete values on quasi-regular semiperfectoid algebras). This follows from reducing modulo p and the derived conjugate filtration. Similarly, $A \mapsto L\Omega^{\geqslant i}_{A/R}$ defines a sheaf of spectra (also in the heart).

Construction 7.9 (The Hodge-completed variant). Let R be a quasi-syntomic ring. On the category of quasi-syntomic R-algebras, the constructions $A \mapsto \widehat{L\Omega_{A/R}}, \widehat{L\Omega^{\geqslant i}_{A/R}}$ defines a sheaf of coconnective spectra, which belongs to the heart of the t-structure (in fact, it takes discrete values on quasi-regular semiperfectoid algebras). This follows from the Hodge filtration.

[†] One also uses here that if $A \to B$ is a map of animated nonnegatively graded rings with $A_0 = B_0 = \mathbb{Z}_p$, then $A \to L\Omega_{B/A}$ is an isomorphism in degree 1; this follows easily from the case of a polynomial algebra.

The cohomology theories of [30], for algebras over a perfectoid base, can be described as 'deformations' of (Hodge-completed) de Rham cohomology, which therefore plays a central role in the theory. This arises as the combination of the following two results. The first (cf. [30, Section 5]) gives a close relationship between de Rham and periodic cyclic homology. Other proofs (which work outside the p-complete context) have been given by Antieau [6], Moulinos–Robalo–Toën [104], and Raksit [107]. For the result, we write $\mathrm{HC}^- = \mathrm{HH}^{h\mathbb{T}}$, $\mathrm{HP} = \mathrm{HH}^{t\mathbb{T}}$.

Theorem 7.10. *Let R be a quasi-syntomic ring, and let A be a quasi-syntomic R-algebra such that $L_{R/A}$ is the suspension of a p-completely flat module (for example, A could be quasi-regular semiperfectoid). Then we have natural isomorphisms*

$$\pi_{2i}\mathrm{HP}(A/R;\mathbb{Z}_p) = \widehat{L\Omega_{A/R}}, \quad \pi_{2i}\mathrm{HC}^-(A/R;\mathbb{Z}_p) = \widehat{L\Omega^{\geqslant i}_{A/R}}.$$

In particular, by quasi-syntomic descent, we obtain multiplicative, convergent exhaustive $\mathbb{Z}$-indexed descending filtrations for any quasi-syntomic R-algebra A, on $\mathrm{HC}^-(A/R;\mathbb{Z}_p)$, $\mathrm{HP}(A/R;\mathbb{Z}_p)$ *with*

$$\mathrm{gr}^i\mathrm{HP}(A/R;\mathbb{Z}_p) = \widehat{L\Omega_{A/R}}[2i], \quad \mathrm{gr}^i\mathrm{HC}^-(A/R;\mathbb{Z}_p) = \widehat{L\Omega^{\geqslant i}_{A/R}}[2i].$$

Remark 7.11. In characteristic zero and for A/R smooth, the analogs of these filtrations are canonically split (for example, by Adams operations), and the connection between periodic cyclic and de Rham cohomology is classical, cf. [94, Section 5.1.12]. However, these filtrations are not canonically split in positive characteristic, and the induced spectral sequences from de Rham cohomology to periodic cyclic homology need not degenerate for smooth projective varieties [7].

The second result, which comes from analyzing the structure of $\mathrm{THH}(R;\mathbb{Z}_p)$, states that TP gives a one-parameter deformation of HP, for algebras over a perfectoid base.

Theorem 7.12 [30, Theorem 7.12]. *Let R be a perfectoid ring, and let A be any R-algebra. Then there is a natural equivalence*

$$\mathrm{TP}(A;\mathbb{Z}_p)/\xi = \mathrm{HP}(A/R;\mathbb{Z}_p). \tag{14}$$

More precisely, we have an equivalence of $\mathbb{E}_\infty$-algebras $\mathrm{TP}(A;\mathbb{Z}_p) \otimes_{\mathrm{TP}(R;\mathbb{Z}_p)} \mathrm{HP}(R/R;\mathbb{Z}_p) = \mathrm{HP}(A/R;\mathbb{Z}_p)$. Using Theorem 6.3 and the surrounding discussion, we have that $\mathrm{TP}_*(R;\mathbb{Z}_p) = A_{\mathrm{inf}}[u^{\pm 1}]$ and $\mathrm{HP}(R/R;\mathbb{Z}_p) = R[u^{\pm 1}]$; the map $\mathrm{TP}_*(R;\mathbb{Z}_p) \to \mathrm{HP}_*(R/R;\mathbb{Z}_p)$ has kernel generated by the element $\xi \in A_{\mathrm{inf}}$. Compare also [9] for a discussion of related results.

By considering (14) for A a quasi-regular semiperfectoid R-algebra, combining with Theorem 7.10, and using the definitions of the motivic filtrations, we find that

$$\widehat{\mathbb{\Delta}}_A/\xi = \widehat{L\Omega_{A/R}}. \tag{15}$$

By quasi-syntomic descent, we obtain (15) for all quasi-syntomic R-algebras A. In particular, $\widehat{\mathbb{\Delta}}_A$ gives a one-parameter deformation of (Hodge-completed) derived de Rham cohomology.

Remark 7.13 (Non-Nygaard complete prismatic cohomology). Given a perfectoid ring R, one can define a 'Nygaard decompleted' version Δ_{-} of $\widehat{\Delta}_{-}$, which deforms derived de Rham cohomology rather than its Hodge completion. Namely, one considers the quasi-syntomic sheaf $\widehat{\Delta}_{-}$ and restricts to formally smooth R-algebras, and then left Kan extends from formally smooth (or p-complete polynomial) R-algebras to all p-complete R-algebras, as functors to (p,ξ)-complete $\mathbb{E}_\infty$-algebras over A_{inf}. This yields a construction $A \mapsto \Delta_A$ which provides a deformation along the parameter ξ of derived de Rham cohomology, that is, one has functorial equivalences $\Delta_A/\xi \simeq L\Omega_{A/R}$, which therefore also restricts to a sheaf on quasisyntomic R-algebras (and belongs to the heart). At least a priori, this construction depends on the choice of the perfectoid ring R mapping to A. However, in [32, Section 7], a purely algebraic construction of Δ_{-} is given (on quasi-regular semiperfectoid rings, from which one can descend) that makes clear that Δ_{-} can genuinely be defined on the whole quasi-syntomic site, without the choice of a perfectoid base. Similarly, Δ_{-} is still equipped with a Nygaard filtration $\{\mathcal{N}^{\geqslant *}\Delta_{-}\}$ such that the completion with respect to this filtration is $\widehat{\Delta_{-}}$; this follows because the associated graded terms of the Nygaard filtration are left Kan extended from p-complete polynomial rings as proved in [8, Corollary 5.21].

We have seen that p-adic derived de Rham cohomology coincides with the 'underived' version for smooth algebras. More generally, there is a similar description in the case of a locally complete intersection singularity (or a quasi-syntomic ring) in terms of the divided power de Rham complex of a polynomial algebra surjecting onto it. This fact is essentially the comparison between crystalline cohomology and derived de Rham cohomology [24] and the classical description (due to Berthelot [21, Section V.2.3]) of crystalline cohomology in terms of the divided power de Rham complex, cf. also [26] for another approach. We do not review the general theory of divided power structures in detail and give an ad hoc construction; cf. also the recent work [99] for a treatment of the derived divided power envelope construction.

Construction 7.14 (Divided power envelopes of free algebras). Let (A, I) be a pair consisting of a p-torsionfree $\mathbb{Z}_{(p)}$-algebra and an ideal $I \subset A$. Suppose that A is a polynomial $\mathbb{Z}_{(p)}$-algebra and $I \subset A$ is the ideal generated by a collection of the polynomial generators, that is, (A, I) is a free object (in the evident sense) in the category of such pairs.

We define the *divided power algebra* $D_I(A)$ to be the subalgebra of $A \otimes \mathbb{Q}$ generated by A and the elements $\frac{y^i}{i!}, y \in I$; this is also the divided power envelope (cf., for instance, [114, Tag 07H7]). We have a descending multiplicative filtration $\{\mathrm{Fil}^{\geqslant *} D_I(A)\}$ defined by the divided powers: $\mathrm{Fil}^{\geqslant r} D_I(A)$ is the ideal generated by all elements $\frac{y_1^{j_1} \ldots y_m^{j_m}}{j_1! \ldots j_m!}$ for $j_1 + \cdots + j_m \geqslant r$ for the $y_k \in I$. On the A-algebra $D_I(A)$, we have a flat connection $d : D_I(A) \to D_I(A) \otimes_A \Omega^1_{A/\mathbb{Z}_{(p)}}$ (extended from $d : A \to \Omega^1_A$, so, for instance, $d(\frac{y_1^i}{i!}) = \frac{y_1^{i-1}}{(i-1)!} dy_1$), and this connection satisfies the Griffiths transversality property: $d(\mathrm{Fil}^{\geqslant r} D_I(A)) \subset \mathrm{Fil}^{\geqslant r-1} D_I(A) \otimes_A \Omega^1_A$. In particular, we can form the *divided power de Rham complex*

$$\Omega^\bullet_{D_I(A)} = D_I(A) \to D_I(A) \otimes_A \Omega^1_A \to D_I(A) \otimes_A \Omega^2_A \to \ldots,$$

and this is in turn equipped with a multiplicative filtration such that

$$\mathrm{Fil}^{\geqslant r} \Omega^\bullet_{D_I(A)} = \mathrm{Fil}^{\geqslant r} D_I(A) \to \mathrm{Fil}^{\geqslant r-1} D_I(A) \otimes_A \Omega^1_A \to \mathrm{Fil}^{\geqslant r-2} D_I(A) \otimes_A \Omega^2_A \to \ldots. \quad (16)$$

Let (A, I) be a free pair as above, and let A_0, I_0 denote the reductions modulo p. We denote by $(-)^{(1)}$ the Frobenius twist along A_0, so $A_0/\varphi(I_0) = (A_0/I_0)^{(1)}$, for instance. Then one checks (cf. [24, Lemma 3.42] and [30, Proposition 8.11]) that $D_I(A)/p$ admits an ascending, exhaustive, multiplicative filtration (an analog of the conjugate filtration) such that

$$\mathrm{gr}^0 D_I(A)/p = A_0/\varphi(I_0) = (A_0/I_0)^{(1)}$$

and in general

$$\mathrm{gr}^i D_I(A)/p = (\Gamma^i_{A_0/I_0}(I_0/I_0^2))^{(1)}. \tag{17}$$

Explicitly, the ith stage of the filtration on $D_I(A)/p$ is the $A_0/\varphi(I_0)$-module generated by $\frac{y_1^{j_1} \dots y_m^{j_m}}{j_1! \dots j_m!}$ for $j_1 + \dots j_m \leqslant pi$.

Construction 7.15 (Derived divided power envelopes). Given any pair (A, I) consisting of a $\mathbb{Z}_{(p)}$-algebra A and an ideal $I \subset A$, we can simplicially resolve the pair in terms of pairs (B, J) which are free in the sense above. Taking simplicial resolutions, we obtain the *derived divided power envelope* $LD_I(A)$ (a priori, an animated ring) equipped with its filtration $\mathrm{Fil}^{\geqslant *} LD_I(A)$ and a connection satisfying Griffiths transversality. By left Kan extension of (17), there exists an analogous increasing, multiplicative, and exhaustive filtration on $LD_I(A)/p$.

Suppose that (A, I) is a pair such that $A, A/I$ are p-torsionfree, the Frobenius on $A_0 = A/p$ is flat (for example, A/p is smooth over a perfect ring), and such that the p-completion of $L_{(A/I)/A}$ is p-completely flat. In particular, it follows from the conjugate filtration (and reducing modulo p) that $LD_I(A)$ is a p-torsionfree, discrete ring. In particular, it is simply the subring of $A \otimes \mathbb{Q}$ generated by the $\frac{y^i}{i!}, y \in I$, and (by taking resolutions), one sees that it is actually the divided power envelope in the usual sense [114, Tag 07H7]. We will only be interested in this case.

We now record the main result. Again, we emphasize that this result is essentially the comparison between crystalline and derived de Rham cohomology as in [24].

Theorem 7.16 ($L\Omega$ via the divided power de Rham complex). *Suppose (A, I) is a pair such that $A, A/I$ are p-torsionfree, A/p is Cartier smooth and has flat Frobenius, and such that the p-completion of $L_{(A/I)/A}$ is p-completely flat. Then there is a natural multiplicative, filtered isomorphism between the p-adic derived de Rham cohomology $L\Omega_{(A/I)/\mathbb{Z}_{(p)}}$ (with the Hodge filtration) and the p-completed divided power de Rham complex $\Omega^\bullet_{D_I(A)}$ (with the filtration (16)).*

Proof. We will use a similar argument as in Theorem 7.5. In fact, it suffices to show that for a pair (A, I) satisfying the above conditions, $\Omega^\bullet_{D_I(A)}$ is quasi-isomorphic modulo p to its left Kan extension from free pairs; for a free pair, the natural map induces a filtered quasi-isomorphism $\Omega^\bullet_{D_I(A)} \to \Omega^\bullet_{(A/I)/\mathbb{Z}_{(p)}}$ by the Poincaré lemma (that is, the divided power Poincaré lemma; this is easy to check by hand, cf. [21, Lemme 2.1.2]). For this, we will produce an appropriate filtration on $\Omega^\bullet_{D_I(A)}/p$.

By our assumptions, $D_I(A)$ is simply the subring of $A \otimes \mathbb{Q}$ generated by the divided powers of I. Thus, we can use the conjugate filtration $\mathrm{Fil}^{\leqslant *}_{\mathrm{conj}} D_I(A)/p$, as in (17), which is defined by left Kan extension from the case of a free algebra. From its definition (and left Kan extension), we see that $A_0 = A/p$-connection on $D_I(A)/p$ is compatible with the conjugate filtration. In particular, we have an ascending, exhaustive, multiplicative filtration on $\Omega^\bullet_{D_I(A)}/p$ such that gr^i is the de Rham complex over A_0 of the A_0-module-with-connection $\Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)}$. It thus suffices to show that this de Rham complex (where we write $I_0 = I/p$)

$$\Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)} \to \Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)} \otimes_{A_0} \Omega^1_{A_0} \to \Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)} \otimes_{A_0} \Omega^2_{A_0} \to \dots \tag{18}$$

as a functor from such pairs (A, I) to $\mathcal{D}(\mathbb{F}_p)$, is left Kan extended from the free objects. Now the A_0-connection on $\Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)}$ is the canonical (Frobenius descent) connections, cf. [24, Lemma 3.44]: explicitly, this follows because any ip-th divided power $\gamma_{ip}(y), y \in I$ is a flat section for this connection. Since this is a Frobenius descent connection, the Cartier isomorphism (valid since A_0 is Cartier smooth) goes into effect: the jth cohomology of (18) is given by $\Gamma^i_{A_0/I_0}(I_0/I_0^2)^{(1)} \otimes_{A_0} (\Omega^i_{A_0})^{(1)}$. Thus, it follows that (18) is left Kan extended from free pairs, whence the result. $\square$

8 | THE RING A_{crys}

In this section, we will construct the functor $A \mapsto \widehat{\mathbb{\Delta}}_A$ together with the Nygaard filtration and divided Frobenii in characteristic p. We describe the construction purely algebraically here, and in the next section will outline the proof that it is compatible with the construction arising from topological Hochschild homology.

We first need the divided power construction for ideals containing p (and where the divided powers are compatible with the canonical divided powers on (p)). The construction is analogous to that of Construction 7.14; however, we will not have the analog of the Hodge filtration.

Construction 8.1 (Derived divided powers for ideals containing p). Let (A, I) be a pair consisting of a $\mathbb{Z}_{(p)}$-algebra and an ideal $I \subset A$ containing p.

Suppose first (A, I) is free: in other words, that A is a polynomial ring and $I \subset A$ is the ideal generated by a subset of the polynomial generators together with p. In this case, we define $D_I(A)$ to be the subring of $A \otimes \mathbb{Q}$ generated by A and $\{\frac{y^i}{i!}\}_{y \in I}$. In general, we define the derived divided powers $LD_I(A)$ by simplicially resolving the pair (A, I) by free objects, and taking the induced simplicial resolution of divided power envelopes.

As before, $LD_I(A)$ defines an animated ring, and its rationalization is simply A. To control it in general, we again use the conjugate filtration. This gives that if (A, I) is a free pair, then $D_I(A)/p$ has an ascending filtration as in (17). In particular, we find that if the pair (A, I) is such that A is p-torsionfree, A/p is Cartier smooth with flat Frobenius, and $L_{(A/I)/(A/p)}$ is the suspension of a flat A/I-module, then $LD_I(A)$ is discrete and p-torsionfree; we will thus simply write $D_I(A)$. In particular, again by taking resolutions and comparing, one verifies the universal property that $D_I(A)$ is actually the divided power envelope of (A, I) compatible with the canonical divided powers on (p).

Definition 8.2 (The rings $A_{\mathrm{inf}}, A_{\mathrm{crys}}$). Let $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$.

(1) The ring $R^\flat$ (the tilt of R) is defined as the inverse limit perfection of R, that is, $R^\flat = \varprojlim_{\varphi} R$. This comes with a natural map $R^\flat \to R$, and our assumption implies that this map is surjective.
(2) The ring $A_{\mathrm{inf}}(R)$ is defined to be $W(R^\flat)$; we have a natural surjective map

$$\theta : W(R^\flat) \to R. \tag{19}$$

In particular, $A_{\mathrm{inf}}(R)$ is the universal p-complete pro-nilpotent thickening of R.
(3) The ring $A_{\mathrm{crys}}(R)$ is defined as the p-complete (derived) divided power envelope of the surjection θ (whose kernel includes (p)).

Remark 8.3 (Properties of A_{crys}). Our assumptions imply that $LD_{\ker\theta}W(R^\flat)$ is p-torsionfree by the conjugate filtration. Therefore, $A_{\mathrm{crys}}(R)$ can equivalently be obtained by taking the subring of $A_{\mathrm{inf}}(R)[1/p]$ generated by the divided powers of $\ker(\theta)$, and then p-adically completing again. In particular, it is actually a discrete p-torsionfree, p-complete ring (and not an animated ring). Moreover, it is the p-completion of the (classical) divided power envelope of θ compatible with divided powers on (p), since the classical and derived divided power envelopes coincide.

Remark 8.4. The choice of the map θ is in some sense arbitrary. Given any perfect $\mathbb{F}_p$-algebra P with a surjection $P \twoheadrightarrow R$, we could instead construct $A_{\mathrm{crys}}(R)$ as the p-complete divided power envelope of $W(P) \twoheadrightarrow R$; this does not change the outcome.

Example 8.5. Let R be the ring $\mathbb{F}_p[x^{1/p^\infty}]/(x)$. Then $A_{\mathrm{crys}}(R)$ is the p-adic completion of the subring $\mathbb{Z}_p[x^{1/p^\infty}, \frac{x^i}{i!}]_{i \geqslant 0} \subset \mathbb{Q}_p[x^{1/p^\infty}]$.

The construction $R \mapsto A_{\mathrm{crys}}(R)$ defines a sheaf of spectra (which actually has image in discrete spectra) on $\mathrm{QRSPerf}_{\mathbb{F}_p}$. Indeed, since A_{crys} takes values in p-complete, p-torsionfree abelian groups, it suffices to observe that $A_{\mathrm{crys}}(R)/p$ defines a sheaf; but this in turn follows from the conjugate filtration as in (17) and descent for the cotangent complex and its wedge powers. In fact, one can explicitly identify its reduction modulo p as a sheaf on $\mathrm{QRSPerf}_{\mathbb{F}_p}$. One can give a proof of this using derived divided powers for $\mathbb{F}_p$-algebras.

We will be interested in the case of certain lci singularities, and first we will need the following result.

Theorem 8.6 (cf. [30, Proposition 8.12]). *For $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$, we have a natural isomorphism $A_{\mathrm{crys}}(R)/p = L\Omega_{R/\mathbb{F}_p}$.*

One can also prove the following closely related result, identifying $A_{\mathrm{crys}}(R)$ with the derived de Rham cohomology of any p-adic lift.

Theorem 8.7. *Let S be a quasi-syntomic ring which is p-torsionfree and such that $R = S/p$ is quasi-regular semiperfect. Then we have a natural isomorphism $A_{\mathrm{crys}}(R) = L\Omega_{S/\mathbb{Z}_p}$.*

Proof. Let $S^\flat$ denote the inverse limit perfection of S/p; we have a map $W(S^\flat) \to S$ which is surjective modulo p by our assumptions, hence surjective. By Theorem 7.16, it follows that

$L\Omega_{S/\mathbb{Z}_p} = L\Omega_{S/W(S^\flat)}$ is the p-complete derived divided power envelope of the surjection $W(S^\flat) \to S$. Similarly, by construction $A_{\mathrm{crys}}(R)$ is the p-complete derived divided power envelope of the surjection $W(S^\flat) \to S/p$ compatible with the divided powers on (p). But it is easy to see that for any pair (A, I) with $A, A/I$ p-torsionfree, the derived divided power envelope of (A, I) and the derived divided power envelope of $(A, (I, p))$ (where the latter is taken compatible with divided powers on p) agree: indeed, this follows by left Kan extension from the polynomial case, when the result is clear. □

By descent, we obtain from $A_{\mathrm{crys}}(-)$ a sheaf of spectra on $\mathrm{QSyn}_{\mathbb{F}_p}$, which is a p-adic lift of the sheaf $L\Omega_{-/\mathbb{F}_p}$. One can show that this is precisely *derived crystalline cohomology*, that is, the functor on $\mathbb{F}_p$-algebras obtained by left Kan extending (absolute) crystalline cohomology. In particular, the basic comparison theorems in crystalline (or de Rham–Witt) theory yields that de Rham cohomology of smooth $\mathbb{F}_p$-algebras admits a p-adic lift given by crystalline cohomology. Left Kan extending to quasi-syntomic $\mathbb{F}_p$-algebras, one obtains a p-adic lift of $L\Omega_{-/\mathbb{F}_p}$, and one shows that this is precisely the above one. In other words:

Theorem 8.8 (cf. [30, Theorem 8.14]). *For a quasi-regular semiperfect $\mathbb{F}_p$-algebra R, there is a natural isomorphism between the derived crystalline cohomology*[†] *of R (or the derived de Rham–Witt cohomology of R), obtained by left Kan extending crystalline cohomology from polynomial $\mathbb{F}_p$-algebras, and the ring $A_{\mathrm{crys}}(R)$.*

In particular, by descent from quasi-regular semiperfect $\mathbb{F}_p$-algebras, the construction of the ring $A_{\mathrm{crys}}(R)$ provides another approach to crystalline cohomology; of course, the approach is not essentially different from the classical one, since the definition of divided powers is fundamental to the crystalline site and the basic construction of crystalline cohomology. A key insight of [30] is that topological Hochschild homology provides a new (and fundamentally different) approach to the construction of crystalline cohomology, where divided powers arise very naturally (instead of being introduced by fiat). Most importantly, this approach has the advantage of working equally well in mixed characteristic, where it reproduces the prismatic cohomology [32]. Before formulating the results, we need one more ingredient.

Definition 8.9 (The Nygaard filtration on A_{crys}, [30, Section 8.2]). Let R be a quasi-regular semiperfect $\mathbb{F}_p$-algebra. We define the *Nygaard filtration* $\{\mathcal{N}^{\geqslant *}A_{\mathrm{crys}}(R)\}$ such that $\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R) \subset A_{\mathrm{crys}}(R)$ consists of those elements $x \in A_{\mathrm{crys}}(R)$ such that $p^i \mid \varphi(x)$, where $\varphi : A_{\mathrm{crys}}(R) \to A_{\mathrm{crys}}(R)$ denotes the endomorphism induced by Frobenius. By construction, since everything involved is p-torsionfree, we have an induced divided Frobenius map

$$\varphi/p^i : \mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R) \to A_{\mathrm{crys}}(R).$$

Example 8.10. Consider the case where $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ as in Example 8.5, so that $A_{\mathrm{crys}}(R)$ is the p-completion of the ring $\mathbb{Z}_p[x^{1/p^\infty}, \frac{x^j}{j!}]_{j\geqslant 0}$. Then $\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$ is the p-completion of the subring generated by $p^{i-j}(\frac{x^j}{j!})$ for all $j \geqslant 0$.

[†] In fact, one can also use the actual crystalline cohomology of R, as one sees by the universal property of A_{crys}.

Example 8.11 (The case of a δ-lift). More generally, let S be a p-torsionfree, p-complete δ-ring such that $R = S/p$ is quasi-regular semiperfect. Then we have the canonical identification $A_{\mathrm{crys}}(R) = L\Omega_S$ (cf. Theorem 8.7), and the Nygaard filtration on $A_{\mathrm{crys}}(R)$ is the tensor product of the Hodge filtration on $L\Omega_S$ and the p-adic filtration on $\mathbb{Z}_p$. In other words, in the p-complete filtered derived category, we have

$$\{\mathcal{N}^{\geqslant *}A_{\mathrm{crys}}(R)\} = \{L\Omega_S^{\geqslant *}\} \otimes \{p^*\mathbb{Z}_p\}. \tag{20}$$

This follows by left Kan extension from the case of p-completed free δ-ring, using [30, Proposition 8.7] (or [28, Proposition 8.3.3] in the setting of Dieudonné complexes). By descent, we obtain (20) for arbitrary p-torsionfree quasi-syntomic δ-rings.

Although the definition of the Nygaard filtration in this manner does not make the claim immediately evident, each $\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$ turns out to define a sheaf of p-complete spectra on $\mathrm{QRSPerf}_{\mathbb{F}_p}$. One way to see this is to use the following proposition, which gives very strong control over the Nygaard filtration:

Proposition 8.12 (cf. [30, Theorem 8.14(2)]). *Let $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$. For each $i \geqslant 0$, the map φ/p^i induces an isomorphism*

$$\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)/\mathcal{N}^{\geqslant i+1}A_{\mathrm{crys}}(R) \xrightarrow{\sim} \mathrm{Fil}_{\mathrm{conj},\leqslant i}(A_{\mathrm{crys}}(R)/p).$$

More precisely, φ/p^i (by construction) gives a well-defined map $\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)/\mathcal{N}^{\geqslant i+1}A_{\mathrm{crys}}(R) \xrightarrow{\sim} A_{\mathrm{crys}}(R)/p$; the claim is that this map has image in the ith stage of the conjugate filtration (which is also the ith stage of the de Rham conjugate filtration under Theorem 8.6), and induces an isomorphism onto its image.

Since the Nygaard filtration gives a sheaf of p-complete spectra on $\mathrm{QRSPerf}_{\mathbb{F}_p}$ (due to Proposition 8.12 and the conjugate filtration), by quasi-syntomic descent we obtain a filtration on the derived crystalline cohomology of quasi-syntomic $\mathbb{F}_p$-algebra (in particular any smooth algebra over a perfect field). This is the classical construction of the Nygaard filtration. To describe this, we review some fundamentals of de Rham–Witt theory, after [78].

Suppose R is a smooth algebra over a perfect field k. In this case, one has the de Rham–Witt complex $W\Omega_R^\bullet$ of [78], which gives an explicit p-torsionfree, p-adically complete commutative differential graded algebra representing the crystalline cohomology of R,

$$WR = W\Omega_R^0 \xrightarrow{d} W\Omega_R^1 \xrightarrow{d} W\Omega_R^2 \to \dots,$$

and equipped with a map of commutative dg-algebras $W\Omega_R^\bullet \to \Omega_{R/k}^\bullet$ which induces a quasi-isomorphism $W\Omega_R^\bullet/p \to \Omega_{R/k}^\bullet$. The object $W\Omega_R^\bullet$ is equipped with operators of graded abelian groups $F, V : W\Omega_R^\bullet \to W\Omega_R^\bullet$ recovering the Witt vector Frobenius and Verschiebung in degree zero and satisfying the identities

$$FV = VF = p, \quad dF = pFd, \quad Vd = pdV,$$

(and $FdV = d$, which is actually a consequence by p-torsionfreeness). Another construction of $W\Omega^{\bullet}_R$ as a 'strict Dieudonné algebra' (and universal property/construction in that category) is given in [28].

The endomorphism φ of the commutative differential graded algebra $W\Omega^{\bullet}_R$ induced by the Frobenius on R is given in degree i by p^iF. Then one can define a descending, multiplicative *Nygaard filtration* on the differential graded algebra $W\Omega^{\bullet}_R$ such that

$$\mathcal{N}^{\geqslant i}W\Omega^{\bullet}_R = p^{i-1}VW\Omega^0_R \to p^{i-2}VW\Omega^1_R \to \cdots \to VW\Omega^{i-1}_R \to W\Omega^i_R \to W\Omega^{i+1}_R \to \dots . \tag{21}$$

By construction, the Frobenius φ (that is, p^dF in degree d) is divisible by p^i on $\mathcal{N}^{\geqslant i}W\Omega^{\bullet}_R \subset W\Omega^{\bullet}_R$ and we can define a divided Frobenius (of cochain complexes)

$$\varphi/p^i : \mathcal{N}^{\geqslant i}W\Omega^{\bullet}_R \to W\Omega^{\bullet}_R.$$

Moreover, one checks by an explicit homological calculation that the map of cochain complexes

$$\varphi/p^i : \mathcal{N}^{\geqslant i}W\Omega^{\bullet}_R/\mathcal{N}^{\geqslant i+1}W\Omega^{\bullet}_R \to W\Omega^{\bullet}_R/p \to \Omega^{\bullet}_{R/k} \tag{22}$$

induces a quasi-isomorphism from the source to the i-truncation of the target (the source lives in degrees $\leqslant i$). All this can also be developed in the generality of strict Dieudonné complexes; compare [28, Section 8].

The Nygaard filtration is somewhat subtle in general. Indeed, a purely crystalline approach to the Nygaard filtration is not expected to exist (in filtration degrees $\geqslant p$).

The basic comparison result is that the Nygaard filtration on $A_{\mathrm{crys}}(R)$ recovers by descent the above Nygaard filtration on crystalline cohomology.

Proposition 8.13 (cf. [30, Theorem 8.14(3)]). *The Nygaard filtration $\{\mathcal{N}^{\geqslant *}A_{\mathrm{crys}}(R)\}$ descends to the above Nygaard filtration (in the derived category) on crystalline cohomology for smooth algebras over a perfect field.*

To summarize, from the above discussion, we have an equivalence between the following two constructions of sheaves on $\mathrm{QSyn}_{\mathbb{F}_p}$. For convenience, we will use the first notation $LW\Omega_{\_}$.

(1) The derived functor $R \mapsto LW\Omega_R$ of the derived de Rham–Witt cohomology on polynomial (or smooth) $\mathbb{F}_p$-algebras, together with its Nygaard filtration $\mathcal{N}^{\geqslant *}LW\Omega_R$ obtained by left Kan extending the Nygaard filtration (21) on polynomial rings.
(2) The quasi-syntomic sheaf $R \mapsto R\Gamma_{\mathrm{QSyn}}(\mathrm{Spec}R, A_{\mathrm{crys}}(-))$ given on the basis $\mathrm{QRSPerf}_{\mathbb{F}_p}$ by the construction $A_{\mathrm{crys}}(-)$, equipped with the Nygaard filtration $\{R\Gamma_{\mathrm{QSyn}}(\mathrm{Spec}R, \mathcal{N}^{\geqslant *}A_{\mathrm{crys}}(-))\}$ obtained by descending the Nygaard filtration on A_{crys} of quasi-regular semiperfect $\mathbb{F}_p$-algebras (Definition 8.9).

In fact, either construction of the Nygaard filtration gives a map in the filtered derived category for any $R \in \mathrm{QSyn}_{\mathbb{F}_p}$

$$\varphi : \mathcal{N}^{\geqslant *}LW\Omega_R \to p^*LW\Omega_R,$$

where the target denotes the p-adic filtration on the derived de Rham–Witt cohomology.

We next need to discuss the completion of this sheaf with respect to the Nygaard filtration. Again, there are two equivalent ways to proceed. The first is simply to take the completion in the filtered derived category of $\{\mathcal{N}^{\geqslant *}LW\Omega_R\}$ for any $R \in \mathrm{QSyn}_{\mathbb{F}_p}$, for example, as constructed by left Kan extension from polynomial algebras; we denote this by $\{\mathcal{N}^{\geqslant *}\widehat{LW\Omega_R}\}$. The second (which we describe below) is to take the Nygaard completion of $A_{\mathrm{crys}}(-)$ for quasi-regular semiperfect $\mathbb{F}_p$-algebras, and then descend.

Construction 8.14 (The Nygaard completion of $A_{\mathrm{crys}}(-)$). Let $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$. Then we define the *Nygaard completion* $\widehat{A_{\mathrm{crys}}}(R)$ to be the completion of the ring $A_{\mathrm{crys}}(R)$ with respect to the Nygaard filtration, $\widehat{A_{\mathrm{crys}}}(R) = \varprojlim_i A_{\mathrm{crys}}(R)/\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$. We also obtain the Nygaard filtration $\mathcal{N}^{\geqslant *}\widehat{A_{\mathrm{crys}}}(R)$ obtained by completion, and the completed divided Frobenius $\varphi/p^i : \mathcal{N}^{\geqslant i}\widehat{A_{\mathrm{crys}}}(R) \to \widehat{A_{\mathrm{crys}}}(R)$ for $i \geqslant 0$.

Proposition 8.15. *For $R \in \mathrm{QRSPerfd}_{\mathbb{F}_p}$, the ring $\widehat{A_{\mathrm{crys}}}(R)$ is p-complete and p-torsionfree. Moreover, we have a natural isomorphism $\widehat{A_{\mathrm{crys}}}(R)/p \simeq \widehat{L\Omega_{R/\mathbb{F}_p}}$ for $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$.*

Proof. First, $A_{\mathrm{crys}}(R)$ is clearly derived p-complete as an inverse limit of modules of bounded torsion. The p-torsionfreeness follows because for $x \in A_{\mathrm{crys}}(R)$, if $px \in \mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$, then $x \in \mathcal{N}^{\geqslant i-1}A_{\mathrm{crys}}(R)$; this is evident from the definition of the Nygaard filtration. Next, one verifies that the map $A_{\mathrm{crys}}(R)/p \to L\Omega_{R/\mathbb{F}_p}$ (as in Theorem 8.6) carries $\mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$ to the ith stage of the Hodge filtration $L\Omega^{\geqslant i}_{R/\mathbb{F}_p}$, for example, by calculating explicitly in the case of $\mathbb{F}_p[x^{1/p^\infty}]/(x)$. The result then follows, cf. [30, Theorem 8.14]. $\square$

When we descend from the quasi-regular semiperfect $\mathbb{F}_p$-algebras, we find that $\widehat{LW\Omega_R}/p \simeq \widehat{L\Omega_{R/\mathbb{F}_p}}$ for $R \in \mathrm{QSyn}_{\mathbb{F}_p}$, that is, Nygaard completed $LW\Omega\_$ gives a p-adic lift of Hodge-completed derived de Rham cohomology. For smooth algebras over a perfect ring, it follows that the Hodge or Nygaard completion does nothing. More generally, by Remark 7.6, this also follows for Cartier smooth algebras:

Corollary 8.16. *Suppose R is a Cartier smooth $\mathbb{F}_p$-algebra (for example, a smooth algebra over a perfect field). Then the map $LW\Omega_R \to \widehat{LW\Omega_R}$ is an equivalence, that is, the Nygaard filtration is automatically complete.*

Proposition 8.17. *If the $\mathbb{F}_p$-algebra R is Cartier smooth, then the map in the filtered derived category*

$$\varphi : \left\{\mathcal{N}^{\geqslant *}LW\Omega_R\right\} \to \{p^* LW\Omega_R\}$$

has the property that on gr^i the map is the truncation $\tau^{\leqslant i}$. (Equivalently, the map exhibits the source as the connective cover in the Beilinson t-structure, cf. [30, Section 5], of the target.)

Proof. Indeed, this follows because on associated graded terms, the divided Frobenius is identified with the map $L(\tau^{\leqslant i}\Omega^\bullet)_R \to L\Omega^\bullet_{R/\mathbb{F}_p}$, due to Proposition 8.12 (and the following discussion) by left Kan extension and descent. The hypothesis of Cartier smoothness implies that this map implements $\tau^{\leqslant i}$-truncation, that is, $L(\tau^{\leqslant i}\Omega^\bullet_R) = \tau^{\leqslant i}\Omega^\bullet_R$. $\square$

9 | THE MOTIVIC FILTRATIONS FOR QUASI-REGULAR SEMIPERFECT $\mathbb{F}_p$-ALGEBRAS

In this section, we describe the identification between the associated graded pieces $\mathcal{N}^{\geqslant i}\mathbb{\Delta}_R\{i\}$ of the motivic filtration on $\mathrm{TC}^-(R;\mathbb{Z}_p)$ and the Nygaard-completed Nygaard pieces of crystalline cohomology, for R a quasi-syntomic $\mathbb{F}_p$-algebra. By the usual descent, we may work with $\mathrm{QRSPerf}_{\mathbb{F}_p}$.

Let $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$ be a quasi-regular semiperfect $\mathbb{F}_p$-algebra. From the cyclotomic spectrum $\mathrm{THH}(R)$, we obtain the following objects.

(1) The spectra $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}}, \mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}$, which have homotopy groups concentrated in even degrees, calculated via the $\mathbb{T}$-homotopy fixed point and Tate spectral sequences.
(2) An identification $\mathrm{TP}(R)/p = \mathrm{HP}(R/\mathbb{F}_p)$ (a special case of (14)).
(3) The cyclotomic Frobenius $\varphi : \mathrm{TC}^-(R) \to \mathrm{TP}(R)$.

Consequently, given a quasi-regular semiperfect $\mathbb{F}_p$-algebra $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$, we obtain the following.

(1) A p-adically complete, p-torsionfree ring $\widehat{\mathbb{\Delta}_R} \overset{\text{def}}{=} \pi_0(\mathrm{TC}^-(R))$ and an endomorphism $\varphi_{\mathrm{cyc}} : \widehat{\mathbb{\Delta}_R} \to \widehat{\mathbb{\Delta}_R}$. induced by the cyclotomic Frobenius.
(2) A descending, multiplicative complete filtration $\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}_R}$ arising from the homotopy fixed point spectral sequence; in particular, $\mathcal{N}^{=i}\mathbb{\Delta}_R = \pi_{2i}\mathrm{THH}(R)$. Moreover, we have $\mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}_R} = x^i\pi_{2i}\mathrm{TC}^-(R) \subset \pi_0\mathrm{TC}^-(R)$.
(3) The property that $\varphi_{\mathrm{cyc}}(\mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_R) \subset p^i\widehat{\mathbb{\Delta}}_R$.
(4) A canonical, multiplicative isomorphism $\widehat{\mathbb{\Delta}}_R/p \simeq \widehat{L\Omega_{R/\mathbb{F}_p}}$.
(5) When R is perfect, $\widehat{\mathbb{\Delta}_R} = W(R)$, the endomorphism φ_{cyc} identifies with the (Witt vector) Frobenius, the filtration $\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}_R}$ is the p-adic filtration.

Theorem 9.1 (cf. [30, Theorem 8.17]). *For $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$, there is a functorial isomorphism of rings $\widehat{A_{\mathrm{crys}}}(R) \simeq \widehat{\mathbb{\Delta}_R}$, carrying the Nygaard filtration on $\widehat{A_{\mathrm{crys}}}(R)$ to $\{\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}}_R\}$. The cyclotomic Frobenius φ_{cyc} on $\widehat{\mathbb{\Delta}_R}$ agrees with the endomorphism induced by the Frobenius $\varphi : R \to R$.*

Corollary 9.2 (The motivic filtration). *For $R \in \mathrm{QSyn}_{\mathbb{F}_p}$, there is a convergent and exhaustive descending $\mathbb{Z}$-indexed multiplicative filtration* $\mathrm{Fil}^{\geqslant *}\mathrm{TC}^-(R), \mathrm{Fil}^{\geqslant *}\mathrm{TP}(R)$ *such that*

$$\mathrm{gr}^i\mathrm{TC}^-(R) = \mathcal{N}^{\geqslant i}\widehat{LW\Omega_R}[2i] \tag{23}$$

$$\mathrm{gr}^i\mathrm{TP}(R) = \widehat{LW\Omega_R}[2i]. \tag{24}$$

With respect to these filtrations, the cyclotomic Frobenius is the divided Frobenius φ/p^i on the ith graded piece.

We will give a proof of these results below, in a slightly different manner than [30]. While there is an explicit topological argument in [30] in the case of certain algebras, we argue instead using the following rigidity property of crystalline cohomology as a p-adic deformation of de Rham homology.

Theorem 9.3 (cf. [28, Theorem 10.1.2]). *Let* $R \mapsto F(R)$, $\mathrm{QRSPerf}_{\mathbb{F}_p} \to \mathrm{Rings}$ *be a functor on quasi-regular semiperfect* $\mathbb{F}_p$*-algebras taking values in* p*-adically complete,* p*-torsionfree rings. Suppose given an isomorphism of ring-valued functors* $F(-)/p \simeq L\Omega_{-/\mathbb{F}_p}$*. Then there is a unique isomorphism* $F(-) \simeq A_{\mathrm{crys}}(-)$ *lifting the specified isomorphism modulo* p*.*

Remark 9.4. Theorem 9.3 has very recently been generalized by Mondal [103]: de Rham cohomology of $\mathbb{F}_p$-algebras admits a unique deformation over any local Artinian ring with residue field $\mathbb{F}_p$ (coming from the base-change of crystalline cohomology). Mondal's work relies on some of the stacky ideas studied by Drinfeld [45].

Proof of Theorem 9.1. To begin with, we cannot directly apply Theorem 9.3 since for a quasi-regular semiperfect $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$, we have $\widehat{\mathbb{\Delta}_R}/p = \widehat{L\Omega_{R/\mathbb{F}_p}}$, that is, we obtain the Hodge completion of the derived de Rham cohomology. We thus need to first 'decomplete'; this will follow Remark 7.13 (in the case where the perfectoid base is $\mathbb{F}_p$). To do this, we define the construction $R \mapsto \{\mathcal{N}^{\geqslant *}\mathbb{\Delta}_R\}$ on $\mathrm{QSyn}_{\mathbb{F}_p}$ by restricting $R \mapsto \{\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}_R}\}$ to finitely generated polynomial $\mathbb{F}_p$-algebras and then left Kan extending to all quasi-syntomic $\mathbb{F}_p$-algebras. By construction (and Theorem 7.5), it follows that $R \mapsto \mathbb{\Delta}_R$ is a p-adic deformation of $R \mapsto L\Omega_{R/\mathbb{F}_p}$; in particular, $\mathbb{\Delta}$ defines a sheaf of spectra on $\mathrm{QSyn}_{\mathbb{F}_p}$. It is easy to see that the completion of the filtered sheaf $\{\mathcal{N}^{\geqslant *}\mathbb{\Delta}_R\}$ is indeed $\{\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}_R}\}$ (since the associated graded terms of $\{\mathcal{N}^{\geqslant *}\widehat{\mathbb{\Delta}_R}\}$, that is, $\pi_{2*}\mathrm{THH}(R)$, are already left Kan extended from their unfolding to polynomial algebras).

It follows from Theorem 9.3 that there is a unique functorial isomorphism (for $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$) $\mathbb{\Delta}_R \simeq A_{\mathrm{crys}}(R)$ for $R \in \mathrm{QRSPerf}_{\mathbb{F}_p}$ compatible with the isomorphism mod p to $L\Omega_{R/\mathbb{F}_p}$; in particular, this isomorphism is compatible the projection maps to R. Moreover, again by left Kan extension the cyclotomic Frobenius defines an endomorphism

$$\varphi_{\mathrm{cyc}} : \mathbb{\Delta}_R \to \mathbb{\Delta}_R$$

carrying $\mathcal{N}^{\geqslant i}\mathbb{\Delta}_R$ into $p^i\mathbb{\Delta}_R$. We observe that $\varphi_{\mathrm{cyc}} : \mathbb{\Delta}_R \to \mathbb{\Delta}_R$ (or $A_{\mathrm{crys}}(R) \to A_{\mathrm{crys}}(R)$) is necessarily the endomorphism induced by functoriality from the Frobenius $\varphi : R \to R$. Note first that this is indeed the case when R is perfect, by Theorem 6.3, and $\mathbb{\Delta}_R = A_{\mathrm{crys}}(R) = W(R)$. It follows that when R is semiperfect, we have by naturality (along $R^\flat \to R$) a commutative diagram

$$\begin{tikzcd} W(R^\flat) = A_{\mathrm{crys}}(R^\flat) \arrow[r, "\varphi_{\mathrm{cyc}}"] \arrow[d] & W(R^\flat) \arrow[d] \\ A_{\mathrm{crys}}(R) \arrow[r, "\varphi_{\mathrm{cyc}}"] & A_{\mathrm{crys}}(R). \end{tikzcd}$$

It follows that φ_{cyc} and $A_{\mathrm{crys}}(\varphi)$ are both endomorphisms of $A_{\mathrm{crys}}(R)$ which agree when restricted to $W(R^\flat)$; now taking divided power envelopes and p-completing again show that they agree.

We have now shown that there is an isomorphism $\mathbb{\Delta}_R \simeq A_{\mathrm{crys}}(R)$, compatible with Frobenii (the cyclotomic Frobenius and the Frobenius induced by functoriality), so we simply write φ. Now $\varphi(\mathcal{N}^{\geqslant i}\mathbb{\Delta}_R) \subset p^i\mathbb{\Delta}_R$. It follows that under the above comparison, we have $\mathcal{N}^{\geqslant i}\mathbb{\Delta}_R \subset \mathcal{N}^{\geqslant i}A_{\mathrm{crys}}(R)$ as submodules of $\mathbb{\Delta}_R = A_{\mathrm{crys}}(R)$: that is, the filtration coming from THH is contained in the Nygaard filtration. It remains to show that both filtrations are actually equal. Given this, it will follow that $\widehat{\Delta}_R = \widehat{A_{\text{crys}}}(R)$, compatible with filtrations and Frobenii (by p-adic continuity).

It suffices to show that the inclusion $\mathcal{N}^{\geqslant i}\Delta_R \subset \mathcal{N}^{\geqslant i}A_{\text{crys}}(R)$ is an equality for each i and for the ring $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. Indeed, the inclusion will then be an equality for any tensor product of such rings. Now any quasi-regular semiperfect $\mathbb{F}_p$-algebra admits a surjection from a tensor product R' of such rings which also induces a surjection on cotangent complexes, from which we see that $\Delta_{R'} \to \Delta_R$ and $A_{\text{crys}}(R') \to A_{\text{crys}}(R)$ induce surjections on filtered pieces (cf. [30, Proposition 8.12] for this argument). Thus we can reduce to the case $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$.

Suppose $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. we know that $\Delta_R = A_{\text{crys}}(R)$ is the p-completion of $\mathbb{Z}_p[x^{1/p^\infty}, \frac{x^j}{j!}]_{j\geqslant 0}$. Indeed, it suffices (due to the explicit description of the Nygaard filtration in this case, and since $p \in \mathcal{N}^{\geqslant 1}\Delta_R$) to see that $\frac{x^i}{i!}$ belongs to the image of $\mathcal{N}^{\geqslant i}\Delta_R \to \Delta_R$ or equivalently maps to zero in the quotient $\Delta_R/\mathcal{N}^{\geqslant i}\Delta_R$. For this, we use a grading argument, also used in [28, Section 10.2]. To this end, we can replace R by $k[x^{1/p^\infty}]/(x)$ for k a perfect field containing a transcendental element t; this admits an automorphism given by sending $x^i \mapsto t^i x^i, i \in \mathbb{Z}[1/p]_{\geqslant 0}$. Now $\frac{x^i}{i!}$ has weight i with respect to the induced grading (from this automorphism), while the weights of $\Delta_R/\mathcal{N}^{\geqslant i}\Delta_R$ are less than i as one sees from comparing $\mathcal{N}^j\Delta_R = \pi_{2j}\mathrm{THH}(R)$, which admits a finite filtration by $\bigwedge^{j'} L_{R/\mathbb{F}_p}[-j']$ for $j' \leqslant j$. □

We now unwind the construction explicitly for Cartier smooth algebras and in particular verify the Segal conjecture. In the context of topological Hochschild homology, the *Segal conjecture* refers to the assertion that the cyclotomic Frobenius $\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ should be an equivalence in sufficiently high degrees after p-completion. The reason for the name comes from the case $R = \mathbb{S}$; in this case, $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$ and the Frobenius (or the unit map) $\mathbb{S} \to \mathbb{S}^{tC_p}$ is actually a p-adic equivalence [63, 91],[†] which is the special case of the Segal Burnside ring conjecture for the group C_p (in general a theorem of Carlsson [40]). In the classical approach to topological cyclic homology, this implies that the genuine C_{p^n}-fixed points of THH agree p-adically with the C_{p^n}-homotopy fixed points for all $n \geqslant 0$, an insight due to [38, 118] and formalized in [105, Corollary II.4.9]. Many cases in which topological cyclic homology has been effectively computed for ring spectra such as [13, 14] are cases where the Segal conjecture holds, and the Segal conjecture for THH seems to play a central role in the theory.

Given this, it would be of interest to better understand the class of quasisyntomic rings for which some version of the Segal conjecture holds; this seems closely related to some type of regularity of the ring. Since everything in sight is endowed with a motivic filtration, one expects to see the Segal conjecture at the level of filtered pieces.

We will describe this in characteristic p. First, if R is any $\mathbb{F}_p$-algebra, we have an equivalence of spectra $\mathrm{THH}(R)^{tC_p} \simeq \mathrm{HP}(R/\mathbb{F}_p) \simeq \mathrm{TP}(R)/p$, cf. [30, Proposition 6.4]. Consequently, for $R \in \mathrm{QSyn}_{\mathbb{F}_p}$, we can define a complete, $\mathbb{Z}$-indexed descending multiplicative motivic filtration on $\mathrm{THH}(R)^{tC_p}$ such that $\mathrm{gr}^i = \widehat{L\Omega_{R/\mathbb{F}_p}}[2i]$. With respect to this, the cyclotomic Frobenius $\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ is a filtered map, and on gr^i it is given by the map φ/p^i :

[†] See also [64] for a recent proof at $p = 2$ using topological Hochschild homology.

$\mathcal{N}^{\geqslant i}\Delta_R/\mathcal{N}^{\geqslant i+1}\Delta_R \to \widehat{\Delta}_R/p = \widehat{L\Omega_{R/\mathbb{F}_p}}$; this follows from the commutative diagram

$$\begin{tikzcd} \mathrm{TC}^-(R) \arrow[r, "\varphi"] \arrow[d] & \mathrm{TP}(R) \arrow[d] \\ \mathrm{THH}(R) \arrow[r, "\varphi"] & \mathrm{THH}(R)^{tC_p} \end{tikzcd}$$

and descent from $\mathrm{QRSPerf}_{\mathbb{F}_p}$, since the cyclotomic Frobenius realizes the divided Frobenius. Note that $\mathrm{TC}^-(R)/x = \mathrm{THH}(R)$ for $x \in \pi_{-2}(\mathrm{TC}^-(\mathbb{F}_p))$ as in Theorem 6.3 and the following discussion.

The next result (for smooth algebras over a perfect field) appears in [30, Corollary 8.18]. The last part had previously appeared in [68].

Corollary 9.5. *For $R/\mathbb{F}_p$ Cartier smooth, the cyclotomic Frobenius* $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ *has the property that on* $\mathrm{gr}^i[-2i]$*, it identifies with the $(-i)$-connective cover $\tau^{\leqslant i}(\Omega^*_{R/\mathbb{F}_p}) \to \Omega^*_{R/\mathbb{F}_p}$. If $\Omega^i_{R/\mathbb{F}_p} = 0$ for $i > d$ (for example, R could be smooth of dimension d over a perfect ring), then* $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ *has $(d-3)$-truncated homotopy fiber.*

Proof. This follows from Proposition 8.17, given the above discussion and description of the map $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ on associated graded pieces. Note that the fiber of the map on gr^i lives in degrees $\leqslant i-2$ for each i; under the second hypothesis, the map φ moreover induces isomorphisms on associated graded gr^i for $i \geqslant d$, which yields the second assertion. □

Question 9.6. Note that the results of [30, Section 9] (as well as [32]) prove the Segal conjecture for smooth algebras over a perfectoid ring. The Segal conjecture does hold for p-complete regular Noetherian rings (with mild finiteness hypotheses), cf. [100, Section 5] for an argument that relies on the Beilinson–Lichtenbaum conjecture for the generic fiber. For smooth algebras over the ring of integers in a p-adic field (for $p > 2$), this was proved (in a purely p-adic fashion, using TR) in [71, 72]. Can one prove a filtered version of the Segal conjecture in such cases?

Remark 9.7. Another approach to the motivic filtration on TP in characteristic p has been given in [10], using the expression $\mathrm{TP}(-;\mathbb{Z}_p) = (\mathrm{TR}(-;\mathbb{Z}_p))^{t\mathbb{T}}$ proved in [10] using that TR_* for a regular $\mathbb{F}_p$-algebra recovers the de Rham–Witt complex [65]. This approach does not seem to recover the filtration in mixed characteristic, though.

10 | THE $\mathbb{Z}_p(i)$: AN EXAMPLE AND SOME QUESTIONS

In this section, we revisit the calculation of the p-adic K-theory (or equivalently the topological cyclic homology) of the dual numbers over a perfect field k of characteristic p. This calculation is due to Hesselholt–Madsen [70], using the methods of equivariant stable homotopy theory, and has since been extended and generalized in various directions (see, for instance, [3, 4, 66, 69]). The calculation (more generally for truncated polynomial algebras) was recently revisited by Speirs [113], who gave another approach using the Nikolaus–Scholze formula (6) for TC.

To begin with, let us discuss some aspects of the motivic filtration on TC in particular.

Construction 10.1 (The motivic filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$). Given an animated $\mathbb{Z}_p$-algebra R, there is a natural complete, descending, multiplicative $\mathbb{Z}_{\geqslant 0}$-indexed filtration $\mathrm{Fil}^{\geqslant *}\mathrm{TC}(R;\mathbb{Z}_p)$ with

$$\mathrm{gr}^i\mathrm{TC}(R;\mathbb{Z}_p) = \mathbb{Z}_p(i)(R)[2i], \quad i \geqslant 0. \tag{25}$$

For quasi-syntomic rings, the motivic filtration is defined using descent as before: it is the double speed Postnikov filtration (in $\mathrm{Shv}(\mathrm{QSyn}, \mathrm{Sp})$). In particular, for a quasi-syntomic ring R, we have the expression

$$\mathbb{Z}_p(i)(R) = \mathrm{fib}(\mathrm{id} - \varphi_i : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_R\{i\} \to \widehat{\mathbb{\Delta}}_R\{i\}). \tag{26}$$

This is a consequence of (6), using that the two terms above are as the graded quotients of $\mathrm{TC}^-(R;\mathbb{Z}_p), \mathrm{TP}(R;\mathbb{Z}_p)$. Now a key feature of $\mathrm{TC}(-;\mathbb{Z}_p)$ (not shared by $\mathrm{TC}^-(-;\mathbb{Z}_p), \mathrm{TP}(-;\mathbb{Z}_p)$) is that it commutes with sifted colimits [43, Theorem G]. Moreover, one can check that the motivic filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$ when defined in the above manner on quasi-syntomic rings is actually left Kan extended from p-complete polynomial algebras, and has the property that $\mathrm{Fil}^{\geqslant i}\mathrm{TC}(R;\mathbb{Z}_p)$ is $(i-1)$-connective, cf. [8, Theorem 5.1]. Using these facts, one can left Kan extend the motivic filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$ from p-complete polynomial algebras to all p-complete animated rings. This will agree with the previous definition on quasi-syntomic rings and, because of the connectivity statement, will converge.

Let us describe the $\mathbb{Z}_p(i)$ in characteristic p.

Construction 10.2 (The $\mathbb{Z}_p(i)$ in characteristic p). For R an animated $\mathbb{F}_p$-algebra, we recall that we have the derived de Rham–Witt cohomology $LW\Omega_R$, the Nygaard filtration $\mathcal{N}^{\geqslant i}$, and the divided Frobenius $\varphi/p^i : \mathcal{N}^{\geqslant i}LW\Omega_R \to LW\Omega_R$. There is a natural isomorphism

$$\mathbb{Z}_p(i)(R) = \mathrm{fib}(\mathrm{id} - \varphi/p^i) : \mathcal{N}^{\geqslant i}LW\Omega_R \to LW\Omega_R. \tag{27}$$

In fact, for $R \in \mathrm{QSyn}_{\mathbb{F}_p}$, this expression after Nygaard completion is a consequence of (26) and the identification of $\widehat{\mathbb{\Delta}}_{-}$ and $\widehat{LW\Omega_{-}}$ on $\mathrm{QSyn}_{\mathbb{F}_p}$. Then the main observation is that the Nygaard completion is actually superfluous in the expression for the $\mathbb{Z}_p(i)$, by a p-adic continuity argument, cf. the proof of [30, Proposition 8.20]. Left Kan extending from finitely generated polynomial algebras, we obtain (27) in general.

Using the above, one can identify the $\mathbb{Z}_p(i)$ on regular $\mathbb{F}_p$-algebras as the pro-étale cohomology of the logarithmic de Rham–Witt sheaves $W\Omega^i_{\log}[-i]$, cf. [30, Corollary 8.21]; as in [83] this can be generalized to Cartier smooth algebras. By the results of [58], this shows that the $\mathbb{Z}_p(i)$ are in fact p-adic étale motivic cohomology for regular $\mathbb{F}_p$-schemes.

Remark 10.3 (The Frobenius action). In general, if R is any $\mathbb{F}_p$-algebra, a key feature of the $\mathbb{Z}_p(i)$ is that the Frobenius $\varphi : R \to R$ acts as multiplication by p^i on $\mathbb{Z}_p(i)(R)$; this is evident from its expression (27). In particular, the motivic filtration on $\mathrm{TC}(R;\mathbb{Z}_p)$ becomes, after rationalization, the eigenspace decomposition based on the Frobenius action, and consequently splits canonically. This is of course analogous to the motivic filtration on K-theory, which rationally diagonalizes the Adams operations.

We will give here a description of the $\mathbb{Z}_p(i)$ of $k[x]/x^2$ (which by the motivic filtration easily gives the description of $\mathrm{TC}(k[x]/x^2)$); the calculation is very close to that of [113]. The key observation is that $k[x]/x^2$ admits a natural lift to a quasisyntomic δ-ring, so we can use the divided power de Rham complex to compute everything. Our strategy is to use the fact (cf. Example 8.11) that if R is a p-torsionfree, p-complete δ-ring in QSyn, then there is a functorial divided Frobenius $\varphi/p^i : L\Omega^{\geqslant i}_{R/\mathbb{Z}_p} \to L\Omega_{R/\mathbb{Z}_p}$, and the Nygaard filtration on $L\Omega_{R/\mathbb{Z}_p} = LW\Omega_{R/p}$ is the tensor product of the p-adic filtration and the Hodge filtration. This is a consequence of the case of p-complete polynomial δ-rings (by a left Kan extension argument), where it follows by a direct comparison between the de Rham complex of R and the de Rham–Witt complex of R/p. Compare [30, Section 8.1.2].

Now let (A, I) be a pair such that $A, A/I$ are equipped with the compatible structure of δ-rings, and suppose both $A, A/I$ are p-complete, p-torsionfree, and quasi-syntomic. Suppose A/p is Cartier smooth and the Frobenius on A/p is flat. As we have seen, the Nygaard filtration on $LW\Omega_{A/(I,p)} = L\Omega_{(A/I)/\mathbb{Z}_p}$ identifies with the tensor product of the Hodge filtration and the p-adic filtration. Using Theorem 7.16, we can identify the de Rham complex (with its Hodge filtration, and Frobenius) of A/I as the divided power de Rham complex of A, with divided powers along I, and with the divided power filtration. In light of this, we obtain an explicit cochain complex representing $\mathbb{Z}_p(i)(A/(I, p))$.

Indeed, we construct the divided power de Rham complex $\Omega^\bullet_{D_I(A)}$ with the (cochain-level) Nygaard filtration $\mathcal{N}^{\geqslant *}\Omega^\bullet_{D_I(A)}$ (defined as the tensor product filtration as above). The Frobenius lift $\varphi : A \to A$ induces a Frobenius lift φ on $\Omega^\bullet_{D_I(A)}$ which becomes divisible by p^i on the subcomplex[†] $\mathcal{N}^{\geqslant i}\Omega^\bullet_{D_I(A)}$, and we can take

$$\mathbb{Z}_p(i)(A/(I, p)) = \mathrm{fib}\left(\mathcal{N}^{\geqslant i}\Omega^\bullet_{D_I(A)} \xrightarrow{\varphi/p^i - 1} \Omega^\bullet_{D_I(A)}\right). \tag{28}$$

To see this, we may reduce by descent to the case where $A/(I, p)$ is quasi-regular semiperfect and A is a perfect δ-ring (where the Frobenius φ is an isomorphism), and then the above is effectively the definition. Thus, we get an expression for $\mathbb{Z}_p(i)(A/(I, p))$ as the mapping fiber of a cochain map between explicit cochain complexes.

In this section, we illustrate the above method by proving the following result. By the motivic filtration on TC, this reproves the result of [70, Theorem 8.2] (and by [101] yields the calculation of $K_*(k[x]/x^2; \mathbb{Z}_p)$ since $K_*(k; \mathbb{Z}_p) = \mathbb{Z}_p$ in degree zero, [74, 85]).

Theorem 10.4. *Let k be a perfect $\mathbb{F}_p$-algebra for $p > 2$. Then for $i > 0$, we have that $\mathbb{Z}_p(i)(k[x]/x^2)$ has no cohomology in degree outside 1. Moreover, $H^1(\mathbb{Z}_p(i)(k[x]/x^2))$ is isomorphic to a direct sum $\bigoplus_{1\leqslant d\leqslant 2i-1,(d,2p)=1} W_{n(i,d)}(k)$ where $n = n(i, d)$ is chosen such that $p^{n-1}d \leqslant 2i - 1 < p^n d$.*

Proof. In the above strategy, we consider the example $(A, I) = (\widehat{W(k)[x]}_p, (x^2))$ where the δ-structure is such that $\delta(x) = 0$, so the Frobenius lift carries $x \mapsto x^p$. It follows from Theorem 7.16

[†] This crucially uses that I is preserved by δ. For example, if $x \in I$, then the element $\frac{x^i}{i!} \in \mathrm{Fil}^{\geqslant i} D_I(A)$ has the property that $\varphi(\frac{x^i}{i!}) = \frac{(x^p + p\delta(x))^i}{i!} = \sum_{a+b=i} \frac{x^{pa} p^b \delta(x)^b}{a!b!}$. Each term in the sum is divisible by p^i in $D_I(A)$.

that $LW\Omega_{k[x]/x^2}$ is given by the p-completion of the divided power de Rham complex

$$W(k)\left[x, \frac{x^{2j}}{j!}\right]_{j\geqslant 0} \to W(k)\left[x, \frac{x^{2j}}{j!}\right]_{j\geqslant 0} dx. \tag{29}$$

The Hodge filtration is (as in Theorem 7.16 again), given by the divided power filtration and the naive filtration. That is, $L\Omega^{\geqslant i}_{W(k)[x]/x^2}$ corresponds to the p-completion of the subcomplex

$$\bigoplus_{j\geqslant i} W(k)\left\{\frac{x^{2j}}{j!}, \frac{x^{2j+1}}{j!}\right\}_{j\geqslant i} \xrightarrow{d} \bigoplus_{j\geqslant i-1} W(k)\left\{\frac{x^{2j}}{j!}, \frac{x^{2j+1}}{j!}\right\}dx. \tag{30}$$

Also, the Frobenius is defined on the complex by sending $x \mapsto x^p$.

It follows now from Example 8.11 that $\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2}$ is given by the p-completion of the complex

$$\bigoplus_{j\geqslant 0} p^{\max(i-j,0)}W(k)\left\{\frac{x^{2j}}{j!}, \frac{x^{2j+1}}{j!}\right\} \xrightarrow{d} \bigoplus_{j\geqslant 0} p^{\max(i-j-1,0)}W(k)\left\{\frac{x^{2j}}{j!}, \frac{x^{2j+1}}{j!}\right\}dx. \tag{31}$$

In particular, due to (28), we obtain that $\mathbb{Z}_p(i)(k[x]/x^2)$ is the mapping fiber of $\varphi/p^i - 1$ from the complex (31) to (29).

Let us evaluate the $\mathbb{Z}_p(i)(k[x]/x^2)$. First, we can evaluate the cohomology of $LW\Omega_{k[x]/x^2}$. Note that (other than the $W(k)$ in internal and cohomological degree zero) only H^1 is nonzero, and it has a natural grading (where $|x| = 1$); with respect to this grading, we have easily from (29)

$$\left(H^1(LW\Omega_{k[x]/x^2})\right)_d = \begin{cases} W(k)/d & d \text{ odd} \\ 0 & d \text{ even.} \end{cases}$$

Explicitly, the generator in degree $d = 2j + 1$ is $\frac{x^{2j}}{j!}dx$.

Similarly, from (31) we see that $\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2}$ has cohomology concentrated in (cohomological) degree 1 other than $p^iW(k)$ in internal and cohomological degree zero,[†] and with respect to the internal grading, we have

$$H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_d = \begin{cases} W(k)/pd & d = 2j+1, j < i \\ W(k)/d & d = 2j+1, j \geqslant i\,. \\ 0 & d \text{ even} \end{cases} \tag{32}$$

Explicitly, the generator in degree $2j + 1$ is $p^{\max(i-j-1,0)}\frac{x^{2j}}{j!}dx$.

Now we need to understand the canonical and divided Frobenius maps.

[†] Alternatively, we could phrase everything in terms of the relative cohomology relative to the ideal (x), and ignore these degree zero terms; in any case they do not contribute to the $\mathbb{Z}_p(i)$ for $i > 0$.

(1) Consider $\varphi/p^i : \mathcal{N}^{\geqslant i} LW\Omega_{k[x]/x^2} \to LW\Omega_{k[x]/x^2}$. This map multiplies the internal grading by p. Suppose $j < i$. Then it carries the class of $p^{i-j-1}\frac{x^{2j}}{j!}dx$ into

$$p^{-i}p^{i-j-1}\frac{x^{p2j}d(x^p)}{j!} = p^{-j}\frac{x^{(2j+1)p-1}}{j!}dx. \tag{33}$$

Now $p^j j!$, $(jp)!$, and $(\frac{(2j+1)p-1}{2})!$ agree up to p-adic units. It follows from this that for $d = 2j+1$ for $j < i$, φ/p^i carries the degree d summand $H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_d$ isomorphically to the degree pd summand of $H^1(LW\Omega_{k[x]/x^2})_{pd}$ (in fact, it carries a generator to the generator due to (33)).

(2) For $j \geqslant i$, the map φ/p^i carries the generator in degree $d = 2j+1$ to a nonunit multiple of the generator in degree pd. More precisely, the generator $\frac{x^{2j}}{j!}dx$ is carried to p^{j-i+1} times a generator (arguing as above).

(3) For $d = 2j+1$ for $j \geqslant i$, the canonical map induces an isomorphism on degree d summands.

Fix an odd integer $d \geqslant 1$ such that d is not divisible by p. In this case, we consider the map of abelian groups

$$\bigoplus_{a\geqslant 0} H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_{p^a d} \xrightarrow{\varphi/p^i - 1} \bigoplus_{a\geqslant 0} H^1(LW\Omega_{k[x]/x^2})_{p^a d} \tag{34}$$

This suffices for the calculation of $\mathbb{Z}_p(i)(k[x]/x^2)$, since we can decompose the map $\varphi/p^i - 1$ over such d.

Let $n = n(i,d)$ be such that $p^{n-1}d \leqslant 2i-1 < p^n d$. We claim that (34) is surjective, and the kernel is $W_{n(i,d)}(k)$ if $d \leqslant 2i-1$ (and zero if $d > 2i-1$). The map

$$\bigoplus_{a\geqslant n} H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_{p^a d} \xrightarrow{\varphi/p^i - 1} \bigoplus_{a\geqslant n} H^1(LW\Omega_{k[x]/x^2})_{p^a d} \tag{35}$$

is seen to be an isomorphism after p-completion since the canonical map is an isomorphism while the divided Frobenius is locally nilpotent, thanks to the calculation in item (2) above. Thus, to prove the claim about (34), it suffices to quotient by the summands for $a \geqslant n$, and to consider the map

$$\bigoplus_{a<n} H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_{p^a d} \xrightarrow{\varphi/p^i - 1} \bigoplus_{a<n} H^1(LW\Omega_{k[x]/x^2})_{p^a d}. \tag{36}$$

In fact, the enumerated statements above show (e.g., by filtering both sides and passing to associated graded) that

$$\bigoplus_{a<n-1} H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_{p^a d} \xrightarrow{\varphi/p^i - 1} \bigoplus_{a<n} H^1(LW\Omega_{k[x]/x^2})_{p^a d}$$

is an isomorphism, whence the kernel of (36) is isomorphic to $H^1(\mathcal{N}^{\geqslant i}LW\Omega_{k[x]/x^2})_{p^{n-1}d} = W_n(k)$ (by (32)), as desired. □

It would be interesting to revisit the various calculations of topological cyclic homology of p-adic rings, traditionally carried out using TR and equivariant stable homotopy theory, using the motivic filtration (and in particular to calculate the $\mathbb{Z}_p(i)$). In principle, the above gives a purely algebraic approach to the calculation for $\mathbb{F}_p$-algebras with lci singularities.

Question 10.5. Can one calculate the $\mathbb{Z}_p(i)$ of $\mathbb{F}_p$-algebras with worse than lci singularities?

A basic example would be the case of a square-zero extension $k \oplus V$, for k a perfect field and V a k-vector space, where the K-theory is calculated in [92]. This calculation has been extended to perfectoid rings by Riggenbach [109], using the approach to TC of [105].

In mixed characteristic, one knows [30, Section 10] that for formally smooth algebras over $\mathcal{O}_C$, the $\mathbb{Z}_p(i)$ are given by the truncated p-adic nearby cycles of the usual Tate twists on the generic fiber; they thus are closely related to integral p-adic Hodge theory. For $i \leqslant p-2$, or when one works up to bounded denominators, it is shown in [8] that that the $\mathbb{Z}_p(i)$ recover 'syntomic cohomology' in a form essentially due to [50, 80]. It would be interesting to carry out more calculations of the $\mathbb{Z}_p(i)$ and TC in mixed characteristic.

Finally, the K-theory of $\mathbb{Z}/p^2$ is only known in a limited range [2, 39].

Question 10.6. Can one compute $\mathbb{Z}_p(i)(\mathbb{Z}/p^n)$ (and thus the K-theory of $\mathbb{Z}/p^n$) for $n > 1$?

The work [25] uses prismatic cohomology to show that $L_{K(1)}K(\mathbb{Z}/p^n) = 0$ for $n \geqslant 1$; this fact (and some generalizations) are also proved by different methods in [87, 100]. However, accessing the p-adic K-groups (or the $\mathbb{Z}_p(i)$) themselves seems to be substantially more difficult. A stacky approach to prismatic cohomology has been proposed by Drinfeld [45] and Bhatt–Lurie, and in particular one expects that the coherent cohomology of the object Σ'' introduced in [45] should be related to the $\mathbb{Z}_p(i)$. We hope that an increased understanding of the structure of Σ'' and of prismatic cohomology in general will also shed some light on these K-theoretic questions. The very recent work of Liu–Wang [93] on calculating $\mathrm{TC}(\mathcal{O}_K; \mathbb{F}_p)$ via descent-theoretic methods is an important step in this direction.

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