# The Algebra of Categorical Spectra

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# Chapter 1

# Introduction

higher category theory. Deeper category theory, deeper algebra.

# Chapter 2

# Background

## 2.1 Preliminaries on category theory

### 2.1.1 $n$-categories and $n$-algebroids

There are a few different ways to work with $(\infty, n)$-categories. In this thesis, we take the *model-independent* approach: this means we let $n\mathsf{Cat}$ be *the* (large $(\infty,1)$-)category of $(\infty, n)$-categories, as well as the larger category $n\mathsf{Algbrd} = n\mathsf{Cat}^{\mathrm{f}}$ of $n$-algebroids (aka. flagged $(\infty, n)$-categories) without choosing a point-set presentation of it. In doing so, we base on some preexisting robust theory of $(\infty, 1)$-categories, such as in [Lur09b]. We start by recalling some definitions about *strict* $\infty$-categories; they will play the role of combinatorial blueprints in the study of weak $\infty$-categories.

**Definition 2.1.1.** A *strict* 0*-category* is a set: $0\mathsf{Cat}^{\mathrm{str}} \coloneqq \mathsf{Set}$. Let $n \geq 0$ and suppose inductively that the $(1, 1)$-category $n\mathsf{Cat}^{\mathrm{str}}$ of strict $n$-categories is already defined. Then a *strict* $(n+1)$*-category* is a strictly $n\mathsf{Cat}^{\mathrm{str}}$-enriched category: $(n + 1)\mathsf{Cat}^{\mathrm{str}} \coloneqq (n\mathsf{Cat}^{\mathrm{str}})\text{-}\mathsf{Cat}^{\mathrm{str}}$. The inclusion $n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow (n+1)\mathsf{Cat}^{\mathrm{str}}$ admits both left and right adjoint, denoted by ${}^{\leq n}(-)$ and $(-)^{\leq n}$, respectively. Let $\infty\mathsf{Cat}^{\mathrm{str}}$ be the colimit $\operatorname{colim}_n n\mathsf{Cat}^{\mathrm{str}} \in \mathsf{Pr}^{\mathsf{L}}$ along the inclusions $n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow (n + 1)\mathsf{Cat}^{\mathrm{str}}$ (or equivalently, the limit in $\widehat{\mathsf{Cat}}$ along the *categorical truncations* $(-)^{\leq n}$).

**Definition 2.1.2.** Let $X$ be a strict $n$-category. The *suspension* $\sigma X$ is a strict $(n + 1)$-category with two objects $\{\perp, \top\}$, $\mathrm{Hom}_{\sigma X}(\perp, \top) = X$, $\mathrm{Hom}_{\sigma X}(\perp, \perp) = * = \mathrm{Hom}_{\sigma X}(\top, \top)$, $\mathrm{Hom}_{\sigma X}(\top, \perp) = \emptyset$ (the compositions are uniquely determined). With the canonical bipointing $* \sqcup * = \sigma\emptyset \to \sigma X$, the functor $n\mathsf{Cat}^{\mathrm{str}} \to (n + 1)\mathsf{Cat}^{\mathrm{str}}_{**}$ is colimit-preserving with the right adjoint $(X, x_0, x_1) \mapsto \mathrm{Hom}_X(x_0, x_1)$.

**Definition 2.1.3.** Let $(X, x_0, x_1)$, $(Y, y_0, y_1)$ be bipointed strict $\infty$-categories. The *wedge sum* $X \vee Y$ is the pushout $((X \sqcup Y)/(x_1 = y_0)$ with bipointing $(x_0, y_1)$.

**Definition 2.1.4.**

- The *$n$-cell* $C_n$ is the strict $n$-category $\sigma^n(*)$. We let $\mathbb{G}_n$ denote the full subcategory $\{C_k \mid 0 \leq k \leq n\} \subset n\mathsf{Cat}^{\mathrm{str}}$ and $\mathbb{G} \coloneqq \mathbb{G}_\infty$.
- The *canonically bipointed theta category* $\Theta^{\mathrm{can}}_{**} \subset \infty\mathsf{Cat}^{\mathrm{str}}_{**}$ is the smallest full subcategory containing the terminal object $C_0$ and closed under the operations of suspension and wedge sum. The (Joyal's) *theta category* $\Theta \subset \infty\mathsf{Cat}^{\mathrm{str}}$ is the image of $\Theta^{\mathrm{can}}_{**}$ under the forgetful functor. Let $\Theta_n = \Theta \cap n\mathsf{Cat}^{\mathrm{str}}$.

*Remark* 2.1.5. The full subcategory $\Theta_n \subset n\mathsf{Cat}^{\mathrm{str}}$ is dense, i.e., the restricted Yoneda embedding $n\mathsf{Cat}^{\mathrm{str}} \to \mathsf{PSh}_{\mathsf{Set}}(\Theta_n)$ is fully faithful with the image characterized by a certain sheaf condition, called the *Segal condition*. One way to state it is as follows: any object $\theta \in \Theta$ admits a canonical colimit representation as the maximal cells (under the inclusion) glued along their shared boundary: $\operatorname{colim}_i C_{n_i} \xrightarrow{\sim} \theta$. Now a presheaf $P : \Theta_n^{\mathrm{op}} \to \mathcal{C}$ satisfies the Segal condition if for any $\theta \in \Theta_n$, the induced diagram $P(\theta) \to \lim_i P(C_{n_i})$ is a limit diagram. In contrast, the full subcategory $\mathbb{G}_n \subset n\mathsf{Cat}^{\mathrm{str}}$ is a set of colimit generators but not dense, i.e., the further restricted Yoneda embedding $n\mathsf{Cat}^{\mathrm{str}} \to \mathsf{PSh}_{\mathsf{Set}}(\mathbb{G}_n)$ is conservative but not fully faithful; it fails to remember compositions (cf. Remark 2.2.1).

Now we give a few descriptions of $n\mathsf{Algbrd}$ (see the references for precise definitions):

- (iterated enrichment, [Lur09a][GH15][Hin20][Ste21]) There is an assignment $X \mapsto \mathrm{Assos}_X$ giving the "many-object-version" of the associative operad for a given groupoid of objects $X \in \mathsf{S}$. For a symmetric monoidal category $\mathsf{V}$, a $\mathsf{V}$-algebroid[1] with the groupoid of objects $X$ is an $\mathrm{Assos}_X$-algebra in $\mathsf{V}$[2]. Unstraightening the contravariant functor $X \mapsto \mathsf{Algbrd}_X(\mathsf{V})$, one obtains the category $\mathsf{Algbrd}(\mathsf{V})$. Let $n\mathsf{Algbrd} := \mathsf{Algbrd}(\dots(\mathsf{Algbrd}(\mathsf{S}))\dots)$ be the $n$-fold iteration (with the cartesian monoidal structure). Let $\infty\mathsf{Algbrd}$ be the colimit of $n\mathsf{Algbrd}$ along the inclusions in $\mathsf{Pr}^{\mathsf{L}}$.
- ($\Theta$-presheaves, [Rez10]) The inclusion $\mathsf{Set} \hookrightarrow \mathsf{S}$ induces $n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow n\mathsf{Algbrd}$, whose restriction $\Theta_n \hookrightarrow n\mathsf{Algbrd}$ is dense, i.e., the restricted Yoneda embedding $n\mathsf{Algbrd} \to \mathsf{PSh}(\Theta_n)$ is fully faithful. The essential image is characterized by the same Segal condition as in 2.1.5.
- (presheaves on a suitable site) More generally, if $\mathcal{S} \subset n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow n\mathsf{Algbrd}$ is dense, then one can study $n\mathsf{Algbrd}$ by describing the localization $\mathsf{PSh}(\mathcal{S}) \to n\mathsf{Algbrd}$ and the combinatorics of $\mathcal{S}$. The localization admits an explicit description using the cell attachment of torsion-free complexes when $\mathcal{S}$ is *suitable* in the sense of [Cam23b, Theorem B].
- If we take the notion of $n$-categories (see below) as the primary one, the notion of $n$-algebroids is equivalent to that of *flagged* $n$-categories of [AF18]: $n\mathsf{Algbrd} \simeq n\mathsf{Cat}^{\mathrm{f}}$. Flagging is an extra structure of an $n$-category. Roughly speaking, it keeps track of the choices of the groupoids of objects ($X$ above) at each stage of enrichment, which is not invariant under categorical equivalence.

Our official definition is the first one; various presheaf presentations will replace the role of models. Notice $n$-algebroids are "evil" as a notion of categories, taken up to isomorphisms instead of equivalences. This is why it contains the $(1,1)$-category of strict $n$-categories, constructed out of strict enrichment. We define $n\mathsf{Cat}$ by fixing this:

- The category of $\mathsf{V}$-categories $\mathsf{V}\text{-}\mathsf{Cat} \subset \mathsf{Algbrd}(\mathsf{V})$ is the localization with respect to *categorical equivalences*, i.e., fully faithful and essentially surjective maps of algebroids. In fact, it suffices to invert $E \to *$, where $E$ is (the base change to $\mathsf{V}$ of) the contractible groupoid with two objects. Let $\mathsf{S} := 0\mathsf{Cat}$, $(n+1)\mathsf{Cat} := (n\mathsf{Cat})\text{-}\mathsf{Cat}$ and $\infty\mathsf{Cat} := \operatorname{colim}(\cdots \hookrightarrow n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat} \hookrightarrow \cdots)$ in $\mathsf{Pr}^{\mathsf{L}}$. The localization $n\mathsf{Cat} \subset n\mathsf{Algbrd}$ are generated by the *Rezk maps* $\sigma^k(E \to *)$, $0 \le k \le n$. The local objects are also called *univalent* or *Rezk-complete*, meaning that the prescribed groupoid of objects of the algebroid is in fact the maximal subgroupoid that is determined by the notion of equivalences *internal* to the algebroid.
- The Rezk maps considered as maps in $\mathsf{PSh}(\Theta)$ yields the localization $\mathsf{PSh}(\Theta_n) \to n\mathsf{Algbrd} \to n\mathsf{Cat}$. This is the original context by Rezk.
- In terms of flagged $n$-categories, the univalent complete objects are those with the *maximal* flags.

The intersection of $n\mathsf{Cat}$ and $n\mathsf{Cat}^{\mathrm{str}}$ in $n\mathsf{Algbrd}$ is $n\mathsf{Gaunt}$, the $(1,1)$-category of *gaunt* $n$-categories:

$$\begin{tikzcd} n\mathsf{Gaunt} \arrow[r, hook] \arrow[d, hook] & n\mathsf{Cat}^{\mathrm{str}} \arrow[r, hook] \arrow[d, hook] & \mathsf{PSh}_{\mathsf{Set}}(\Theta_n) \arrow[d, hook] \\ n\mathsf{Cat} \arrow[r, hook] & n\mathsf{Algbrd} \arrow[r, hook] & \mathsf{PSh}(\Theta_n) \end{tikzcd} \tag{2.1}$$

Each inclusion in the diagram is right adjoint to an ($\omega$-accessible) localization; the top row is the 0-truncated parts of the bottom row, whereas the passage to the middle and the left column from the right is by Segal conditions and univalence. In particular, all of these are compactly generated with compact generators $\mathbb{G}_n$. In most of what follows, the distinction between $n\mathsf{Cat}$ and $n\mathsf{Cat}^{\mathrm{f}} = n\mathsf{Algbrd}$

[1]These are called *categorical algebras* in [GH15]. In [Ste21], where the author took the notation, more general setting with the *category* of objects is considered. It is also worth mentioning that, since the relevant symmmetric monoidal structure on $\mathsf{V}$ are cartesian and $\mathsf{V}$ contains $\mathsf{S}$, $\mathsf{V}$-algebroids can also be defined as simplicial objects $X : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{V}$ satisfying $X_0 \in \mathsf{S}$ and the Segal condition $X_n \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

[2]When $X$ is a set and $\mathsf{V}$ is a $(1,1)$-category, we recover the notion of stirctly $\mathsf{V}$-enriched categories.

does not play much role, so we write $n\mathsf{Cat}^{(\mathrm{f})}$ when it does not matter. We write ${}^{\leq n}(-)$, $(-)^{\leq n}$ for the inclusion $n\mathsf{Algbrd} \to \infty\mathsf{Algbrd}$ (**right adjoint preserves homotopically truncated objects but still potentially conflict with strict one?**). The category of $\infty$-categories is a fixed point of enrichment [Ste21, Remark 3.6.12][Gol23] **check for algebroids**:

$$\infty\mathsf{Cat} \simeq (\infty\mathsf{Cat})\text{-}\mathsf{Cat}, \quad \infty\mathsf{Algbrd} \simeq \mathsf{Algbrd}(\infty\mathsf{Algbrd}).$$

*Remark* 2.1.6. There are also a few combinatorial presentations of the categories $n\mathsf{Cat}$ and $n\mathsf{Algbrd}$ using marked simplicial and cubical sets [Ver08][CKM20]. While these are useful for many purposes (allowing explicit combinatorial constructions), its category theory is not as throughly developed as the $n = 1$ case of quasi-categories. The twist is that the combinatorial building blocks (simplices) are not fibrant and the combinatorial complexity of their fibrant replacements (orientals) grows exponentially. One advantage of these models is that they work as relative categories by design. See [Lou22] for a proof that the model category for $n$-complicial sets indeed model $n\mathsf{Cat}$ described above. The result in this paper does not depend on these theories.

### 2.1.2 Suspension

Just as in the strict case, suspension increases the category level by one and plays an important role in $n$-category theory.

The suspension can be defined either in terms of enriched categories or $\Theta$-presheaves:

**Definition 2.1.7.**
- ([Ste21, Example 3.3.6]?) Let $X \in n\mathsf{Algbrd}$. Then the (unreduced) suspension of $X$ is the initial bipointed $\mathsf{V}$-algebroid $\{\perp, \top\} \to \sigma X$ equipped with $X \to \mathrm{Hom}_{\sigma X}(\perp, \top)$. It defines a colimit-preserving functor $\sigma : n\mathsf{Algbrd} \to (n+1)\mathsf{Algbrd}_{**}$ whose right adjoint is $(\{y_0, y_1\} \to Y) \mapsto \mathrm{Hom}_Y(y_0, y_1)$ on objects. Note that this suspension restricts to $n\mathsf{Cat}^{\mathrm{str}} \to (n+1)\mathsf{Cat}^{\mathrm{str}}_{**}$ and agrees with 2.1.2. **check**
- By our definition of $\Theta$, the suspension of Definition 2.1.2 restricts to a functor $\sigma : \Theta_n \to (\Theta^{\mathrm{can}}_{**}) \cap (n+1)\mathsf{Cat}_{**}$. Let $\tilde{\sigma} : \mathsf{PSh}(\Theta_n) \to \mathsf{PSh}(\Theta_{n+1})_{**}$ be the unique colimit-preserving extension. This descends to a colimit-preserving functor $\sigma : n\mathsf{Algbrd} \to (n+1)\mathsf{Algbrd}_{**}$.

The two definitions of $\sigma$ are the same as both are colimit-preserving and agrees on the dense subcategory. These are compatible for different $n$. Note that $\sigma$ valued in bipointed $\infty$-algebroids is fully faithful. If $X \in n\mathsf{Algbrd}$ is univalent, so is $\sigma X$, so they restrict to different versions of categories.**rewrite**

### 2.1.3 Duality

All the categories in the diagram 2.1 have the same (0-truncated) group of automorphisms:

**Proposition 2.1.8.** *Let $\mathsf{C}$ denote one of $\mathsf{Gaunt}$, $\mathsf{Cat}^{\mathrm{str}}$, $\mathsf{Cat}$, $\mathsf{Algbrd}$ and $0 \leq n \leq \infty$. Then any automorphism of $n\mathsf{C}$ preserves the subcategories $\mathbb{G}_n$, $\Theta_n$ and the restriction $\mathrm{Aut}(n\mathsf{C}) \to \mathrm{Aut}(\mathbb{G}_n) \simeq (\mathbb{Z}/2)^n$ is an equivalence.*

When $n$ is finite, the proposition is [BS21, Theorem 4.13, Lemma 10.2] for $n\mathsf{Gaunt}$, $n\mathsf{Cat}$ and the same argument works for $n\mathsf{Cat}^{\mathrm{str}}$, $n\mathsf{Algbrd}$. The key idea is to characterize the $n$-cell $C_n$ as an object of the abstract category $n\mathsf{Gaunt}$, $n\mathsf{Cat}$, etc. as the smallest generator (i.e., an object corepresenting a conservative functor) with respect to the retract relation. It is easy to check $\mathrm{Aut}(\mathbb{G}_n) \simeq (\mathbb{Z}/2)^n$, each copy of $\mathbb{Z}/2$ corresponding to flipping the cosource and the cotarget maps $s, t : C_{k-1} \to C_k$ for $0 < k \leq n$. These automorphisms uniquely extend to $\mathsf{PSh}(\Theta_n)$ and fixes all relevant subcategories.

The same idea does not seem to apply directly when $n = \infty$ because the infinite cell $C_\infty$ is a proper retract of itself. However, one can inductively reconstruct the subcategories $n\mathsf{C} \subset \infty\mathsf{C}$ from abstract category $\infty\mathsf{C}$ as follows[3]:

**Lemma 2.1.9.**
- *$0\mathsf{C} \subset \infty\mathsf{C}$ is the colimit-closure of the terminal object.*

[3]This idea is based on the post [htt] characterizing posets inside $\mathrm{ho}\,\mathsf{Cat}_{(1,1)}$.

- *Let $n\mathsf{C}' \subset \infty\mathsf{C}$ be the full subcategory spanned by the objects $X$ satsifying the following condition:*

  *for any (homotopically) 0-truncated object $Y \in (\infty\mathsf{C})_{\leq 0}$, the map $\mathrm{Map}(Y, X) \to \mathrm{Map}(Y^{\leq n-1}, X)$ in $\mathsf{S}$ induced by the counit $Y^{\leq n-1} \to Y$ of the colocalization $(\infty\mathsf{C})_{\leq 0} \to ((n-1)\mathsf{C})_{\leq 0}$ is mono.*

  *Then $n\mathsf{C}$ is the colimit-closure of $n\mathsf{C}' \subset \infty\mathsf{C}$.*

*Proof.* The first point is clear. For the second point, it suffices to prove $\mathbb{G}_n \subset n\mathsf{C}' \subset n\mathsf{C}$. First we show $\mathbb{G}_n \subset n\mathsf{C}'$ by induction on $n$. $C_0 \in n\mathsf{C}'$ is clear. Let $X = C_k$ for $1 \leq k \leq n$. If $k < n$, by hypothesis $Y^{\leq n-2} \to Y$ and $Y^{\leq n-2} \to Y^{\leq n-1}$ both induce injections (of sets), so $\mathrm{Map}(Y, C_k) \to \mathrm{Map}(Y^{\leq n-1}, C_k)$ is mono. It remains to show the restriction map $\mathrm{Map}(Y, C_n) \to \mathrm{Map}(Y^{\leq n-1}, C_n)$ is injective. This is because $f : Y^{\leq n-1} \to C_n$ extends to $Y$ precisely when for any nondegenerate $n$-cell $y : C_n \to Y$, the boundary $f \circ y|_{\partial C_n}$ can be extended to $C_n$, and such extension is unique if exists.

Next asuume $X \in n\mathsf{C}'$. To see $X \in n\mathsf{C}$, observe that for $k \geq n$, the $(n-1)$-categorical truncation of the maps $\partial C_{k+1} \hookrightarrow C_{k+1} \twoheadrightarrow C_k$ are identities on $\partial C_n$. The assumption implies that the induced maps $\mathrm{Map}(C_k, X) \to \mathrm{Map}(C_{k+1}, X) \to \mathrm{Map}(\partial C_{k+1}, X)$ are mono. The composite is the diagonal of the restriction map $\mathrm{Map}(C_k, X) \to \mathrm{Map}(\partial C_{k-1}, X)$ (which is mono when $k = n$), so by induction on $k \geq n$ we see $\mathrm{Map}(C_k, X) \xrightarrow{\sim} \mathrm{Map}(C_{k+1}, X) \xrightarrow{\sim} \mathrm{Map}(\partial C_{k+1}, X)$, i.e., any $k$-cell for $k > n$ is degenerate. $\square$

*Remark* 2.1.10. The category $n\mathsf{Gaunt}'$ is the category of so-called $n$-posets; by convention, a $(-1)$-poset is the terminal object, and $n$-poset are those enriched in $(n-1)$-posets.

In particular, any automorphism of $\infty\mathsf{C}$ preserves the subcategory $n\mathsf{C}$ and therefore restricts to $\mathbb{G}_n$ for any $n \geq 0$, so the proposition follows. The folloing copy of $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{Aut}(\infty\mathsf{Algbrd})$ is of dimension-independent importance:

**Definition 2.1.11.** The *odd dual* (resp. *even dual*) flips $s, t : C_{k-1} \to C_k$ for $k$ odd (resp. even). We denote them by $(-)^{\mathrm{op}}$, $(-)^{\mathrm{co}}$, respectively. The *total dual* flips the cells of all dimensions, i.e. $(-)^{\mathrm{coop}}$. We denote the total dual also by $(-)^{\circ}$.

### 2.1.4 The cubes and the Gray tensor product

The Gray tensor product is a monoidal structure on $\infty\mathsf{Algbrd}$ behaves additively on categorical levels. Steiner's theory gives a neat way to define the Gray tensor product of strict $\infty$-categories:

**Proposition 2.1.12.** *There exists a unique monoidal structure, called the* oplax Gray tensor product *on $\infty\mathsf{Cat}^{\mathrm{str}}$ which promotes $\nu|_{\mathsf{adCh}^{\mathrm{Ste}}} : \mathsf{adCh}^{\mathrm{Ste}} \hookrightarrow \infty\mathsf{Cat}^{\mathrm{str}}$ to a monoidal functor, which we denote by $\otimes^{\mathrm{oplax}}$. The* lax Gray tensor product *is the reversed monoidal structure $(\otimes^{\mathrm{oplax}})^{\mathrm{rev}} = \otimes^{\mathrm{lax}}$ which we will simply denote by $\otimes$. (I think $\lambda$ is strong monoidal).*

**Definition 2.1.13.** The *cube category* $\square \subset \infty\mathsf{Cat}^{\mathrm{str}}$ is the monoidal category generated by the interval $\square^1$. We denote the $n$-th tensor power by $\square^n = (\square^1)^{\otimes n}$. In terms of Steiner's theory, we have $\square^n = \lambda((\cdots \to 0 \to \mathbb{Z} \xrightarrow{(1,-1)} \mathbb{Z} \oplus \mathbb{Z})^{\otimes n})$.

*Remark* 2.1.14. A monoidal structure $\otimes'$ on $\square$ such that $\square^n \otimes' \square^m = \square^{n+m}$ is completely characterized by the bifunctor $\otimes' : \square \times \square \to \square$; the coherence data are uniquely deteremined because $\mathrm{Aut}(\square^n) = *$ (see 3.1.12).

*Remark* 2.1.15. Since the Gray tensor product is *semicartesian* (i.e., the unit object of the is terminal), there are morphisms $X \otimes Y \to X \times Y$ functorial in $X, Y$. In fact, the identity functor promotes to a lax monoidal functor $(\infty\mathsf{Algbrd}, \times) \to (\infty\mathsf{Algbrd}, \otimes)$. This is a general fact about semicartesian monoidal structures, but the author does not know a proof in the literature, so let us include the argument here; suppose $\mathcal{C}^{\otimes} \to \Delta^1$ is semicartesian monoidal category. One can construct the opposite monoidal category $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ by postcomposing the involution $(-)^{\mathrm{op}} : \mathsf{Cat} \to \mathsf{Cat}$ to the Segal object $\Delta^{\mathrm{op}} \to \mathsf{Cat}$ it classifies. Then $\mathcal{C}^{\otimes}$ is semicartesian if and only if $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ is *unital*. Let $(-)^{\mathrm{sym}} : \mathsf{Mon}(\mathsf{Cat}) \to \mathsf{CMon}(\mathsf{Cat})$ be the left adjoint to the forgetful functor. Then constructing a lax monoidal functor $\mathcal{C}^{\times} \to \mathcal{C}^{\otimes}$

which is identity on the underlying categories is equivalent to constructing a lax monoidal functor $((\mathcal{C}^{\mathrm{op}})^{\otimes})^{\mathrm{sym}} \to (\mathcal{C}^{\mathrm{op}})^{\amalg}$ which is identity on underlying categories. Now [Lur17, p. 2.4.3.9] says that a lax monoidal functor from a unital operad to a cocartesian operad is determined by the underlying functor.

*Remark* 2.1.16. I think in fact this $\infty\mathsf{Algbrd}^{\times} \to \infty\mathsf{Algbrd}^{\otimes}$ is a subcategory inclusion. Therefore being an algebroid enriched in cartesian product is a property of Gray-enriched categories. I moreover claim that there is a left adjoint. Is it something like a operadic Kan extension? Or dou I need to formulate this in terms of relative categories or in models like comical sets?

**Proposition 2.1.17.** *The total dual functor promotes to a monoidal endofunctor $\infty\mathsf{Algbrd}^{\otimes} \to \infty\mathsf{Algbrd}^{\otimes}$.*

*Proof.* Notice the total dual restricts to $\square$ and admits a unique monoidal structure $(\square^n)^\circ \otimes (\square^m)^\circ = \square^{n+m} = (\square^n \otimes \square^m)^\circ$ (which is the identity on objects, and the naturality is easily checked in $\mathsf{adCh}$). We get the monoidal structure on the total dual on $\infty\mathsf{Algbrd}$ by (operadic?) Kan extension. $\square$

### 2.1.5 The Gray suspension is the suspension

The following lemma claims that the self-enrichment

**Lemma 2.1.18.** *There are functors*

$$P, P^\circ : \infty\mathsf{Algbrd} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})$$

*whose values $P(X), P^\circ(X)$ are pushout squares with the following specification (only the bottom arrows and the 2-cells are not predetermined):*

$$\begin{tikzcd} \partial\square^1 \otimes X \arrow[r, "\partial\square^1\otimes(X\to *)"] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] & X \otimes \partial\square^1 \arrow[r] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] \\ \square^1 \otimes X \arrow[r] & \sigma X, & X \otimes \square^1 \arrow[r] & \sigma(X^\circ). \end{tikzcd} \tag{2.2}$$

*The diagrams restrict to $X \in \infty\mathsf{Cat}, \infty\mathsf{Cat}^{\mathrm{str}}, \mathsf{Gaunt}$.*

The left pushout formula is [Cam23b, Lemma 3.8] (based on the strict $\infty$-category case [DG20, Cor. B.6.6]). The right pushout formula seems new and requires some care regarding the duality introduced. We will record a detailed proof after the following corollary.

**Corollary 2.1.19.** *There exist canonical identifications functorial in $X$:*

$$\Sigma X \simeq \vec{S}^1 \owedge X, \quad X \owedge \vec{S}^1 \simeq \Sigma X^\circ.$$

*In particular, this provides a natural isomorphism*

$$\tau_X : \vec{S}^1 \owedge X \simeq \Sigma X \simeq X^\circ \owedge \vec{S}^1.$$

*Proof.* Consider the following diagram:

$$\begin{tikzcd} & * \otimes * \arrow[dl] \arrow[dd] & & * \otimes \partial\square^1 \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & \partial\square^1 \arrow[dl] \arrow[dd] \\ X \otimes * \arrow[dd] \arrow[dr, phantom, "(1)"] & & X \otimes \partial\square^1 \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dr, phantom, "(2)"] & & \partial\square^1 \arrow[dd] \arrow[dr, phantom, "(3)"] & \\ & * \otimes \vec{S}^1 \arrow[dl] & & * \otimes \square^1 \arrow[ll] \arrow[rr] \arrow[dl] & & S* \arrow[dl] \\ X \otimes \vec{S}^1 & & X \otimes \square^1 \arrow[ll] \arrow[rr] & & SX^\circ & \end{tikzcd}$$

All the arrows from back to front are induced by the basepoint $* \to X$. The front-right and the back-right faces are the pushout diagrams of **??**, whereas front-left and back-left faces are the usual pushout description of $\vec{S}^1 \simeq B\mathbb{N}$. Since all the faces in front and back are pushouts, we have the induced equivalences on the total cofibers

$$\vec{S}^1 \oslash X \simeq \mathrm{cofib}(1) \leftarrow \mathrm{cofib}(2) \to \mathrm{cofib}(3) \simeq \Sigma X^\circ.$$

The other equivalence $\Sigma X \simeq X \oslash \vec{S}^1$ follows from the other pushout diagram of **??**. $\square$

*Proof of Lemma 2.1.18.* Note that $P(\emptyset)$, $P^\circ(\emptyset)$ are both the unique (identity) square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \partial\square^1 \arrow[d] \\ \emptyset \arrow[r] & \partial\square^1, \end{tikzcd}$$

so $P$, $P^\circ$ must lift to functors $\infty\mathsf{Algbrd} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$. These are necessarily colimit-preserving because the colimit in the codomain is computed componentwise (in each coslice category)[4]. Let $\mathsf{Fun}^{\mathrm{po}}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$ denote the (colimit-closed) full subcategory of the codomain consisting of pushout squares. Notice that, once we have functors $P^{(\circ)}|_\square : \square \to \mathsf{Fun}^{\mathrm{po}}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$ with the components as specified, their unique colimit-preserving extensions to $\mathsf{PSh}(\square)$ automatically descend to $\infty\mathsf{Algbrd}$ and satisfy the requirement:

$$\begin{aligned} \mathsf{Fun}(\square, \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}) &\simeq \mathsf{LFun}(\mathsf{PSh}(\square), \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}) \\ &\hookleftarrow \mathsf{LFun}(\infty\mathsf{Algbrd}, \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}) \end{aligned}$$

They descend to $\infty\mathsf{Algbrd}$ because each component of the square does (which is a consequence of the existence of Gray tensor product), and the extended functor lands in pushout squares if the original functor does. We provide $P|_\square$ and $P^\circ|_\square$ in two steps: (1) construct the squares valued in $\infty\mathsf{Cat}^{\mathrm{str}}$ and (2) check that they are pushouts in $\infty\mathsf{Algbrd}$. Note that the (weak) tensor product of objects in $\widetilde{\square}$, as well as the suspension and the total dual, can be computed in strict $\infty$-categories.

(1) Since our definition of the Gray tensor product involves Steiner theory, so does the construction of the squares $P(\square^n)$, $P^\circ(\square^n)$. We define the functors $\tilde{P}, \tilde{P}^\circ : \mathsf{adCh}^{\mathrm{Ste}} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{adCh}^{\mathrm{Ste}})$ with

$$\tilde{P}(C) = \begin{tikzcd} C \otimes \lambda(\partial\square^1) \arrow[r] \arrow[d] & \lambda(\partial\square^1) \arrow[d] \\ C \otimes \lambda\square^1 \arrow[r] & \sigma(C), \end{tikzcd} \qquad \tilde{P}^\circ(C) = \begin{tikzcd} \lambda(\partial\square^1) \otimes C \arrow[r] \arrow[d] & \lambda(\partial\square^1) \arrow[d] \\ \lambda\square^1 \otimes C \arrow[r] & \sigma(C^\circ). \end{tikzcd}$$

As diagrams of graded $\mathbb{Z}$-modules, two diagrams are identical, and the bottom map is given by (under the notation $\lambda\square^1 = (e\mathbb{Z} \xrightarrow{\binom{-1}{1}} \bot\mathbb{Z} \oplus \top\mathbb{Z})$)

$$\begin{aligned} C_q \otimes \bot\mathbb{Z} \oplus C_q \otimes \top\mathbb{Z} \oplus C_{q-1} \otimes e\mathbb{Z} &\xrightarrow{(0,0,1)} C_{q-1} \quad (q > 0), \\ C_0 \otimes \bot\mathbb{Z} \oplus C_0 \otimes \top\mathbb{Z} &\xrightarrow{(\varepsilon,\varepsilon)} \bot\mathbb{Z} \oplus \bot\mathbb{Z} \quad (q = 0). \end{aligned}$$

In both diagrams, it is straightforward to check that it defines a map of augmented directed complexes functorial in $C$. With the other three obvious maps, it defines a (strictly) commutative squares $\tilde{P}(C)$, $\tilde{P}^\circ(C)$ natural in $C$. Through the antimonoidal equivalence $\mathsf{Gaunt}^{\mathrm{Ste}} \simeq \mathsf{adCh}^{\mathrm{Ste}}$ and restricting to $\square \cup \{\emptyset\}$, we obtain $P^{(\circ)} : \square \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{Gaunt}^{\mathrm{Ste}})_{P(\emptyset)/} \subset \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$.

[4] Recall that the colimit of $p : X \to Y_{y/}$ is (almost by definition) the colimit of the corresponding cone $\bar{p} : X^\triangleleft \to Y$.

(2) It remains to show that $P^{(\circ)}(\Box^n)$ are pushout squares. We say $X \in \infty\mathsf{Algbrd}$ is good (resp.$\circ$-good) if $P(X)$ (resp. $P^\circ(X)$) is a pushout. The terminal category $*$ is clearly ($\circ$-)good, so it suffices to show that if a strong Steiner $\infty$-category $X$ is ($\circ$-)good, then so is $X \otimes \Box^1$ (resp. $\Box^1 \otimes X$). By assumption that $P^{(\circ)}(X)$ is a pushout (which is preserved by tensoring $\Box^1$), we have the following factorizations of $P(X \otimes \Box^1)$ and $P^\circ(\Box^1 \otimes X)$:

$$\begin{tikzcd} \partial\Box^1 \otimes X \otimes \Box^1 \arrow[r] \arrow[d, hook] & \partial\Box^1 \otimes \Box^1 \arrow[r] \arrow[d, hook] & \partial\Box^1 \arrow[d, hook] \\ \Box^1 \otimes X \otimes \Box^1 \arrow[r] & (\sigma X) \otimes \Box^1 \arrow[r] & \sigma(X \otimes \Box^1), \end{tikzcd} \qquad \begin{tikzcd} \Box^1 \otimes X \otimes \partial\Box^1 \arrow[r] \arrow[d, hook] & \Box^1 \otimes \partial\Box^1 \arrow[r] \arrow[d, hook] & \partial\Box^1 \arrow[d, hook] \\ \Box^1 \otimes X \otimes \Box^1 \arrow[r] & \Box^1 \otimes \sigma(X^\circ) \arrow[r] & \sigma((\Box^1 \otimes X)^\circ). \end{tikzcd}$$

The composite rectangles are pushouts if and only if the right squares are. To ease the notation, we replace $X^\circ$ with $X$ (and $(\Box^1 \otimes X)^\circ$ with $(\Box^1)^\circ \otimes X$) in the right diagram. Factor the right squares into $(a) + (b)$ of the next diagram by factoring the left inclusions as

$$\partial\Box^1 \otimes \Box^1 \hookrightarrow \big((\sigma X \otimes \{0\}) \vee (\{\top\} \otimes \Box^1)\big) \sqcup \big((\{\bot\} \otimes \Box^1) \vee (\sigma X \otimes \{1\})\big) \to (\sigma X) \otimes \Box^1,$$
$$\Box^1 \otimes \partial\Box^1 \hookrightarrow \big((\Box^1 \otimes \{\bot\}) \vee (\{1\} \otimes \sigma X)\big) \sqcup \big((\{0\} \otimes \sigma X) \otimes (\Box^1 \otimes \{\top\})\big) \to \Box^1 \otimes (\sigma X).$$

(note that these wedge sums are both strict and weak: see [Cam23a, Theorem 2.31]):

$$\begin{tikzcd} & \partial\Box^1 \otimes \Box^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & * \sqcup * \arrow[d] \\ \sigma X \sqcup \sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & \sigma X \vee \Box^1 \sqcup \Box^1 \vee \sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \sigma X \sqcup \sigma X \arrow[d] \\ \sigma(X \otimes \Box^1) \arrow[r] & (\sigma X) \otimes \Box^1 \arrow[r] & \sigma(X \otimes \Box^1) \end{tikzcd}$$

$$\begin{tikzcd} & \Box^1 \otimes \partial\Box^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & * \sqcup * \arrow[d] \\ \sigma X \sqcup \sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & \sigma X \vee \Box^1 \sqcup \Box^1 \vee \sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \sigma X \sqcup \sigma X \arrow[d] \\ \sigma((\Box^1)^\circ \otimes X) \arrow[r] & \Box^1 \otimes \sigma X \arrow[r] & \sigma((\Box^1)^\circ \otimes X) \end{tikzcd}$$

Since $(a)$ is a pushout, it suffices to show that there exist pushout squares $(c)$ such that the horizontal compositions of $(b) + (c)$ are the identities. One can construct the sections $(c)$ of $(b)$ again at the level of augmented directed complexes; $(b)$ is an identity in degrees greater than 1. Degree 0 parts are characterized by that each summand are bipointed maps with source-sink bipointing. In degree 1, ...

Now $(a)$ and $(b) + (c)$ are pushouts by inspection, so it suffices to show that the squares $(c)$ are pushouts. In fact, these are pushouts in $\mathsf{PSh}(\Theta)$, i.e., it is a pushout after plugging into $\mathrm{Hom}(\theta, -)$ for $\theta \in \Theta$; in fact, by casework about the image of the vertices, it reduces to checking that $\mathrm{Hom}_{**}(\sigma\theta', \sigma(X \otimes \Box^1)) \to \mathrm{Hom}_{**}(\sigma\theta', (\sigma X) \otimes \Box^1)$ is a bijection; since $\mathbb{G}$ is colimit-generating, we may assume $\theta' = C_n \in \mathbb{G}$. In this case one can explicitly check (in Steiner complexes side) that any bipointed map $C_n \to (\sigma X) \otimes \Box^1$ must factor through $\sigma(C_{n-1} \to X \otimes \Box^1)$.

$\Box$

By colimit-extending the diagrams $(c)$ (in the category under the value of $X = \emptyset$), we get the folloing pushout formula as a biproduct:

**Corollary 2.1.20.** *The squares $(c)$ in the proof of the lemma is a pushout diagram in $\mathsf{PSh}(\Theta)$ for any $X \in \mathsf{PSh}(\Theta)$.*

### 2.1.6 Gray categories

Todo:

(1) Cartesian enrichment is a property of Gray enrichment

(2) Any Gray category has an underlying 2-category

(3) Gray cylinder is enough to recorver a Gray enrichemnt

## 2.2 Appendix: Steiner's theory for strict $\infty$-categories

In this appendix, we give a summary of Steiner's theory, which provides equivalence between a class of strict $\infty$-categories and a class of chain complexes with a certain positivity structure. It is a powerful computational tool in combinatorics of strict $\infty$-categories, especially when confusing dualities are involved. In what follows, "category" without specification will mean $(1,1)$-*category*. References for this section include [Ste04], [Ara+23], [DG20], [OR23].

*Remark* 2.2.1. The strict $\infty$-categories defined in Definition 2.1.1 admits the following more explicit description:

(1) for each integer $n \geq 0$ a set $C_n$, called the set of $n$-cells,

(2) for each $p > q \geq 0$ a structure of a category with objects $C_q$ and morphisms $C_p$, i.e.,

  (i) the ($q$-)source and the ($q$-)target maps $s_q, t_q : C_p \to C_q$,

  (ii) the identity map $i_p : C_q \to C_p$,

  (iii) the composition map $*_p : {C_p}_{s_q}\times_{C_q}{}_{t_q}C_p \to C_p$ satisfying associativity and unitality,

which are compatible in the sense that for $p > q > r \geq 0$ the above data defines a 2-category structure on $(C_p, C_q, C_r)$, i.e., they satisfy the globularity conditions $s_p s_q = s_p$, $t_p t_q = t_p$ and the "interchange law" $(f *_q g) *_r (h *_q k) = (f *_r h) *_q (g *_r k)$ for $f, g, h, k \in C_p$. The data (1) and (2)(i)(ii) are precisely the data of reflexive globular sets (i.e., presheaves on $\mathbb{G}$) and (iii) is the structure required to extend it to presheaves on $\Theta$ satisfying the Segal conditions.

*Remark* 2.2.2. For any category $\mathcal{C}$ with finite limits, the obvious modification of the above defines the notion of *strict $\omega$-category objects in $\mathcal{C}$*, which describes $\mathcal{C}$-valued presheaves on $\Theta$ satisfying the Segal conditions.

### 2.2.1 Steiner's adjunction

We first recall relevant definitions on the chain complex side.

**Definition 2.2.3.** (1) An *augmented directed chain complex* (ADC for short) is a triple $(A, A^+, \epsilon) = ((A_\bullet, \partial_\bullet), A^+_\bullet, \epsilon)$, where $A \in \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ is a nonnegatively graded chain complex, $\epsilon : A_0 \to \mathbb{Z}$ is an augmentation and $A^+_n \subset A_n$ is a sub-$\mathbb{N}$-module (aka. submonoid) for each $n$ (called the *positivity submonoid* and is not necessarily closed under differential). We often omit $\partial_\bullet$, $A^+$ and $\epsilon$ from notation when it is not confusing.

(2) A map $A \to B$ of ADCs is a chain map $f : A \to B$ that commute with augmentations and satisfies $f(A^+) \subset B^+$. Let $\mathsf{adCh}$ denote the category of ADCs.

(3) A *basis* of an ADC $(A_\bullet, A^+_\bullet)$ is a graded subset $\{B_q \subset A^+_q\}_{q \geq 0}$ which is both a $\mathbb{Z}$-basis of $A$ and a $\mathbb{N}$-basis[5] of $A^+$. If $A$ admits a basis:

[5] A basis of an $\mathbb{N}$-module is unique if exists, so it makes sense to talk about "the" basis of an ADC. In fact, any isomorphism $f : \mathbb{N}^{\oplus I} \to \mathbb{N}^{\oplus J}$ is induced by a bijection $I \to J$ ($f$ and $f^{-1}$ are injective, so they do not decrease the sum of components).

(i) Any element $a = \sum_{b \in B_q} \lambda_b b \in A_q$ is uniquely a difference $a = a^+ - a^-$ with $a^+, a^- \in A^+$. We write $\partial_q^{\pm}(a) \coloneqq (\partial_q(a))^{\pm}$. Also, define $\mathrm{supp}(a) \subset B_q$ as the set of $b \in B_q$ with $\lambda_b \neq 0$.

(ii) The basis $\{B_q\}$ is *unital* if for every $q \geq 0$ and $b \in B_q$ we have $\epsilon \circ \partial_0^+ \circ \cdots \circ \partial_{q-1}^+(b) = 1 = \epsilon \circ \partial_0^- \circ \cdots \circ \partial_{q-1}^-(b)$.

(iii) Consider the preorder on $\bigsqcup_{q \geq 0} B_q$ generated by the relation

$$\{(a,b) \mid b \in B_q, a \in \mathrm{supp}(\partial_{q-1}^- b)\} \cup \{(a,b) \mid a \in B_q, b \in \mathrm{supp}(\partial_{q-1}^+ a)\}.$$

The basis is *strongly loop-free* if this preorder is a partial order.

(4) A *strong Steiner complex* is an ADC with with a strongly loop-free unital basis.

*Remark* 2.2.4. The category $\mathsf{adCh}$ is cocomplete and colimits can be computed degreewise. More precisely, the forgetful functor $\mathsf{adCh} \to \mathsf{grMod}_{\mathbb{Z}} \times \mathsf{grMod}_{\mathbb{N}}$ creates colimits.

**Example.** For any CW complex $X$ with chosen orientations of cells, we regard the augmented cellular chain complex $C_\bullet(X) \to \mathbb{Z}$ as an ADC with the basis consisting of the cells. In particular, let $D^p \subset \mathbb{R}^p \subset \mathbb{R}^\infty$ be the unit $n$-disk with the CW structure $D^p = \left(\bigsqcup_{q \leq p-1}(e_+^q \sqcup e_-^q)\right) \sqcup e^p$, where $e_\pm^q = \{(*, \dots, *, \pm(\text{at } q\text{th}), 0, \dots, 0)\} \subset D^p$. The cellular chain $\{C_\bullet(D^q)\}_{q \geq 0}$ extends to an $\omega$-category object in $\mathsf{adCh}^{\mathrm{op}}$; for $p > q \geq 0$,

- the co-source and the co-target map $s^q, t^q : C_\bullet(D^q) \to C_\bullet(D^p)$ are induced by the inclusions $D^q \hookrightarrow D^p$ with the image $e_-$ and $e_+$,
- the co-identity map $i^q : C_\bullet(D^p) \to C_\bullet(D^q)$ is induced by the projection $D^p \to D^q$,
- The co-composition map $*^p : C_\bullet(D^p) \to \mathrm{colim}(C_\bullet(D^p) \leftarrow C_\bullet(D^q) \to C_\bullet(D^p)) \cong C_\bullet(D^p \sqcup_{D^q} D^p)$ is induced by the $q$-fold unreduced suspension of the pinch map $D^{p-q} \to D^{p-q} \vee D^{p-q}$.

In other words, $\{C_\bullet(D^q)\} : \mathbb{G} \to \mathsf{adCh}$ extends to a functor $\Theta \to \mathsf{adCh}$ satisfying the Segal conditions, so the following definition makes sense:

**Definition 2.2.5.** Steiner's adjunction

$$\begin{tikzcd} \omega\mathsf{Cat}^{\mathrm{str}} \arrow[r, shift left, "\lambda"] \arrow[r, phantom, "\perp"] & \mathsf{adCh} \arrow[l, shift left, "\nu"] \end{tikzcd}$$

is the restricted Yoneda extension adjunction of $\{C_\bullet(D^q)\}$[6].

Now we explain the notion corresponding to strong Steiner complexes on the category side:

**Definition 2.2.6.** Let $C$ be a strict $\omega$-category.

(1) A set of cells $E = \bigsqcup E_n$ of $C$, where $E_n \subset C_n$, is a *polygraphic basis* if the following diagram is a pushout for any $n$:

$$\begin{tikzcd} \bigsqcup_{E_q} S[q-1] \arrow[r] \arrow[d, hook] & \varpi_{q-1} C \arrow[d, hook] \\ \bigsqcup_{E_q} D[q] \arrow[r] & \varpi_q C \end{tikzcd}$$

$C$ is called a *polygraph* or a *computad* if it admits a polygraphic basis. [makkai 05, AGOR 2.4] shows that if a basis exists, it must be the set of nondegenerate indecomposables.

(2) Let $E$ be a polygraphic basis. For $c \in C_q$, define $\mathrm{supp}(c) \subset E_q$ be the set of factors of $c$.

[6]For more combinatorial descriptions of these functors, see e.g. [OR23, §2.3].

(3) Consider the preorder on $E$ generated by the relation

$$\bigcup_{p<q}\{(a,b)\in E_p\times E_q \mid a\in \mathrm{supp}(s_p b)\}\cup\bigcup_{p>q}\{(a,b)\in E_p\times E_q \mid b\in \mathrm{supp}(t_q a)\}.$$

A polygraphic basis $E$ is *strongly loop-free* if the preorder is a partial order.

(4) A strict $\omega$-category is *strong Steiner* if it admits a strongly loop-free polygraphic basis[7].

*Remark* 2.2.7. Any polygraph is Gaunt. **check**

**Theorem 2.2.8** ([Ste04], [Ara+23])**.** *The adjunction $\lambda \dashv \nu$ restricts to an equivalence between the category of strong Steiner categories and the category of strong Steiner complexes. Moreover, $\lambda C$ is strong Steiner iff $C$ is, and similarly for $\nu$.*

We will use the following observation (maybe not, will remove):

**Corollary 2.2.9.** *Let $\{A_i \to A\}_{i\in I}$ be a colimit cone in $\mathsf{adCh}$ with each $A_i$ and $A$ strong Steiner. Then $\{\nu A_i \to \nu A\}_{i\in I}$ is a coimit cone in $\omega\mathsf{Cat}^{\mathrm{str}}$.*

*Proof.* Let $C$ be the colimit of the diagram $\{\nu A_i\}$ in $\omega\mathsf{Cat}^{\mathrm{str}}$ and $f : C \to \nu A$ be the comparison functor. By applying $\lambda$, we see that $\{A_i = \lambda\nu A_i \to \lambda C\}$ is a colimit cone in $\mathsf{adCh}$, so $\lambda f : \lambda C \to \lambda\nu A \simeq A$ is an isomorphism. Therefore $\lambda C$ is a strong Steiner complex and $C$ is a Strong Steiner strict $\omega$-category. Now $\nu A$ and $C$ are both colimits of $\nu A_i$ in the full subcategory of strong Steiner objects, the comparison map $f$ must be an isomorphism, i.e., $\nu A$ is the colimit also in the whole category $\omega\mathsf{Cat}^{\mathrm{str}}$. $\square$

### 2.2.2 Operations on augmented directed complexes

**Definition 2.2.10.** The *suspension* $\sigma : \mathsf{adCh} \to \mathsf{adCh}$ sends an augmented directed complex

$$\cdots \to A_n \xrightarrow{\partial_n^A} A_{n-1} \to \cdots \to A_0 \xrightarrow{\varepsilon^A} \mathbb{Z}$$

to

$$\cdots \to (\sigma A)_n = A_{n-1} \xrightarrow{\partial_{n-1}^A} (\sigma A)_{n-1} = A_{n-2} \to \cdots \to (\sigma A)_1 = A_0 \xrightarrow{\binom{-\varepsilon^A}{\varepsilon^A}} (\sigma A_0) = \bot\mathbb{Z}\oplus\top\mathbb{Z} \xrightarrow{\varepsilon^{\sigma A}=(1,1)} \mathbb{Z}$$

with the positivity submonoids $(\sigma A)_n^+ = A_{n-1}^+$ for $n\geq 1$ and $(\sigma A)_0^+ = \bot\mathbb{N}\oplus\top\mathbb{N}$.

**Definition 2.2.11.** For $\tau\in(\mathbb{Z}/2)^{\mathbb{Z}_{\geq 1}}$, define the *$\tau$-dual* functor $\mathsf{adCh}\to\mathsf{adCh}$ by sending $(A_n, \partial_n : A_n \to A_{n-1}, \varepsilon, A^+)$ to $(A_n, (-1)^{\tau(n)}\partial_n, \varepsilon, A^+)$. When $\tau$ is constantly 1, we call the $\tau$-dual the *total dual* and denote by $(-)^\circ$.

**Definition 2.2.12.** (1) The usual symmetric monoidal structure on $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$ with Koszul sign rule induces a symmetric monoidal structure on the category $\mathsf{Ch}_{\geq 0}(\mathbb{Z})_{/\mathbb{Z}}$ of *augmented* complexes by $\epsilon_{A\otimes B} : (A\otimes B)_0 \simeq A_0\otimes B_0 \xrightarrow{\epsilon_A\otimes\epsilon_B} \mathbb{Z}\otimes\mathbb{Z}\simeq\mathbb{Z}$.

(2) Let $A = (A, A^+, \epsilon_A)$ and $B = (B, B^+, \epsilon_B)$ be ADCs. We define the tensor product $A\otimes B$ as $(A\otimes_{\mathbb{Z}} B, A^+\otimes_{\mathbb{N}} B^+, \epsilon_{A\otimes B})$. This tensor product canonically extends to a monoidal structure.

*Remark* 2.2.13. The positivity structure in $\mathsf{adCh}$ prevent the tensor product from being symmetric: if $A$ and $B$ are both free ADCs on a single basis element in degree 1, say $a$ and $b$, then the symmetry morphism $a\otimes b\mapsto -b\otimes a$ does not preserve the positivity submonoid.

Suspension enjoys the following universal property:

**Lemma 2.2.14.** $S : \omega\mathsf{Cat}^{\mathrm{str}} \to \omega\mathsf{Cat}^{\mathrm{str}}_{**}$ : Hom

**Lemma 2.2.15** (2.12 of OR)**.** *There is an isomorphism $\nu\Sigma A\cong\sigma\nu A$ natural in $A\in\mathsf{adCh}$.*

[7]This formulation is due to AGOR and not the original definition.

## 2.3 Background on categorical spectra

Todo:

(1) definition

(2) examples

### 2.3.1 Basic Definitions

In this section, we review the basics of categorical spectra. We refer the reader to [Ste21, Chapter 13] for more detail.

**Definition 2.3.1.** Let $\mathsf{C}$ be a presentable category (with the cartesian monoidal structure). A *pointed object* in $\mathsf{C}$ is an object of the category $\mathsf{C}_* := \mathsf{C}_{1/} \simeq \mathsf{Alg}_{\mathbb{E}_0}(\mathsf{C}) \simeq \mathsf{S}_* \otimes \mathsf{C}$. Pointed objects in $\infty\mathsf{Cat}$ and $\infty\mathsf{Algbrd}$ are called pointed $\infty$-categories and pointed $\infty$-algebroids, respectively. Explicitly, it is a pair $(X, x)$ of an $\infty$-category or algebroid $X$ and an object $x \in X$. We will omit the basepoint if there is no risk of confusion.

There is a projection $\mathsf{Algbrd}(\mathsf{C})_* \simeq \mathsf{Algbrd}(\mathsf{C}) \times_{\mathsf{S}} \mathsf{S}_* \to \mathsf{S}_*$, whose fiber over $* \in \mathsf{S}_*$ is $\mathsf{Mon}(\mathsf{C}) \simeq \mathsf{Alg}(\mathsf{C})$. Since the projection is a cartesian fibration, the inclusion $\mathsf{Mon}(\mathsf{C}) \hookrightarrow \mathsf{Algbrd}(\mathsf{C})_*$ admits a right adjoint given by $\Omega_{\mathsf{C}} : \mathsf{Algbrd}(\mathsf{C})_* \to \mathsf{Mon}(\mathsf{C}); (X, x) \mapsto \mathsf{End}_X(x)$.

**Definition 2.3.2.** The *loop functor* on the pointed $\infty$-algebroids is the composite

$$\Omega : \infty\mathsf{Algbrd}_* \simeq \mathsf{Algbrd}(\infty\mathsf{Algbrd})_* \xrightarrow{\Omega_{\infty\mathsf{Algbrd}}} \mathsf{Mon}(\infty\mathsf{Algbrd}).$$

It restricts to the loop functor $\Omega : \infty\mathsf{Cat}_* \to \mathsf{Mon}(\infty\mathsf{Cat})$. The right adjoint is the *delooping* $\mathrm{B} : \mathsf{Mon}(\infty\mathsf{Cat}) \to \infty\mathsf{Cat}_*$.

The underlying endofunctor $\infty\mathsf{Cat}_* \to \mathsf{Mon}(\infty\mathsf{Cat}) \xrightarrow{\text{forget}} \infty\mathsf{Cat}_*$ is still denoted by $\Omega$. It has a left adjoint $\Sigma : \infty\mathsf{Cat}_* \xrightarrow{\mathrm{Free}_{\mathbb{E}_1}} \mathsf{Mon}(\infty\mathsf{Cat}) \xrightarrow{\mathrm{B}} \infty\mathsf{Cat}_*$, which we call the *(reduced) suspension*.

Informally, $\Omega(X, x)$ is the monoidal category of endomorphisms of the basepoint $x \in X$ equipped with the composition monoidal sturcture and the basepoint $\mathrm{id}_x$.

*Remark* 2.3.3. The loop $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ preserves filtered colimits, or equivalently, the suspension $\Sigma$ preserves compact objects. It follows from the fact that the filtered colimit commutes with taking the hom object in the enriched categories **reference, Muro?** but also from the formula of suspension

**Definition 2.3.4.** The category of *categorical spectra* is the limit of categories

$$\mathsf{CatSp} := \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*).$$

Its object, a categorical spectrum $X \in \mathsf{CatSp}$, is a sequence $(X_n, x_n)_{n \in \mathbb{N}}$ of pointed categories equipped with identifications $f_n : (X_n, x_n) \xrightarrow{\sim} (\mathsf{End}_{X_{n+1}}(x_{n+1}), \mathrm{id}_{x_{n+1}})$. We will often supress $x_n$ and $f_n$ in the notation.

*Remark* 2.3.5. The above limit defining $\mathsf{CatSp}$ can be taken in $\mathsf{Pr}^{\mathsf{R}}{}_{\omega}$. In particular, $\mathsf{CatSp}$ is compactly generated and $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ preserves filtered colimits. One can also define the flagged version of categorical spectra: $\mathsf{CatSp}^{\mathrm{f}} := \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Algbrd}_* \xrightarrow{\Omega} \infty\mathsf{Algbrd}_*)$.

### 2.3.2 Connectivity

**Definition 2.3.6.** A category is 0-connective if it is nonempty. For $n \geq 1$, a category $X$ is $n$-connective if the canonical map $\mathrm{B}^n\Omega^n(X, x) \to X$ is an equivalence for any $x \in X$.

**Proposition 2.3.7.** *The following are equivalent:*

(1) *$X$ is $n$-connective.*

(2) *$^{\leq 0}X$ is an $n$-connective groupoid.*

The following

**Definition 2.3.8.** Suppose $X$ is $\infty$-connective, i.e., $^{\leq 0}X$ is contractible. Then $X$ itself is a terminal category.

**Proposition 2.3.9** (Freudenthal suspension theorem)**.** *Let $X$ be $n$-connective. Then the map $X \to \Omega\Sigma X$ is $2n$-connective.*

*Proof.* Since the localization $\infty\mathsf{Cat} \to \mathsf{S}$ preserves finite product [Ste21, Proposition 3.6.13]. □

**Definition 2.3.10.** A categorical spectrum $X$ is *$n$-connective* if $X[-n]$ lies in the essential image of the functor $\mathrm{B}^{\infty} : \mathsf{CMon}(\infty\mathsf{Cat}) \to \mathsf{CatSp}$.

**Proposition 2.3.11.** *The following are equivalent.*

(1) *$X$ is $n$-connective.*

(2) *For any $k > n$, $^{\leq 0}(X_k)$ is connected.*

(3) *For any $k$, $X_k$ is $(k+n)$-connective*

(4) *what else? inductive characterization?*

right orthogonal class of the connective categorical spectra; truncated factorization system?

# Chapter 3

# Tensor product of categorical spectra

Todo:

(1) half-central structure

(2) existence Proof

(3) formulas

(4) is infinite product exact?

(5) examples

(6) linear version?

(7) various freeness

(8) underlying 2-category

## 3.1 Half-central structure of $\vec{S}^1$

This section perhaps contains technically the most important technical ingredient of this thesis: the half central structure on the directed circle. This is the higher categorical incarnation of the Koszul sign rule in the usual homotopy theory.

In this section, we will define the notion of "half-center" of $\infty\mathsf{Cat}_*$ and prove that the directed circle $\vec{S}^1 = \mathrm{B}\mathbb{N}$ admits a unique half-central structure (Theorem 3.1.7). This will be the key technical input for the construction of the tensor product of categorical spectra in section 3.3.

### 3.1.1 Half-center

We begin by recalling the notion of center of a monoidal category.

**Definition 3.1.1.** Let $A$ be an $\mathbb{E}_1$-algebra object of $\mathsf{V} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$. The *center* $\mathcal{Z}_{\mathsf{V}}(A)$ of $A$ (in $\mathsf{V}$) is the object $\mathsf{End}_{{}_A\mathsf{BMod}_A(\mathsf{V})}(A)$. A *central structure* on $s \in A$ ($s : 1_{\mathsf{V}} \to A$?) is a lift of $s$ along the forgetful functor $\mathcal{Z}_{\mathsf{V}}(A) = \mathsf{End}_{{}_A\mathsf{BMod}_A}(A) \to \mathsf{End}_{\mathsf{RMod}_A}(A) \simeq A$.

We will omit $\mathsf{V}$ from the notation when it is not confusing nor important. There is another forgetful functors $\mathsf{End}_{{}_A\mathsf{BMod}_A}(A) \to \mathsf{End}_{\mathsf{LMod}_A}(A) \simeq A^{\mathrm{rev}}$.

*Remark* 3.1.2. As we will see below, the center is just another name for the Hochshild cohomology of $A$, seen as a bimodule over itself. The notion of center defined above is sometimes called the $\mathbb{E}_2$-center, because it admits a canonical structure of an $\mathbb{E}_2$-algebra object of $\mathsf{V}$, with which the center is characterized by a universal property (see [Lur17, §5.3]). This $\mathbb{E}_2$-structure on the center can be naturally understood from Morita theory. Namely, there is a Morita 2-$\mathsf{V}$-category**check ref** $\mathfrak{Alg}(\mathsf{V})$ of $\mathbb{E}_1$-algebras and bimodules in $\mathsf{V}$. In this category, $A$ as an algebra is an object and $A$ as a bimodule is the identity morphism of an object $A$, so the center admits a description

$$\mathcal{Z}_{\mathsf{V}}(A) \simeq \Omega(\Omega(\mathfrak{Alg}(\mathsf{V}), A), \mathrm{id}_A),$$

from which the $\mathbb{E}_2$-structure is clear.

According to 2.1.19, it seems natural to expect that $\vec{S}^2 = \vec{S}^1 \oslash \vec{S}^1$ lifts to the center, which would imply that $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ lifts to a $\infty\mathsf{Cat}^\otimes$-bimodule homomorphism. However, the lift to the center requires infinitely many coherence data, which is a priori hard to deal with. $\vec{S}^1$ is gaunt, i.e., homotopy-theoretically more trivial than $\vec{S}^2$ is, and it turns out that it is easy to deal with the coherence of the finer "half-central" structure for $\vec{S}^1$.

To define the notion of the half-center (with respect to an involution $D$), let $D : A \to A$ be a monoidal endofunctor equipped with an equivalence $D \circ D \simeq \mathrm{id}_A$. Recall that there is a locally fully faithful functor $\mathsf{Alg}(\mathsf{V}) \to \mathfrak{Alg}(\mathsf{V})$ which is identity on objects and regards algebra homomorphism as bimodules. Explicity, an algebra homomorphism $f : A \to B$ is sent to an $(A, B)$-bimodule ${}^f B$, whose underlying right $B$-module is $B$ itself and the left action of $A$ is provided by $f$. By abuse of notation, we denote the $(A, A)$-bimodule ${}^D A$ also by $D$.

**Definition 3.1.3.** The *half-center* of $A$ with respect to $D$ is $\mathcal{Z}_{\mathsf{V}}(A, D) \coloneqq \mathrm{Hom}_{{}_A\mathsf{BMod}_A}(A, D)$.

*Remark* 3.1.4. The following diagram commutes (note $A = D = D \otimes_A D$ after forgetting to $\mathsf{RMod}_A$):

$$\begin{tikzcd}
\mathcal{Z}(A, D) \simeq \mathrm{Hom}_{{}_A\mathsf{BMod}_A}(A, D) \arrow[r, "D\otimes_A(-)", "\sim"'] \arrow[d, "\text{forget}"'] & \mathrm{Hom}_{{}_A\mathsf{BMod}_A}(D, D \otimes D) \arrow[d, "\simeq"] \\
A \simeq \mathsf{End}_{\mathsf{RMod}_A}(A, A) & \mathrm{Hom}_{{}_A\mathsf{BMod}_A}(D, A) \arrow[l, "\text{forget}"']
\end{tikzcd}$$

Thus lifting $x \in A$ to $\mathcal{Z}(A, D)$ in fact gives a simultaneous lift to $\mathrm{Hom}(A, D)$ and $\mathrm{Hom}(D, A)$. In particular, a half-central structure on $x$ induces a central structure on $x \otimes x$ by composition $\mathrm{Hom}(A, D) \times \mathrm{Hom}(D, A) \to \mathrm{Hom}(A, A) = \mathcal{Z}(A)$.

### 3.1.2 Cyclic bar construction and the Hochshild cohomology

Here we review the standard resolution of a bimodule into free ones, called the *cyclic bar construction* and the resulting description of the half-center $\mathcal{Z}_{\mathsf{V}}(A, D)$ as the Hochshild cohomology of an $(A, A)$-bimodule $D$ (cf. [BFN10]).

Let $A, B$ be $\mathbb{E}_1$-algebras in $\mathsf{V}$ and let $M, N$ be $(A, B)$-bimodules. Our goal is find a convenient description of $\mathrm{Hom}_{{}_A\mathsf{BMod}_B(\mathsf{V})}(M, N)$. The equivalence ${}_A\mathsf{BMod}_B(\mathsf{V}) \simeq \mathsf{LMod}_A(\mathsf{RMod}_B(\mathsf{V}))$ gives the adjunction with the comonad $T = A \otimes (-)$:

$$\begin{tikzcd}
\mathsf{LMod}_A(\mathsf{RMod}_B(\mathsf{V})) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{RMod}_B(\mathsf{V}) \arrow[l, shift left, "A\otimes(-)"']
\end{tikzcd}.$$

The associated comonad resolution $M \simeq \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}(T^{n+1}M)$ of the bimodule $M$ gives

$$\mathrm{Hom}_{{}_A\mathsf{BMod}_B}(M, N) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{{}_A\mathsf{BMod}_B}(A^{\otimes(n+1)} \otimes M, N) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{RMod}_B}(A^{\otimes n} \otimes M, N).$$

Now we apply this to the case $A = B$, $M = A$, $N = D = {}^D A$, where $D : A \to A$ be an involution as in the last subsection. As the right $A$-module structure on $D$ is the same as $A$, we see

$$\mathcal{Z}_{\mathsf{V}}(A, D) \coloneqq \mathrm{Hom}_{{}_A\mathsf{BMod}_A}(A, D) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{RMod}_A}(A^{\otimes n} \otimes A, A) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A).$$

*Remark* 3.1.5. On the right-hand side, the data of the involution $D$ is encoded in the cosimplicial structure. Explicitly, the structure map $d^i : \mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathsf{V}}(A^{\otimes(n+1)}, A)$ sends $f : A^{\otimes n} \to A$ to

$$d^i f : x_0 \otimes \cdots \otimes x_n \mapsto \begin{cases} D(x_0) f(x_1 \otimes \cdots \otimes x_n) \quad (i = 0), \\ f(x_0 \otimes \cdots \otimes x_{i-1}x_i \otimes \cdots \otimes x_n) \quad (1 \leq i \leq n), \\ f(x_0 \otimes \cdots \otimes x_{n-1})x_n \quad (i = n+1). \end{cases}$$

### 3.1.3 Half-central structure on $\vec{S}^1$

Now we specialize to our case of interest:

**Notation 3.1.6.** Let $\mathsf{V} = \mathsf{Pr}_\omega$, $A = \infty\mathsf{Cat}_*^{\oslash}$, $D = (-)^\circ : A \to A$ be the total dual (monoidal) functor, which flips the cells of all dimensions (Proposition 2.1.17). We continue to denote the associated bimodule ${}^{D}A$ by $D$.

The goal of this section is to prove the following:

**Theorem 3.1.7.** *$\vec{S}^1 \in A$ and $\Sigma \simeq \vec{S}^1 \oslash (-) \in \mathsf{End}_{\mathsf{RMod}_A}(A)$ uniquely lifts along the forgetful functor $\mathcal{Z}_{\mathsf{V}}(A, D) \to \mathsf{End}_{\mathsf{RMod}_A}(A) \simeq A$.*

**Corollary 3.1.8.** *The category $\mathsf{CatSp}$ and the functor $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ lifts to ${}_{\infty\mathsf{Cat}_*}\mathsf{BMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}_\omega)$.*

Unpacking the obstruction theory of the totalization of cosimplicial objects, we see that the data of half-central structure on $\vec{S}^1$ amounts to the following:

- An object $\vec{S}^1 \in \omega\mathsf{Cat}_*$,
- A natural isomorphism $\tau_X : \vec{S}^1 \oslash X \xrightarrow{\sim} X^\circ \oslash \vec{S}^1$,
- A natural homotopy $\theta_{X,Y}$ filling the triangle

$$\begin{tikzcd} & X^\circ \oslash \vec{S}^1 \oslash Y \arrow[dr, "X^\circ \oslash (\tau_Y)"] & \\ \vec{S}^1 \oslash X \oslash Y \arrow[ur, "(\tau_X) \oslash Y"] \arrow[rr, "\tau_{X \oslash Y}"] & & X^\circ \oslash Y^\circ \oslash \vec{S}^1 \end{tikzcd}$$

- A natural homotopy filling the 3-simplex

$$\begin{tikzcd} \vec{S}^1 \oslash X \oslash Y \oslash Z \arrow[r] \arrow[d] \arrow[dr] & X^\circ \oslash Y^\circ \oslash Z^\circ \oslash \vec{S}^1 \\ X^\circ \oslash \vec{S}^1 \oslash Y \oslash Z \arrow[r] \arrow[ur] & X^\circ \oslash Y^\circ \oslash \vec{S}^1 \oslash Z \arrow[u] \end{tikzcd}$$

  whose boundary is filled by homotopies $\tau$ and $\theta$.
- and so on.

It turns out that the all the diagrams live in contractible components, and after providing the natural equivalence $\tau$, there cannot be any nontrivial choice of the coherence data at each step. With this in mind, we have more direct argument:

**Lemma 3.1.9.** *Let $X = X^\bullet$ be a cosimplicial groupoid and $x \in X^0$ be a point. Suppose that*

(1) *the connected component of $(d^0)^n(x) \in X^n$ is contractible for all $n \geq 0$, and*

(2) *there is a path $d^0 x \simeq d^1 x \in X^1$.*

*Then the fiber of $\mathrm{Tot}(X^\bullet) \to X^0$ over $x$ is contractible.*

*Proof.* By left Kan extension along $\{[0]\} \hookrightarrow \mathbf{\Delta}$, the data of $x \in X^0$ is equivalent to a natural transformation $x : I \to X$, where $I$ is the tautological cosimplicial set $\mathbf{\Delta} \hookrightarrow \mathsf{Set} \hookrightarrow \mathsf{S}$. In this way, the map $\mathrm{Tot}(X^\bullet) \to X^0$ is corepresented by $I \to *$, where $*$ is terminal cosimplicial groupoid. Consider the image factorization $I \twoheadrightarrow Y^\bullet \hookrightarrow X^\bullet$ of $x$, so $Y^n \subset X^n$ is the union of connected components of the images of $x$ by the structure maps $X^0 \to X^n$. The conditions (1), (2) ensures $Y^\bullet \xrightarrow{\sim} *$, so the factorization $I \to * \simeq Y^\bullet \to X^\bullet$ gives the canonical point in the fiber of $\mathrm{Tot}(X^\bullet) \to X^0$ over $x$. Conversely, any factorization $I \twoheadrightarrow * \to X$ uniquely factors through $Y$ and is an image factorization, so the groupoid of such factorizations is contractible. □

In the following, let $\mu_n$ generically denote the $n$-ary multiplication map of algebra objects (in particular, monoidal categories).

**Lemma 3.1.10.** *For $n \geq 0$, the connected component of $\Sigma \circ \mu_n$ in the underlying groupoid of $\mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A) = \mathsf{LFun}_\omega(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*)$ is contractible.*

The proof of this lemma will occupy the next section.

**Lemma 3.1.11.** *There exists a (unique) equivalence of endofunctors $\tau : (-) \oslash \vec{S}^1 \xrightarrow{\sim} \vec{S}^1 \oslash (-)^\circ$.*

*Proof.* The uniqueness follows from the $n = 1$ case of Lemma 3.1.10. The existence is Corollary 2.1.19. □

*Proof of Proposition 3.1.7.* Apply 3.1.9 to $X^\bullet = \mathrm{Hom}_{\mathsf{V}}(A^{\otimes \bullet}, A)$ with the cosimplicial structure described in 3.1.5. The conditions (1) and (2) are Lemmas 3.1.10 and 3.1.11, respectively (note that $\mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A)$ is a full subcategory of $\mathsf{LFun}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*)$). □

### 3.1.4 Proof of Lemma 3.1.10

Consider the composition

$$j : \square^{\times n} \to \mathsf{PSh}(\square)^{\otimes n} \to \infty\mathsf{Cat}^{\otimes n} \xrightarrow{(-)_+} \infty\mathsf{Cat}_*^{\otimes n}.$$

The first functor is the Yoneda embedding, the second is the tensor power of the localization $\mathsf{PSh}(\square) \to \omega\mathsf{Cat}$, and the last is the base change along $\mathsf{S} \to \mathsf{S}_*$ in $\mathsf{Pr}$. From the universal property of each functor, $j$ induces a fully faithful embedding

$$\mathsf{LFun}_\omega(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*) \subset \mathsf{LFun}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*) \simeq \mathsf{LFun}(\infty\mathsf{Cat}^{\otimes n}, \infty\mathsf{Cat}_*) \hookrightarrow \mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*).$$

In particular, the connected component of $F := \Sigma \circ \mu_n \circ j \in \mathsf{LFun}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*)$ is equivalent to that of $\Sigma \circ \mu_n \in \mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*)$. Now we have the following commutative diagram:

$$\begin{tikzcd}
\infty\mathsf{Cat}_* \otimes \cdots \otimes \infty\mathsf{Cat}_* \arrow[r, "\mu_n"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \\
\infty\mathsf{Cat} \otimes \cdots \otimes \infty\mathsf{Cat} \arrow[u, "(-)_+"] \arrow[r, "\mu_n"] & \infty\mathsf{Cat} \arrow[u, hook] \arrow[r, "\mathrm{Free}_{\mathbb{E}_1}"] & \mathsf{Mon}(\infty\mathsf{Cat}) \arrow[u, "\mathrm{B}"'] \\
\square \times \cdots \times \square \arrow[u, hook] \arrow[r, "\mu_n"] \arrow[uu, bend left, "j"] & \square \arrow[u, hook] \arrow[r, "\mathrm{Free}_{\mathbb{E}_1}"] & \mathsf{Mon}(\mathsf{Gaunt}) \arrow[u, hook]
\end{tikzcd}$$

The lower-left square commutes by the characterization of Gray tensor product. Free $\mathbb{E}_1$-algebra functors restricted to $\square$ lands in the gaunt monoidal categories by the explicit formula $\mathrm{Free}_{\mathbb{E}_1}(\square^n) \simeq \coprod_{k \geq 0} \square^{kn}$. Therefore the connected component of $F$ in $\mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*)$. is equivalent to the connected component of $\mathrm{Free}_{\mathbb{E}_1} \circ \mu_n \in \mathsf{Fun}(\square^{\times n}, \mathsf{Mon}(\mathsf{Gaunt}))$. The latter is a $(1,1)$-category, so the connected component is equivalent to a delooping of a monoid in $\mathsf{Set}$. We must show that the (ordinary) group $\mathrm{Aut}(F)$ of the invertible object of that monoid is trivial. Recall we have the following equalizer diagram of sets:

$$\mathrm{Aut}(F) \longrightarrow \textstyle\prod_{x \in \square^{\times n}} \mathrm{Aut}_{\mathsf{Mon}(\mathsf{Gaunt})}(Fx) \rightrightarrows \textstyle\prod_{x,y \in \square^{\times n}} \mathrm{Hom}_{\mathsf{Mon}(\mathsf{Gaunt})}(Fx, Fy).$$

We claim that for any $x = (\square^{k_1}, \dots, \square^{k_n})$ the group $\mathrm{Aut}(Fx) = \mathrm{Aut}(\mathrm{Free}_{\mathbb{E}_1}(\square^{k_1+\cdots+k_n}))$ is trivial. Note that the natural map $\mathrm{Aut}_{\mathsf{Gaunt}}(\square^m) \to \mathrm{Aut}_{\mathsf{Mon(Gaunt)}}(\mathrm{Free}(\square^m))$ is a bijection; the inverse is given by the restiction to the indecomposable part[1]. Now the lemma is reduced the following:

**Lemma 3.1.12.** *$\square^n$ is a strong Steiner category and the partial order of Item 3 on its polygraphic basis is a linear order. In particular, $\mathrm{Aut}(\square^n)$ is trivial.*

*Proof.* The second part follows from the first because any automorphism must preserve the order of the basis. First consider the case $n = 1$. $\square^1$ corresponds to the ADC $\lambda\square^1 = (\mathbb{Z}\underline{?} \xrightarrow{(-1,1)} \mathbb{Z}\underline{0} \oplus \mathbb{Z}\underline{1})$ in degree $[0, 1]$ with the basis $\{\underline{0}, \underline{1}, \underline{?}\}$ and the augmentation $\epsilon(\underline{0}) = \epsilon(\underline{1}) = 1$. In this case, the order on the basis is the total order $\underline{0} < \underline{?} < \underline{1}$.

Now we consider the general case. $\square^n$ is by definition the strong Steiner category corresponding to the ADC $(\lambda\square^1)^{\otimes n}$. A basis element corresponds to a "binary string" $\vec{a} = a_1a_2\cdots a_n \coloneqq a_1{\otimes}a_2{\otimes}\cdots{\otimes}a_n$ where $a_i \in \{\underline{?}, \underline{0}, \underline{1}\}$. Let $\vec{a}$, $\vec{b}$ be two basis elements such that $a_i = b_i$ for $i < k$ and $a_k \neq b_k$. Unwinding the definition, we see that the order on the basis is the "signed lexicographic order," i.e., $\vec{a} < \vec{b}$ if and only if either

- $a_1, \dots, a_{k-1}$ contains even number of $\underline{?}$ and $a_k < b_k$ (in the totally ordered set $\{\underline{0} < \underline{?} < \underline{1}\}$), or
- $a_1, \dots, a_{k-1}$ contains odd number of $\underline{?}$ and $a_k > b_k$.

We spell out a sample case to give an idea. If $a_1, \dots, a_{k-1}$ contains even number of $\underline{?}$, then one sees by the Koszul sign rule that the maximal among those starting with $a_1 \cdots a_{k-1}\underline{0}$ is $a_1 \cdots a_{k-1}\underline{01}\cdots\underline{1}$ and the minimal among those starting with $a_1 \cdots a_{k-1}\underline{?}$ is $a_1 \cdots a_{k-1}\underline{?1}\cdots\underline{1}$, but we also have $a_1 \cdots a_{k-1}\underline{01}\cdots\underline{1} \in \mathrm{supp}(\partial^-(a_1 \cdots a_{k-1}\underline{?1}\cdots\underline{1}))$, so $a_1 \cdots a_{k-1}\underline{01}\cdots\underline{1} < a_1 \cdots a_{k-1}\underline{?1}\cdots\underline{1}$. Similar argument for other cases shows that only the first different entry matters when comparing two strings. This signed lexicographic order clearly is a linear order. $\square$

## 3.2 obstruction theory for totaliztion of cosimplicial spaces

**not sure if I should include this section** This section is not logically necessary, but serves to give the description of the half-central structure as in 3.1.3. The proof is included in order to provide a concise model-independent account for the classical theory of [Bou89].

**Notation 3.2.1.** For a functor $f : \mathcal{C} \to \mathcal{D}$, we denote the restriction $\mathsf{Fun}(\mathcal{D}, \mathsf{S}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{S})$ by $f^*$ and the right Kan extension by $f_*$, so we have an adjunction $f^* \dashv f_*$. Recall the equivalence $\lim f_*F \xrightarrow{\sim} \lim F$.

Let $X = X^\bullet$ be a cosimplicial object in $\mathsf{S}$. Our goal is to understand when a point $x_0 \in X^0$ lifts to the totalization $\mathrm{Tot}\, X \coloneqq \lim X^\bullet$. Let $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ denote the full subcategory spanned by $[k]$ for $k \leq n$, let $X_{\leq n} : \mathbf{\Delta}_{\leq n} \to \mathsf{S}$ be the restriction of $X$ and $\mathrm{Tot}_n X \coloneqq \lim X_{\leq n}$. We have the following tower of canonical maps:

$$\mathrm{Tot}\, X \to \cdots \to \mathrm{Tot}_2 X \to \mathrm{Tot}_1 X \to \mathrm{Tot}_0 X = X^0.$$

This is a limit diagram because the natural map $\mathrm{colim}_n\, \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ is an equivalence, and the limit of an $\mathsf{S}$-valued functor is the space of cocartesian sections of its unstraightening (cf. [**MathOverflowDecomposingColimit**]). Now our task is to understand the fiber of $\mathrm{Tot}_{n+1} X \to \mathrm{Tot}_n X$ at a given point $x_n \in \mathrm{Tot}_n X$.

Let $R^nX : \mathbf{\Delta}_{\leq n+1} \to \mathsf{S}$ be the right Kan extension of $X|_{\mathbf{\Delta}_{\leq n}}$ and $M^nX \coloneqq (R^nX)^{n+1}$ (called the *$n$-th matching object* of $X$). The limit of the unit natural transformation $\alpha : X_{\leq n+1} \to R^nX$ is the map in question: $\mathrm{Tot}_{n+1} X \to \lim R^nX \xrightarrow{\sim} \mathrm{Tot}_n X$.

Let $\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1} \subset \mathbf{\Delta}_{\leq n+1}$ denote the subcategory with the same objects but only with injective morphisms. The forgetful functor from the category of sub-simplices $u : (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1})_{/[n+1]} \to \mathbf{\Delta}_{\leq n+1}$

[1] For a monoidal $\infty$-category $\mathcal{M}$, its indecomposable part can be defined as the pullback of $\mathrm{indec}(\pi_0({}^{\leq 0}\mathcal{M})) \hookrightarrow \pi_0({}^{\leq 0}\mathcal{M}) \leftarrow \mathcal{M}$, where $\infty\mathsf{Cat} \xrightarrow{\leq 0(-)} \mathsf{S} \xrightarrow{\pi_0} \mathsf{Set}$ are (product-preserving) left adjoints to the inclusions. Indecomposables are only functorial in monoidal equivalences.

is coinitial (see e.g. [Lur17, p. 1.2.4.17]), so $\lim X \simeq \lim u^* X$. Let $\mathcal{C} = (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n})_{/[n+1]}$ so that $\mathcal{C}^{\triangleright} = (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1})_{/[n+1]}$, and let $\mathcal{C} \xrightarrow{i} \mathcal{C}^{\triangleright} \xleftarrow{j} \{[n+1]\}$ be the inclusions. For any functor $F : \mathcal{C}^{\triangleright}$, the functor $j_* j^* F$ is constant at $F([n+1])$ and the counit $F \to i_* i^* F$ replaces the value $F[n+1]$ by the point $*$. It follows that the following square in $\mathsf{Fun}(\mathcal{C}^{\triangleright}, \mathsf{S})$ is cartesian:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\eta"] & j_* j^* F \arrow[d, "\eta'"] \\ i_* i^* F \arrow[r] & i_* i^* j_* j^* F. \end{tikzcd}$$

Recall that the limit of a constant diagram is given by the cotensoring with the geometric realization (groupoidification) of the diagram shape. Also note that the geometric realization of $\mathcal{C}$ and $\mathcal{C}^{\triangleright}$ are $S^n$ and $*$ because, as a simplicial set, $\mathcal{C}$ is the barycentric subdivision of $\partial \Delta^{n+1}$. As a result, the limit of the above square over $\mathcal{C}^{\triangleright}$ is the following cartesian square in $\mathsf{S}$:

$$\begin{tikzcd} \lim F \arrow[r] \arrow[d, "\eta"] & F[n+1] \arrow[d, "\eta'"] \\ \lim i^* F \arrow[r] & (F[n+1])^{S^n}. \end{tikzcd}$$

Plugging $\alpha : X_{\leq n+1} \to R^n X$ into $F$, we get a cartesian cube (i.e. the cube is a limit diagram, cf. [Lur17, section 6.1.1]) $\eta(\alpha) \Rightarrow \eta'(\alpha)$. Comparing the initial vertices of $\eta(\alpha)$, $\eta'(\alpha)$ with the pullback of the rest of the squares (and since $i^*\alpha$ is an equivalence), we see that the following square in $\mathsf{S}$ is cartesian:

$$\begin{tikzcd} \mathrm{Tot}_{n+1} X \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\ \mathrm{Tot}_n X \arrow[r] & (X^{n+1})^{S^n} \times_{(M^n X)^{S^n}} M^n X \end{tikzcd}$$

Tracing the construction, one sees that the map $S^n \to X^{n+1}$ corresponding to an element $x_n \in \mathrm{Tot}_n X$ sends the basepoint of $S^n$ (coming from $0 \in \partial\Delta^{n+1}$) to $(d^0)^{n+1}(x_0)$, where $x_0$ is the element of $X^0$ that underlies $x_n$. Now we can show the following result:

**Proposition 3.2.2.** *A given point $x_n \in \mathrm{Tot}_n X$ lifts to $x_{n+1} \in \mathrm{Tot}_{n+1} X$ if and only if the induced map $o(x_n) : S^n \to X^{n+1}$ is trivial in $\pi_n(X^{n+1}, (d^0)^{n+1}(x_0))$. When a lift exists, the space of lifts is equivalent to $\Omega^{n+1}(N^n(X), (d^0)^{n+1}(x_0))$, where $N^n(X)$ is the fiber of $X^{n+1} \to M^n(X)$ at the image of $x_n$.*

*Proof.* The image of $x_n$ in $(X^{n+1})^{S^n} \times_{(M^n X)^{S^n}} M^n X$ is equivalent to the data of the following commutative square

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\ * \arrow[r] \arrow[ur, dashed, "x_{n+1}"] & M^n X \end{tikzcd}$$

and the data of the lift $x_{n+1}$ is equivalent to the dashed arrow with two homotopies filling the triangles. The lower-left triangle can always be filled by composing some $* \to S^n$ with the square, so the triviality of $[x_n] \in \pi_n(X^{n+1})$ suffices. Now assume the triviality of $[x_n]$. The space of fillers is equivalent to the space of nullhomotopies of $S^n \to N^n X$. This is $\Omega^{n+1}(N^n X)$. $\square$

*Remark* 3.2.3. By a similar argument one can show that, for any $X^\bullet$ satisfying $X^m = *$ for $m \neq n$, the totalization is either empty or $\Omega^n X^n$ (cf. [Lur17, Corollary 1.2.4.18]).

## 3.3 Construction of the tensor product

Theorem 3.1.7 allows us to lift the defining colimit diagram in $\mathsf{Pr}_\omega$ to

$$\infty\mathsf{Cat}_* \xrightarrow{\Sigma} \infty\mathsf{Cat}_* \xrightarrow{\Sigma} \cdots \to \mathsf{CatSp}$$

a telescope in the category of $(\infty\mathsf{Cat}_*, \infty\mathsf{Cat}_*)$-bimodules in $\mathsf{Pr}_\omega$:

$$A \xrightarrow{\Sigma} D \xrightarrow{\Sigma} A \xrightarrow{\Sigma} D \xrightarrow{\Sigma} \cdots \to A_\Sigma = \mathsf{CatSp}.$$

In particular, $\Sigma_\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ canonically lifts to a map of $(A, A)$-bimodules. We denote $\mathsf{Alg}({}_A\mathsf{BMod}_A(\mathsf{V}))$ by $\mathsf{Alg}_A(\mathsf{V})$. We can now state and prove the main theorem:

**Theorem 3.3.1.** *(1) $\vec{S}^1 \in A$ acts invertibly (from left and right) on the bimodule $A_\Sigma$.*

(2) *The map $\Sigma^\infty : A \to A_\Sigma$ exhibits $\mathsf{CatSp}$ as idempotent $\mathbb{E}_0$-algebra of ${}_A\mathsf{BMod}_A(\mathsf{Pr}_\omega)$.*

(3) *$\mathsf{CatSp}$ admits a unique lift to $\mathsf{Alg}_A(\mathsf{V})$. The lax monoidal forgetful functor ${}_A\mathsf{BMod}_A(\mathsf{V}) \to \mathsf{V}$ induces the underlying presentably monoidal structure on $\mathsf{CatSp}$, which is characterized by promoting $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ to a monoidal functor.*

(4) *The telescope $A_\Sigma$ is a monoidal inversion $A[\vec{S}^{-1}]$. That is, the induced map*

$$\mathrm{Hom}_{\mathsf{Alg}_A(\mathsf{V})}(A_\Sigma, B) \to \mathrm{Hom}_{\mathsf{Alg}_A(\mathsf{V})}(A, B)$$

*is an inclusion of the connected components of functors which sends $\vec{S}^1$ to a (two-sided) invertible object. It also has a few universal property as a module on which $s$ acts invertibly (see (3) and (4) of the next Proposition).*

The theorem is a consequence of the following more general observations:

**Proposition 3.3.2.** *Let $\mathsf{V} = \mathsf{Pr}_\omega$[2] and let $A^\otimes \in \mathsf{Alg}(\mathsf{Pr}_\omega)$ be a monoidal category (not necessarily $\infty\mathsf{Cat}_*$). Let $s \in \mathcal{Z}(A)$ be an object with a central structure (in particular $\tau : l_s = s \otimes (-) \xrightarrow{\sim} r_s = (-) \otimes s$), with which we regard $l_s : A \to A$ as a morphism in ${}_A\mathsf{BMod}_A(\mathsf{V})$. Let $A_s \coloneqq \mathrm{colim}(A \xrightarrow{l_s} A \xrightarrow{l_s} \cdots) \in {}_A\mathsf{BMod}_A(\mathsf{V})$ be its telescope. Assume moreover that $\tau_s : s \otimes s \to s \otimes s$ is equivalent to $\mathrm{id}_{s\otimes s}$[3]. Then:*

(1) *The left- and right- actions of $s \in A$ on $A_s$ are equivalent and both invertible.*

(2) *The canonical functor $\eta : A \to A_s$ exhibits $A_s$ as an idempotent $\mathbb{E}_0$-algebra object of ${}_A\mathsf{BMod}_A(\mathsf{V})$. In particular, it uniquely lifts to an idempotent $\mathbb{E}_1$-algebra object in ${}_A\mathsf{BMod}_A(\mathsf{V})$.*

(3) *For any $B \in \mathsf{Alg}(\mathsf{V})$, the forgetful functor ${}_{A_s}\mathsf{BMod}_B(\mathsf{V}) \to {}_A\mathsf{BMod}_B$ (resp. ${}_B\mathsf{BMod}_{A_s}(\mathsf{V}) \to {}_B\mathsf{BMod}_{A_s}(\mathsf{V})$) is a fully faithful with the essential image consisting of bimodules $M$ on which $s$ acts invertibly from the left (resp. right).*

(4) *There is a monoidal localization $A_s \otimes (-) \otimes A_s : {}_A\mathsf{BMod}_A(\mathsf{V})^\otimes \to {}_{A_s}\mathsf{BMod}_{A_s}(\mathsf{V})^\otimes$ lifting the forgetful functor ${}_{A_s}\mathsf{BMod}_{A_s}(\mathsf{V}) \to {}_A\mathsf{BMod}_A(\mathsf{V})$.*

---

[2]This choice is not important here except that the monoidal structure on $A$ should commute with sequential colimits.

[3]This is a noncommutative variant of an argument usually attributed to Voevodsky, see [Voe98][Rob15][Nik17], but the way of complication is quite different. The main point of Voevodsky's argument was that when we want to invert an object $s \in A$ by telescoping (in a symmetric monoidal category), it suffices to check that the cyclic permutation $(123)$ on $s^3$ is homotopic to the identity. In our argument, such condition is trivially satisfied (we use double suspension to begin with, and there is no nontrivial automorphism of $s^2$). We use the telescope of *left* multiplications, which would intuitively invert the left action of $x$, except that we need central structure to make sense of it. The right action is inverted essentially because it can be identified with the left action through the central structure, and the triviality of transposition is used here again.

(5) *The monoidal localization of (4) induces the localization* $\mathsf{Alg}({}_A\mathsf{BMod}_A(\mathsf{V})) \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{Alg}({}_A\mathsf{BMod}_A(\mathsf{V}))$ *For* $B \in \mathsf{Alg}({}_A\mathsf{BMod}_A(\mathsf{V}))$*, the fact that the unit* $B \to LRB \simeq A_s \otimes B \otimes A_s$ *is an algebra homomorphism characterizes the given algebra structure on* $A_s \otimes B \otimes A_s$

*Proof.* (1) Its left (resp. right) action on $A_s$ are given as the colimit of the telescope (in horizontal direction) of the following commutative squares:

$$\begin{tikzcd} A \arrow[r, "l_s"] \arrow[d, "l_s"'] \arrow[dr, phantom, "\swarrow\alpha"] & A \arrow[d, "l_s"] \\ A \arrow[r, "l_s"] & A, \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} A \arrow[r, "l_s"] \arrow[d, "r_s"'] \arrow[dr, phantom, "\swarrow\beta"] & A \arrow[d, "r_s"] \\ A \arrow[r, "l_s"] & A \end{tikzcd}.$$

Here the natural equivalences $\alpha, \beta$ are provided by regarding $l_s$ as left (resp. right) module morphisms, so $\beta$ is just the associator $\beta_x : (s \otimes x) \otimes s \xrightarrow{\sim} s \otimes (x \otimes s)$, whereas $\alpha$ is the composition of the central structure of the horizontal morphisms with the associator, i.e.,

$$\alpha_x : s_b \otimes (s_a \otimes x) \xrightarrow{s_b \otimes \tau_x} s_b \otimes (x \otimes s_a) \xrightarrow{\sim} (s_b \otimes x) \otimes s_a \xleftarrow{\tau_{s_b} \otimes x} s_a \otimes (s_b \otimes x),$$

where we wrote $s_a$, $s_b$ to avoid confusion between $s$ corresponding to the horizontal and vertical arrows. Notice $\alpha_x \simeq (\tau_{s_b} \otimes x)^{-1} \simeq \mathrm{id}$; the second is by assumption and the first via $\theta_{s_b^2, x}$ (as in 3.1.3, but for $s$). So we have the factorization of the 2-cell:

$$\begin{tikzcd} A \arrow[rr, "l_s"] \arrow[d, "l_s"'] & & A \arrow[d, "l_s"] \\ A \arrow[rr, "l_s"'] \arrow[urr, "f=\mathrm{id}" description] & & A. \end{tikzcd}$$

By the cofinality of $\mathbb{N} \xrightarrow{+1} \mathbb{N}$, the map $A_s \to A_s$ that $f$ induces is an inverse to the map induced by $l_s$, so the left action of $s$ is invertible.

It remains to verify that the right action is invertible. It suffices to show the existence of the invertible 3-cell filling the following cylinder in $\mathsf{V}$ (subscript $s$ of $l, r$ are omitted), because then its telescope exhibits that $\tau$ induces the equivalence between left- and right- action on $A_s$:

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, bend right, "r"'] \arrow[d, bend left, "l"] \arrow[d, phantom, "\Leftarrow\tau"] \arrow[dr, phantom, "\swarrow\alpha"] & A \arrow[d, "l"] \\ A \arrow[r, "l"'] & A, \end{tikzcd} \simeq \begin{tikzcd} A \arrow[r, "l"] \arrow[d, "r"'] \arrow[dr, phantom, "\swarrow\beta"] & A \arrow[d, bend right, "r"'] \arrow[d, bend left, "l"] \arrow[d, phantom, "\Leftarrow\tau"] \\ A \arrow[r, "l"'] & A \end{tikzcd}$$

The existence of the 3-cell can be verified as follows: unpacking the description of $\alpha$ as above, it reduces to providing the following equivalence of 2-cells (unmarked equivalences are associators):

$$(ll \xrightarrow{l\tau} lr \simeq rl \xrightarrow{r\tau} ll) \simeq (ll \xrightarrow{\tau l} rl \simeq lr \xrightarrow{\tau r} rr).$$

Applying the coherence of the center and the assumption on the right-hand side, we have $\tau l(x) = \tau_{s \otimes x} \xrightarrow[\theta_{s,x}]{\sim} (s \otimes s \otimes x \xrightarrow{\tau_s \otimes x} s \otimes s \otimes x \xrightarrow{s \otimes \tau_x} s \otimes x \otimes s) \simeq s \otimes \tau_x \simeq l\tau_x$ and similarly $\tau r(x) \simeq \tau_x \otimes s \simeq r\tau_x$, so we are done.

(2) The morphisms $A \otimes_A A_s \xrightarrow{\eta \otimes A_s}$ and $A_s \otimes_A A \xrightarrow{A_s \otimes \eta}$ are the colimits of the telescope along the endomorphism $l_s \otimes A_s$ and $A_s \otimes l_s$, which are left and right action of $s$ on $A_s$, so they are invertible. Any idempotent $\mathbb{E}_0$-algebra lifts uniquely to an idempotent $\mathbb{E}_1$-algebra by Proposition 3.3.5

(3) The unit transformation $M \to A_s \otimes_A M$ is idempotent, so the free-forgetful adjunction is a localization. The unit is an equivalence iff $s$ acts invertibly on $M$ from the left.

(4) It is a special case of Proposition 3.3.6.

(5) It is a special case of Remark 3.3.7.

∎

*Proof of 3.3.1.* Apply 3.3.2 for $A = \infty\mathsf{Cat}_*^{\oslash}$, $s = \vec{S}^2$. To check $\tau_{\vec{S}^2} : \vec{S}^2 \oslash \vec{S}^2 \xrightarrow{\sim} \vec{S}^2 \oslash \vec{S}^2$ is homotopic to the identity, it suffices to observe that the monoid of endomorphisms of $\vec{S}^4 = \mathrm{B}^4\,\mathrm{Free}_{\mathbb{E}_4}$ is $\mathrm{Free}_{\mathbb{E}_4} = \bigsqcup_{n\in\mathbb{N}} \mathbb{E}_4(n)/\Sigma_n$ and so $\mathrm{Aut}(\vec{S}^4) \simeq *$. ∎

### 3.3.1 Idempotent $\mathbb{E}_1$-algebras

If $\mathcal{C}$ is a symmetric monoidal category, [Lur17, p. 4.8.2.9] that the forgetful functor $\mathsf{CAlg}^{\mathrm{idem}}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathcal{C})$ is an equivalence. We spell out (straightforward but not completely obvious) verifications of the $\mathbb{E}_1$-variant of the related statements. Let $\mathcal{C}^{\otimes}$ be a $\mathbb{E}_1$-monoidal category with the unit 1.

**Definition 3.3.3.** An $\mathbb{E}_0$-algebra $\eta : 1 \to E$ in $\mathcal{C}$ is said to be *idempotent* if the maps $E \simeq 1 \otimes E \xrightarrow{\eta\otimes E} E \otimes E$ and $E \simeq E \otimes 1 \xrightarrow{E\otimes\eta} E \otimes E$ are equivalences. An $\mathbb{E}_1$-algebra $E$ is *idempotent* if the multiplication map $E \otimes E \to E$ is idempotent, or equivalently, the underlying $\mathbb{E}_0$-algebra is idempotent.

By definition, an $\mathbb{E}_0$-algebra $\eta : 1 \to C$ is idempotent if and only if the functor $L^l_E = E \otimes (-) : \mathcal{C} \to \mathcal{C}$ is a localization (as in [Lur09b, Prop. 5.2.7.4]) if and only if $L^r_E = (-) \otimes E : \mathcal{C} \to \mathcal{C}$ is a localization. The composition $L_E = E \otimes (-) \otimes E : \mathcal{C} \to \mathcal{C}$ is also a localization. Now [Lur17, Prop. 2.2.1.9] implies the existence of a unique $\mathbb{E}_1$-monoidal structure on $E\mathcal{C}E = L_E(\mathcal{C})$ promoting $L : \mathcal{C} \to E\mathcal{C}E$ to a monoidal functor. The tensor unit of $E\mathcal{C}E$ is $E$, and since $L_E\mathcal{C}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ is an operad inclusion by construction, $E$ lifts to an idempotent $\mathbb{E}_1$-algebra object in $\mathcal{C}$. Note that $E \otimes \eta \simeq \eta \otimes E$ because both are inverse to the multiplication map $E \otimes E \to E$.

*Remark* 3.3.4. If $\mathcal{C}$ is given a symmetric monoidal structure, any idempotent $\mathbb{E}_1$-algebra automatically upgrades to an $\mathbb{E}_\infty$-algebra, so this section is only relevant if $\mathcal{C}$ itself is noncommutative. This observation also implies that $\mathsf{S} \hookrightarrow n\mathsf{Cat}$ is not idempotent in $\mathsf{Pr}$ for $n \geq 0$ because they have noncommutative monoidal structures (namely, the lax Gray tensor products) on these categories, and similarly for $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{CatSp}$.

**Proposition 3.3.5.** *The forgetful functor $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_1}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathcal{C})$ is an equivalence of posets.*

*Proof.* The above argument shows the essential surjectivity. For the rest, suppose there exists a $\mathbb{E}_0$-algebra map $f : A \to B$ between idempotent $\mathbb{E}_1$-algebras. We wish to show $\mathrm{Hom}_{\mathbb{E}_0}(A, B)$, $\mathrm{Hom}_{\mathbb{E}_1}(A, B)$ are both contractible. Since $\eta_B \otimes B \simeq (1 \otimes B \xrightarrow{\eta_A\otimes B} A \otimes B \xrightarrow{f\otimes B} B \otimes B)$ is an equivalence, $B$ is a retract of $A \otimes B$ and therefore $L^l_A$-local, i.e., $B \simeq A \otimes B'$ for some $B'$. In fact, $(B \xrightarrow{\eta_A\otimes B} A \otimes B) \simeq A \otimes B' \xrightarrow{\eta_A\otimes A\otimes B'} A \otimes A \otimes B'$ is an equivalence. Now consider the square $(f : A \to B) \otimes (\eta_A : 1 \to A)$; the diagonal is $(B \otimes \eta_A) \circ f \simeq (f \otimes A) \circ (A \otimes \eta_A) \simeq (f \otimes A) \circ (\eta_A \otimes A) \simeq \eta_B \otimes A$, so $f \simeq (B \otimes \eta_A)^{-1} \circ (\eta_B \otimes A)$ is uniquely determined, i.e., $\mathrm{Hom}_{\mathbb{E}_0}(A, B) \simeq *$. The symmetric argument implies $B \simeq B \otimes A$, so $B$ lies in $\mathsf{Alg}_{\mathbb{E}_1}(E\mathcal{C}E) \subset \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})$. It follows that $\mathrm{Hom}_{\mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})}(A, B)$ is contractible because $A$ is initial (tensor unit) in $\mathsf{Alg}_{\mathbb{E}_1}(E\mathcal{C}E)$. ∎

**Proposition 3.3.6.** *Let $E$ be an idempotent $\mathbb{E}_1$-algebra. Then*

(1) *The forgetful functor $\mathsf{LMod}_E(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathsf{RMod}_E(\mathcal{C}) \to \mathcal{C}$) is an equivalence to the full subcategory $E\mathcal{C}$ (resp. $\mathcal{C}E$).*

(2) *the forgetful functor ${}_E\mathsf{BMod}_E(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is an equivalence to the full suboperad $E\mathcal{C}^{\otimes}E \subset \mathcal{C}^{\otimes}$.*

*Proof.* We spell out the left module case; the others are similar. The unit law $M \simeq 1 \otimes M \xrightarrow{\eta_A\otimes M} A \otimes M \xrightarrow{a} M$ implies $M$ is a retract of $A \otimes M$, so $M$ is $L^l_A$-local and therefore $\eta_A \otimes M = \eta_A \otimes A \otimes M'$ (for some $M'$) is an equivalence. This means $a$, the counit map of the free-forgetful adjunction, is also an equivalence. Also, on these local objects, the unit map $A \otimes X \xrightarrow{\eta\otimes A\otimes X} A \otimes A \otimes X$ of the free-forgetful adjunction is equivalence, so the adjunction induces the stated equivalence. ∎

*Remark* 3.3.7. Since $L : \mathcal{C}^{\otimes} \rightleftarrows {}_E\mathsf{BMod}_E(\mathcal{C})^{\otimes} : R$ is a monoidal localization, the argument of [GGN16, Lemma 3.6] shows that there is a localization $L' : \mathsf{Alg}(\mathcal{C}) \to \mathsf{Alg}({}_E\mathsf{BMod}_E(\mathcal{C}))$ commuting with $L$ via the forgetful functors, and moreover, for $A \in \mathsf{Alg}(\mathcal{C})$, the algebra structure on the underlying object of $RLA$ is characterized by that the unit map $A \to RLA$ extends to a morphism of algebras.

## 3.4 Formula for the tensor product

### 3.4.1 action of $\vec{S}^1$

Let us examine the left and right action of $\vec{S}^1$ on $\mathsf{CatSp}$ by unpacking the (half-center analog of) proof of 3.3.2.

Recall that the left action $l_{\vec{S}^1} : \mathsf{CatSp} \to \mathsf{CatSp}$ is given by the limit of the following diagram:

$$\begin{tikzcd} A \arrow[r, "l_{\vec{S}^1}"] \arrow[d, "l_{\vec{S}^1}"'] & D \arrow[r, "l_{\vec{S}^1}"] \arrow[d, "l_{\vec{S}^1}"'] & A \arrow[r, "l_{\vec{S}^1}"] \arrow[d, "l_{\vec{S}^1}"'] & \cdots \\ A \arrow[r, "l_{\vec{S}^1}"] & D \arrow[r, "l_{\vec{S}^1}"] & A \arrow[r, "l_{\vec{S}^1}"] & \cdots , \end{tikzcd}$$

where the commuting 2-cell is given by the half-central structure $\tau_{\vec{S}^1} : \vec{S}^1 \oslash \vec{S}^1 \xrightarrow{\sim} (\vec{S}^1)^{\circ} \oslash \vec{S}^1$. Since $(\vec{S}^1)^{\circ} \simeq (\vec{S}^1)$ and $\mathrm{Aut}(\vec{S}^1 \oslash \vec{S}^1) \simeq (\mathrm{Free}_{\mathbb{E}_2}(*))^{\times} \simeq *$, the diagram is (equivalent to) simply

$$\begin{tikzcd} \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\not\not"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\not\not"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \cdots \\ \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \cdots , \end{tikzcd}$$

Which induces the shift functor $\Sigma = [1] : \mathsf{CatSp} \to \mathsf{CatSp}$ by definition. Moreover, the half-central structure on $\vec{S}^1 \in \infty\mathsf{Cat}_*$ induces a half-central structure on $\Sigma^{\infty}\vec{S}^1 = \mathbb{F}[1] \in \mathsf{CatSp}$.

$D : \mathsf{CatSp} \to \mathsf{CatSp}$ is a levelwise $D$.

To compute the right action, let $\Omega^{\circ}$ be the

### 3.4.2 A formula

For a spectrum $X$, a fundamental observation is that $X$ is the colimit of the suspension spectra of its components: $\mathrm{colim}_n \Sigma^{\infty-n}\Omega^{\infty-n}X \xrightarrow{\sim} X$. The formula remains valid for categorical spectra by the same formal reason:

**Proposition 3.4.1.** *Let $\{\mathcal{C}_n\} \in \mathsf{Fun}(\mathbb{N}^{\triangleright}, \mathsf{Pr}_{\omega})$ be a colimit diargram of compactly generated categories whose structure morphisms $\mathcal{C}_n \to \mathcal{C} \coloneqq \mathcal{C}_{\infty}$ is denoted by $L_n \dashv R_n$. Then there is a colimit diagram $L_0R_0 \to L_1R_1 \to \cdots \to \mathrm{id}_{\mathcal{C}}$ in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$ induced by the counit maps.*

*Proof.* Let $\tilde{\mathcal{C}} \to \mathbb{N}^{\triangleright}$ be the cartesian and cocartesian fibration unstraightening the diagram $\{\mathcal{C}_n\}$. By cartesian transport and because $\{\mathcal{C} \xrightarrow{R_n} \mathcal{C}_n\}$ is a limit cone, $\mathcal{C}$ is equivalent to the categoary of cartesian sections on $\mathbb{N}$ [Lur09b, Prop. 3.3.3.1]: $\mathcal{C} \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{cart}}_{/\mathbb{N}^{\triangleright}}(\mathbb{N}^{\triangleright}, \tilde{\mathcal{C}}) \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{cart}}_{/\mathbb{N}}(\mathbb{N}, \tilde{\mathcal{C}}|_{\mathbb{N}})$. Composing with the cocartesian transport $\tilde{\mathcal{C}} \to \mathcal{C}$ (i.e., the left adjoint of the inclusion), we get the diagram in question $\mathcal{C} \to \mathsf{Fun}(\mathbb{N}^{\triangleleft}, \tilde{\mathcal{C}}) \to \mathsf{Fun}(\mathbb{N}^{\triangleleft}, \mathcal{C})$. Let $f : \mathrm{colim}(L_0R_0 \to L_1R_1 \to \cdots) \to \mathrm{id}$ be the comparison map. The above equivalence also implies $R_n : \mathcal{C} \to \mathcal{C}_n$ are jointly conservative, so to show that $f$ is an equivalence, it suffices to check $R_nf$ is an equivalence for each $n \in \mathbb{N}$. This follows from that $R_m$ preserves filtered colimits and that for each $m \geq n$, the map $R_n\varepsilon_m : R_nL_mR_m \to R_n$ induced by the counit map is equivalent to $R_{n,m}R_m\varepsilon_m \simeq \mathrm{id}_{R_n}$. □

**Corollary 3.4.2.** *Let $X = (X_n)$ be a categorical spectrum. Then there are canonical equivalences $\mathrm{colim}_n \Sigma^{\infty-n}X_n \xrightarrow{\sim} \mathrm{colim}_n \mathrm{B}^{\infty-n}X_n \xrightarrow{\sim} X$.*

**Corollary 3.4.3.** *Let $X = (X_n)$, $Y = (Y_n)$ be categorical spectra. Then we have*

$$X \otimes Y \simeq \operatorname*{colim}_{i,j}(\Sigma^{\infty-i-j}(D^j X_i) \oslash Y_j) \simeq \operatorname*{colim}_{n}(\Sigma^{\infty-4n}(X_{2n} \oslash Y_{2n})).$$

**Corollary 3.4.4.** *corresponding formula for the function categorical spectra.*

Recall that for a spectrum $X \in \mathsf{Sp}$, the following conditions are equivalent:

(1) $X$ is *finite*, i.e., $X \simeq \Sigma^{\infty-n}Y$ for some natural number $n$ and a finite pointed CW complex $Y$.

(2) $X$ is *perfect*, i.e., $X$ belongs to the smallest stable subcategory which contains $\mathbb{S}$ and is closed under retracts.

(3) $X$ is *compact*, i.e., $\mathrm{Map}_{\mathsf{Sp}}(X, -) : \mathsf{Sp} \to \mathsf{S}$ preserves filtered colimits.

(4) $X$ is *dualizable*, i.e., the functor $X \otimes (-) : \mathsf{Sp} \to \mathsf{Sp}$ admits left or right (equivalently, both) adjoints.

**Definition 3.4.5.** A (pointed) $\infty$-category is *finite* if it belongs to the smallest subcategory $\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)} \subset \infty\mathsf{Cat}_{(*)}$ which contains the (pointed) cells $\mathbb{G} = \{C_{n(,+)}\}_{n\geq 0}$ and closed under finite colimits **or generated from the unit and the partially lax colimits?**. **what are the compact objects of $\infty$Cat etc.?** A categorical spectrum is *finite* if it is of the form $\Sigma^{\infty-n}X$ for some integer $n$ and finite pointed $\infty$-category $X$.

*Remark* 3.4.6. Finite $\infty$-categories are compact and the inclusion $\infty\mathsf{Cat}^{\mathrm{fin}} \to \infty\mathsf{Cat}$ preserves finite colimits. By [Lur09b, Proposition 5.3.5.11, Example 5.3.6.8], the left Kan extension $\mathrm{Ind}(\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)}) \to \infty\mathsf{Cat}_{(*)}$ is fully faithful and colimit preserving. Since the cells generate $\infty\mathsf{Cat}$ under colimits, we have $\mathrm{Ind}(\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)}) \xrightarrow{\sim} \infty\mathsf{Cat}_{(*)}$. In particular, every (pointed) $\infty$-category is canonically a filtered colimit of finite ones. **better if I can say more explicitly that it is a colimit of finite *sub*categories**

**Example.** Any finite torsion-free complex is a finite $\infty$-category by [Cam23a, Theorem B](**I want to say finite computad but idk if the pushout diagram is weak**). In particular, any strong Steiner $\infty$-category corresponding to a finite-dimensional strong Steiner complex is finite; examples include the objects of $\Theta$, lax cubes and orientals.

*Remark* 3.4.7. closure of finite stuff under Gray tensor and lax pushout, partially lax colimits?

**Corollary 3.4.8.** *Any categorical spectrum is a filtered colimit of finite spectra. More precisely, the inclusion of the subcategory $\mathsf{CatSp}^{\mathrm{fin}} \subset \mathsf{CatSp}$ induces an equivalence $\mathrm{Ind}(\mathsf{CatSp}^{\mathrm{fin}}) \xrightarrow{\sim} \mathsf{CatSp}$.*

*Proof.* Since $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ preserves filtered colimits, finite categorical spectra are compact **it is a bimodule hom so I should be able to say more?**. By [Lur09b, Proposition 5.3.5.11], the left Kan extension $\mathrm{Ind}(\mathsf{CatSp}^{\mathrm{fin}}) \to \mathsf{CatSp}$ is fully faithful. To show that it is an equivalence, we must check that the smallest full subcategory of $\mathsf{CatSp}$ containing $\mathsf{CatSp}^{\mathrm{fin}}$ and closed under filtered colimits is $\mathsf{CatSp}$ itself, which follows from Corollary 3.4.2 and Remark 3.4.6. $\square$

*Remark* 3.4.9. The analogous statement seems false for *conective* categorical spectra; $\mathsf{CatSp}^{\mathrm{fin,cn}} \simeq \mathrm{Free}(\mathsf{CMon}(\omega\mathsf{Cat}_*))$ probably needs *sifted* colimits to generate $\mathsf{CatSp}^{\mathrm{cn}}$.

**Corollary 3.4.10.** *Compact objects of $\mathsf{CatSp}$ are precisely the retracts of some finite categorical spectrum.*

**can I remove "retracts of" similarly to the case of spectra?** guess: need algebraic invariants that tells me the recipe of how to construct catgories out of cells and conservative enough for highly connected categories. use the space of cells? first I think I should characterize the finite categories using the space of cells.

## 3.5 comparison

In this section, we compare our tensor product with known monoidal structures of relevant objects.

### 3.5.1 additivity on category levels and comparison to spectra

Similarly to the Gray tensor product, the tensor product of categorical spectra behaves additively on category levels.

**Proposition 3.5.1.** *For $m, n \in \mathbb{Z} \cup \{\pm\infty\}$, the essential image of $m\mathsf{CatSp} \otimes n\mathsf{CatSp} \subset \mathsf{CatSp} \otimes \mathsf{CatSp} \to \mathsf{CatSp}$ is $(m+n)\mathsf{CatSp}$. Here the convention is $\infty + (-\infty) = -\infty$.*

*Proof.* The category $n\mathsf{CatSp}$ is the colimit-closure of the set $\{\Sigma_+^{\infty-i}\square^j \mid 0 \le j \le n+i \text{ or } j = 0\}$. Since $D\square^j \simeq \square^j$, we have $\Sigma_+^{\infty-i}\square^j \otimes \Sigma_+^{\infty-k}\square^l \simeq \Sigma_+^{\infty-i-k}(D^k\square^j) \otimes \square^l \simeq \Sigma_+^{\infty-i-k}\square^{j+l}$. □

**exponential ideal?**

We isolate the case when $m$ or $n$ is $-\infty$:

**Proposition 3.5.2.** *The subcategory $\mathsf{Sp} \to \mathsf{CatSp}$ is an $\otimes$-ideal. The localization is smashing with the sphere spectrum $\mathbb{S}$.*

### 3.5.2 additivity on connectivity and comparison to symmetric monoidal $\infty$-categories

Recall we denoted the image of the embedding $\mathsf{CMon}(\infty\mathsf{Cat}) \hookrightarrow \mathsf{CatSp}$ by $\mathsf{CatSp}^{\mathrm{cn}}$ and said its objects are *connective*. A categorical spectrum $X$ is *$n$-connective* if the natural map $\mathrm{B}^{\infty+n}\Omega^{\infty+n}X$ is an equivalence.

**Proposition 3.5.3.** *The following are equivalent:*

1. *$X$ is $n$-connective.*
2. *$X_k$ is equivalent to an algebroid with a single $j$-morphism up thorough $j < n + k$.*
3. *$X_k^{\le 0}$ is a $n + k$-connective space for any $k >> 0$.*
4. *$\mathbb{G}_m(X) \in \mathsf{Sp}$ is $n$-connective.*
5. *All objects of $X_k$ are equivalent to one another for $k \ge -n$.*

**Proposition 3.5.4.** *Let $X, Y$ be a categorical spectrum which is $m, n$-connective but not $(m+1)$, $(n+1)$-connective, respectively. Then $X \otimes Y$ is $(m+n)$-connective but not $(m+n+1)$-connective.*

The following proposition shows that our tensor product restricts to what should be called the lax Gray tensor product of symmetric monoidal $\infty$-categories.

**Proposition 3.5.5.** *The inclusion $\mathrm{B}^\infty : \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{\sim} \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ exhibits $\mathsf{CMon}(\infty\mathsf{Cat})$ as a monoidal subcategory. The monoidal structure is characterized by the fact that the functor $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat} \to \mathsf{CMon}(\infty\mathsf{Cat})$ promotes to a monoidal functor with respect to the Gray tensor product of the domain.*

*Proof.* The tensor unit $\mathbb{F}$ is in $\mathsf{CatSp}^{\mathrm{cn}}$. To prove the first statement, by [Lur17, Proposition 2.2.1.1] it suffices to check that for any $X, Y \in \mathsf{CMon}(\infty\mathsf{Cat})$, the tensor product $(\mathrm{B}^\infty X) \otimes (\mathrm{B}^\infty Y)$ lies in the image of $\mathrm{B}^\infty$. Write $X \simeq \mathrm{colim}_i\, \mathrm{Free}_{\mathbb{E}_\infty}(\mathcal{C}_i)$, $Y \simeq \mathrm{colim}_j\, \mathrm{Free}_{\mathbb{E}_\infty}(\mathcal{D}_j)$. Now we have

$$(\mathrm{B}^\infty X) \otimes (\mathrm{B}^\infty Y) \simeq (\operatorname*{colim}_i \Sigma_+^\infty(\mathcal{C}_i)) \otimes (\operatorname*{colim}_j \Sigma_+^\infty(\mathcal{D}_j)) \simeq \operatorname*{colim}_{i,j}(\Sigma_+^\infty(\mathcal{C}_i) \otimes \Sigma_+^\infty(\mathcal{D}_j)) \simeq \operatorname*{colim}_{i,j} \Sigma_+^\infty(\mathcal{C}_i \otimes \mathcal{D}_j).$$

Since $\mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ is closed under colimits, the last colimit stays inside $\mathsf{CatSp}^{\mathrm{cn}}$. The characterization is also clear from this computation. □

Recall that by [GGN16] $\mathsf{CMon} \in \mathsf{Pr}^{\mathsf{L}}$ is an idempotent commutative algebra and we have the associated symmetric monoidal localization $\mathsf{CMon} : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ to *semiadditive* presentable categories. In particular, it induces $\mathsf{CMon} : \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ and $\mathsf{CMon} : \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$. We denote the symmetric monoidal structure on $\mathsf{CMon}(\infty\mathsf{Cat}, \times)$ by $\circledast$ and the monoidal structure on $\mathsf{CMon}(\infty\mathsf{Cat}, \otimes)$ by $\otimes$. The lax monoidal functor $\mathrm{id} : (\infty\mathsf{Cat}, \times) \to (\infty\mathsf{Cat}, \otimes)$ of Remark 2.1.15 induces the lax monoidal structure on $\mathsf{CMon}(\infty\mathsf{Cat})^{\circledast} \to \mathsf{CMon}(\infty\mathsf{Cat})^{\otimes}$.

**Corollary 3.5.6.** *The functor* $\mathrm{B}^\infty$ *induces a functor* $\mathsf{Rig}_{\mathbb{E}_1}(\infty\mathsf{Cat}) \to \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{CatSp})$.

description of internal hom, limit of lax natural transformation category of deloopings?

## 3.6 Examples

$(-) \otimes \mathbb{S}$ is the group completion/spectral localization. $\mathbb{S}$ is compact, fitting into the cofiber sequence $\mathbb{F} \xrightarrow{\Delta} \mathbb{F} \oplus \mathbb{F} \xrightarrow{(1,-1)} \mathbb{S}$ but not dualizable.

## 3.7 projective, flat and invertible objects

**Proposition 3.7.1.** *compact Projective objects in* $\mathsf{CatSp}^{\mathrm{cn}}$ *are the finite free objects, i.e.,* $\mathbb{F}^{\oplus n}$ *for some* $n$. *(I think it's also true for non-finite projectives?)*

*Proof.* I think this is a generalities on algebraic theories and compact-projective generation? □

**Proposition 3.7.2.** *Dualizable objects in* $\mathsf{CatSp}^{\mathrm{cn}}$ *are precisely the finite projective ones (so finite free).*

*Proof.* Because $\mathbb{F}$ is compact projective. □

**Proposition 3.7.3.** $\mathrm{Pic}(\mathsf{CatSp}^{\otimes}) = \{\mathbb{F}[n] \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}$.

*Proof.* Let $X$ be an invertible object with the monoidal inverse $X^{-1}$. Since $\mathbb{F}$ is compact, $X$ is also compact, so it is a retract of a finite categorical spectrum, which in particular is $n$-connective for some $n \in \mathbb{Z}$. By shifting if necessary **and using $\infty$-connective is contractible** we may assume that $X$ is 0-connective but not 1-connective. In this case, $X^{-1}$ must be also connective by the previous section (**ref**). In this case, $X$ must be finite free, so $\mathrm{Pic}(\mathbb{N}, +) = \{0\}$ implies $X \simeq \mathbb{F}$. □

# Chapter 4

# absolute colimits in categorical spectra

Todo:

(1) dualizable implies compact implies Finite

(2) three-periodic sequence

(3) relation to stable 2-categories literature?

(4) lax fiber and lax cofiber, some sample computation

(5) sample calculation of duality

(6) lax pushouts are absolute?

## 4.1 Comma and Cocomma squares

### 4.1.1 weighted (co)limit description

Let $\mathcal{C}$ be an object in $\mathsf{LMod}_{\infty\mathsf{Cat}^{\otimes}}(\mathsf{Pr}^{\mathsf{L}})$. Let $J = \bullet \leftarrow \bullet \rightarrow \bullet = \Lambda^2_0 = \mathrm{Sd}(\square^1)$ be the walking cospan category. We define $W_{\bullet\bullet} : J^{\mathrm{op}} \otimes J^{\mathrm{op}} \to \mathsf{Cat}$ by the commutative diagram

$$\begin{tikzcd}
\square^0 \arrow[r, "="] \arrow[d, "="] & \square^0 \arrow[d, "0"] & \emptyset \arrow[l] \arrow[d] \\
\square^0 \arrow[r, "0"] & \square^1 & \square^0 \arrow[l, "1"'] \\
\emptyset \arrow[u] \arrow[r] & \square^0 \arrow[u, "1"'] & \square^0, \arrow[l, "="] \arrow[u, "="']
\end{tikzcd}$$

This induces the following adjunction between the category of spans and cospans (with natural transformations).

$$\mathsf{Fun}(J, \mathcal{C}) \underset{\lim^W}{\overset{\mathrm{colim}^W}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Fun}(J^{\mathrm{op}}, \mathcal{C})$$

Explicitly, the left adjoint takes a span $B \leftarrow A \rightarrow C$ to the cocomma object $B \rightarrow B\vec{\amalg}_A C \leftarrow C$ which is the colimit of the following diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr, "0"] & & A \arrow[dl, "1"'] \arrow[dr] & \\
B & & \square^1 \otimes A & & C
\end{tikzcd}$$

and the right adjoint takes a cospan $B \to D \leftarrow C$ to the cocomma object $B \leftarrow B\vec{\times}_D C \to C$ Which is the limit of the following diagram

$$\begin{tikzcd} B \arrow[dr] & & D^{\square^1} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & C \arrow[dl] \\ & D & & D & \end{tikzcd}$$

This adjoint factors through the (1-)category of (lax) squares (check):

$$\mathsf{Fun}(J, \mathcal{C}) \underset{\text{res}}{\overset{\text{lke}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(\square^2, \mathcal{C}) \underset{\text{rke}}{\overset{\text{res}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(J^{\mathrm{op}}, \mathcal{C})$$

To see this, recall the decomposition

$$\square^2 = \Delta^{\{00,10,11\}} \cup_{\Delta^{\{00,11\}}=\sigma(\{0\})} \sigma\square^1 \cup_{\sigma(\{1\})=\Delta^{\{00,11\}}} \Delta^{\{00,01,11\}}.$$

Using this, we see that the data of the lax square $\square^2 \to \mathcal{C}$ which we depict as

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is (functorially) equivalent to either of the following commutative diagrams:

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "i_0"] & B \arrow[dd] \\ A \arrow[r, "i_1"] \arrow[d] & \square^1 \otimes A \arrow[dr, "\alpha"] & \\ C \arrow[rr] & & D, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"] \arrow[dd] & & B \arrow[d] \\ & D^{\square^1} \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & D \\ C \arrow[r] & D & \end{tikzcd}$$

*Remark* 4.1.1. Another interpretation using profunctors?

$$\mathsf{Fun}(J^{\mathrm{op}}, \mathcal{C}) \underset{\perp}{\rightleftarrows} \mathsf{Morita}(\mathcal{C}) \underset{\perp}{\rightleftarrows} \mathsf{Fun}(J, \mathcal{C})$$

fibered over $\mathcal{C} \times \mathcal{C}$.

The analog in the lax setting of the pasting law of the pullback and pushout squares is rather restrictive:

**Proposition 4.1.2.** *Suppose we have the following diagrams*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\beta"] & E \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\gamma"] & D \arrow[r] \arrow[d] & F \\ G \arrow[r] & H & \end{tikzcd}$$

*where $\alpha$ is a cocomma square and $\beta$, $\gamma$ are invertible. Then*

(1) *$\beta * \alpha$ is a cocomma square if and only if $\beta$ is a pushout square.*

(2) *$\gamma * \alpha$ is a cocomma square if and only if $\gamma$ is a pushout square.*

*Similar assertion holds for comma and pullback squares.*

*Proof.* It follows from the pasting law of pullback squares together with the fact that the colimit of $C \leftarrow A \to \square^1 \otimes A \leftarrow A \to B$ can be computed by forming two pushout squares. $\square$

Notice that even when $\mathcal{C} = \infty\mathsf{Cat}$, the construction of comma and cocomma squares is not 2-functorial. However, we have the following:

**Proposition 4.1.3.** *Let $\mathcal{C}$ be a pointed category, so that it is in $\mathsf{LMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}^{\mathsf{L}})$. The loop and suspension functors $\Sigma : \mathcal{C} \to \mathcal{C}$ induces $\mathsf{Fun}(\square^2, \mathcal{C}) \to \mathsf{Fun}((\square^2)^{2\text{-op}}, \mathcal{C})$.*

In the pointed case, we denote the action by $\oslash : \infty\mathsf{Cat}_* \otimes \mathcal{C} \to \mathcal{C}$. To apply the previous considerations, note that $X \in \infty\mathsf{Cat}$ acts by $X_+ \oslash (-)$. The point of the above proposition is $\Sigma(\square^1_+ \oslash X) \simeq (\vec{S}^1 \oslash \square^1_+) \oslash X \xrightarrow{\sim} ((\square^1)^{\mathrm{op}}_+ \oslash \vec{S}^1) \oslash X$, using the half-central structure of $\vec{S}^1$. This is reflected in the classical definition of triangulated category; namely, this is the negative sign introduced when we rotate the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ to $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$; since we do not have negatives in our categorical setting, we need to be more careful about the classical sign issues.

**Lemma 4.1.4.** *If $F : \mathcal{C} \to \mathcal{C}$ is a morphism in $\mathsf{LMod}_{\infty\mathsf{Cat}_*^{\oslash}}(\mathsf{Pr}^{\mathsf{L}})$, then we have a canonical equivalence $F\Sigma \simeq \Sigma F : \mathcal{C} \to D \otimes \mathcal{C}$ in $\mathsf{LMod}_{\infty\mathsf{Cat}_*^{\oslash}}(\mathsf{Pr}^{\mathsf{L}})$. If $\vec{S}^1$ acts invertibly on $\mathcal{C}$ (so $\mathcal{C}$ is a $\mathsf{CatSp}$-module), then $\Omega F \Sigma$ is a $\infty\mathsf{Cat}_*$-homomorphism. What I really want to say here is that $\Omega F \Sigma$ is an $\infty\mathsf{Cat}_*^{\oslash}$-enriched functor and $\eta$ is a 2-natural transformation. Does it work?*

## 4.2 Cone vs Path Space

Let $I \in \omega\mathsf{Cat}_*$ be the interval $\square^1$ with the basepoint 0. It is the lax cofiber of $\mathrm{id} : S^0 \to S^0$, i.e. the pushout of $* \leftarrow \{0\}_+ \oslash S^0 \to \square^1_+ \oslash S^0$.

The special case of the fiber vs cofiber is the following:

**Proposition 4.2.1.** *There are the folloiwng diagram functorial in $X$:*

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathrm{Cone}(X) \arrow[r] \arrow[d] \arrow[ur, phantom, "\Nearrow\beta"] & \mathrm{B}X \arrow[d, "\mathrm{id}"] \\
& 0 \arrow[r] \arrow[ur, phantom, "\Nearrow\gamma"] & \mathrm{B}X,
\end{tikzcd}$$

*where $\alpha$ is a cocomma square, $\beta$ is a bicartesian square, $\gamma$ is a comma square.*

We first construct the universal case $X = \mathbb{F}$ unstably:

$$\begin{tikzcd}
S^0 \arrow[r, "\mathrm{id}"] \arrow[d] & S^0 \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\Nearrow\alpha"] & I \arrow[r] \arrow[d] \arrow[ur, phantom, "\Nearrow\beta"] & \vec{S}^1 \arrow[d, "\mathrm{id}"] \\
& * \arrow[r] \arrow[ur, phantom, "\Nearrow\gamma"] & \vec{S}^1,
\end{tikzcd}$$

Applying $\Sigma^\infty(-) \otimes X$, (since tensoring from the right preserves lax squares) we have the desired squares, with cocomma $\alpha$ and cocartesian $\beta$. Since $\gamma * \beta$ is a bicomma square, it suffices to show that $\gamma$ is a comma square. Notice that the lax fibers are the cotensoring by $I$. In particular, $\gamma$ induces the following comparison map:

$$\mathrm{Cone}(X) \coloneqq \Sigma^\infty I \otimes X \to \mathrm{Path}(X) = [\Sigma^\infty I, X[1]] = [\Sigma^\infty I[-1], X]$$

We claim that this is an equivalence. In fact, we will show the following:

**Proposition 4.2.2.** $\Sigma^\infty I[-1]$ *is the left dual of* $\Sigma^\infty I$ *in the monoidal category* $\mathsf{CatSp}$ *via the counit map* $\varepsilon : \Sigma^\infty I[-1] \otimes \Sigma^\infty I \to \mathbb{F}$ *induced by the above comparison map.*

*Remark* 4.2.3. This should be thought of as the analog of Spanier-Whitehead duality. The shift degree 1 should account for the dimension of 1 of the interval...

*Remark* 4.2.4. I think it is true that $I^{\oslash n} = \sigma\Delta^n$ with the source vertex pointing (didn't check the duality).

*Remark* 4.2.5. Is the unpointed interval $\Sigma^\infty_+ \square^1$ also dualizable? If so, we have a large supply of dualizable objects by tensoring etc...

*Remark* 4.2.6. $\Sigma^\infty I = \mathrm{B}^\infty \operatorname{Free}_{\mathbb{E}_\infty}(I)$ is the symmetric monoidal category freely generated by an object $c$ and a morphism $1 \to c$, or in other words, a single $\mathbb{E}_0$-algebra. In particular, we have an explicit description $\operatorname{Free}_{\mathbb{E}_\infty}(I) = \operatorname{Env}(\mathbb{E}_0) = \mathsf{Fin}^{\mathrm{inj}}$. The inclusion $\mathsf{Fin}^{\simeq} = \operatorname{Env}(\mathrm{Triv}) \to \operatorname{Env}(\mathbb{E}_0) = \mathsf{Fin}^{\mathrm{inj}}$ is the unit map $\mathbb{F} \to \Sigma^\infty I$.

*Proof.* Unpacking the construction, the morphism $\vec{S}^1 \otimes \varepsilon : \Sigma^\infty(I \oslash I) \to \mathbb{F}[1]$ is induced by the map $I \oslash I \to \vec{S}^1 \simeq \mathrm{B}\mathbb{N}$ depicted as

$$\begin{tikzcd} 00 \arrow[r, equal] \arrow[d, equal] & 10 \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ 01 \arrow[r] & 11 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} * \arrow[r, equal, "0"] \arrow[d, equal, "0"'] & * \arrow[d, "1"] \arrow[dl, phantom, "/\!/"] \\ * \arrow[r, "1"] & * \end{tikzcd}$$

We define the unit map $\eta : \mathbb{F} \to I \otimes I[-1]$ so that the composition

$$\vec{S}^1 \otimes \mathbb{F} \xrightarrow{\vec{S}^1 \otimes \eta} \vec{S}^1 \otimes \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \xrightarrow[\sim]{\tau_I \otimes \mathrm{id}} \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^1 \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \simeq \Sigma^\infty(I^{\mathrm{op}} \oslash I).$$

is induced by the loop $r \circ s \in \Omega(I^{\mathrm{op}} \oslash I)$, where $r, s$ are the maps depicted as follows (we identify $I^{\mathrm{op}}$ with the interval $0 \to 1$ with the vertex 1 marked):

$$\begin{tikzcd} 00 \arrow[r, equal] \arrow[d, "s"] & 10 \arrow[d, equal] \arrow[dl, phantom, "\swarrow"] \\ 01 \arrow[r, "r"] & 11. \end{tikzcd}$$

Using the first coherence data of the half-central structure of $\vec{S}^1$, one can check that the composition

$$\mathbb{F} \otimes \vec{S}^1 \xrightarrow{\eta \otimes \vec{S}^1} \Sigma^\infty I \otimes \vec{S}^{-1} \otimes I \otimes \vec{S}^1 \xrightarrow{\mathrm{id} \otimes \tau_I^{-1}} \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \vec{S}^1 \otimes \Sigma^\infty I^{\mathrm{op}} \simeq \Sigma^\infty(I \otimes I^{\mathrm{op}}).$$

is induced by the loop $r' \circ s' \in \Omega(I \oslash I^{\mathrm{op}})$, where $r', s'$ are the maps depicted as

$$\begin{tikzcd} 00 \arrow[r, "s'"] \arrow[d, equal] & 10 \arrow[d, "r'"] \arrow[dl, phantom, "\swarrow"] \\ 01 \arrow[r, equal] & 11. \end{tikzcd}$$

checking the triangle identities are straightforward but tedious... ∎

*Remark* 4.2.7. Suppose $L$ is the left dual of $R$ in $\mathsf{CatSp}$ (so that $L \otimes (-) \dashv [L, -] \simeq R \otimes (-)$). By postcomposing the adjoint inverses [1] and [−1], one sees $L^\circ[\pm 1] \dashv R[\mp 1]$ and $L[\pm 1] \dashv R^\circ[\mp 1]$. In particular, we have the four-periodic cycle

$$I \dashv I^{\mathrm{op}}[-1] \dashv I^{\mathrm{op}} \dashv I[-1] \dashv I.$$

### 4.2.1 fiber vs cofiber

Let $W$ be a functor $\square^1 \to \mathsf{CatSp}$ depicted as $\mathbb{F} \to \Sigma^\infty I$. Note that $W$ is the weight for lax fiber and cofiber: $\overrightarrow{\mathrm{cof}}(X \to Y) = \mathrm{colim}^W(X \to Y)$, $\overrightarrow{\mathrm{fib}}(X \to Y) = \lim^W(X \to Y)$.

**Theorem 4.2.8.** *There is an adjunction*

$$\begin{tikzcd} \mathsf{CatSp} \arrow[r, shift left, "{W[-1]\otimes(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\square^1, \mathsf{CatSp}) \arrow[l, shift left, "\mathrm{colim}^W"] \end{tikzcd} .$$

*In particular, there is an equivalence* $\mathrm{colim}^W \simeq \lim$

*Proof.* Note that both functors respect the right $\mathsf{CatSp}$-module structures, so it suffices to construct the unit and counit and check the triangle identities for the generators (i.e., corepresentable functors $\otimes\mathbb{F}$).

(1) We first compute $\mathrm{colim}^W(W[-1])$:

$$\begin{aligned} \mathrm{colim}(\Sigma^\infty I \otimes \mathbb{F}[-1] \leftarrow \mathbb{F} \otimes \mathbb{F}[-1] \to \mathbb{F} \otimes \Sigma^\infty I[-1]) &\simeq \mathrm{colim}\, \Sigma^{\infty-1}(I^{\mathrm{op}} \leftarrow S^0 \to I) \\ &\simeq (I^{\mathrm{op}} \sqcup_{S^0} I)[-1]. \end{aligned}$$

So, to define the unit $\eta_{\mathbb{F}} : \mathbb{F} \to \mathrm{colim}^W(W[-1])$, it suffices to define $s : \vec{S}^1 \to I^{\mathrm{op}} \sqcup_{S^0} I$. Note that the codomain is a free category on the graph (where $*$, $\bullet$ denotes the basepoint and a non-basepoint object, respectively)

$$\begin{tikzcd} * \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} .$$

We define $s$ by picking up the loop of the graph; more precisely, it is the map classified by the element $1 \in \Omega(I^{\mathrm{op}} \sqcup_{S^0} I) \simeq \mathbb{N}$. Then general $\eta_Z : Z \to \mathrm{colim}^W(Z[-1] \to \Sigma^{\infty-1} I \otimes Z)$ for $Z \in \mathsf{CatSp}$ is $\eta_{\mathbb{F}} \otimes Z$.

(2) Next let $f : X \to Y \in \mathsf{CatSp}$. For the counit, we must define a commutative square

$$\varepsilon_f : \quad \begin{tikzcd} \mathbb{F}[-1] \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ I[-1] \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r] & Y \end{tikzcd}$$

naturally in $f$. Consider the following diagram:

$$\begin{tikzcd} & X \arrow[rr] \arrow[dl] \arrow[dd] & & Y \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\ Y \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\ & 0 \arrow[rr] \arrow[dl] & & \overrightarrow{\mathrm{cof}}\, f \arrow[rr] \arrow[dl] & & X[1] \arrow[dl] \\ 0 \arrow[rr] & & \overrightarrow{\mathrm{cof}}(\mathrm{id}_Y) \arrow[rr] & & Y[1] & \end{tikzcd}$$

Both in the back and front faces, the left is a lax cofiber square and the right is a cofiber square, which pastes to the large lax pushout (loop-suspension) rectangle. Now note that the map $\overrightarrow{\mathrm{cof}}(\mathrm{id}_Y) = I \otimes Y \to Y[1]$ factors into $I \otimes Y \xrightarrow{\sim} [I, Y[1]] \to Y[1]$, where the first map is induced by the counit map of the duality $I[-1] \dashv I$, i.e., $I \oslash I \to \vec{S}^1$ mapping the two nondegenerate 1-cells to the generating loop. We define the counit $\varepsilon_f$ as the transpose of the square

$$\begin{tikzcd} \overrightarrow{\mathrm{cof}}\, f \arrow[r] \arrow[d] & X[1] \arrow[d, "{f[1]}"] \\ {[I, Y[1]]} \arrow[r] & Y[1] \end{tikzcd}$$

(3) We must check the triangle identities for the $\mathsf{CatSp}$-right module generators, i.e., $\mathbb{F} \in \mathsf{CatSp}$ and $(0 \to \mathbb{F}), (\mathbb{F} \xrightarrow{=} \mathbb{F}) \in \mathsf{Fun}(\square^1, \mathsf{CatSp})$. This is straightforward but somewhat confusing, so we spell out the detail:

(i) The first triangle identity is (after a suspension)

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "s"] \arrow[d] & \mathbb{F} \otimes (I^{\mathrm{op}} \sqcup_{S^0} I)[-1] \arrow[r, "r"] \arrow[d] & \mathbb{F} \arrow[d] \\
I \arrow[r, "I \otimes s"] & I \otimes (I^{\mathrm{op}} \sqcup_{S^0} I)[-1] \arrow[r] & I
\end{tikzcd}$$

After another suspension, the top row is unstably $\vec{S}^1 \to (I^{\mathrm{op}} \sqcup_{S^0} I) \to \vec{S}^1$, where the latter map is the quotient by $I$, so the composite is the identity. Similarly, the bottom row is $I \to I \otimes (I^{\mathrm{op}} \sqcup_{S^0} I)[-1] \to I \otimes (I^{\mathrm{op}} \otimes I)[-1] \simeq I \otimes I \otimes I[-1] \to \vec{S}^1 \otimes I[-1] = I$, where the second map is depicted as

$$I \otimes \left( \begin{tikzcd} & * \\ * \arrow[r, "I"] & \bullet \arrow[u, "I^{\mathrm{op}}"'] \end{tikzcd} \hookrightarrow \begin{tikzcd} * \arrow[r, equal] & * \\ * \arrow[u, equal] \arrow[r, "I"] \arrow[ur, phantom, "\nearrow"] & \bullet \arrow[u, "I^{\mathrm{op}}"'] \end{tikzcd} \right) [-1].$$

Tensoring $\vec{S}^1$ from the right, it reduces to checking that the composite $I \owedge \vec{S}^1 \to I \owedge I \owedge I^{\mathrm{op}} \to \vec{S}^1 \owedge I^{\mathrm{op}} \simeq I \owedge \vec{S}^1$ is the identity. Presenting as the quotients of the cubes, the maps are

$$\left( \begin{tikzcd} 00 \arrow[r, equal] & 01 \\ 10 \arrow[u, equal] \arrow[r, "l"] \arrow[ur, phantom, "\alpha \nearrow"] & 11 \arrow[u, equal] \end{tikzcd} \right) \to \left( \begin{tikzcd} 000 \arrow[r, equal] \arrow[d, equal] & 010 \arrow[r, equal] \arrow[d, "e_1"] & 011 \arrow[d, equal] \\ 100 \arrow[r, "e_2"] \arrow[ur, phantom, "\beta \nearrow"] & 110 \arrow[r, "e_3"] \arrow[ur, phantom, "\gamma \nearrow"] \arrow[d, phantom, "\Downarrow"] & 111 \\ 100 \arrow[r, "e_2"] \arrow[dr, equal] & 110 \arrow[r, "e_3"] \arrow[d, phantom, "\Downarrow\delta"] & 111 \arrow[dl, equal] \\ & 101 & \end{tikzcd} \right) \to \left( \begin{tikzcd} 00 \arrow[r, equal] \arrow[d, "l'"] & 01 \arrow[d, equal] \\ 10 \arrow[r, equal] \arrow[ur, phantom, "\nearrow \alpha'"] & 11 \end{tikzcd} \right)$$

where $l \mapsto e_3 \circ e_2$, $e_1 \mapsto l'$, $e_2 \mapsto l'$, $e_3 \mapsto \mathrm{id}_*$, $\alpha \mapsto \gamma * \beta$, $\beta \mapsto \mathrm{id}_{l'}$, $\gamma \mapsto \alpha'$, $\delta \mapsto \alpha'$, so the composite is the identity.

(ii) The other triangle identity (after a suspension) is that for $f : X \to Y$, the bottom composition of the following diagram is an identity:

$$\begin{tikzcd}
& \mathbb{F} \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\varepsilon_f){[1]}"] & X[1] \arrow[d, "f{[1]}"] \\
& I \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r] \arrow[d, "\overleftarrow{\mathrm{cof}}"] & Y[1] \arrow[d, "\overleftarrow{\mathrm{cof}}"] \\
\vec{S}^1 \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r, "s \otimes \mathrm{id}"] & (I^{\mathrm{op}} \sqcup_{S^0} I) \otimes \overrightarrow{\mathrm{cof}}(f) \arrow[r] & \overrightarrow{\mathrm{cof}}(f)[1],
\end{tikzcd}$$

where the vertical arrows are oplax cofiber sequence (note that the suspension switches lax and oplax squares).

- For $f = f_0 : 0 \to \mathbb{F}$, we have $\overrightarrow{\mathrm{cof}}(f) = \mathbb{F}$ and the composite is $\vec{S}^1 \xrightarrow{s} I^{\mathrm{op}} \sqcup_{S^0} I \to \vec{S}^1$, where the latter map is the quotient by $I^{\mathrm{op}}$, so the composite is the identity.

- For $f = f_1 : \mathbb{F} \xrightarrow{=} \mathbb{F}$, we have $\overrightarrow{\mathrm{cof}}(f) = I$ and the middle-horizontal map is $I \oslash I \to \vec{S}^1$, the one used in the definition of the counit in (2). The composite is

$$\left(\begin{tikzcd} 00 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha \nearrow"] & 01 \arrow[d, "l"] \\ 10 \arrow[r, equal] & 11 \end{tikzcd}\right) \to \left(\begin{tikzcd} * \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\beta \nearrow"] & * \arrow[d, "e_1"] \\ * \arrow[r, "e_2"] \arrow[d, equal] \arrow[dr, phantom, "\gamma \nearrow"] & \bullet \arrow[d, "e_1^{\mathrm{op}}"] \\ * \arrow[r, equal] & * \end{tikzcd}\right) \to \left(\begin{tikzcd} 00 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha \nearrow"] & 01 \arrow[d, "l"] \\ 10 \arrow[r, equal] & 11 \end{tikzcd}\right)$$

, where $\alpha \mapsto \beta * \gamma$, $l \mapsto e_1^{\mathrm{op}} \circ e_1$, $e_1 \mapsto l$, $e_2 \mapsto l$, $\beta \mapsto \mathrm{id}_l$, $e_1^{\mathrm{op}} \mapsto \mathrm{id}_*$, $\gamma \mapsto \alpha$, so the composite is the identity.

Since any map $f : X \to Y$ is a pushout of $f_0 \otimes Y \leftarrow f_0 \otimes X \to f_1 \otimes X$, the triangle identity is proven.

□

*Remark* 4.2.9. The counit square in (2) is equivalent to a lax square

$$\begin{tikzcd} \overrightarrow{\mathrm{cof}}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & X[1] \arrow[d, "f[1]"] \\ 0 \arrow[r] & Y[1]. \end{tikzcd}$$

The theorem claims that it is a comma square, i.e., the induced comparison map $\overrightarrow{\mathrm{cof}}(f) \to \overrightarrow{\mathrm{fib}}(f[1])$ is an equivalence.

### 4.2.2 duality of building blocks

$\square^1_+$

# Chapter 5

# Application to TQFT

Todo:

(1) categorical spectra with adjoints

(2) cobordism hypothesis with singularities

(3) cobordism hypothesis, adding 0-handles?

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