Henselian pair §6 Comparisons of K & TC

§6.1 Étale K-thry is TC (mod p)

- Geisser - Hesselholt TC of schemes

  R : smooth / k perfect field of char p

  $\Rightarrow$ the p-adic Étale K-theory of R = TC(R)

We extend this :

Thm 6.1 (R, m, k) sHen local char $k = p > 0$

$$\Rightarrow K^{inv}(R)/p = 0$$

Proof · by DGM we can assume R is discrete (?)

· by the main thm

$$K^{inv}(R)/_p \xrightarrow{\sim} K^{inv}(k)/_p$$

$\rightsquigarrow$ assume $R = k$: sep. cl. field of char p

k : ind-smooth $\mathbb{F}_p$-algebra

(k / $\mathbb{F}_p[X]$

$\mathbb{F}_p$ = perfect
take trans. basis

Thm 4.28 R : ind-smooth $\mathbb{F}_p$-alg, $n \geq 0$

functorial
(1) $\exists\ \pi_n(K(R)/p) \to \nu^n(R)$, isom if R : local

(2) $\exists\ 0 \to \tilde{\nu}^{n+1}(R) \to \pi_n(TC(R)/_p) \to \nu^n(R) \to 0$

$$\begin{array}{ccc} & & \\ \uparrow \text{DGM} & & \nearrow (1) \\ \pi_n(K(R)/p) & & \end{array}$$

(3) R : local $\Longrightarrow$ $\pi_n(K^{inv}(R)/p) \simeq \tilde{\nu}^{n+2}(R)$ functorially

$\rightsquigarrow$ enough to show $\tilde{\nu}^n(k) = 0$ $(n \geq 2)$

$(K^{inv}(k)/p$: connective ??)

$\Longleftarrow$

$1 - C^{-1} : \Omega^n_k \longrightarrow \Omega^n_k / d\Omega^n_k$ is surjective

$\cup$

$\omega = a\, dx_1 \cdots dx_n \in \Omega^n_k$

$\underline{\underline{u \in k}}$

$$(1 - C^{-1})(u\omega) = u\omega - (ua)^p x_1^{p-1} \cdots x_n^{p-1} dx \cdots dx_n$$

$$= (u - u^p(ax_1 \cdots x_n)^{p-1})\, \omega$$

$1 = u - A \cdot u^p$ $\quad\exists u$: solution as k : sep closed.

$\uparrow$ separable polynomial (diff $= 1$)

$\boxed{//}$

Def 4.27

$$\Omega^n_k \xrightarrow{1-C^{-1}} \Omega^n_k / d\Omega^{n-1}_k \longrightarrow \tilde{\nu}^n(k)$$

Need to show $1-C^{-1}$ is surjective

Def 2.23 The inverse Cartier operator $(R/\mathbb{F}_p)$

$$C^{-1} : \Omega^n_R \longrightarrow H^n(\Omega^*_R) \subset \Omega^n_R / d\Omega^{n-1}_R$$

multiplicative, $\begin{cases} C^{-1}(a) = a^p \\ C^{-1}(db) = b^{p-1}\, db \end{cases}$

$d(xy) \mapsto x^{p-1}y^{p-1}\, d(xy)$
$= (x^{p-1}dx)\cdot y^p$
$+ x^p\cdot(y^{p-1}dy)$

$x\,dy + y\,dx \mapsto$

$$\begin{tikzcd}
R & & \\
& R^\varphi \arrow[ul, "f"'] & R \arrow[l] \arrow[ull, bend right, "\mathrm{Fab}"'] \\
& \mathbb{F}_p \arrow[u] \arrow[uul] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{F}_p \arrow[l, "\simeq", "\mathrm{Frob}"'] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
x^p \arrow[d, phantom, "\in" description] & & \\
\mathbb{F}_p[x] & x \arrow[ul, mapsto] & \\
& \mathbb{F}_p[x] \arrow[ul] \arrow[u, phantom, "\ni" description] & \mathbb{F}_p[x] \arrow[l] \\
& \sum a_i^p x^i \arrow[u, phantom, "\in" description] & \sum a_i x^i \arrow[l, mapsto] \arrow[u, phantom, "\in" description]
\end{tikzcd}$$

$$\begin{tikzcd}
R^\varphi \arrow[r, "f"] \arrow[d, "d"'] & R \arrow[d, "d"] \\
\Omega_{R^\varphi} \arrow[r, "f / p"'] & \Omega_R
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & x^p \arrow[d, mapsto] \\
dx \arrow[r, mapsto, "0"'] & p\cdot x^{p-1}dx
\end{tikzcd}$$

$dx \longmapsto x^{p-1}dx$

Leibniz rule ✓

$$d(x+y) \mapsto (x+y)^{p-1}(dx+dy)$$
$$-\big)\ dx+dy \mapsto x^{p-1}dx + y^{p-1}dy$$

$$\left.\begin{aligned} &\sum_{k=1}^{p-2} \binom{p-1}{k-1} x^{k-1}y^{p-k}\ dx \\ &+ \sum_{k=1}^{p-1} \binom{p-1}{k} x^k\cdot y^{p-k-1}\ dy \end{aligned}\right] = \sum \frac{1}{p}\binom{p}{k}\big(d(x^k y^{p-k})\big)$$

$$d\left(x^k y^{p-k}\right) = k\,(dx)\,x^{k-1}\cdot y^{p-k} + (p-k)\,x^k\cdot(dy)\,y^{p-k-1}$$

Alternative argument (rather than using $\tilde{D}^n$ description)

Suslin's argument

$\rightsquigarrow K^{inv}(-)/p$ invariant under extension of sep cl. fields

$\rightsquigarrow$ reduced to $\overline{\mathbb{F}}_p$

$\Downarrow$

direct calc.

<u>Prop 6.2</u> $F : \mathrm{Ring} \longrightarrow \mathrm{Ab}$

assume: (1) F comm. with filt colim

henselization $\rightsquigarrow$ étale nbhd

(2) rigidity ($F(R) \simeq F(R/I)$ for a henselian pair)

geometric pt $\Downarrow$ (strict) henselization

$\Rightarrow L/k$, $L = L^{sep}$, $k = k^{sep}$

$\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k$

$\Rightarrow F(k) \simeq F(L)$

<u>Proof</u> <u>Claim</u> $q : X \longrightarrow \mathrm{Spec}\, k$ connected smooth affine k-sch of finite type

$\forall$ sections x, x_0

$x^* = x_0^* : F(X) \xrightarrow{\ } F(k)$ (section q^*)

[056U] $\{x \in X : \text{closed}, k = k(x)\} \subset X$ is dense

sep cl. $\quad$ k-pt is dense

x^* the same on

$\Uparrow$

<u>proof</u> <u>Fix x_0</u> $: \mathrm{Spec}\, k \longrightarrow X$ $\quad$ enough to show $\exists U$: Zariski nbhd of x_0

s.t. $\forall x : \mathrm{Spec}\, k \longrightarrow U$,

$\forall \alpha \in F(X)$, $(x^* - x_0^*)(\alpha) = 0$.

$(x^* - x_0)(\alpha) = x^*(\alpha - q^* x_0 \alpha)$

$\rightsquigarrow$ replacing α by $\alpha - q^* x_0 \alpha$.

$\Downarrow$

connectedness + Zariski density of k-pts of X

[04QM stacks]

We may assume $x_0^* \alpha = 0$ (want to show $x^* \alpha = 0$)

By the assumption of rigidity

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "x_0"] \arrow[dr] & X \\ & (X_{x_0})^h \arrow[u, "\text{henselization}"'] \\ \exists\, Y \arrow[u] \arrow[uur] \arrow[r, "\text{image}"] & U \arrow[uu] \end{tikzcd}$$

étale nbd of x_0

$$\begin{tikzcd} Y \arrow[d] & \\ \mathrm{Spec}\, k \arrow[ur, dashed, "\tilde{x}"] \arrow[r, "x"] & U \arrow[r] & X \end{tikzcd}$$

by $k = k^{sep}$

$0 \leftarrow\!\shortmid\ \alpha$ $\quad$ U

$$\begin{tikzcd} 0 & \alpha \arrow[l, mapsto] \\ Fk & FX \arrow[l] \arrow[dl] \\ F(X_0^h) \arrow[u, "\text{rigidity}\ \cong"] & \\ \mathrm{colim}\, F(S) \arrow[u, "\wr"] & (S_q, q) \end{tikzcd}$$

$\exists\, F(S)$ $\quad$ S: étale R-alg $\qquad$ $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ (étale $\Rightarrow$ open)

Stacks 04GV $\quad$ R: ring. $\quad \mathfrak{p} \in \mathrm{Spec}\, R$.

Consider the category of pairs $(S, \mathfrak{q})$ ← filtered

$$\begin{array}{ccc} R & \to & S \\ \circlearrowleft & & \circlearrowleft \\ \mathfrak{p} & \leftarrow\!\shortmid & \mathfrak{q} \end{array}$$

étale

s.t. $\kappa(\mathfrak{p}) \cong \kappa(\mathfrak{q})$

$$\rightsquigarrow (R_{\mathfrak{p}})^h = \mathrm{colim}_{(S,\mathfrak{q})} S = \mathrm{colim}_{(S,\mathfrak{q})} S_{\mathfrak{q}}$$

// Claim

<u>proof of prop</u> may assume $k = k^{sep}$ $\quad k = \mathbb{F}_p$ or $\mathbb{Q}$

(k : perfect $\Longleftrightarrow$ k^{sep} : alg. closed)

k : perfect $\Rightarrow$ $k \to L$ ind-smooth

$$L = \operatorname{colim}_{\text{filt.}} A_\alpha$$

<u>injectivity</u>

$$\begin{tikzcd} k \arrow[r, "i"] \arrow[loop left, "id"] & A_\alpha \arrow[l, bend left, "r"] \end{tikzcd} \quad \text{smooth}$$

$\exists$ retraction because k = sep cl. (again by the Zariski density of k-pts?)

$$\begin{tikzcd} F(k) \arrow[r, "i"] \arrow[dr, "\text{injective}"'] & F(A_\alpha) \arrow[d] \\ & \operatorname{colim} F(A_\alpha) \simeq F(L) \end{tikzcd}$$

$\rightsquigarrow F(k) \xrightarrow{i} F(A_\alpha)$ injective

✓

<u>Surjectivity</u> may assume $\operatorname{Spec} A_\alpha$: connected.

$$\begin{tikzcd} \operatorname{Spec} A_\alpha \otimes_k L \arrow[r] \arrow[d] & \operatorname{Spec} A_\alpha \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

connected by k = alg closed

$$\begin{tikzcd} F(A_\alpha \otimes_k L) \arrow[d, "\text{indep of } A_\alpha \otimes_k L"] \\ F(L) \end{tikzcd} \qquad \begin{tikzcd} & A_\alpha \otimes_k L \arrow[d] \\ A_\alpha \arrow[r] \arrow[ur] & L \end{tikzcd}$$

$$\begin{tikzcd} F(k) \arrow[r, "i"] & F(A_\alpha) \arrow[r, "r"] \arrow[d] & F(k) \\ & F(L) & \end{tikzcd}$$

$\ni u \mapsto \tilde{u}$

$\exists \tilde{u}$

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[dr] & L \\ & k \arrow[u] \end{tikzcd}$$

(both k-linear)

✓

Rephrasing 6.1 in terms of homotopy group of sheaves

X : scheme

$\pi_n(\mathcal{K}/p) :=$ étale sheafification of $\pi_n(K(-)/p)$

$\pi_n(\mathcal{TC}/p) :=$ ———— $\pi_n(TC(-)/p)$

$$\begin{tikzcd} \mathrm{Shv}^{ét}(X) \arrow[r, shift right] & \mathrm{PShv}(X) \arrow[l, shift right] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

$$\pi_n(\mathcal{K}/p) \longleftarrow\!\shortmid \pi_n(K(-)/p)$$

$$\pi_n(\mathcal{TC}/p) \longleftarrow\!\shortmid \pi_n(TC(-)/p)$$

$$\begin{tikzcd} X_{\mathbb{F}_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \mathbb{F}_p \arrow[r] & \mathbb{Z} \end{tikzcd} \qquad i : (X_{\mathbb{F}_p})_{ét} \longrightarrow X_{ét}$$

closed inclusion

what does it mean?

$\rightsquigarrow$ Thm 6.1 ($+$ $TC(R)/p = 0$ if $p \in R^\times$)

$$\Longleftrightarrow \quad \pi_n(\mathcal{TC}/p) \xrightarrow{\simeq} i_* i^* \pi_n(\mathcal{K}/p)$$

$$i^*\left(\pi_n(\mathcal{TC}/p) \longrightarrow \pi_n(\mathcal{K}/p)\right)$$

Thm X: proper / $\mathrm{Spec}\, R$, $(R, (p))$ henselian

$$\pi_n(\mathcal{TC}/p)(X) \longrightarrow \pi_n(\mathcal{K}/p)(X)$$

$$\begin{tikzcd} \pi_n \mathcal{TC}/p \arrow[r, "\sim"] \arrow[d] & i_* i^* \pi_n(\mathcal{K}/p) \\ i_* i^* \pi_n(\mathcal{TC}/p) \arrow[ur] & \end{tikzcd}$$

$$\pi_n(TC/p) \simeq i_* i^* \pi_n(K/p)$$

check stalkwise $\quad X_{\mathbb{F}_p, \text{ét}} \overset{i}{\hookrightarrow} X_{\text{ét}}$

$$\begin{array}{ccc} \mathrm{Shv}_{Ab}(X_{\mathbb{F}_p,\text{ét}}) & \underset{i_*}{\overset{i^*}{\rightleftarrows}} & \mathrm{Shv}_{Ab}(X_{\text{ét}}) \\ & & \downarrow x^* \\ & & Ab \end{array}$$

<u>x: geometric point</u>

if $\begin{cases} x \notin X_{\mathbb{F}_p}, \text{ then } & x^* i_* = 0 \quad \text{check} \\ x \in X_{\mathbb{F}_p} \text{ then } & x^* \pi_n(TC/p) \quad\quad x^* i_* i^* \pi_n(K/p) \end{cases}$

$x^* i_* i^*$

$$\begin{array}{ccc} x^* \pi_n(TC/p) & & x^* i_* i^* \pi_n(K/p) \\ \| & \nwarrow & \uparrow \simeq \\ \lim \pi_n(TC(U)/p) & & x^* \pi_n(K/p) \\ & & \| \\ \text{on strictly henselian?} & & \lim_{\text{étale}} \pi_n K(U)/p \end{array}$$

is it true more generally that $Z \overset{i}{\hookrightarrow} X$ closed ?

$$\begin{cases} x^* i_* = 0 & \text{if } x \notin Z \\ x^* i_* i^* \xleftarrow{\sim} x^* & \text{if } x \in Z \end{cases}$$

?

§6.2 Asymptomatic Comparison of K & TC (mod p)

<u>Thm</u> R: comm. ring, p: prime

(1) $(R, (p))$ henselian

(2) R/p has finite Krull dimension

$$d := \max\left\{1, \left\lceil \sup_{x \in \operatorname{Spec}(R/p)} \log_p [k(x) : k(x)^p] \right\rceil\right\}$$

Then $K(R)/p^r \longrightarrow TC(R)/p^r$ is $\pi_{\geq d}$-equivalence $\forall r$

<u>Proof</u>

$$\begin{array}{ccc} K(R)/p^r & \longrightarrow & K(R/(p))/p^r \\ \downarrow & \lrcorner & \downarrow \\ TC(R)/p^r & \longrightarrow & TC(R/(p))/p^r \end{array}$$

$\rightsquigarrow$ We may assume R: $\mathbb{F}_p$-algebra

$$K(R)/p^r \xrightarrow[\text{cover}]{\text{conn}} \mathbb{K}(R)/p^r \longrightarrow TC(R)$$

$\deg \geq 1$ equiv

$\downarrow$

cofiber $/p^r$: (-1)-truncated] 0-truncated

replace K by $\mathbb{K}$

$$\mathbb{S} \xrightarrow{p^r} \mathbb{S} \longrightarrow \mathbb{S}/p^r$$

- Thomason–Trobaugh: Nisnevich descent for $\mathbb{K}$ $(/p^r)$
- Blumberg–Mandell: Nisnevich descent for TC $(/p^r)$

$\dim R < \infty \Rightarrow (\operatorname{Spec} R)_{\mathrm{Nis}}$ has locally of homotopy dim $< \infty$ ($\Rightarrow$ Postnikov complete)

$\Rightarrow \operatorname{Shv}^{\mathrm{Nis}}_{\mathrm{Sp}}(\operatorname{Spec} R)$ is left complete

$\mathcal{F}$ "

$\rightsquigarrow$ descent ss for $\mathcal{F}$

$$H^p(X_{Nis}; \pi_q^{Nis}\mathcal{F}) \Rightarrow \pi_{q-p}\mathcal{F}^{Nis}(X)$$

Nisnevich descent for a weakly localizing invariant

Def $R \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \in \mathrm{Mod}_R(\mathrm{Cat}_\infty^{\mathrm{perf}})$
fully faithful on the underlying ∞-cats

$\rightsquigarrow$ The Verdier quotient

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, hook] \arrow[d] & \mathcal{A}_2 \arrow[d] \\ \{0\} \arrow[r] & \mathcal{A}_2/\mathcal{A}_1 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

A weakly localizing invariant of R-linear ∞-cats with values in $\mathcal{D}$ ($\in \mathrm{Pr}_{st}$) is a functor

$$F : \mathrm{Mod}_R(\mathrm{Cat}_\infty^{\mathrm{perf}}) \longrightarrow \mathcal{D}$$

Verdier quot. $\mapsto$ cofib seq.

X : qcqs spectral algebraic space

$\rightsquigarrow \mathrm{QCoh}(X) \in \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, st})$
SAG 9.6.1.1

$\mathrm{QCoh}(X)^\omega = \mathrm{Perf}(X)$ $(= \{\text{dualizable objects}\})$

$\rightsquigarrow \mathrm{Perf}(X) \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$

$f : Y \to X \quad \rightsquigarrow f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ (sym mon)

$\rightsquigarrow f^* : \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(Y)$

Def $X_{\text{ét}} = \{ U \longrightarrow X \text{ étale}, U : \text{qc} \}$ (étale site)
+ the étale topology

Thm Suppose $F : \mathrm{Mod}_{\mathrm{Perf}(X)}(\mathrm{Cat}_\infty^{\mathrm{perf}}) \longrightarrow \mathcal{D}$ is weakly localizing

$T(n)$-local for some implicit p, n

Then
$$\begin{aligned} X_{\text{ét}} &\longrightarrow \mathcal{D} \\ (U \to X) &\longmapsto F(\mathrm{Perf}(U)) \end{aligned}$$
is a sheaf

Nisnevich descent $\Longleftrightarrow$ Nisnevich excision