RESEARCH STATEMENT:
ALGEBRA AND GEOMETRY OF CATEGORICAL SPECTRA

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1. Overview

I study higher categorical algebra, i.e., algebraic structure in higher categories[1]. The goal is to build a theory that is robust enough to carry out a sort of algebraic geometry. Specifically, I study *categorical spectra*. It is a novel higher categorical generalization of the notion of spectra in homotopy theory[2]. Primary use of this notion is in functorial field theories; as such, it sees the rich interplay between higher algebra and mathematal physics. Aside from it, I expect it to see deeper structures in noncomutative algebraic geometry: current noncommutative geometry seems to be closed at 1-categorical level, but this might be an artifact of avoiding the laxness, new subtleties in higher categorical algebras. It is worth mentioning as a potential sourse of interacting field that my original inspiration comes from papers by Connes and Consani [CC20] in search of absolute algebra, even though the current direction has diverted from their work.

The following table of analogy is to give an idea about the context where categorical spectra fits in:

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\mathsf{Set} = \mathsf{Cat}_{(0,0)}$	spaces/groupoids $\mathsf{S} = \mathsf{Cat}_{(\infty,0)}$	∞-categories $\infty\mathsf{Cat} = \mathsf{Cat}_{(\infty,\infty)}$
—	homotopy n-type	n-category
$(1,1)$-categories $\mathsf{Cat}_{(1,1)}$	categories $\mathsf{Cat} = \mathsf{Cat}_{(\infty,1)}$	∞-categories $\infty\mathsf{Cat}$
Cartesian product $\times$	Cartesian product $\times$	lax Gray tensor product $\otimes$
abelian groups Ab	spectra Sp	categorical spectra CatSp
	grouplike $\mathbb{E}_\infty$-space $\simeq \mathsf{Sp}^{\mathrm{cn}}$	symmetric monoidal ∞-categories
abelian categories	(pre)stable categories	(stable $\infty\mathsf{Cat}^{\otimes}$-bimodules?)
—	loop $\Omega(X,x) = \mathrm{Aut}_X(x)$	$\Omega(X,x) = \mathsf{End}_X(x)$
—	suspension $\Sigma = (-) \wedge B\mathbb{Z}$	$\Sigma = \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1} = (-) \oslash B\mathbb{N}$
free functor $\mathsf{Set} \to \mathsf{Ab}$	suspension spectra $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$	$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$	finite set spectrum $\mathbb{F}$
tensor product $\otimes_\mathbb{Z}$	tensor (smash) product $\otimes_\mathbb{S}$	tensor product $\otimes_\mathbb{F}$

The theory of spectra, themselves being the natural homotopical generalization of abelian groups, has seen huge success: Waldhausen's *brave new algebra* philosophy was to take spectra seriously as a building block of algebra. Much of the current development in homotopy theory and its neighborhood, including spectral algebraic geometry and algebraic K-theory, crucially relies on it. I propose to take also categorical spectra seriously as a fundamental algebraic notion. New feature, which is also a big challegne, in the categorical context is the *noninvertibilty* of cells and the resulting *asymmetry*. Higher algebra comes with hierarchy (the third row of the table). In homotopical theory, operations that let objects leak out of their homotopical hierarchy was the *colimits*. Now the categorical hierarchies are closed under limits and colimits, and *lax colimits* plays the similar role to colimits in homotopy theory. The noninvertibility of higher cells in the constructed object causes the asymmetry and makes them

[1] We take everything to be homotopical by default: n-category means (∞,n)-category, including the case $n = \infty$. "higher-categorical" means (∞,n)-categorical for $n > 1$, whereas "homotopical" replaces the more usual usage of "higher-categorical," and "higher" can mean the both. Besides this point, we mostly follow the notations of [Lur09a][Lur17]

[2] It was introduced independently by at least a few groups of people [Hor18][Ste21][Reu23][Joh23b]. At first I called them ∞-spectra, but I adopted more descriptive *categorical spectra* from Stefanich's thesis, which spends a chapter on formal foundation of categorical spectra and utilizes it to package interesting functoriality of higher quasicoherent sheaves. Johnson-Freyd, Reutter and Scheimbauer call them *towers*.

harder to control. For this reason, lax constructions are dreaded and often shunned; to say the least, it makes it a challenging task to construct a roboust theory, even though it seems unavoidable if we try to retain noninvertible information. However, the algebra in this world is not a barren wasteland. In fact, there are emerging evidence that interesting algebra can be done with categorical spectra. For instance, Johnson-Freyd and Reutter has recently developed a version of Galois theory and announced in [Joh23a] that the Galois-closedness (in characteristic 0) characterizes the "universal physical targets of TQFT," capturing non-classical algebraic extensions starting with the category of super vector spaces, along the lines of Freed and Hopkins' paper [FH21] on reflection positivity.

If we hope to *do algebra* with categorical spectra, construction of tensor product is a fundamental problem. Based on the lax Gray tensor product on $\infty\mathsf{Cat}$, I proved the following:

Theorem 1.1. *There exists a unique (noncommutative) tensor product on* CatSp *promoting* $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ *to a monoidal functor with respect to the lax Gray tensor product. The unit for the tensor product* $\mathbb{F} = \mathrm{B}^\infty\mathsf{Fin}^\simeq$ *is the delooping of symmetric monoidal groupoid of finite sets. It acts additively on the categorical level, which takes* $\mathbb{Z}\cup\{\pm\infty\}$*-values. Moreover, the tensor product localizes to the full subcategory of categorical spectra with duals, i.e., those with the stable cells admitting adjoints in all reasonable dimensions*[3].

Recall the statement of cobordism hypothesis: for a symmetric monoidal n-category X with duals, there is a canonical equivalence

$$\mathrm{ev}_* : \mathsf{Fun}^\otimes(n\mathsf{Bord}^{\mathrm{fr}}, X) \xrightarrow{\sim} X^\simeq.$$

Colloquially, the framed cobordism category is the free symmetric monoidal n-category on a single fully dualizable object. Note that the natural transformations of TQFTs only recover the underlying space of X. An important corollary to the theorem is that we can formally refine the cobordism hypothesis:

Corollary 1.2. *In the same setting as cobordism hypothesis, if we use the internal hom of categorical spectra, we have*

$$[n\mathsf{Bord}^{\mathrm{fr}}, X] \xrightarrow{\sim} X.$$

This should be regarded as a typical example where "lax" consideration recovers noninvertible information, and as a good indication that the defined tensor product is something useful. It allows, for instance, to lift $O(n)$- (or $\mathrm{PL}(n)$-)action on the underlying groupoid of X to the level of category.

2. Categorical spectra

First, we go straight to the definition of categorical spectra. Let $\infty\mathsf{Cat}$ be the category of ∞- (aka. (∞,∞)-) categories[4]. A *pointed* ∞*-category* (X, x) is an ∞-category X with a distinguished object (basepoint) x. We often surpress the basepoint from notation.

Definition 2.1 ([Mas21][Ste21])**.** The *loop* functor $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ sends a pointed ∞-category (X, x) to $\Omega(X, x) := (\mathsf{End}_X(x), \mathrm{id}_x)$, the (monoidal) ∞-category of endomorphisms. It admits a left adjoint called the *suspension* Σ. A *categorical spectrum* is a sequence $X = (X_n)_{n\in\mathbb{N}}$ of pointed ∞-categories with equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. More precisely, the category of categorical spectra is the limit

$$\mathsf{CatSp} := \lim(\cdots \to \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*) \quad \in \mathsf{Pr}^\mathsf{R}.$$

We denote the functor $X \mapsto X_0$ by $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$, which has the left adjoint Σ^∞. We also have *shift* autofunctors $(-)[n]$ with $\Omega = [-1]$ and $\Sigma = [1]$.

While it is a *natural* definition, it is a priori unclear how *useful* the notion is. Part of the goal of this statement is to persuade the reader that it is not a meaningless generalization but a interesting object to study. We first note that this is a common generalization of symmetric monoidal ∞-categories and spectra: the categories $\infty\mathsf{SMCat} := \mathsf{CMon}(\infty\mathsf{Cat})$ and Sp both fully faithfully embeds into CatSp.

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[3] There are two essentially distinct versions of this. See section 4 for more detail.

[4] We exclusively work with the colimit of the inclusions $n\mathsf{Cat} \hookrightarrow (n+1)\mathsf{Cat}$ in Pr^L, not Pr^R. It is characterized as being initial among the homotopy fixed points of enrichment endofunctor $(-)$-$\mathsf{Cat} : \mathsf{Pr}^\mathsf{L} \to \mathsf{Pr}^\mathsf{L}$ [Gol23]. ω is traditionally used instead of ∞, but the fixed point property lets us to be less specific and avoid awkward notations like "$(k+\omega)$-category."

The former embedding is the *infinite delooping* $\mathrm{B}^\infty : \infty\mathsf{SMCat} \hookrightarrow \mathsf{CatSp}$, whose image should be regarded as *connective categorical spectra* $\mathsf{CatSp}^{\mathrm{cn}}$; as a consequence of Baez-Dolan delooping hypothesis, commutative monoid objects in $\infty\mathsf{Cat}$ are precisely the infinite loop objects, so the limit tower in the above definition factors through the forgetful functor $\infty\mathsf{SMCat} \to \infty\mathsf{Cat}_*$ to give the limit diagram of right adjoints

$$\mathsf{CatSp} \xrightarrow{\sim} \lim(\cdots \to \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat} \xrightarrow{\Omega} \infty\mathsf{SMCat}),$$

with the left adjoint B^∞. We have $\Sigma^\infty = \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty}$. It also follows that CatSp is semiadditive (i.e., has biproducts $\oplus$).

The subcategory $\mathsf{Sp} \subset \mathsf{CatSp}$ consists of categorical spectra $(X_n)_n$ whose components X_n are all groupoids (aka. spaces or anima, $\mathsf{S} = 0\mathsf{Cat} \subset \infty\mathsf{Cat}$). The inclusion $\mathsf{Sp} \hookrightarrow \mathsf{CatSp}$ has both left and right adjoints: the localization (left adjoint) $(-)^{\mathrm{gp}}$ is the group completion functor, which levelwise inverts cells and group completes, whereas the colocalization (right adjoint) $\mathbb{G}_m$[5] takes levelwise the maximal Picard subgroupoid. Sp is the $n = -\infty$ case of the following categorical hierarchy:

Definition 2.2 ([Ste21, Notation 13.2.21])**.** Let $-\infty \le n \le \infty$. The category $n\mathsf{CatSp} \subset \mathsf{CatSp}$ of *n-categorical spectra* consists of objects $X = (X_k)_{\in \mathbb{N}}$ such that X_k is a $\max\{0, n+k\}$-category.

CatSp is the cases $n = \infty$. All intermediate cases are shifts of another. The inclusion from one to another admits both left and right adjoints, similarly to $m\mathsf{Cat} \hookrightarrow n\mathsf{Cat}$ for $m \le n$.

Notice $\mathsf{Sp} \cap \infty\mathsf{SMCat} = \mathsf{Sp}^{\mathrm{cn}} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S})$. The above fact that CatSp is the limit of $\infty\mathsf{SMCat}$ corresponds to that Sp is the stabilization of $\mathsf{Sp}^{\mathrm{cn}}$. While Ω^∞ restricts to the one for spectra, Σ^∞ does not: the relation is $\Sigma^\infty_{\mathsf{Sp}} \simeq (\Sigma^\infty)^{\mathrm{gp}}$. The free object on a point in categorical spectra is the symmetric monoidal groupoid of finite sets $\mathbb{F} \coloneqq \mathrm{B}^\infty \mathsf{Fin}^{\simeq} = \Sigma^\infty_+(*)$, while in spectra it is the sphere $\mathbb{S}$. The fact $\mathbb{S} = \mathbb{F}^{\mathrm{gp}}$ is known as the Barratt-Priddy-Quillen theorem[6][BP72]. Baez-Dolan delooping hypothesis itself is a categorical version of May's recognition principle for n-fold loop spaces[May72], whose $n = \infty$ case is often used to motivate the notion of spectra. Note that we work *without* group completion, which is somewhat difficult to analyze and sometimes quite destructive. For instance, May's recognition principle can be separated into the delooping hypothesis and a less formal fact about group completion.

The following important formula for spectra remains valid:

Lemma 2.3. *Let $X = (X_n)_{n\in\mathbb{N}}$ be a categorical spectrum. Then we have $X = \mathrm{colim}_n \Sigma^{\infty - n} X_n$.*

Before closing this section, we list a few more typical examples from [Ste21, §13.3].

Example.
(1) Let $\mathcal{C}$ be a category with finite limits. $n\mathsf{Span}(\mathcal{C})$ is the n-category with the same objects as $\mathcal{C}$, a morphism from x to y is a span $x \leftarrow z \to y$, a 2-morphisms are spans of spans, and so on, up thourgh n-morphisms. Symmetric monoidal structure is given by objectwise Cartesian product, so the unit is the terminal object. Then $\{n\mathsf{Span}(\mathcal{C})\}$ forms a categorical spectrum.
(2) n-modules and presentable categorical spectrum: Let $\mathsf{CAT}^{\mathrm{ccpl}}$ be the very large category of large categories with small colimits and functors that preserves them. Let κ_0 be the first large cardinal. Consider the endorfunctor $\mathsf{Mod}^{\mathrm{pr}}_{(-)} : \mathsf{CAlg}(\mathsf{CAT}^{\mathrm{ccpl}}) \to \mathsf{CAlg}(\mathsf{CAT}^{\mathrm{ccpl}})$ given by $\mathsf{V} \mapsto \mathsf{Mod}_{\mathsf{V}}(\mathsf{CAT}^{\mathrm{ccpl}})^{\kappa_0}$, assigning κ_0-compact cocomplete modules to a cocompletely symmetric monoidal category. It is shown that $\mathsf{Mod}^{\mathrm{pr}}_{\mathsf{V}} \simeq \mathsf{Mod}_{\mathsf{V}}(\mathsf{Pr}^{\mathsf{L}})$ when V is presentable. By iteration we define $n\mathsf{Mod}_{\mathsf{V}} \coloneqq (\mathsf{Mod}^{\mathrm{pr}}_{(-)})^n(\mathsf{V})$; iteratedly using that V-modules are naturally V-enriched, it can be enhanced to an n-V-category, whose underling n-categories forms a large 1-categorical spectrum $\underline{\mathsf{V}} \coloneqq \{n\mathsf{Mod}_{\mathsf{V}}\}$. In particular, we set $n\mathsf{Pr} \coloneqq n\mathsf{Mod}_{\mathsf{S}}$, $n\mathsf{Pr}_{\mathrm{st}} \coloneqq n\mathsf{Mod}_{\mathsf{Sp}}$. If A is an $\mathbb{E}_\infty$-ring, we define $\underline{A} \coloneqq \Omega\underline{\mathsf{Mod}_A(\mathsf{Sp})}$. It is an object of $\mathsf{PrCatSp}_{\mathrm{st}} \coloneqq \lim(\cdots \xrightarrow{\Omega} n\mathsf{Pr}_{\mathrm{st}} \xrightarrow{\Omega} (n-1)\mathsf{Pr}_{\mathrm{st}} \xrightarrow{\Omega} \cdots)$**Germán pointed me out that it's not clear if the loop preserves presentability. I'll need to think about that**.

[5]This is taken from [Joh23a]. I used Pic until recently, but now adopting the 0-th level notation for consistency.

[6]It is sometimes rephrased as $K(\mathbb{F}_1) = \mathbb{S}$ because the finite sets can be considered as perfect modules over the mythical absolute base field $\mathbb{F}_1$, which seems to suggest that spectral algebraic geometry sees some $\mathbb{F}_1$-geometric information.

(3) [Hau17][JS17] Morita categorical spectrum: Let $\mathcal{C}$ be a symmetric monoidal n-category (with good relative tensor products). Then one can construct a symmetric monoidal $(n+k)$-category $\mathsf{Morita}_k(\mathcal{C})$, whose objects are $\mathbb{E}_k$-algebras in $\mathcal{C}$, a morphism $A \to B$ is a $\mathbb{E}_{k-1}$-algebra object is (A, B)-bimodules, and so on. Then $\{\mathsf{Morita}_k(\mathcal{C})\}$ forms a n-categorical spectrum $\mathsf{Morita}(\mathcal{C})$.

3. Tensor product

All the commutative algebra and algebraic geometry is built on the notion of tensor product of abelian groups. It is characterized as the unique presentably symmetric monoidal structure that the free functor $\mathsf{Set} \to \mathsf{Ab}$ is symmetric monoidal. The tensor product (aka. smash product) of spectra is similar, but historically, it demanded more sophisticated groundwork. While Boardman[Boa65] provided arguably the best definition $(1,1)$-categorically possible then, even humble desiderata were shown to be incompatible with the point-set approach[Lew91], and several eclectic point-set constructions followed, e.g., [Elm+07], but it had to wait until Lurie's solid foundation of $(\infty,1)$-categories[Lur09a] for a truely canonical construction. As an example of microcosm principle[7], Lurie first contstructed a symmetric monoidal structure $\otimes$ on Pr^{L}, the large category of presentable categories, characterized by the following properties: presheaf functor $\mathcal{P} : \mathsf{Cat} \to \mathsf{Pr}^{\mathsf{L}}$ is symmetric monoidal, and $\otimes$ distributes over colimits. A (commutative) algebra object in Pr^{L} is precisely a presentabe (symmetric) monoidal category whose monoidal product is distributive over colimits.

Theorem 3.1 ([Lur17]). *$\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ is an idempotent $\mathbb{E}_0$-algebra in Pr^{L}, i.e., $\Sigma^\infty_+ \otimes \mathrm{id} : \mathsf{Sp} \simeq \mathsf{S} \otimes \mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$ is an equivalence. Since the forgetful functor $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_\infty}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ is an equivalence, Sp uniquely promotes to a object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ so that Σ^∞_+ is symmetric monoidal. It is the unit object of the monoidal subcategory $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}} \subset \mathsf{Pr}^{\mathsf{L}}$ of stable presentable categories.*

This robust implementation (together with tho whole ∞-categorical setup) unlocked the explosive development of spectral algebraic geometry and algebraic K-theory in the past 15 years or so. Given the importance of tensor products of building blocks of algebra, the following question is natural:

Question 1. *Can we equip CatSp with a natural presentably (symmetric) monoidal structure?*

The answer turns out to be tricky. We would like a characterization similar to Theorem 3.1, but we cannot expect $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \xrightarrow{\Sigma^\infty_+} \mathsf{CatSp}$ to be idempotent as an $\mathbb{E}_0$-algebra in Pr^{L}. In fact, the category $\infty\mathsf{Cat}$ is already not idempotent over S, as objects of $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^{\mathsf{L}})$ have only one compatible $\mathbb{E}_1$-structure, which uniquely promotes to an $\mathbb{E}_\infty$-structure, but as we will see, $\infty\mathsf{Cat}$ admits an asymmetric monoidal structure with the terminal unit. More reasonable question is whether $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ is idempotent, but to make sense of it, we must choose an algebra structure on $\infty\mathsf{Cat}$. The obvious first choice is the Cartesian monoidal structure, but the suspension fails to be a module homomorphism over it for a simple reason:[8]: if X, Y are m, n-categories respectively, then $X \wedge \Sigma Y$ is a $\max\{m, n+1\}$-category, while $\Sigma(X \wedge Y)$ is a $\max\{m, n\} + 1$-category, so we have $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$ in general. In other words, suspension is not given by smashing with $\vec{S}^1 \coloneqq \mathrm{B}\mathbb{N} = \Sigma S^0$.

To fix this problem, we adopt a monoidal structure called the *(lax Gray) tensor product*, that acts additively on the category levels. Recall the *gaunt ∞-categories*; Gaunt is the full subcategory spanned by 0-truncated objects in $\infty\mathsf{Cat}$ [BS21]. It is the intersection of $\infty\mathsf{Cat}$ and the $(1,1)$-category of strict ω-categories. Steiner's theory is an important technical tool that provides an adjunction between strict ω-categories and the category adCh of augmented chain complexes equipped with a graded sub $\mathbb{N}$-modules. It restricts to an equivalence between the subcategories of *strong Steiner* objects (automatically gaunt for categories): $\mathsf{Gaunt}_{\mathrm{Ste}} \simeq \mathsf{adCh}_{\mathrm{Ste}}$. The category adCh inherits a monoidal structure from the usual tensor product of chain complexes. It is noncommutative because the positivity structure, i.e., the sub $\mathbb{N}$-modules is incompatible with Koszul sign rule. The category of *cubes* $\square = \{\square^n \mid n \geq 0\} \subset$

[7] Microcosm principle tells you that, to talk about an object with certain structure (e.g. a commutative monoid), you must first equip the ambient category with the corresponding structure (e.g. a symmetric monoidal structure).

[8] However, the connective part $\mathsf{CatSp}^{\mathrm{cn}}$ can be easily given a symmetric monoidal structure; as in [GGN16], for any $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$, one has $\mathsf{CMon}(\mathcal{C}) \simeq \mathsf{CMon}(\mathsf{S}) \otimes \mathcal{C}$ and $\mathsf{CMon}(\mathsf{S}) \in \mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^{\mathsf{L}})$, so a unique symmetric monoidal structure $\circledast$ making $\mathrm{Free}_{\mathbb{E}_\infty} : \mathcal{C}^\times \to \mathsf{CMon}(\mathcal{C})^\circledast$ symmetric monoidal. This product $\circledast$ does not commute with delooping.

$\mathsf{Gaunt}_{\mathrm{Ste}} \simeq \mathsf{adCh}_{\mathrm{Ste}}$ is the monoidal full sub category generated by (the cellular chain complex of) $\square^1$; as categories, the n-cube $\square^n = (\square^1)^{\otimes n}$ has a noninvertible atomic cells for each combinatorial faces of the n-cube. The (lax Gray) *tensor product* $\otimes$ on $\infty\mathsf{Cat}$ is a presentably monoidal structure extending the monoidal structure on $\square$, which is unique, if exists, by the density of $\square \subset \infty\mathsf{Cat}$ [Cam22]. The internal hom of the tensor product is the ∞-category of functors and lax natural transformations: $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \mathsf{Fun}^{\mathrm{lax}}(Y, Z))$.

Theorem 3.2 ([Ver08][VRO23][Lou23][9])**.** *The tensor product on* $\infty\mathsf{Cat}$ *exists.*

We denote the pointed version ("lax smash" product) by $\oslash$. The suspension functor can be identified with $(-) \oslash \vec{S}^1$, which is clearly a left $\infty\mathsf{Cat}^{\otimes}$-module morphism. However, to mimic Lurie's strategy, we must promote it to a bimodule homomorphism. The left modules over a noncommutative algebra do not inherit a monoidal structure. It is in fact a key technical point: giving the structure of a bimodule homomorphism to $(-) \oslash S : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ is equivaent to lifting S to the *center* (aka. Hochshild cohomology) of $\infty\mathsf{Cat}_*^{\oslash}$, and in general higher coherence can be difficult to spell out. In our case, one can show that $\vec{S}^1$ turns out to be *half-central* in the following sense: let $D : \mathcal{A} \to \mathcal{A}$ be a monoidal functor such that $D^2 \simeq \mathrm{id}$. As an algebra morphism is a (pro)functor between the deloopings, D can be seen as a $\mathcal{A}$-bimodule; explicitly, it is the identity bimodule $\mathcal{A}$ except that the left action is twisted by D. We define the *half-center* of $\mathcal{A}$ with respect to D as $\mathrm{Hom}_{\mathsf{BMod}_A}(A, D) \simeq \mathrm{Hom}_{\mathsf{BMod}_A}(D, A)$. The following theorem is relatively formal after showing (1):

Theorem 3.3.
(1) $\vec{S}^1 = \mathrm{B}\mathbb{N} \in \infty\mathsf{Cat}_*$ *canonically lifts to the half-center with respect to the total dual (which flips the domain and codomain of all cells). In particular,* Σ^2 *canonically promotes to a* $\infty\mathsf{Cat}^{\otimes}$*-bimodule morphism.*
(2) *With the induced bimodule structure from above,* $\Sigma^{\infty}_{+} : \infty\mathsf{Cat} \to \mathsf{CatSp}$ *is an idempotent* $\mathbb{E}_0$*-algebra in* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}})$*. In particular, it uniquely promotes to an* $\mathbb{E}_1$*-algebra object.*
(3) *The presentably monoidal structure on* CatSp *given by forgetting along the lax monoidal functor* $\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Pr}^{\mathsf{L}}$ *satisfies the universal property of* $\infty\mathsf{Cat}^{\otimes}[(\vec{S}^1)^{-1}]$.
(4) *Categorical filtration makes* CatSp *into a filtered monoidal category, i.e., The tensor product of an* n*-categorical spectrum and an* m*-categorical spectrum is a* $(m+n)$*-categorical spectrum. In particular,* $0\mathsf{CatSp}$ *is a monoidal subcategory.*
(5) $\mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ *is a monoidal subcategory. It follows that* $\infty\mathsf{SMCat}$ *admits a unique* $\mathbb{E}_1$*-monoidal structure* $\otimes$ *that makes* $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat}^{\otimes} \to \infty\mathsf{SMCat}^{\otimes}$ *monoidal.*

Remark 3.4.
(1) In (4), we take $-\infty + \infty = -\infty$. In other words, $\mathsf{Sp} \subset \mathsf{CatSp}$ is a tensor-ideal. The localization is smashing by the sphere spectrum $\mathbb{S}$. Since $\mathsf{Sp} \subset \mathsf{CatSp}$ is a monoidal subcategory, we have the inclusion $\mathsf{Alg}(\mathsf{Sp}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.
(2) From footnote 8, $\infty\mathsf{SMCat}$ also has a symmetric monoidal structure $\circledast$. The identity functor $\infty\mathsf{SMCat}^{\circledast} \to \infty\mathsf{SMCat}^{\otimes}$ is lax monoidal and the two monoidal structures agree on $\mathsf{CMon}(\mathsf{S})$. In particular, we we have the inclusion $\mathsf{Rig}_{\mathbb{E}_1}(\mathsf{S}) \hookrightarrow \mathsf{Alg}(\mathsf{CatSp})$.

Similarly natural question is to compare $\underline{R} \otimes \underline{S}$ with $\underline{R \otimes S}$ for $R, S \in \mathsf{CAlg}(\mathsf{S})$. I think $\mathsf{PrCatSp}$ must be given a monoidal structure that restricts to $\mathsf{Pr}^{\mathsf{L}^{\otimes}}$ in level one but above that nonsymmetric, compatibly to Gray tensor product. Is it reasonable to expect for $\mathsf{PrCatSp}_{\mathrm{st}}$ to end up being symmetric monoidal? It has -1 at the top level. Anyway, $\mathsf{CAlg}(\mathsf{Sp}) \to \mathsf{Alg}(\mathsf{PrCatSp})$ (or even $\mathsf{CRig}(\mathsf{Sp})$) will be lax monoidal at least? **make this part more precise**

4. Categorical spectra with duals

One context where categorical spectra naturally appears is the study of (T)QFTs. Recall the cobordisms hypothesis:

[9] Loubaton proved the equivalence of $\infty\mathsf{Cat}$ and a combinatorial model called complicial sets, where the Gray tensor product was constructed by Verity. It is not a priori clear if the transferred tensor product satisfies this characterization, but it follows from the fact that the 0-truncation commutes with the Verity's Gray tensor product and [Lou23, Theorem 4.3.3.26] that the gaunt categories are closed under Gray cylinders in $\infty\mathsf{Cat}$, as communicated by Loubaton. Less model-dependent approach is taken by [CM23] for 2-categories, but extension to ∞-category is not straightforward. We will only use the model-independent characterization.

Theorem 4.1 (Cobordism hypothesis, framed version. [Lur09b][GP22])**.** *Let X be a symmetric monoidal n-category. Then we have $\mathrm{ev}_* : \mathsf{Fun}^{\otimes}(n\mathsf{Bord}, X) \to (X^{\mathrm{fd}})^{\simeq}$. That is, the category of framed n-dimensional TQFT with target X and the natural transformations is equivalent to underying groupoid of fully dualizable objects in X.*

An n-category has adjoints for k-morphsims if any k-morphism admits left and right adjoints. We simply say that an n-category has adjoints if it has adjoints for k-morphisms for $k < n$. Note that it is not reasonable to ask for an n-category X to have adjoints for n-morphisms; for the top dimensional cells, adjoints are the same as inverses, as the unit and counit must be invertible. In particular, the statement "$\mathcal{C}$ is an n-category with adjoints" depends on n and implies that either X is an n-cateogry but not a $(n-1)$-category, or X is a groupoid. This leads to the following category level dependent definition of *categorcial spectra with duals*:

Definition 4.2. An *n-categorical spectrum with duals*[10] is a categorical specrum $X = (X_k)_{k\geq 0} \in n\mathsf{CatSp}$ where X_k is an $(n+k)$-category with adjoints.

Previous consideration implies that $n\mathsf{CatSp}^{\mathrm{dual}} \cap m\mathsf{CatSp}^{\mathrm{dual}} = \mathsf{Sp} = -\infty\mathsf{CatSp}^{\mathrm{dual}}$ for any $m \neq n$. Since one can write the condition of having adjoints as being local with respect to a certain set of morphisms corepresenting adjunctions, $n\mathsf{CatSp}^{\mathrm{dual}} \hookrightarrow \mathsf{CatSp}$ has the localization L_n. There are essentially two cases: when $n = \infty$ and $n = 0$. Recall that $0\mathsf{CatSp}$ is a monoidal subcategory. The following essentially follows from a computation of the tensor product $\square^1 \otimes \mathsf{Adj} \in 3\mathsf{Cat}$ of the interval and the generic adjunction:

Theorem 4.3. *The localizations $L : \mathsf{CatSp} \to \mathsf{CatSp}^{\mathrm{dual}}$ and $L_0 : 0\mathsf{CatSp} \to 0\mathsf{CatSp}^{\mathrm{dual}}$ are compatible with the tensor product, i.e., the tensor product preserves $L_{(0)}$-equivalence in each variable. Therefore, there exist unique $\mathbb{E}_1$-monoidal structures $\otimes^L$ and $\otimes^{L_0}$ on $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$ making L and L_0 monoidal.*

For finite n, there is a natural embedding $\mathrm{B}^{\infty-n} : n\mathsf{SMCat} \hookrightarrow 0\mathsf{CatSp}$; it puts the symmetric monoidal category at the n-th level and fill the lower and higher level with loop and deloop, respectively. For $n = \infty$, the natural embedding is B^{∞} as before. These are in fact convenient: in all cases, we have $n\mathsf{SMCat}^{\mathrm{dual}} = n\mathsf{SMCat} \cap (0)\mathsf{CatSp}^{\mathrm{dual}}$ in $(0)\mathsf{CatSp}$. The cobordism hypothesis states that the n-dimensional framed bordism category is a 0-categorical spectrum with duals freely generated on a $(-n)$-cell: $\mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}} \simeq L_0(\mathbb{F}[-n])$. This makes it clear that the "point" in $\mathsf{Bord}_n^{\mathrm{fr}}$ is secrectly given an n-framing. Using tensor algebra, it can even be combined into a single equation $\mathsf{Bord}_{\bullet}^{\mathrm{fr}}[-\bullet] \coloneqq \bigoplus_{n\geq 0} B^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}} = L_0\,\mathrm{Tens}(\mathbb{F}[-1])$, which gives the graded $\mathbb{E}_1$-rig structuree on bordism categories given by cartesian product of manifolds; one can think this as encoding various compactifications of field theories at once. Note that the tensor unit of $0\mathsf{CatSp}^{\mathrm{dual}}$ is still $\mathbb{F} = \mathrm{B}^{\infty}\mathsf{Bord}_0$. The theorem above have the following formal but important consequence.

Corollary 4.4. *We have the following refinement of cobordism hypothesis*[11]*: $[B^{\infty-n}n\mathsf{Bord}, X] \simeq X_n^{\mathrm{fd}}$, where $[-,-]$ is the internal hom of categorical spectra and $(-)^{\mathrm{fd}} : 0\mathsf{CatSp} \to 0\mathsf{CatSp}^{\mathrm{dual}}$ is the right adjoint to inclusion***check again***. In other words, we can recover the* subcategory *of fully dualizable objects and adjointable morphisms, instead of the space of fully dualizable objects.*

Similar corollary applies to the case $n = \infty$, but I do not know if a geometric constuction of $L\mathbb{F}$ is given in the literature. I expect it to be the bordism category of stably framed manifolds.**ask around** In a sense, it is more important than individual Bord_n because it is the tensor unit. The ring structure should again come from the cartesian product of manifolds. This is the rig category structure speculated in [Yua], which is the category level incarnation of the ring structure of sphere spectrum.

Part of the reasons to be interested in categorical spectra with duals is the potential to restore some commutativity of tensor products: adding adjoints is a milder version of adding inverses. More

[10] this terminology reflects the fact that the existence of adjoints in X_{n+1} forces the existence of duals in the symmetric monoidal ∞-category $X_n \simeq \Omega X_{n+1}$.

[11] The content of this statement is independent of the cobordism hypothesis: one can read $n\mathsf{Bord}^{\mathrm{fr}}$ as an abstract free symmetric monoidal n-category on a fully dualizable object.

concretely, passage to adjoints and mates gives an equivalence $X \to X^{\mathrm{op}}$ so we get $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$[12]. I hope to upgrade this into a braiding of $(0)\mathsf{CatSp}^{\mathrm{dual}}$:

Conjecture 1. The $\mathbb{E}_1$-monoidal structure on $\mathsf{CatSp}^{\mathrm{dual}}$ and $0\mathsf{CatSp}^{\mathrm{dual}}$ promotes to an $\mathbb{E}_2$-structure.

Ideally, we would like an $\mathbb{E}_\infty$ structure, but it is possible that the choice of $X \to X^{\mathrm{op}}$, for example whether I use left or right adjoint, affect the canonicity of braiding, so $\mathbb{E}_2$ seems to be a sensible conjecture for now.

5. Open Directions

Because of the novel nature of this project, there are plethora of interesting problems to be asked.

5.1. More on cobordism categories.
The discussion in the previous section suggests that $0\mathsf{CatSp}^{\mathrm{dual}}$ is a home to cobordism categories with unstable tangential structure, whose spectra localizations are Madsen-Tillmann spectra, and $\mathsf{CatSp}^{\mathrm{dual}}$ is a home to cobordism categories with stable tangential (or normal) structure, whose spectra localizations are Thom spectra.

Problem 2. *Give a categorical spectral construction of the "categorical Thom spectra" along the lines of [And+14] and understand the relation to the Pontrjagin-Thom construction. What about the "categorical Madsen-Tillmann spectra" and Galatius-Madsen-Tillmann-Weiss theorem?*

It will be necessary to work on parametrized categorical spectra. On $(\infty, 1)$-topos, this should be easy. It is not clear yet how much generality I will need. Parametrizing categorical spectra will have another application, namely incorporating geometric structure (that are not necessarily tangential; examples are Riemanninan metric and G-bundle with a flat connection) by working over the site of cartesian spaces, as in [GP23]. Yet another to-do item is to formulate cobordism hypothesis with singularities and tangle hypothesis in our setting. Proof of cobordism hypothesis itself can't be really formal, but do we now have enough technology to make the reduction (outlined in [Lur09b, §4.4]) of tangle hypothesis to cobordism hypothesis with sigularities rigorous?

5.2. Higher semiadditivity and finite path integral.
For an n-category $\mathcal{C}$ (for finite n), [Lur09b, §3.2] outlines the definition of $\mathsf{Fam}_n(\mathcal{C})$. When $\mathcal{C} = *$, $\mathsf{Fam}_n(*) = n\mathsf{Span}(\mathsf{S}_{\mathrm{fin}})$ is the n-category of spans in π-finite spaces. Roughly speaking, the functor $\mathsf{Fam}_n(\mathcal{C}) \to \mathsf{Fam}_n(*)$ exhibits $\mathsf{Fam}_n(\mathcal{C})$ as the category of spans of π-finite spaces on whose n-cell is given a local system of n-cells of $\mathcal{C}$ in a compatible way. Let $\mathsf{Fam}_n^k(\mathcal{C})$ denote the The importance of this category is described in [Fre+09]: it classifies classical field theory valued in $\mathcal{C}$, and it provides a natural formalism for quantization. An example of classical field theory is the $\mathsf{Bord}_n \to n\mathsf{Span}(\mathsf{Mfld}^{\mathrm{op}}) \xrightarrow{\mathrm{Map}(-,X)} \mathsf{Fam}_n(*)$ for $X \in \mathsf{S}_{\mathrm{fin}}$. The quantization procedure in this language is just composing with the canonical "finite path integral functor" $\int : \mathsf{Fam}_n(\mathcal{C}) \to \mathcal{C}$. This morphism is characterized as the universal ∞-semiadditive category mapping from $\mathcal{C}$, as proven by [Har20] in the $n = 1$ case and the general case is announced by [Sch23].

Let $X = (X_n)$ be a categorical spectrum. It is immediate from the definition that $\mathsf{Fam}(X) := \{\mathsf{Fam}_n(X_n)\}$ again forms a categorical spectrum. In the same vein (or hopefully cleaner), we expect a univeral characterization of $\{\mathsf{Fam}_n(X_n)\}$.

5.3. Brauer categorical spectrum and higher etale topology.
Reference: [GL21]

Recall from the example 2 that if A is an $\mathbb{E}_\infty$ ring, we can associate a large categorical spectrum $\underline{A} \in \mathsf{PrCatSp}_{\mathrm{st}}$ **(?)**. $\mathbb{G}_m(\underline{A})$ contains valuable informations about A: we have $\Omega^{\infty-2}\mathbb{G}_m(\underline{A}) = \mathrm{Br}(A)$ is the Brauer space of A. At least when A is connective, the structure is well-understood by [AG14]. The key was the etale local triviality of Br. But nothing has been done beyond degree 3.

Problem 3. *Give the "categorically-etale" topology on presentable stable categorical spectra in a way that $A \mapsto \underline{A}$ and $A \mapsto \mathbb{G}_m(\underline{A}$ are sheaves. Do we need to generalize the notion of topos and sheaves?*

Announced in [Joh23a]: if X contains $\mathbb{Q}$ and "Galois-closed," then $\mathbb{G}_m(X) = I_{\mathbb{Q}/\mathbb{Z}}$.

This seems to describe the categorical etale behaviour of this higher Brauer spectrum. Classical algebraically closed fields like $\mathbb{C}$ is not algebarically closed in this world: 0-categorical algebraic closedness is only the first level of infinite hierarchy...

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[12] $(-)^{\mathrm{op}} : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ flips all odd dimensional cells and is antimonoidal. It induces a similar functor on CatSp.

5.4. **algebraic K-theory.** It is an interesting problem to define the K-theory of suitably compactly generated or dualizable presentable stable categorical spectra, including the class of $\underline{\mathsf{Mod}_R}$, in a way that $K(\underline{\mathsf{Mod}_R})$ either recovers or enhances the classical $K(R)$, and is also related to secondary K-theory and higher. It should admit a trace map to higher hochshild homology; this should appear as the value of the tori of the TQFT defined using the theory of higher quasicoherent sheaves of [Ste21] (the B-model). Optimistically, one hopes that the redshift phenomena gets some illuminating description here, in light of "category level vs chromatic level" picture.

Anish said something about etaleness in TTG and seemed related. Very optimistic hope is that redshift gets more illuminating proof here, in light of category level vs chromatic level picture.

Another point worth mentioning is that, in the proof of

This is the crucial result that motivates many new development in the field.

Another point: in the proof of Land-Tamme's excision, appearance of lax pullback seems to be considered kind of random and not very well understood. I want to understand it using the language of higher categorical algebra and in relation to the noncommutative tensor product.

5.5. **t-structures. categorical chain complex.** One major difficulty working with categorical spectra is the lack of robust reduction principle, corresponding to the t-structure in stable categories and spectral sequences. The root of the problem seems to be that a categorical hierarchy is something more closed than the homotopical hierarchy: it is easy to "leak out" from one homotopical level to higher, generically by colimit-type constructions. Descent works precisely this way. Since one categorical level is closed both under limit and colimit, similar ideas inevitably fail. It seems that the idea that replaces are lax colimits. I didn't have time to check the relation to categorical chain complexes, but seems related in the case with duals. Thinking of Dold-Kan, there should be some truncation structure induced by truncated oriental objects.

when there are enough duals Lurie's categorical chain complexes may be used but I need to check how it really works and it may be totally irrelevant... At least it is a kind of reduction principle that allows to slice a symmetric monoidal n-category into 1-categorical level.

5.6. **stability of higher categories.** Theorem 3.3 implies that being a CatSp-module is a property of $\infty\mathsf{Cat}^{\otimes}$-bimodule. This corresponds to the fact that a Sp-module structure is actually a property of being stable presentable. In $(\infty,1)$-category theory, the stability have intrinsic characterizations. It would be useful to have ones for CatSp-modules. The difficulty here is understanding $\infty\mathsf{Cat}^{\otimes}$-bimodule structure itself. The most important data will be the Gray cylinder. The rest should be a sort of coherence data (morphisms in $\boxed{\,}$) and conditions (localization along $\mathcal{P}(\boxed{\,}) \to \infty\mathsf{Cat}$).

Let $\mathcal{C}$ be an ∞-category. We may be able to see this as a Gray-bimodule in the following way, if it has enough $\infty\mathsf{Cat}$-weighted limits and colimits: Because $\mathcal{C}$ itself can be viewed as a category enriched in $\infty\mathsf{Cat}$, it makes sense to think about $\infty\mathsf{Cat}$-weighted (co)limits, and in particular about (partially) (op)lax (co)limits (in a similar fashion to [Ber20]).Then characterizing stability there seems to be a meaningful question.

5.7. **Directed homotopy theory, Boolean and natural cohomology.** Grothendieck's homotopy hypothesis identifies groupoids with the weak homotopy types of topological spaces. In the same spirit, ∞-categories can be thought of as homotopy types of some sorts of *directed* spaces. Categorical spectrum can be seen as a (co)homology theory on such. In this viewpoint, we have the folloing conceptual problem:

Problem 4. *Give an excisive-functor style characterization of categorical spectra.*

This is closely related to a characterization of stability we speculated in the last section. It is an unavoidable question if we hope to develop (at least the first-order) Goodwillie calculus and apply it to deformation theory. In any way, it gives an invariant of directed spaces. Directed topological spaces have found many applications, including practical ones, so computing cohomology of those spaces is be a problem of interest. Cohomology theory given by ordinary spectra does not see any information of directedness: if X is a directed space (i.e., ∞-category) and E is a spectrum, $E^*(X) = [\Sigma^\infty_+ X, E] \simeq [\Sigma^\infty_+ X \otimes \mathbb{S}, E] \simeq E^*(|X|)$, where $|X|$ is a groupoidification of X. The next basic example would be the Eilenberg-Mac Lane spectra of semifields. Finite semifields are either a finite field $\mathbb{F}_p$ or the boolian semifield $\mathbb{B}$, whose addition is idempotent; $1+1=1$. This "characteristic one" linear over $\mathbb{B}$ is developed in [CC19]. It is also proven that in [Gus+23]that a projective $\mathbb{B}$-module valued 1-dimensional TQFT corresponds to nondeterministic finite state automaton. Analogous to [Mil58], the following problem is of fundamental computational interest:

Problem 5. *Compute the Steenrod algebra $[H\mathbb{B}, H\mathbb{B}]$ and the dual Steenrod algebra $H\mathbb{B} \otimes H\mathbb{B}$. What about the natural cohomology $H\mathbb{N}$?*

The latter amounts to understanding the resolution of $\mathbb{N}$ in terms of $\mathsf{Fin}^{\simeq}$, i.e, "resolving the 20000-year old mistake."

5.8. **deformation theory.** In spectral algebraic geometry, deformation-theoretic technique is overwhelmingly important. This is because it allows us to induct on the truncation level. This was possible because, heuristically, square-zero extension flows out from the homotopical level. Now categorical level looks more "closed" and the only way to get out of it is by some lax construction. We will likely to need some first order Goodwillie derivative type technology for this, and understanding excisiveness is the first thing to do.

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