Finiteness properties (rings are connective)

LMod_A (§2.7) | CAlg_A (§4.1)

a perf + fin Tor-amplitude

" perfect $\xleftarrow{M = L_{B/A}}$ locally of fin pres

(of finite presentation $\Rightarrow$ locally of fin pres; $\pi_0 B/\pi_0 A$ fin pres)

$\Downarrow$ | $\Downarrow$

almost perfect $\longleftarrow$ almost of fin pres $\Longleftarrow$ A, B Noetherian

$\Downarrow$ lim | $\Downarrow$

$\vdots$ | $\vdots$

$\Downarrow$ | $\Downarrow$

perfect to order $n+1$ $\longleftarrow$ of fin gen to ord $n+1$

$\Downarrow$ $\quad \tau_{\leq n} M \in (\mathrm{LMod}_{\leq n})^{\omega}$ | $\Downarrow$ $\quad \tau_{\leq n} B \in (\tau_{\leq n} \mathrm{CAlg}_A)^{\omega}$

perfect to order n $\longleftarrow$ of fin gen to ord n

$\Downarrow$ | $\Downarrow$

$\vdots$ | $\vdots$

perfect to order 1 $\longleftarrow$ of fin gen to ord 1

$\Downarrow$ | $\Downarrow$ $\quad \pi_0 B/\pi_0 A$ fin pres

perfect to ord 0 $\Longleftrightarrow$ $\pi_0 M/\pi_0 A$ finite | of fin gen to ord 0

if M: connective | $\Longleftrightarrow$ $\pi_0 B/\pi_0 A$ fin gen

$\Downarrow$

$\vdots$

$\Downarrow$ colim

almost connective

§2.7 Finiteness properties of modules

$A \in \mathrm{Alg}^{cn}$, $M \in \mathrm{LMod}_A$

Def 2.7.0.1 / **Prop 2.7.0.4** M is <u>perfect to order n</u> if it satisfies the following equivalent conditions :

(1) $\forall \{N_\alpha \in (\mathrm{LMod}_A)_{\le n}\}$ filtered diagram

$$\varinjlim \mathrm{Ext}^i_A(M, N_\alpha) \longrightarrow \mathrm{Ext}^i_A(M, \varinjlim N_\alpha) \text{ is } \begin{cases} \text{inj} & i = 0 \\ \text{bij} & i < 0 \end{cases}$$

$$\Leftrightarrow \mathrm{fib}\left(\varinjlim \mathrm{Hom}_A(M, N_\alpha) \longrightarrow \mathrm{Hom}_A(M, \mathrm{colim}\, N_\alpha)\right) \in \mathrm{Sp}_{\le -1}$$

$$\Leftrightarrow \mathrm{fib}\left(\varinjlim \mathrm{Map}_A(M, N_\alpha) \longrightarrow \mathrm{Map}_A(M, \mathrm{colim}\, N_\alpha)\right) \simeq *$$

(-1)-truncated ?

(2) $\forall \{N_\alpha\}$ as above & transition maps are injective on π_n

$$\varinjlim \mathrm{Map}_A(M, N_\alpha) \xrightarrow{\simeq} \mathrm{Map}_A(M, \varinjlim N_\alpha)$$

(3) M : almost connective (i.e. connective up to a shift $\leftarrow$ 6.2.5.2)

and

$\forall \{N_\alpha\}$: filtered diagram in $\mathrm{LMod}_A^\heartsuit$ with injective transition maps

$$\varinjlim \mathrm{Map}_A(M, N_\alpha[n]) \xrightarrow{\simeq} \mathrm{Map}_A(M, N_\alpha[n])$$

<u>Rmk</u> • by (2), M: perf to ord $(n+1)$

$$\Downarrow$$

$$\tau_{\le n} M \in (\mathrm{LMod}_{A\,\le n})^{\omega} \Longrightarrow M: \text{almost connective}$$

$$\operatorname*{colim}_{k\to-\infty} \tau_{\ge k}(\tau_{\le n} M) = \tau_{\le n} M$$

$$\exists k \quad \Uparrow \ \ \tau_{\le n} M$$

$$\Downarrow$$

M: perf to ord n

- M: $(n+1)$-connective $\Rightarrow$ perfect to ord n
- M: perfect to ord n $\Longleftrightarrow$ $M[k]$: perfect to ord $n+k$
- M: almost perfect $\Longleftrightarrow$ $\forall n\in\mathbb{Z}$ M: perf to ord n

$$((\mathrm{LMod}_A)_{\le n})^{\omega} = \bigcup_{k\in\mathbb{Z}} (\mathrm{LMod}_A)^{\omega}_{\le n} \cap (\mathrm{LMod}_A)_{\ge -k} = \bigcup_k (\mathrm{LMod}_{A\,[-k,n]})^{\omega}$$

$(\Rightarrow)$

$$M \in \mathrm{LMod}_A^{\mathrm{aperf}} \Longleftrightarrow \exists k\in\mathbb{Z}\ \forall n\ge 0 \ \underbrace{\tau_{\le n}(M[k])}_{=\,(\tau_{\le n-k}M)[\cancel{k}]} \in (\underbrace{\tau_{\le n}\mathrm{LMod}_R^{cn}}_{=\,(\mathrm{LMod}_R)_{[0,n]}^{\cancel{k}\ n-k}})^{\omega} \quad \text{t-str vs } \tau_{\le n}$$

$$\forall n\ge 0\ \exists k\in\mathbb{Z}$$

$(\Leftarrow)$

<u>M: k-connective</u> $\quad \forall n\in\mathbb{Z}$ M: perf to ord n

$$\Longleftrightarrow \tau^{st}_{\le n} M \in ((\mathrm{LMod}_A)_{\cancel{\le n}\ \cancel{\ge k}\ [k,n]})^{\omega} = \tau_{\le n-k}(\mathrm{LMod}_A)_{\ge k}^{\omega}$$

$$\tau_{\le n-k} M$$

- $M' \to M \to M''$, M': perf to ord n
  $\Rightarrow$ (M: perf to ord n $\Longleftrightarrow$ M'': perf to ord n)

$$\left(\begin{array}{ccccc} \operatorname{colim} \mathrm{Map}_A(M''[-n], N_\alpha) & \to & \operatorname{colim} \mathrm{Map}_A(M[-n], N_\alpha) & \to & \operatorname{colim} \mathrm{Map}(M'[-n], N_\alpha) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \mathrm{Map}_A(M''[-n], \operatorname{colim} N_\alpha) & \to & \mathrm{Map}_A(M[-n], \operatorname{colim} N_\alpha) & \to & \mathrm{Map}(M'[-n], N_\alpha) \end{array}\right)$$

- $A \simeq \prod_{i=1}^{n} A_i \rightsquigarrow \mathrm{Mod}_A \simeq \prod \mathrm{Mod}_{A_i}$ $\quad$ M: perf to ord n
  $M \simeq \prod M_i$ $\quad \Longleftrightarrow \forall M_i$: perf to ord n

Proof

(1) $\Rightarrow$ (2) injection on π_n $\Longleftrightarrow$ n-truncated cofiber

$$\begin{tikzcd}
N_\alpha \arrow[r] \arrow[d, bend right, "\varinjlim \Omega^\infty \mathrm{Hom}(M,-)"'] & \operatorname{colim} N_\alpha \arrow[r] \arrow[d, phantom, "\shortparallel"] & K_\alpha \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
 & N & (\mathrm{LMod}_A)_{\le n}
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim_\alpha \mathrm{Map}_A(M,N_\alpha) \arrow[r, "\simeq"'] & \mathrm{Map}_A(M,N) \arrow[r, "{*\simeq}"'] & \varinjlim_\alpha \mathrm{Map}_A(M,K_\alpha)
\end{tikzcd}$$

iff

fiber sequence

Applying (1) for $\{K_\alpha\}$

$$\underset{\sim}{\Omega^\infty}\mathrm{fib}\Big(\underbrace{\varinjlim_\alpha \mathrm{Hom}_A(M,K_\alpha) \longrightarrow}_{\varinjlim \mathrm{Map}_A(M,K_\alpha)\,\simeq\,*.} \underset{\simeq\, *}{\mathrm{Hom}_A(M, \overset{0}{\overset{\shortparallel}{\varinjlim K_\alpha}})}\Big) \simeq *$$

(2) $\Rightarrow$ (3) M: almost connective follows from

$(\tau_{\le n} M) \in ((\mathrm{LMod}_A)_{\le n})^\omega$, $\operatorname{colim} \tau_{\ge -k}\tau_{\le n} M \simeq \tau_{\le n} M$

(3) $\Rightarrow$ (1) Take $\{N_\alpha\}$: filtered diagram in $(\mathrm{LMod}_A)_{\le n}$

M: $(-k)$-connective ($\exists\, k \gg 0$)

$\rightsquigarrow$ Enough to consider the case when N_α: $(-k)$-conn.

Assume $N_\alpha \in (\mathrm{LMod}_A)_{[n-i,n]}$, proceed by induction on i.

$\underline{i=0 \text{ case}}$ $N_\alpha \in \mathrm{LMod}_A^\heartsuit[n]$, $N := \operatorname{colim} N_\alpha$

- Consider $N'_\alpha := \mathrm{Im}(N_\alpha \to N) \rightsquigarrow \{N'_\alpha \in \mathrm{LMod}_A^\heartsuit[n]\}$ with injective tr maps, $N \simeq \operatorname{colim} N'_\alpha$

$$\varinjlim_\alpha \mathrm{Map}_A(M,N'_\alpha) \xrightarrow{\sim} \mathrm{Map}_A(M,N) \quad \text{by (3)}$$

- $\underset{\mathrm{Ker}(N_\alpha \to N)}{\overset{\shortparallel}{K_\alpha}} \rightarrowtail N_\alpha \twoheadrightarrow N'_\alpha$, $K_{\alpha,\beta} := \mathrm{Ker}(N_\alpha \to N_\beta)$ for $\beta \ge \alpha$

$$\rightsquigarrow \{K_{\alpha,\beta}\}_{\beta\geq\alpha} \xrightarrow{\text{colim}} K_\alpha$$

$$\overset{(3)}{\rightsquigarrow} \quad \underset{\alpha}{\underrightarrow{\lim}} \; \underset{\beta\geq\alpha}{\underrightarrow{\lim}} \operatorname{Map}_A(M, K_{\alpha,\beta}) \xrightarrow{\sim} \underset{\alpha}{\underrightarrow{\lim}} \operatorname{Map}_A(M, K_\alpha)$$

$$\begin{array}{c} \underset{\alpha}{\underrightarrow{\lim}} \operatorname{Map}_A(M, K_\alpha) \\ \downarrow \\ \underset{\alpha}{\underrightarrow{\lim}} \operatorname{Map}_A(M, N_\alpha) \\ \downarrow \\ \underset{\alpha}{\underrightarrow{\lim}} \operatorname{Map}_A(M, N'_\alpha) \simeq \operatorname{Map}_A(M, N) \end{array}$$

- $\underset{\substack{(\alpha,\beta)\\ \beta\geq\alpha}}{\underrightarrow{\lim}} \operatorname{Map}_A(M, K_{\alpha,\beta}) \simeq *$ as $\begin{array}{c} K_{\alpha,\beta} \\ \downarrow \simeq 0 \\ K_{\alpha',\beta'} \end{array}$ if $\alpha' \geq \beta$

$\lrcorner$

<u>Inductive step</u> $N_\alpha : (n-i-1)$-connective

<u>$i \rightsquigarrow (i+1)$</u>

$$\begin{array}{ccccc} * & \longrightarrow & \text{fib} & \longrightarrow & * \\ \downarrow \text{IH} & & \downarrow & & \downarrow n=0 \text{ case} \\ \underrightarrow{\lim} \operatorname{Map}_A(M, \tau_{\geq 2n-i} N_\alpha) & \longrightarrow & \underrightarrow{\lim} \operatorname{Map}_A(M, N_\alpha) & \xrightarrow[\Omega^{i+1}]{} & \underrightarrow{\lim} \operatorname{Map}_A(M, \pi_{n-i-1} N_\alpha[n-i-1]) \underset{[n]}{} \quad : \text{fib seq} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Map}_A(M, \tau_{\geq 2n-i} N) & \longrightarrow & \operatorname{Map}_A(M, N) & \longrightarrow & \operatorname{Map}_A(M, \pi_{n-i-1} N[n-i-1]) \quad : \text{fib seq} \end{array}$$

$\boxed{\text{h}}$

⑤ Finitely n-presented modules

Def 2.7.1.1 M is finitely n-presented $\overset{\text{def}}{\iff}$ n-truncated and perfect to order $(n+1)$

$$\left(\iff M \in \left((\mathrm{LMod}_A)_{\le n}\right)^{\omega} \right)$$

Cor below

Remark • $f : M \to M'$ in LMod_A, $\pi_n f$: surj, $\pi_{<n} f$: isom

$\rightsquigarrow$ $\forall N \in (\mathrm{LMod}_A)_{\le n}$

$$\mathrm{Hom}_A(\underset{\in \mathrm{Sp}_{\le -1}}{\mathrm{cof}(f)}, N) \longrightarrow \mathrm{Hom}_A(M', N) \longrightarrow \mathrm{Hom}_A(M, N)$$

($n+1 \le$ conn. of $\mathrm{cof}(f)$, N: $\le n$)

$\rightsquigarrow$ $\mathrm{Ext}^i_A(M', N) \longrightarrow \mathrm{Ext}^i_A(M, N)$ $\begin{cases} \text{inj} & i = 0 \\ \text{isom} & i < 0 \end{cases}$

• If M : perfect to order n. $\Rightarrow$ M' : perfect to order n

∵ $\{N_\alpha\}$ filtered diagram in $(\mathrm{LMod}_A)_{\le n}$

$\rightsquigarrow$

$$\begin{tikzcd} \mathrm{colim}\, \mathrm{Ext}^i(M', N_\alpha) \arrow[r, "a"] \arrow[d, "b"] & \mathrm{colim}\, \mathrm{Ext}^i(M, N_\alpha) \arrow[d, "c"] \\ \mathrm{Ext}^i(M, \mathrm{colim}\, N_\alpha) \arrow[r, "d"] & \mathrm{Ext}^i(M, \mathrm{colim}\, N_\alpha) \end{tikzcd}$$

$\underline{i = 0}$ a, c, d inj $\Rightarrow$ b : inj

$\underline{i < 0}$ a, c, d isom $\Rightarrow$ b : isom

Cor • M : perfect to order $(n+1)$ $\Rightarrow$ $\tau_{\le n} M$: finitely n-presented

• $M \in (\mathrm{LMod}_{A \le n})^{\omega}$ $\Rightarrow$ $(\mathrm{LMod}_A)_{\le n} \overset{\tau_{\le n}}{\underset{}{\rightleftarrows}} \mathrm{LMod}_A$ $\quad$ M : retract of $\tau_{\le n} M'$ (fin. n-pres)

$\exists M'$: perfect

$\Rightarrow$ M : fin n-pres

• M : fin n-pres $\Rightarrow$ $M = \tau_{\le n} M \in (\mathrm{LMod}_{A \le n})^{\omega}$

Prop 2.7.1.5 A : discrete $\Rightarrow$ TFAE :

(1) M is fin. 0-presented, $\forall N \in \mathrm{RMod}_A^\heartsuit$, $\pi_1(N \otimes_A M) = 0$

(2) M is perfect, Tor-amplitude ≤ 0

Proof (2) $\Rightarrow$ (1) Enough to show $M \in \mathrm{LMod}_{\le 0}$.

$$A \in \mathrm{RMod}_A^\heartsuit \underset{\text{Tor-amp} \le 0}{\rightsquigarrow} M \simeq A \otimes_A M \in \mathrm{Sp}_{\le 0}.$$

(1) $\Rightarrow$ (2) M is $(-n)$-connective for $\exists n \gg 0$

induction on n

$\underline{n=0}$ $M \in \mathrm{LMod}_A^\heartsuit$, $\mathrm{Tor}_A^1(N, M) = 0$ $\forall N \in \mathrm{RMod}_A^\heartsuit$

$\Rightarrow M$: flat $/A$ $\underset{M:\text{connective}}{\Longrightarrow}$ Tor-amplitude ≤ 0

perfect to order 1 $\Rightarrow \underset{M=}{\pi_0 M} \in (\mathrm{LMod}_A)_{\le 0}^\omega$

$\Rightarrow M \in (\mathrm{LMod}_A^\heartsuit)^\omega$ fin presented $\Rightarrow$ perfect

$\underline{(n-1) \to n}$ M : $(-n)$-connective for $n > 0$

$M[n]$: perfect to order $n+1$ $(\overset{\text{to}}{\Rightarrow}$ ord $1)$

$\rightsquigarrow \underset{\underset{\pi_{-n} M}{\shortparallel}}{\tau_{\le 0} M[n]} \in (\mathrm{LMod}_A)_{\le 0}^\omega$ fin pres. A-mod

$\rightsquigarrow$ Take $\alpha : A^k[-n] \longrightarrow M$ s.t. $\pi_{-n}\alpha$: surj

$$M \to \mathrm{cofib}(\alpha) : (n-1)\text{-connective}$$

- Perf to ord 1 : closed under cofibers
- $M, \Sigma^{-n+1} A^k \in (\mathrm{LMod}_A)_{\le 0} \Rightarrow \mathrm{cofib}(\alpha) \in (\mathrm{LMod}_A)_{\le 0}$
- $\forall N \in \mathrm{RMod}_A^\heartsuit$, $\pi_1(N \otimes_A \mathrm{cofib}(\alpha)) = 0$

$\pi_1(N \otimes_A M)$ $\qquad$ $\pi_0(N \otimes_A \Sigma^{-n+1} A^k)$

$\Rightarrow \mathrm{cofib}(\alpha)$ perfect, Tor-amplitude ≤ 0 $\Rightarrow$ also for M

$\Sigma^{-n} A^k$

$\square$

2.7.2 Alternate Characterization

• $\underset{\text{(almost) perfect}}{\widetilde{A^k}} \xrightarrow{\phi} M \rightsquigarrow$ M: perf to ord n $\iff$ $\mathrm{cofib}(\phi)$ perf to ord n

$\iff$ $\mathrm{fib}(\phi)$ perf to ord $(n-1)$

Prop If M : connective, then

M : perf to ord 0 $\iff$ $\pi_0 M$: finite $\pi_0 A$-mod

Proof

$(\Leftarrow)$ $\pi_0\phi$: surj $\Rightarrow$ $\mathrm{fib}\,\phi$: connective $\Rightarrow$ $\mathrm{fib}\,\phi$: perf to ord (-1)

$\iff$ M : perf to ord 0

$(\Rightarrow)$ Use (3) of def of "perf to ord 0"

$\{N_\alpha\}$: filtered system of finite $\pi_0 A$-submod of $\pi_0 M$

$$\Rightarrow \operatorname*{colim}_\alpha \mathrm{Map}_A(\underset{\pi_0 M}{\underset{\wr\wr}{\tau_{\le 0}M}}, N_\alpha) \xrightarrow{\sim} \mathrm{Map}_A(\underset{\pi_0 M}{\underset{\wr\wr}{\tau_{\le 0}M}}, \pi_0 M)$$

$$\begin{tikzcd} \pi_0 M \arrow[r] \arrow[dr, "\mathrm{id}"'] & N_{\alpha_0} \arrow[d, hook] \\ & \pi_0 M \end{tikzcd} \longmapsto \underset{\mathrm{id}}{\in}$$

$N_{\alpha_0} = \pi_0 M$. $\square$

Cor TFAE for $M \in \mathrm{LMod}_A$, $n \in \mathbb{Z}$

2.7.2.2

(1) M perf to ord n

(2) $\exists P$: perfect with Tor-amp $\le n$

$\exists \phi : P \to M$ s.t. $\mathrm{fib}\,\phi$ is n-connective

(3) $\exists P$: perfect

$\exists \phi : P \to M$ s.t. $\mathrm{fib}\,\phi$ is n-connective

Proof (2) $\Rightarrow$ (3) obvious

(3) $\Rightarrow$ (1) $P \xrightarrow{\phi} M \longrightarrow \mathrm{cofib}\,\phi$: $(n+1)$-connective

$\ddot{\text{perf to}}$ ord n $\Rightarrow$ [perf to ord n $\Leftarrow$ perf to ord n]

(1) $\Rightarrow$ (2) M : $(-k)$-connective $\exists k \gg 0$.

- M : perf to order n $\Leftrightarrow$ $M[k]$: perf to ord $(n+k)$
- P : perfect with Tor-amp $\leq n$

  $\Leftrightarrow$ $P[k]$: perf with Tor-amp $\leq n+k$
- $\mathrm{fib}(P \to M)$: n-conn. $\Leftrightarrow$ $\mathrm{fib}(P[k] \to M[k])$: $(n+k)$-conn.

$\rightsquigarrow$ replacing M by $M[k]$, n by $n+k$

we may assume M : connective.

induction on n : If $n < 0$ Take $P = 0$

$\rightsquigarrow \mathrm{fib}(0 \to M) = M[-1]$: (-1)-conn.

If $n \geq 0$ Consider a π_0-surj $A^m \xrightarrow[\alpha]{} M$.

connective,

$\rightsquigarrow$ $\mathrm{fib}(\alpha)$: perfect to ord $(n-1)$

induction hypothesis : $\exists \psi : Q \to \mathrm{fib}(\alpha)$

$$\begin{cases} Q : \text{perf, Tor-amp} \leq n-1 \\ \mathrm{fib}\,\psi : (n-1)\text{-connective} \end{cases}$$

$$\begin{tikzcd}
\mathrm{fib}\,\psi \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{fib}\,\phi \arrow[d] \\
Q \arrow[r] \arrow[d, "\psi"'] & A^m \arrow[r, "\mathrm{cofib}"'] \arrow[d, equal] & P \arrow[r] \arrow[d, "\phi"] & Q[1] \\
\mathrm{fib}\,\alpha \arrow[r] & A^m \arrow[r] & M
\end{tikzcd}$$

$\mathrm{fib}\,\phi = \mathrm{fib}\,\psi[1]$: n-connective

P : perf, Tor-amp $\leq n$

Cor 2.7.2.3 Let A: left Noetherian (i.e. $\pi_n A$: left Noetherian $\pi_0 A$-mod)

Then M: perfect to order n iff

(1) $\pi_i M = 0$ for $i \ll 0$

(2) For each $m \le n$, $\pi_m M$ is fin. $\pi_0 A$-mod

Proof M: perf to ord $n \Rightarrow$ (1)

$\rightsquigarrow$ It suffices to prove M: perf to ord $n \Longleftrightarrow$ (2)

assuming M: connective

Induction on n:

$\underline{n<0}$ empty conditions

$\underline{n=0}$: 2.7.2.1

$\underline{n>0}$ Take $\underset{\substack{\Uparrow \\ \text{connective}}}{\mathrm{fib}\,\phi} \longrightarrow A^m \xrightarrow[\phi]{} M$

$$\underbrace{(\pi_{i+1} A)^m}_{} \longrightarrow \pi_{i+1} M \longrightarrow \pi_i\, \mathrm{fib}\,\phi \longrightarrow \underbrace{(\pi_i A)^m}_{\text{fin. gen. } \pi_0 A\text{-mod}}$$

$\pi_{i+1} M$: fin. gen $\Longleftrightarrow \pi_i\, \mathrm{fib}\,\phi$: fin gen

$(i \le n-1)$ $\qquad (i \le n-1)$

$\Updownarrow$ induction hypothesis

M: perfect to order n $\Longleftrightarrow$ $\mathrm{fib}\,\phi$: perfect to order $(n-1)$

$\boxed{\text{q.e.d.}}$

Cor 2.7.2.4 $M \in \mathrm{LMod}_A^{cn}$, $n \ge 0$ TFAE:

(a) M is perfect to order n

(b) $\exists M_\bullet \in (\mathrm{LMod}_A^{cn})^{\Delta^{op}}$ s.t. $|M_\bullet| = M$,

M_k: fin.gen free for $0 \le k \le n$

(c) $\exists M_\bullet \in (\mathrm{LMod}_A^{cn})^{\Delta^{op}}$ s.t. $|M_\bullet| = M$,

M_k: perfect to order $(n-k)$ for $0 \le k \le n$

Proof (b) $\Rightarrow$ (c) ✓

(c) $\Rightarrow$ (a) HA 1.2.4.1 $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{LMod}_A) \simeq \mathrm{Fun}(\Delta^{op}, \mathrm{LMod}_A)$

$$X(n) = |\mathrm{sk}_n X_\bullet| \longleftarrow\!\shortmid \; X_\bullet$$

$$\operatorname{colim} X(n) \simeq |X_\bullet|$$

Take $M_\bullet$ as in (c)

$$\begin{tikzcd} \arrow[r, squiggly, "DK"'] & P_0 \arrow[r] \arrow[drrrr] & P_1 \arrow[r] \arrow[drrr] & P_2 \arrow[r] \arrow[drr] & \cdots \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \\ & & & & & M & \end{tikzcd}$$

P_i : perfect to order n for $i \leq n$

∵ induction on i :

$\underline{i=0}$ $P_0 \simeq M_0$

$\underline{i-1 \to i}$ $P_{i-1} \longrightarrow P_i \longrightarrow \Sigma^i N$

?. N : i-th term of the normalized chain complex of M.

$\rightsquigarrow$ direct summand of M_i

$P_n \longrightarrow M$ has $(n+1)$-connective cofiber

$\rightsquigarrow M$: perfect to order n ⌟

(a) $\Rightarrow$ (b) inductively construct

$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r] \arrow[drr, "\phi_0"'] & P_1 \arrow[r] \arrow[dr, "\phi_1"] & P_2 \arrow[r] \arrow[d, "\phi_2"] & \cdots \\ P_{-1} \arrow[u, equal] & & & M & \end{tikzcd}$$

s.t. $\mathrm{Cofib}\, \phi_m$: $(m+1)$-connective

$\left.\begin{array}{l} \Sigma^{-m}(\mathrm{cofib}\, P_{m-1} \to P_m) : \text{finite free} \\ P_m : \text{perfect} \end{array}\right\}$ for $m \leq n$

$\underline{m=0}$ $\quad \exists\, P_0 \xrightarrow[\phi_0]{} M$ $\quad \begin{cases} P_0 : \text{finite free}, \\ \pi_0 \phi_0 : \text{surjective} \end{cases}$

$\underline{m-1 \rightsquigarrow m}$ $(m \leq n)$

$$\underset{\text{perfect}}{P_{m-1}} \xrightarrow{\phi_{m-1}} \underset{\text{perf to ord } n}{M} \longrightarrow \mathrm{cofib}(\phi_{m-1}) : \begin{array}{l} m\text{-conn.} \\ \text{perf to order } m \end{array}$$

$$\exists \psi \uparrow$$

$$\Sigma^m F \ : \text{finite free}$$

s.t. $\pi_m \psi : \text{surj}$

$$\begin{tikzcd}
P_{m-1} \arrow[r, equal] \arrow[d] & P_{m-1} \arrow[d, "\phi_{m-1}"] & \\
P_m \arrow[r, "\phi_m"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & M \arrow[r] \arrow[d] & \mathrm{cofib}\, \phi_m \arrow[d, equal] \\
\Sigma^m F \arrow[r, "\psi"'] & \mathrm{cofib}\, \phi_{m-1} \arrow[r] & \mathrm{cofib}\, \phi_m
\end{tikzcd}$$

$\underline{m > n}$ $\quad P_m := M$

$M_\bullet$: corresponding simplicial object

M_k : direct sum of finitely many copies of

P_0 and $\Sigma^n \mathrm{cof}(P_{n-1} \to P_n)$ for $n \leq k$

$\rightsquigarrow$ finite free $\quad \square$

2.7.3 Extension of Scalars

Consider $f : A \to B$ in Alg^{cn}

<u>Prop.</u> $M \in \mathrm{LMod}_A$ is perf to ord n $\Rightarrow$ $B \otimes_A M \in \mathrm{LMod}_B$ is perf to ord n

($\Leftarrow$ if B : faithfully flat right A-mod)

<u>proof</u>

($\Rightarrow$) M : $(-m)$-connective $\exists m \gg 0$

Replacing M by $M[m]$ and n by $n+m$,

we may assume M is connective. ($\rightsquigarrow B \otimes_A M$: connective)

$n < 0$: trivial

<u>induction on n</u> when $n \geq 0$

Take $\exists A^k \xrightarrow{\phi} M$ with $\mathrm{fib}\,\phi$: connective and perf. to ord $(n-1)$

$$\rightsquigarrow \underbrace{B \otimes_A \mathrm{fib}\,\phi}_{\text{connective}} \to B^k \longrightarrow \underbrace{B \otimes_A M}_{\text{fin. gen.}} \qquad \underline{n = 0 \checkmark}$$

inductive hypothesis $\Rightarrow B \otimes_A \mathrm{fib}\,\phi$: perf to ord $(n-1)$

$\Rightarrow B \otimes_A M$: perf to ord n

($\Leftarrow$ when faithfully flat)

Assume $B \otimes_A M$: perfect to order n, $(-m)$-connective

$A \to B$ flat $\rightsquigarrow \pi_i(B \otimes_A M) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i M) \underset{i<-m}{\simeq} 0$

$\Downarrow$ faithfully flat

i.e. M : $(-m)$-connective. $\qquad \pi_i M \underset{i<-m}{\simeq} 0$

Replacing M by $M[m]$ and n by $n+m$, We may assume M : connective.

$n<0$: trivial

$n=0$: $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$: fin. gen $\pi_0 B$-mod

$\Longrightarrow$ $\pi_0 M$: fin gen $\pi_0 A$-mod

(∵ $\{b_i \otimes m_i\}_{1\le i\le k}$ generates $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$

$\rightsquigarrow$ $(\pi_0 A)^k \xrightarrow{\phi} \pi_0 M$ induces $(\pi_0 B)^k \to \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$

$(0,\dots,1,\dots,0) \longmapsto m_i$ surjection

$\xrightarrow{\text{faithfully flat}}$ ϕ : surjection)

<u>$n>0$</u> induction

<u>$n=0$</u> $\Rightarrow \exists\ \mathrm{fib}(\phi) \longrightarrow A^k \xrightarrow{\phi} M$ with $\mathrm{fib}(\phi)$: connective

$$B\otimes_A \mathrm{fib}(\phi) \longrightarrow B^k \longrightarrow B\otimes_A M$$

: perf to ord $(n-1)$ $\Longleftarrow$: perf to ord n

$\Downarrow$ IH

$\mathrm{fib}(\phi)$ perf to ord $(n-1)$ $\Longrightarrow$ M : perf to ord n ▨

<u>Prop 2.7.3.2</u> Assume $\pi_0 A \twoheadrightarrow \pi_0 B$ with nilpotent kernel I

and $M \in \mathrm{LMod}_A^{cn}$, $M' := B\otimes_A M \in \mathrm{LMod}_B^{cn}$

Then: (a) M : perf to ord n $\Longleftrightarrow$ M' : perf to ord n

(b) M : almost perfect $\Longleftrightarrow$ M' : almost perfect

(c) M has Tor-amp $\le k$ $\Longleftrightarrow$ M' has Tor-amp $\le k$

(d) M : perfect $\Longleftrightarrow$ M' : perfect

(e) M : n-connective $\Longleftrightarrow$ M' : n-connective

Proof (a) ($\Rightarrow$) ✓

($\Leftarrow$) Assume M': perfect to ord n.

$n<0$ M: connective

$n=0$ M: connective

$\Rightarrow \pi_0 M' = \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M) = (\pi_0 M)/I(\pi_0 M)$.

fin gen $\pi_0 B \simeq \pi_0 A/I$ - module

$\Rightarrow \pi_0 M$: fin gen $\pi_0 A$ - module

$I \subset \pi_0 A$ nilpotent, N: discrete $\pi_0 A$-mod

- $N/IN = 0 \Rightarrow N = IN = I^2 N = \cdots = 0$
- N/IN generated by $\bar{x}_1, \dots, \bar{x}_m$

$\Rightarrow N$ generated by $(x_1, \dots, x_m) =: N_0$

☺ $(N/N_0)/I(N/N_0) = 0$

$n>0$ $\mathrm{fib}\,\phi \longrightarrow A^k \xrightarrow[\phi]{} M$

: connective

M': perf to ord n

$\Rightarrow \mathrm{fib}(B \otimes_A \phi)$: perf to ord $(n-1)$

$\overset{IH}{\Rightarrow} \mathrm{fib}\,\phi$: perf to ord $(n-1)$

$\Rightarrow M$: perf to ord n ⌟

(b) follows from (a)

(c) ($\Rightarrow$) $N \in \mathrm{RMod}_B^{\heartsuit} \rightsquigarrow N \otimes_B M' = N \otimes_A M$: k-truncated

($\Leftarrow$) $N \in \mathrm{RMod}_A^{\heartsuit}$

$\rightsquigarrow N \supset IN \supset I^2 N \supset \cdots \supset 0$

$$\left.\begin{array}{ccc} N & IN & I^2N \\ \downarrow & \downarrow & \downarrow \\ N/IN & IN/I^2N & I^2N/I^3N \end{array}\right\} \in \mathrm{RMod}_B^{\heartsuit}$$

$$\xrightarrow{\otimes_A M}\quad N\otimes_A M \supset IN\otimes_A M \supset I^2N\otimes_A M \supset \cdots \supset 0$$

$$\left.\begin{array}{ccc} N/IN \otimes_A M & (IN/I^2N)\otimes_A M & \\ \wr\| & \wr\| & \cdots \\ N/IN \otimes_B M' & (IN/I^2N)\otimes_B M' & \end{array}\right\}\ k\text{-truncated}$$

$$\rightsquigarrow N\otimes_A M \in Sp_{\le k}$$

(d) <u>HA 7.2.4.23</u> almost perfect & has finite Tor-amplitude $\iff$ perfect

(e) $(\Rightarrow)$, $(\Leftarrow, n\le 0)$: trivial

$n>0$: induction

M': n-connective $\xRightarrow{\text{IH}}$ M: $(n-1)$-connective

$$\begin{array}{ccc} \pi_0(M'[1-n]) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, & \pi_0 M[1-n]) & \\ \wr\| & \wr\| & \\ 0 \simeq \pi_{n-1}M' \qquad \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, & \pi_{n-1}M) = (\pi_{n-1}M)/I(\pi_{n-1}M) & \end{array}$$

nilp $\Downarrow$

$\pi_{n-1}M = 0$. $\square$

<u>Prop 2.7.3.3</u> $f: A\to B$ in Alg^{cn}, $n\ge 0$. TFAE:

(1) B: left A-mod (via f) which is perfect to order n

(2) $\forall M\in \mathrm{LMod}_B^{cn}$, perfect to order n (as a left B-mod)
$\Rightarrow$ M: perfect to order n as a left A-mod

<u>proof</u> $(2)\Rightarrow(1)$: $M=B$

$(1)\Rightarrow(2)$ induction on n

<u>$n=0$</u> $\left.\begin{array}{l}\pi_0 M : \text{fin.gen} / \pi_0 B, \\ \pi_0 B : \text{fin.gen} / \pi_0 A\end{array}\right\} \Rightarrow \pi_0 M : \text{fin.gen} / \pi_0 A$

$\underline{n>0}$ $\quad \mathrm{fib}\,\phi \longrightarrow B^k \longrightarrow M$

connective

$\mathrm{fib}\,\phi$: perfect to order $(n-1)$ $/B$

$\Longrightarrow$ $(\quad \_\_\_ \quad .. \quad \_\_\_ \quad)$ $/A$

$B^k \longrightarrow M \longrightarrow \mathrm{fib}\,\phi[1]$

perfect to order n

$\Longrightarrow$ M : perfect to order n $\quad \boxed{S}$

2.7.4. Fiberwise Connectivity Criterion $\quad A \in \mathrm{CAlg}^{cn}$

Classically : Let M : fin. gen A-mod , $x \in |\mathrm{Spec}\, A|$.

$\Uparrow n=0$ $\qquad M_x = 0 \xRightarrow[\text{Nakayama}]{} \exists U \ni x$ open $\quad M|_U = 0$

Prop 2.7.4.1 Let $M \in \mathrm{LMod}_A$, perfect to order n $\quad (n \geq 0)$

$K = A_p / pA_p$ for some $p \in \mathrm{Spec}\, A$

TFAE : (a) $\exists a \in \pi_0 A \setminus p$ s.t. $M[a^{-1}]$: $(n+1)$-connective

(b) $K \otimes_A M$ is $(n+1)$-connective

proof (a) $\Rightarrow$ (b)

$$K \otimes_A M = K \underset{A[a^{-1}]}{\otimes} M[a^{-1}] \quad : (n+1)\text{-connective}$$

connective $\quad \left(\begin{array}{l} S \subset \pi_0 A \text{ multiplicative} \\ \rightsquigarrow M \mapsto S^{-1}M \ : \ t\text{-exact} \\ \qquad\qquad\qquad \text{HA 7.2.3.25} \end{array} \right)$

(b) $\Rightarrow$ (a)

Claim : $\forall k \leq n \ \exists a \in \pi_0 A \setminus p$ s.t. $M[a^{-1}]$: $(k+1)$-connective.

true for $k \ll 0$ as M is almost connective

$\rightsquigarrow$ induction on k

Replace $A \rightsquigarrow A[b^{-1}]$ $\rightsquigarrow$ We may assume M is k-connective
$M \rightsquigarrow M[b^{-1}]$

$\exists$ by $(k-1)$-case

$M[-k]$ perf to ord $(n-k) \geq 0$, connective

$\pi_0 M$: fin gen

$$\rightsquigarrow \pi_0(K \otimes_A M[-k]) \simeq \mathrm{Tor}_0^{\pi_0 A}(K, \pi_0 M[-k]) \overset{(b)}{\simeq} 0$$

Nakayama $\Downarrow$

$$\exists a \notin p \quad \pi_0(A[a^{-1}] \otimes_A M[-k]) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 A[a^{-1}], \pi_0 M[-k]) \simeq 0$$

$$\| \qquad \pi_k(M[a^{-1}])$$

$\square$

Cor 2.7.4.$\frac{2}{3}$ $n \geq 0$, $M \in \mathrm{Mod}_A$, then

- M: $(n+1)$-connective $\iff$ M: perfect to order n and $\forall K$: residue field of A, $K \otimes_A M$ is $(n+1)$-connective
- $M \simeq 0$ $\iff$ M: almost perfect and $\forall K$ $K \otimes_A M \simeq 0$

Proof ($\Rightarrow$) ✓

($\Leftarrow$) M: locally $(n+1)$-connective, almost connective
necessary?

$\Rightarrow$ $\exists A \longrightarrow B$ faithfully flat

$B \otimes_A M$: $(n+1)$-connective

$\Rightarrow$ M: $(n+1)$-connective

M: k-connective, $k \leq n$ $\qquad$ $B \otimes_A M$ $(n+1)$-conn.

$\Rightarrow$ $\pi_k(B \otimes_A M) \underset{\text{flat}}{\simeq} \pi_0 B \underset{\pi_0 A}{\otimes} \pi_k M \simeq 0$

$\underset{\substack{\text{faithfully} \\ \text{flat}}}{\Longrightarrow}$ $\pi_k M \simeq 0$, i.e. M: $(k+1)$-connective

◻

4.1 Finiteness conditions on comm alg.

classically $\phi : A \longrightarrow B$ is (i) of finite type

(ii) of finite presentation

For connective E_∞-rings $\phi : A \to B$

(i') of finite type $\iff \pi_0 \phi$ is of finite type

$\iff \exists\ A\{x_1, \dots, x_n\} \longrightarrow B$ π_0 surjection

(ii') of finite presentation $\iff B \in$ (smallest full subcat of $CAlg_A$ containing $A\{x\}$ and closed under finite colimits)

Problems

- (ii') is not local w.r.t. Zariski topology (on A, B)

  $\rightsquigarrow$ locally of fin pres ($\iff$ compact)

  this is local w.r.t. étale topology on A and B

- When A, B discrete, Noetherian

  (ii) $\not\Rightarrow$ (ii') (or even locally of fp)

  $\Downarrow$ almost of fp

  i.e. $\forall n \geq 0$ $(C_n) : \tau_{\leq n} B \in \left(\tau_{\leq n} CAlg_A^{cn}\right)^{\omega}$

Analog to the cell complex (can be made precise)

$$\begin{array}{ccc}
\text{generator} & \longleftrightarrow & 0\text{-cell} \\
\text{relation} & \longleftrightarrow & 1\text{-cell} \\
\text{relation between relations} & \longleftrightarrow & 2\text{-cell} \\
 & \vdots & \\
\text{of finite type} & \longleftrightarrow & \text{finitely many } 0\text{-cells} \\
\text{of fin gen of order } n & \vdots & \text{———} \quad 0, \cdots, n\text{-cells} \\
\text{almost of fin pres} & \longleftrightarrow & \text{finitely many cells in each dim} \\
\text{of finite presentation} & \longleftrightarrow & \text{finite \# of cells in total}
\end{array}$$

Def $n \geq 0$

$\phi : A \longrightarrow B$ mor of connective calg

is of finite generation to order n if

closed under colim

(FG_n) : $\forall \{ C_\alpha \in \tau_{\leq n} CAlg^{cn} \}_\alpha$: filtered diagram s.t. $\pi_n C_\alpha \longrightarrow \pi_n C_\beta$ injective

$$\varinjlim_\alpha Map_{CAlg_A}(B, C_\alpha) \xrightarrow{\sim} Map_{CAlg_A}(B, \varinjlim_\alpha C_\alpha)$$

$$\begin{array}{cc}
\text{IS} & \text{IS} \\
\varinjlim_\alpha Map_{\tau_{\leq n} CAlg_A^{cn}}(\tau_{\leq n} B, C_\alpha) & Map_{\tau_{\leq n} CAlg_A^{cn}}(\tau_{\leq n} B, \varinjlim C_\alpha)
\end{array}$$

$(FG_0) \overset{4.1.1.3}{\Longleftrightarrow} \pi_0 \phi$ is of fin type

$\Uparrow$

$(C_0) \Longleftrightarrow \pi_0 \phi$ is of fin pres

$(C_0) \Rightarrow (FG_0)$

$(FG_1) \Rightarrow (C_0)$

$(C_1) \Rightarrow (FG_1)$

$(FG_2) \Rightarrow (C_1)$

$\vdots$

$\left. \begin{array}{c} (C_0) \\ (C_1) \\ \vdots \end{array} \right\} (AFP)$

$\pi_0 \phi$ is of fin pres $\Rightarrow$ (AFP) if A, B Noetherian (Hilbert basis theorem)

(AFP)

$\Uparrow$

(LFP)

$\Uparrow$

(FP)

Rmk 4.1.1.5 $(FG_{n+1}) \Rightarrow (C_n) \Rightarrow (FG_n)$: by def

$(AFP) \Longleftrightarrow \forall n \ (C_n) \Longleftrightarrow \forall n \ (FG_n)$

Prop 4.1.1.3 For $\phi: A \to B$ in CAlg^{cn}, TFAE:

(a) (FG_0), i.e., $\{C_\alpha\}$ filtered diagram of discrete calg /A. with injective $C_\alpha \to C_\beta$.

$$\Rightarrow \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}_A}(B, \varinjlim C_\alpha)$$

(b) $\pi_0\phi$ is of finite type (in the classical sense)

(We say ϕ is of finite type)

Proof (a)$\Rightarrow$(b) : $\underset{\substack{C_\alpha \subset \pi_0 B \\ \text{fin.gen}/\pi_0 A}}{\mathrm{colim}} C_\alpha \xrightarrow{\sim} \pi_0 B$.

by (a) $\varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B)$

$\rightsquigarrow \exists \alpha$ s.t.
$$\begin{tikzcd} C_\alpha \arrow[r, hook] & \pi_0 B \\ & B \arrow[u, "\text{canonical map}"'] \arrow[ul, dashed] \end{tikzcd}$$

$\rightsquigarrow C_\alpha \xhookrightarrow{\sim} \pi_0 B$: fin gen $/\pi_0 A$.

(b)$\Rightarrow$(a): Take any $\{C_\alpha\}$ as in (a). Regard $C_\alpha \hookrightarrow \varinjlim C_\alpha =: C$ as a subalgebra

$$\begin{tikzcd} \varinjlim \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \arrow[r, "\rho"] \arrow[d, phantom, "\cong" description] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \arrow[d, phantom, "\cong" description] \\ \varinjlim \mathrm{Map}_{\mathrm{CAlg}^\heartsuit_{\pi_0 A}}(\pi_0 B, C_\alpha) \arrow[r, "\bar\rho"] & \mathrm{Map}_{\mathrm{CAlg}^\heartsuit_{\pi_0 A}}(\pi_0 B, C) \end{tikzcd}$$

$\uparrow$ bijection

Remark 4.1.1.6 $\{f_\alpha : A_\alpha \to B_\alpha\}$ finite collection

$f = \prod_\alpha f_\alpha$

?

$\rightsquigarrow$ f is FG_n (resp. aofp, lofp)

$\Longleftrightarrow$ $\forall \alpha$ f_α : —— " ————

Remark 4.1.1.7 $B \to B'$ in $CAlg_A^{cn}$: surj on π_n
isom on $\pi_{<n}$

Then $B : FG_n \Rightarrow B' : FG_n$

$\{C_\alpha \in CAlg_A\}$: filtered diagram, n-truncated

$\downarrow$ colim injective on π_n

C

$$\begin{tikzcd}
\mathrm{colim}\, Map_A(B', C_\alpha) \arrow[r] \arrow[d] & Map_A(B', C) \arrow[d] \\
\mathrm{colim}\, Map_A(B, C_\alpha) \arrow[r, "\sim"] & Map_A(B, C)
\end{tikzcd}$$

$$\begin{tikzcd}
Map_A(\tau_{\le n} B', C_\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & Map_A(\tau_{\le n} B', C) \arrow[d] \\
Map_A(\tau_{\le n} B, C_\alpha) \arrow[r] & Map_A(\tau_{\le n} B, C)
\end{tikzcd}$$

??

$\longrightarrow \tau_{\le n} B \to \tau_{\le n} B'$

Remark 4.1.1.8/9

- $A \to B$: FG_{n+1} $\Rightarrow$ $A \to \tau_{\le n} B$: FG_{n+1}
- $A \to B$: FG_{n+1} and n-truncated $\Longleftrightarrow$ $B \in (\tau_{\le n} CAlg_A^{cn})^\omega$

($\Leftarrow$) $\tau_{\le n} CAlg_A^{cn} \rightleftarrows CAlg_A^{cn}$ cptly gen

$\exists B'$: cpt s.t. B : is a retract of $\tau_{\le n} B'$

($\Rightarrow$) $FG_{n+1} \Rightarrow B \simeq \tau_{\le n} B \in (\tau_{\le n} CAlg_A^{cn})^\omega$

4.1.2 Differential Characterization

HA 7.4.3.18 $f : A \to B$ in CAlg^{cn} is almost of fp
locally of fp

iff (1) $\pi_0 B$: fin. pres / $\pi_0 A$

refinement (2) $L_{B/A} \in \mathrm{Mod}_B$ is almost perfect
perfect

Prop 4.1.2.1 $n > 0$. f is of finite generation to order n

iff (1) $\pi_0 B$: fin pres / $\pi_0 A$

(2) $L_{B/A} \in \mathrm{Mod}_B$: perfect to order n

Review : Cotangent complex

$\exists$ Adjunction

(absolute) cotangent complex

$$\begin{array}{ccccc} & (A, L_A) & \longleftarrow & A & \\ & \cap & & \cap & \\ \mathrm{Mod}(\mathrm{Sp}) = & \mathrm{Mod} & \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} & \mathrm{CAlg} & = \mathrm{CAlg}(\mathrm{Sp}) \\ & \cup & & \cup & \\ & (B, M) & \longmapsto & B \oplus M & \end{array}$$

: trivial square-zero ext

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_B, M) \simeq \mathrm{Der}(B, M) \simeq \mathrm{Map}_{\mathrm{CAlg}}(B, B \oplus M)$$

Relative version : $A \to B$ in CAlg
(A : connective) $\Rightarrow B \otimes_A L_A \to L_B \to L_{B/A}$ in Mod_B

relative cotangent complex

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, M) \simeq \mathrm{Der}_A(B, M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A//B}}(B, B \oplus M)$$

Another way to define:

$$\begin{array}{ccccc} \mathrm{Mod}(\mathrm{Mod}_A) \simeq & \mathrm{Mod}|_{\mathrm{CAlg}_A} & \underset{\longrightarrow}{\overset{L_{(-)/B}}{\overset{\longleftarrow}{\perp}}} & \mathrm{CAlg}_A \simeq \mathrm{CAlg}(\mathrm{Mod}_A) \\ & \cup & & \cup \\ & (C, M) & \longmapsto & C \oplus M \end{array}$$

Tangent Correspondence $\mathcal{M}^T(\mathrm{CAlg}_A) \xrightarrow{p} \Delta^1$:

Cart & cocart fib classifying this adjunction

i.e. $p^{-1}(0) = \mathrm{CAlg}_A \ni B$

$p^{-1}(1) = \mathrm{Mod}|_{\mathrm{CAlg}_A} \ni (C, M)$

morphism $B \longrightarrow (C, M)$ in $\mathcal{M}^T$

$(B, L_{B/A}) \to (C, M)$ $\quad$ $B \to C$ in CAlg_A + $C \otimes_B L_{B/A} \longrightarrow M$ in Mod_C

$B \longrightarrow C \oplus M$ in CAlg_A

<u>Def</u>

$\left(\begin{smallmatrix}\mathrm{fib}\,\phi \\ \shortparallel \\ M\end{smallmatrix} \longrightarrow\right) \tilde{C} \xrightarrow{\phi} C$ in CAlg_A is a square-zero extension

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}} \exists \eta : L_{C/A} \longrightarrow M[1]$ derivation

s.t.
$$\begin{tikzcd} \tilde{C} \arrow[r, "\phi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "\eta"] \\ (C, 0) \arrow[r, "0"'] & (C, M[1]) \end{tikzcd}$$
in $\mathcal{M}^T$

$$\left(\Longleftrightarrow \begin{tikzcd} \tilde{C} \arrow[r, "\phi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "d_\eta"] \\ C \arrow[r, "d_0"'] & C \oplus M[1] \end{tikzcd} \text{ in } \mathrm{CAlg}_A \right)$$

<u>HA 7.4.1.28</u> $B \in \mathrm{CAlg}_A^{cn}$ $\rightsquigarrow$ Postnikov tower is a sequence of square-zero extensions

η can be chosen functorially (by 7.4.1.26)

Prop 4.1.2.1 $n > 0$. f is of finite generation to order n

iff (1) $\pi_0 B$: fin pres / $\pi_0 A$

(2) $L_{B/A} \in \mathrm{Mod}_B$: perfect to order n

Proof ($\Rightarrow$) (1) ✓

(2) Prop 2.7.0.4 : M is perf to ord n

$\Longleftrightarrow$ (i) M : k-connective for $\exists k$

(ii) $\forall \{N_\alpha\}$ filtered diagram of injective maps in $\mathrm{LMod}_A^{\heartsuit}$

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{LMod}_A}(M, \Sigma^n N_\alpha) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{LMod}_A}(M, \varinjlim_\alpha \Sigma^n N_\alpha)$$

(i) HA 7.4.3.2 $\mathrm{cofib}(f)$ is n-connective $(n \geq 0)$

$\Rightarrow$ $L_{B/A}$ is n-connective.

(ii)

$$\begin{tikzcd}
\operatorname{colim}_\alpha \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^n N_\alpha) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^n \operatorname{colim}_\alpha N_\alpha) \arrow[d, "\simeq"] \\
\operatorname{colim}_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, B \oplus \Sigma^n N_\alpha) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, B \oplus \Sigma^n \operatorname{colim}_\alpha N_\alpha) \arrow[d] \\
\operatorname{colim}_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq n} B \oplus \Sigma^n N_\alpha) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq n} B \oplus \Sigma^n \operatorname{colim}_\alpha N_\alpha)
\end{tikzcd}$$

Pullback

filt colim of $\mathrm{Map}\left(B, \begin{tikzcd} B \oplus \Sigma^n N_\alpha \arrow[r] \arrow[d] & B \oplus \Sigma^n \operatorname{colim} N_\alpha \arrow[d] \\ \tau_{\leq n} B \oplus \Sigma^n N_\alpha \arrow[r] & \tau_{\leq n} B \oplus \Sigma^n \operatorname{colim} N_\alpha \end{tikzcd}\right)$

$\lrcorner$ of mod $\Rightarrow$ $\lrcorner$ of calg

B : fin gen to ord n

$\Rightarrow$ the bottom map is an equivalence

($\Leftarrow$) Take a filtered diagram $\{C_\alpha \in \tau_{\le n}\mathrm{CAlg}_A^{cn}\}$ with transition maps injective on π_n, $C := \mathrm{colim}\, C_\alpha$

$$\begin{tikzcd}
C_\alpha = \tau_{\le n} C_\alpha \arrow[r] \arrow[d] & \tau_{\le n} C = C \arrow[d] \\
\tau_{\le n-1} C_\alpha \arrow[r] \arrow[d, dashed] & \tau_{\le n-1} C \arrow[d, dashed] \\
\pi_0 C_\alpha = \tau_{\le 0} C_\alpha \arrow[r] & \tau_{\le 0} C = \pi_0 C
\end{tikzcd}$$

We'll prove

$\rightsquigarrow$ $\theta_i : \mathrm{colim}_\alpha \mathrm{Map}_A(B, \tau_{\le i} C_\alpha) \xrightarrow{\sim} \mathrm{Map}_A(B, \tau_{\le i} C)$

by induction on $0 \le i \le n$

$\underline{i=0}$ follows from (1)

$\underline{(i-1) \to i}$ $\Sigma^i \pi_i C \to \tau_{\le i} C \to \tau_{\le i-1} C$ is a square-zero ext,

i.e. $\exists \eta : \tau_{\le i-1} C \to (\tau_{\le i-1} C, \Sigma^{i+1} \pi_i C)$ in $\mathcal{M}^T$

s.t.

$$\begin{tikzcd}
\tau_{\le i} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tau_{\le i-1} C \arrow[d, "\exists \eta"] \\
(\tau_{\le i-1} C, 0) \arrow[r] & (\tau_{\le i-1} C, \Sigma^{i+1} \pi_i C)
\end{tikzcd} \quad \text{in } \mathcal{M}^T$$

$\rightsquigarrow$ fiber seq

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i} C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i-1} C) \arrow[d] \\
* \simeq \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, 0) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^{i+1} \pi_i C)
\end{tikzcd}$$

Similarly for C_α

$\rightsquigarrow$

$$\begin{tikzcd}
\mathrm{colim}_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i} C_\alpha) \arrow[r, "\theta_i"] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i} C) \arrow[d] \\
\mathrm{colim}_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i-1} C_\alpha) \arrow[r, "\theta_{i-1}", "\simeq_{IH}"'] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\le i-1} C) \arrow[d] \\
\mathrm{colim}_\alpha \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^{i+1} \pi_i C_\alpha) \arrow[r, "\phi"'] & \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, \Sigma^{i+1} \pi_i C)
\end{tikzcd}$$

Let $D_\alpha := \mathrm{fib}(\Sigma^{n+1}\pi_n C_\alpha \longrightarrow \Sigma^{n+1}\pi_n C)$: n-truncated

$$\mathrm{fib}\Big(\underset{\substack{\cap\\ \mathrm{Sp}_{\le -1}}}{\mathrm{colim}\,\mathrm{Hom}_B(L_{B/A}, D_\alpha)} \longrightarrow \mathrm{Hom}_B(L_{B/A}, \underset{\substack{\wr\wr\\ 0}}{\mathrm{colim}\, D_\alpha})\Big) \in \mathrm{Sp}_{\le -1}$$

by (2)

fib seq

$$\underset{\substack{\cap\\ \mathrm{Sp}_{\le -1}}}{\mathrm{colim}_\alpha\,\mathrm{Hom}_B(L_{B/A}, D_\alpha)} \longrightarrow \mathrm{colim}_\alpha\,\mathrm{Hom}_B(L_{B/A}, \Sigma^{n+1}\pi_n C_\alpha) \longrightarrow \mathrm{Hom}_B(L_{B/A}, \Sigma^{n+1}\pi_n C)$$

$\rightsquigarrow \phi$ is (-1)-truncated

$\rightsquigarrow \theta_n$: ho. eq. $\square$

4.1.3 Persistence of Finite Generation $\quad$ $n \geq 0$

Prop 4.1.3.1 Let
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$
in CAlg^{cn}, f is of FG to order n

Then g is of FG to order n $\Leftrightarrow$ $g\circ f$ is of FG to order n

Proof $\underline{n=0}$ $\quad \pi_0 B \,/\, \pi_0 A$: fin pres (fin type)

$\pi_0 C \,/\, \pi_0 B$: fin pres. (fin type) $\Leftrightarrow$ $\pi_0 C \,/\, \pi_0 A$: fin pres (fin type)

$\underline{n>0}$ $\quad \underbrace{C \otimes_B L_{B/A}}_{\text{perfect to order } n} \to L_{C/A} \to L_{C/B}$

$L_{C/A}$ perf to ord n $\Leftrightarrow$ $L_{C/B}$ perf to ord n

$\boxed{//}$

Prop 4.1.3.2
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$
$B' \simeq A' \otimes_A B$

in CAlg^{cn}. $\rightsquigarrow$ f is of fg to order n $\Downarrow$ f' is of fg. to order n

Proof

$\underline{n=0}$ $\quad \pi_0(A' \otimes_A B) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 A', \pi_0 B)$ $\;/\; \pi_0 A'$: fin.gen (fin pres)

if $\pi_0 B \,/\, \pi_0 A$ fin.gen (fin pres)

$\underline{n>0}$ $\quad L_{B'/A'} \simeq B' \otimes_B \underbrace{L_{B/A}}_{\text{perf to order } n}$: perf to ord n

Cor 4.1.3.3 $R \in \mathrm{CAlg}^{cn}$,
$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ & A' \arrow[r] & B' \end{tikzcd}$$
in CAlg^{cn}_R

Then if A, A', B are of fg to ord n $/R$

$\Longrightarrow$ B' is of fg to order n $/R$

Prop 4.1.3.4 $A \xrightarrow{f} B$ in CAlg^{cn}, $n \geq 0$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ \pi_0 A \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

Then f is of FG to ord n iff f_0 is of FG to ord n

Cor 4.1.3.5 : f is aofp $\iff$ f_0 is aofp

Proof $\pi_0 B_0 \simeq \pi_0(\pi_0 A \otimes_A B) \simeq \pi_0 B$

$\rightsquigarrow$ $f : \pi_0 B_0 / \pi_0 A$ fin gen (resp. fin pres)

iff $f_0 : \pi_0 B / \pi_0 A$ — " — (—— " —)

$\uparrow$ $n=0$ $\quad$ $\uparrow$ $n>0$

$n>0$ $L_{B/A}$ perf to ord n as a B-mod

$\iff \pi_0 B \otimes_B L_{B/A}$ perf to ord n as a $\pi_0 B$-mod

$\quad \wr\!\wr$

$\pi_0 B_0 \otimes_{B_0} L_{B_0/\pi_0 A}$

$\iff L_{B_0/\pi_0 A}$ perf to ord n as a B_0-mod

$$\begin{tikzcd} B \arrow[d] \\ \pi_0 B \simeq \pi_0 B_0 \\ B_0 \arrow[u] \end{tikzcd}$$

Prop. 2.7.3.2

$\blacksquare$

4.1.4 Local Nature of Finite Generation

HA 7.5.4.4 $f : A \to B$ in CAlg^{cn}, étale $\Rightarrow$ f: locally of fin pres (i.e. $B \in (\mathrm{CAlg}_A)^{\omega}$)

(or SAG B.1.1.3)

Prop 4.1.4.1 Let $A \xrightarrow{f} B$ in CAlg^{cn}. Then :

(1) f is of FG to ord n $\Rightarrow$ $\forall B \xrightarrow[g]{} B'$ étale, $g \circ f$ is of FG to order n

(2) $\{B \to B_i\}$ finite étale cover, $A \to B \to B_i$ is (FG_n) (i.e. $B \to \prod_{i=1}^{k} B_i$ faithfully flat) $\Rightarrow$ $A \to B$ is (FG_n)

Proof (1) ✓

(3) in the same situation
$A \to B \to B_i$: lofp $\Longrightarrow$ $A \to B$ lofp

(2)
Stacks project 35.11 (02KJ)

35.11.1 $R \to A \to B$ ring maps, $R \to B$ is of fp (resp. fin type)
$A \to B$ faithfully flat, of fin pres
$\Longrightarrow$ $R \to A$ is of fp (resp. fin type)

$$\Longrightarrow \quad \underbrace{\pi_0 A \to \pi_0 B \to \pi_0 B'}_{\pi_0 f \qquad \text{faithfully flat, étale}} \qquad (B' := \prod_{i=1}^{k} B_i)$$

fp or ft $\Rightarrow$ $\pi_0 f$: fp or ft
$n>0$ $\quad$ $n=0$

It remains to show $L_{B/A}$: perfect to order n

$$B' \otimes_B L_{B/A} \xrightarrow{\simeq} L_{B'/A} \longrightarrow L_{B'/B}$$

perfect to ord n $\quad$ $L_{B'/B} = 0$ étale

$B_\alpha \otimes_B L_{B/A} \to L_{B_\alpha/A}$

$\underset{2.7.3.1}{\Longrightarrow}$ $L_{B/A}$: perfect to ord n

(3) from (2), B is almost of finite presentation

It suffices to show $L_{B/A}$: perfect B-mod

perfect $\underset{\text{HA 7.2.4.23}}{\Longleftrightarrow}$ almost perfect & finite Tor-amplitude

flat local $\quad$ §2.8

: flat local

$B_\alpha \otimes_B L_{B/A} \simeq L_{B_\alpha/B}$: perfect

∎

Prop 4.1.4.3 $f: A \to B$ in CAlg^{cn}, Suppose $\exists \{A \to A_i\}_{1 \le i \le k}$ s.t.

(1) $A \to \prod_i A_i$ faithfully flat

(2) $\forall i$. $A_i \xrightarrow[f_i]{} A_i \otimes_A B$ is FG_n (resp. aofp, lofp)

Then f is FG_n (resp. aofp, lofp)

Proof On π_0 : loft, lofp : fpqc local on the target

(stacks project 02KY, 02KZ affine : 00QP, 00QQ)

$\rightsquigarrow$ It suffices to show $L_{B/A}$ is perf. to ord n

(resp. almost perfect, perfect)

by (2) & (4.1.1.6), $\prod A_i \to \prod A_i \otimes_A B$ is FG_n (aofp, lofp)

$\prod A_i = A'$, $\prod A_i \otimes_A B = B'$

$B' \otimes_B L_{B/A} \simeq L_{B'/A'}$: perf to ord n (aperf, perf)

B' : faithfully flat $\overset{2.8.4.2}{\Longrightarrow}$ $L_{B/A}$: —— " ——

$\square$