"Higher alg K of Derived categories & schemes" [TT90]

"Algebraic K-theory and Etale Cohomology" by Thomason

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Outline

1. Overview
2. A Crash Course in Topos and Etale Cohomology
3. Spectra-Valued Hypercohomology and Descent Spectral sequences
4. Inverting the Bott element
5. Proof Sketch of the Main Theorem
6. Applications

K-theory of Schemes ⤳ scheme analogue to top K

$\mathcal{E}$: exact or Waldhausen category $\rightsquigarrow K(\mathcal{E}) \in \mathbf{Sp}_{(\geq 0)}$ by Quillen Q-construction $\Omega BQ\mathcal{E}$ or Waldhausen $\mathcal{S}_\bullet$-construction $\Omega|\mathcal{S}_\bullet\mathcal{E}|$.

Definition

For a (qcqs) scheme X,

loc free $\mathcal{O}_X$-mod of fin rank

$\sim$ [TT90]

- $\mathbf{Coh}_X = \{$coherent $\mathcal{O}_X$-module$\}$: exact $\rightsquigarrow \underline{G(X)}$, K'
- $\mathbf{VB}_X = \{$(algebraic) vector bundles on $X\}$: exact $\rightsquigarrow K^{\mathrm{naive}}(X)$
- $\mathbf{Perf}_X = \{$perfect complexes of $\mathcal{O}_X$-modules$\}$: Waldhausen $\rightsquigarrow K(X)$

- A *perfect complex* is a complex of $\mathcal{O}_X$-module locally q-iso to a bounded complex of vector bundles. This is (roughly) characterized by compactness or by dualizability in the derived category of X.
- K and K^{naive} are contravariant in X, so is G for noetherian X.
- K and K^{naive} comes equipped with E_∞-ring structure by $\otimes$.
- $\exists$Nonconnective refinement K^B with connective cover $K(X) \to K^B(X)$ (for regular noetherian separated scheme negative K's vanish).

Bass

- $K^{\text{naive}}(X) \cong K(X)$ if X has ample family of line bundles, but in general $K(X)$ is better-behaved.
- $G(X) \cong K^{\text{naive}}(X) \cong K(X)$ if X is regular noetherian separated.
- G-theory is "less sensitive" (e.g. homotopy invariance, nilinvariance).

So at least the definition of $K(X)$ is analogous to the topological K-theory (e.g. both K_0 recovers the grothendieck group of "vector bundles").

Question

How far does it behaves similarly to topological K? e.g. does it behaves like a "cohomology theory," satisfing local-to-global principles such as Mayer-Vietoris, AHSS, etc...?

The answer is, **Yes** for Zariski (or Nisnevich) topology, and **No** for étale topology (which is more analogous to usual topology). Our main result says that this problem disappears if we invert a "Bott element" β.

Main Theorem

Let X be a scheme (noetherian, Krull dim $< \infty$), $\ell^v \in \mathbb{Z}$ a prime power invertible in $\mathcal{O}_X$ (If $\ell = 2$ also suppose $\sqrt{-1} \in \mathcal{O}_X$). Assume $[\exists d \in \mathbb{Z}, \forall x \in X, \mathrm{cd}_\ell^{\text{ét}}(k(x)) \leq d]$ and $[\forall x \in X, \exists$ a "Tate-Tsen filtration" for $\overline{k(x)}/k(x)]$. For $F = K/\ell^v(-)[\beta^{-1}] : X_{\text{ét}} \to \mathbf{Sp}$ ("Bott-inverted" mod-ℓ^v K-theory),

? ? $\xrightarrow{\pi_t}$ Ab

1. F satisfies étale hypercohomology descent;

   ? ?

$$F(X) \xrightarrow{\sim} \mathbb{H}^\bullet(X_{\text{ét}}; F),$$

$\mu_{\ell^v} \to \mathbb{G}_m \xrightarrow{\ell^v} \mathbb{G}_m$

2. The sheafification $\tilde{\pi}_t F$ of $\pi_t \circ F : X_{\text{ét}} \to \text{Ab}$ is given by

$$\tilde{\pi}_t F \cong \underbrace{\mathbb{Z}/\ell^v(t/2)}_{\text{Tate twist}} := \begin{cases} \mu_{\ell^v}^{\otimes t/2} & t : \text{even}, \\ 0 & t : \text{odd}. \end{cases}$$

$\mu_{\ell^v}(R)$ $= \{\ell^v\text{-th roots of unity in } R\}$

3. There is a strongly convergent SS (analogous to AHSS)

$$E_2^{s,t} = H^s_{\text{ét}}(X; \mathbb{Z}/\ell^v(t/2)) \Rightarrow K/\ell^v_{-s-t}(X)[\beta^{-1}] := \pi_{-s-t} F(X).$$

cf.) $H^s(X; KU^*) \Rightarrow KU^*(X)$

$KU^* = KU_{-*}$

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Sheaves on topological spaces

Let X be a topological space.

Definition

- A *presheaf* on X is a contravariant functor from the category of open sets and inclusions to **Set**; denote $\mathrm{Psh}(X) := \mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set})$.
- A presheaf F is a *sheaf* if for any $U \in \mathbf{Open}(X)$ and its open cover $\mathcal{U} = \{U_i \hookrightarrow U\}_{i \in I}$, the following diagram is a equalizer:

$$FU \to \prod_i FU_i \rightrightarrows \prod_{i,j} F(U_i \times_U U_j).$$

$$\cong \lim \qquad \rightrightarrows \prod F(U_i \times U_j \times U_k)$$

Let $\mathrm{Sh}(X) \subset \mathrm{Psh}(X)$ denote the full subcategory of sheaves.

Grothendieck Topology

Here the fact that a cover consists of inclusions is inessential:

Definition

Let $\mathcal{C}$ be a (small) category with finite limits.

- A *Grothendieck topology* τ on $\mathcal{C}$ consists of a collection of *covers* $\mathrm{Cov}(U) \ni \mathcal{U} = \{U_i \to U\}_{i \in I}$ for each $U \in \mathcal{C}$ satisfying axioms (an isom is a cover, a cover of a cover is a cover, the pullback of a cover along a map is a cover). A pair $(\mathcal{C}, \tau)$ is called a *site*.
- *Presheaves* and *sheaves* on $(\mathcal{C}, \tau)$ are defined in the same way as before. We have categories $\mathrm{Sh}(\mathcal{C}, \tau) \subset \mathrm{Psh}(\mathcal{C})$.
- A category equivalent to $\mathrm{Sh}(\mathcal{C}, \tau)$ for some site $(\mathcal{C}, \tau)$ is called a *(Grothendieck) topos*.

Geometric Morphisms and Points

$$f: \mathcal{E} \underset{f_*}{\overset{f^*:\text{lex}}{\rightleftarrows}} \mathcal{F}$$

- $\mathrm{Psh}(\mathcal{C}) = \mathrm{Sh}(\mathcal{C}, \mathrm{triv})$ where covers in triv are only isomorphisms.
- $i : \mathrm{Sh}(\mathcal{C}, \tau) \hookrightarrow \mathrm{Psh}(\mathcal{C})$ admits a left adj $\mathbf{a} : F \mapsto \tilde{F}$. This is lex[1] $\rightsquigarrow$ preserves (abelian) group objects, ring objects, etc. (a: lex)
- An adjunction between topoi with lex left adjoint (e.g. $\mathbf{a} \dashv i$) is called a *geometric morphism*. Denote the (2-)category of topoi by **Topos**.
- $\mathbf{Top}^{\mathrm{sober}} \hookrightarrow \mathbf{Topos}$ via $(X \xrightarrow{f} Y) \mapsto (f_* : \mathrm{Sh}(X) \leftrightarrows \mathrm{Sh}(Y) : f^*)$.
- A point $* \xrightarrow{x} X$ of $X \in \mathbf{Top}^{\mathrm{sober}}$ corresponds to the "stalk $\dashv$ skyscraper" geom mor $x_* : \mathbf{Set} = \mathrm{Sh}(*) \leftrightarrows \mathrm{Sh}(X) : x^*$. In general, we define a *point* p of a topos $\mathcal{E}$ by a geometric morphism $p_* : \mathbf{Set} \leftrightarrows \mathcal{E} : p^*$. ($\mathcal{F}_x \leftarrow \mathcal{F}$)
- $\mathcal{E}$ is said to have *enough points* if $f : X \to Y$ is isom iff $p^* f$ is isom. $\forall p$: points

[1] lex = left exact = finite limit preserving

Topology on Schemes

Let X be a scheme.

- The *Zariski site* $X_{\mathrm{Zar}} \subset \mathbf{Sch}_X$ is the full subcategory of open inclusions $U \hookrightarrow X$ with open covers in Zariski topology.
- The *étale site* $X_{\text{ét}} \subset \mathbf{Sch}_X$ is the full subcat[2] of étale maps $U \to X$ locally of finite presentation. $\{U_i \to U\}_{i \in I}$ is a cover if $\coprod_i U_i \to U$ is surjective (as sets).
  faithfully flat
- The *Nisnevich site* X_{Nis} again consists of schemes étale over X, but covers are more restrictive.

inclusions of sites are "continuous," so induce geometric morphisms

$$\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\mathrm{Nis}}) \to \mathrm{Sh}(X_{\mathrm{Zar}}).$$

These sites have enough points, which bijectively correspond to points of the underlying space. Stalks are $\mathcal{O}_{X,x}, \mathcal{O}^{\mathrm{h}}_{X,x}, \mathcal{O}^{\mathrm{sh}}_{X,x}$.
Any representable functor $\mathrm{Hom}_X(-, Z)$ for $Z \in \mathbf{Sch}_X$ is an étale sheaf.
fppf descent

[2] morphisms are automatically étale.

Abelian Sheaf Cohomology

Let X_τ be a site (in this notation the terminal object is often denoted by X). $\mathrm{Sh}_{\mathrm{Ab}}(X_\tau) \cong \mathrm{Ab}(\mathrm{Sh}(X_\tau))$ is an abelian category with enough injectives. The global section $\Gamma : \mathrm{Sh}_{\mathrm{Ab}}(X_\tau) \to \mathrm{Ab}$ is lex.

Definition

For $F \in \mathrm{Sh}_{\mathrm{Ab}}(X_\tau)$, define $H^*(X_\tau; F) := R^*\Gamma(F)$.

$H^*(X_\tau; F)$ is also denoted as $H^*_\tau(X; F)$, e.g. $H^*_{\text{ét}}(X; F)$.

Some facts about étale cohomology

Unlike Zariski cohomology, étale cohomology has the following properties:

- If X: variety over $\mathbb{C}$, then $H^*(X_{\text{ét}}; A) \cong H^*_{\text{Sing}}(X^{\text{an}}; A)$ $A = \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}_\ell,$
- Analogous results to singular cohomology, such as Künneth theorem, fundamental class, Poincaré duality, Lefschetz fixed point formula, etc.
- étale cohomology often can be computed by Čech cohomology;
  X: quasiprojective over a noetherian ring, F: additive presheaf of abelian groups, then $\check{H}^p_{\text{ét}}(X; F) \xrightarrow{\cong} H^p_{\text{ét}}(X; \tilde{F})$.
- $H^*_{\text{ét}}(\operatorname{Spec} k; F) \cong H^*(\operatorname{Gal}(\bar{k}/k); \lim_{k'} F(k'))$ for a field k, where k' runs finite subextensions of a separable closure $k \subset k' \subset \bar{k}$, RHS is the Galois cohomology.

$$\frac{k'/k \quad \text{fin sep}}{\{\operatorname{Spec} k' \longrightarrow \operatorname{Spec} k\}}$$

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Čech Hypercohomology

Coh = derived Γ "=" holim $(-)$
Γ for sh lim(Čech nerve)

Let $F \in \mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$. For a cover $\mathcal{U} = \{U_i \to U\}_{i \in I}$ define the *Čech nerve* $F_{\mathcal{U}}^\bullet \in \mathbf{Sp}^\Delta$ by

$$F_{\mathcal{U}}^n = \prod_{i_0, \dots, i_n \in I^{[n]}} F(U_{i_0} \times_U \cdots \times_U U_{i_n}) \leftarrow FU$$

(with δ_i, σ_i maps induced by projections/diagonals of $\times_U$-factors).

Definition

The *Čech hypercohomology* is $\check{\mathbb{H}}^\bullet(\mathcal{U}; F) := \mathrm{holim}_\Delta F_{\mathcal{U}}^\bullet$.

- $\exists$ natural augmentation $FU \to \check{\mathbb{H}}^\bullet(\mathcal{U}; F)$.
- $\check{\mathbb{H}}^\bullet(-; F)$ is functorial; $\mathrm{Cov}(X) \to \mathrm{ho}(\mathbf{Sp})$.
- $\mathcal{A} = \{\mathcal{U}_\alpha \in \mathrm{Cov}(X)\}$: filtered system
  $\rightsquigarrow \check{\mathbb{H}}^\bullet(\mathcal{A}; F) := \mathrm{colim}\, \check{\mathbb{H}}^\bullet(\mathcal{U}_\alpha; F)$.
- $\mathcal{A}$ is cofinal in $\mathrm{Cov}(X) \rightsquigarrow \check{\mathbb{H}}^\bullet(X_\tau; F) := \check{\mathbb{H}}^\bullet(\mathcal{A}; F)$.

Sheaf Hypercohomology

Suppose X_τ has enough points with set of points $\mathcal{P}$.

- Can still define "stalk $\dashv$ skyscraper" adjunction $p_* : \mathbf{Sp} \leftrightarrows \mathrm{Psh}_{\mathbf{Sp}}(X_\tau) : p^*$ for $p \in \mathcal{P}$ by the same formula (Kan ext/(co)ends/explicitly) as for **Set**-valued preheaves.
- Define the monad T on $\mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$ by $TF(U) = \prod_{p \in \mathcal{P}} (p_* p^* F)(U)$.
- The *Godement resolution* of $F \in \mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$ is an augmented cosimplicial spectrum $F \to T^\bullet F$ given by "monad resolution". Note that this only depends on stalks and satisfies $\pi_q(T^\bullet F) = T^\bullet(\pi_q F) = T^\bullet(\tilde{\pi}_q F)$.

$$\mathrm{id} \xrightarrow{\eta} T$$

$$T^n F \rightleftarrows T^{n+1} F$$

$$T \xleftarrow{\mu} T^2$$

Definition

The sheaf hypercohomology is $\mathbb{H}^\bullet(X_\tau; F) := \operatorname{holim}_\Delta T^\bullet F$.

Spectral Sequence from a Cosimplicial Spectrum

- For a Reedy fibrant cosimplicial spectrum $F^\bullet : \Delta \to \mathbf{Sp}$, we have a tower of fibrations of spectra

$$\mathrm{Tot}\, F := \int_{n \in \Delta} (F^n)^{\Delta^n} \to \cdots \xrightarrow{\phi_3} \mathrm{Tot}^2 F \xrightarrow{\phi_2} \mathrm{Tot}^1 F \xrightarrow{\phi_1} \mathrm{Tot}^0 F \to *,$$

where $\mathrm{Tot}^n F := \mathrm{Tot}(\mathrm{cosk}^n F)$.

- Reedy fibrancy $\rightsquigarrow$ $\mathrm{holim}\, \mathrm{Tot}^n F \simeq \mathrm{Tot}\, F \simeq \mathrm{holim}_\Delta F$.
- LES of cofiber sequences $[\mathrm{fib}\, \phi_n \to \mathrm{Tot}^n F \xrightarrow{\phi_n} \mathrm{Tot}^{n-1} F]$ gives rise to the folloing exact couple:

$$\bigoplus \pi_{s+t}(\mathrm{Tot}^t F) \xrightarrow{\phi} \bigoplus \pi_{s+t}(\mathrm{Tot}^t F) \to \bigoplus \pi_{s+t}(\mathrm{fib}\, \phi_t).$$

- Half-plane conditionally convergent SS with entering diffenentials

$$E_1^{s,t} = \pi_{s-t}(\mathrm{fib}\, \phi_t) \Rightarrow \lim_n \pi_{s-t}(\mathrm{Tot}^n F).$$

- The target fits into the Milnor sequence

$$0 \to \lim_n{}^1 \pi_{s-t-1}(\mathrm{Tot}^n F) \to \pi_{s-t}(\operatorname*{holim}_\Delta F^\bullet) \to \lim_n \pi_{s-t}(\mathrm{Tot}^n F) \to 0.$$

Hypercohomology Spectral Sequence

Under suitable conditions $\lim^1 = 0$ and the SS is strongly convergent. Applying to cosimplicial spectra $T^\bullet F$ and $F^\bullet_{\mathcal{U}}$ for $F \in \mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$ and a cover $\mathcal{U}$, we can identify the E_1-pages with relevant complexes to get

$$E_2^{s,t} := H^s(X_\tau; \tilde{\pi}_{-t}F) \Rightarrow \pi_{-s-t}\mathbb{H}^\bullet(X_\tau; F), \quad \sim F(X)$$
$$E_2^{s,t} := \check{H}^s(\mathcal{U}; \pi_{-t}F) \Rightarrow \pi_{-s-t}\check{\mathbb{H}}^\bullet(\mathcal{U}; F),$$

(similar SS for $\check{\mathbb{H}}^\bullet(\mathcal{A}; F)$ and $\check{\mathbb{H}}^\bullet(X_\tau; F)$). Here $\tilde{\pi}_{-t}F$ denotes the sheafification of $X_\tau \ni U \mapsto \pi_{-t}F(U) \in \mathrm{Ab}$.

$\left(F : X_\tau \xrightarrow{F} Sp \xrightarrow{\pi_{-t}} Ab\right)^\sim$

e.g. $X = BG$ ⇝ ho fixed pt SS

$X = \mathcal{M}_{ell} \leftarrow \mathcal{O}$ ⇒ TMF_*

Cohomological Descent

To get a AH type SS from this, we need to understand

1. Is the augmentation $F(X) \to \mathbb{H}^\bullet(X_\tau, F)$ a weak equivalence?
2. Can we identify the "sheaf of coefficients" $\tilde{\pi}_{-t}F$ with something computable?

First question can be dealt with by an organized way:

Definition

- $F \in \mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$ satisfies *(sheaf) cohomological descent* if $FU \xrightarrow{\sim} \mathbb{H}^\bullet((X_\tau)_{/U}; F)$ for any U.
- $F \in \mathrm{Psh}_{\mathbf{Sp}}(X_\tau)$ satisfies *Čech cohomological descent* for $\mathcal{U} \in \mathrm{Cov}(U)$ (resp. $\mathcal{A}$) if $FU \xrightarrow{\sim} \check{\mathbb{H}}^\bullet(\mathcal{U}; F)$ (resp. $\check{\mathbb{H}}^\bullet(\mathcal{A}; F)$).

Under some conditions (boundedness on $\pi_q F$ or cohomological dimension), sheaf cohomological descent implies Čech cohomological descent for all filtered system of covers (use spectra version of Cartan-Leray SS).

What's Next?

Does K satisfies descent for topologies on schemes?

- **Yes** for Zariski and Nisnevich.
- **No** for étale, but **Yes** if we invert "Bott element."

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The Bott element in $K/(\ell^v)$

Let $K/\ell^v(X)$ be a cofiber of $K(X) \xrightarrow{\ell^v} K(X)$. This cofiber sequence yields a long exact sequence

$$\cdots \to K_n(X) \to K_n(X) \to (K/\ell^v)_n(X) \xrightarrow{\partial} K_{n-1}(X) \to \cdots .$$

$\beta \in (K/\ell^v)_2(X) \to K_1(R) \xrightarrow{\ell^v} K_1(R)$

1

- First suppose a ring R has a primitive ℓ^v-th root of unity. In this case $K_1(R)$ also has a primitive root by $R^\times \hookrightarrow K_1(R)$. Choose an element $\beta \in (K/\ell^v)_2(R)$ such that $\partial\beta$ is a primitive l^v-th root of unity.
- Though $\mathbb{Z}$ has no primitive ℓ^v-th root of unity, one can use Bockstein SS to argue that $\beta^{(\ell-1)\ell^{n-1}}$ "lives in" $K/\ell^v(\mathbb{Z})$.
- Using $K(\mathbb{Z})$-algebra structure on $K(X)$ for any X we have a (some power of) Bott element x in $K/\ell^v(X)$.

Thus we can invert x to get $K/\ell^v(X)[\beta^{-1}]$.

This is known to be equivalent to either of the following;

- For odd ℓ, there is an Adams self-map $v_1 : \Sigma^{2\ell-2}(\mathbb{S}/\ell) \to \mathbb{S}/\ell$ which is a KU_*-equivalence (for $\ell = 2$ similar map $\Sigma^8(\mathbb{S}/2) \to \mathbb{S}/2$). Define

$$T(1) := \operatorname{hocolim}\left(\mathbb{S}/\ell \xrightarrow{v_1} \Sigma^{-(2\ell-2)}\mathbb{S}/\ell \xrightarrow{v_1} \cdots\right)$$

  and $K(X) \wedge T(1)$.
- Bousfield localize $K/\ell^v(X)$ by $K(1)$ or KU.

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Main Theorem (repeated)

Let X be a scheme (noetherian, Krull dim $< \infty$), $\ell^v \in \mathbb{Z}$ a prime power invertible in $\mathcal{O}_X$ (If $\ell = 2$ also suppose $\sqrt{-1} \in \mathcal{O}_X$). Assume $[\exists d \in \mathbb{Z}, \forall x \in X, \mathrm{cd}_\ell^{\text{ét}}(k(x)) \leq d]$ and $[\forall x \in X, \exists$ a "Tate-Tsen filtration" for $\overline{k(x)}/k(x)]$. For $F = K/\ell^v(-)[\beta^{-1}] : X_{\text{ét}} \to \mathbf{Sp}$,

1. F satisfies étale hypercohomology descent;
$$F(X) \xrightarrow{\simeq} \mathbb{H}^\bullet(X_{\text{ét}}; F),$$
2. The sheafification $\tilde{\pi}_t F$ of $\pi_t \circ F : X_{\text{ét}} \to \mathrm{Ab}$ is given by
$$\tilde{\pi}_t F \cong \mathbb{Z}/\ell^v(t/2),$$
3. There is a strongly convergent SS (analogous to AHSS)
$$E_2^{s,t} = H^s_{\text{ét}}(X; \mathbb{Z}/\ell^v(t/2)) \Rightarrow K/\ell^v_{-s-t}(X)[\beta^{-1}] := \pi_{-s-t}F(X).$$

We saw that the spectral sequence follows from the first two points.

Fundamental Theorems in K-theory

We need the following theorems of K-theory.

1. *localization theorem*: For X: qcqs scheme, U: qc open, $Z = X \setminus U$. the sequence $K^B(X \text{ on } Z) \to K^B(X) \to K^B(U)$ is a fiber sequence. Here the first one is the K-theory of perfect complexes which are acyclic on U. In good cases (such as everything is regular) $K^B(X \text{ on } Z)$ is equivalent to $K(Z)$.
2. *Rigidity theorem*: Let (R, I) be a hensel local ring with $1/\ell \in R$. Then $R \to R/I$ induces an equivalence $K(R/I) \xrightarrow{\sim} K(R)$.
3. *Continuity*: If $X = \lim X_\alpha$ is the limit of inverse system of schemes with affine bonding maps, then $\operatorname{colim}_\alpha K(X_\alpha) \xrightarrow{\sim} K(X)$.

Step-by-Step Reduction to Field Cases

1. For open $U, V \subset X$, consider the following square; it is homotopy cartesian because maps between fibers are w.e (by directly showing that the derived categories of perfect complexes are equivalent). The same is true for F because the construction preserves holim.
2. Zariski descent follows from this *Mayer-Vietoris property*. We have descent SS $E_2^{s,t} = H^s_{\mathrm{Zar}}(X; \tilde{pi}_{-t}F) \Rightarrow \pi_{-s-t}F(X)$.
3. This SS allows us to reduce étale descent for X to étale descent for each $\operatorname{Spec} \mathcal{O}_{X,x}$ because
4. By induction on $N = \dim X$, we may assume $\dim \mathcal{O}_{X,x} = N$. Consider $Z = \operatorname{Spec}(k(x)) \hookrightarrow X \hookleftarrow U = X \setminus Z$, so $\dim Z = 0$, $\dim U = N - 1$.
   $N-1$
   0 By localization long exact sequence and five lemma we see that the augmentation is weak equivalence.
5. So enough to treat 0-dimensional local rings, i.e. Artin local rings. The étale site of an Artin local ring are equivalent to that of its residue field.

One can instead use Nisnevich descent to reduce to henselian local rings, then use Gabber's rigidity theorem to reduce to fields.

Descent for Field Cases

This is hard and technical. Since for spectrum of fields étale cohomology is a Galois cohomology (and the SS is homotopy fixed point SS), we can apply some equivariant techniques. In particular, "wrong-way" direction (transfer) functoriality for finite étale morphisms play an important role.

Identification of Coefficients in E_2 term

from [TT90]
(different from [Thomason '85])

- The inclusion of subgroups generated by β^i induces maps of étale sheaves of abelian groups $\mathbb{Z}/\ell^\nu(t/2) \to \tilde{\pi}_t K/\ell^\nu(-)[\beta^{-1}]$.
- To show this is isomorphism of sheaves, it suffice to show that on stalks. By continuity, stalk at x is given by evaluation at corresponding strict henselian local rings $(\pi_t K/\ell^\nu(-)[\beta^{-1}])_x \cong K/\ell^\nu(\mathcal{O}^{\mathrm{sh}}_{X,x})[\beta^{-1}]$.

$\operatorname{Spec} \mathcal{O}^{sh}_{X,x} = \varprojlim U$

$\left\{ * \xrightarrow{x} U \xrightarrow{\text{ét}} X \right\}$

Étale nbd's of x

- By Rigidity theorem, $K/\ell^\nu(\mathcal{O}^{\mathrm{sh}}_{X,x}) \cong K/\ell^\nu\left(\overline{k(x)}\right)$, so we can reduce it to the case of separably closed fields.
- This follows from Suslin's result: When $\bar{k}$ is a separably closed field of characteristic not ℓ, the choice of β determines a graded ring isomorphism $(K/\ell^\nu)_*(\bar{k}) \cong \mathbb{Z}/\ell^\nu[\beta]$.

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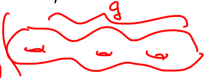
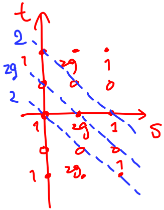
Computation for Curves over separably closed field

Let $\bar{k}$ be a separably closed field of characteristic not ℓ, $\bar{X}$ be a connected proper smooth curve over $\bar{k}$ of genus g. Then

cf. H^*_{sing}( ; $\mathbb{Z}/\ell^v$)



$$H^s_{\text{ét}}(\bar{X}; \mathbb{Z}/\ell^v(k)) \cong \begin{cases} \mathbb{Z}/\ell^v(k) \simeq \mathbb{Z}/\ell^v & s = 0, \\ \mathrm{Pic}(X)[\ell^v](k-1) \simeq (\mathbb{Z}/\ell^v)^{2g} & s = 1, \\ \mathbb{Z}/\ell^v(k-1) \simeq (\mathbb{Z}/\ell^v) & s = 2, \\ 0 & s \geq 3. \end{cases}$$

(depending on a choice of ℓ^v-th root of 1)

These are free $\mathbb{Z}/\ell^v$-module of rank $1, 2g, 1$ (as expected!). The SS collapses at E_2-page and we get

$$(K/\ell^v)_n(C)[\beta^{-1}] \cong \begin{cases} (\mathbb{Z}/\ell^v)^{\oplus 2} & n : \text{even}, \\ (\mathbb{Z}/\ell^v)^{\oplus 2g} & n : \text{odd}. \end{cases}$$

2-periodic

☺ (C contains ℓ^v-th roots of unity)

Relation with zeta functions

This paper was an early attempt for the Quillen-Lichtenbaum conjecture (proven by Rost-Voevodski). The related results proven here are:

- Assume $\gcd(q, l) = 1$, and assume further $4|q-1$ if $l = 2$. Let $X = \operatorname{Spec} \mathcal{O}_K$ be a function field (finitely generated of transcendence degree 1) $K/\mathbb{F}_q$ such that $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is a connected smooth curve over $\mathbb{F}_q$. Then we have the following formula for special values of the Hasse-Weil ζ function:

  Very sketchy in the paper; see the next page

$$\left| \frac{\# K_{2n-2}(X)[\beta^{-1}]_\ell\hat{}}{\# K_{2n-1}(X)[\beta^{-1}]_\ell\hat{}} \right|_\ell = |\zeta(X, 1-n)|_\ell \quad (n \geq 2).$$

- Analogously, if F is a totally real number field with ring of integers $\mathcal{O}_F$, then up to powers of 2, we have the following formula for the value of the Dedekind ζ for $n \geq 1$:

$$\zeta_F(1-2n) = \frac{\# K_{4n-2}(\mathcal{O}_F)[\beta^{-1}]}{\# K_{4n-1}(\mathcal{O}_F)[\beta^{-1}]}.$$

$\mathcal{A} = \{\mathcal{U}_n := \{X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n} \to X\}\}$ filtered system of single-map cov of X

étale cov (pullback of $\{\mathbb{F}_{q^n} \to \mathbb{F}_q\}$)

$$E_2^{s,t} = \check{H}^s(\mathcal{A}; \underbrace{(K/\ell^\nu)_{-t}(-)[\beta^{-1}]}_{F_{-t}}) \Rightarrow (K/\ell^\nu)_{-s-t}(X)[\beta^{-1}]$$

(sheaf coh descent $\Rightarrow$ Čech coh descent)

|| def

$$\varinjlim_n \check{H}^s(\mathcal{U}_n; F)$$

$$\| \longleftarrow (X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})^{\otimes_X n} \simeq X \otimes_{\mathbb{F}_q} (\mathbb{F}_{q^n}{}^{\otimes_{\mathbb{F}_q} n})$$

$$\varinjlim_n \check{H}^s(\{\mathbb{F}_{q^n} \to \mathbb{F}_q\}; F(X \otimes -))$$

def || (cofinal)

$$\overbrace{\varprojlim \mathrm{Gal}\, \mathbb{F}_{q^n}/\mathbb{F}_q}^{\mathbb{Z}/n\mathbb{Z}} \simeq \hat{\mathbb{Z}} \ni 1 = \boxed{\text{Frobenius } \varphi : x \mapsto x^q}$$

$$\check{H}^s_{ét}(\mathbb{F}_q; F(X \otimes -)) \simeq H^s(\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q); \underbrace{\varinjlim F(X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})}_{= F(\bar{X})})$$

$$(\langle\varphi\rangle \hookrightarrow \hat{\mathbb{Z}})^*, \quad \langle\varphi\rangle \cong \mathbb{Z}$$

(isom $s = 0, 1$; inj $s = 2$)

$$H^s(\langle\varphi\rangle; F(\bar{X})) \circlearrowleft \varphi$$

$$\| $$

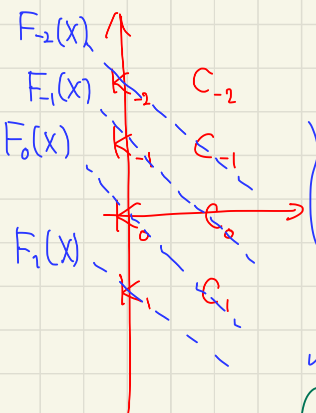
$$\mathrm{Ext}^s_{\mathbb{Z}[\varphi^\pm]}(\mathbb{Z}; F(\bar{X}))$$

$\boxed{\text{free resol } 0 \to \mathbb{Z}[\varphi^\pm] \xrightarrow{1-\varphi} \mathbb{Z}[\varphi^\pm] \to \mathbb{Z} \to 0}$

$$(K/\ell^\nu)_{-t}(\bar{X})[\beta^{-1}] \circlearrowleft \varphi$$

$$H^s(0 \to \overset{\deg\, 0}{F(\bar{X})} \xrightarrow{1-\varphi} \overset{1}{F(\bar{X})} \to 0)$$

$$= [\overset{0}{\mathrm{Ker}(1-\varphi)}, \overset{1}{\mathrm{Cok}(1-\varphi)}]$$



$$\underline{\text{LES}} \quad \cdots \to F_n(\bar{X}) \xrightarrow{1-\varphi} F_n(\bar{X}) \to F_{n-1}(X) \to \cdots$$

Recall: finite free mod $/\mathbb{Z}/\ell^\nu$ — also finite

Mittag-Leffler $\Rightarrow$ $\cdots \to K_n(\bar{X})[\beta^{-1}]^\wedge_\ell \xrightarrow{1-\varphi} K_n(\bar{X})[\beta^{-1}]^\wedge_\ell \to K_{n-1}(X)[\beta^{-1}]^\wedge_\ell \to \cdots$

injective $\Leftrightarrow$ eigenvalue of $\varphi \neq 1$

follows from special case of Deligne's thm (for curves by Weil) $H^i_{ét}(\bar{X}; \mathbb{Q}_\ell(j/2)) \circlearrowleft \varphi$ eigenvalues are alg # of abs value $q^{\left(\frac{i+j}{2}\right)}$

$$\rightsquigarrow \underline{\text{SES}} \quad 0 \to K_n(\bar{X})[\beta^{-1}]^\wedge_\ell \xrightarrow{1-\varphi} K_n(\bar{X})[\beta^{-1}]^\wedge_\ell \to K_{n-1}(X)[\beta^{-1}]^\wedge_\ell \to 0$$

fin gen free $\mathbb{Z}_\ell$-mod

finite (by str. thm of fin gen mod / PID)

of order $|\det(1-\varphi)|$

$\left(\# \prod_i (\mathbb{Z}_\ell / \ell^{v_i} \mathbb{Z}_\ell) \rightsquigarrow \text{power of } \ell \right)$

$$\rightsquigarrow \left| \frac{\# K_{2n-2}(\bar{X})[\beta^{-1}]^\wedge_\ell}{\# K_{2n-1}(\bar{X})[\beta^{-1}]^\wedge_\ell} \right| = \frac{\det(1-\varphi \mid H^0(\bar{X};\mathbb{Q}_\ell(1-n)) \oplus H^2(\bar{X};\mathbb{Q}_\ell(-n)))}{\det(1-\varphi \mid H^1(\bar{X};\mathbb{Q}_\ell(-n)))}$$

Compare $\varphi = Fr_q^{-1}$ & poincaré duality

Lefschetz trace formula

$$\zeta(X,t) = \frac{\det(1-t\,Fr_q \mid H^1_c(\bar{X};\mathbb{Q}_\ell))}{\det(1-t\,Fr_q \mid H^0_c(\bar{X};\mathbb{Q}_\ell) \oplus H^2_c(\bar{X};\mathbb{Q}_\ell))}$$

The End

Thank You