Logistic Regression

Data Science for Marketing Decision Making
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Introduction

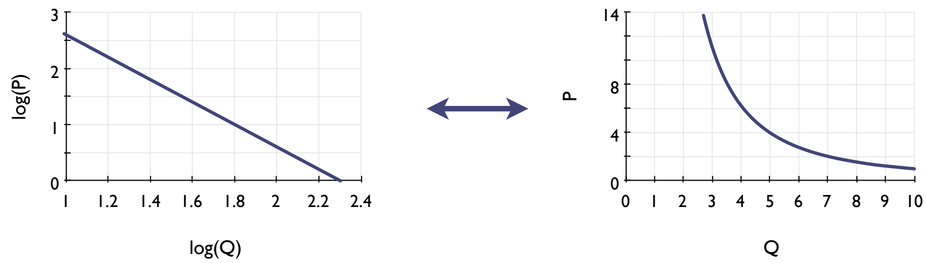
- Many applications of marketing analytics involve *discrete outcomes* and *classification problems*
- Examples
  - We send a catalog to a customer in our data base. Will the customer respond?
  - Is a customer in our data base likely to cancel her account in the next three months?
  - Is a household likely to default on a loan?
  - Does online exposure of a consumer to display advertising lead to a sale?

The regression model we used so far

Linear regression model:

$$\mathbb{E}(Y|X) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

Predicts an output that varies continuously in the inputs



Limited in usefulness to predict discrete outcomes, such as the examples discussed before:

- 1 = "buy from catalog", 0 = "do not buy from catalog"
- 1 = "cancel account", 0 = "do not cancel account"
- 1 = "default on loan", 0 = "do not default on loan"

Logistic regression

- Logistic regression is a widely applicable method to predict choices at the individual (consumer) level or to classify outcomes into discrete categories
- The outcome in a *binary logistic regression* model is discrete, and either 0 or 1
  - Extension: Methods for multinomial outcomes (1, 2, 3, ...)
- Allows us to predict the outcomes in our examples (purchase response, account cancellation, ...)

Logistic regression predicts the conditional (on the inputs) probability of the two possible outcomes ($Y = 0, 1$) from the independent variables:

$$\Pr\{Y = 0|X\}, \Pr\{Y = 1|X\} \longleftarrow X_1, X_2, \ldots, X_p$$

Why predict the probability of an outcome, not the outcome itself?

- Example: Predict default = 1 if default is more likely than no default
- Will predict default = 1 both if chance of default is 90% and if chance of default is 51%
- Predicting probability is more informative than predicting 0 or 1

Probability prediction

Is it possible to predict the outcome probability ($Y = 1$) using a linear regression model?

$$\Pr\{Y = 1|X\} = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

This is called a *linear probability model*

Problems with the linear probability model:

- Predicted probability may be smaller than 0 or larger than 1— may yield poor prediction

Alternative:

- Transform prediction to take values between 0 and 1 using some known function
- Generalized linear model (GLM)

Probability prediction using logistic function

Use *logistic function* for transformation:

$$\Lambda(z) = \frac{\exp(z)}{1+\exp(z)}, \qquad 0 < \Lambda(z) < 1$$

Logistic regression model:

- Index for level of predicted probability:

$$z = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

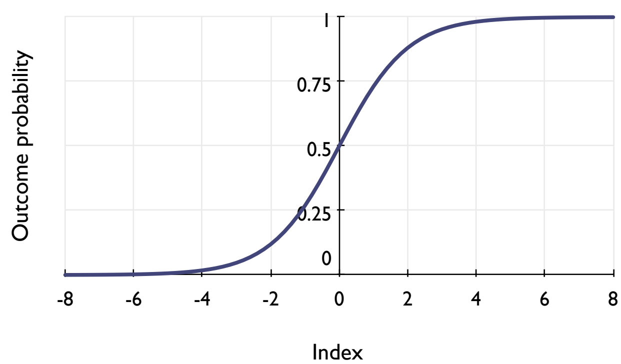
- Transform index to yield probability prediction between 0 and 1

$$\begin{aligned}\Pr\{Y=1|X) &= \frac{\exp(z)}{1+\exp(z)} \\ &= \frac{\exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p)}{1+\exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p)}\end{aligned}$$

Exploring the logistic regression model

Index: $z = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$

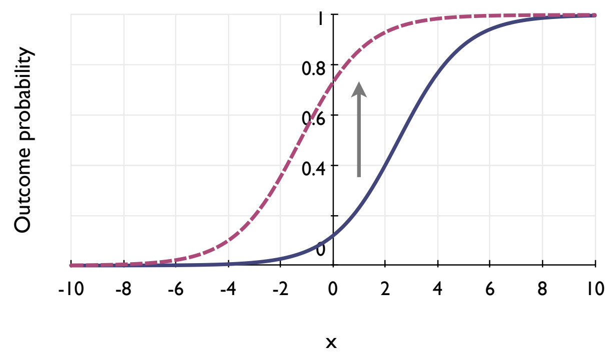
Logistic curve:



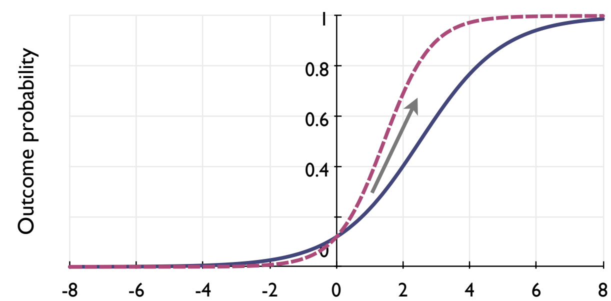
- Predicted outcome probability close to 0 for small (negative) index values
- Predicted outcome probability close to 1 for large index values
- $\Lambda(0) = 0.5$

Note: Variable X_K (not index z) on x-axis

- Effect of increasing β_0 ("intercept")



- Effect of increasing β_k ("slope parameter")



Remarks

- Terminology
  - In marketing (and economics) the logistic regression model for binary outcomes is often called the *logit model* or *binary logit model*
- If we replace the logistic function $\Lambda(z)$ with $F(z)$, the cumulative distribution function for a standard normal distribution, we obtain the *probit model* or *probit regression*
  - The probit model is similar to logistic regression, but does not have a closed-form solution for the outcome probabilities

Choice-theoretic foundation for the logit model

- Behavioral model of choice with two options, 0 and 1
  - Can be applied to product or brand purchase, voting behavior, or choice of a marriage partner
- We denote the index z by u, the *indirect utility* from a choice:

$$u(X) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

  - The inputs could be price and promotions, political positions of a candidate, or attributes of a potential partner
- Choice of $Y = 1$ versus option $Y = 0$ is based on utility maximization: Choose option 1 if (and only if)

$$U_1 = u(X) + \epsilon > 0 = U_0$$

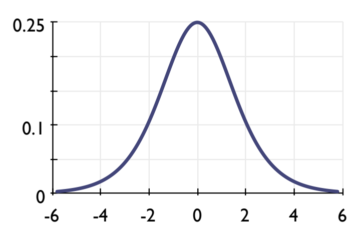
  - ϵ is an unobserved (to the data scientist) component of indirect utility
  - $U_0 = 0$ is simply a convenient normalization

Choice model prediction

- Because ϵ is unobserved, we do not know the exact value of U_1
- We assume ϵ has a standard logistic distribution. The cumulative distribution function of the standard logistic distribution is given by

$$\Lambda(z) = \frac{\exp(z)}{1 + \exp(z)}$$

- The probability density function of the standard logistic distribution:



probability density

Probability of choice

- If ϵ has a standard logistic distribution, then the probability that the decision maker choose option 1 over 0 is

$$\begin{aligned}\Pr\{u(X)+\epsilon>0|X\} &= \frac{\exp(u(X))}{1+\exp(u(X))} \\ &= \frac{\exp(\beta_0+\beta_1X_1+\cdots+\beta_pX_p)}{1+\exp(\beta_0+\beta_1X_1+\cdots+\beta_pX_p)}\end{aligned}$$

- Note the role of the unobserved component in indirect utility: It allows us to rationalize all observed choices in the data.
  - Example: $u(X)$ may be very large, but we many nonetheless observe that option 0 is chosen. This is due to some unmeasured utility component captured by ϵ

Proof

$$\begin{aligned}\Pr\{Y=0|X\} &= \Pr\{u(X)+\epsilon\le 0|X\} \\ &= \Pr\{\epsilon\le -u(X)\} \\ &= \Lambda(-u(X)) \\ &= \frac{\exp(-u(X))}{1+\exp(-u(X))} \\ &= \frac{\exp(-u(X))}{1+\exp(-u(X))}\cdot\frac{\exp(u(X))}{\exp(u(X))} \\ &= \frac{1}{1+\exp(u(X))} \\ &= \frac{1}{1+\exp(\beta_0+\beta_1X_1+\cdots+\beta_pX_p)}\end{aligned}$$

Hence

$$\begin{aligned}\Pr\{Y=1|X\} &= 1-\Pr\{Y=0|X\} \\ &= \frac{\exp(\beta_0+\beta_1X_1+\cdots+\beta_pX_p)}{1+\exp(\beta_0+\beta_1X_1+\cdots+\beta_pX_p)}\end{aligned}$$

Estimation

- Example: Demand for a potential partner when using an online dating site
- Online dating data:
  - Observe various user attributes, and if a site user sends a first-contact (unsolicited) e-mail to another user after viewing his or her profile
  - The decision to send an e-mail is a binary choice
    - $1 =$ send e-mail, $0 =$ ignore the other user
  - Because of sample size we only look at choices among opposite-sex partners, e.g. decisions of women to send first-contact e-mails to men
- Data in `Online-Dating.RData`

Variable	Description
`profile_gender`	Gender of person in profile, `male` or `female`
`first_contact`	1 = first-contact e-mail sent, 0 = otherwise
`age`	Age of the person in the profile, in years
`age_older`	1 = potential mate in profile is at least 5 years older
`age_younger`	1 = potential mate in profile is at least 5 years younger
`looks`	Numerical looks rating
`height`	Inches
`height_taller`	1 = potential mate at least 2 inches taller
`height_shorter`	1 = potential mate at least 2 inches shorter
`bmi`	Body mass index
`yrs_education`	Years of education
`educ_more`	1 = potential mate has at least 2 more years of education
`educ_less`	1 = potential mate has at least 2 years less of education
`income`	$1,000 annual income
`diff_ethnicity`	1 = potential mate has different ethnicity than browser

Estimation using R

Model:

- Utility of sending a first contact-email ($Y = 1$) depends on utility from a potential match
- Utility of potential match depends on attributes of the mate, such as age, looks, etc. $(X_1, X_2, \ldots)$

Estimation using the glm function:

```
fit = glm(y ~ x1 + x2 + ..., family = binomial(), data = DT)
summary(fit)
```

- Binary 0/1 outcome variable `y`
- GLM = generalized linear model
- Use `binomial(link = "probit")` to estimate a probit model

Predict if women send a first-contact e-mail after viewing the profile of a potential male partner

```
library(data.table)
load("./Data/Dating-Data.RData")

women_DT = dating_DT[profile_gender == "male",
                     -"profile_gender", with = FALSE]

womens_choices = glm(first_contact ~ .,
                     family = binomial(),
                     data = women_DT)
```

```
Call:
glm(formula = first_contact ~ ., family = binomial(), data = women_DT)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.7989  -0.4184  -0.3644  -0.3115   3.0393

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept)    -7.5251886  0.4706216 -15.990  < 2e-16 ***
age             0.0150776  0.0020255   7.444 9.77e-14 ***
age_older      -0.2693740  0.0324052  -8.313  < 2e-16 ***
age_younger    -0.3078933  0.0364871  -8.438  < 2e-16 ***
looks           0.5236808  0.0285258  18.358  < 2e-16 ***
height          0.0417912  0.0058863   7.100 1.25e-12 ***
height_taller   0.3470403  0.0690557   5.026 5.02e-07 ***
height_shorter -0.3188366  0.1272128  -2.506   0.0122 *
bmi             0.0360348  0.0058656   6.143 8.07e-10 ***
yrs_education   0.0133780  0.0075301   1.777   0.0756 .
educ_more      -0.1812976  0.0338545  -5.355 8.55e-08 ***
educ_less      -0.2295514  0.0395019  -5.811 6.20e-09 ***
income          0.0025459  0.0002513  10.130  < 2e-16 ***
diff_ethnicity -0.4960341  0.0761361  -6.515 7.26e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



```
(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 41036  on 79999  degrees of freedom
Residual deviance: 40129  on 79986  degrees of freedom
AIC: 40157

Number of Fisher Scoring iterations: 6
```

```
Call:
glm(formula = first_contact ~ ., family = binomial(), data = men_DT)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.8318  -0.4696  -0.4157  -0.3480   3.0594

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept)     0.0517899  0.4060820   0.128 0.898517
age             0.0061920  0.0018682   3.314 0.000918 ***
age_older      -0.1792139  0.0384492  -4.661 3.15e-06 ***
age_younger    -0.0734181  0.0304971  -2.407 0.016067 *
looks           0.4837548  0.0214545  22.548  < 2e-16 ***
height         -0.0153335  0.0052140  -2.941 0.003273 **
height_taller  -0.5151478  0.1042932  -4.939 7.84e-07 ***
height_shorter  0.1107209  0.0541539   2.045 0.040898 *
bmi            -0.0620012  0.0044635 -13.891  < 2e-16 ***
yrs_education  -0.0204628  0.0065302  -3.134 0.001727 **
educ_more      -0.0611685  0.0318745  -1.919 0.054979 .
educ_less      -0.1431284  0.0329015  -4.350 1.36e-05 ***
income          0.0002289  0.0003797   0.603 0.546576
diff_ethnicity -0.2465653  0.0421174  -5.854 4.79e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Logistic regression output

- Coefficient estimate: Shows how each variable enters $u(X)$, and thus the probability of choosing option 1
- z-statistic: Standard normal test statistic used to test for statistical significance of coefficient estimate
  - For all practical purposes use like a t-statistic
- p-value: As in standard regression model

- AIC: Akaike information criterion
  - AIC value not directly interpretable
  - Useful to select a model among many candidate models, and also allows us to compare models with a different number of independent variables
  - Approach:
    1. Estimate each candidate model and record the AIC
    2. Select model with lowest AIC

Estimation: Mathematical background

How does the computer estimate the logistic regression model coefficients?

- In regression analysis, the idea is to make the predicted outcome as close as possible to the actual outcome
- The same idea is at work here: the software selects parameter values to match the predicted probability of an outcome (between 0 and 1) to the actual outcome (0 or 1)

- The estimation method used is called *maximum likelihood*

$$\max_{\boldsymbol{\beta}} \log(l(\mathit{data} = (\boldsymbol{y}, \boldsymbol{X})|\boldsymbol{\beta})) = \sum_{i=1}^{n} \log(\Pr\{Y = y_i|\boldsymbol{x}_i; \boldsymbol{\beta}\})$$

Interpretation of the estimates

- The estimated coefficients cannot be interpreted in the same way as the coefficient estimates in the linear regression model!
  - In the example before, the estimate on the dummy variable `height_taller` (men making choices) was -0.515
  - This does not mean that the probability of a first contact decreases by 51.5 percent if men choose between taller women and women of similar height

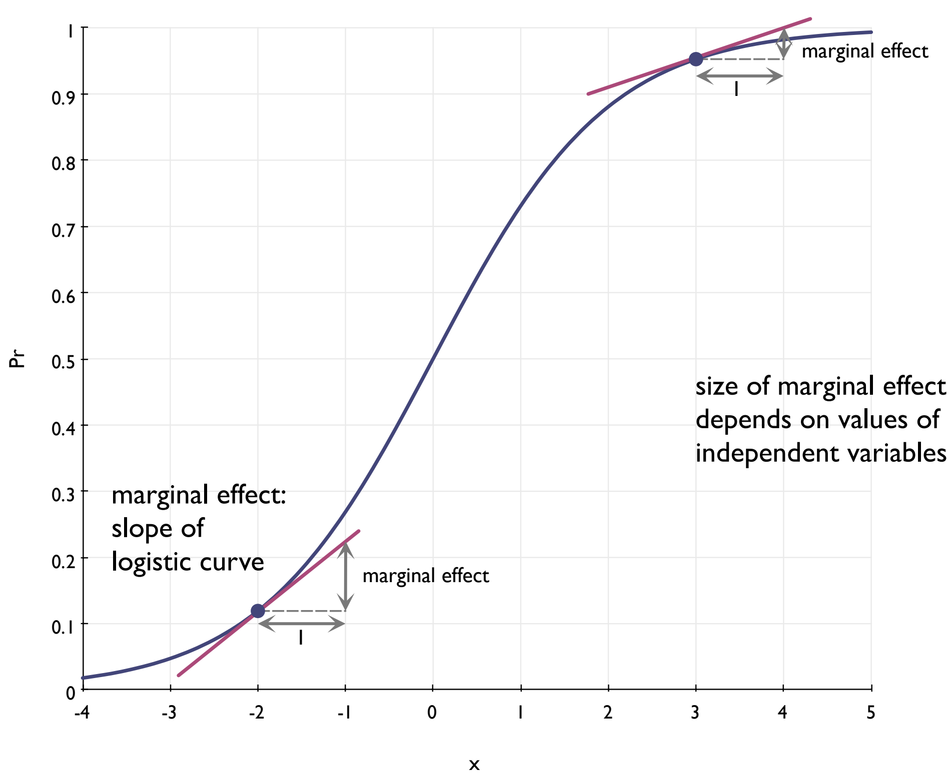
- Even if we take the log of a non-categorical independent variable the coefficient estimate is not an elasticity
  - Note: Generally, there is no benefit of taking the log of the independent variables in the logistic regression model

Interpretation of the estimates:Marginal effects

- The marginal effect of the independent variable X_k is the change in the outcome probability $\Pr\{Y = 1|X\}$ relative to an increase in X_k by a "small" unit

$$\text{marginal effect} = \frac{d\Pr\{Y = 1|X\}}{dX_k} \approx \frac{\Delta\Pr\{Y = 1|X\}}{\Delta X_k}$$

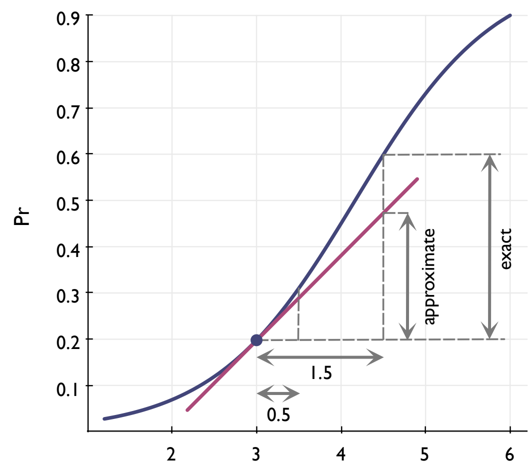
- ΔX_k is the "small" increase and $\Delta\Pr\{Y = 1|X\}$ is the corresponding change in the outcome probability
- Mathematically, the marginal effect is the derivative of the outcome probability with respect to X_k



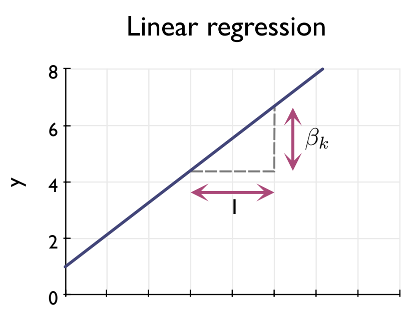
Using the marginal effects

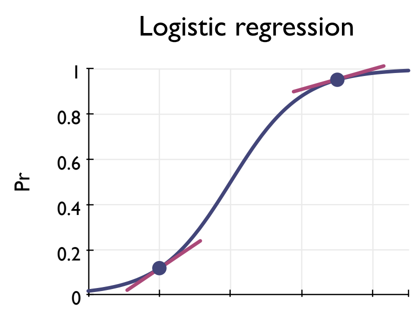
- We would like to predict by how much the outcome probability $\Pr\{Y = 1|X\}$ changes for a change in X_k by ΔX_k units

$$\Pr\{Y = 1|X\} \approx \Delta X_k \cdot \text{marginal effect}$$



Marginal effects: Linear vs. logistic regression model





- In linear regression model, marginal effect $=$ regression coefficient $= \beta_k$. Marginal effect does not depend on values of inputs
- In logistic regression, marginal effect $=$ slope of the logistic curve. Depends on values of inputs

Marginal effect concept is simple in linear regression, hence we did not previously discuss it

Calculation of marginal effects

1. Decide for what values of the independent variables you want to calculate the marginal effect
   - Observed or mean of each input
2. Calculate the outcome probability at the chosen values of the independent variables:

$$p = \Pr\{Y = 1|X\} = \frac{\exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p)}{1 + \exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p)}$$

3. The marginal effect of increasing X_k is

$$\text{marginal effect} = \beta_k \cdot p \cdot (1 - p)$$

Exact effect of increasing X_k

Compare predictions at $X = X_0$ and $X = X_1$:

$$\Delta \Pr\{Y = 1\} = \Pr\{Y = 1|X_1\} - \Pr\{Y = 1|X_0\}$$

Best done using the `predict` command in R

Calculating marginal effects in R

- Install and load the package `erer`
- Estimate the model and then calculate the marginal effects:

```
fit = glm(y ~ x1 + x2 + ..., family = binomial(), x = TRUE, data = DT)
maBina(fit, digits = 6)
```

- Need to set the `x = TRUE` option in the `glm` function!
- Use the `digits` option to choose the number of digits in the output
- By default, `maBina` calculates the marginal effects at the means of all independent variables
- To calculate the average marginal effects across all observations in the data, use

```
maBina(fit, x.mean = FALSE, digits = 6)
```

- For dummy variables, `maBina` predicts the exact effect of increasing the variable from $X_k = 0$ to $X_k = 1$


```
womens_choices = glm(first_contact ~ ., family = binomial(),
                     x = TRUE, data = women_DT)
library(erer)
maBina(womens_choices, x.mean = FALSE, digits = 6)

                  effect    error    t.value  p.value
(Intercept)    -0.491205 0.030399 -16.158449 0.000000
age             0.000984 0.000132   7.475458 0.000000
age_older      -0.016271 0.001910  -8.516871 0.000000
age_younger    -0.017896 0.001989  -8.998352 0.000000
looks           0.034183 0.001825  18.734088 0.000000
height          0.002728 0.000383   7.121283 0.000000
height_taller   0.019068 0.003350   5.691873 0.000000
height_shorter -0.017346 0.006045  -2.869600 0.004111
bmi             0.002352 0.000382   6.161249 0.000000
yrs_education   0.000873 0.000491   1.777149 0.075548
educ_more      -0.010938 0.001993  -5.488605 0.000000
educ_less      -0.013511 0.002211  -6.111024 0.000000
income          0.000166 0.000016  10.175925 0.000000
diff_ethnicity -0.025411 0.003176  -8.002155 0.000000

mean(women_DT$first_contact)     # For comparison

[1] 0.0711
```

Calculating exact effects of increasing X_k in R

- Predict outcome probabilities:

```
Pr = predict(fit, type = "response")
Pr = predict(fit, newdata = new_DT, type = "response")
```

- If the `newdata` option is not specified probabilities will be predicted for the data set used to estimate the model
- Make sure the `data.table` or data frame in `newdata` includes all variables used in the original logistic regression model

Example: Change (increase) in first-contact probabilities if all men become millionaires

```
Pr_0 = predict(womens_choices, type = "response")

# Copy data and set income to 1000 (= $1,000,000)
millionaires_DT = women_DT
millionaires_DT[, income := 1000]

# Predict first-contact probabilities at high income data
Pr_1 = predict(womens_choices, newdata = millionaires_DT,
               type = "response")

Delta_Pr = Pr_1 - Pr_0
summary(Delta_Pr)

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
0.07189 0.30870 0.35270 0.34880 0.39580 0.53020
```

Historical background

- Logistic regression—the logit model—is a specific instance of a *discrete choice model*
- Discrete choice models were first developed and used to explain choice behavior in the 1970s
- Discrete choice models have revolutionized marketing — much of the work conducted at Booth and other schools during the last thirty years would not have been possible without these models
- Jim Heckman (University of Chicago) and Dan McFadden (UC Berkeley) received the Nobel Prize in Economics in 2000 for their work on statistical methods to capture individual choices

Summary

- Logistic regression predicts the probability of the two possible outcomes $Y = 0, 1$ conditional on the values of the independent variables $X_1, \ldots, X_p$
- The logistic regression model can be interpreted as a choice model, where the choice of 1 versus 0 is based on a comparison of the corresponding indirect utilities
- Estimating the logistic regression model is as simple as estimating the linear regression model, but interpreting the coefficient estimates is more difficult
  - Marginal effects: Change in outcome probability relative to a "small" change in one of the independent variables