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An Extension of the Black-Litterman Model: Letting the Data Speak

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An Extension of the Black-Litterman Model: Letting the Data Speak

The Black-Litterman model is a popular approach for asset allocation by blending an investor's proprietary views with the view of the market. However, their model ignores the data-generating process whose dynamics can have significant impact on future portfolio returns. This paper extends the Black-Litterman model to allow Bayesian learning to exploit all available information– the equilibrium view, differing market views as well as the data. The framework allows practitioners to combine the insights from the Black and Litterman (1992) model with the data to generate potentially more reliable trading strategies and more robust portfolios. Further, we show that many Bayesian learning tools can now be readily applied to practical portfolio selections in conjunction with the Black and Litterman (1992) model.

Since Markowitz's (1952) seminal work, the mean-variance framework is widely used in practice. However, as is the case with any model, the true parameters (the expected returns and risks) are unknown and have to be estimated from data. Based on standard estimates, the optimized solution has, however, a well known problem of taking unusually large long and short positions when no portfolio constraints are imposed, and taking many zero positions when no-short-sell constraints are imposed. Given that parameter estimation errors are nontrivial, and that the economic intuition of these extreme positions is not obvious, it is hence difficult in practice to implement the optimized mean-variance solution.

The Black-Litterman model (the BLM henceforth), originated by Black and Litterman (1992), provides a novel solution to the problem. It assumes that the investor starts with the view of the market, then updates his view via the Bayesian rule. In a typical case, the market view is taken as the equilibrium expected returns implied by the CAPM. If the investor has no views different from the market, and if the investor treats the equilibrium expected returns as the true returns, he should hold the equilibrium or value-weighted market portfolio. When he does have a view on the relative performance of the assets, the BLM provides a way for the investor to tilt his optimal portfolio away from the market according to the strength of his view. Hence, the key insight of the BLM is to blend an investor's proprietary views with that of the market to construct a portfolio that is optimal in a certain sense. Ever since its publication, the BLM has generated wide applications and a number of studies. Recently, for example, Fabozzi and Focardi, and Kolm (2006) incorporate various trading strategies into the model; Jones, Lim and Zangari (2007) provide a practical implementation of the framework for structured equity portfolios; and Martellini and Ziemann (2007) adapt it to hedge funds.

The theoretical basis of the BLM is Bayesian learning. The investor starts with the view of the market, then updates this view to his own via the Bayesian rule to balance his view and that of the market as represented by an equilibrium model. This is different from the usual Bayesian decision making where the learning often takes place in light of the data– the observed asset returns which tell us how they have moved in the past and how they might likely move in the future. Theoretically, if one has an infinite number of data, one can learn about the true expected returns, making the use of the equilibrium model unnecessary.

In addition, an equilibrium model may not necessarily be consistent with the data. For example, the CAPM is rejected by Gibbons, Shanken and Ross (1989) and Zhou (1991), among others. On the other hand, with a limited amount of data in practice, estimates of the model parameters can be noisy and an equilibrium model can help to provide useful information to tie down the parameters. In short, the BLM model makes a clever use of an equilibrium model, but it ignores the data or the data-generating process of the asset returns. Subsequent studies of the BLM, to the authors knowledge, also ignore the role of the data. The relation to data is an important missing link of the BLM, because the data are both economically and statistically crucial in understanding the dynamics of the asset returns, including in particular their expected returns and risks.

We extend here the BLM to allow the Bayesian learning to use all available information– the views, the equilibrium model, and the data. For easy understanding of our approach, we first review the portfolio choice problem and discuss the BLM, and then provide a simple extension of the BLM. Next, we use a numerical example to illustrate the basic ideas. Finally, prior to concluding, we present somewhat more complex generalizations of the BLM.

Mean-Variance Portfolio

Consider the case of an investor who allocates funds among N risky assets and a riskless asset. Let R_t be an N-vector of returns on the risky assets in excess of the riskless asset at time t. The standard assumption on the probability distribution of R_t is that R_t is independent, identical, and normally distributed over time, with mean μ and covariance matrix Σ.

In the mean-variance framework, it is well known that the investor's portfolio choice problem of maximizing the expected return for a given level of risk is equivalent to select portfolio weights w to maximize his mean-variance utility,

$$U(w) = E[R_{T+1}] - \frac{\gamma}{2}\operatorname{Var}[R_{T+1}] = w'\mu - \frac{\gamma}{2}w'\Sigma w, \tag{1}$$

where T is today's time, $T+1$ is the future period, and γ is the investor's coefficient of relative risk aversion. The problem has the standard solution,

$$w = \frac{1}{\gamma}\Sigma^{-1}\mu. \tag{2}$$

This is the optimal allocation per dollar of funds across the N risky assets, with the balance invested in the riskless asset.

However, μ and V have to be estimated to compute w in practice. They are usually estimated by using the sample mean and sample covariance matrix of the asset returns, available from period 1 to T. The problem is that, when the estimates are treated as if they were the true parameters, and are plugged into Equation 2, the resulting optimal portfolio weights, the optimized solution, often have unusually large long or short positions when no portfolio constraints are imposed, and many zero positions when no-short-sell constraints are imposed. This is documented widely in the literature, as also demonstrated here by the example provided later.

Black-Litterman Model

Black and Litterman (1992) provide an elegant solution to the extreme position problem of the plug-in portfolio weights. Their idea is to start from the value-weighted index of the risky assets which we refer to as stocks here for simplicity. In equilibrium, if the CAPM is true, and if all investors hold the same view and have the same risk aversion, then their demand for the risky assets should exactly be equal to the outstanding supply, and hence their portfolio weights should be given by the optimal portfolio weights formula, Equation 2. This implies that the equilibrium expected excess returns (or the equilibrium risk premium often referred in the literature) must satisfy

$$\mu^e = \gamma\Sigma w_e, \tag{3}$$

where γ is the risk-aversion typically specified as a number close to three; w_e is the equilibrium portfolio weights computed based on the market value of the stocks; Σ is the covariance matrix of asset returns often estimated by the sample covariance matrix or by a risk model. In practice, the equilibrium expected returns are easily evaluated from Equation 3.

Black and Litterman (1992) take a Bayesian approach, referred as "mixed estimation strategy" in their paper, on learning the value of the true expected excess return μ, which is the critical input for any portfolio choice decision. Recall that a Bayesian regards all parameters as random variables. Hence, μ is naturally assumed to be normally distributed with mean μ^e,

$$\mu = \mu^e + \epsilon^e, \quad \epsilon^e \sim N(0, \tau\Sigma), \tag{4}$$

where ϵ^e is the deviation of μ from μ^e that is normally distributed with zero mean and covariance matrix $\tau\Sigma$, and τ is a scalar indicating the degree of belief in how close μ is to the equilibrium value. In the absence of any views on future stock returns, and in the special case of $\tau = 0$, the investors' portfolio weights must equal to w_e, the value-weighted index.

However, an active portfolio manager or an informative investor is likely to have views on μ that are different from μ^e in a substantial way. Black and Litterman (1992) illustrate that views on the relative performance of the stocks can be represented mathematically by a single vector equation,

$$P\mu = \mu^v + \epsilon^v, \quad \epsilon^v \sim N(0, \Omega), \tag{5}$$

where P is a $K \times N$ matrix summarizing K views, μ^v is a K-vector summarizing the prior means of the view portfolios, and ϵ^v is the residual vector. The covariance matrix of the residuals, Ω, measures the degree of confidence the investor has on his views.

Applying the Bayesian rule to both the equilibrium relationship and the view equation, Black and Litterman (1992) show that the Bayesian updated expected returns and risks are

$$\bar{\mu}_{\text{BL}} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu^e + P'\Omega^{-1}\mu^v], \tag{6}$$

$$\bar{\Sigma}_{\text{BL}} = \Sigma + [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}. \tag{7}$$

Plugging these two updated estimates into the earlier optimal portfolio formula, we obtain the BLM solution to the portfolio choice problem.

Note that the Black-Litterman expected return, $\bar{\mu}_{\text{BL}}$, is a weighted average of the equilibrium expected return and views. The less confident the investor is in his views, the closer it to the equilibrium value, and, intuitively, the closer the Black-Litterman portfolio to w_e. This is indeed the case as shown mathematically by He and Litterman (1999). Hence, the BLM tilts the investor's optimal portfolio away from the market portfolio according to the strength of his views. Since the market portfolio is a reasonable one without any extreme positions, any suitably controlled tilt should also yield a portfolio without any extreme positions. This is perhaps the reason that makes the BLM popular in practice.

An Extension

The BLM model combines both the equilibrium model and the investor's views, but it ignores entirely the data-generating process of the asset returns. In particular, the sample means of the asset returns play no roles in the Bayesian updating. To see how the data can be important, imagine a case in which one stock has an almost zero correlation with the rest of the market. Assume $\gamma = 3$ and the stock has an annual standard deviation of 20% with 1% market share. Then the (annual) equilibrium expected return is 1.2%. If the sample mean is 1% in the past 50 years, the equilibrium expected return is likely to be reasonable. On the other hand, if the sample mean is 10% in the past 50 years, the equilibrium expected return is most likely to under-estimate the true expected return substantially. When the views are not strong, the Black-Litterman expected return is close to the equilibrium expected return, and so it will under-estimate the true expected return as well. Now, even if the views are strong about the stock's outperformance relative to the market, errors in estimating the expected returns of other stocks can make the relative performance view quite different from what is intended. For example, a view that stock A outperforms stock B by 5% can be exaggerated to 9% if the expected return of A is over-estimated by 4%. In short, the historical average returns contain important information on future expected returns and should not be ignored almost entirely.

From an asset pricing perspective, the equilibrium relationship is subject to specification errors for which the data have a lot to say. For example, in terms of the stock market, the equilibrium relation hedges on the validity of the CAPM. But the CAPM is strongly rejected by the data. It is now well accepted that Fama and French (1993) three-factor model is a much better choice, and including the momentum factor, as shown by Carhart (1997), can make the model even better. Based on these findings, the equilibrium expected returns that rely on holding a portfolio proportional to the market capitalization are likely to be incorrect. Certain adjustments based on the return data or past history should be able to improve portfolio selection.

By statistical decision theory, the data and the associated model for the data (data-generating process) are fundamental in Bayesian learning. A typical Bayesian inference is to update a given prior by the data, as demonstrated by Shanken (1987) and Harvey and Zhou (1990) in the context of analyzing the validity of the CAPM. What is the prior in the BLM? Although it is not explicitly stated in either Black and Litterman (1992) or elsewhere, we can interpret that the mixed estimation strategy is an update of the prior equilibrium relationship by the views. Since their residuals in the BLM are independent, reversing the order of the interpretation will yield the same result. However, we here take the equilibrium relation as the starting prior, then it seems more natural that this is updated by the additional information, the views. To link the equilibrium prior, the views and the data all together, we have to view the first two as priors. In contrast to the standard Bayesian updating, we now have not just one single prior, but two priors of both the equilibrium relationship and the views. Nevertheless, the Bayesian learning mechanism does allow a sequence of priors, and one can simply update them sequentially (see, e.g., Geweke (2005)). Armed with these observations, we are ready to extend the BLM to account for the information from data.

For simplicity, we maintain the same assumption underlying the BLM that Σ is a known matrix in the Bayesian updating process. Then, the Black-Litterman estimates, which are update of the equilibrium relationship by the views, can be further updated by the likelihood function of the data to obtain

$$\bar{\mu} = [\Delta^{-1} + (\Sigma/T)^{-1}]^{-1}[\Delta^{-1}\bar{\mu}_{\mathrm{BL}} + (\Sigma/T)^{-1}\hat{\mu}], \tag{8}$$

$$\bar{\Sigma} = \Sigma + [\Delta^{-1} + (\Sigma/T)^{-1}]^{-1}. \tag{9}$$

where $\hat{\mu}$ is the sample mean of the asset returns, T is the sample size, and $\Delta = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$ is the covariance matrix of the Black-Litterman estimate $\bar{\mu}_{\mathrm{BL}}$.

The data-updated expected return $\bar{\mu}$ is a weighted average of Black-Litterman estimate $\bar{\mu}_{\mathrm{BL}}$ and the sample mean $\hat{\mu}$. If we have a lot of data so that T is large, then $\bar{\mu}$ will be determined more by $\hat{\mu}$. In the limit when we have an infinity amount of data, it must converge to the true expected returns. This is the desired consistency property of the Bayesian decision making. When there are ample data, we should be able to make the right decisions. Any non-dogmatic priors should have no impact on the inference whatsoever.

On the other hand, if T is small or if one has a strong belief either in the equilibrium relationship or in the views, $\bar{\mu}$ will be determined primarily by $\bar{\mu}_{\mathrm{BL}}$. They offer additional information that should help to tie down the correct true parameter values. In bypassing, it might be useful to note that the BLM says nothing about how to choose τ and Ω that represents our beliefs or confidence in the equilibrium relationship or in the views. These are matters of empirical calibration and portfolio management experience.

An Example

To illustrate the extension, we consider the allocation of funds into the seven major industrial countries– Australia, Canada, France, Germany, Japan, U.K. and the U.S., with monthly dollar returns available from Global Financial Data from January 1970 to December 2007. The second column of Table 1 reports the equilibrium expected excess returns (annualized) on the seven countries, computed based on Equation 3, and the fifth column reports the equilibrium/value-weighted weights (in percentage points). Unlike the earlier years studied by He and Litterman (1999), France's stock market now has a market share of 7.9% versus 5.2%, and Germany has a share of 6.6% versus 5.5%. So, France surpasses Germany in terms

Table 1. Asset Returns and Portfolio

	μ_e	$\hat{\mu}$	μ_{BL}	w_e	$\hat{w}$	w_{BL}
Australia	5.1	8.2	6.1	4.0	27.2	3.7
Canada	5.2	6.1	6.5	6.8	-19.1	43.0
France	5.5	8.3	6.5	7.9	34.5	21.8
Germany	4.5	5.1	4.5	6.5	-23.3	-8.6
Japan	4.2	6.7	4.7	13.5	25.6	12.2
U.K.	5.9	8.7	6.4	12.0	23.4	10.9
U.S.	4.8	5.9	4.6	49.2	60.1	7.9

of the stock market value, though it has only about half of Germany's GDP. The third column of Table 1 reports the country sample average excess returns. In comparison with the equilibrium returns, there are two interesting facts. First, the cross-country differences are much greater. Second, the rankings of the countries in terms of the returns are quite different. For example, Japan has the least return of 4.2%, but now ranked fourth with a return of 6.7%. Clearly, the optimal portfolio weights based on the sample averages, reported in the sixth column and computed from Equation 2, are totally different from the equilibrium weights, with huge short positions of 23.3% and 19.1% on Germany and Canada, respectively. Interestingly, though, the long position in the U.S. is only 60%, not too far away from an earlier value of 49.2%.

Now, let us assume that we have two views, similar to the case examined by He and Litterman (1999). The first view is that France will outperform Germany by 2% with a standard error of 2%. The second view is that Canada will outperform the U.S. by 2% with a standard error of 4%. Then the BLM estimates of the expected returns are obtained from Equations 6 and 7, and reported in the fourth column of Table 1. The expected return differences between France and Germany, and between Canada and the U.S. are updated from 1% to 2%, and from 0.4% to 1.9%, respectively. Correspondingly, the relative portfolio weights, reported in the last column, are substantially adjusted according to the new expected returns. For example, the equilibrium holding of 6.5% in Germany now becomes a short position of 8.6%. Both France and Canada have increased their long positions substantially from 7.9% to 21.8%, and from 6.8% to 43.0%, respectively.

To see the role of data, image that the sample mean and covariance matrix are those obtained from a large sample size so that they are reliable estimates of the true expected returns and risk. Since they are quite different from the equilibrium returns, the latter must be inaccurate. Hence, a combination of them with the supposed correct views will not necessarily yield outstanding performance. Even when they are not plentiful, Bayesian learning suggests that the data nevertheless provide useful information on the moments of the returns as well as their dynamic behavior. It is thus of interest to blend the BLM solution with the data. To do so, suppose that the previous sample mean and covariance matrix are obtained with hypothetical sample sizes of $T = 60, 120$ and 240, respectively, which correspond to 5, 10 and 20 years of monthly data. We examine how the expected returns and the weights will be updated with such typical sample sizes. Table 2 reports the results, the expected blended returns, $\hat{\mu}_T$s computed from Equation 8, and the portfolio weights, $\hat{w}_T$s computed from Equation 2 based on $\hat{\mu}_T$s and the covariance matrix from Equation 9.

Table 2. Black-Litterman Model Blended with Data

	$\hat{\mu}_{60}$	$\hat{\mu}_{120}$	$\hat{\mu}_{240}$	$\hat{w}_{60}$	$\hat{w}_{120}$	$\hat{w}_{240}$
Australia	7.2	7.6	7.9	17.7	21.3	23.8
Canada	6.1	6.1	6.1	-0.1	-7.1	-12.2
France	7.7	7.9	8.1	32.6	33.3	33.8
Germany	4.7	4.9	5.0	-20.1	-21.3	-22.1
Japan	5.9	6.2	6.4	20.6	22.5	23.8
U.K.	7.8	8.1	8.3	18.7	20.4	21.7
U.S.	5.4	5.6	5.7	46.6	51.6	55.2

It is seen that $\hat{\mu}_T$ approaches $\hat{\mu}$ quickly as T increases. Even with $T = 60$, the expected returns are close to $\hat{\mu}$ already. The most striking impact is on the portfolio weights. The 43.0% position in Canada now becomes a short position of 0.1%. This is drastic, but nevertheless a move in the right direction since the sample weight is a short position of 19.1%. With $T = 240$, the portfolio weights are not much different from $\hat{w}$ any more. It should be emphasized that the data-generating process here is quite simple, so that the speed of the sample size being able to adjust the prior is high. But in a more realistic situation when the data-generating process is complex, as discussed in the next section, the speed can be much slower. Alternatively, if the confidence in the BLM is high as reflected by a small Δ matrix (or by a small hyperparameter as discussed in the next section), the speed will be lower too. Regardless of the sample size, though, the blending or updating is consistent with Bayesian decision theory that it will be optimal to do it given the data at hand and given the assumed data-generating process.

A simple way to see the effectiveness of the updating is to assume that the sample estimates were the true parameters. This is of course biased against the BLM model, but is still useful to see that how fast one can go from the BLM model, assumed incorrect, to the true model. In this case, the certainty-equivalent return based on the true weights $\hat{w}$ is 5.00%, while the w_{BL} has only 3.89%. But the updated certainty-equivalent returns are 4.89%, 4.95% and 4.99% with the sample sizes 60, 120, and 240, respectively. Interestingly, while $\hat{\mu}_{60}$, half way from w_{BL} to $\hat{w}$, is not nearly as extreme as $\hat{w}$, it obtains almost most of the optimal expected portfolio return. The implication is that, if the data represent the truth, the blending of the BLM even with a small amount of data can quickly approach the level of the optimal expected portfolio return. In general, given enough data and given the true data-generating process, there is no guarantee that the BLM solution will converge to any limit, but the Bayesian blending must converge to the true optimal solution.

Further Generalizations

Our extension of the BLM is kept simple to allow the framework and the associated ideas be revealed without being hindered by technical details. The simple updating framework can, however, be generalized in a number of important ways. First, like the BLM, the uncertainty in Σ has been ignored so far. This is not a problem when N is small, because the sample covariance matrix is usually an accurate estimate of Σ. However, the uncertainty can matter substantially when N is large, as shown by Kan and Zhou (2007).

To capture the uncertainty in estimating Σ, we can specify an inverted Wishart prior distribution,

$$p_0(\Sigma) \propto |\Sigma|^{-\frac{\nu}{2}} exp\left\{-\frac{1}{2} tr \Sigma^e \Sigma^{-1}\right\}, \tag{10}$$

where Σ^e is the earlier constant estimate of the covariance matrix in the equilibrium relationship of the BLM, and ν is a parameter capturing the degree of confidence in the estimate. This form of prior can be easily updated via the Bayesian rule. The greater the ν, the more weight we put on the prior. When ν is large enough, the updated Σ becomes virtually Σ^e, the same as the earlier case where Σ is treated as constant. However, with reasonable views, the data should help to learn about the distribution of Σ.

Denote by $\Phi_0(\mu)$ the normal prior density with the Black-Litterman estimate as the mean and $\eta\Delta$ as the covariance matrix, where η is another hyperparameter to reflect one's degree of belief in the BLM ($\eta = 1$ in the earlier extension). Based on the two informative priors, the posterior distribution of the parameters becomes then

$$p(\mu, \Sigma) \propto p_0(\Sigma)\Phi_0(\mu)\mathcal{L}(\mu, \Sigma), \tag{11}$$

where $\mathcal{L}(\mu, \Sigma)$ is the likelihood function of the data. It can be shown that the posterior mean of the expected return is roughly a weighed average of the prior mean and the sample mean of the data. The more the data, the more we learn from them to change our prior belief on the parameters.

Once the uncertainty in estimating Σ is modeled, there are a couple of complexities. While updating on the expected returns can be carried out based on Equation 11 by combining the Black-Litterman estimate with the likelihood function of the data, their posterior distribution will no longer be normal, but be t-distributed. Furthermore, the Bayesian portfolio choice decision should now be determined by maximizing the expected utility under the predictive distribution of the data. Fortunately, the tools provided by Pástor (2000), for example, can be readily adapted to the implementation here.

Our second generalization of the BLM is to allow the expected returns be explained by a number of factors. The return data-generating process can then be written as,

$$R = \alpha + \beta_1 f_t + \cdots + \beta_K f_K + \epsilon, \tag{12}$$

where $f_1, \ldots, f_K$ are the factors and K is the number of factors. The factor model is the usual market model regression if the market portfolio is taken as the single factor. Since one-factor model is usually rejected for stock returns, the Fama and French (1993) three-factor model is a popular alternative. Another popular choice is to argument the three-factor model by the momentum factor of Carhart (1997). Under the factor model, the Bayesian portfolio choice can be carried almost exactly as before. Moreover, since the factors provide additional information on the asset returns, the estimated expected returns now should be more accurate than before.

The factor model also permits an investor to express his prior belief on the degree of validity of a given asset pricing model. If the investor believes dogmatically about the Fama and French (1993) three-factor model, the alphas should be identically zero. In general, a normal prior can be imposed on the alphas with a multiplier parameter on the covariance matrix to reflect the degree of belief on how close to zero they are. This prior can clearly be easily combined with $\Phi_0(\mu)$. In addition, the investors can compare and assess competing asset pricing models. Pástor and Stambaugh (2000) have pioneered studies of this kind. Their insights and tools are readily applied here in conjunction with the BLM.

The factors above are assumed observable. That is, we know what they are and can collect data on them. Alternatively, we can assume that they are latent factors, which are completely unobservable and unknown, except for the value of K. In this case, the Bayesian learning is much more complex than before. The methods developed by Geweke and Zhou (1994), and Nardari and Scruggs (2007) are well suited for the purpose, however.

Our third generalization of the BLM is to allow informational variables to forecast the future expected returns, so that the portfolio decision varies with time-varying economic conditions. In this case, the data-generating process can be modeled as,

$$R_t = a_0 + BZ_{t-1} + v_t, \tag{13}$$

where Z_{t-1} is a vector of M predictive variables available at time t, v_t is a vector of the residuals. The predictive variables can be allowed to follow a VAR(1) process,

$$Z_t = C_0 + C_1 Z_{t-1} + u_t, \tag{14}$$

with residuals u_t. In finance, there is a huge literature on the predictability of stock returns. Avramov (2004), among others, show how the predictability as well as asset pricing models can be used to improve one's portfolio selection significantly. All of the existing tools developed in this area can be adapted here to combine the BLM in the same way as before.

Finally, much of the exiting studies rely on normality assumption on the asset returns. Tu and Zhou (2004) show that normality is rejected with P-values virtually zero for US stock returns. On the other hand, the t-distribution can model the data well despite of skewness of the data, for the chance of observing a large skewness from a symmetric t is very high. Given a t distribution for the data, the portfolio allocations can be drastically different from those under normality, but there are not much differences in terms of the utilities. However, beyond stationary models for the stock returns, one can capture business cycles and structural breaks of the economy. In general, not only the portfolio allocations will be different, but also the utilities. For example, Tu (2007) provides a regime-switching model to capture the changing nature of bull and bear markets, and finds that the utility gains are substantial in comparison with the case of simply ignoring the dynamics of the data. This, too, can be combined with the BLM here based on Equation 11.

Conclusion

Black and Litterman (1992) provide an ingenious and yet simple solution to the optimal portfolio choice problem by blending an investor's proprietary views with the view of the market to construct a portfolio that is meaningful in practice. Their model in particular offers a useful framework for beating the benchmark index by over- and under-weighting certain assets. However, their model ignores the data-generating process whose dynamics can have significant impact on future portfolio returns. Our paper provides a simple way to blend further the Black and Litterman (1992) model with the data. There are three implications. First, theoretically, if the data represent the truth and if there are enough of them, there is no guarantee that the BLM solution will converge to any limit, but the Bayesian blending of the BLM with data must converge to the true optimal portfolio. Second, our framework allows practitioners to combine the insights from the Black and Litterman (1992) model with the data to generate potentially more reliable trading strategies and more robust portfolios. Third, many existing Bayesian learning tools, that are developed in the finance literature to update priors via the likelihood function of the data, can now be readily applied to practical portfolio selections in conjunction with the Black and Litterman (1992) model.

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