Portfolio Management Review Session: Week 4

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Find MVP and Optimal Portfolio

- We have two portfolios and want to combine them to find the minimum variance portfolio and optimal portfolio.
- MVP: find weights $w = (w_1, w_2)$ such that portfolio has minimum possible variance.
- Optimal Portfolio: Portfolio of N risky assets and risk-free which maximizes utility.
- Tangency Portfolio: Portfolio of N risky assets which is combined with risk-free asset in optimal portfolio.
- $E[r_1] = 13\%, \sigma_1 = 0.15$
- $E[r_2] = 10\%, \sigma_2 = 0.12$
- $\sigma_{1,2} = -0.05$
- $r_f = 3\%$

MVP

- Portfolio weights, w_{MVP}, which minimize $\sigma_p^2 = w^{'} V w$.
- V: covariance matrix of N risk assets.
- Solution:

$$w_{MVP} = \frac{V^{-1} i}{i' V^{-1} i}$$

- i: i is an $N \times 1$ column vector of ones.
- Need to find V^{-1} then plug in.

Inverse of 2x2 Matrix

- General Solution for matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{detA} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- This problem:

$$V = \begin{bmatrix} (0.15)^2 & -0.05 \\ -0.05 & (0.12)^2 \end{bmatrix} \Rightarrow V^{-1} = \frac{1}{detV} \begin{bmatrix} (0.12)^2 & 0.05 \\ 0.05 & (0.15)^2 \end{bmatrix}$$

$$= \frac{1}{(0.15)^2(0.12)^2 - (-0.05)^2} \begin{bmatrix} (0.12)^2 & 0.05 \\ 0.05 & (0.15)^2 \end{bmatrix}$$

$$= \begin{bmatrix} -6.176 & -22.9779 \\ -22.9779 & -10.3401 \end{bmatrix}$$

Inverse of 2x2 Matrix

- $w_{MVP} = \frac{V^{-1}i}{i'V^{-1}i}$
- Plug and chug:

$$w_{MVP} = \frac{\begin{bmatrix} -6.176 & -22.9779 \\ -22.9779 & -10.3401 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -6.176 & -22.9779 \\ -22.9779 & -10.3401 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} -29.5956 \\ -33.3180 \end{bmatrix}}{-62.9136} = \begin{bmatrix} 0.4704 \\ 0.5296 \end{bmatrix}$$

- Put 47% in portfolio one and 52% in portfolio 2 to obtain minimum variance.

Tangency Portfolio

- $w_{TP} = \frac{V^{-1}E}{i' V^{-1} E}$
- $E = \begin{bmatrix} E[r_1] - r_f \\ E[r_2] - r_f \\ \dots \\ E[r_N] - r_f \end{bmatrix}$
- For this problem $E = \begin{bmatrix} 0.13 - 0.03 \\ 0.10 - 0.03 \end{bmatrix}$
- Plug in and get:

$$w_{TP} = \begin{bmatrix} 0.4290 \\ 0.5710 \end{bmatrix}$$

- Put 42.9% in portfolio one, and 57.1% in portfolio 2 for tangency portfolio. Then combine optimally with risk-free asset based on utility.

Tangency Portfolio: Rationality?

- TP is a mathematical concept, and for an individual investor does not depend on rationality of market.
- TP is the maximization problem for given investor **taking as given** the actions of others.
- If every investor is rational, TP is market portfolio (CAPM)
- True TP, Ex-post TP, Ex-ante TP
- How to test for TP?

$$R_{i,t} - R_{f,t} = \alpha + \beta(R_{p,t} - R_{f,t}) + \epsilon_{i,t}$$

- If P is TP, then $\alpha = 0$ for every asset i.

Bayesian Alpha

- Adjusting β is common practice.
- Why not adjust α?
- If we have reason to believe CAPM is true, we want to incorporate that knowledge into our estimated α.
- Therefore weight between α estimate and prior belief:

$$\alpha^* = w\hat{\alpha} + (1 - w)\alpha_0$$

- As prior uncertainty increases, increase w. And vice versa.

Covariance Matrix Shrinking

- With large number of assets (N), it becomes computationally difficult to invert matrix.
- Diagonal matrices are easy to invert.
- Shrink V to a diagonal matrix with variances on diagonal.
- How? Some weighted average of $\hat{V}$ and D.
- Problem Set!

Investment Restrictions: Why?

- In solutions above, no restriction on w.
- Optimal weights could have $w << 0$, i.e. short-sell constraints.
- In practice, there are short-sell constraints (e.g. $w \geq 0$).
- Additionally, organization may put limit on exposure to certain asset classes (e.g. Harvard Management Company)
- We want to incorporate these restrictions into our optimization.

Investment Restrictions: How?

- Maximize Utility subject to constraints:

$$\max_{w} U = Ep - \frac{\gamma}{2}\sigma_p^2 = w^{'}E - \frac{\gamma}{2}w^{'}Vw$$

$$\text{s.t.} \sum_{i=1}^{N} w_i = 1$$

$$\text{Other Constraints (e.g } w_j \geq 0)$$

- Need to know risk aversion, γ.

Investment Restrictions: Alternative Forumlation

- What if we don't know (or want to estimate) γ?
- Pick target return ($\bar{E}$) and find optimal portfolio (i.e. minimum variance), subject to constraints:

$$\min_{w} \sigma_p^2 = w^{'} V w$$
$$\text{s.t.} Ep = w^{'} E = \bar{E}$$
$$\sum_{i=1}^{N} w_i = 1$$
$$\text{Other Constraints (e.g } w_j \geq 0)$$

- Example: HMC has bounds on exposure to different assets.

Investment Restrictions: Effects

- Restrictions shift in efficient frontier.
- Without restrictions, we can do everything in world with restrictions **plus** additional strategies.
- We must do at least as well (if we're smart...)