Lecture 1: Properties of Returns

In this lecture, we will

- define returns, simple and continuously compounded
- analyze distributions of returns
- construct multiperiod returns
- compute shortfall probabilities
- discuss simulation approaches

$\boxed{\textbf{One-Period Returns}}$

- Simple net return is defined as

$$R_{t+1} \;=\; \frac{D_{t+1} + P_{t+1}}{P_t} - 1 = \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

Two components:

– Income yield $= \frac{D_{t+1}}{P_t}$

– Capital gain/loss $= \frac{P_{t+1} - P_t}{P_t}$

Moments of Returns

- View returns R_t, $t = 1, \ldots, T$, as random variables

First moment: Expected return

$$E = \mathrm{E}(R_t)$$

Second moment: Variance

$$\sigma^2 = \mathrm{E}\left[(R_t - E)^2\right]$$

Third moment: Skewness

$$skewness = \mathrm{E}\left[(R_t - E)^3\right] \;/\; \sigma^3$$

Fourth moment: Kurtosis

$$kurtosis = \mathrm{E}\left[(R_t - E)^4\right] \;/\; \sigma^4$$

Estimating Moments of Returns

- Take a sample of realized returns: $R_1, R_2, \ldots, R_T$
- You can estimate E by the sample mean $\bar{R}$:

$$\bar{R} = \frac{1}{T} \sum_{t=1}^{T} R_t$$

- You can estimate σ^2 by the sample variance s^2:

$$s^2 = \frac{1}{T-1} \sum_{t=1}^{T} \left(R_t - \bar{R}\right)^2$$

- There are also sample versions of *skewness* & *kurtosis*

  – *In MATLAB, arrange returns in a* $T \times 1$ *vector* $R = [R_1; R_2; \ldots; R_T]$, *and compute mean(R), var(R), std(R), skewness(R), kurtosis(R).*

  – *In Assignment 1, download the data files from Chalk into your current directory and type*

    $>> x = load('returns\_annual.txt');$

    $>> Rstock = x(:, 2);$

    $>> Rbond = x(:, 3);$

    $>> mean(Rstock)$

    $ans = 0.1303$

    $>> std(Rstock)$

    $ans = 0.1741$

    $>> [skewness(Rstock) \;\; kurtosis(Rstock)]$

    $ans = -0.4233 \;\; 2.9944$

    $>> [mean(Rbond) \;\; std(Rbond) \;\; skewness(Rbond) \;\; kurtosis(Rbond)]$

    $ans = 0.0602 \;\; 0.1105 \;\; 0.8491 \;\; 4.0099$

  – *How do average returns and std's of stocks and bonds compare?*

Confidence Intervals

- Assume that returns $\{R_t\}$ are normally distributed,

$$R_t \sim N(E, \sigma^2)$$

- Confidence intervals for any one R_t:

$$\begin{aligned}
\text{Prob}\,(E - 1.00\sigma \ < \ R_t \ < \ E + 1.00\sigma) &= 68\% \\
\text{Prob}\,(E - 1.65\sigma \ < \ R_t \ < \ E + 1.65\sigma) &= 90\% \\
\text{Prob}\,(E - 1.96\sigma \ < \ R_t \ < \ E + 1.96\sigma) &= 95\%
\end{aligned}$$

- In practice, replace the unobserved parameters E and σ by their estimates $\bar{R}$ and s. When T is reasonably large, the probabilities are not significantly affected:

$$\begin{aligned}
\text{Prob}\left(\bar{R} - 1.65s \ < \ R_t \ < \ \bar{R} + 1.65s\right) &\approx 90\% \\
\text{Prob}\left(\bar{R} - 1.96s \ < \ R_t \ < \ \bar{R} + 1.96s\right) &\approx 95\%
\end{aligned}$$

  - *In Assignment 1, the 95% confidence interval for next year's stock return is*

    $>> [mean(Rstock) - 1.96 * std(Rstock) \ \ mean(Rstock) + 1.96 * std(Rstock)]$

    $ans = -0.2110 \ \ 0.4715$

  - *The 95% confidence interval for next year's bond return is*

    $>> [mean(Rbond) - 1.96 * std(Rbond) \ \ mean(Rbond) + 1.96 * std(Rbond)]$

    $ans = -0.1563 \ \ 0.2768$

  - *How do these intervals compare?*

- Confidence intervals for E:

  $\text{Prob}\left(\bar{R} - 1.65s(\bar{R}) \ < \ E \ < \ \bar{R} + 1.65s(\bar{R})\right) \approx \ 90\%$

  $\text{Prob}\left(\bar{R} - 1.96s(\bar{R}) \ < \ E \ < \ \bar{R} + 1.96s(\bar{R})\right) \approx \ 95\%$

- How do we compute $s(\bar{R})$, the standard error of $\bar{R}$?
  - $\bar{R}$ is normally distributed: $\bar{R} \sim N(E, \frac{\sigma^2}{T})$
  - The standard error of $\bar{R}$ is therefore

$$s(\bar{R}) = \frac{s}{\sqrt{T}}$$

- *In Assignment 1, 95% confidence interval for the average stock return over the following 30 years is*

  $>> [mean(Rstock) - 1.96 * std(Rstock)/sqrt(30) \ \ mean(Rstock) + 1.96 * std(Rstock)/sqrt(30)]$

  $ans = 0.0680 \quad 0.1926$

- *The 95% confidence interval for the average 30-year bond return is*

  $>> [mean(Rbond) - 1.96 * std(Rbond)/sqrt(30) \ \ mean(Rbond) + 1.96 * std(Rbond)/sqrt(30)]$

  $ans = 0.0207 \quad 0.0998$

- *How do these intervals compare to those from the previous page?*

- To test the hypothesis that a given asset offers expected return $E = \mu$, construct the t-statistic

$$t = \frac{\bar{R} - \mu}{s(\bar{R})} = \sqrt{T}\frac{\bar{R} - \mu}{s}$$

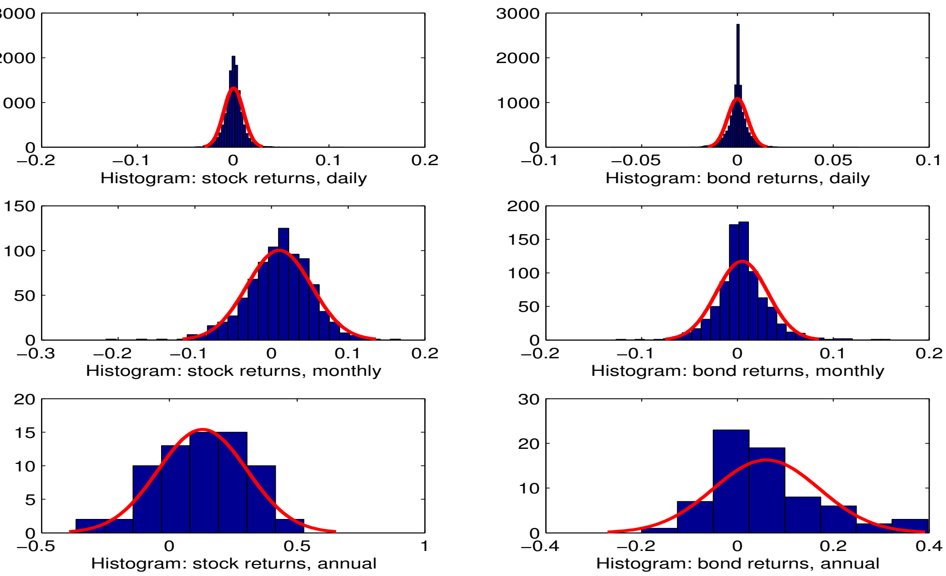
  - if the absolute value of the t-statistic exceeds 2, then we reject $E = \mu$ with 95% confidence
  - or see if the confidence interval for E includes μ

Distributions of Returns

- Most common assumption: normal distribution

  Is this assumption appropriate?

- Plot the histograms of stock and bond returns:
  - *In MATLAB, we can simply type hist(Rstock) or histfit(Rstock)*
  - *The results from Assignment 1:*



  - *What do we conclude from these graphs?*

- For normal distribution, $skew = 0$ and $kurt = 3$
  - *In our data,*

	Stocks			*Bonds*		
	daily	*monthly*	*annual*	*daily*	*monthly*	*annual*
Skewness	*-0.57*	*-0.42*	*-0.42*	*0.10*	*0.56*	*0.85*
Kurtosis	*22.57*	*4.53*	*2.99*	*11.34*	*7.13*	*4.01*

  - *Jarque-Bera test rejects normality, except for annual stock returns*
  - *We'll keep this in mind, and proceed using normality at first*

Shortfall Probability

- What is the probability that our portfolio will fail to achieve a minimum return threshold next period?
  - $\text{Prob}(R < K) = ?$, where K is a fixed threshold
- Recall that $R \sim N(E, \sigma^2) \Rightarrow z = \frac{R-E}{\sigma} \sim N(0,1)$
- Therefore,

$$\begin{aligned}\text{Prob}(R < K) &= \text{Prob}(\frac{R-E}{\sigma} < \frac{K-E}{\sigma}) \\ &= \text{Prob}(z < \frac{K-E}{\sigma})\end{aligned}$$

- $\text{Prob}(z < Z)$ can be computed using a standard normal table from your statistics course (see last page)
- **Example:** What is the probability that the S&P 500 index will outperform the 5% T-bill rate next year?

$$\text{Prob}(R > 0.05) =$$

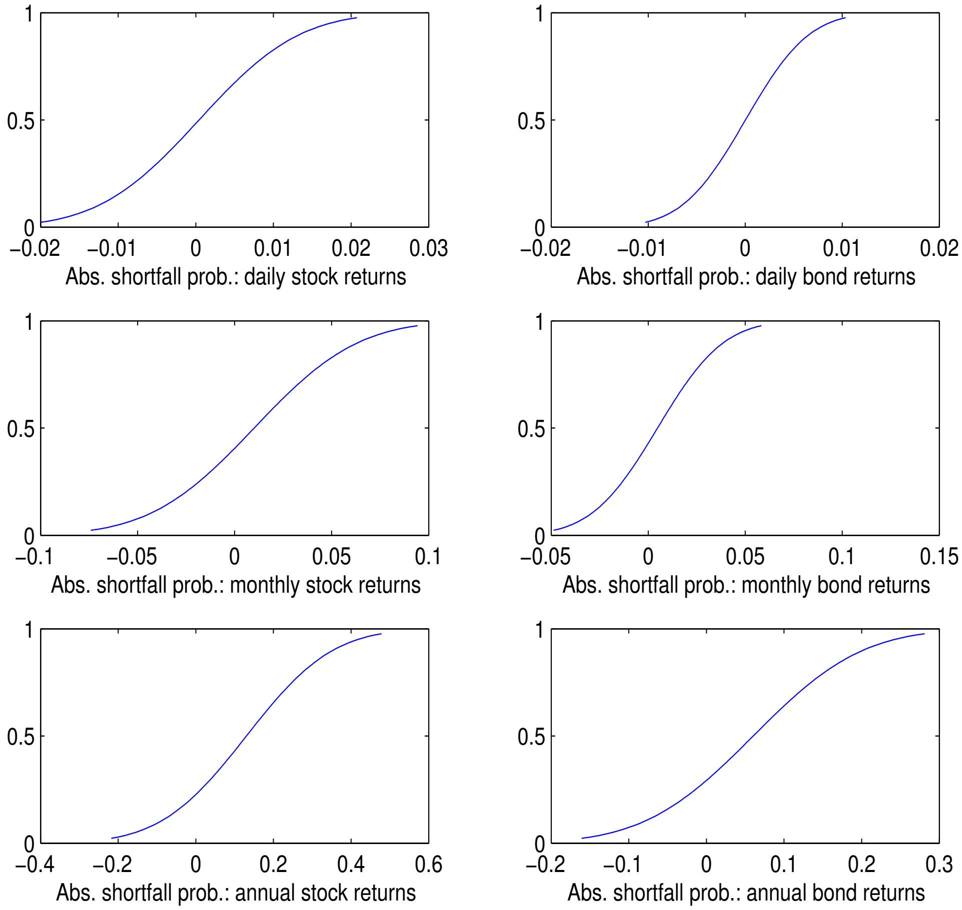
  - *Computing* $\text{Prob}(z < Z)$ *in MATLAB:*

    $>> ProbRbigger = 1 - normcdf((0.05 - mean(Rstock))/std(Rstock))$

    $ProbRbigger = 0.6777$

- How do shortfall probabilities vary across stocks and bonds? Across different investment horizons?

  – *In Assignment 1, we can use the normcdf command for a range of threshold levels, and obtain the following:*

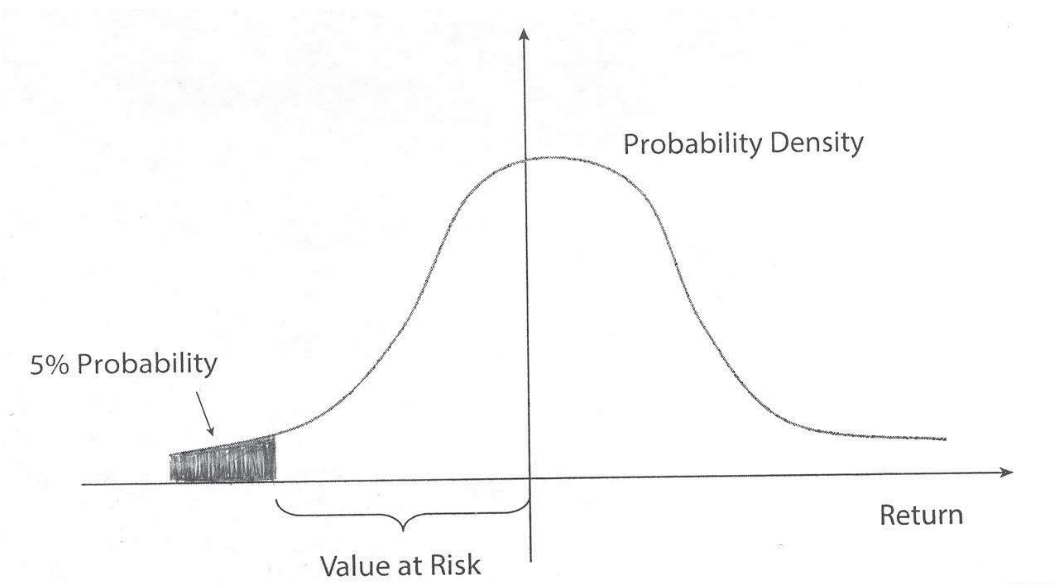


  – *How do we interpret these graphs?*

Value at Risk (VaR)

- Popularized in 1994 by JP Morgan through its free RiskMetrics service

- Instead of considering a given threshold return, consider a given shortfall probability (e.g., 5%)



- For a given investment, denote
  - V: the current dollar value of the investment
  - ΔV: change in V over a given future horizon
- The return on investment is then $R = \Delta V / V$
- Assuming normal distribution, $R \sim N(E, \sigma^2)$,

$$\begin{aligned} \text{Prob}\,(R < E - 1.65\sigma) &= 0.05 \\ \text{Prob}\,(\Delta V < V\,(E - 1.65\sigma)) &= 0.05, \end{aligned}$$

so there is 5% probability of a dollar loss greater than

$$-V\,(E - 1.65\sigma)$$

- **Example:** For a \$1 million investment in S&P 500,

Horizon	E	σ	$-V\,(E - 1.65\sigma)$
1 day	0.041%	1.02%	\$16,384
1 month	1.01	4.21	59,357
1 year	13.03	17.41	156,991

- For short horizons (such as 1 day), we often simply set $E = 0$ and compute

$$\text{VaR} = V \times 1.65\sigma$$

  – A particularly simple measure of risk! But…

- What are the drawbacks of VaR?

  –

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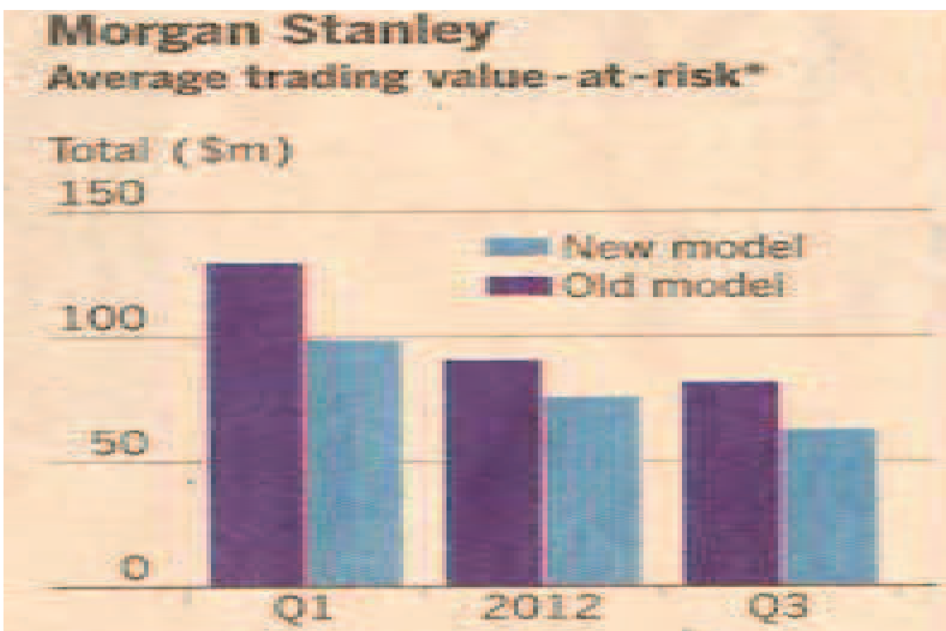
- **Example:**



- So... should we continue using VaR?

Covariance and Correlation

- How do asset returns comove?
- For two assets i and j, covariance is defined as
$$\sigma_{ij} = \mathrm{Cov}(R_i, R_j) = \mathrm{E}\left[(R_i - E_i)(R_j - E_j)\right],$$
and correlation as
$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}, \quad -1 \leq \rho_{ij} \leq 1$$
- To estimate covariance and correlation, compute
$$\hat{\sigma}_{ij} = \frac{1}{T} \sum_{t=1}^{T} (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$
$$\hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{s_i s_j}$$
  - *In MATLAB, use cov and corrcoef. In Assignment 1,*

```
>> cov([Rstock  Rbond])
ans = 0.0303   -0.0002
     -0.0002    0.0122
>> corrcoef([Rstock  Rbond])
ans = 1.0000   -0.0121
     -0.0121    1.0000
```

- Correlation squared indicates what fraction of return variance on asset i is explained by asset j's return
  - If you regress R_i on R_j (or vice versa), the R-squared from this regression is ρ_{ij}^2

Relative Performance

- What is the probability that one asset will outperform another? $\text{Prob}(R_i < R_j) = ?$
  - E.g., outperforming a benchmark index
- Denote $D = R_i - R_j$. Then

$$\begin{aligned}
\text{E}(D) &= \text{E}(R_i - R_j) = E_i - E_j \\
\sigma^2(D) &= \text{Var}(R_i - R_j) \\
&= \text{Var}(R_i) + \text{Var}(R_j) - 2\text{Cov}(R_i, R_j) \\
&= \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\rho_{ij}
\end{aligned}$$

- Therefore,

$$\begin{aligned}
\text{Prob}(R_i < R_j) &= \text{Prob}(D < 0) \\
&= \text{Prob}\left(\frac{D - \text{E}(D)}{\sigma(D)} < \frac{-\text{E}(D)}{\sigma(D)}\right) \\
&= \text{Prob}\left(z < \frac{-\text{E}(D)}{\sigma(D)}\right) \\
&= \text{Prob}\left(z < \frac{E_j - E_i}{\sqrt{\sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\rho_{ij}}}\right)
\end{aligned}$$

  - *In MATLAB, two equivalent ways of computing* $\text{Prob}(D < 0)$*:*

    $>> D = Rstock - Rbond;$

    $>> ProbDneg = normcdf(-mean(D)/std(D))$

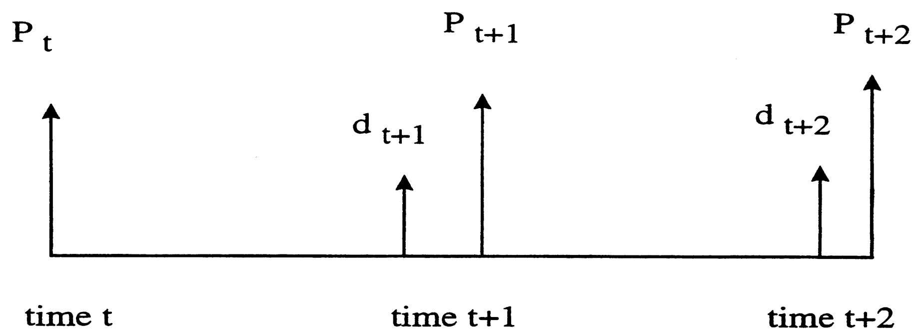
    $ProbDneg = 0.3677$

    $>> ProbDneg = normcdf(0, mean(D), std(D))$

    $ProbDneg = 0.3677$

Multiperiod Returns

- How do we compound returns over time?



$$1 + R_{t+1} = \frac{d_{t+1} + P_{t+1}}{P_t} \quad \ldots \quad \text{return from } t \text{ to } t+1$$

$$1 + R_{t+2} = \frac{d_{t+2} + P_{t+2}}{P_{t+1}} \quad \ldots \quad \text{return from } t+1 \text{ to } t+2$$

Then the two-period return between t and $t+2$ is

$$\begin{aligned} 1 + R_{t \to t+2} &= \frac{d_{t+2} + P_{t+2} + d_{t+1}\left(\frac{d_{t+2}+P_{t+2}}{P_{t+1}}\right)}{P_t} \\ &= \left(\frac{d_{t+2} + P_{t+2}}{P_{t+1}}\right) . \left(\frac{d_{t+1} + P_{t+1}}{P_t}\right) \\ &= (1 + R_{t+1}) \,.\, (1 + R_{t+2}) \end{aligned}$$

- Compounding is easier with continuously compounded returns (summation instead of multiplication)

Continuously Compounded Returns

- Let R_t be the simple return in period t.
  The continuously compounded return r_t is then
  $$r_t = \ln(1 + R_t),$$
  where ln denotes the natural logarithm.
- The reverse relation is also worth spelling out:
  $$R_t = \exp(r_t) - 1 = e^{r_t} - 1,$$
  where $e = 2.71828\ldots$ is the Euler number.
- Why is this rate called continuously compounded?

- We will assume that r_t are i.i.d. over time, with
  $$\begin{aligned} \mathrm{E}(r_t) &= \mu \\ \mathrm{Var}(r_t) &= \sigma^2, \end{aligned}$$
  and that r_t are normally distributed:
  $$r_t \sim N(\mu, \sigma^2)$$
  - Same assumptions as in the Black-Scholes model
  - Does this make more sense than $R_t \sim N(\mu, \sigma^2)$?

Future Value of Investment

- If you initially invest \$1, the value of your investment after T periods is

$$V_T = (1 + R_1)(1 + R_2) \dots (1 + R_T)$$

- Taking logs of both sides, we obtain

$$\ln(V_T) = \ln(1 + R_1) + \ln(1 + R_2) + \ldots + \ln(1 + R_T),$$

so that we can also write

$$\begin{aligned} \ln(V_T) &= r_1 + r_2 + \ldots + r_T \\ V_T &= \exp(r_1 + r_2 + \ldots + r_T) \end{aligned}$$

- Since a sum of normally distributed random variables is also normal, $\ln(V_T)$ is normal, with

$$\begin{aligned} \mathrm{E}\left(\ln(V_T)\right) &= \mathrm{E}(r_1 + r_2 + \ldots + r_T) \\ &= \mathrm{E}(r_1) + \mathrm{E}(r_2) + \ldots + \mathrm{E}(r_T) \\ &= T\mu \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\left(\ln(V_T)\right) &= \mathrm{Var}(r_1 + r_2 + \ldots + r_T) \\ &= \mathrm{Var}(r_1) + \mathrm{Var}(r_2) + \ldots + \mathrm{Var}(r_T) \\ &= T\sigma^2 \end{aligned}$$

(Note: All return covariances are zero due to our assumption that r_t are independent over time (i.i.d.).)

- Thus, the mean and variance of $\ln(V_T)$ are both proportional to the length of the investment horizon, T

- **Example:** Suppose you invest \$1m in S&P, whose continuously compounded returns have mean 10% and standard deviation 20%. What is the distribution of the log future value of your investment after 10 years?

  – *In MATLAB, you can solve this problem as follows:*

    $>> r = log(1 + Rstock);$

    $>> ElnVT = 10 * mean(r)$

    $ElnVT = 1.0977$

    $>> VarlnVT = 10 * var(r)$

    $VarlnVT = 0.2706$

  What is the distribution of V_T? See next page.

- The average continuously compounded return is

$$\bar{r} = \frac{1}{T}(r_1 + r_2 + \ldots + r_T) = \frac{1}{T}\ln(V_T).$$

  Therefore,

$$\begin{aligned} \mathrm{E}(\bar{r}) &= \frac{1}{T}\mathrm{E}(\ln(V_T)) = \mu \\ \mathrm{Var}(\bar{r}) &= \left(\frac{1}{T}\right)^2 \mathrm{Var}(\ln(V_T)) = \frac{1}{T}\sigma^2 \end{aligned}$$

  – Note: The variance of $\bar{r}$ shrinks as T increases

The Lognormal Distribution

- The random variable x has lognormal distribution if $y = \ln(x)$ is normal

  $\Longrightarrow$ Since $\ln(V_T)$ is normal, V_T is lognormal

- From the properties of the lognormal distribution,

$$\begin{aligned}
\mathrm{E}\,(V_T) &= \exp\left[T(\mu + \sigma^2/2)\right] \\
\mathrm{Median}\,(V_T) &= \exp(T\mu) \\
\mathrm{Var}\,(V_T) &= \left[\exp(T\sigma^2) - 1\right] \exp\left[2T(\mu + \sigma^2/2)\right]
\end{aligned}$$

- **Example:**
  What is the distribution of V_T in the S&P example?

$$\mathrm{E}\,(V_T) =$$

$$\mathrm{Median}\,(V_T) =$$

$$\mathrm{Var}\,(V_T) =$$

Absolute Shortfall Risk

- What is the probability that the final investment value V_T will be below a given threshold level K?
  - $\text{Prob}(V_T < K) =?$
- Since $\ln(V_T) \sim N(T\mu, T\sigma^2)$, we have

$$\begin{aligned}
\text{Prob}(V_T < K) &= \text{Prob}(\ln(V_T) < \ln(K)) \\
&= \text{Prob}\Big(\frac{\ln(V_T) - T\mu}{\sqrt{T}\sigma} < \frac{\ln(K) - T\mu}{\sqrt{T}\sigma}\Big) \\
&= \text{Prob}\Big(z < \frac{\ln(K) - T\mu}{\sqrt{T}\sigma}\Big),
\end{aligned}$$

  which we can look up in the standard normal table
- **Example:** You have \$20m invested in a portfolio with $\mu = 1\%$ and $\sigma = 4\%$ per month. What is the probability that your investment will be worth less than \$25m in 3 years?

$$\text{Prob}(V_T < \frac{25}{20}) =$$

  - *In MATLAB, this is a one-liner:*
    $>> ProbShortfall = normcdf(log(25/20), 36 * 0.01, 6 * (0.04))$
    $ProbShortfall = 0.2843$

- **Example:** What is the probability that a risky investment whose continuously compounded return has mean μ and variance σ^2 will underperform a risk-free investment with continuously compounded return r?

$$\begin{aligned}
\text{Prob}(V_T < e^{Tr}) &= \text{Prob}(\ln(V_T) < Tr) \\
&= \text{Prob}(z < \frac{Tr - T\mu}{\sqrt{T}\sigma}) \\
&= \text{Prob}(z < \sqrt{T}\frac{r - \mu}{\sigma})
\end{aligned}$$

As T grows, this probability goes to zero (if $\mu > r$).

Relative Shortfall Risk

- What is the probability that one risky asset will outperform another over T periods?
- Consider two assets, S and B, whose contin. comp. returns are i.i.d. normal with correlation ρ_{SB}:

$$\begin{aligned}
r_{S,t} &\sim N(\mu_S, \sigma_S^2) \\
r_{B,t} &\sim N(\mu_B, \sigma_B^2)
\end{aligned}$$

- What is the probability that stocks will underperform bonds over T periods?

$$\text{Prob}(V_{ST} < V_{BT}) = ?$$

- First, note that

$$\begin{aligned}\text{Prob}(V_{ST} < V_{BT}) &= \text{Prob}(\ln(V_{ST}) < \ln(V_{BT})) \\ &= \text{Prob}(\underbrace{\ln V_{ST} - \ln V_{BT}}_{\Delta} < 0)\end{aligned}$$

- Second, note that

$$\begin{aligned}\Delta &= (r_{S,1} + r_{S,2} + \ldots + r_{S,T}) - (r_{B,1} + r_{B,2} + \ldots + r_{B,T}) \\ &= (r_{S,1} - r_{B,1}) + (r_{S,2} - r_{B,2}) + \ldots + (r_{S,T} - r_{B,T}),\end{aligned}$$

so that Δ is normal with

$$\begin{aligned}\text{E}(\Delta) &= T\ \text{E}(r_{S,t} - r_{B,t}) = T(\mu_S - \mu_B) \\ \sigma^2(\Delta) &= T\ \text{Var}(r_{S,t} - r_{B,t}) = T(\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B)\end{aligned}$$

- **Example:** What is the probability that stocks will underperform bonds over 5 years? Assume $\mu_S = 0.10$, $\mu_B = 0.05$, $\sigma_S = 0.20$, $\sigma_B = 0.08$, $\rho_{SB} = 0.12$; annual

$$\begin{aligned}\text{E}(\Delta) &= 5(0.10 - 0.05) = 0.25 \\ \sigma^2(\Delta) &= 5\left[(0.20)^2 + (0.08)^2 - 2(0.12)(0.20)(0.08)\right] = 0.21\end{aligned}$$

$$\begin{aligned}\text{Prob}(V_{ST} &< V_{BT}) = \text{Prob}(\Delta < 0) \\ &= \text{Prob}\left(\frac{\Delta - \text{E}(\Delta)}{\sigma(\Delta)} < \frac{-\text{E}(\Delta)}{\sigma(\Delta)}\right) \\ &= \text{Prob}\left(z < \frac{-\text{E}(\Delta)}{\sigma(\Delta)}\right) = \text{Prob}\left(z < \frac{-0.25}{\sqrt{0.21}}\right) \\ &= 0.29\end{aligned}$$

- Note that

$$\frac{-\mathrm{E}(\Delta)}{\sigma(\Delta)} = -\frac{T(\mu_S - \mu_B)}{\sqrt{T}\underbrace{\sqrt{\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B}}_{c}} = -\sqrt{T}\frac{\mu_S - \mu_B}{c}$$

  approaches either plus or minus infinity as $T \to \infty$, so $\mathrm{Prob}(V_{ST} < V_{BT})$ approaches zero or one as $T \to \infty$

- Specifically, as long as $\mu_S > \mu_B$, then the probability of stocks underperforming bonds approaches zero

- In general, the asset with the higher expected continuously compounded return will outperform the other asset almost surely as $T \to \infty$

How do we solve such a relative shortfall problem in MATLAB?

Begin by constructing a series of differences $r_{S,t} - r_{B,t}$*:*

$>> dif = log(1 + Rstock) - log(1 + Rbond);$

Estimate $\mathrm{E}(\Delta)$ *and* $\sigma^2(\Delta)$*:*

$>> EDel = 5 * mean(dif);$

$>> VDel = 5 * var(dif);$

Compute $\mathrm{Prob}(\Delta < 0)$ *(two equivalent ways):*

$>> Prob = normcdf(-EDel/sqrt(VDel))$

$Prob = 0.2613$

$>> Prob = normcdf(0, EDel, sqrt(VDel))$

$Prob = 0.2613$

Simulation Approaches

- Simulation is a powerful technique that allows us to solve a variety of problems in portfolio management
- Basic steps
  1. Simulate T random draws R_t, $t = 1, \ldots, T$, using one of the two methods described below
  2. Use these draws to construct one draw of V_T as
  $$V_T = (1 + R_1)(1 + R_2) \ldots (1 + R_T)$$
  3. Repeat steps 1 and 2, say, 10,000 times
  4. Use the resulting 10,000 simulated draws of V_T to answer various questions about V_T
- Example: Absolute shortfall risk
  - To compute Prob($V_T < K$), take the number of V_T's for which $V_T < K$, and divide it by 10,000
- Example: Relative shortfall risk
  - To compute Prob($V_{ST} < V_{BT}$), first simulate 10,000 values of both V_{ST} and V_{BT}
  - Then count the number of simulations for which $V_{ST} < V_{BT}$, and divide it by 10,000
- We work with R_t, but you could also work with r_t

1. The Known-Distribution Method

- Assume that returns $\{R_t\}$ are normally distributed,
$$R_t \sim N(E, \sigma^2)$$
  - Other distributions can be used as well, e.g. fat-tailed Student t (simulate in MATLAB: *trnd*)
- To simulate one draw of V_T,
  1. Draw a value z from standard normal distribution
  2. Construct a draw of R_t as $R_t = E + \sigma z$
  3. Repeat steps 1 and 2, T times
  4. Use these draws of R_t to construct one draw of V_T

In MATLAB, you can construct one draw of V_T *as follows:*
$>> E = 0.10; sigma = 0.20; T = 50;$
$>> z = randn(T, 1);$
$>> R = E + sigma * z;$
$>> V = prod(1 + R);$
Here, V *is one draw of* V_T*.*

Constructing 10,000 draws of V_T *is no more difficult:*
$>> E = 0.10; sigma = 0.20; T = 50; N = 10000;$
$>> z = randn(T, N);$
$>> R = E + sigma * z;$
$>> V = prod(1 + R);$
Here, V *is a* $1 \times 10,000$ *row of* V_T*'s.*

To compute $\text{Prob}(V_T < 10)$*, simply type*
$>> ProbVTless10 = sum(V < 10)/10000$
$ProbVTless10 = 0.1191$

- An alternative MATLAB approach uses the "for" loop. This approach involves more lines of code, but it is simpler conceptually, and instructive in any event:

```
clear all
E = 0.10; sigma = 0.20; T = 50; N = 100000;
V = zeros(N,1);             % initialize V
for n = 1:N                 % loop over simulations
    R = zeros(T,1);         % initialize R
    for t = 1:T             % loop over time periods
        z = randn;
        R(t) = E + sigma*z;
    end
    V(n) = prod(1+R);
end
ProbVTless10 = sum(V < 10)/N
```

- This code can be written in MATLAB's (or other) editor and saved in the current directory with an extension ".m" (e.g., as "simulation.m"). Then you can run the code by typing its name at the command line:

```
>> simulation

ProbVTless10 = 0.1154
```

- If you are working with r_t instead of R_t, replace $V_T = (1 + R_1) \ldots (1 + R_T)$ by $V_T = \exp(r_1 + \ldots + r_T)$

  In MATLAB, you get exactly the same answer from

```
>> V = prod(1 + R);
```

  and

```
>> r = log(1 + R);
>> V = exp(sum(r));
```

2. The Resampling Method

- Main advantage:
  No need to assume a specific distribution for returns
- Assume: R_t is generated from historical distribution
- To simulate one draw of V_T,
  1. Randomly select R_t from the historical return series on the same investment (with replacement)
  2. Repeat step 1, T times
  3. Use these draws of R_t to construct one draw of V_T

In MATLAB, you can construct one draw of R_t from its historical distribution (e.g., from a 69×1 column Rstock) as follows:

$>> a = ceil(69 * rand);$

$>> R = Rstock(a);$

To construct T draws of R_t, type

$>> a = ceil(69 * rand(T, 1));$

$>> R = Rstock(a);$

Here, R is a $T \times 1$ column of return draws. To cumulate these draws to construct one draw of V_T, type

$>> V = prod(1 + R);$

Constructing 10,000 draws of V_T using resampling is no more difficult:

$>> a = ceil(69 * rand(T, 10000));$

$>> R = Rstock(a);$

$>> V = prod(1 + R);$

Alternatively, you can use a "for" loop.

Table For $N(x)$ When $x \geq 0$

This table shows values of $N(x)$ for $x \geq 0$. The table should be used with interpolation. For example

$$\begin{aligned} N(0.6278) &= N(0.62) + 0.78[N(0.63) - N(0.62)] \\ &= 0.7324 + 0.78 \times (0.7357 - 0.7324) \\ &= 0.7350 \end{aligned}$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9986	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000