Are Momentum Profits Robust to Trading Costs?

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Abstract

This paper tests whether momentum-based strategies remain profitable after considering market frictions, in particular price concessions induced by trading. Alternative measures of price impact are estimated and applied to alternative momentum-based trading rules. The performance of traditional momentum strategies, in addition to strategies designed to reduce the cost of trades, is evaluated. We find that, after taking into account the price impact induced by trades, as much as 5 billion dollars (relative to December 1999 market capitalization) may be invested in some momentum-based strategies before the apparent profit opportunities vanish. Other, extensively studied, momentum strategies are not implementable on a large scale. The persistence of momentum returns exhibited in the data remains an important challenge to the asset-pricing literature.

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1 Introduction

There is a growing literature on the predictability of stock returns based on the information contained in past returns. At very short horizons, such as a week or a month, returns are shown to have negative serial correlation (reversal), while at three to twelve month horizons, they exhibit positive serial correlation (momentum). During longer horizons, such as three to five years, stock returns again exhibit reversals.[1] The momentum of individual stocks is extensively examined by Jegadeesh and Titman (1993, 2001). They show that one can obtain superior returns by holding a zero-cost portfolio that consists of long positions in stocks that have out-performed in the past (*winners*), and short positions in stocks that have under-performed during the same period (*losers*). The evidence suggests that most of the returns to a momentum trading strategy are due to losers rather than to winners (see, e.g., Grinblatt and Moskowitz (1999) and Hong, Lim, and Stein (2000)). The momentum anomaly is made more puzzling by the existence of pronounced seasonal patterns. The usual momentum return continues from February to November, increases in December, and then changes to a strong reversal in January (Jegadeesh and Titman (1993) and Grinblatt and Moskowitz (1999)). The seasonality in momentum is not, however, solely a January phenomenon (Heston and Sadka (2002)). To date, no measures of risk have been found that completely explain momentum returns. Fama and French (1996) find that a three factor asset pricing model can explain the returns of the long-horizon reversal portfolios, but not of the intermediate-term momentum portfolios. Grundy and Martin (2001) study the risk sources of momentum strategies and conclude that while factor models can explain most of the variability of momentum returns, they fail to explain their mean returns (also see Jegadeesh and Titman (2001)). Momentum has also been shown to be robust across national financial markets (see, e.g., Rouwenhorst (1998) and Bhojraj and Swaminathan (2001)). Some view this unexplained persistence of momentum returns throughout the last several decades as one of the most serious challenges to the asset-pricing literature (Fama and French (1996)).

In the absence of a risk premium-based explanation for momentum profits, an important question is whether there are significant limits to arbitrage (Shleifer and Vishny (1997)) that prevent investors from trading to such an extent that would drive away the apparent profits. While limits to arbitrage will not explain the underlying causes for the *existence* of profitable momentum strategies, they may explain their *persistence.* We investigate the effect of trading costs, including price impact, on the profitability of particular momentum strategies. In particular, we estimate the size of a momentum-based fund that could be achieved before abnormal returns are either statistically insignificant or driven to zero. We investigate several trading cost models and momentum portfolio strategies and find that the estimated excess returns of some momentum strategies disappear only after \$4.5 to over \$5.0 billion (relative to market capitalization in December 1999)

[1] For evidence on short horizon reversal, see Poterba and Summers (1988), and Jegadeesh (1990); for momentum and long run reversal, see DeBondt and Thaler (1985), Jegadeesh and Titman (1993, 2001), and Grinblatt and Moskowitz (1999).

is engaged (by a single fund) in such strategies. The statistical significance of these excess returns disappear only after \$1.1 to \$2.0 billion is engaged (by a single fund) in such strategies. Therefore, transaction costs, in the form of spreads and price impacts of trades, do not fully explain the return persistence of past winner stocks exhibited in the data. Thus, this anomaly remains an important puzzle.

There are several components of trading costs that differ dramatically in size and in ease of measurement. The components that can be measured with the least error are the explicit trading costs of commissions and bid/ask spreads. When trading an institutional size portfolio these proportional costs can be swamped by the additional non-proportional cost of price impact and the "invisible costs" of post-trade adverse price movement (Treynor (1994)). The nature of the price impact of trades has been the subject of extensive theoretical and empirical study (for example, Kyle (1985), Easley and O'Hara (1987), Glosten and Harris (1988), Hasbrouck (1991 a,b), Huberman and Stanzl (2000), and Breen, Hodrick, and Korajczyk (2002)). The economic importance of price impact is demonstrated by Loeb (1983), Keim and Madhavan (1996, 1997), and Knez and Ready (1996), who show that transaction costs increase substantially as the size of an order increases. Incorporating the explicit trading costs (commissions and spreads) into portfolio returns has occurred in the literature for some time. For example, Schultz (1983) and Stoll and Whaley (1983) investigate the effect of commissions and spreads on size-based trading strategies. A number of studies investigate the effects of trading costs on prior-return based (momentum and contrarian) trading strategies. Ball, Kothari, and Shanken (1995) show that microstructure effects, such as bid/ask spreads, significantly reduce the profitability of a contrarian strategy. Grundy and Martin (2001) calculate that at round-trip transactions costs of 1.5%, the profits on a long/short momentum strategy become statistically insignificant. At round-trip transactions costs of 1.77%, they find that the profits on the long/short momentum strategy are driven to zero. Incorporating non-proportional price impacts of trades into trading strategies has only recently received significant attention. Knez and Ready (1996) study the effects of price impact on the profitability of a trading strategy based on the weekly autocorrelation and cross-autocorrelation of large-firm and small-firm portfolios. They find that the trading costs swamp the excess returns on strategy. Mitchell and Pulvino (2001) incorporate commissions and price impact costs into a merger arbitrage portfolio strategy. They find that the trading costs reduce the profits of the strategy by 300 basis points per year.

Sadka (2001) examines single-month momentum strategies at the turn of the year, since momentum strategies exhibit the highest excess returns during December and January. That paper concludes, however, that after considering costs of price impact, only a small amount can be invested before the apparent profit opportunities vanish. This paper extends the analysis of Sadka (2001) in several dimensions. We investigate momentum strategies over the entire year rather than at the turn of the year, and form momentum-based portfolios optimized to take into account the differences, across assets, in price impact. We conclude that momentum-based strategies are indeed exploitable (i.e., statistically significant) with investment amounts up to \$1.5 billion, and that abnormal returns are not driven to zero until the investment size reaches \$5.0 billion. Chen, Stanzl, and Watanabe (2001) estimate the maximal fund size attainable before price impacts eliminate profits on size, book-to-market, and momentum strategies. They find that maximal fund sizes are small for all strategies. We believe that our results differ from theirs mainly because we construct strategies designed to reduce price-impact costs.

We study the profitability of long positions in winner-based momentum strategies after accounting for the cost of trading. We incorporate several models of trading costs, including proportional and non-proportional costs. Two proportional cost models are based on quoted and effective spreads. We study two alternative price impact models (non-proportional costs): one based on Glosten and Harris (1988) and the other from Breen, Hodrick, and Korajczyk (2002). In addition to value-weighted and equal-weighted trading strategies commonly found in the literature, we derive a liquidity-weighted portfolio rule that maximizes post-price impact expected return on the portfolio, as well as strategies that combine the liquidity-weighted strategy and value-weighted (buy and hold) strategies. The liquidity-weighted portfolio is derived through a static optimization problem, rather than a fully dynamic portfolio setting. The fact that portfolio weights that are convex combinations of value weights and liquidity weights occasionally out-perform both value-weighted and liquidity-weighted strategies indicates that those combinations are closer to those from a dynamic optimization. For the price impact models, trading costs are non-proportional, and therefore the percentage costs grow with the size of the portfolio being traded. We calculate the size of the portfolio that (1) eliminates the statistical significance of the portfolio abnormal return, (2) drives the level of abnormal return to zero, and (3) drives the portfolio Sharpe ratio to that of the maximal Sharpe ratio obtained from combinations of the Fama and French (1993) market, size, and book-to-market portfolios.

In Section 2 we discuss the momentum literature and the particular portfolio strategies that we investigate. In Section 3 we introduce measures of proportional and non-proportional (price impact) trading costs. A trading model that incorporates price impacts is developed in Section 4. In Section 5 an optimal trading strategy with forecastable price impacts is derived. The performance of various momentum strategies is evaluated in Section 6. Concluding remarks are presented in Section 7.

2 Momentum Trading Strategies

Following Jegadeesh and Titman (1993), we define momentum-based strategies by the length of the period over which past returns are calculated, J, and the length of time the position is held, K. This paper, and much of the literature, uses monthly data, so J and K are measured in months. For example, with $J = 12$ and $K = 3$, the strategy would rank stocks at time t by the cumulative return from the end of month $t-12$ to the end of month t. "Winners" are those firms with the highest ranking-period returns and "losers" are those stocks with the lowest ranking-period returns. In much of the literature, stocks with the top 10% ranking-period returns are defined as "winners" and stocks with the lowest 10% ranking-period returns are defined as "losers," and we follow this convention. Some studies assume that the momentum trading strategy is implemented at the end of month t and held for K months. Others, in order to avoid microstructure effects, wait a certain period of time before implementing a trading strategy. We call this waiting period a "skip" period and denote its length as S. The triplet (J, S, K) describes the momentum strategies. Jegadeesh and Titman (1993) implement strategies with $J = \{3, 6, 9, 12\}$, $S = \{0, 0.25\}$ (i.e., no skip period, and a skip period of one week), and $K = \{3, 6, 9, 12\}$. Jegadeesh and Titman (1993, Table I) report the returns on the losers' decile, on the winners' decile, and on the zero-cost strategy of taking a long position in the winners' decile and a short position in the losers' decile. They report that all of the zero-cost momentum portfolios have positive returns, all, except one, have statistically significant returns, and the most profitable long/short strategy is the $J = 12/S{=}0.25/K = 3$ strategy. Grundy and Martin (2001) study a $J = 6/S = 1/K = 1$ strategy and find that it yields significant abnormal returns.

While the momentum anomaly is the existence of significant returns to winners in excess of losers, past research has found that most of the returns to a long/short momentum trading strategy are due to losers rather than to winners. For example, Hong, Lim, and Stein (2000, Table III) find that between 73% and 100% of the long winners/short losers momentum portfolio excess return is determined by the return difference between the loser portfolio (bottom 30% of past returns) and middle return portfolio (middle 40% of past returns) for size deciles 2 to 9. Grinblatt and Moskowitz (1999, Table 1) find a stronger relation between returns and past returns (for a $J = 12/S = 1/K = 1$ strategy) for losers versus winners. Jegadeesh and Titman (2001, Table IV) find larger abnormal returns (in absolute value) for loser portfolios than winner portfolios.

Despite the evidence that greater momentum profits are obtained from past losers versus past winners, we limit our analysis to winners alone. The reason stems from the asymmetry of trading costs between engaging in a long position and short-selling. The nature of short-selling execution, especially large positions, involves additional costs, not fully captured by our measure of price impact. For example, losers are stocks that have extremely under-performed in the past, and as such they are biased to small firms, which may be difficult to short-sell. We show below that losers are much less liquid than winners, as shown by the high price impact coefficients of losers relative to winners. Although there is evidence that costs of short-selling are not sufficient to eliminate momentum profits (Geczy, Musto, and Reed (2002)), we choose the more conservative approach of studying past winner-based portfolio strategies. The persistence of winners is an important anomaly on its own, since the excess returns of winners exhibited in the data are statistically significant. Although restricting the analysis to winners and to long strategies would potentially bias toward not finding significant post-transactions costs return, we do in fact find significant returns.

Our sample consists of all stocks included in the Center for Research in Security Prices (CRSP) monthly data files from February 1967 to December 1999. From 1967 to 1972, the CRSP data files include New York Stock Exchange (NYSE) and American Stock Exchange (AMEX) stocks; after 1972, NASDAQ stocks are added to the sample. Table 1 contains average returns of portfolios of past winners (top decile) and losers (bottom decile) for ranking periods (J) of two, five, eight, and eleven months, skip periods (S) of one month, and holding periods (K) of one, three, and six months. The previous literature typically uses equal weights (EW) or value (measured by market capitalization) weights (VW) to form portfolios. In Table 1, we use the same EW and VW strategies. We discuss alternative weighting schemes below. Each month after the first month only a $\frac{1}{K}$ fraction of the portfolio is rebalanced.[2] We conduct the analysis first using only NYSE-listed stocks and subsequently using the entire universe of stocks (NYSE, AMEX, and NASDAQ) available on CRSP. The results for EW strategies are reported in Panel A of Table 1, separately for winners and losers. Similar to Jegadeesh and Titman (1993), we conclude that, ignoring price impacts, the most profitable strategies for equal-weighted long positions in winners and short positions in losers are 11/1/1 and 11/1/3. The same is true for value-weighted strategies with the exception that the long 8/1/1 strategy is slightly more profitable for the NYSE/AMEX/NASDAQ sample. The frequently studied 5/1/6 trading strategy also exhibits high mean return. Since the 11/1/3 and 5/1/6 strategies are profitable and closest to those most extensively studied in the literature, we will focus on these strategies. Without considering price concessions and using only NYSE-listed stocks, these winners-based strategies earn 1.71% and 2.13% per month for 11/1/3 VW and EW, respectively, and 1.49% and 1.93% per month for 5/1/6 VW and EW, respectively. Their Sharpe ratios are 0.20, 0.24, 0.17, and 0.22, respectively (for comparison, the mean return of the Standard & Poors (S&P) 500 portfolio over the sample period is 1.1% per month with a Sharpe ratio of 0.13).[3]

3 Measures of Trading Costs

We study the effects on the profitability of the past winner-based momentum strategies implied by four alternative measures of trading costs. Two of the measures are proportional trading cost models, and

[2] Alternatively, one might consider strategies that require rebalancing only once, at the end of the holding period, instead of rebalancing a fraction of the portfolio every month, as required by the strategies above. We have analyzed such strategies and found them to underperform the strategies above, after including price impacts.

[3] Since momentum arbitrage strategies exhibit a reversal during January, one might consider altering our investment strategies accordingly. We note that the January reversal is mainly a loser phenomenon (see, e.g., Sadka (2001)), and has little effect on winners. The average returns during January are as follows: Equal-weighted strategies earn 3.87% (11/1/3) and 4.05% (5/1/6) for winners and 8.56% (11/1/3) and 8.08% (5/1/6) for losers. Value-weighted strategies earn 1.99% (11/1/3 winners), 2.03% (5/1/6 winners), 3.64% (11/1/3 losers), and 3.32% (5/1/6 losers). We proceed to investigate strategies based on long winners throughout the entire year.

therefore, are independent of the size of the portfolio traded. These are based on the quoted and effective spreads. The remaining two measures are non-proportional trading cost models and reflect the fact that the price impact of trading increases in the size of the position traded. The price impact measures are based on Glosten and Harris (1988) and Breen, Hodrick, and Korajczyk (2002). Our momentum strategies cover a much longer time period than that covered by the TAQ data, and therefore we first describe the in-sample estimation of the different trading models and then introduce a method of estimating them outside the initial estimation period.

3.1 In-Sample Estimation

Effective spreads are measured as the absolute value of the relative difference of transaction price and mid-point of quoted bid and ask. Quoted spread is measured as the ratio (minute by minute) between the quoted bid-ask spread and the mid-point (half the quoted spread is considered as cost). Monthly estimates of these two measure are obtained as their simple average throughout the month. We denote k^E and k^Q as the effective and quoted spreads, respectively.

For non-proportional trading costs we use two alternative specifications of the price impact function. One is the price impact estimated in Breen, Hodrick, and Korajczyk (2002). This measure posits a proportional relation between percentage returns and net share turnover over 30-minute duration time periods:

$$\frac{\Delta p_{i,t}}{p_{i,t-1}} = \lambda_i^{BHK} \times Turnover_{i,t} \tag{1}$$

where $p_{i,t}$ is the last transaction price of asset i in time period t, $\Delta p_{i,t} = p_{i,t} - p_{i,t-1}$ is the price impact associated with the transactions in period t, λ_i^{BHK} is asset i's price impact coefficient, and $Turnover_{i,t}$ is the net number of shares traded (trades are signed according to the buy or sell classification introduced by Lee and Ready (1991); quantities are multiplied by 1000) divided by the number of shares outstanding for firm i. Buyer-initiated trades correspond to positive values of $Turnover_{i,t}$ and seller-initiated trades correspond to negative values. This specification is motivated by the linear pricing rule of Kyle (1985), which expresses price changes as a linear function of net volume. Breen, Hodrick, and Korajczyk (2002) motivate the use of scaled measures (i.e., net turnover rather than net volume and returns rather than price changes) as a means of obtaining more meaningful cross-sectional and time series comparisons of price impact. Using returns rather than price changes does induce non-linearity in the price impact. In fact, as demonstrated in Section 3, this specification induces a convex price impact function.

Our second specification for the price impact function is from Glosten and Harris (1988, Equation (5)). The Glosten and Harris (1988) specification allows a decomposition of the price impact into fixed and variable components. The regression model is:

$$\Delta p_{i,t} = \alpha_i + \lambda_i^{GH} q_{i,t} + \Psi_i \Delta d_{i,t} + \varepsilon_{i,t} \tag{2}$$

where $\Delta p_{i,t}$ is the price change of stock i from trade $t-1$ to trade t as a consequence of a (signed) trade of $q_{i,t}$ shares of the stock. Every trade is classified as a "buy" or a "sell" according to the classification scheme of Lee and Ready (1991). The sign of a trade is denoted $d_{i,t}$ and is assigned a value of +1 for a "buy"and -1 for a "sell". The difference between the sign of a current trade and the previous trade is denoted $\Delta d_{i,t}$. The regression coefficient λ_i^{GH} represents the variable cost of trading, while Ψ_i represents the fixed costs.

Theoretical research argues that the permanent component of the price impact function should be linear (e.g., Kyle (1985) and Huberman and Stanzl (2000)). Empirical studies find concave price impact functions (see, e.g., Hasbrouck (1991a), Hausman, Lo, and MacKinlay (1992), and Keim and Madhavan (1996)). We believe that the use of linear and convex price impact functions is reasonable in our case for several reasons. First, the choice of trade size is endogenous. Those large trades that researchers observe in the data are likely to be ones for which the price impact is low (for example, due to credible signalling that the trader is uninformed). Otherwise, the trade would be broken into smaller trades (Bertsimas and Lo (1998)). It is not plausible to assume that the naive momentum trading strategies discussed in the literature could be executed under these favorable conditions. Second, concave empirical price impact functions may be observed in the data due to leakage of information while a block trade is being "shopped" (see, e.g., Nelling (1996)). That is, the measured price impact for a block under-estimates the true price impact, thus leading to unattainable concavity in the measured price impact function. Last, if the true price impact functions are concave, then our results may be regarded as conservative, since we over-estimate the costs of trading for large trades.

The measure of time differs across the two price impact specifications. In the Breen, Hodrick, and Korajczyk (2002) formulation, Equation (1), trades are aggregated over 30-minute intervals so that $\Delta p_{i,t}$ is the change in the last transaction price from time interval $t-1$ to time interval t, and $Turnover_{i,t}$ is the signed (net) turnover in time interval t. In the Glosten and Harris (1988) formulation, Equation (2), time is defined in terms of trades. That is, $q_{i,t}$ is the signed size of trade t, and $\Delta p_{i,t}$ is the price change of stock i from trade $t-1$ to trade t. For an illustration of the different trading cost functions see Figure 1.

We use intra-day data to estimate the price impact coefficient each month, $\tau = 1, ..., T$, for our cross-section of firms. This provides a time series of coefficients, $\lambda_{i,\tau}^{BHK}$, $\lambda_{i,\tau}^{GH}$, and $\Psi_{i,\tau}$. We use the intra-day trade Trade and Quotation (TAQ) database of the New York Stock Exchange (NYSE). The TAQ data are available beginning January 1993 and we estimate the time series of monthly coefficients from January 1993 to May 1997. These data are matched with the firm characteristics described below. The resulting sample consists of 6,513 firms, not all of which have data for each month. For the average month there are 3,699 firms with data. Approximately two-thirds of the firms trade on the NYSE and AMEX while one-third of the firms trade on NASDAQ. We estimate $\lambda_{i,\tau}^{BHK}$ separately for NYSE/AMEX and NASDAQ firms. For computational reasons we estimate $\lambda_{i,\tau}^{GH}$ and $\Psi_{i,\tau}$ using NYSE firms only.

3.2 Out-of-Sample Estimation

Since our momentum strategies cover a much longer time period than that covered by the TAQ data, we need a method of estimating the coefficients outside the initial estimation period. We do this by estimating the cross-sectional relation (over January 1993 to May 1997) between the coefficients, $\lambda_{i,\tau}^{BHK}$, $\lambda_{i,\tau}^{GH}$, and $\Psi_{i,\tau}$, and effective and quoted spreads, and a set of predetermined firm-specific variables meant to be proxies for market-making costs (due to adverse selection and carrying costs) and shareholder heterogeneity. We use this cross-sectional relation to estimate price impact in the out-of-sample period using the firm-specific predetermined variables that are observable in the the out-of-sample period.

For example, for the Breen, Hodrick, and Korajczyk (2002) specification, (1), let $\widehat{\Gamma}_\tau$ be the estimated vector of coefficients from the cross-sectional relation:

$$\widehat{\lambda}_\tau^{BHK} = X_{\tau-1}\Gamma_\tau + \upsilon_\tau \tag{3}$$

where $\widehat{\lambda}_\tau^{BHK}$ is the $N_\tau \times 1$ vector of price impact coefficients of N_τ firms estimated for month τ, and $X_{\tau-1}$ is the $N_\tau \times k$ matrix of predetermined variables for the cross-section of firms with $X_{i,\tau-1} = (1, X_{1,i,\tau-1}, ..., X_{9,i,\tau-1})$:

$X_{1,i,\tau}$ = market cap of firm i at the end of month t divided by the average market cap of CRSP firms, minus one

$X_{2,i,\tau}$ = total volume for firm i from month $t-2$ to month t divided by the total volume, over the same period, for the average NYSE firm, minus one

$X_{3,i,\tau}$ = firm i's stock price at the end of month t divided by the price at the end of month $t-6$, minus one

$X_{4,i,\tau}$ = absolute value of $X_{3,i,\tau}$

$X_{5,i,\tau}$ = dummy variable equal to unity if the firm is included in the S&P500 index

$X_{6,i,t}$ = dividend yield = Compustat item#20/$p_{\tau-1}$

$X_{7,i,\tau}$ = R^2 of firm i's returns regressed on returns of the NYSE index over the preceding 36 months

$X_{8,i,\tau}$ = dummy variable equal to unity if the firm is traded on NYSE

$X_{9,i,\tau}$ = inverse of stock price of the previous month.

As in Fama and MacBeth (1973), we use the time-series average of the monthly estimates, $\widehat{\Gamma}_\tau$, to estimate the cross-sectional relation, $\widehat{\Gamma} = \frac{\widehat{\Gamma}_1+\widehat{\Gamma}_2+...+\widehat{\Gamma}_T}{T}$. To estimate the price impact for firm i over month τ we calculate the product of $\widehat{\Gamma}$ and $X_{i,\tau-1.}$

$$\widehat{\lambda}_{i,\tau}^{BHK} = X_{i,\tau-1}\widehat{\Gamma}. \tag{4}$$

While the coefficient $\widehat{\Gamma}$ is estimated over the 1993-1997 time period, the predetermined variables are observable before the momentum trading strategy is implemented. The predetermined variables are constructed to avoid scale differences across the time period. For example, while the market capitalization of a large firm in 1967 is very different from the market capitalization of a large firm in 1997, a large firm will always have a high relative market capitalization, used to calculate $X_{1,i,\tau}$. Breen, Hodrick, and Korajczyk (2002) note that the cross-sectional coefficients are quite stable over the 1993-1997 sample period. They also compare out-of-sample the predicted price impact to the actual price impact for a set of institutional trades. They find that the predicted price impact is high relative to the actual price impact. Thus the predicted impact accommodates the possibility that trading costs are higher earlier in the sample period.

The same type of approach is taken to estimate the coefficients from the Glosten and Harris (1998) model, $\lambda_{i,\tau}^{GH}$ and $\Psi_{i,\tau}$, and effective and quoted spreads, k_E and k_Q. Sample statistics for firm characteristics, $X_{\tau-1}$, are reported in Table 2. The results of the cross-sectional regressions, Equation (3), are reported in Table 3. The cross-sectional model has much more explanatory power for the fixed component of trading costs, Ψ, and the spread variables, than for the variable component, λ. Table 4 presents details of the distribution of the predicted spread and price impact measures obtained from the cross-sectional regressions, similar to Equation (4) for $\widehat{\lambda}_{i,\tau}^{BHK}$. Panel A of Table 4 compares, for all firms in the sample, the predicted parameters between the in-sample period (over which the TAQ data are used to estimate $\widehat{\lambda}_{i,\tau}^{BHK}$, $\lambda_{i,\tau}^{GH}$, $\Psi_{i,\tau}$, k_E and k_Q (left panel A-1)) and the full sample (right panel A-2). The mean predicted parameters are similar across the in-sample and out-of-sample periods. Panel B of Table 4 compares the parameters for the winner decile and loser decile for the 11/1/1 strategy. By every metric, the loser stocks are less liquid, on average, than the winner stocks. Panel C presents an equivalent comparison of winners and losers for the 5/1/1 strategy. As in Panel B, the loser stocks are less liquid, on average, than the winner stocks.

4 Trading Models with Price Impacts

The typical momentum strategies investigated in the literature are not optimized to take into account the price impact costs of trading. To incorporate transaction costs of trades, we first develop the formulation of the total cost of a trade. We start the discussion of the cost of execution of trades with a general derivation. Denote the prevailing market price of an asset by p. A purchase of q units of this asset would cost a total of x as follows

$$pq + \int_0^q f(p, q)\, dq = x \tag{5}$$

where $f(p, q)$ is the price impact cost function, and the price acts as a state variable that could influence the cost function. The price impact function must induce positive costs for both buy and sell investment positions, and therefore admissible functions are such that $f(p, 0) \geq 0$, and $f(p, q)\, q \geq 0$, $\forall q \neq 0, \forall p$. This formulation implicitly assumes that the trade of q shares is divided into many infinitesimal trades (as in Bertsimas and Lo (1998)) and that over the trading period there is no price reversion.[4]

The Breen, Hodrick, and Korajczyk (2002) model assumed for price impacts generates an exponential price-impact function as follows. Using the definition of $Turnover_{i,t}$ as

$$Turnover_{i,t} \equiv \frac{\Delta q_{i,t}}{(Shares\ Outstanding)_{i,t}} \tag{6}$$

where $\Delta q_{i,t}$ is the number of shares bought/sold of asset i at period t in month τ. Substituting (6) in Equation (1) we have

$$\frac{\Delta p_{i,t}}{p_{i,t}} = \frac{\lambda_{i,\tau}^{BHK}}{(Shares\ Outstanding)_{i,\tau}} \Delta q_{i,t} \tag{7}$$

By defining

$$\overline{\lambda}_{i,\tau}^{BHK} \equiv \frac{\lambda_{i,\tau}^{BHK}}{(Shares\ Outstanding)_{i,\tau}} \tag{8}$$

Equation (7) is further simplified to

$$\frac{\Delta p_{i,t}}{\Delta q_{i,t}} = \overline{\lambda}_{i,\tau}^{BHK} p_{i,t} \tag{9}$$

and therefore, in the limit when $\Delta q_{i,t} \to 0$, the supply function is expressed by

$$\overline{p_{i,t}} = p_{i,t} e^{\overline{\lambda}_{i,\tau}^{BHK} q_{i,t}} \tag{10}$$

where $\overline{p_{i,t}}$ is the price of asset i after a trade of $q_{i,t}$ shares has been executed with a starting price of $p_{i,t}$ at time t. In the context of Equation (5) the price impact cost function is expressed as

$$f(p,q) = p\left(e^{\overline{\lambda} q} - 1\right). \tag{11}$$

Therefore, the total price impact of a trade of $q_{i,t}$ shares is calculated through $\int_0^q f(p,x)\,dx$, and explicitly

$$\int_0^{q_{i,t}} p_{i,t}\left(e^{\overline{\lambda}_{i,\tau}^{BHK} x} - 1\right) dx = p_{i,t}\frac{1}{\overline{\lambda}_{i,\tau}^{BHK}}\left[e^{\overline{\lambda}_{i,\tau}^{BHK} q_{i,t}} - 1\right] - p_{i,t}q_{i,t} = \frac{MVE_{i,t}}{\lambda_{i,\tau}^{BHK}}\left[e^{\frac{\lambda_{i,\tau}^{BHK}}{MVE_{i,t}} p_{i,t} q_{i,t}} - 1\right] - p_{i,t}q_{i,t}. \tag{12}$$

The last step of the derivation above used the definition of the market value of equity (MVE)

$$MVE_{i,t} = p_{i,t} \times SharesOutstanding_{i,t}. \tag{13}$$

An illustration of the price impact function is provided in Figure 1.

The price impact coefficients $\lambda_{i,\tau}^{BHK}$ are assumed to be known at time t. Define x_t as the value of the portfolio at time t, before rebalancing, and $\overline{x}_t$ as the value after rebalancing. The momentum-based trading

[4]The assumption of no price reversion throughout the trading process somewhat relaxes the need to define the time horizon of the trade, as long as the time horizon for expected return begins after the trade is fully executed. This assumption is plausible for market orders and especially for situations in which a trade must be executed as soon as possible.

strategy, consisting of purchasing the stocks in the past winners' decile, implicitly defines which stocks are included in the portfolio. The stocks that need to be traded at time t are divided into two mutually exclusive sets as follows:

$$\begin{aligned} I_{1,t} &= \{i : \omega_{i,t} > 0, \omega_{i,t-1} \geq 0\} \\ I_{2,t} &= \{i : \omega_{i,t} = 0, \omega_{i,t-1} > 0\} \end{aligned} \tag{14}$$

where $\omega_{i,t}$ is the portfolio weight associated with asset i at time t. $I_{1,t}$ consists of all stocks held at time t, which could include ones also held at time $t-1$ or those that are added to the pool of winners at time t. $I_{2,t}$ consists of stock that were held at time $t-1$ but are no longer in the winners' decile at t, and therefore need to be sold. The portfolio weights are percentages of the actual investment after price impacts, $\overline{x}_t$. The purpose of defining $I_{2,t}$ is to be able to include trading rules that require liquidation of assets, as an input to an optimization problem defined later in Section 5. Also, short-sale constraints are imposed, since we only consider strategies consisting of long positions. Denoting the (raw) return[5], without price impacts, of stock i for the period beginning at t until $t+1$ as $R_{i,t+1}$, the following recursive relations hold:

$$\begin{aligned} x_t &= \overline{x}_{t-1} \sum_{i \in I_{1,t} \cup I_{2,t}} \omega_{i,t-1} \left(1 + R_{i,t}\right) \\ E_t\left[x_{t+1}\right] &= \overline{x}_t \sum_{i \in I_{1,t}} \omega_{i,t} \left(1 + E_t\left[R_{i,t+1}\right]\right). \end{aligned} \tag{15}$$

Assume that the portfolio is rebalanced at time t. At the beginning of time t, prior to trading for rebalancing purposes, the number of shares of each stock is given by

$$q_{i,t} = \frac{\omega_{i,t-1}\overline{x}_{t-1}\left[1 + R_{i,t}\right]}{p_{i,t}}. \tag{16}$$

A trading strategy specifies the allocation of assets after rebalancing at time t, by assigning the weights $\omega_{i,t}$. Therefore, the number of shares of each asset required after trading at time t is expressed as[6]

$$\overline{q}_{i,t} = \frac{\omega_{i,t}\overline{x}_t}{p_{i,t}}. \tag{17}$$

To solve for the post-trade portfolio value, $\overline{x}_t$, notice that the sum of the post-trade value and the total price impact must equal the pre-trade value, x_t. Explicitly, the following equality must hold

$$\overline{x}_t + \sum_{i \in I_{1,t} \cup I_{2,t}} \left[\frac{1}{b_{i,t}}\left[e^{b_{i,t}p_{i,t}\left[\overline{q}_{i,t} - q_{i,t}\right]} - 1\right] - p_{i,t}\left[\overline{q}_{i,t} - q_{i,t}\right]\right] = x_t \tag{18}$$

[5]The return is assumed to be adjusted for dividends and stock splits that may have occurred during time t (in our empirical work, this corresponds to the returns recorded on CRSP).

[6]Notice that $q_{i,t+1}$ may not necessarily equal $\overline{q}_{i,t}$. For example, stock splits may change the number shares in the portfolio. Therefore, the periodic total return is used to calculate the number of shares available at the end of every investment period. This return complies with the return data available on CRSP database. Notice that this return is adjusted for dividend payments as well, and therefore we implicitly assume that dividends are reinvested without price impacts.

where $b_{i,t}$ is the price impact coefficient (t is any time during month τ), adjusted for firm size

$$b_{i,t} \equiv \frac{\lambda_{i,\tau}^{BHK}}{MVE_{i,\tau}}. \tag{19}$$

Equation (18) is a budget constraint to the investment. Notice, however, that Equation (18) holds in equality, rather than weak inequality, because of the implicit assumption that all available funds must be allocated. Therefore, the investor must plan the investment strategy so that, after considering the price impact of the trades, all the funds are allocated. To simplify the budget constraint, define

$$a_{i,t} \equiv \omega_{i,t-1}\overline{x}_{t-1}\left(1+R_{i,t}\right) \tag{20}$$

That is, $a_{i,t}$ is the monetary amount invested in stock i at the end of the previous investment period. The budget constraint translates to

$$\overline{x}_t + \sum_{i\in I_{1,t}} \left[\frac{1}{b_{i,t}}\left[e^{b_{i,t}[\omega_{i,t}\overline{x}_t - a_{i,t}]} - 1\right] - [\omega_{i,t}\overline{x}_t - a_{i,t}]\right] + \sum_{i\in I_{2,t}} \left[\frac{1}{b_{i,t}}\left[e^{-b_{i,t}a_{i,t}} - 1\right] + a_{i,t}\right] = x_t \tag{21}$$

Equation (21) divides the summation on the right hand side so that the assets liquidated due to change in the set of feasible assets, are separated from the rest of the assets. This is done because the summation associated with forced liquidation acts as a constant term. Notice that to obtain reasonable values for $\overline{x}_t$, the restriction $0 \leq \overline{x}_t < x_t$ must be imposed. The constraint implies that (a) price impact costs are positive ($\overline{x}_t < x_t$) and (b) price impact costs do not exceed the amount traded ($0 \leq \overline{x}_t$).[7] However, since the total price impact is always positive, for any amount of a nonzero trade, the restriction $\overline{x}_t < x_t$ holds by construction. Thus, only $\overline{x}_t \geq 0$ is imposed.

Given $\overline{x}_t$ from (21), and expected returns $E_t[r_{i,t+1}]$, we use Equation (15) to find $E_t[x_{t+1}]$. Finally, the net expected return to a trading strategy, after price impacts, is found by definition

$$E_t[r_{p,t+1}] = \frac{E_t[x_{t+1}]}{x_t} - 1. \tag{22}$$

For an illustration of the time-series process of the portfolio value see Figure 2.

For the Glosten and Harris (1988) specification, we only state the final results. The complete derivation of the trading model for linear price-impact costs is provided in Sadka (2002). The trading costs due to the variable cost λ^{GH} may be described by $f(p,q) = \lambda^{GH} q$, and the fixed costs as $f(p,q) = \Psi p$. Thus, by re-defining $b_{i,t} \equiv \lambda_{i,t}^{GH}/p_{i,t}^2$, and defining $\overline{\Psi}_{i,t} = \Psi_{i,t}/p_{i,t}$, Equation (21) translates to

$$\overline{x}_t + \frac{1}{2}\sum_{i\in I_{1,t}} b_{i,t}\left[\omega_{i,t}\overline{x}_t - a_{i,t}\right]^2 + \sum_{i\in I_{1,t}} \overline{\Psi}_{i,t}\left|\omega_{i,t}\overline{x}_t - a_{i,t}\right| + \frac{1}{2}\sum_{i\in I_{2,t}} b_{i,t}a_{i,t}^2 + \sum_{i\in I_{2,t}} \overline{\Psi}_{i,t}a_{i,t} = x_t. \tag{23}$$

[7] The upper bound for $\overline{x}_t$ may be viewed as a budget constraint. The lower bound is imposed to assure a positive investment amount. Notice that the lower bound does not act as a short-sell constraint. The latter may be achieved by imposing nonnegativity constraints on the weights $\omega_{i,t}$.

and the expected return to a trading strategy is again calculated by Equation (22).

Similar to the fixed costs in the Glosten-Harris model, proportional trading costs may be expressed as $f(p,q) = kp$, where k is a constant proportional cost (in our study, k^E and k^Q are the effective and quoted spreads, respectively). Under these assumptions Equation (21) translates to

$$\overline{x}_t + \sum_{i \in I_{1,t}} k^E_{i,t} \left| \omega_{i,t}\overline{x}_t - a_{i,t} \right| + \sum_{i \in I_{2,t}} k^E_{i,t} a_{i,t} = x_t. \tag{24}$$

Notice that the formulation in Equation (24) is effectively independent of the initial amount of investment (this can be proved through recursive induction).

5 Liquidity Tilted Portfolio Formation with Price Impacts

In the framework developed above, an investment strategy at any given time t is entirely defined by the assets' weights and the actual investment amount. Therefore, the static problem of finding the strategy with the highest expected return every period (with the Breen, Hodrick, and Korajczyk (2002) specification of the price impact function) may be expressed as follows:[8,9]

$$\max_{\omega_t} \sum_{i \in I_{1,t}} \omega_{i,t}\overline{x}_t \left(1 + E_t\left[R_{i,t+1}\right]\right) \tag{25}$$

$$s.t.$$

$$\overline{x}_t + \sum_{i \in I_{1,t}} \left[\frac{1}{b_{i,t}} \left[e^{b_{i,t}[\omega_{i,t}\overline{x}_t - a_{i,t}]} - 1 \right] - \left[\omega_{i,t}\overline{x}_t - a_{i,t}\right] \right] + \sum_{i \in I_{2,t}} \left[\frac{1}{b_{i,t}} \left[e^{-b_{i,t}a_{i,t}} - 1 \right] + a_{i,t} \right] = x_t \tag{26}$$

$$\sum_{i \in I_{1,t}} \omega_{i,t} = 1$$

$$\omega_{i,t} \geq 0 \tag{27}$$

$$\overline{x}_t \geq 0 \tag{28}$$

The nonnegativity constraints are imposed to ensure that the optimal weights are reasonable, i.e., to avoid cases of extreme long and short positions. This is similar to the nonnegativity constraints added to the problem of finding the tangency portfolio in the classical mean and standard deviation setting (see Jagannathan and Ma (2001)).

[8] We focus only on the maximization of expected returns, without considering any control for second moments. In the context of the classical mean/variance portfolio selection literature, that problem would translate to minimizing variance holding expected return constant and subject to a budget constraint (i.e., all wealth is fully invested after considering price impacts (see Equation (26))). This problem can only be solved numerically, which is computationally challenging, especially when several hundred stocks are considered. We leave this for future work.

[9] Treating the optimization as a static, one-period problem does not take into account the multi-period nature of momentum trading strategies and the consequent possibility of minimizing trading costs through a buy-and-hold policy. Below, we investigate the performance of trading strategies that are convex combinations of liquidity tilted and buy-and-hold strategies.

To simplify the formulation of the problem, denote the following contemporaneous auxiliary variable:

$$A_t \equiv \sum_{i \in I_{1,t}} \frac{1}{b_{i,t}} + \sum_{i \in I_{2,t}} \frac{1}{b_{i,t}} \left[1 - e^{-b_{i,t} a_{i,t}}\right]. \tag{29}$$

The budget constraint (26) translates to

$$\sum_{i \in I_{1,t}} \frac{1}{b_{i,t}} e^{b_{i,t}[\omega_{i,t} \overline{x}_t - a_{i,t}]} = A_t \tag{30}$$

where $a_{i,t} = 0$ if asset i has not been included in the investment portfolio last period. Furthermore, to reduce dimensionality, it is preferable to use levels of investment rather than relative portfolio weights. For this reason, define the monetary amount $y_{i,t}$ invested in stock $i \in I_{1,t}$ as

$$y_{i,t} \equiv \omega_{i,t} \overline{x}_t. \tag{31}$$

Notice that this definition implies that

$$\overline{x}_t = \sum_{i \in I_{1,t}} y_{i,t}. \tag{32}$$

So far no upper bound to investment has been imposed. However, in general, such constraints may be required. Therefore, we add an upper bound, $d_{i,t}$, to the investment allowed in each asset i. In most cases, the lower bound on an investment in asset i is set to zero; however, we solve the problem for the general case where the lower bound is set to $c_{i,t}$. Suppressing the time index t, the static optimization problem translates to

$$\max_{y} \sum_{i \in I_1} y_i \left(1 + E\left[R_i\right]\right) \tag{33}$$

$$s.t.$$

$$\sum_{i \in I_1} \frac{1}{b_i} e^{b_i (y_i - a_i)} \leq A \tag{34}$$

$$c_i \leq y_i \leq d_i. \tag{35}$$

Notice that the budget constraint has been changed to a weak inequality in order to formulate a convex optimization problem. Nevertheless, at the optimum, the budget constraint is binding.

The optimal solution is characterized in Theorem 1, a more general version of which is proven in the Appendix. (For the version of Theorem 1 for the Glosten and Harris (1988) price impact function, see Sadka (2002)).

Theorem 1 *There exists a unique solution to the optimization problem above. Ignoring the upper- and lower-bounds, the optimal trading strategy is characterized by*

$$\forall i \in I_1 \qquad y_i^* = \frac{1}{b_i} \ln \left[\frac{\left(1 + E\left[R_i\right]\right) A}{\sum_{i \in I_1} \frac{1+E[R_i]}{b_i}} \right] + a_i.$$

If our initial position at time zero is one with an endowment of x_0, none of which is invested ($a_{i,0} = 0$), the optimal strategy at $t = 0$ is obtained by implementing the following specifications:

$$\begin{aligned} A_t &= x_0 + \sum_{i \in I_1} \frac{1}{b_{i,0}} \\ a_{i,0} &= 0 \qquad \forall i \in I_1. \end{aligned} \tag{36}$$

To simplify the application of the liquidity tilted portfolio rule of Theorem 1, we add the simplifying assumption that all assets in the trading strategy (all firms in the top past winners decile in the empirical work below) have the same expected return. Clearly, a better model of cross-sectional expected returns would lead to better performance of the trading strategy.

Corollary 1 *Assume that all assets in the restricted set of assets chosen by the trading strategy have the same expected returns, and there are no upper bounds to investment. Then, the optimal weights at time $t = 0$ are given by*

$$\omega_i = \frac{\frac{1}{b_i}}{\sum_{i \in I_1} \frac{1}{b_i}} \qquad \forall i \in I_1. \tag{37}$$

Adding the assumption that all assets have identical price impact coefficients, $\lambda_i = \lambda$, yields market values as the optimal weights, since $\frac{1}{b_i} = \frac{MVE_i}{\lambda}$.

The proof of Corollary 1 may be found in the Appendix. This result provides additional theoretical justification for the frequent empirical use of market-value weighted portfolios. Numerous empirical papers use market values as weights to adjust for the liquidity of assets in a portfolio. For example, large firms are more liquid than small firms and therefore the use of market capitalization weights shifts more weight to the return of the more liquid assets. Corollary 1 shows that market values are optimal portfolio weights under the assumption that price impacts and expected returns are equal across firms included in the trading strategy. In our empirical work below, we assume that all stocks in the winners decile have the same expected return. However, we allow the price impact coefficients to differ across firms.

6 Performance Evaluation of Momentum Strategies

We wish to evaluate the performance of various momentum-based trading strategies. For proportional transactions cost models, a trading strategy's performance is independent of the size of the portfolio. For non-proportional price impact transactions costs, the performance of the trading strategy declines with the size of the portfolio. Therefore, we are interested in determining the amount that a single portfolio manager could invest before the performance of momentum strategies breaks even with that of the benchmark.

6.1 Benchmark Asset Pricing Model

We compute Sharpe ratios and abnormal returns (α) relative to the three-factor model of Fama and French (1993) for different investment levels. Using the Fama-French (1993) three-factor model we estimate the time-series regression

$$R_{W,t} - R_{f,t} = \alpha_W + \beta_{W,t} R_{M,t} + s_{W,t} SMB_t + h_{W,t} HML_t + \varepsilon_{W,t} \tag{38}$$

where $R_{W,t} - R_{f,t}$ is the monthly return of the momentum portfolio ($W \equiv (J, S, K)$), in excess of the one-month Treasury bill return ($R_{f,t}$), of a past winner, momentum-based, portfolio; $R_{M,t}$ is the return on the CRSP value-weighted market portfolio, in excess of $R_{f,t}$; SMB_t is the average return on three small-capitalization portfolios minus the average return on three large capitalization portfolios; and HML_t is the average return on two value (high book-to-market equity) portfolios minus the average return on two growth (high book-to-market equity) portfolios.[10] The conditional exposures of the momentum portfolio to the three factors are denoted by $\beta_{W,t}$, $s_{W,t}$, and $h_{W,t}$.

Given that the composition of momentum-based portfolio strategies, by definition, is based on past returns, it is also based partially on conditional factor risk. For example, if the return on the market is high over the ranking period, our winner portfolio will tend to include high market risk assets. Conversely, if the return on the market is low over the ranking period, our winner portfolio will tend to include low market risk assets. This time variation in conditional systematic risk is discussed in a number of papers (e.g., Chopra, Lakonishok, and Ritter (1992), Jones (1993), and Grundy and Martin (2001)). Grundy and Martin (2001) derive a model in which momentum-based portfolios have conditional factor risk exposures that are linear functions of the ranking period factor portfolio returns. While other effects, such as leverage effects (Chopra, Lakonishok, and Ritter (1992)), may make the relation more complex, we rely on the results of Grundy and Martin (2001) and model the momentum portfolio's conditional factor risk as a linear function of the ranking period factor returns. That is:

$$\begin{aligned} \beta_{W,t} &= a_\beta + b_\beta R_{M,W,t} + c_\beta SMB_{W,t} + d_\beta HML_{W,t} \qquad (39) \\ s_{W,t} &= a_s + b_s R_{M,W,t} + c_s SMB_{W,t} + d_s HML_{W,t} \\ h_{W,t} &= a_h + b_h R_{M,W,t} + c_h SMB_{W,t} + d_h HML_{W,t} \end{aligned}$$

where $R_{M,W,t}$, $SMB_{W,t}$, and $HML_{W,t}$ are the average cumulative (excess) returns of the factors over the K overlapping ranking periods of length J used to define the momentum strategy. They are calculated as follows:

[10] The factors are available from Ken French at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

$$
\begin{aligned}
R_{M,W,t} &= \frac{1}{K}\sum_{z=0}^{K-1}\left[\prod_{i=t-J-S-z}^{t-S-z-1}(1+R_{M,i}+R_{f,i})-\prod_{i=t-J-S-z}^{t-S-z-1}(1+R_{f,i})\right]\\
SMB_{W,t} &= \frac{1}{K}\sum_{z=0}^{K-1}\left[\prod_{i=t-J-S-z}^{t-S-z-1}(1+S_i)-\prod_{i=t-J-S-z}^{t-S-z-1}(1+B_i)\right]\\
HML_{W,t} &= \frac{1}{K}\sum_{z=0}^{K-1}\left[\prod_{i=t-J-S-z}^{t-S-z-1}(1+H_i)-\prod_{i=t-J-S-z}^{t-S-z-1}(1+L_i)\right].
\end{aligned}
$$

As defined in Fama and French (1993), S and B denote small and large capitalization; L, M, and H denote low, medium, and high book-to-market equity ratios; S/L denotes the particular combination of portfolios (small capitalization and low book-to-market in this case); and:

$$
\begin{aligned}
S_t &= \frac{1}{3}\left(S/L_t+S/M_t+S/H_t\right)\\
B_t &= \frac{1}{3}\left(B/L_t+B/M_t+B/H_t\right)\\
H_t &= \frac{1}{2}\left(S/H_t+B/H_t\right)\\
L_t &= \frac{1}{2}\left(S/L_t+B/L_t\right).
\end{aligned}
$$

Plugging the formulation of the conditional factor loadings Equation (39) into Equation (38), we have the following regression model:

$$
\begin{aligned}
R_{W,t}-R_{f,t} = \; & \alpha_W\\
& +a_\beta R_{M,t}+b_\beta R_{M,t}R_{M,W,t}+c_\beta R_{M,t}SMB_{W,t}+d_\beta R_{M,t}HML_{W,t}\\
& +a_s SMB_t+b_s SMB_t R_{M,W,t}+c_s SMB_t SMB_{W,t}+d_s SMB_t HML_{W,t}\\
& +a_h HML_t+b_h HML_t R_{M,W,t}+c_h HML_t SMB_{W,t}+d_h HML_t HML_{W,t}\\
& +\varepsilon_{W,t}.
\end{aligned}
$$

Figure 3 plots the estimated time-varying factor risk exposures, $\widehat{\beta}_{W,t}$, $\widehat{s}_{W,t}$, and $\widehat{h}_{W,t}$, for the 11/1/3 winner portfolio along with the unconditional factor sensitivity. The figure also includes the ranking period factor return. As predicted by the analysis of Grundy and Martin (2001), there is significant time variation in risk that is related to ranking period factor returns, as in Equation (39). Although we estimate $\widehat{\beta}_{W,t}$, $\widehat{s}_{W,t}$, and $\widehat{h}_{W,t}$ as functions of $R_{M,W,t}$, $SMB_{W,t}$, and $HML_{W,t}$, the figure only plots the own-factor ranking-period return (i.e., $R_{M,W,t}$ for $\widehat{\beta}_{W,t}$, $SMB_{W,t}$ for $\widehat{s}_{W,t}$, etc.). The 11/1/3 equal-weighted winner portfolio has estimated factor loadings that range from 0.73 to 1.48 for the market factor, from 0.19 to 2.13 for the size factor, and from -0.68 to 0.47 for the book-to-market factor. For comparison, the unconditional factor loadings are 1.05, 0.97, and -0.09, respectively. The unconditional factor loadings are similar to the values of 1.13, 0.68, and 0.04 found for a 11/1/1 strategy by Fama and French (1996, Table VII).

6.2 Abnormal Momentum Profits with Proportional Costs

Our analysis is restricted to 11/1/3 and 5/1/6 strategies, since they exhibit significant performance before price impacts (see Table 1) and are extensively studied in the literature. The results for VW and EW momentum portfolios with proportional transactions costs are shown in Table 5 for NYSE listed stocks. The estimated abnormal returns, $\widehat{\alpha}$, for zero transactions costs are 80 and 57 basis points per month for the EW and VW, 11/1/3 momentum strategies, respectively. The value for the EW strategy is higher than the 59 basis points found with an unconditional three-factor model by Fama and French (1996, Table VII) for a 11/1/1 strategy. For the 5/1/6 strategy the abnormal returns are 59 and 33 basis points per month for the EW and VW strategies. These are smaller than the 148 basis point abnormal return found by Grundy and Martin (2001 Table 1B) (for an EW 6/1/1 strategy). All four abnormal returns (EW and VW for 11/1/3 and 5/1/6). are statistically significant. With proportional transactions costs equal to the effective spread, $\widehat{\alpha}$ is 61 and 45 basis points with t-statistics of 6.86 and 3.59 for EW and VW 11/1/3 momentum strategies, respectively. For the 5/1/6 strategy the abnormal returns are 41 and 22 basis points per month for the EW and VW strategies, with t-statistics of 5.60 and 2.31. For proportional transactions costs implied by the quoted spread, $\widehat{\alpha}$ is 54 and 40 basis points with t-statistics of 6.08 and 3.17 for EW and VW, 11/1/3 momentum strategies, respectively. For the 5/1/6 strategy the abnormal returns are 35 and 17 basis points per month for the EW and VW strategies, with t-statistics of 4.72 and 1.82. The results indicate that the proportional costs used here do not drive away the statistical significance of momentum profits (with the exception of using quoted spreads for the 5/1/6 VW strategy).

Another way to evaluate the performance of momentum strategies is to test whether the inclusion of such portfolios in the feasible set of assets improves the investment frontier. This is done by calculating the slope of the tangency portfolio, with and without momentum strategies. In our sample, an investment frontier spanned by the three Fama-French factors has a slope of 0.23 (this is the maximum attainable Sharpe ratio with these three assets). As shown in Table 5, adding the 11/1/3 EW momentum strategy as a fourth asset, increases the attainable slope to 0.44, without considering transaction costs. When effective and quoted spreads are considered as proportional trading costs, slopes of 0.38 and 0.35, respectively, are achieved. Notice that both 11/1/3 and 5/1/6 (EW and VW) improve the investment frontier, even after considering trading costs. We conclude that proportional trading costs do not eliminate the observed profitability of momentum strategies.

6.3 Abnormal Momentum Profits with Price-Impact Costs

We now turn to the non-proportional cost, price impact models. In addition to calculating the performance of value-weighted (VW) and equal weighted (EW) momentum portfolios, we also investigate liquidity-weighted

(LW) momentum portfolios when assuming the existence of non-proportional, price impact costs. The LW portfolios are constructed using the simplifying assumption of Corollary 1, that all assets in the winner portfolio have the same expected return.[11] Additionally we investigate the performance of portfolios whose weights are convex combinations of the VW and LW weights. We study the performance of these above strategies as we vary the initial amount invested in each at the beginning of February 1967.[12] Every month, the portfolios are rebalanced according to the rules dictated by the trading strategies. These rules define the stocks to be included in the portfolio (according to the different ranking and holding periods) and their weight in the portfolio. The portfolios are self-financing, since no additional funds are added to or removed from the portfolio during the entire investment period, which ends at the end of December 1999. The net returns are calculated using the trading model developed above, assuming that the price impact coefficients are known. Also, since the set of firm characteristics used to predict price impact, X_{t-1}, is predetermined at time t, the strategies are adapted to the information set available at the time of each trade, and therefore these strategies are admissible (however, for much of the sample, Γ is estimated with future data).

6.3.1 Breen, Hodrick, and Korajczyk (2002) Price-Impact Specification

We first investigate the performance after price impacts implied by the specification of Breen, Hodrick, and Korajczyk (2002) in Equation (1). The results for NYSE traded firms are given in Figure 4. In Figure 4a we plot the estimated portfolio abnormal returns, $\widehat{\alpha}$, for a number of momentum strategies as a function of the level of initial investment (expressed in terms of December 1999 market capitalization). Price impact quickly drives away the profitability of equal-weighted strategies. Abnormal returns are driven to zero with investment portfolios larger than \$2 billion for value weighted strategies. However, for the liquidity weighted (LW) strategy, or the 50/50 weighting of the LW and VW strategies, α is driven to zero only after approximately \$5 billion is invested. Figure 4b is a plot of the t-statistic for $\widehat{\alpha}$ as the level of investment varies. A horizontal line is drawn at the the value 1.96, the approximate cutoff for significance at the 5% level. The EW and LW strategies become insignificant at very low levels of investment. The VW strategy is statistically significant until the size of the investment is slightly less than \$1 billion, while the combined LW/VW strategy is significant until approximately \$1.1 billion is invested.

The LW strategy provides high levels of abnormal return (as can be seen in Figure 4a) but low levels of

[11] The definition of LW differs across the BHK and GH price-impact models. Corollary 1 directly addresses the BHK case, and therefore we use $1/b_i$ as liquidiy-weights in this case. For the GH case we use a similar weighting scheme in spirit. Since there are fixed and variable costs in that model, LW are calculated as the average between weights generated from $1/b_i$ and from $1/\overline{\Psi}_i$ (notice the definition of b_i differs between BHK and GH).

[12] The dollar amounts reported throughout the paper are expressed relative to market capitalization at the end of December 1999. That is, we report the dollar amount at the end of 1999 that constitutes the same fraction of total market capitalization as the initial invetment in February 1967.

statistical significance (Figure 4b) because the LW portfolio tends to be less well diversified than the VW or combined LW/VW portfolios. Figure 4c provides one estimate of the monthly value creation ($\widehat{\alpha}$ times the level of investment) for different levels of investment. For the LW and the combined LW/VW portfolios, value creation is maximized with portfolios investing approximately \$2.5 billion. In Figure 4d we plot the maximal Sharpe ratio attainable through combinations of Treasury bills, the three Fama/French factor portfolios and the winner momentum portfolio. A horizontal line (at value of 0.23) is drawn at the maximal Sharpe ratio attainable through combinations of Treasury bills, the three Fama/French factor portfolios.[13] These results mirror those in Figure 4a: the EW Sharpe ratio drops to that of the factor portfolios for low levels of investment; the VW Sharpe ratio drops to that of the factor portfolios for a level of investment around \$2 billion; and the LW and LW/VW Sharpe ratios drop to that of the factor portfolios for a level of investment around \$5 billion.

When price impacts are considered, the LW strategy earns higher excess returns for larger initial investments. The LW strategy is designed to maximize post-trading returns without regard to diversification. The results in Figure 4 are consistent with this. Figure 4a shows that the LW strategy provides superior average returns to the VW strategy after price impact is taken into account, except for very small investment amounts. Figure 4d shows that the LW strategy leads to more concentrated (less well diversified) portfolios than the VW strategy since the Sharpe ratio is smaller for the LW strategy until large investments levels.

6.3.2 Glosten and Harris (1988) Price-Impact Specification

We now turn to performance assuming price impacts implied by the specification of Glosten and Harris (1988), Equation (2). The results for NYSE traded firms are given in Figure 5. The basic patterns are similar to those in Figure 4. In Figure 5a we plot the estimated portfolio abnormal returns, $\widehat{\alpha}$, for a number of momentum strategies as a function of the level of initial investment. As with the previous specification, price impact quickly drives away the profitability of equal-weighted strategies. Abnormal returns are driven to zero with investment portfolios larger than \$3 billion for value weighted strategies. However for the liquidity weighted (LW) strategy α is driven to zero only after over \$5 billion is invested. For the 50/50 weighting of the LW and VW strategies, α is driven to zero after approximately \$4.5 billion is invested. Figure 5b is a plot of the t-statistic for $\widehat{\alpha}$ as the level of investment varies. The EW and VW strategies become insignificant at levels of investment less than \$1 billion. The LW strategy and the combined LW/VW strategy are significant until approximately \$1.5 billion is invested. Figure 5c provides one estimate of the monthly value creation ($\widehat{\alpha}$ times the level of investment) for different levels of investment. As before, for the LW and

[13] Notice all performance lines associated with the different momentum strategies are above or equal to 0.23. This is due to the fact that we only consider here strategies that long winners, and therefore we restrict the weight assigned to the momentum portfolio as part of the tangency portfolio to be nonnegative.

the combined LW/VW portfolios, value creation is maximized with portfolios investing approximately \$2.5 billion. In Figure 5d we plot the maximal Sharpe ratio attainable through combinations of Treasury bills, the three Fama/French factor portfolios, and the winner momentum portfolio. A horizontal line is drawn at the maximal Sharpe ratio attainable through combinations of Treasury bills and the three Fama/French factor portfolios. These results mirror those in Figure 5a: the EW Sharpe ratio drops to that of the factor portfolios for low levels of investment; the VW Sharpe ratio drops to that of the factor portfolios for a level of investment around \$2 billion; and the LW and LW/VW Sharpe ratios drop to that of the factor portfolios for a level of investment around \$4.5 to \$5 billion.

6.3.3 Extending Admissible Set of Assets

We now apply the price impacts model implied by the specification of Breen, Hodrick, and Korajczyk (2002), Equation (1), to a strategy that invests in AMEX and NASDAQ stocks, in addition to NYSE stocks. Expanding the sample in this manner has two offsetting effects. First, the newly included firms are likely to be smaller and less liquid, on average, than the NYSE stocks. This would tend to reduce the break-even investment amounts for the expanded sample. Second, with a larger sample of firms, a portfolio investing a given amount of funds can spread those funds across a greater number of firms. Since the investment in any one firm is lower, the price impact is lower. This would tend to increase the break-even investment amounts for the expanded sample. Figure 6 compares the performance of both EW and VW versions of 11/1/3 and 5/1/6 strategies. After price impacts, VW strategies dominate EW strategies and the 11/1/3 strategy dominated the 5/1/6 strategy.

Because of the dominance of VW 11/1/3 strategies in Figure 6, in Figure 7 we look at the performance of VW, LW, and LW/VW 11/1/3 strategies. In comparing Figures 7a, 7b, and 7d to Figures 4a, 4b, and 4d, all three strategies have larger break-even investment amounts with the expanded sample. This is true in terms of the portfolio size that drives $\widehat{\alpha}$ to zero, that drives the t-statistic of $\widehat{\alpha}$ to 2.0, and that drives the maximal Sharpe ratio to that of the Fama and French factors. Turning to panel C of Figures 4 and 7, the fund size that creates maximal value is also larger for the expanded sample.

7 Conclusions

This paper tests whether momentum-based strategies that previously have been shown to earn high abnormal returns remain profitable after considering price impact induced by trading. The paper develops a methodology to include liquidity in a trading model and demonstrates the importance of such measures to the performance evaluation of trading strategies. In summary, we find that, when price impact is ignored, the 11/1/3 and 5/1/6 strategies earn significant abnormal returns relative to the Fama and French (1993) three-factor asset pricing model. The strategies remain profitable when transaction costs are proportional costs equal to the effective and quoted spreads. The 11/1/3 strategy outperforms the 5/1/6 strategy and equal-weighted strategies outperform value-weighted strategies. In contrast to the results ignoring price impact costs, both 11/1/3 and 5/1/6 momentum strategies perform better, post price impact, using market capitalization weights rather than equal weights. For example, the zero-α break-even point is 200 million dollars for the 11/1/3 EW strategy, while it is more than 2 billion dollars for a 11/1/3 VW strategy. This is due to the fact that value-weighting is concentrated in more liquid stocks than equal-weighting. Equal-weighted portfolios have higher price impact costs. Trading costs are crucial for equally weighted strategies since their performance measures decrease dramatically even when a relatively small investment is considered. These results are especially important in light of recent momentum literature, which concentrates on equally weighted strategies. For example, Fama and French (1996), Grundy and Martin (2001), and Yao (2001), study 11/1/1, 6/1/1, and 6/0/6 equally weighted strategies, respectively. These strategies seem to be less tractable in the context of transaction costs. The results are consistent across two alternative measures of price impact from Glosten and Harris (1988) and Breen, Hodrick, and Korajczyk (2002), with the Glosten and Harris (1988) measure leading to slightly larger break-even points. We attempt to construct superior momentum strategies by taking price impacts into account while choosing the portfolio weights. Our liquidity weighted (LW) strategies provide higher post-price impact abnormal returns relative to VW strategies, although the lack of diversification in the LW strategies can lead to lower t-statistics for the abnormal returns. Portfolio strategies which have weights that are convex combinations of the LW and VW weights often provide abnormal returns similar to the LW portfolios and larger t-statistics for the abnormal returns.

We calculate the break-even size of a momentum based portfolios with monthly rebalancing. We find that the estimated excess returns of some momentum strategies disappear only after \$4.5 to over \$5.0 billion (relative to market capitalization in December 1999) is engaged (by a single fund) in such strategies. The statistical significance of these excess returns disappears only after \$1.1 to \$2.0 billion is engaged (by a single fund) in such strategies.

There are several reasons to believe that our break-even fund sizes are conservative (that is, too small). First, our strategies are implemented at the end of each month without any attempt to "trickle out" the trades beyond the impact estimation interval. Since the Breen, Hodrick, and Korajczyk (2002) price impact coefficients are measured over a 30-minute interval, we are implicitly assuming that the month-end rebalancing takes place over a 30-minute interval. The Glosten and Harris (1988) price impact coefficients are measured on a trade-by-trade basis, so in that case, we are implicitly assuming that the month-end rebalancing takes place in a single trade. Certainly an astute portfolio manager might choose to transact in a more patient fashion, thereby reducing price impact costs. For example, Breen, Hodrick, and Korajczyk (2002) compare the predicted price impact obtained from fitted values of cross-sectional regressions like (3) to the actual price impact experienced by a sample of institutional traders. The predicted price impacts were higher than the actual price impacts, on average. Second, while we derive optimal weights for any set of expected returns, our empirical results rely on the simplifying assumption of Corollary 1, that expected returns are the same for all assets in our "winner" momentum portfolio. Performance could be improved by a better model of expected returns.[14] Third, for reasons outlined above, we assume either linear or convex price impact functions rather than concave price impacts. If the concavity observed empirically is obtainable, rather than being due to information leakages or credible signalling by uninformed traders, then we should be able to invest larger quantities profitably. Our LW strategy is based on maximizing post-price impact returns rather that maximizing a return-to-risk ratio. A strategy that incorporates risk, in addition to returns, should lead to higher Sharpe ratios. Finally, the liquidity-based portfolios that we have examined are based only on partial optimization, since only a single investment period is considered in the optimization problem. An extension of the static optimization to a dynamic setting, should result in superior performance of strategies under price impacts.

Whether the break-even fund sizes calculated here are large or small is somewhat in the eye of the beholder. A break-even fund size of \$4.5 to over \$5.0 billion (where $\widehat{\alpha}$ is driven to zero) is small relative to the total market capitalization of the NYSE (\$11.7 trillion). Chen, Stanzl, and Watanabe (2001, Table 13) report data on hedge fund sizes by investment style. From the sample of hedge funds in the TASS data set, a break-even fund size of \$5.0 billion represents 2.7% of total value of hedge funds, 8.9% of total value of "Arbitrage" style hedge funds and 34.2% of "Trend Follower" style hedge funds (see Chen, Stanzl, and Watanabe (2001, Table 13)). As noted above, there are reasons to believe that attainable break-even funds sizes are larger than those calculated here. Hence they would represent larger fractions of the hedge fund universe. The profitability of the strategies is in addition to the profits already earned by momentum-based investors in the market over the sample period.

While time variation in expected liquidity is considered in our analysis, we do not consider liquidity risk in the factor model which is used to evaluate abnormal returns nor in the portfolio selection problem. We document that winners are more liquid than losers and have higher momentum-based expected returns so a simple, direct relation between momentum-based expected returns and liquidity is not likely to explain the anomaly. However, winners and losers may have different exposures to systematic shifts in liquidity (e.g., Chordia, Roll, and Subrahmanyam (2000) and Pástor and Stambaugh (2001)) and, therefore, different

[14] We have estimated expected returns using several momentum-based models. The strategies based on these expected returns under-perform those that assume equal expected returns. This is due to the failure of the models to explain either the level of expected returns, or the cross-sectional variation of expected returns, or both. Note that not only is the cross-sectional variation an important input to our model, but also the actual level of expected returns, since the ratio of expected return over price impact is a crucial input to our model.

premiums to systematic liquidity risk than might be suggested by the absolute levels of liquidity. We leave these issues to future research.

Accounting for the "hidden costs" of trading, the price impact, leads to a large decline in the apparent profitability of some previously studied momentum-based strategies. Given the size of the break-even portfolios and the likelihood that they are underestimated, transaction costs do not appear to fully explain the return persistence of past winner stocks exhibited in the data. Thus, this anomaly remains an important puzzle.

Appendix

Theorem 1 (general version)

Define the sets

$$\begin{aligned} Z^c &= \left\{ \{z_i^c\}_{i\in I_1} : z_i^c \equiv \frac{1+E[R_i]}{e^{b_i(c_i-a_i)}} \right\} \\ Z^d &= \left\{ \left\{z_i^d\right\}_{i\in I_1} : z_i^d \equiv \frac{1+E[R_i]}{e^{b_i(d_i-a_i)}} \right\} \\ Z &= Z^c \cup Z^d. \end{aligned} \tag{40}$$

Rank all assets in $Z = \left\{z_{(1)}, z_{(2)}, ...\right\}$ and for each index define the following sets of indexes:

$$\begin{aligned} I_{(z)} &= \left\{ i : i \in I_1 \ , \ z_i^c \geq Z_{(z)} > z_i^d \right\} \\ J_{(z)} &= \left\{ j : j \in I_1 \ , \ Z_{(z)} > z_j^c \right\} \\ K_{(z)} &= \left\{ k : k \in I_1 \ , \ z_k^d \geq Z_{(z)} \right\}. \end{aligned} \tag{41}$$

There exists a unique solution to the static optimization problem above. The optimal trading strategy is characterized by

$$\begin{aligned} \forall i &\in I_{(z^*)} \qquad y_i^* = \frac{1}{b_i} \ln \left[\frac{1+E[R_i]}{\lambda_{(z^*)}} \right] + a_i \\ \forall j &\in J_{(z^*)} \qquad y_j^* = c_j \\ \forall k &\in K_{(z^*)} \qquad y_k^* = d_k \end{aligned} \tag{42}$$

where z^* and λ satisfy

$$Z_{(z^*)} \geq \lambda_{(z^*)} > Z_{(z^*-1)} \tag{43}$$

$$\lambda_{(z^*)} = \frac{\sum_{i\in I_{(z^*)}} \frac{1+E[R_i]}{b_i}}{A - \sum_{j\in J_{(z^*)}} \frac{1}{b_j} e^{b_j(c_j-a_j)} - \sum_{k\in K_{(z^*)}} \frac{1}{b_k} e^{b_k(d_k-a_k)}}. \tag{44}$$

Proof. The Lagrange formulation of the maximization problem is given by

$$\mathcal{L} = \sum_{i\in} y_i \left(1+E[R_i]\right) + \lambda \left(A - \sum_{i\in I_1} \frac{1}{b_i} e^{b_i(y_i-a_i)} \right) - \sum_{i\in I_1} \mu_i \left(y_i - c_i\right) - \sum_{i\in I_1} \gamma_i \left(d_i - y_i\right) \tag{45}$$

The first order conditions, along with the complementary slackness conditions, are given by

$$\frac{\partial \mathcal{L}}{\partial y_i} = 1 + E[R_i] - \lambda e^{b_i(y_i-a_i)} - \mu_i + \gamma_i = 0 \tag{46}$$

$$\lambda \left(\sum_{i\in I_1} \frac{1}{b_i} e^{b_i(y_i-a_i)} - A \right) = 0 \qquad \lambda \geq 0 \tag{47}$$

$$\mu_i \left(y_i - c_i\right) = 0 \qquad \mu_i \leq 0 \tag{48}$$

$$\gamma_i \left(d_i - y_i\right) = 0 \qquad \gamma_i \leq 0. \tag{49}$$

In general, the solution requires the division of the set of assets, I_1, into three mutually disjoint sets I, J, and K (some of which may be empty) as follows:

$$
\begin{aligned}
I &= \{i : \mu_i = 0 \wedge \gamma_i = 0 \quad (c_i \leq y_i \leq d_i)\} && (50)\\
J &= \left\{j : \mu_j < 0 \wedge \gamma_j = 0 \quad (y_j = c_j)\right\} \\
K &= \{k : \mu_k = 0 \wedge \gamma_k < 0 \quad (y_k = d_k)\}\,. && (51)
\end{aligned}
$$

The first order condition (46) implies that

$$\lambda e^{b_i(y_i - a_i)} = 1 + E\left[R_i\right] - \mu_i + \gamma_i. \tag{52}$$

Applying Equation (52) to each of the sets above yields

$$
\begin{aligned}
\forall i &\in I & 1 + E\left[R_i\right] - \lambda e^{b_i(y_i - a_i)} = 0 &\quad\Longrightarrow\quad \lambda = \frac{1 + E\left[R_i\right]}{e^{b_i(y_i - a_i)}} && (53)\\
\forall j &\in J & \mu_j = 1 + E\left[R_j\right] - \lambda e^{b_j(c_j - a_j)} < 0 &\quad\Longrightarrow\quad \lambda > \frac{1 + E\left[R_j\right]}{e^{b_j(c_j - a_j)}} && (54)\\
\forall k &\in K & \gamma_k = 1 + E\left[R_k\right] - \lambda e^{b_k(d_k - a_k)} > 0 &\quad\Longrightarrow\quad \frac{1 + E\left[R_k\right]}{e^{b_k(d_k - a_k)}} > \lambda. && (55)
\end{aligned}
$$

Also note that the upper bound and lower bound for every $i \in I_1$ satisfies $c_i \leq d_i$ by definition. This implies that

$$\forall i \in I_1 \qquad \frac{1 + E\left[R_i\right]}{e^{b_i(c_i - a_i)}} \geq \frac{1 + E\left[R_i\right]}{e^{b_i(d_i - a_i)}}. \tag{56}$$

Notice that the proposition includes definitions to the sets I, J, and K using the index z^* (see Equation (42)). One may verify that at optimum, the definitions of these sets must coincide with their corresponding definitions given in the proposition. The set I contains the assets that are traded as an interior solution, between the lower and upper bounds; the set J corresponds to the assets traded at their lower bound; and the set K corresponds to the assets traded at their upper bound.

To solve for λ, multiply the budget constraint (Equation (34) with equality) by λ, and plug in it the left hand side of the first order condition (Equation (52)). This procedure results in:

$$\lambda = \frac{\sum_{i \in I_1} \frac{1 + E[R_i] - \mu_i + \gamma_i}{b_i}}{A}. \tag{57}$$

Using the above expressions for μ_j and γ_k (Equation (53)), we obtain the following equation:

$$\lambda = \frac{\sum_{i \in I} \frac{1 + E[R_i]}{b_i} - \lambda \sum_{j \in J} \frac{1}{b_j} e^{b_j(c_j - a_j)} - \lambda \sum_{k \in K} \frac{1}{b_k} e^{b_k(d_k - a_k)}}{A}. \tag{58}$$

Solving for λ from Equation (58) yields the following expression:

$$\lambda = \frac{\sum_{i \in I} \frac{1 + E[R_i]}{b_i}}{A - \sum_{j \in J} \frac{1}{b_j} e^{b_j(c_j - a_j)} - \sum_{k \in K} \frac{1}{b_k} e^{b_k(d_k - a_k)}}. \tag{59}$$

Finally, the second order conditions for maximum must be satisfied:

$$\frac{\partial^2 \mathcal{L}}{\partial y_i^2} = -\lambda b_i e^{b_i(y_i - a_i)}. \tag{60}$$

By construction, $\lambda > 0$. Thus, we conclude that $\partial^2 \mathcal{L}/\partial y_{i,t}^2 \leq 0$, and therefore the necessary conditions for optimality are satisfied. The optimization problem is of a convex nature and thus the solution found above satisfies necessary and sufficient conditions of optimality.

The interpretation of Proposition 1 follows basic economic principles. The value z_i^c is the ratio of marginal return and marginal cost for the first dollar, above the lower bound c_i, invested in asset i. Similarly, z_i^d is the ratio of marginal return and marginal cost for the last dollar, below the upper bound, invested in asset i. The ratio of marginal return and marginal cost may be viewed as the marginal net return. Due to increasing marginal costs and constant marginal returns, the marginal net return for any asset i decreases between z_i^c and z_i^d. For this reason the extreme marginal net returns for all assets are sorted in a descending fashion. Then, funds are allocated to the assets according to the latter ordering. The allocation is stopped when the budget constraint is met. This is controlled by the multiplier $\lambda_{(z^*)}$. ■

Proof of Corollary 1

Using the guidelines of the Remark 1, and omitting the expressions associated with the lower and upper bounds, we have:

$$y_i = \frac{1}{b_i} \ln \left[\frac{x_0 + \sum_{i \in I_1} \frac{1}{b_i}}{\sum_{i \in I_1} \frac{1}{b_i}} \right] \tag{61}$$

and therefore the weights are calculated as

$$\omega_i = \frac{y_i}{\sum_{i \in I_1} y_i} = \frac{\frac{1}{b_i}}{\sum_{i \in I_1} \frac{1}{b_i}}. \tag{62}$$

Since $b_i = \lambda_i / MVE_i$, assuming that all price impact coefficients are equal produces:

$$\omega_i = \frac{MVE_i}{\sum_{i \in I_1} MVE_i}. \tag{63}$$

This provides an additional theoretical foundation to the use of market capitalizations as weights of assets in a portfolio, as they minimize price impacts of trading the portfolio.

■

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Table 1

Average Returns to Momentum Strategies

A momentum strategy is defined by the triplet (*J*,*S*,*K*), where *J* is the ranking period (according to past *J*-month cumulative return), S is a skip period (set to one month in all the strategies below), and *K* is the holding period. Every month stocks are sorted according to the chosen ranking period (*J*), then after skipping one month (*S*), portfolios are formed using stocks in the top decile (winners) and in the lower decile (losers). The portfolios are held for *K* months. This process is repeated every month, while a 1/*K* fraction of each portfolio is rebalanced. The time-series means of momentum portfolio monthly returns, as well as the associated *t*-statistics (two-digit numbers), are presented below for various ranking and holding periods. The analysis is performed separately using NYSE-listed stocks, and using all NYSE,AMEX, and NASDAQ stocks. Panel A uses equal weighted for each stock while forming the portfolios, and Panel B uses value (market capitalization) weights. The analysis uses data for the period February 1967 to December 1999 (395 months).

Panel A: Equal-weighted strategies

Winners

	NYSE						NYSE+AMEX+NASDAQ				
J	K					J	K				
	1	3	6	9	12		1	3	6	9	12
2	0.0151	0.0155	0.0161	0.0165	0.0165	2	0.0130	0.0137	0.0142	0.0147	0.0147
	4.85	4.96	5.17	5.29	5.23		3.87	4.15	4.32	4.44	4.42
5	0.0181	0.0185	0.0193	0.0191	0.0177	5	0.0162	0.0164	0.0167	0.0165	0.0150
	5.78	5.76	5.87	5.92	5.77		4.87	4.87	4.84	4.90	4.74
8	0.0211	0.0209	0.0207	0.0194	0.0178	8	0.0187	0.0184	0.0179	0.0163	0.0147
	6.54	6.46	6.35	5.98	5.51		5.62	5.50	5.33	4.88	4.43
11	0.0223	0.0213	0.0199	0.0186	0.0171	11	0.0200	0.0187	0.0169	0.0154	0.0138
	6.75	6.42	6.04	5.67	5.24		5.83	5.50	5.03	4.59	4.15

Losers

	NYSE						NYSE+AMEX+NASDAQ				
J	K					J	K				
	1	3	6	9	12		1	3	6	9	12
2	0.0108	0.0103	0.0101	0.0101	0.0104	2	0.0109	0.0095	0.0096	0.0101	0.0106
	2.85	2.80	2.81	2.83	2.96		2.69	2.35	2.41	2.55	2.71
5	0.0082	0.0081	0.0082	0.0080	0.0097	5	0.0100	0.0090	0.0092	0.0091	0.0105
	2.14	2.12	2.21	2.27	2.78		2.37	2.13	2.19	2.29	2.74
8	0.0080	0.0074	0.0072	0.0085	0.0102	8	0.0103	0.0092	0.0086	0.0098	0.0116
	1.97	1.86	1.86	2.21	2.66		2.31	2.08	2.02	2.30	2.75
11	0.0065	0.0067	0.0080	0.0095	0.0111	11	0.0081	0.0083	0.0095	0.0110	0.0128
	1.63	1.68	2.01	2.40	2.81		1.84	1.88	2.18	2.53	2.97

Panel B: Value-weighted strategies

Winners

	NYSE						NYSE+AMEX+NASDAQ				
J	K					J	K				
	1	3	6	9	12		1	3	6	9	12
2	0.0123	0.0127	0.0125	0.0131	0.0130	2	0.0138	0.0144	0.0136	0.0140	0.0137
	4.32	4.67	4.65	4.89	4.79		4.36	4.73	4.58	4.75	4.62
5	0.0134	0.0140	0.0149	0.0149	0.0139	5	0.0152	0.0154	0.0156	0.0154	0.0141
	4.67	4.89	5.17	5.28	5.14		4.78	4.83	4.90	4.96	4.82
8	0.0164	0.0167	0.0168	0.0159	0.0145	8	0.0185	0.0181	0.0175	0.0162	0.0145
	5.63	5.74	5.74	5.44	5.00		5.85	5.73	5.50	5.12	4.66
11	0.0170	0.0171	0.0159	0.0152	0.0140	11	0.0183	0.0182	0.0164	0.0150	0.0137
	5.62	5.69	5.31	5.06	4.75		5.55	5.56	5.08	4.71	4.37

Losers

	NYSE						NYSE+AMEX+NASDAQ				
J	K					J	K				
	1	3	6	9	12		1	3	6	9	12
2	0.0088	0.0088	0.0087	0.0080	0.0079	2	0.0030	0.0041	0.0051	0.0050	0.0052
	2.70	2.83	2.88	2.68	2.68		0.82	1.21	1.54	1.53	1.62
5	0.0071	0.0076	0.0067	0.0057	0.0072	5	0.0000	0.0014	0.0028	0.0023	0.0044
	2.09	2.36	2.15	1.92	2.44		0.03	0.44	0.85	0.77	1.37
8	0.0067	0.0060	0.0050	0.0059	0.0073	8	0.0010	0.0012	0.0012	0.0026	0.0050
	1.87	1.74	1.50	1.78	2.24		0.25	0.31	0.32	0.71	1.36
11	0.0028	0.0039	0.0051	0.0062	0.0075	11	-0.0030	-0.0011	0.0013	0.0035	0.0060
	0.80	1.11	1.48	1.82	2.24		-0.75	-0.29	0.35	0.92	1.55

Table 2

Firm Characteristics

Time-series means of the cross-sectional monthly distributional statistics of firm characteristics are presented below. The analysis is presented separately for NYSE/AMEX and NASDAQ. Panel A includes only data of a subset, the period January 1993 to May 1997 (53 periods). Panel B expands the sample to include the entire data for the period February 1967 to December 1999 (395 periods).

Panel A: Estimation Period (Jan 1993 - May 1997)						Panel B: Sample Period (Feb 1967 - Dec 1999)					
NYSE/AMEX						NYSE/AMEX					
Variable	Mean	Std	Minimum	Median	Maximum	Variable	Mean	Std	Minimum	Median	Maximum
X_1	0.90	6.74	-1.00	-0.70	138.65	X_1	0.67	6.60	-1.00	-0.74	172.77
X_2	-0.18	1.89	-1.00	-0.78	32.53	X_2	-0.22	1.65	-1.00	-0.75	29.09
X_3	0.07	0.31	-0.88	0.03	4.70	X_3	0.07	0.30	-0.77	0.03	4.29
X_4	0.19	0.25	0.00	0.12	4.70	X_4	0.23	0.25	0.00	0.17	4.29
X_5	0.12	0.32	0.00	0.00	1.00	X_5	0.12	0.33	0.00	0.00	1.00
X_6	0.02	0.04	0.00	0.01	0.83	X_6	0.03	0.05	0.00	0.01	1.62
X_7	0.04	0.05	0.00	0.02	0.40	X_7	0.06	0.06	0.00	0.04	0.40
X_8	0.77	0.42	0.00	1.00	1.00	X_8	0.71	0.45	0.00	1.00	1.00
X_9	0.14	0.43	0.00	0.06	12.30	X_9	0.14	0.36	0.00	0.06	9.56
NASDAQ						NASDAQ					
Variable	Mean	Std	Minimum	Median	Maximum	Variable	Mean	Std	Minimum	Median	Maximum
X_1	-0.75	1.52	-1.00	-0.94	62.72	X_1	-0.77	2.80	-1.00	-0.95	125.86
X_2	-0.50	1.66	-1.00	-0.86	45.15	X_2	-0.62	1.08	-0.99	-0.86	25.48
X_3	0.08	0.48	-0.95	0.03	9.55	X_3	0.06	0.43	-0.83	0.00	6.24
X_4	0.30	0.39	0.00	0.21	9.55	X_4	0.29	0.36	0.00	0.21	6.25
X_5	0.01	0.11	0.00	0.00	1.00	X_5	0.01	0.08	0.00	0.00	0.82
X_6	0.01	0.03	0.00	0.00	1.15	X_6	0.01	0.03	0.00	0.00	0.92
X_7	0.03	0.05	0.00	0.02	0.37	X_7	0.04	0.05	0.00	0.03	0.34
X_9	0.33	1.06	0.00	0.11	29.74	X_9	0.47	1.16	0.01	0.14	15.41

X_1 = market cap at the end of last month divided by the average market cap of CRSP, minus one
X_2 = total volume during the last three months divided by the average firm volume on NYSE, minus one
X_3 = stock price at the end of last month divided by the price six month prior, minus one
X_4 = absolute value of X_3
X_5 = dummy variable equal to unity if the firm is included in the S&P500 index
X_6 = dividend yield
X_7 = R^2 of returns regressed on NYSE index, monthly returns over the last 36 months
X_8 = dummy variable equal to unity if the firm is traded on NYSE
X_9 = inverse of stock price of the previous month

Table 3
Transaction costs and Firm Characteristics

Estimates of the average cross-sectional relation between different transaction costs and firm-specific predetermined variables are provided below (these relations are estimates of Γ, as explained in the paper). Two non-proportionate transaction costs are considered. The Breen-Hodrick-Korajczyk (2002) measure is based on the model $\Delta p_{i,t}/p_{i,t} = \lambda_{BHK,i}\Delta q_{i,t}$, where $\Delta p_{i,t}/p_{i,t}$ is the relative price improvement of stock i as a result of trading a net total of $q_{i,t}$ (signed) shares in a 30-minute interval (t). The Glosten-Harris (1988) measure is based on the model $\Delta p_{i,t} = \lambda_{GH,i}\Delta q_{i,t} + \psi_i \Delta d_{i,t}$, where $\Delta p_{i,t}$ is the relative price improvement as a result of trading $q_{i,t}$ (signed) shares at time t (here t represents event time), and $d_{i,t}$ is an indicator for buyer-initiated (+1) or seller-initiated (-1) trade. For proportionate costs we consider effective and quoted spreads. Effective spreads are measured as the absolute price improvement relative to mid-point of quoted bid and ask. Quoted spread is measured as the ratio between the quoted bid-ask spread and the mid-point. All transaction costs are estimated on a monthly basis. The analysis is separated for NYSE/AMEX and NASDAQ for the Breen-Hodrick-Korajczyk measure, and includes only NYSE for all other measures. The analysis used data for the period January 1993 to May 1997.

	Non-Proportionate Costs								Proportionate Costs			
	Breen-Hodrick-Korajczyk				Glosten-Harris				Effective Spread		Quoted Spread	
	NYSE/AMEX		NASDAQ		NYSE				NYSE		NYSE	
Variable	$\lambda^{BHK} \times 10^5$	T-statistic	$\lambda^{BHK} \times 10^5$	T-statistic	$\lambda^{GH} \times 10^6$	T-statistic	$\psi \times 10^4$	T-statistic	$k^E \times 10^4$	T-statistic	$k^Q \times 10^4$	T-statistic
Intercept	2.48	28.40	1.38	30.20	0.454	12.55	26.60	20.72	35.47	22.41	108.29	24.75
X_1	0.25	13.60	0.19	10.80	0.005	5.14	-0.14	-8.13	-0.03	-1.50	-0.07	-1.47
X_2	-0.61	-15.50	-0.15	-12.70	-0.069	-12.93	-0.71	-7.85	-1.93	-15.04	-6.01	-19.33
X_3	-0.63	-3.99	-0.14	-2.04	-0.060	-0.68	-54.13	-12.78	-57.00	-14.44	-158.17	-16.76
X_4	0.34	2.13	-0.21	-2.51	0.250	1.46	59.83	12.30	70.74	13.10	192.43	17.79
X_5	-1.68	-20.90	-0.47	-10.20	-0.201	-4.89	-14.13	-26.24	-18.44	-28.13	-48.09	-28.57
X_6	4.15	4.84	17.70	12.80	-0.796	-2.47	19.13	1.52	-51.56	-3.60	-315.34	-8.45
X_7	-3.42	-9.23	-2.46	-16.00	-0.540	-2.81	-27.91	-7.89	-36.08	-7.32	-86.14	-7.11
X_8	-0.26	-5.23										
X_9	3.23	15.70	3.06	15.00	0.566	5.56	121.92	14.07	168.64	14.38	363.96	12.46
R^2	7.10%		6.90%		6.49%		58.44%		66.13%		63.37%	

X_1 = market cap at the end of last month divided by the average market cap of CRSP, minus one
X_2 = total volume during the last three months divided by the average firm volume on NYSE, minus one
X_3 = stock price at the end of last month divided by the price six month prior, minus one
X_4 = absolute value of X_3
X_5 = dummy variable equal to unity if the firm is included in the S&P500 index
X_6 = dividend yield
X_7 = R^2 of returns regressed on NYSE index, monthly returns over the last 36 months
X_8 = dummy variable equal to unity if the firm is traded on NYSE
X_9 = inverse of stock price of the previous month

Table 4

Estimated Measures of Liquidity

Time-series means of cross-sectional diagnostics of different liquidity measures are presented below. Two non-proportionate transaction costs are considered. The Breen-Hodrick-Korajczyk (2002) measure is based on the model $\Delta p_{i,t}/p_{i,t} = \lambda_{BHK,i}\Delta q_{i,t}$, where $\Delta p_{i,t}/p_{i,t}$ is the relative price improvement of stock *i* as a result of trading a net total of $q_{i,t}$ (signed) shares in a 30-minute interval (*t*). The Glosten-Harris (1988) measure is based on the model $\Delta p_{i,t} = \lambda_{GH,i}\Delta q_{i,t} + \psi_i \Delta d_{i,t}$, where $\Delta p_{i,t}$ is the relative price improvement as a result of trading $q_{i,t}$ (signed) shares at time *t* (here *t* represents event time), and $d_{i,t}$ is an indicator for buyer-initiated (+1) or seller-initiated (-1) trade. For proportionate costs we consider effective and quoted spreads. Effective spreads are measured as the absolute price improvement relative to mid-point of quoted bid and ask. Quoted spread is measured as the ratio between the quoted bid-ask spread and the mid-point (half the quoted spread are provided below). All transaction costs are estimated on a monthly basis. The estimation analysis includes NYSE-listed stocks for the period January 1993 to May 1997. Using cross-sectional relation between the different liquidity measures and pre-determined firm-characteristics (see Table 2), the liquidity measures are re-estimated for the entire sample period, February 1967 to December 1999. Panel A includes all the firms in our sample for two different time periods: the period January 1993 to May 1997 (53 periods), and the period February 1967 to December 1999 (395 periods). Panels B and C include only stocks in the top (winners) and bottom (losers) deciles according to 11/1/1 and 5/1/1 equal-weighted momentum strategies, respectively (see Table 1 for a description of momentum strategies). The analyses in Panels B and C use data for the period February 1967 to December 1999.

Panel A: All Firms

Panel A-1: Jan 1993 - May 1997						Panel A-2: Feb 1967 - Dec 1999					
Variable	Mean	Std	Minimum	Median	Maximum	Variable	Mean	Std	Minimum	Median	Maximum
$\lambda^{BHK} \times 10^5$	2.89	1.88	0.01	2.79	46.59	$\lambda^{BHK} \times 10^5$	2.82	1.93	0.02	2.75	52.65
$\lambda^{GH} \times 10^6$	0.49	0.26	0.00	0.50	7.16	$\lambda^{GH} \times 10^6$	0.48	0.24	0.01	0.49	5.03
$\psi \times 10^3$	4.20	4.86	0.08	3.62	148.27	$\psi \times 10^3$	4.36	4.22	0.14	3.77	106.99
$k^E \times 10^4$	5.44	6.59	0.07	4.71	203.72	$k^E \times 10^4$	5.59	5.60	0.16	4.84	141.00
$\frac{1}{2} k^Q \times 10^4$	7.33	7.39	0.09	6.51	223.14	$\frac{1}{2} k^Q \times 10^4$	7.54	6.38	0.23	6.70	154.06

Panel B: 11/1/1 (Feb 1967 - Dec 1999)

Winners						Losers					
Variable	Mean	Std	Minimum	Median	Maximum	Variable	Mean	Std	Minimum	Median	Maximum
$\lambda^{BHK} \times 10^5$	2.40	1.27	0.02	2.45	11.47	$\lambda^{BHK} \times 10^5$	4.36	3.59	0.09	3.59	31.03
$\lambda^{GH} \times 10^6$	0.51	0.19	0.02	0.53	1.39	$\lambda^{GH} \times 10^6$	0.77	0.57	0.08	0.64	3.88
$\psi \times 10^3$	3.62	2.04	0.69	3.36	19.29	$\psi \times 10^3$	11.62	12.01	2.65	8.33	82.68
$k^E \times 10^4$	5.01	2.82	0.78	4.67	26.54	$k^E \times 10^4$	14.97	16.07	3.11	10.37	108.01
$\frac{1}{2} k^Q \times 10^4$	6.92	3.31	1.15	6.60	30.91	$\frac{1}{2} k^Q \times 10^4$	18.58	17.64	4.16	13.72	118.53

Panel C: 5/1/1 (Feb 1967 - Dec 1999)

Winners						Losers					
Variable	Mean	Std	Minimum	Median	Maximum	Variable	Mean	Std	Minimum	Median	Maximum
$\lambda^{BHK} \times 10^5$	2.44	1.46	0.02	2.43	14.26	$\lambda^{BHK} \times 10^5$	4.17	3.40	0.07	3.51	30.44
$\lambda^{GH} \times 10^6$	0.55	0.21	0.02	0.57	1.60	$\lambda^{GH} \times 10^6$	0.73	0.53	0.06	0.63	3.72
$\psi \times 10^3$	3.87	2.55	0.83	3.49	23.80	$\psi \times 10^3$	11.47	10.99	3.77	8.46	79.87
$k^E \times 10^4$	5.49	3.58	0.98	4.95	33.05	$k^E \times 10^4$	14.50	14.69	4.16	10.36	103.88
$\frac{1}{2} k^Q \times 10^4$	7.41	4.06	1.38	6.90	37.50	$\frac{1}{2} k^Q \times 10^4$	18.31	16.09	5.62	13.89	114.26

Table 5

Performance Under Proportionate Transaction Costs

We evaluate the performance of momentum trading strategies according to the trading model developed here, using proportionate transaction costs. We form a portfolio in the beginning of February 1967 according to a chosen momentum strategy (see Table 1 for a description of momentum strategies) with a certain initial monetary amount of investment. The portfolio is rebalanced on a monthly basis, following the trading rule of the chosen strategy, until the end of December 1999. The proportionate costs considered here include effective and quoted spreads. Effective spreads are measured as the absolute price improvement relative to mid-point of quoted bid and ask. Quoted spread is measured as the ratio between the quoted bid-ask spread and the mid-point (half the quoted spread is considered as trading cost). Transaction costs are estimated on a monthly basis, using NYSE-listed stocks for the period January 1993 to May 1997. Then, using cross-sectional relation between the different liquidity measures and pre-determined firm-characteristics (see Table 2), the spreads are re-estimated for the entire sample period, February 1967 to December 1999. Assuming that the estimated price spreads are perfectly foreseeable, we rebalance the portfolio every month while keeping it self-financing, after considering the execution costs of trades. For every momentum-based trading strategy we calculate the time series of monthly returns, net of transaction costs. Three performance measures are reported: (1) The intercept (Alpha) of the conditional Fama and French (1993) regressions (as explained in this paper), (2) The *t*-statistic associated with Alpha, and (3) the slope of the investment frontier of a set consisting four assets: the three Fama and French (1993) portfolios and the momentum portfolio (this is calculated as the maximum attainable Sharpe ratio of a combination of the four assets). Since we use proportionate transaction costs, all performance measures are invariant to the initial investment. The analysis uses monthly returns of all NYSE stocks available on CRSP.

Panel A: 11/1/3 Strategy

	Equal-Weighted		
	Alpha	T-Stat of Alpha	Slope of Investment Frontier
Raw Return	0.0080	8.92	0.44
Return net of Effective	0.0061	6.86	0.38
Return net of Quoted	0.0054	6.08	0.35

	Value-Weighted		
	Alpha	T-Stat of Alpha	Slope of Investment Frontier
Raw Return	0.0057	4.54	0.32
Return net of Effective	0.0045	3.59	0.29
Return net of Quoted	0.0040	3.17	0.28

Panel B: 5/1/6 Strategy

	Equal-Weighted		
	Alpha	T-Stat of Alpha	Slope of Investment Frontier
Raw Return	0.0059	8.07	0.41
Return net of Effective	0.0041	5.60	0.34
Return net of Quoted	0.0035	4.72	0.32

	Value-Weighted		
	Alpha	T-Stat of Alpha	Slope of Investment Frontier
Raw Return	0.0033	3.46	0.29
Return net of Effective	0.0022	2.31	0.26
Return net of Quoted	0.0017	1.82	0.25

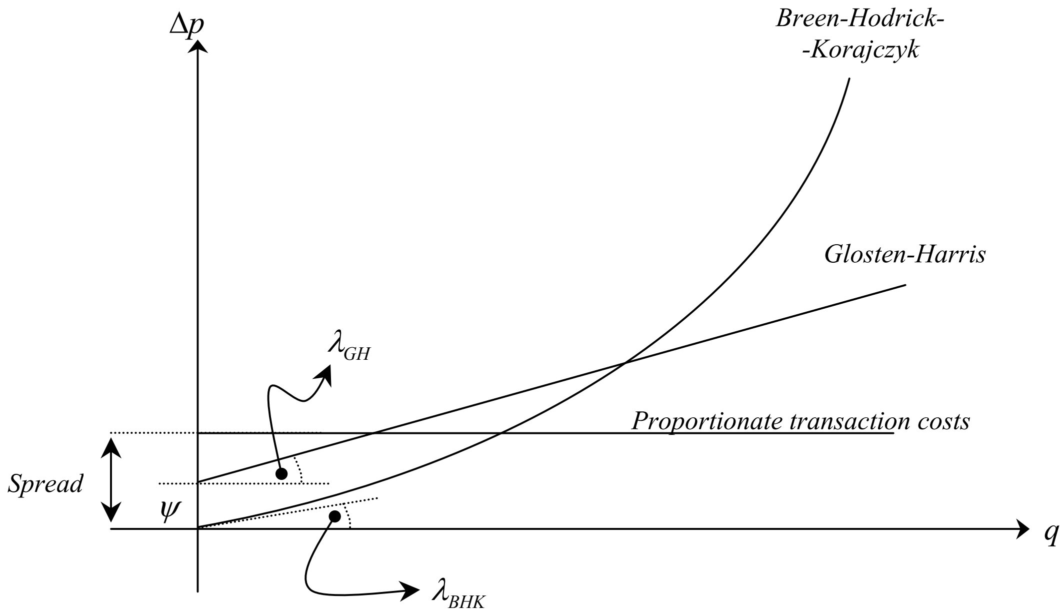


Figure 1. Transaction cost functions. In this paper we consider four different measures of transaction costs: Two non-proportionate costs, the Breen-Hodrick-Korajczyk (2002) measure, and the Glosten-Harris (1988) measure, and two proportionate costs, effective spreads and quoted spreads. The Breen-Hodrick-Korajczyk measure is based on the model $\Delta p_{i,t}/p_{i,t} = \lambda_{BHK,i}\Delta q_{i,t}$, where $\Delta p_{i,t}/p_{i,t}$ is the relative price change of stock *i* as a result of trading a net total of $q_{i,t}$ (signed) shares in a 30-minute interval (*t*). The Glosten-Harris (1988) measure is based on the model $\Delta p_{i,t} = \lambda_{GH,i}\Delta q_{i,t} + \psi_i \Delta d_{i,t}$, where $\Delta p_{i,t}$ is the absolute price change as a result of trading $q_{i,t}$ (signed) shares at time *t* (here *t* represents event time), and $d_{i,t}$ is an indicator for buyer-initiated (+1) or seller-initiated (-1) trade. Effective spreads are measured as the absolute price change relative to mid-point of quoted bid and ask. Quoted spread is measured as the ratio between the quoted bid-ask spread and the mid-point (half the quoted spread is considered as cost). The figure above illustrates these different functions.

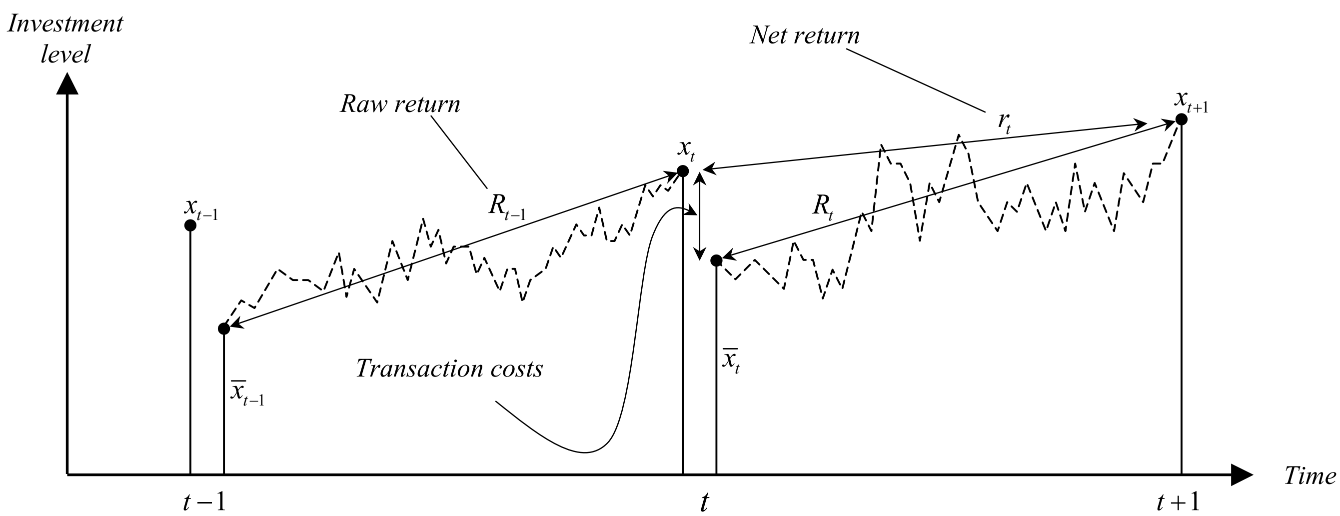
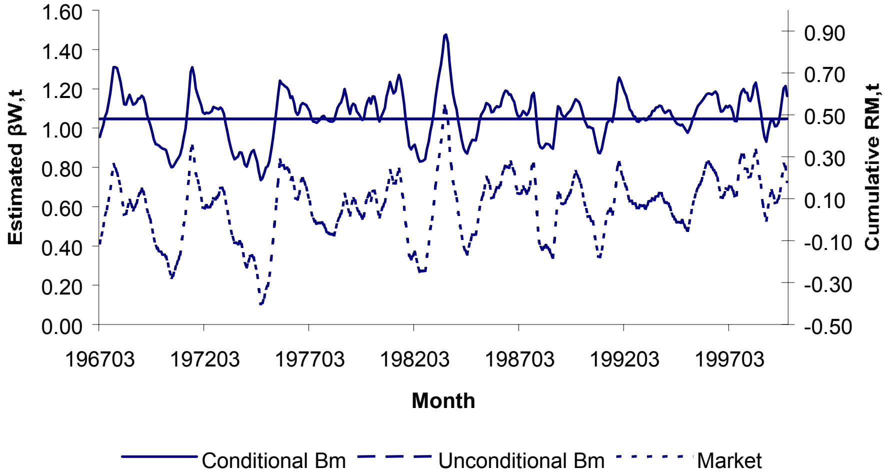
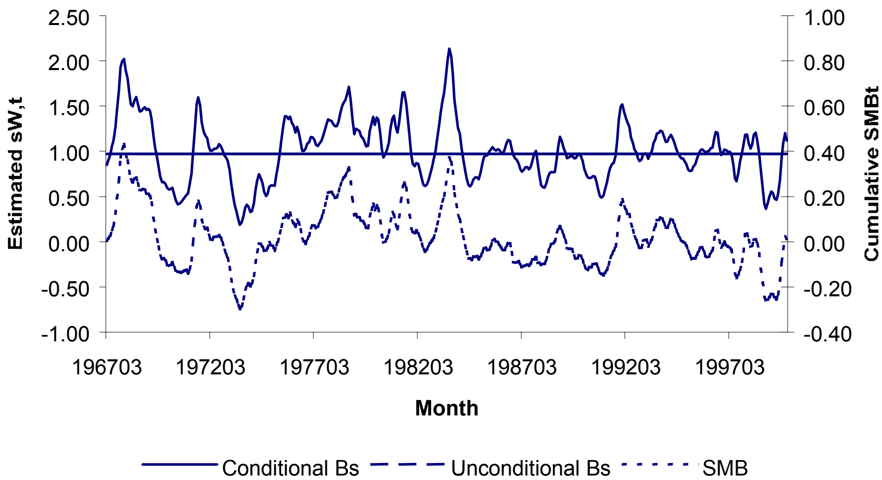


Figure 2. The process of investment level. The figure above illustrates the innovation of level of investment according to the trading model assumed in this paper. At time *t*, just before rebalancing, the total amount invested in the portfolio is x_t. Due to transaction costs induced by rebalancing, the actual amount invested after rebalancing drops to $\overline{x}_t$. Consequently, the expected returns, denoted by R_t, drop to r_t.





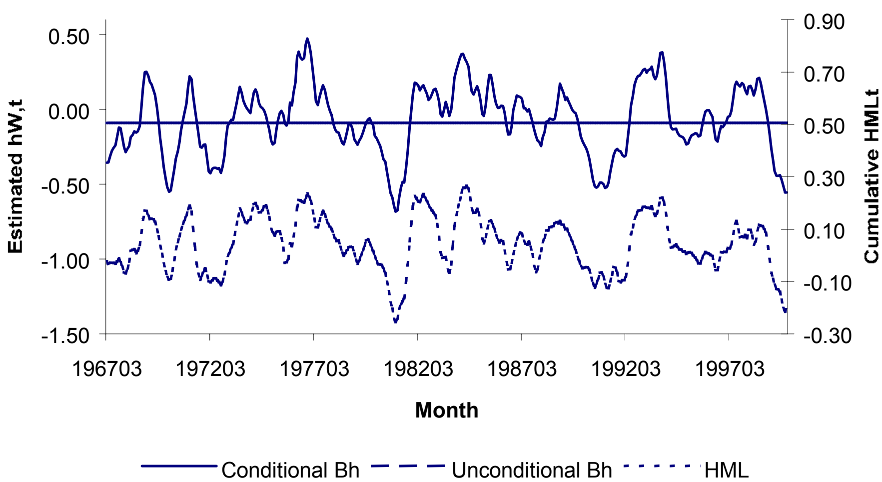


Figure 3. Conditional Factor Loadings of Momentum (11/1/3 equal-weighted). Factor loadings are estimated through the time-series regression

$$\begin{aligned} R_{W,t} - R_{f,t} = \alpha \quad & + a_\beta R_{M,t} && + b_\beta R_{M,t} R_{M,W,t} && + c_\beta R_{M,t} SMB_{W,t} && + d_\beta R_{M,t} HML_{W,t} \\ & + a_s SMB_t && + b_s SMB_t R_{M,W,t} && + c_s SMB_t SMB_{W,t} && + d_s SMB_t HML_{W,t} \\ & + a_h HML_t && + b_h HML_t R_{M,W,t} && + c_h HML_t SMB_{W,t} && + d_h HML_t HML_{W,t} \quad + \quad \varepsilon_t \end{aligned}$$

where $R_{W,t} - R_{f,t}$ is the monthly excess return of the 11/1/3 equally weighted momentum portfolio, $R_{M,t}$, SMB_t, and HML_t are the Fama and French (1993) factors, and $R_{M,W,t}$, $SMB_{W,t}$, and $HML_{W,t}$ are the corresponding cumulative (excess) returns of the factors (see description in the paper). Conditional factor loadings $\beta_{W,t}$, $s_{W,t}$, and $h_{W,t}$ are then calculated through

$$\beta_{W,t} = a_\beta + b_\beta R_{M,W,t} + c_\beta SMB_{W,t} + d_\beta HML_{W,t} \qquad s_{W,t} = a_s + b_s R_{M,W,t} + c_s SMB_{W,t} + d_s HML_{W,t} \qquad h_{W,t} = a_h + b_h R_{M,W,t} + c_h SMB_{W,t} + d_h HML_{W,t}$$

The time-series of the conditional factor loadings as well as the cumulative (excess) returns of the factors, are plotted above, separately for each factor. Unconditional loadings of the momentum strategy are obtained via a standard Fama and French time-series regression, i.e., constraining all the coefficients above, except for α, a_β, a_s, and a_h, at zero, and are also plotted above. The analysis uses monthly returns of all NYSE, AMEX, and NASDAQ stocks available on CRSP for the period March 1967 until December 1999.

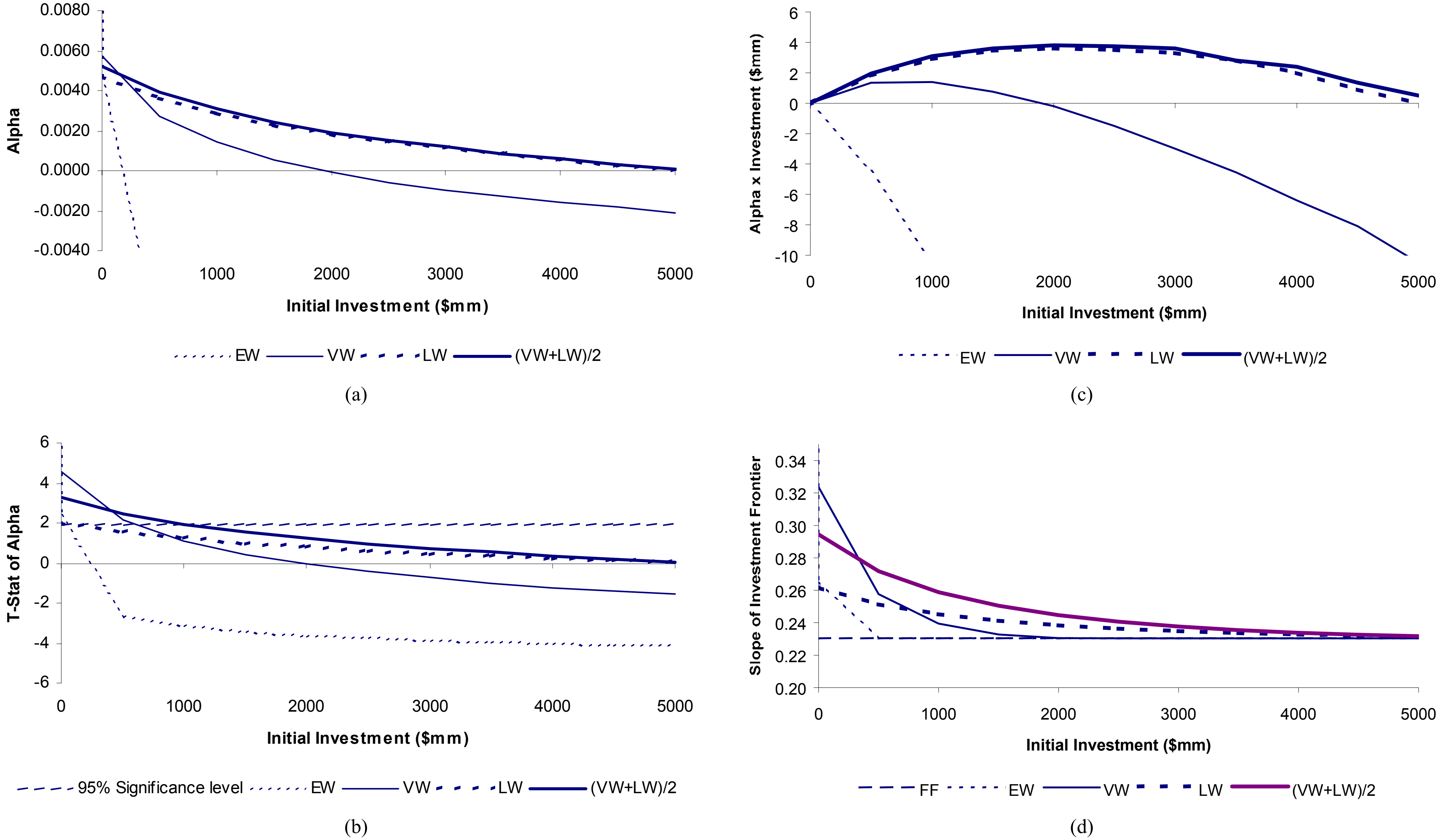


Figure 4. Performance Evaluation of Momentum Strategies (NYSE, Breen-Hodrick-Korajczyk). We evaluate the performance of momentum trading strategies using the trading model developed here. Specifically, we implement the 11/1/3 strategy (see description in Table 1) using various weighting schemes (equal weights (EW), value weights (VW), liquidity weights (LW), and a convex combination (VW+LW)/2). We form a portfolio in the beginning of February 1967 according to a chosen momentum strategy with a certain initial monetary amount of investment. We rebalance the portfolio on a monthly basis, following the trading rule of that strategy, until the end of December 1999. The execution costs of trading any stock *i* are assumed to follow the model $\Delta p_i/p_i = \lambda_i \Delta q_i$, where $\Delta p_i/p_i$ is the relative price improvement as a result of trading Δq_i (signed) shares. The price impact coefficients λ_i are calculated as the fitted values of cross-sectional regressions of measured price impacts on firm characteristics. These regressions used the Trades and Quotes data for the period January 1993 until May 1997. Assuming that the estimated price impacts are perfectly foreseeable, we rebalance the portfolio every month while keeping it self-financing, after considering the price impact of trades. For every momentum-based trading strategy and initial investment we calculate the time series of monthly returns, net of price impacts. Four performance measures are reported: (a) The intercept (Alpha) of the conditional Fama and French (1993) regressions (as explained in this paper), (b) The *t*-statistic associated with Alpha, (c) Alpha multiplied by the amount of investment, and (d) the slope of the investment frontier of a set consisting four assets: the three Fama and French (1993) portfolios and the momentum portfolio (this is calculated as the maximum attainable Sharpe ratio of a combination of the four assets). The initial investment is quoted relative to market capitalization of December 1999. The analysis uses monthly returns of all NYSE stocks available on CRSP.

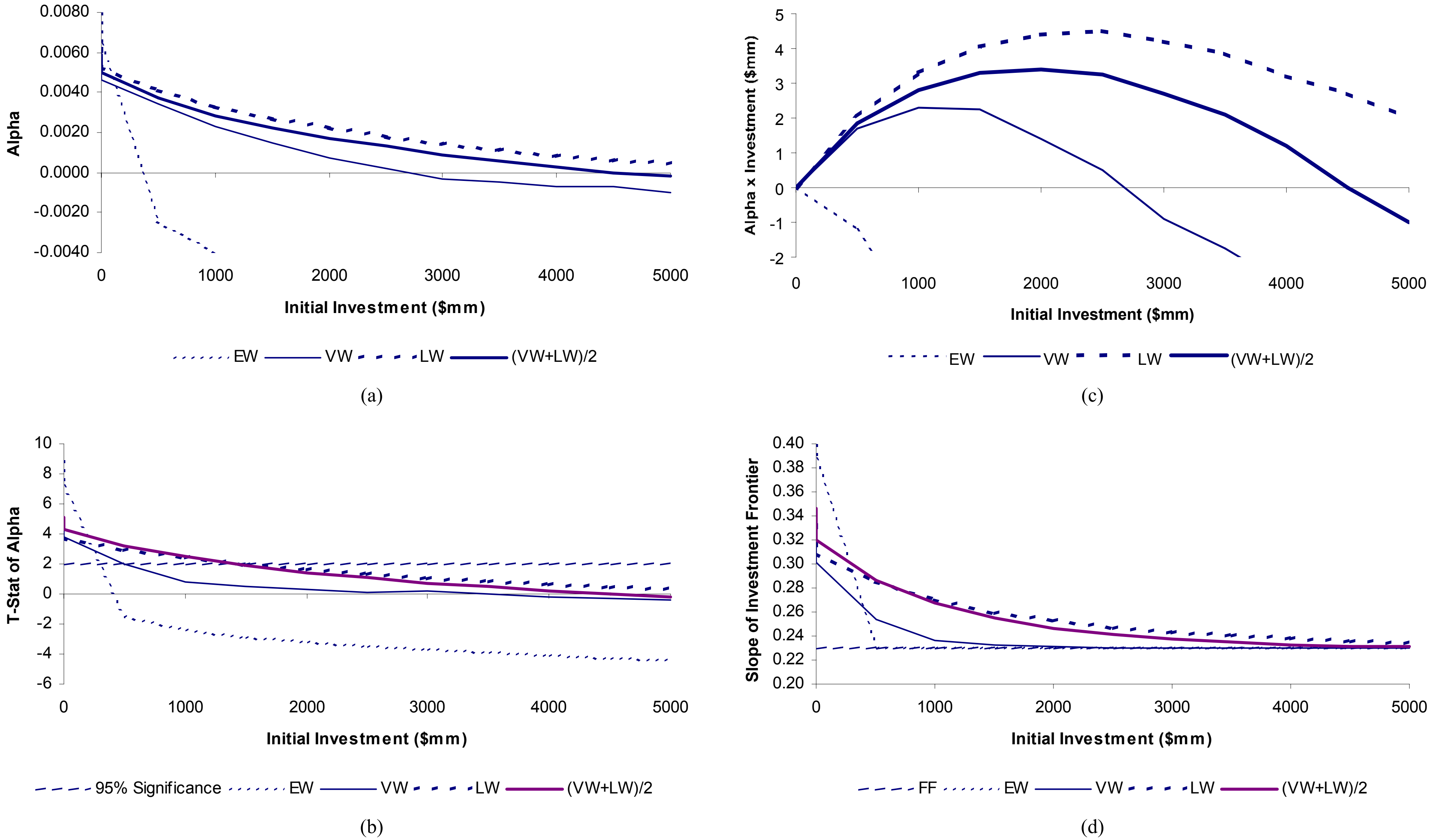
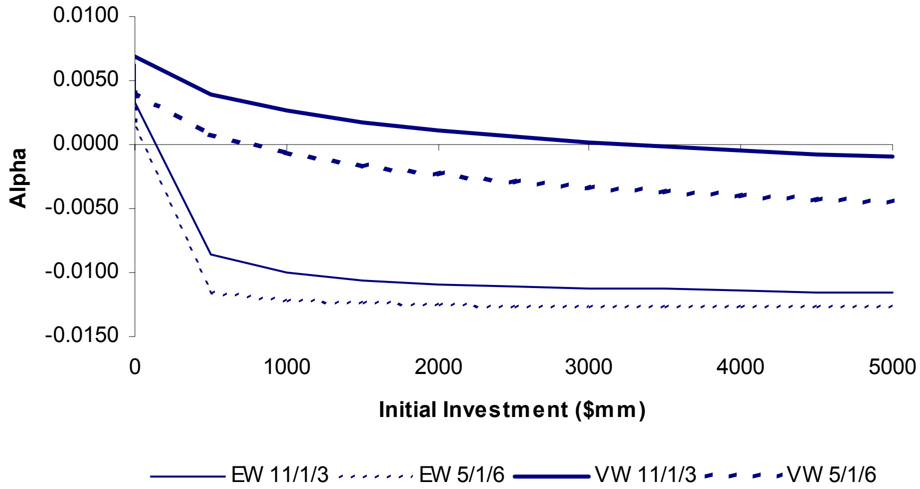
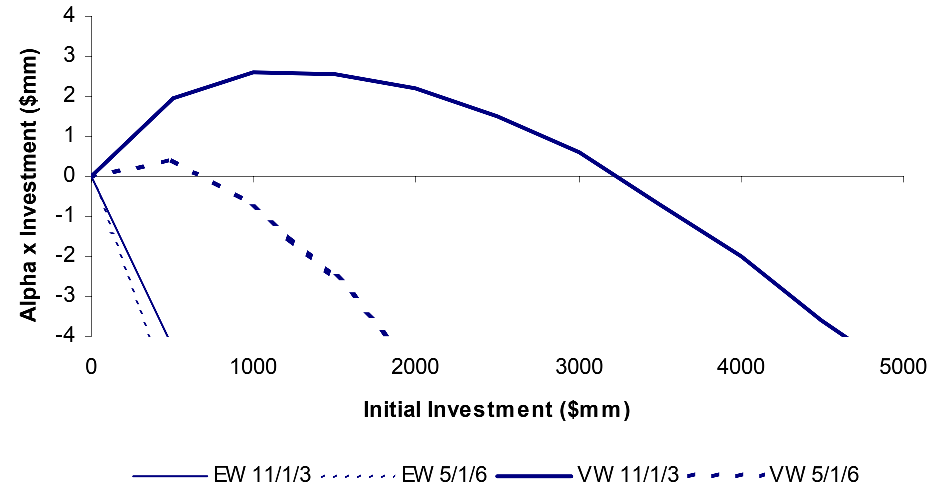


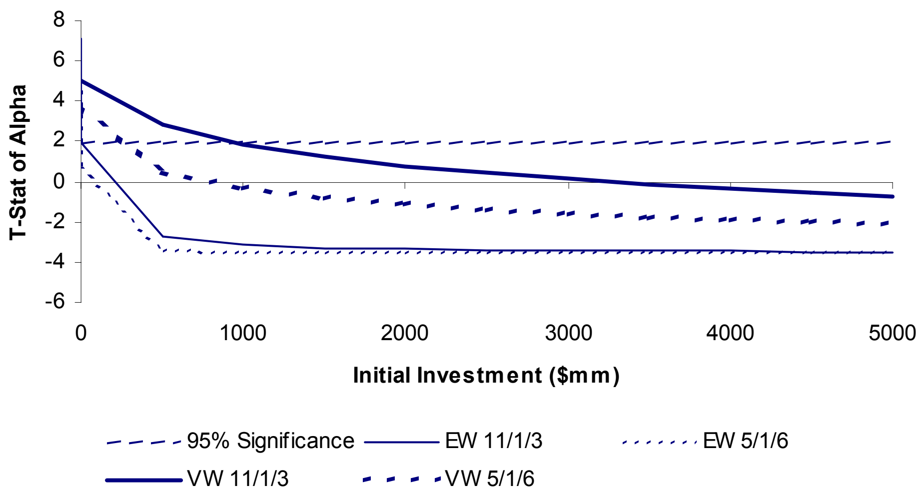
Figure 5. Performance Evaluation of Momentum Strategies (NYSE, Glosten-Harris). We evaluate the performance of momentum trading strategies using the trading model developed here. Specifically, we implement the 11/1/3 strategy (see description in Table 1) using various weighting schemes (equal weights (EW), value weights (VW), liquidity weights (LW), and a convex combination (VW+LW)/2). We form a portfolio in the beginning of February 1967 according to a chosen momentum strategy with a certain initial monetary amount of investment. We rebalance the portfolio on a monthly basis, following the trading rule of that strategy, until the end of December 1999. The execution costs of trading any stock *i* are assumed to follow the model $\Delta p_{i,t} = \lambda_i \Delta q_{i,t} + \psi_i \Delta d_{i,t}$, where $\Delta p_{i,t}$ is the relative price improvement as a result of trading $q_{i,t}$ (signed) shares at time *t* (here *t* represents event time), and $d_{i,t}$ is an indicator for buyer-initiated (+1) or seller-initiated (-1) trade. The price impact coefficients λ_i and ψ_i are calculated as the fitted values of cross-sectional regressions of measured price impacts on firm characteristics. These regressions used the Trades and Quotes data for the period January 1993 until May 1997. Assuming that the estimated price impacts are perfectly foreseeable, we rebalance the portfolio every month while keeping it self-financing, after considering the price impact of trades. For every momentum-based trading strategy and initial investment we calculate the time series of monthly returns, net of price impacts. Four performance measures are reported: (a) The intercept (Alpha) of the conditional Fama and French (1993) regressions (as explained in this paper), (b) The *t*-statistic associated with Alpha, (c) Alpha multiplied by the amount of investment, and (d) the slope of the investment frontier of a set consisting four assets: the three Fama and French (1993) portfolios and the momentum portfolio (this is calculated as the maximum attainable Sharpe ratio of a combination of the four assets). The initial investment is quoted relative to market capitalization of December 1999. The analysis uses monthly returns of all NYSE stocks available on CRSP.



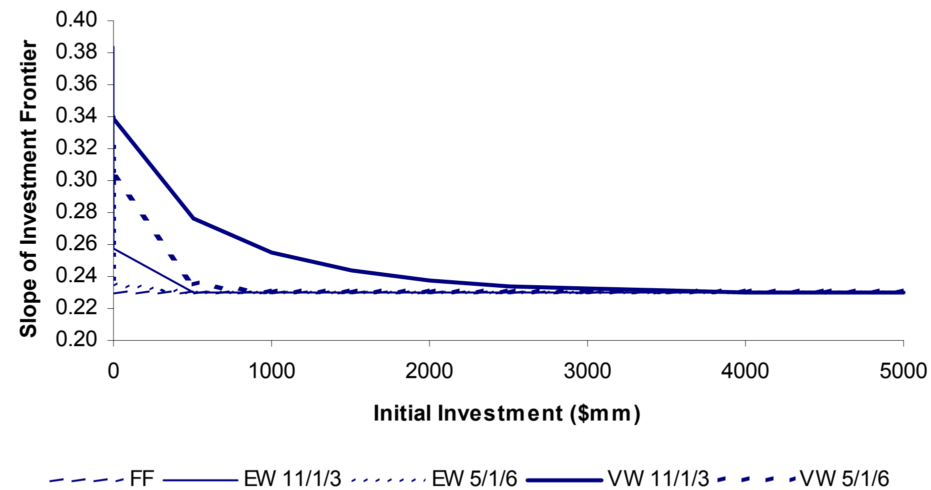
(a)



(c)



(b)



(d)

Figure 6. Performance Evaluation of Momentum Strategies (NYSE, AMEX, and NASDAQ, Breen-Hodrick-Korajczyk). We evaluate the performance of momentum trading strategies using the trading model developed here. Specifically, we implement the 11/1/3 and 5/1/6 strategies (see description in Table 1) using various weighting schemes (equal weights (EW), and value weights (VW)). We form a portfolio in the beginning of February 1967 according to a chosen momentum strategy with a certain initial monetary amount of investment. We rebalance the portfolio on a monthly basis, following the trading rule of that strategy, until the end of December 1999. The execution costs of trading any stock *i* are assumed to follow the model $\Delta p_i/p_i = \lambda_i \Delta q_i$, where $\Delta p_i/p_i$ is the relative price improvement as a result of trading Δq_i (signed) shares. The price impact coefficients λ_i are calculated as the fitted values of cross-sectional regressions of measured price impacts on firm characteristics. These regressions used the Trades and Quotes data for the period January 1993 until May 1997. Assuming that the estimated price impacts are perfectly foreseeable, we rebalance the portfolio every month while keeping it self-financing, after considering the price impact of trades. For every momentum-based trading strategy and initial investment we calculate the time series of monthly returns, net of price impacts. Four performance measures are reported: (a) The intercept (Alpha) of the conditional Fama and French (1993) regressions (as explained in this paper), (b) The *t*-statistic associated with Alpha, (c) Alpha multiplied by the amount of investment, and (d) the slope of the investment frontier of a set consisting four assets: the three Fama and French (1993) portfolios and the momentum portfolio (this is calculated as the maximum attainable Sharpe ratio of a combination of the four assets). The initial investment is quoted relative to market capitalization of December 1999. The analysis uses monthly returns of all NYSE, AMEX, and NASDAQ stocks available on CRSP.

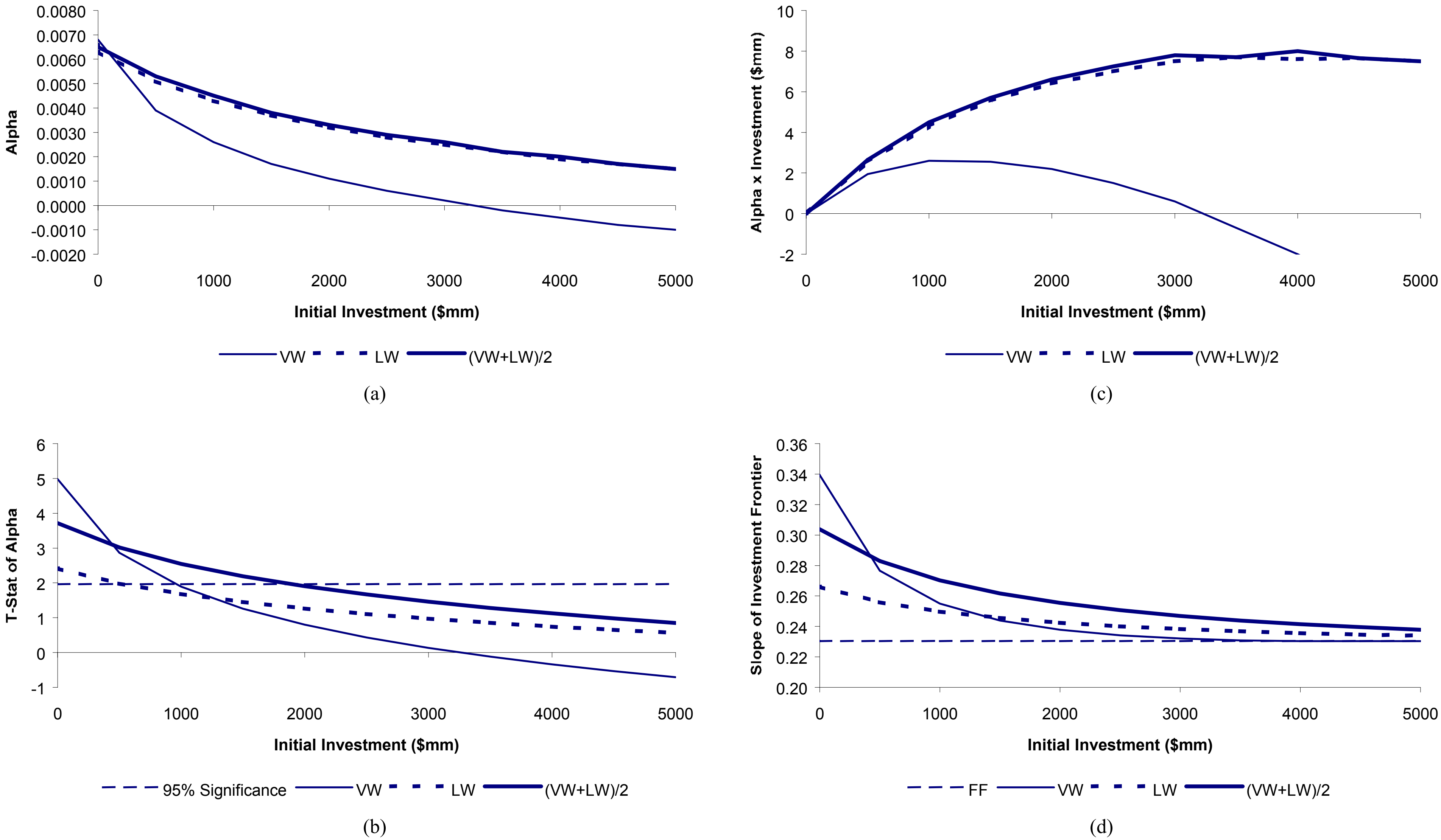


Figure 7. Performance Evaluation of Momentum Strategies (NYSE, AMEX, and NASDAQ, Breen-Hodrick-Korajczyk). We evaluate the performance of momentum trading strategies using the trading model developed here. Specifically, we implement the 11/1/3 strategy (see description in Table 1) using various weighting schemes (value weights (VW), liquidity weights (LW), and a convex combination (VW+LW)/2). We form a portfolio in the beginning of February 1967 according to a chosen momentum strategy with a certain initial monetary amount of investment. We rebalance the portfolio on a monthly basis, following the trading rule of that strategy, until the end of December 1999. The execution costs of trading any stock *i* are assumed to follow the model $\Delta p_i/p_i = \lambda_i \Delta q_i$, where $\Delta p_i/p_i$ is the relative price improvement as a result of trading Δq_i (signed) shares. The price impact coefficients λ_i are calculated as the fitted values of cross-sectional regressions of measured price impacts on firm characteristics. These regressions used the Trades and Quotes data for the period January 1993 until May 1997. Assuming that the estimated price impacts are perfectly foreseeable, we rebalance the portfolio every month while keeping it self-financing, after considering the price impact of trades. For every momentum-based trading strategy and initial investment we calculate the time series of monthly returns, net of price impacts. Four performance measures are reported: (a) The intercept (Alpha) of the conditional Fama and French (1993) regressions (as explained in this paper), (b) The *t*-statistic associated with Alpha, (c) Alpha multiplied by the amount of investment, and (d) the slope of the investment frontier of a set consisting four assets: the three Fama and French (1993) portfolios and the momentum portfolio (this is calculated as the maximum attainable Sharpe ratio of a combination of the four assets). The initial investment is quoted relative to market capitalization of December 1999. The analysis uses monthly returns of all NYSE, AMEX, and NASDAQ stocks available on CRSP.