Lecture 4: Portfolio Optimization

- In this lecture, we will
  - review portfolio mathematics
  - compute optimal portfolio weights
  - discuss problems with mean-variance analysis
  - introduce Bayesian techniques
  - consider numerical optimization to handle investment constraints

Review of Portfolio Mathematics

- Consider **two assets** with simple returns R_1 and R_2, whose means and variances are given by

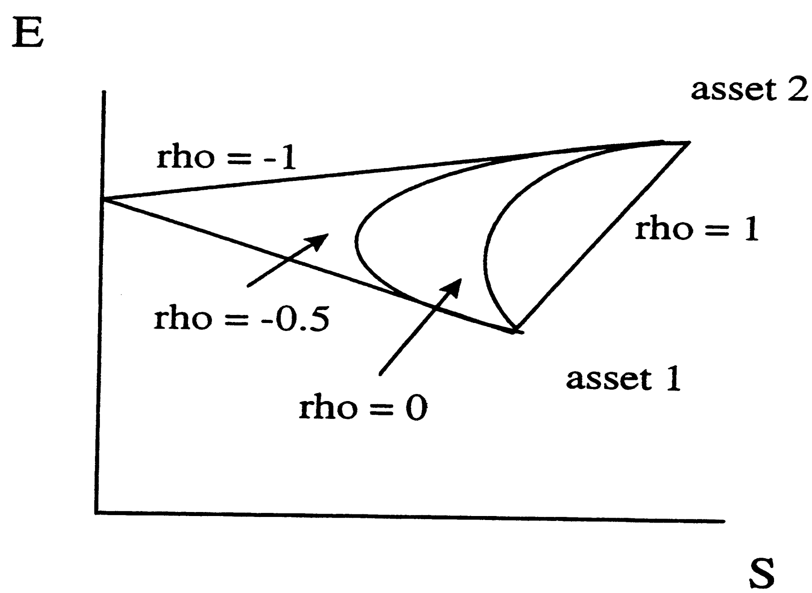
$$\begin{aligned} E_1 &= \mathrm{E}(R_1), \quad \sigma_1^2 = \mathrm{Var}(R_1) \\ E_2 &= \mathrm{E}(R_2), \quad \sigma_2^2 = \mathrm{Var}(R_2), \end{aligned}$$

  whose covariance is σ_{12}, and whose correlation is ρ_{12}

- Portfolio P that puts weight w in asset 1 and $1-w$ in asset 2 has the following expected return and variance:

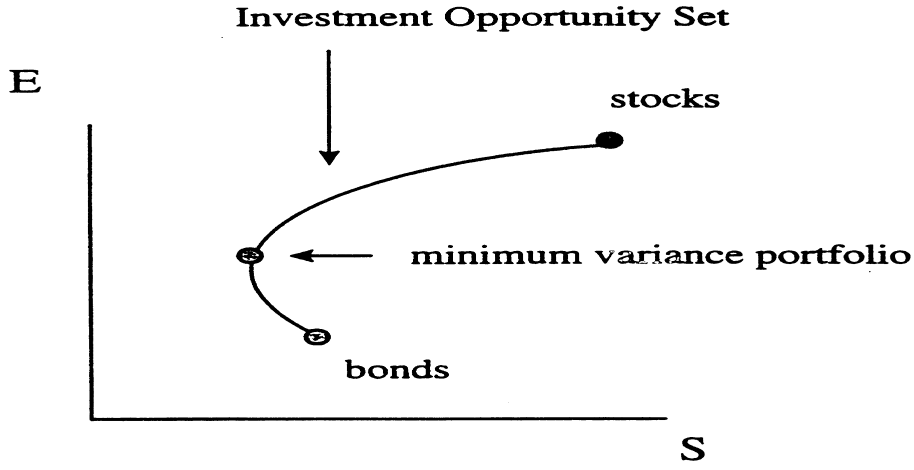
$$\begin{aligned} E_P &= wE_1 + (1-w)E_2 \\ \sigma_P^2 &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \end{aligned}$$

- Relation between E_P and σ_P depends on ρ_{12}:



  – Imperfect correlation allows **diversification**

- The **minimum variance portfolio** (MVP):



– Rearrange the formula for portfolio variance:

$$\begin{aligned}\sigma_P^2 &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \\ &= w^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) + 2w(\sigma_{12} - \sigma_2^2) + \sigma_2^2\end{aligned}$$

– To find w that gives the smallest σ_P^2, set the first derivative of σ_P^2 with respect to w equal to zero:

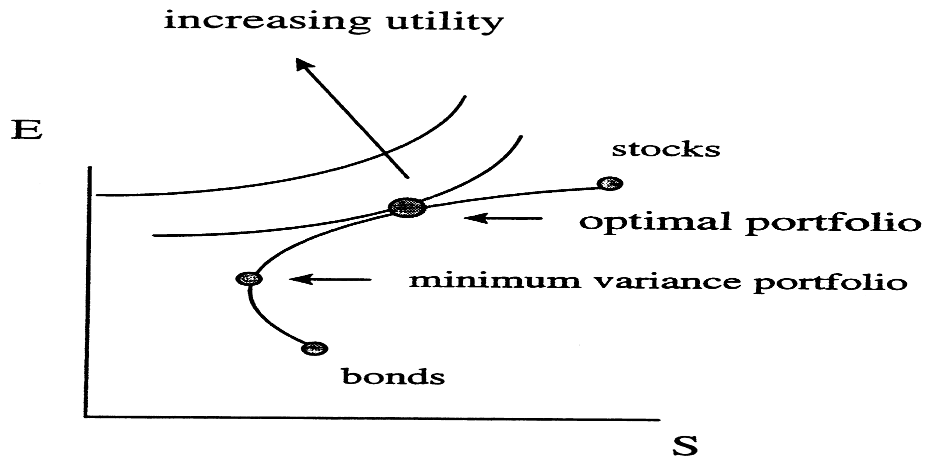
$$2w^*(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) + 2(\sigma_{12} - \sigma_2^2) = 0$$

$$\Rightarrow \boxed{w^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}}$$

- **Example:** For diversified stock and bond portfolios, we have $\sigma_1 = 0.20$, $\sigma_2 = 0.10$, and $\sigma_{12} = 0.004$, so

$$w^* = \frac{(0.10)^2 - 0.004}{(0.10)^2 + (0.20)^2 - 2(0.004)} = 14.3\% \text{ in stocks}$$

- The **optimal portfolio** (OP):



- Suppose investor has a mean-variance utility function:

$$U = E_P - \frac{\gamma}{2}\sigma_P^2,$$

where E_P and σ_P^2 are the mean and variance of the portfolio's returns, and γ is relative risk aversion

- In this two-asset case, we have

$$\begin{aligned} U &= [wE_1 + (1-w)E_2] \\ &\quad -\frac{\gamma}{2}\left[w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}\right] \\ &= w^2\left[-\frac{\gamma}{2}(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})\right] \\ &\quad +w\left[E_1 - E_2 - \gamma(\sigma_{12} - \sigma_2^2)\right] \ + \ \text{const} \end{aligned}$$

- To find w that maximizes utility, set the first derivative of U with respect to w equal to zero:

$$2w^*(-\frac{\gamma}{2}(\sigma_1^2+\sigma_2^2-2\sigma_{12}))+E_1-E_2-\gamma(\sigma_{12}-\sigma_2^2) = 0$$

- Solving for w^*,

$$\Rightarrow \boxed{w^* = \frac{(E_1 - E_2) - \gamma(\sigma_{12} - \sigma_2^2)}{\gamma(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})}}$$

  – What happens when $\gamma \to \infty$?

- **Example:** Consider stock and bond portfolios with $E_1 = 0.12$, $E_2 = 0.06$, $\sigma_1 = 0.20$, $\sigma_2 = 0.10$, and $\sigma_{12} = 0.004$. For an investor with risk aversion $\gamma = 4$,

$$\begin{aligned} w^* &= \frac{(0.12 - 0.06) - 4(0.004 - (0.10)^2)}{4(0.20^2 + 0.10^2 - 2(0.004))} \\ &= 0.5 = 50\% \ \text{ in stocks (asset 1).} \end{aligned}$$

Special case: Asset 2 is **risk-free**

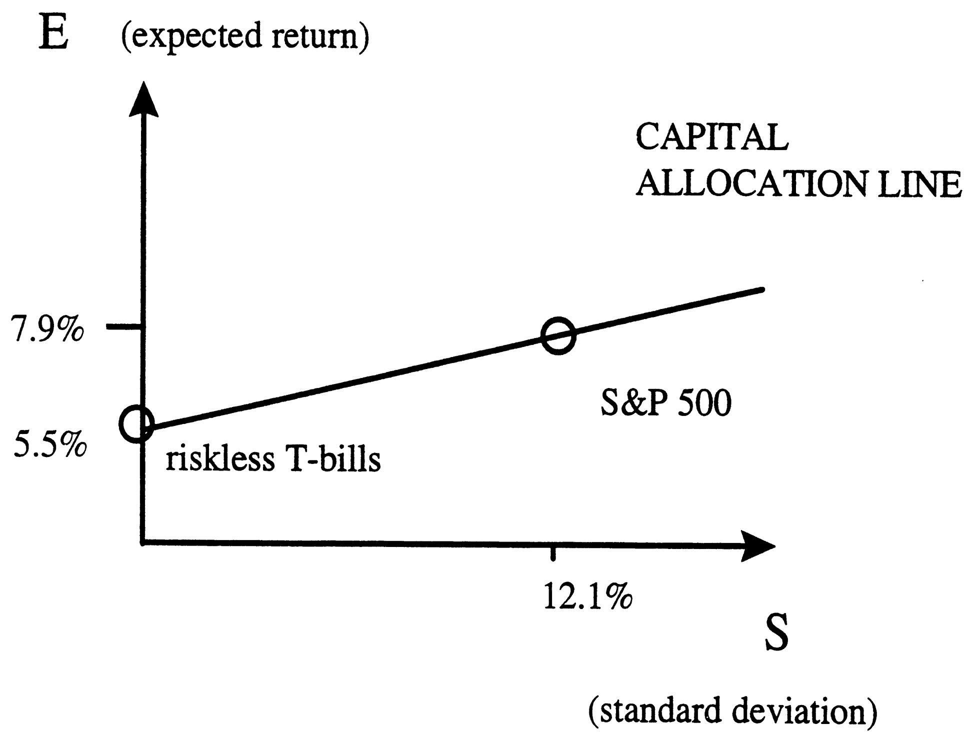
- With $E_2 = r_f$, $\sigma_2 = 0$, and $\sigma_{12} = 0$, the above formula simplifies into

$$\Rightarrow \boxed{w^* = \frac{E_1 - r_f}{\gamma\sigma_1^2}}$$

- **Example:** Suppose your risk aversion is $\gamma = 4$, and you are combining a T-bill paying 5.5% with the S&P 500 index whose $E_1 = 7.9\%$ and $\sigma_1 = 12.1\%$:

$$w^* = \frac{0.079 - 0.055}{4(0.121)^2} = 0.41 = 41\% \ \text{ in the S\&P.}$$

- Recall that all relevant portfolios that involve the S&P index and the T-bill lie on a straight line:



- The slope of this line is the **Sharpe ratio** of the S&P
- All combinations of the S&P index and the T-bill have the same Sharpe ratio:

$$
\begin{aligned}
E_P &= wE_1 + (1-w)r_f = r_f + w(E_1 - r_f) \\
\sigma_P^2 &= w^2\sigma_1^2,
\end{aligned}
$$

so the Sharpe ratio of portfolio P is

$$
S_P = \frac{E_P - r_f}{\sigma_P} = \frac{w(E_1 - r_f)}{w\sigma_1} = \frac{E_1 - r_f}{\sigma_1},
$$

which is the Sharpe ratio of the S&P index (asset 1).

General Case: Many Assets

- Consider **N assets** with simple returns $R_1, \ldots, R_N$
- Denote the expected return on asset i by E_i, for $i = 1, \ldots, N$, and arrange expected returns in a vector E:

$$E = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{pmatrix}$$

  This is an $N \times 1$ vector.

- Let V denote the covariance matrix of asset returns:

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \ldots & \sigma_N^2 \end{pmatrix}$$

  This is an $N \times N$ symmetric matrix.

- Consider portfolio P that puts weight w_1 in asset 1, w_2 in asset 2, all the way to weight w_N in asset N
  - Arrange these weights in an $N \times 1$ vector w:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

- Portfolio P's expected return is

$$E_P = w_1 E_1 + w_2 E_2 + \ldots + w_N E_N = w' E,$$

where w' is a $1 \times N$ transpose of w:

$$w' = (w_1 \ w_2 \ \ldots \ w_N)$$

- Portfolio P's variance is

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = w' V w$$

- Note: Matrix notation simplifies things a lot!

MATLAB is particularly useful for manipulating vectors and matrices. Try this:

$>> x = load('returns\_annual.txt');$

$>> RSB = x(:, 2:3);$

$>> E = mean(RSB)'$

$E = 0.1303$

$\quad 0.0602$

$>> V = cov(RSB)$

$V = 0.0303 \quad -0.0002$

$\quad -0.0002 \quad 0.0122$

To obtain the expected return and covariance matrix for a 70-30 split between stocks and bonds, all you need to type is

$>> w = [0.7; 0.3];$

$>> EP = w' * E$

$EP = 0.1093$

$>> VP = w' * V * w$

$VP = 0.0159$

- The **minimum variance portfolio**
  - Portfolio weights w that minimize $\sigma_P^2 = w'Vw$
  - The solution turns out to be given by

$$w_{MVP} = \frac{V^{-1}i}{i'V^{-1}i},$$

where V^{-1} is the inverse of V and i denotes an $N \times 1$ column of ones:

$$i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad i' = (1\ 1\ \ldots 1)$$

  - Note that you really only need to compute $V^{-1}i$ and then scale the weights to sum to one

MATLAB computes the MVP weights in one line:

$>> wMVP = (inv(V)*ones(2,1))/(ones(1,2)*inv(V)*ones(2,1))$

$wMVP = 0.2894$

$\quad 0.7106$

That is, the MVP puts 29% in stocks and 71% in bonds.

Note how easy it is to invert a matrix in MATLAB:

$>> V = cov(RSB)$

$V = \begin{matrix} 0.0303 & -0.0002 \\ -0.0002 & 0.0122 \end{matrix}$

$>> Vi = inv(V)$

$Vi = \begin{matrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{matrix}$

– **Example:** Consider stock and bond portfolios with standard deviations $\sigma_1 = 0.1741$ and $\sigma_2 = 0.1105$, and with a covariance $\sigma_{1,2} = -0.0002$ (from data). The covariance matrix looks like this:

$$V = \begin{pmatrix} 0.0303 & -0.0002 \\ -0.0002 & 0.0122 \end{pmatrix}$$

The inverse of V:

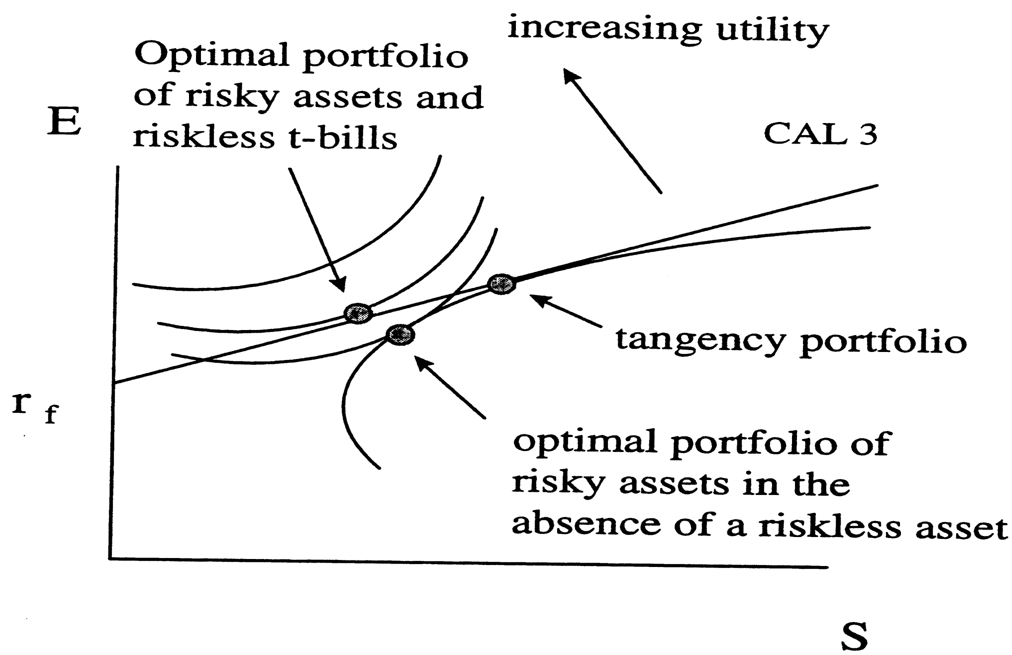
$$V^{-1} = \begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}$$

The MVP weights are therefore

$$\begin{aligned} w_{MVP} &= \frac{V^{-1}i}{i'V^{-1}i} = \frac{\begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{(1\ 1)\begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\ &= \frac{\begin{pmatrix} 33.6221 \\ 82.5682 \end{pmatrix}}{116.1903} \\ &= \begin{pmatrix} 0.2894 \\ 0.7106 \end{pmatrix} \end{aligned}$$

Hence, MVP puts 29% in stocks and 71% in bonds. Check that the same result is obtained using the formula derived in the case of two risky assets.

- **Optimal portfolio (with a risk-free asset)**
  - Find the portfolio of N risky assets and a risk-free asset that maximizes utility
  - Recall that this is a combination of the **tangency portfolio** and the risk-free asset:



  - Redefine E as vector of expected *excess* returns:

$$E = \begin{pmatrix} E(R_1) - r_f \\ E(R_2) - r_f \\ \vdots \\ E(R_N) - r_f \end{pmatrix}$$

  - The weights in the tangency portfolio are

$$w_{TP} = \frac{V^{-1}E}{i'V^{-1}E}$$

– **Example:** Consider the same stock and bond portfolios, for which the inverse of V is given by

$$V^{-1} = \begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}.$$

Sample estimates of expected excess returns:

$$E = \begin{pmatrix} 0.1303 - 0.03 \\ 0.0602 - 0.03 \end{pmatrix} = \begin{pmatrix} 0.1003 \\ 0.0302 \end{pmatrix}$$

The tangency portfolio weights are

$$\begin{aligned} w_{TP} &= \frac{V^{-1}E}{i'V^{-1}E} = \frac{\begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}\begin{pmatrix} 0.1003 \\ 0.0302 \end{pmatrix}}{(1\ 1)\begin{pmatrix} 32.9928 & 0.6293 \\ 0.6293 & 81.9388 \end{pmatrix}\begin{pmatrix} 0.1003 \\ 0.0302 \end{pmatrix}} \\ &= \frac{\begin{pmatrix} 3.3279 \\ 2.5396 \end{pmatrix}}{5.8675} \\ &= \begin{pmatrix} 0.5672 \\ 0.4328 \end{pmatrix} \end{aligned}$$

Hence, TP puts 57% in stocks and 43% in bonds.

After redefining E, MATLAB computes the TP weights in one line:

$>> E = mean(RSB)' - 0.03;$

$>> wTP = (inv(V) * E)/(ones(1, 2) * inv(V) * E)$

$wTP = 0.5672$

$\quad\quad 0.4328$

- Conceptual issues regarding the tangency portfolio
  - Rationality vs. irrationality
    * Suppose that Wall Street investors are fools who make lots of mistakes. Is the TP useful?
    * Yes! TP is a useful concept regardless of whether markets are rational or irrational
    * Choosing TP is a reflection of your rationality, not the rationality of others
    * In the special case where every investor constructs the TP using the same E and V, the TP is the market portfolio – this is the CAPM!
  - True TP vs. ex-post TP vs. ex-ante TP
    * True TP: Based on true values of E and V
    * Ex-post TP: Based on historical $\hat{E}$ and $\hat{V}$
    * Ex-ante TP: Based on our best ex-ante estimates of E and V
  - How do we test whether a given portfolio P is the true TP?
    * Regress excess asset returns on the excess returns of portfolio P:

$$R_{it} - R_{ft} = \alpha + \beta(R_{Pt} - R_{ft}) + \epsilon_{it}$$

    * If P really is the true TP, then $\alpha = 0 \Rightarrow$ test if α is significantly different from zero for all assets

- How do we go from the tangency portfolio to the optimal portfolio, which maximizes utility

$$U = E_P - \frac{\gamma}{2}\sigma_P^2 \text{ ?}$$

- Compute the tangency portfolio's mean and variance of returns, and solve the optimal allocation between one risky asset (TP) and one risk-free asset
- Result: The **optimal portfolio** weights in the N risky assets are given by

$$w_{OP} = \frac{1}{\gamma} V^{-1} E$$

  - These weights do not generally sum to one; the weight in the T-bill is one minus the sum of w_{OP}
  - Note: $\gamma \uparrow \Rightarrow$ risky weights $\downarrow$, T-bill weight $\uparrow$

In MATLAB, the optimal portfolio weights are computed simply as

$>> wOP = inv(V) * E/6$

$wOP = 0.5546$

$\quad\quad 0.4233$

With risk aversion of 6, you invest 55.5% in stocks, 42.3% in bonds, and 2.2% in T-bills.

$>> wOP = inv(V) * E/4$

$wOP = 0.8320$

$\quad\quad 0.6349$

With risk aversion of 4, you invest 83.2% in stocks, 63.5% in bonds, and -46.7% in T-bills (!). Note that the proportion of stocks to bonds is unchanged, regardless of risk aversion.

Choosing Inputs for Mean-Variance Analysis.

- Optimal portfolio weights are proportional to $V^{-1}E$

  $\Rightarrow$ Two inputs, E and V, need to be specified

- Most obvious estimates: **sample estimates**, $\hat{E}$ and $\hat{V}$, which we have been using so far
  - Advantage: unbiased estimators of E and V
  - Disadvantages:
    * We often (think we) have more information about E and V than just past sample estimates
    * Method works poorly when N is large or when T (length of the sample period) is small

- **Example:** Simulation (*tangency1.m*).
  - Simulate T returns on N assets from a normal distribution with known E and V
  - Estimate $\hat{E}$ and $\hat{V}$ using simulated returns
  - Construct TP weights based on $\hat{E}$ and $\hat{V}$; compare them to the "true" TP weights (based on E, V)
  - What do you find?

  - Experiment with various N, T, ρ

- **Example:** The Harvard Management Company case

- **Problem:**
  - There is large estimation error in $\hat{E}$ (and $\hat{V}$)
  - This error is magnified when $\hat{E}$ is multiplied by the inverse of $\hat{V}$, which is often close to non-invertible
  - Thus, portfolios based on $\hat{E}$ and $\hat{V}$ often contain extreme weights and perform poorly out-of-sample
  - Judge this for yourself in Assignment 4!
- One solution to the problem: Bayesian techniques
- Case study: How does the HMC estimate E and V?

Bayesian Approaches to Portfolio Choice

- Bayesian approach: Combine data with prior beliefs

$$
\begin{array}{ccc}
 & & \text{Prior beliefs} \\
 & & \Downarrow \\
\text{Data} & \Longrightarrow & \text{Revised (posterior) beliefs} \\
 & & \Downarrow \\
 & & \text{Investment Decisions}
\end{array}
$$

- Where do prior beliefs come from?
  - Judgment (e.g., beliefs about relative mispricing)
  - Another dataset (e.g., foreign markets)
  - Economic theory (e.g., CAPM)

- Bayesian approaches allow us to *combine information from various sources* in the most efficient way
- Bayesian approaches allow us to *include our own judgment* efficiently
- How do we compute posterior (updated) beliefs?
  - Consider unknown parameter p (e.g., E or V)
  - Suppose prior beliefs about p are centered at p_0, and the sample estimate of p from the data is $\hat{p}$
  - Our updated estimate of p (the "posterior mean") is a weighted average of p_0 and $\hat{p}$, where the weight on $\hat{p}$ is large (and the weight on p_0 is small) if
    - The data are very informative about p (i.e., $\hat{p}$ has a low standard error)
    - Our prior beliefs about p are not strong (i.e., large prior uncertainty around p_0)
- Prior beliefs about **expected returns** E
  - Typical prior distribution: $E \sim N(E_0, V_0)$
    - V_0 reflects the strength of our prior beliefs (strong beliefs $\Rightarrow$ small V_0, and vice versa)
    - E_0 can come from judgment, other data, or from a model such as the CAPM

- **Example:** The **Black-Litterman approach** (Goldman Sachs)
  - This approach combines judgment about the securities' expected future returns with the equilibrium/CAPM estimates of expected returns
  - Deviate from the value-weighted market portfolio in the direction that we subjectively judge to be appropriate

- **Example:** The **Bayesian alphas and betas** (Merrill Lynch, others)

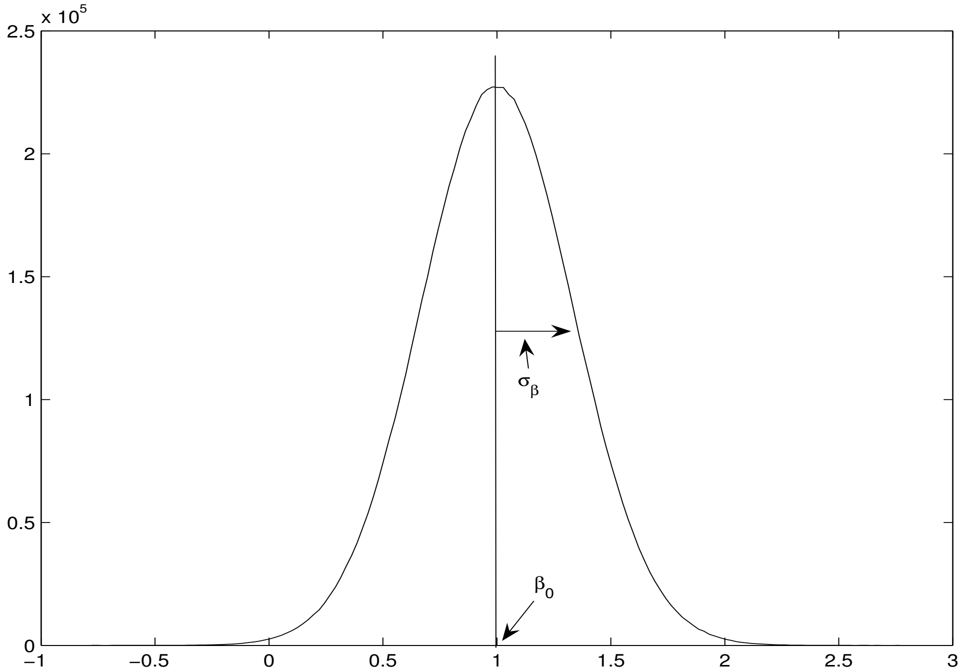
$$R_{Pt} - R_{ft} = \alpha + \beta(R_{Mt} - R_{ft}) + \epsilon_t,$$

  - Historical OLS regression estimates of alpha and beta, $\hat{\alpha}$ and $\hat{\beta}$, are often imprecise
  - Bayesian approaches help us increase this precision by including additional (prior) information

Bayesian Beta

- Suppose your beliefs about beta are given by

$$\beta \sim N(\beta_0, \sigma_\beta^2)$$



  - What are some sensible values for β_0?
  - What are some sensible values for σ_β?

- Bayesian beta is a weighted average of β_0 and $\hat{\beta}$:

$$\beta^* = w\hat{\beta} + (1 - w)\beta_0$$

  - What determines w?

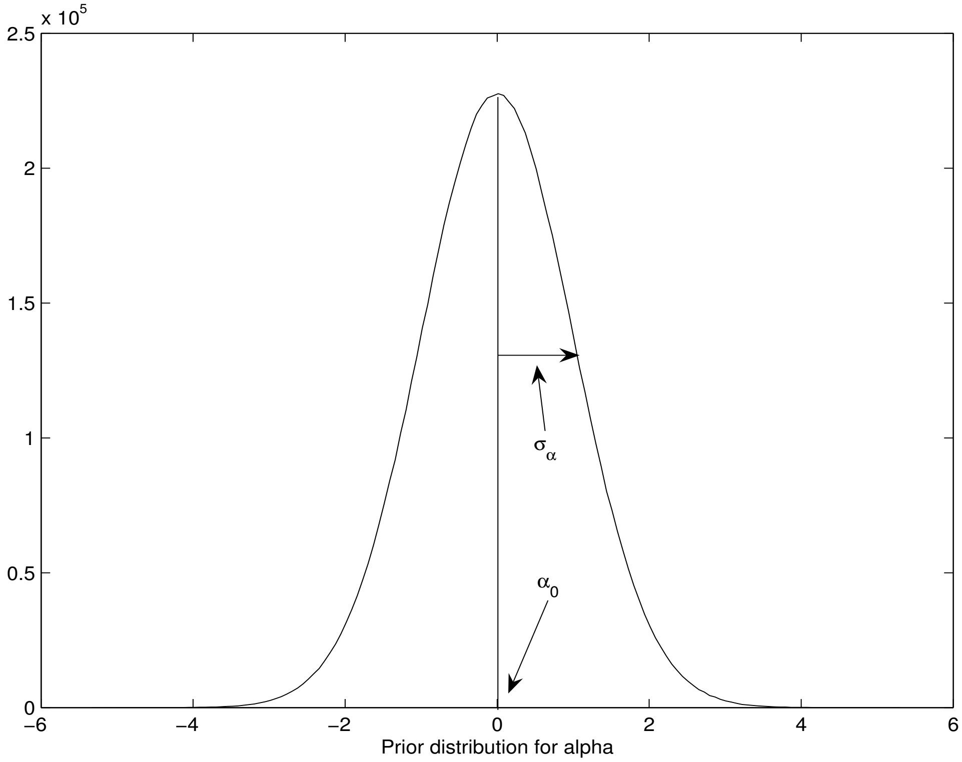
- The "Merrill Lynch adjusted beta":

$$\beta^{ML} = (2/3)\hat{\beta} + (1/3)1$$

Bayesian Alpha

- Suppose your beliefs about alpha are given by

$$\alpha \sim N(\alpha_0, \sigma_\alpha^2)$$



  – Most natural value: $\alpha_0 = 0$ (CAPM)

    perfect confidence about the CAPM $\implies \sigma_\alpha = 0$
    perfect skepticism about the CAPM $\implies \sigma_\alpha = \infty$

  – If you think the asset is *overpriced*, use $\alpha_0 < 0$
  – If you think the asset is *underpriced*, use $\alpha_0 > 0$

- Bayesian alpha is a weighted average of α_0 and $\hat{\alpha}$:

$$\alpha^* = w\hat{\alpha} + (1 - w)\alpha_0$$

  – $w \uparrow$ when $\sigma_\alpha \uparrow$, $T \uparrow$, or $\sigma_\epsilon \downarrow$

- If $\alpha_0 = 0$, σ_α captures the degree of belief in CAPM

  Complete confidence in the CAPM $\implies \sigma_\alpha = 0$

  $\implies w = 0$, so $\alpha^* = 0$

  $\implies$ Invest 100% in the market portfolio

  Complete skepticism about the CAPM $\implies \sigma_\alpha = \infty$

  $\implies w = 1$, so $\alpha^* = \hat{\alpha}$

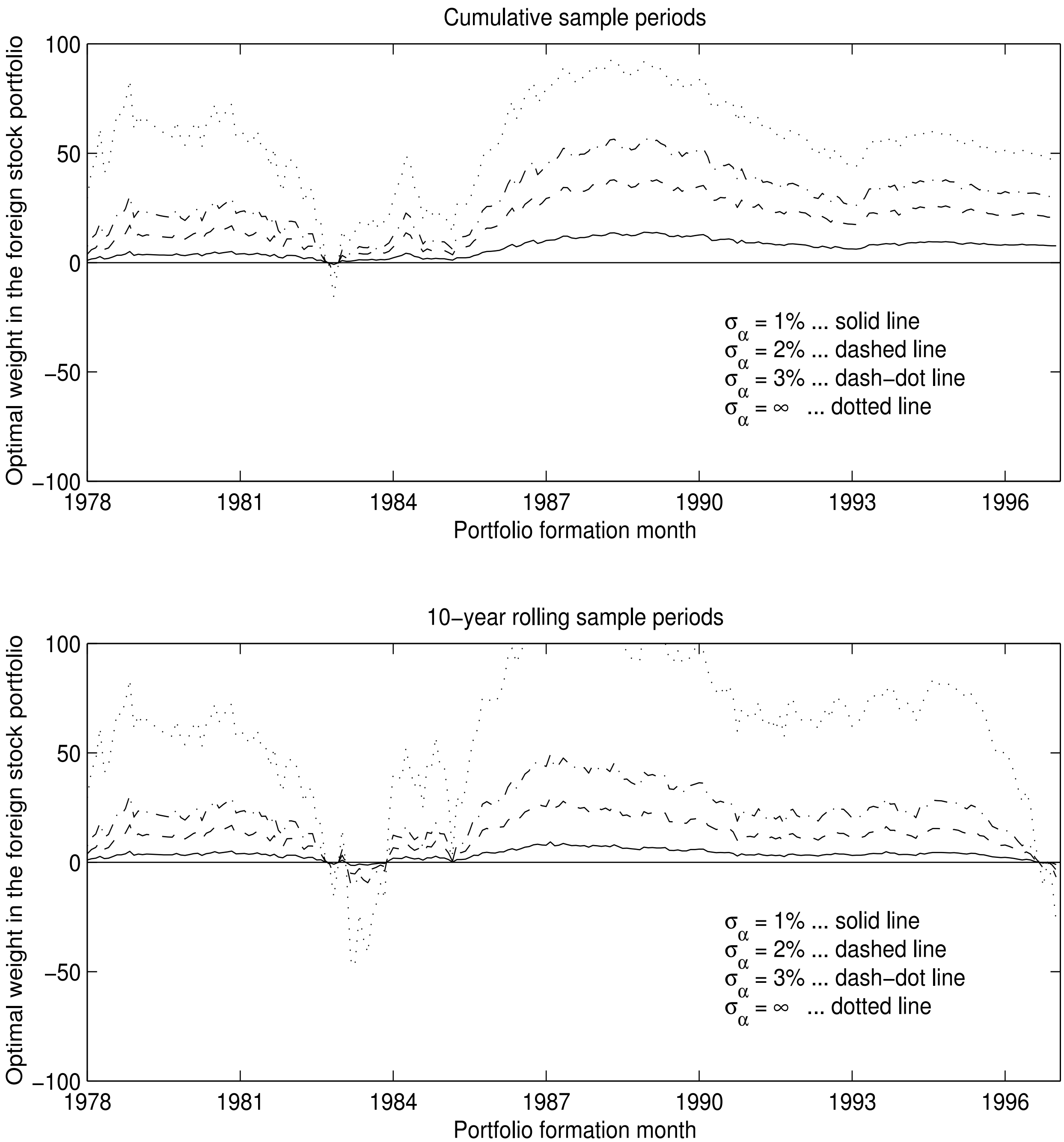
  $\implies$ Invest based on historical estimates $\hat{E}, \hat{V}$

  Sensible approach: Use a small but positive σ_α (e.g., $\sigma_\alpha = 1\%$ per year)

  $\implies$ Stable portfolio weights, deviate from the market a bit in the direction of apparent mispricing

- What do the portfolio weights of the Bayesian strategies with beliefs about alpha look like? (next page)

• Optimal Weight in Foreign Stocks in a Two-Asset Portfolio with the U.S. Market:



Source: Pastor (Journal of Finance, 2000)

- How do these strategies perform out of sample?
- Sharpe ratios for two-asset portfolio strategies:

Asset Combined with the U.S. Market	Prior Standard Deviation of α (σ_α) 0	1%	2%	3%	5%	∞
WXUS (Foreign Stocks) (Jan 1978 – Dec 1996)	0.1665	0.1693	0.1743	**0.1766**	0.1727	0.1538
HML (Value-Growth) (Jul 1937 – Dec 1996)	0.1476	**0.1537**	0.1444	0.1100	0.0393	0.0231
SMB (Small-Big) (Jul 1937 – Dec 1996)	0.1476	**0.1564**	-0.0264	-0.0190	0.0217	0.0159
DFA 9-10 (Small Stocks) (Jan 1987 – Dec 1996)	**0.1912**	0.1899	0.1851	0.1757	0.1510	0.0733

- In this approach, the posterior mean of E is a weighted average of the CAPM-based expected return ($E_0 = \hat{\beta}\,\underbrace{\mathrm{E}(R_{At} - R_{ft})}_{\hat{\mu}_M}$) and the sample estimate ($\hat{E}$):

$$\tilde{E} \;=\; (1 - w_{\hat{\alpha}}) \; \underbrace{(\hat{\beta}\hat{\mu}_M)}_{\text{CAPM}} \;+\; w_{\hat{\alpha}} \; \underbrace{(\hat{\alpha} + \hat{\beta}\hat{\mu}_M)}_{\text{sample avg } \hat{E}}$$

- This method generally works better than using $\hat{E}$
  - Judge this for yourself in Assignment 4!

- Prior beliefs about **the covariance matrix** V
  - When N is large relative to T, the $N \times N$ sample covariance matrix $\hat{V}$ cannot be inverted
  - Solution 1: Use a factor model
  - Solution 2: "Shrink" $\hat{V}$ to a diagonal matrix D
    * That is, use a weighted average of $\hat{V}$ and D
    * As $N \uparrow$, the weight on D must also $\uparrow$

$$D = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}$$

    * For σ_i^2, use values that are reasonable a priori; e.g. setting all σ_i^2 equal to each other works well (D is then proportional to the identity matrix)
    * The resulting average matrix has off-diagonal elements that are closer to zero, which makes the matrix easier to invert and better behaved
    * Drawback: May understate correlations
  - This method generally works better than using $\hat{V}$
    * Judge this for yourself in Assignment 4!
  - Some models explicitly model time-varying volatility (e.g., GARCH); we won't talk about them here

Investment Restrictions

- So far, we have analyzed the optimal portfolio choice problem with no investment constraints
  - This problem has an analytical solution
- However, in many real-world situations, investors face restrictions, such as short-sales constraints
  - Then, **numerical techniques** must be used
- Optimization problem: Choose w to

$$\begin{aligned}&\text{maximize } U = E_P - \frac{\gamma}{2}\sigma_P^2 = w'E - \frac{\gamma}{2}w'Vw \qquad (1)\\ &\text{subject to } \sum_{i=1}^{N} w_i = 1\\ &+ \text{ additional constraints, such as all } w_i \geq 0\end{aligned}$$

- Sometimes it is not clear what value γ should take on, but there is a desired level of expected return. Then

$$\begin{aligned}&\text{minimize } \sigma_P^2 = w'Vw \qquad (2)\\ &\text{subject to } E_P = w'E = \bar{E}\\ &\qquad\qquad \sum_{i=1}^{N} w_i = 1\\ &+ \text{ additional constraints, such as all } w_i \geq 0\end{aligned}$$

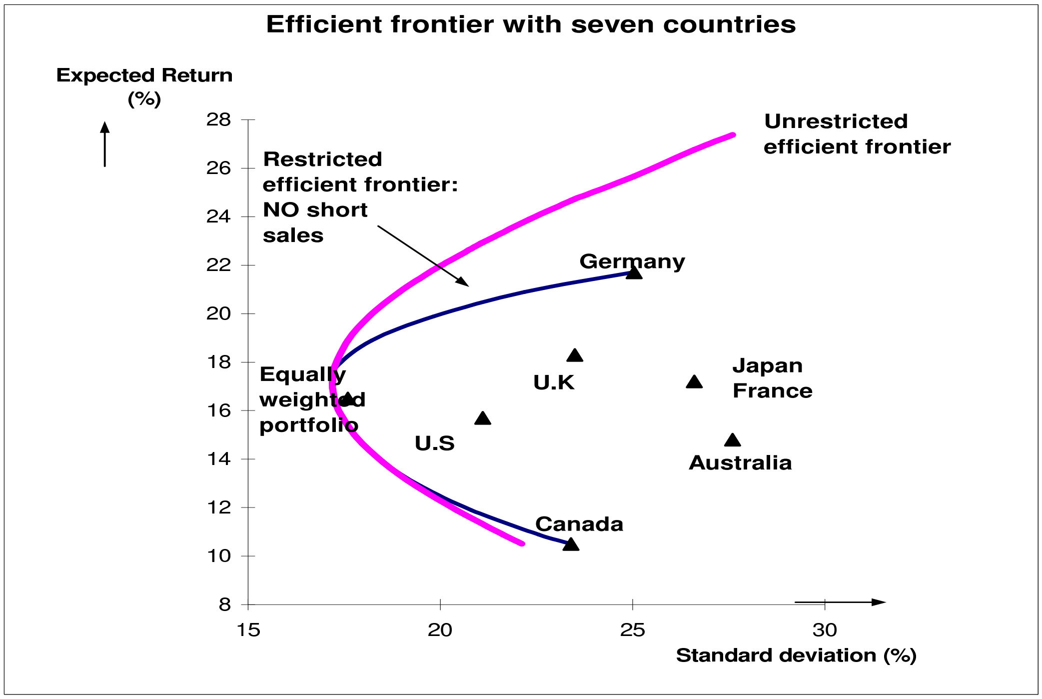
- Both problems can be solved using Matlab's *quadprog*

- We will use *quadprog* here for illustration
  - $X = quadprog(H, f, A, b, C, d, LB, UB)$ finds x that

$$\begin{aligned} \text{minimizes } & \frac{1}{2}x'Hx + f'x \\ \text{subject to } & Ax \leq b \\ & Cx = d \\ & LB \leq x \leq UB \end{aligned}$$

  - To map this into our portfolio problem (1), use $quadprog(\gamma V, -E, [\,], [\,], ones(1, N), 1, [\,], [\,])$
  - To map this into our portfolio problem (2), use $quadprog(V, [\,], [\,], [\,], [E';\ ones(1, N)], [\bar{E};\ 1], [\,], [\,])$
- **Example:** The Harvard Management Company case
  - Is HMC solving problem (1) or (2)?
  - Can we reconstruct Exhibits 5 and 6 in the case?
  - Why is HMC's proposed policy portfolio (Exhibit 8) different from those in Exhibits 5 and 6?

- Investment restrictions shift the efficient frontier in the south-east direction:



- Restrictions tie our hands, so they make us worse off
- Restrictions are sometimes self-imposed as an ad-hoc way of obtaining sensible-looking portfolio weights