The Impact of Skewness and Fat Tails on the Asset Allocation Decision

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The authors modeled the non-normal returns of multiple asset classes by using a multivariate truncated Lévy flight distribution and incorporating non-normal returns into the mean-conditional value at risk (M-CVaR) optimization framework. In a series of controlled optimizations, they found that both skewness and kurtosis affect the M-CVaR optimization and lead to substantially different allocations than do the traditional mean–variance optimizations. They also found that the M-CVaR optimization would have been beneficial during the 2008 financial crisis.

Although numerous alternatives to the mean–variance optimization (MVO) framework have appeared in the literature, no clear leader has emerged. The lack of an agreed-upon alternative to MVO has slowed the development of practitioner-oriented tools. Moreover, the difficulty in estimating the required inputs—returns, standard deviations, and correlations—for MVO is well known, a problem that can be substantially more difficult with more advanced techniques. The future is hard to predict accurately, especially in detail.

Asset class return distributions are not normally distributed, but the typical Markowitz MVO framework that has dominated the asset allocation process for more than 50 years relies on only the first two moments of the return distribution. Equally important, considerable evidence shows that investor preferences go beyond mean and variance to higher moments: skewness and kurtosis. Investors are particularly concerned about significant losses—that is, downside risk, which is a function of skewness and kurtosis.

Recent research suggests that higher moments are important considerations in asset allocation. Patton (2004) showed that knowledge of both skewness and asymmetric dependence (higher correlations in downside markets) leads to economically significant gains, in particular, with no shorting constraints. Harvey, Liechty, Liechty, and Müller (2010) proposed a method for optimal portfolio selection involving a Bayesian decision

theoretical framework that addresses both higher moments and estimation error. They suggested that incorporating higher-order return distribution moments in portfolio selection is important.

The financial crisis of 2008 has led many investors to search for tools that minimize downside risk. In our study, we explored one of the promising alternatives to MVO that incorporates non-normal return distributions: mean-conditional value at risk (M-CVaR) optimization.

Modeling Non-Normal Returns

Empirically, almost all asset classes and portfolios have returns that are not normally distributed. Many assets' return distributions are asymmetrical. In other words, the distribution is skewed to the left (or occasionally the right) of the mean (expected) value. In addition, most asset return distributions are more leptokurtic, or fatter tailed, than are normal distributions. The normal distribution assigns what most people would characterize as meaninglessly small probabilities to extreme events that empirically seem to occur approximately 10 times more often than the normal distribution predicts.

Many statistical models have been put forth to account for fat tails. Well-known examples are the Lévy stable hypothesis (Mandelbrot 1963), the Student's *t*-distribution (Blattberg and Gonedes 1974), and the mixture-of-Gaussian-distributions hypothesis (Clark 1973). The last two models possess finite variance and fat tails, but they are unstable, which implies that their shapes change at different time horizons and that distributions at different time horizons do not obey scaling relations.[1]

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Our preferred method is based on an enhancement to the Lévy stable distribution model (Lévy 1925). In 1963, Mandelbrot modeled cotton prices with a Lévy stable process, an approach that was later supported by Fama (1965). A Lévy stable distribution model can have skewness and fat tails and obeys scaling properties. Unfortunately, the Lévy stable distribution has infinite variance, which violates empirical observations and logic that dictates finite return variance. Infinite variance significantly complicates the task of risk estimation and limits the practical application of the stable distribution.

A simple enhancement that addresses this shortcoming of the Lévy stable distribution is to truncate the extreme tails of the stable distribution, which results in the truncated Lévy flight (TLF) distribution (see Mantegna and Stanley 2000). The TLF distribution is particularly well suited to financial modeling because it has four finite moments that empirically fit the data exceptionally well; perhaps most important for financial modelers seeking an elegant modeling solution, it "scales" appropriately over time. More specifically, it *approximately* obeys Lévy stable scaling relations at small time intervals and slowly converges to a Gaussian distribution at very large time intervals, both of which are consistent with empirical observations.

Xiong (2010) demonstrated that the TLF model provides an excellent fit for a variety of asset classes, including the mean, asymmetries (skewness), and the thickness of the tails (kurtosis), as well as the minimum and maximum returns determined through the truncation process. Because we can parametrically control skewness and kurtosis (in addition to mean and variance) with a multivariate version of the TLF model, it is ideal for not only simulating asset class returns but also studying the impact of incorporating skewness and fat tails into the asset allocation decision through controlled optimizations. Thus, in our controlled optimizations, in which we systematically varied the skewness and kurtosis of the asset classes, we used a multivariate TLF model as the basis for generating asset returns and ultimately estimating a portfolio's conditional value at risk (CVaR).

Conditional Value at Risk

The precursor to CVaR—and a slightly better-known measure of downside risk—is the standard value at risk (VaR), which estimates the loss that is expected to be *exceeded* with a given level of probability over a specified period. CVaR is closely related to VaR and is calculated by taking a probability-weighted average of the possible losses *conditional*

on the loss being equal to or exceeding the specified VaR. Other terms for CVaR include mean shortfall, tail VaR, and expected tail loss. CVaR is a comprehensive measure of the entire part of the tail that is being observed and for many is the preferred measurement of downside risk. In contrast with CVaR, VaR is a statement about only one particular point on the distribution. Studies have shown that CVaR has more attractive properties than VaR (see, e.g., Rockafellar and Uryasev 2000; Pflug 2000).

Artzner, Delbaen, Eber, and Heath (1999) demonstrated that one of the desirable characteristics of a "coherent measure of risk" is subadditivity—that is, the risk of a combination of investments is at most as large as the sum of the individual risks. VaR is not always subadditive, which means that the VaR of a portfolio with two instruments may be greater than the sum of the individual VaRs of those two instruments. In contrast, CVaR is subadditive.

In the most basic case, if one assumes that returns are normally distributed, both VaR and CVaR can be estimated by using only the first two moments of the return distribution (see, e.g., Rockafellar and Uryasev 2000).

$$\text{VaR}_P = \mu_P - 1.65\sigma_P, \tag{1}$$

and

$$\text{CVaR}_P = \mu_P - 2.06\sigma_P, \tag{2}$$

where μ_P and σ_P are the mean and the standard deviation of the portfolio, respectively. For our study, we fixed the probability level for the VaR and the CVaR at 5 percent (corresponding to a confidence level of 95 percent). For example, for a portfolio with $\mu_P = 10$ percent and $\sigma_P = 20$ percent, the VaR and the CVaR of the portfolio are $\text{VaR}_P = 23.0$ percent and $\text{CVaR}_P = -31.2$ percent of the portfolio's starting value. To investors, this -31.2 percent is the average tail loss when a loss exceeds a threshold—in our example, the worst 5th percentile of the return distribution.

Introducing skewness (or asymmetry) and kurtosis into a portfolio's return distribution complicates the calculation of CVaR and brings us to Equation 3:

$$\text{CVaR}_P = \mu_P - f(c, s, k)\sigma_P, \tag{3}$$

where $f(c,s,k)$ is a function of confidence level c, skewness s, and kurtosis k.[2] Unfortunately, the function $f(c,s,k)$ is complicated and generally has no closed-form solution. With Monte Carlo simulations based on the TLF distribution, however, we can model non-normal returns and ultimately estimate Equation 3.

M-CVaR Optimization

Traditional MVO leads to an efficient frontier that maximizes return per unit of variance or, equivalently, minimizes variance for a given level of return. In contrast, M-CVaR maximizes return for a given level of CVaR or, equivalently, minimizes CVaR for a given level of return.

The M-CVaR process that we used in our study takes non-normal return characteristics into consideration and, in general, prefers assets with positive skewness, small kurtosis, and low variance. If the returns of the asset classes are normally distributed or if the method used to estimate the CVaR considers only the first two moments, both MVO and M-CVaR optimization lead to the same efficient frontier and, thus, the same asset allocations. To understand the implications of skewness and kurtosis for portfolio selection, one must estimate CVaR in a manner that captures the important non-normal characteristics of the assets in the opportunity set and how those non-normal characteristics interact when combined into portfolios.

Armed with a measure of CVaR that accounts for skewness and kurtosis, we studied the impact of skewness and kurtosis on asset allocation in a series of five comparisons. We started with a simple hypo-

thetical example and eventually moved on to a more applicable 14-asset-class example. More specifically, Scenarios 1–4 involved four simple hypothetical examples, whereas Scenario 5 consisted of a real-world opportunity set of 14 asset classes.

In Scenario 1, in which we assumed normal returns, we used a traditional quadratic optimization routine to determine the optimal portfolios. In Scenarios 2–4, which involved non-normal return distributions, we used simulation-based optimizations for both MVO and M-CVaR.[3] We simulated asset returns by using multivariate TLF distribution models. Such models result in return distributions that incorporate variance, skewness, and kurtosis into the CVaR estimate. Appendix A provides additional details on the simulation.

Hypothetical Asset Classes

To isolate the impact of skewness and kurtosis on M-CVaR optimization, we ran a controlled experiment with a small asset universe so that we could easily see the insights. We assumed four simple hypothetical assets—Assets A, B, C, and D. The expected returns, standard deviations, and correlation matrix are shown in **Table 1** (Panels A and B). Panel C of Table 1 identifies the skewness and kurtosis for the

Table 1. Capital Market Assumptions with Higher Moments

	Expected Return	Standard Deviation		
A. Expected returns and standard deviations				
Asset A	5%	10%		
Asset B	10	20		
Asset C	15	30		
Asset D	13	30		
	Asset A	Asset B	Asset C	Asset D
B. Correlation matrix				
Asset A	1.00			
Asset B	0.34	1.00		
Asset C	0.32	0.82	1.00	
Asset D	0.32	0.82	0.71	1.00
C. Skewness and kurtosis				
Scenario 1				
Skewness	0	0	0	0
Kurtosis	3	3	3	3
Scenario 2				
Skewness	0	0	0	0
Kurtosis	3.5	3.5	6	3.5
Scenario 3				
Skewness	0	0	–0.5	–0.3
Kurtosis	6	6	6	6
Scenario 4				
Skewness	0	0	–0.5	–0.3
Kurtosis	3.5	3.5	6	3.5

four assets used in the first four scenarios. The normal distribution has zero skewness and a kurtosis of 3. A kurtosis greater than 3 indicates a fatter tail than that of the normal distribution. The multivariate TLF model can take the inputs shown in Table 1 and generate the corresponding multivariate returns that we used for the first four scenario analyses.

Assets A, B, and C have the same ratio of return to risk (standard deviation), 0.5. Asset D has a slightly lower return-to-risk ratio, 0.43. The correlation between Asset A and the other assets is "low," whereas the correlations among Assets B, C, and D are "high." One can think of Asset A as a bond index and Assets B, C, and D as equity indices.

We analyzed the asset allocations as we varied the skewness and kurtosis of the four assets. As can be seen in Panel C of Table 1, by varying the skewness and kurtosis of Asset C relative to the other assets, we were able to use Asset C as our primary guinea pig. We selected a skewness of –0.5 and a kurtosis of 6 (Panel C) in such a way that they are typical values for equity asset classes.[4] Because MVO ignores higher moments, the optimal allocations are nearly the same for the four scenarios based on MVO.[5] In contrast, one would expect the M-CVaR optimizations to lead to different allocations.

Scenario 1. Zero Skewness and Uniform Tails

Zero skewness means symmetrical distributions, and uniform tails means that the kurtosis for all assets in a universe is the same. In our simplest scenario, the returns of all four hypothetical assets are assumed to be multivariate and normally distributed, with a kurtosis of 3. **Table 2** shows eight optimal allocations, ranging from low return (low risk) to high return (high risk). For both MVO and M-CVaR, the only constraint is no shorting. In our study, we used expected return as the basis of MVO

versus M-CVaR optimization comparisons, given our two definitions of risk. Note that the optimal allocations for the MVO and the M-CVaR optimization are the same. Returning to Equation 2, we can see that this intuitive result flows directly from the fact that minimizing CVaR (when estimated under Equation 2) is equivalent to minimizing either σ_P or σ_P^2 for a given μ_P.

If all four assets have symmetrical return distributions but uniformly fat tails with a kurtosis greater than 3, the MVO and the M-CVaR optimization lead to very similar allocations, as reported in Table 2. The reason is that when the kurtoses for all the assets are uniformly higher, their individual contributions to the overall portfolio's CVaR will be higher in absolute value, but not on a relative basis. In other words, the relative (percentage) contribution to the portfolio's CVaR for each asset remains the same. The equivalency of the asset allocations provides empirical evidence that fat tails alone do not invalidate MVO.[6]

Scenario 2. Zero Skewness and Mixed Tails

In Scenario 2, the skewness for all four assets is assumed to be approximately zero and the amount of kurtosis associated with each asset varies.[7] Assets A, B, and D have relatively thin tails with a kurtosis of 3.5, and Asset C has a fat tail with a kurtosis of 6. In the absence of skewness, Scenario 2 tests whether various levels of kurtosis affect asset allocations.

Table 3 shows the results. As one might expect, at an equivalent expected return, the M-CVaR optimizations have significantly lower allocations to Asset C—the only fat-tailed asset in this scenario—than do the MVO allocations. The M-CVaR optimization lowers the portfolio's CVaR by decreasing allocations to Asset C.

Table 2. Scenario 1: Zero Skewness and Normal Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	70.9%	70.9%	54.5%	54.5%	37.3%	37.3%	16.4%	16.4%
Asset B	18.3	18.3	8.4	8.4	0.0	0.0	0.0	0.0
Asset C	10.9	10.9	30.6	30.6	49.3	49.3	65.7	65.7
Asset D	0.0	0.0	6.5	6.5	13.4	13.4	17.9	17.9
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. dev.	11.2%	11.2%	14.9%	14.9%	19.4%	19.4%	24.4%	24.4%
Skewness	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Kurtosis	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
VaR	–11.5%	–11.5%	–15.6%	–15.6%	–21.1%	–21.1%	–27.3%	–27.3%
CVaR	–16.1%	–16.1%	–21.7%	–21.7%	–29.0%	–29.0%	–37.3%	–37.3%

Table 3. Scenario 2: Zero Skewness and Mixed Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	71.0%	68.2%	52.8%	47.3%	35.8%	25.7%	16.9%	4.5%
Asset B	18.8	24.6	13.9	23.2	5.1	23.4	0.6	22.4
Asset C	10.1	7.1	28.8	20.6	46.6	33.2	63.5	45.9
Asset D	0.0	0.1	4.5	8.9	12.5	17.8	18.9	27.2
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. dev.	11.2%	11.3%	14.8%	14.9%	19.3%	19.6%	24.2%	24.6%
Skewness	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Kurtosis	3.34	3.26	4.0	3.6	4.6	3.9	4.9	4.0
VaR	–11.36%	–11.43%	–14.9%	–15.3%	–19.9%	–20.8%	–25.4%	–27.0%
CVaR	–16.63%	–16.57%	–23.4%	–23.0%	–32.4%	–31.8%	–42.2%	–41.3%

Scenario 2 also highlights one of the weaknesses of VaR. For an expected return of 13 percent, the CVaR is 0.9 percentage point less extreme for the M-CVaR optimization than for the MVO. In contrast, for the same two portfolios, the VaR is 1.6 percentage points more extreme for the M-CVaR than for the MVO. We can observe similar behaviors for all four expected returns shown in Table 3, which is equivalent to saying that attempts to lower the VaR inadvertently increase the CVaR. This unwanted result is rooted in the fact that VaR is not a coherent measure of risk.

Scenario 3. Non-Zero Skewness and Uniformly Fat Tails

In Scenario 3, Assets A and B have zero skewness and Assets C and D have (bad) skewness of –0.5 and –0.3, respectively. All four assets are assumed to have a kurtosis of 6. Scenario 3 tests the impact of various levels of skewness when tails are uniform.

The optimization results are displayed in **Table 4**. In the M-CVaR optimization, the allocations to Asset C range from 3.9 to 9.4 percentage points lower than the corresponding MVO allocations. This outcome suggests that when kurtosis is controlled, the M-CVaR optimization tends to avoid the negatively skewed assets in order to minimize the CVaR. The more negative the skewness, the higher the tail loss, or CVaR. In other words, when all assets in the universe have uniformly fat tails, the net impact is from skewness.

Scenario 4. Non-Zero Skewness and Mixed Tails

The skewness in Scenario 4 is the same as in Scenario 3 (Assets A and B with 0 and Assets C and D with –0.5 and –0.3, respectively); however, the kurtosis of the assets varies. Assets A, B, and D all have

a kurtosis of 3.5, whereas Asset C has a disadvantageous kurtosis of 6. Scenario 4 tests the impact of various levels of skewness and kurtosis on asset allocations.

As expected, **Table 5** reports that in the M-CVaR optimization, the allocations to Asset C range from 5.6 to 20.7 percentage points lower than those in the MVO, an additional reduction of 2.6–3 percentage points compared with Scenario 2, in which all the assets have zero skewness. Among the four scenarios, Asset C has the lowest allocations. For an expected return of 13 percent, the portfolio's standard deviation is 0.5 percentage point higher for the M-CVaR optimization but its CVaR, or expected tail loss, is reduced by 1.3 percentage points.

Summarizing Scenarios 1–4

Figure 1 summarizes the impact of skewness and kurtosis on the asset allocation differences that result from MVO and M-CVaR optimization as measured by the allocation to Asset C, our guinea pig asset. Across all four scenarios, MVO led to similar asset allocations at each of the corresponding expected return points. Again, the observed differences are due to sampling errors—that is, slight differences in the return vector, standard deviation vector, and correlation matrix for Scenarios 2–4. In contrast, the M-CVaR optimization incorporated skewness and kurtosis into the asset allocation decision, which led to different optimal mixes—the allocations to Asset C varied by 20 percentage points when changes were made to skewness and kurtosis. Scenario 2 suggests that kurtosis with mixed tails has a significant impact even though the asset return distributions are symmetrical. Scenario 3 implies that skewness has a significant impact when kurtosis is controlled. Scenario 4 shows that the combination of skewness and kurtosis with mixed tails has the largest impact.

Table 4. Scenario 3: Non-Zero Skewness and Uniformly Fat Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	72.1%	68.8%	56.0%	52.5%	37.7%	33.6%	17.2%	13.1%
Asset B	16.2	22.6	6.1	11.9	0.5	6.9	0.0	5.1
Asset C	11.4	7.5	31.0	25.5	48.7	41.3	65.2	55.8
Asset D	0.3	1.2	6.9	10.1	13.1	18.2	17.7	26.0
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. dev.	11.3%	11.3%	14.9%	15.0%	19.5%	19.5%	24.4%	24.6%
Skewness	–0.1	–0.1	–0.3	–0.3	–0.4	–0.3	–0.4	–0.3
Kurtosis	4.2	4.1	4.7	4.5	5.2	4.9	5.4	5.1
VaR	–12.1%	–12.3%	–16.4%	–16.6%	–22.2%	–22.5%	–28.8%	–29.3%
CVaR	–19.1%	–19.0%	–26.9%	–26.7%	–37.2%	–37.0%	–48.2%	–47.9%

Table 5. Scenario 4: Non-Zero Skewness and Mixed Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	71.3%	66.0%	53.4%	44.2%	35.8%	22.9%	16.7%	2.9%
Asset B	18.1	28.6	11.9	28.9	4.4	27.8	0.1	23.8
Asset C	10.6	5.0	29.1	17.3	46.6	29.4	63.1	42.4
Asset D	0.0	0.4	5.6	9.6	13.3	20.0	20.1	30.9
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. dev.	11.3%	11.3%	14.8%	15.1%	19.3%	19.8%	24.3%	24.8%
Skewness	–0.1	–0.1	–0.3	–0.2	–0.4	–0.3	–0.4	–0.3
Kurtosis	3.3	3.2	4.0	3.5	4.6	3.7	4.8	3.9
VaR	–11.9%	–12.0%	–16.2%	–16.7%	–22.2%	–23.1%	–28.7%	–29.9%
CVaR	–17.6%	–17.4%	–25.7%	–25.1%	–36.0%	–34.9%	–46.8%	–45.5%

Figure 1. Allocations to Asset C in the Efficient Frontier with Portfolio Return of 11 Percent for the Four Scenarios

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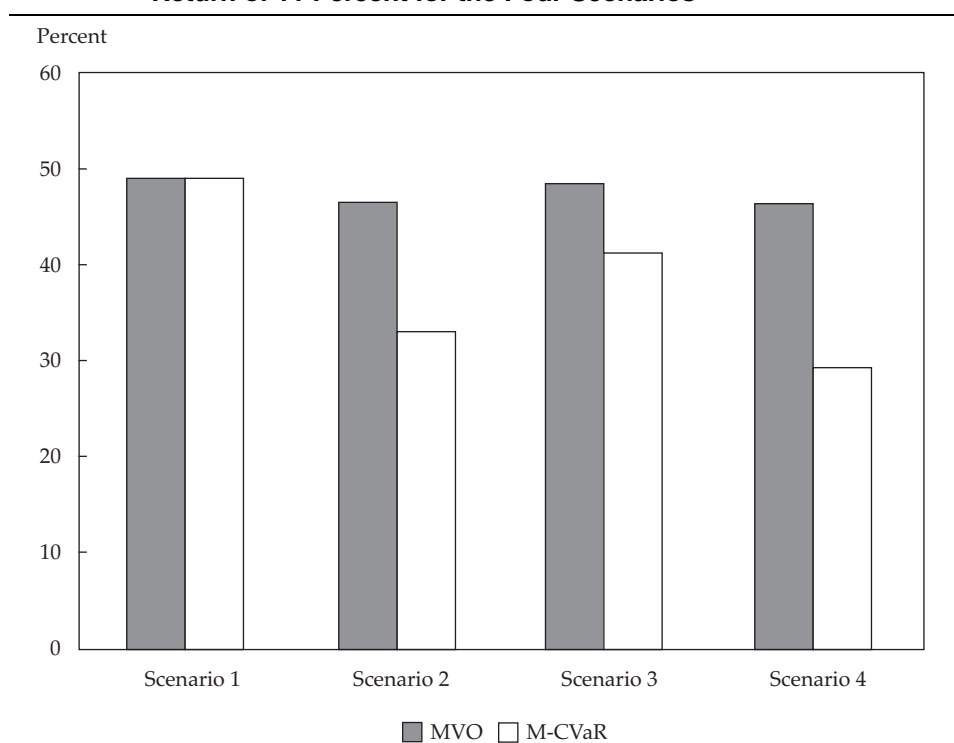
These four scenarios provide useful insights. In an asset universe with mixed tails, information about skewness and kurtosis can significantly affect the optimal allocations in the M-CVaR optimization. In these cases, the CVaR, or expected tail loss, can be reduced by performing the M-CVaR optimization but not the MVO. The reduced CVaR for the two optimizations depends on the distributions of skewness and kurtosis in the asset universe shown in Panel C of Table 1. In the M-CVaR optimization, wider ranges of skewness and kurtosis among the assets lead to a greater reduction in the portfolio's CVaR.

Scenario 5. The 14 Asset Classes

In our final example, Scenario 5, we move away from our four asset classes and apply MVO and M-CVaR to a robust 14-asset-class opportunity set that is typical for a sophisticated investor. The 14 asset classes and various historical descriptive statistics are shown in **Table 6**.[8] We measured the historical mean, standard deviation, skewness, kurtosis, Sharpe ratio, CVaR, and "CVaR ratio" from February 1990 to May 2010. The CVaR ratio measures risk-adjusted performance in the same way as the Sharpe ratio, except that the denominator is CVaR. In Table 6, we can see that emerging markets have the largest tail loss, U.S. bonds have the best Sharpe and CVaR ratios, and non-U.S. developed equities have the worst Sharpe and CVaR ratios.

In contrast to our previous four scenarios—in which we used the multivariate TLF distribution (parameterized on the basis of the capital market assumptions in Table 1) to estimate CVaR—in Scenario 5, we switched to a nonparametric bootstrapping analysis based on historical data. This approach allows other researchers to duplicate this portion of our analysis because few practitioners have a workable version of the multivariate TLF distribution.[9]

Rather than simply use pure historical returns, we used the reverse optimization procedure based on the capital asset pricing model—the starting point for the Black–Litterman model—to infer the expected future return for each asset class (shown in the second column of **Table 7**).[10] We used the capitalization-based weights (as of May 2010) in the first column to back out the expected returns for the 14 asset classes. We then "shifted" the location of the historical return distribution by either adding or subtracting a constant (the difference between the reverse-optimized return and the historical average return) to or from each historical return. By adding or subtracting an asset-class-specific constant to or from each historical return series, we found that the standard deviation, skewness, kurtosis, and correlation matrix of the adjusted returns remain the same as those of the historical returns.

The expected Sharpe ratios and the expected CVaR ratios in Table 7 are quite different from their corresponding historical ratios in Table 6 owing to the substantial differences between expected returns and historical returns. The emerging markets asset class continues to have the largest expected tail loss. Non-U.S. developed equities has the largest expected Sharpe and CVaR ratios. U.S. bonds has the lowest expected Sharpe ratio, whereas U.S. Treasury Inflation-Protected Securities (TIPS) has the lowest expected CVaR ratio.

Next, we performed bootstrapping analyses by randomly drawing from these "shifted" historical return distributions. Conceptually, the starting

Table 6. Historical Descriptive Statistics for the 14 Asset Classes, February 1990–May 2010

	Mean	Std. Dev.	Skewness	Kurtosis	Sharpe Ratio	CVaR	CVaR Ratio
Large value	10.24%	14.70%	–0.82	5.06	0.44	–36.37%	0.18
Large growth	9.34	17.40	–0.64	4.19	0.32	–41.46	0.13
Small value	12.62	17.07	–0.86	5.16	0.51	–42.84	0.20
Small growth	9.47	23.23	–0.41	3.84	0.24	–51.71	0.11
Non-U.S. dev. equities	5.85	17.45	–0.52	4.29	0.11	–40.90	0.05
Emerging markets	13.29	24.26	–0.74	4.72	0.39	–56.11	0.17
Commodities	7.13	15.58	–0.57	6.67	0.21	–35.00	0.09
Non-U.S. REITs	8.05	20.66	–0.22	5.03	0.20	–46.44	0.09
U.S. REITs	15.31	20.36	–0.76	10.51	0.56	–49.50	0.23
U.S. TIPS	8.19	5.54	–0.89	8.27	0.78	–11.74	0.37
U.S. bonds	7.22	3.82	–0.31	3.72	0.89	–6.77	0.50
Non-U.S. bonds	7.67	8.76	0.17	3.54	0.44	–15.89	0.24
Global high yield	10.89	10.59	–1.60	12.50	0.67	–27.13	0.26
Cash	3.85	1.98	–0.26	2.26	0	0.03	0

Table 7. Expected Descriptive Statistics for the 14 Asset Classes

|	Capitalization Weights	Expected Mean	Sharpe Ratio	Expected CVaR	CVaR Ratio
Large value	8.70%	8.94%	0.36	−36.62%	0.14
Large growth	8.67	9.54	0.34	−41.29	0.14
Small value	0.83	9.12	0.32	−43.66	0.12
Small growth	0.76	10.71	0.31	−51.23	0.14
Non-U.S. dev. equities	16.01	10.53	0.40	−39.50	0.18
Emerging markets	4.82	11.88	0.35	−56.30	0.15
Commodities	5.80	6.33	0.16	−35.16	0.07
Non-U.S. REITs	7.98	11.31	0.38	−45.41	0.17
U.S. REITs	3.51	9.24	0.27	−50.97	0.11
U.S. TIPS	0.84	4.78	0.15	−12.64	0.06
U.S. bonds	23.12	4.49	0.13	−7.49	0.07
Non-U.S. bonds	16.04	5.51	0.18	−16.44	0.10
Global high yield	1.92	7.10	0.31	−28.07	0.12
Cash	0.98	4.00	0	0.10	0

shifted historical return distribution can be thought of as the best guess for the unknown "population" distribution in which each random draw results in one possible sample distribution that could be the best representation of the unknown true future distribution. The bootstrapping, or resampling, method simultaneously accounts for input uncertainty and addresses the issues of estimation error, input sensitivity, and highly concentrated asset allocations.

Because CVaR is a tail measure, each drawn sample needed to contain adequate tail information, and thus, the number of data points in each draw could not be too small. We simply set the number of draws equal to the available historical data period: February 1990–May 2010, or 244 months. Therefore, the left tail contained about 12 data points (= 5 percent × 244) each time a new sample distribution was created. We repeated the M-CVaR optimization and the MVO 500 different times for 500 different sample distributions. All the results were recorded and averaged to derive the average optimal asset allocations over the 500 possible realized futures.

To ensure diversification, we limited the maximum allocation for each asset class to 30 percent during each optimization. To mitigate the issue of "optionality" associated with long-only constraints in resampling (see Scherer 2002), we allowed short sales and limited shorting to 30 percent for each asset class. Optionality means that assets are either in or out but are never negative in long-only resampled asset allocations, and thus, higher-volatility assets tend to receive higher allocations (on average) relative to long–short resampled asset allocations.

Figure 2 shows the skewness and kurtosis for the 14 asset classes over the last 20 years (1990–2010). Note that the relationship between skewness and kurtosis is somewhat linear for all 14 asset classes. A higher kurtosis is often accompanied by more extreme negative skewness. Also note that global high yield, U.S. REITs, and U.S. TIPS appear in the bottom right of Figure 2, which suggests that they have high kurtosis and more extreme negative skewness. From this perspective, these three assets have characteristics that are similar to those of Asset C in Scenario 4. Empirically, these assets seem to produce *relatively* stable returns during normal times, but they can suffer severely negative returns during extraordinary events. For U.S. REITs, this outcome is perhaps due to larger-than-normal amounts of leverage coupled with unusually smooth returns because of the large dividend component of returns. For U.S. TIPS and global high yield, we performed an "exceedence correlation" analysis to estimate asymmetrical correlations and found that their correlation with the U.S. equity market is asymmetrically much higher in severe downside markets than in normal markets or up markets.[11] This finding is directly related to negative skewness and higher kurtosis.

Table 8 shows the optimal asset allocations for both the MVO and the M-CVaR optimization from the bootstrapping of the 14 asset classes. As before, we identified efficient asset allocations from the MVO and M-CVaR optimization approaches based on expected returns. We selected four asset mixes (AM1–AM4), with expected returns of 7 percent, 9 percent, 11 percent, and 13 percent for each bootstrapping, respectively. Many of the different asset allocations between the M-CVaR and the MVO are significant at the 5 percent level. For example, the

Figure 2. Skewness and Kurtosis for the 14 Asset Classes, February 1990–May 2010
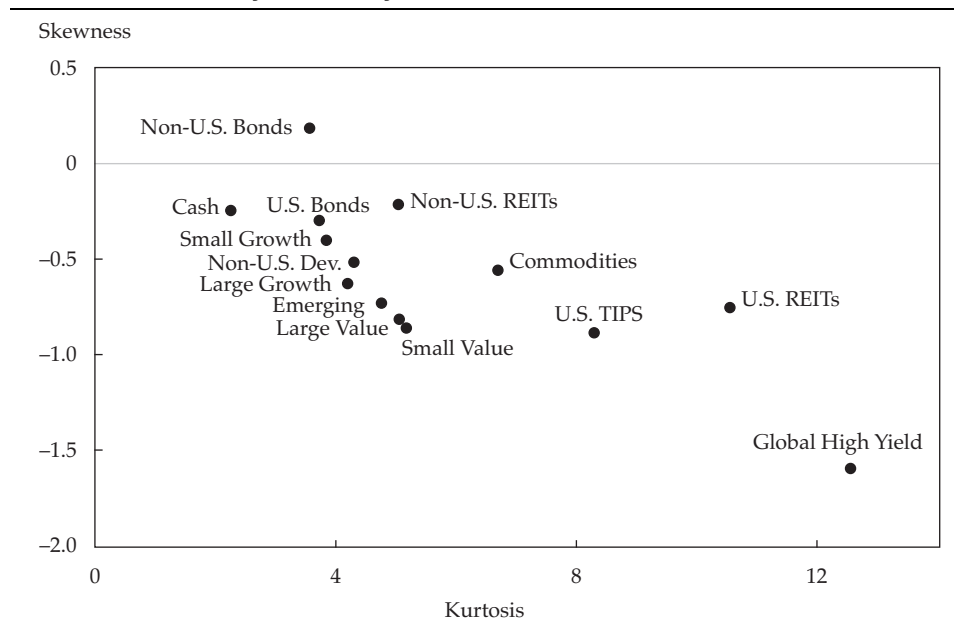
Table 8. Bootstrapped Optimal Allocations and Statistics for the 14 Asset Classes

	AM1		AM2		AM3		AM4	
	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Large value	1.13%	0.59%	3.52%*	2.15%*	5.79%*	3.80%*	8.88%*	6.87%*
Large growth	3.29*	4.63*	3.79	4.52	4.93	5.70	7.67	8.04
Small value	7.54*	8.51*	5.48	6.03	4.52	4.77	5.28	5.79
Small growth	–2.35*	–1.37*	–1.36*	–0.04*	0.08*	1.74*	1.91*	4.01*
Non-U.S. dev. equities	2.82*	–0.45*	5.25*	2.47*	8.54*	7.08*	12.05	11.36
Emerging markets	1.31	1.53	1.56	1.18	2.94	2.35	5.48	4.94
Commodities	3.62*	1.12*	4.03*	1.38*	4.71*	2.17*	4.85*	2.59*
Non-U.S. REITs	–2.83	–2.55	0.01*	2.33*	3.49*	6.19*	7.75*	10.70*
U.S. REITs	–2.55*	–4.36*	–0.74*	–3.42*	1.75*	–1.69*	4.73*	0.98*
U.S. TIPS	15.55	15.17	13.46*	15.51*	11.89*	14.42*	8.10*	10.33*
U.S. bonds	28.18	28.30	22.91*	24.11*	17.35*	20.26*	11.16*	14.46*
Non-U.S. bonds	9.27*	15.62*	9.01*	14.10*	8.82*	12.27*	8.08*	11.02*
Global high yield	5.01*	3.29*	5.22*	2.26*	5.01*	1.53*	4.86*	0.72*
Cash	30.00	29.97	27.84	27.40	20.19*	19.40*	9.19*	8.19*
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. dev.	4.0%	4.6%	6.0%	6.7%	8.5%	9.2%	10.8%	11.5%
Skewness	–0.4	0.3	–0.3	0.3	–0.3	0.1	–0.4	0.0
Kurtosis	5.0	3.7	4.9	3.9	5.0	4.2	5.1	4.3
VaR	–4.8%	–4.7%	–7.6%	–7.5%	–11.4%	–11.1%	–14.8%	–14.7%
CVaR	–7.3%	–6.0%	–11.4%	–9.8%	–16.9%	–15.1%	–22.1%	–20.4%

*Significant at the 5 percent level.

M-CVaR optimization allocates significantly higher amounts, by about 4.5 percentage points, to non-U.S. government bonds and significantly lower amounts, by about 3 percentage points, to global high yield for all the expected return levels.

These two asset classes are the polar points along the skewness axis in Figure 2.

Compared with the MVO, the M-CVaR optimization monotonically underweights global high yield, U.S. REITs, and commodities because of their

more extreme negative skewness and higher kurtosis, and it overweights non-U.S. government bonds, U.S. nominal bonds, and non-U.S. REITs because of their more attractive combined skewness and kurtosis. Small growth receives higher weights in the M-CVaR optimization owing to its attractive upper-left position in Figure 2 (higher skewness and lower kurtosis), even though the weights are negative (short sales) for AM1 and AM2. Non-U.S. developed equities closely matches small growth in all four moments, as shown in Table 6, but it is located below and to the right of small growth and thus receives less weight in the M-CVaR than in the MVO.

Note that the M-CVaR, compared with the MVO, allocates 4.8 percentage points less weight to U.S. TIPS in the asset mix with an expected return of 5 percent (not shown in Table 8) but allocates 2.2 percentage points more weight to U.S. TIPS in AM4, with an expected return of 13 percent. The allocations to TIPS are reduced as the expected return is increased for both the M-CVaR and the MVO, but the reduced amount is less for the M-CVaR (5.9 percentage points) than for the MVO (13.0 percentage points) when the expected return is increased from 5 percent to 13 percent. The main reason is that the M-CVaR is relatively less sensitive to expected return than is the MVO, and the CVaR measure puts more weight on skewness and kurtosis.

A portfolio's skewness or kurtosis is not simply the linear combination of individual asset classes' skewness or kurtosis. Because the M-CVaR minimizes a portfolio's CVaR, or tail loss, an individual asset class's higher-moment information should not be considered entirely separately. This point reinforces the most important lesson of modern portfolio theory: Although individual asset class characteristics are important, what really matters is the portfolio's overall characteristics.

In general, these observations are consistent with our previous discussions of Scenarios 2–4: The M-CVaR optimization tends to pick positively skewed (or less negatively skewed) and thin-tailed assets, whereas the MVO ignores the information from skewness and kurtosis. At the portfolio level, the skewness is higher, the kurtosis is lower, and the CVaR is lower for the M-CVaR optimization. For example, as shown in the bottom of Table 8 for AM4, the expected volatility is increased by 0.7 percentage point, but the skewness is increased from –0.4 to 0, the kurtosis is lowered by 0.8, and the CVaR is lowered by 1.7 percentage points.

Note that bootstrapping allows us to obtain some estimates on sampling error for the CVaR. As mentioned earlier, CVaR is a tail-loss measure that can suffer from estimation errors. For example, for AM4 in Table 8, the mean of the estimated CVaRs over the 500 samples is –20.4 percent (last row, last column), but the variations of those CVaRs are 18 percent in terms of standard deviation. That is, the bulk of the CVaRs range from –2.4 percent to –38.4 percent, which is quite wide. In terms of standard deviation, the average variations of optimal allocations for each asset are about 25 percent over the 500 sample multivariate distributions. For AM1, the corresponding CVaRs range from –4.0 percent to –8.0 percent, which is much narrower than the range for AM4. The average variations of optimal allocations for each asset are about 12 percent for AM1. A larger sample can reduce CVaR estimation error, but the trade-off is that a larger sample is less efficient for resampling. More specifically, a larger sample increases the likelihood that the sample parameters will converge to the starting inputs (population parameters), which will increase the similarity of the optimal asset allocations from different bootstrapped samples.

M-CVaR vs. MVO in the Financial Crisis of 2008

To test whether the M-CVaR optimization, compared with the MVO, would have helped investors during the financial crisis of 2008, we ran an out-of-sample bootstrapping analysis.[12] The bootstrapping procedure is the same as the one described earlier except that it was performed in August 2008, right before the onset of the most dramatic part of the financial crisis. The historical skewness and kurtosis from February 1990 to August 2008 are shown in the second and third columns of **Table 9**. Note that absent the data from September 2008 on, the values for skewness are higher and the values for kurtosis are lower for most equity classes, REITs, and commodities. In other words, the 2008 crisis significantly shifted their left tails further to the left. In particular, commodities were positively skewed before the crisis but were significantly negative after the crisis. In sharp contrast, the crisis made the skewness of U.S. bonds less negative, which suggests that the crisis triggered the flight to safety.

As before, we selected four asset mixes (AM1–AM4) for each resampled portfolio, with expected returns of 7 percent, 9 percent, 11 percent, and 13 percent, respectively. We set the minimum and maximum allocations for each asset class at –30 percent and 30 percent, respectively. We repeated the M-CVaR optimization and the MVO 500 times and recorded the average optimal asset allocations. The differences in average allocations between the M-CVaR and the MVO for the four asset mixes are shown in Table 9 (from column 5 to the last column).

Table 9.   Out-of-Sample Test for M-CVaR and MVO in the Financial Crisis of 2008

	Skewness[a]	Kurtosis[a]	Gain/Loss in Crisis	AM1[b]	AM2[b]	AM3[b]	AM4[b]
Large value	–0.53	4.48	–39.98%	0.68	–0.32	–1.06	–1.19
Large growth	–0.52	4.17	–34.54	1.63	1.64	1.06	0.02
Small value	–0.74	4.56	–42.47	0.08	1.30	1.41	0.73
Small growth	–0.32	3.91	–41.91	–0.33	0.09	0.97	1.91
Non-U.S. dev. equities	–0.29	3.44	–41.26	–2.44	–1.83	–0.52	0.46
Emerging markets	–0.70	4.45	–39.62	–0.63	–1.99	–2.87	–2.40
Commodities	0.13	3.58	–48.47	–0.17	0.08	0.64	0.99
Non-U.S. REITs	0.01	4.50	–46.27	1.14	2.53	3.25	2.93
U.S. REITs	–0.30	3.81	–58.39	–0.90	–2.52	–2.91	–2.88
U.S. TIPS	–0.33	4.67	–2.06	1.53	4.14	3.77	3.48
U.S. bonds	–0.38	3.67	3.29	–1.71	–0.96	–0.13	–0.19
Non-U.S. bonds	0.18	3.56	0.87	3.95	2.44	1.92	1.79
Global high yield	–1.22	10.36	–20.76	–2.72	–3.68	–4.46	–4.53
Cash	–0.20	2.45	0.44	–0.12	–0.91	–1.06	–1.12
M-CVaR return less MVO return				1.11	1.44	1.15	0.84

[a]Skewness and kurtosis were measured from February 1990 to August 2008.
[b]Columns AM1–AM4 show the optimal allocation differences, measured in percentage points, between M-CVaR and MVO for each asset class for the four asset mixes.

A negative sign means that the asset class has a lower weighting based on the M-CVaR optimization. Overall, the allocation differences shown in Table 9 are similar to those shown in Table 8, except for U.S. bonds and commodities. The opposite changes in skewness for U.S. bonds and commodities owing to the crisis resulted in higher allocations to commodities and lower allocations to U.S. bonds for the M-CVaR optimization in Table 9 compared with that in Table 8.

The duration of our out-of-sample testing period was six months (September 2008–February 2009). No rebalancing occurred during these six months. For the 14 asset classes, the losses or gains during the six months are shown in the fourth column of Table 9: U.S. REITs suffered the highest loss (–58.39 percent), followed by commodities (–48.47 percent). The outperformance of the M-CVaR optimization over the MVO is shown in the last row of Table 9. Note that the M-CVaR outperformed the MVO in all four asset allocation mixes, with excess returns ranging from 0.84 percentage point to 1.44 percentage points.

Across the four asset mixes, on average, the M-CVaR optimization overweights non-U.S. bonds over global high yield, as well as non-U.S. REITs over U.S. REITs. It also underweights non-U.S. developed equities and emerging markets. Two adverse allocations made by the M-CVaR optimization are the overweighting in commodities and the underweighting in U.S. bonds relative to the

MVO because the asset class commodities (U.S. bonds) has favorable (unfavorable) pre-crisis skewness. The majority of these allocation differences, however, turn out to be effective, which leads to the overall outperformance for the M-CVaR. This outperformance suggests that the higher-moment information embedded in historical returns had some predictive power in the crisis.[13]

Conclusion

Although practitioners are well aware that asset returns are not normally distributed and that investor preferences often go beyond mean and variance, the implications for portfolio choice are not well known.

In a series of controlled traditional MVOs and M-CVaR optimizations, we gained insights into the ramifications of skewness and kurtosis for optimal asset allocations. In our first four scenarios, prior to running the optimizations, we used the multivariate TLF distribution model to simulate a large number of returns with appropriate variance, skewness, and kurtosis, which, in turn, enabled us to more accurately measure the downside risk of a portfolio by using the CVaR.

In our first example, in which returns are symmetrically distributed and have uniform tails, the MVO and the M-CVaR lead to the same results. When there are varying levels of skewness and kurtosis in the opportunity set of assets, the MVO and the M-CVaR lead to significantly different asset

allocations. In particular, the combination of a negative skewness and a fat tail has the greatest impact on the optimal asset allocation weights. Intuitively, the M-CVaR prefers assets with higher positive skewness, lower kurtosis, and lower variance.

Over the last 20 years, global high yield, U.S. REITs, U.S. TIPS, and value stocks had significant negative skewness, whereas non-U.S. government bonds had positive skewness. The kurtosis for global high yield, U.S. REITs, and U.S. TIPS was higher than it was for other asset classes. In a 14-asset-class bootstrapping analysis, the M-CVaR, relative to the MVO, led to significantly higher allocations to non-U.S. government bonds and U.S. nominal bonds and lower allocations to global high yield, U.S. REITs, and commodities.

An out-of-sample test showed that the M-CVaR outperformed the MVO in the financial crisis of 2008, with excess gains ranging from 0.84 percentage point to 1.44 percentage points across the efficient frontier. This outperformance suggests that higher-moment information embedded in historical returns had some predictive power in the crisis.

Although we are just beginning to understand the impact of higher moments on asset allocation policy and further study is needed, these optimizations drive home a critical implication of modern portfolio theory: What matters is the overall impact on the portfolio's characteristics.

This article qualifies for 1 CE credit.

Appendix A. Multivariate TLF Distribution

We generated multivariate Lévy stable distributed returns through a numerical software package written by John Nolan (2009; for details, see http://academic2.american.edu/~jpnolan/stable/stable.html). Next, we applied a truncation method to these returns with infinite variance so that each return series followed a truncated Lévy flight model and preserved the desired pairwise correlations.

The multivariate Lévy stable returns are generated by the software's multivariate stable functions with discrete spectral measure. The inputs to the function are a matrix that specifies the locations of the point masses as columns, a lambda vector, a beta vector, a location vector, and α. Conceptually, this matrix specification plays a role similar to that of the covariance matrix that drives a traditional multivariate normal simulation. The lambda vector is proportional to the volatility vector. The beta vector specifies the skewness for each asset. The location vector corresponds to an expected return vector. The α indicates the heaviness of the tails.

Note that the disadvantage of the multivariate stable—and thus the TLF distribution—is that it requires the same tail index (α) for each marginal or individual distribution. In a typical portfolio, stocks and bonds tend to have slightly different tail indices. The tail index (α) is closely related to the kurtosis and is the parameter used to control the kurtosis. The lower the tail index (α), the higher the kurtosis, and vice versa.

In Scenarios 2 and 4, we assumed two different kurtosis levels (kurtosis of 6 for Asset C and 3.5 for Assets A, B, and D). To implement these more complicated scenarios, we called the multivariate TLF function twice for each of the two kurtoses. We then replaced the Asset C returns in the first set of multivariate TLF returns (kurtosis = 3.5) with the Asset C returns in the second set of multivariate TLF returns (kurtosis = 6). The correlation matrix was slightly distorted by this replacement, but the distortion is unimportant because we are interested in comparing the MVO with the M-CVaR optimization, given the same multivariate returns.

Notes

1. Scaling relations imply that one can model the return distribution of different time intervals (e.g., from a week to a month) with the same model parameters.
2. Hallerbach (2002) provided a formula for VaR for non-normal distributions, which can be straightforwardly extended to CVaR.
3. Developed by Rockafellar and Uryasev (2000), the M-CVaR optimization algorithm that we used can be easily implemented with simulated stochastic returns.
4. See Table 6.

5. The slight variations among the MVO results in Scenarios 2–4 arise from the simulation procedure. As the number of trials increases, the MVO results approach those from a non-simulation-based quadratic programming technique.
6. Another case in which the MVO and the M-CVaR optimization are equivalent is the multivariate elliptical distribution with fat tails (see, e.g., Bradley and Taqqu 2003).
7. Although the parameters of the simulation are precise, we use the word "approximately" to emphasize that in any given simulation, the "moment" may vary from the "target."

8. The indices that we used as proxies for the 14 asset classes are the Russell 1000 Value Index, Russell 1000 Growth Index, Russell 2000 Value Index, Russell 2000 Growth Index, MSCI World ex USA, MSCI Emerging Markets Index, Ibbotson Associates Commodity Composite Index, FTSE EPRA/NAREIT Global ex U.S. Index, FTSE EPRA/NAREIT U.S. Index, Barclays Capital Real U.S. Treasury TIPS Index, Barclays Capital U.S. Aggregate Bond Index, Citigroup Non-USD World Global Bond Index, Barclays Capital High-Yield Index, and Citigroup 3-Month U.S. Treasury Bill Index. The Barclays Capital Real U.S. Treasury TIPS Index was constructed from February 1990 to November 1997 by Ibbotson Associates because the existing series starts from December 1997.

9. In practice, we believe that one should use expected returns coupled with expected standard deviations, correlations, skewness, and kurtosis to generate the multivariate TLF returns and then use simulation-based optimization to derive the MVO and M-CVaR efficient frontiers. Our simulation-based optimization results for multivariate TLF returns for the 14 asset classes are generally consistent with our bootstrapping results.

10. The excess return reverse optimization formula is $\mu = \lambda \Sigma w$, where λ is the risk aversion coefficient, Σ is the covariance matrix, and w is the capitalization weights.

11. We used measures called "exceedence correlations" (see Longin and Solnik 2001) to determine the asymmetric correlation between the U.S. equity market and both U.S. TIPS and global high yield.

12. A thorough out-of-sample test for both crisis and noncrisis periods was beyond the scope of our study. It would also require a long history of data because the tail information must be estimated.

13. To check for robustness, we repeated the out-of-sample test with pure historical returns (instead of expected returns) from February 1990 to August 2008. In this test, the means were different, whereas the standard deviations, skewness, and kurtosis remained the same. The M-CVaR outperformances were –0.07 percentage point, 0.23 percentage point, 0.97 percentage point, and 0.86 percentage point for the four asset mixes, respectively. On average, the outperformances were positive but slightly worse than those reported in Table 9.

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