Computational Astrophysics Exercises — Assignments Set 3

Exercise 1: Kepler's equation and root finding

Kepler's equation can be used to determine the position of a planet in its orbit around the Sun at a given point in time. It reads

$$M(t) = \frac{2\pi}{P}(t - t_0) = E(t) - e \sin E(t).$$

Here, M denotes the mean anomaly, P the orbital period, E the eccentric anomaly and e the eccentricity (see figure 1). The transit at the perihel (closest approach to the Sun) is at $t = t_0$ with $E = 0$. Hence, the aphel of the orbit is at $t - t_0 = P/2$ and $E = \pi$. Kepler's equation can only be solved numerically.

The most simplest scheme to solve the equation iteratively is the fixed-point iteration (or relaxation)

$$E_{i+1} = M + e \sin E_i, \qquad i = 0, 1, 2, \ldots.$$

Another scheme is the Newton iteration (also known as Newton-Raphson method)

$$E_{i+1} = E_i - \frac{g(E_i)}{g'(E_i)}, \qquad i = 0, 1, 2, \ldots,$$

where

$$g(E) = E - e \sin(E) - M \qquad \text{and} \qquad g'(E) = \frac{dg}{dE} = 1 - e \cos E.$$

(Hint: for eccentricities smaller than 0.8, $E_0 = M$ is a good initial value for the iteration, for $e > 0.8$, $E_0 = \pi$ is better.) The so-called true anomaly f (see figure 1) is given by

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right).$$

The distance to the Sun r is given by

$$r = \frac{a(1-e^2)}{1 + e \cos f},$$

where a denotes the semi-major axis.

The position of the planet in a x, y coordinate system with the Sun in the center (the focus of the ellipse) is

$$x = r \cos f \qquad \text{and} \qquad y = r \sin f.$$

i) Calculate and plot the orbit of a body around the Sun (in x, y coordinates) for eccentricity $e = 0.205$, $a = 0.39$ au (Mercury) and $e = 0.967$, $a = 17.8$ au (Halley's comet). Calculate x, y for 256 points (equally distributed in M around one orbit) and state the total number of iterations for both methods (your own fixed point and Newton-Raphson implementation) which were needed to achieve an accuracy of $\Delta E = E_{i+1} - E_i \leqslant 10^{-9}$.

ii) The positions of planet Earth and Mars on their orbits on January 1 2000 were given by

Planet	a (au)	e	ϕ_0 (deg)	λ (deg)
Earth	1.000	0.0167	102.95	100.46
Mars	1.524	0.0934	336.04	355.45

Here, λ denotes the mean longitude of the orbit, which is defined by

$$\lambda = M + \phi_0.$$

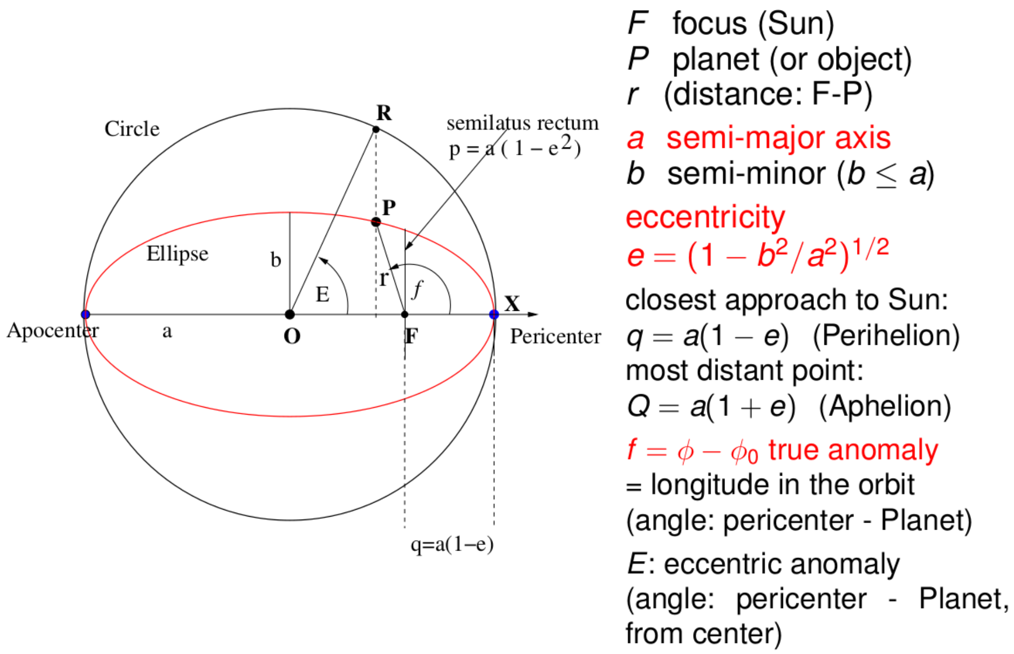

Figure 1: Kepler two body problem (sketch from the lecture notes).

The period ratio of the two planets can be calculated using Kepler's third law with $\mathrm{P}_{\mathrm{Earth}} = 1$ year. Calculate and plot the distance between Earth and Mars for the period from 1.1.1985 until 29.5.2024 with following assumptions: co-planar orbits of the two planets, both orbits are undisturbed Kepler-ellipses, which means the both planets do not interact gravitationally. Proceed as follows: Use the data from the table to determine the mean anomalies M_0 of both planets on 1.1.1985

$$\mathrm{M}_0(\text{planet}) = \lambda(\text{planet}) - \phi_0(\text{planet}) - 15\text{years} \cdot \frac{2\pi}{\mathrm{P}(\text{planet})},$$

where the period P of the particular planet has to be given in years.
Solve Kepler's equation with the mean anomaly at time t

$$\mathrm{M}(\text{planet}) = 2\pi \frac{\mathrm{t}}{\mathrm{P}(\text{planet})} + \mathrm{M}_0(\text{planet}).$$

and calculate with the solution the positions of both planets and plot their mutual distance on each day in the period from 1.1.1985 until 21.5.2025. Hint for python developers: Take a look at module `datetime` to get the number of days between two dates*.

(0b1000 points)

Exercise 2: The Lagrange point

There is a magical point between Earth and the Moon, called L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit, see figure 2.

*It's a little bit quirkier in C, see, e.g., `https://t.ly/adnN-`.

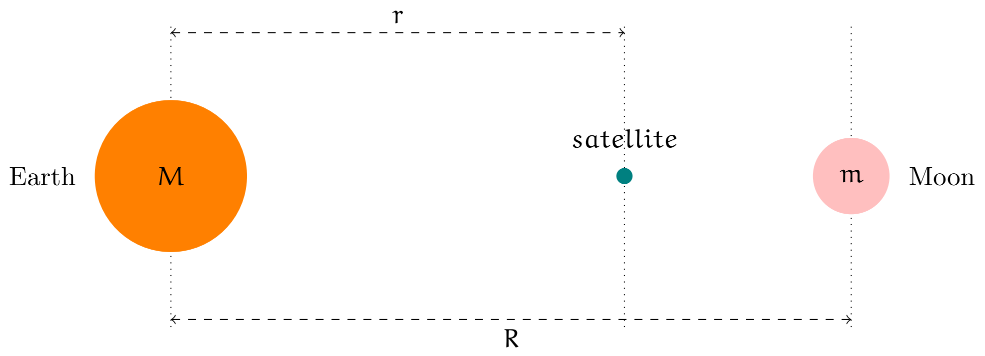

Figure 2: The location of the satellite at the Lagrange L_1 point.

Assuming circular orbits, and assuming that $M_{\text{Earth}} = M \gg M_{\text{Moon}} = m \gg M_{\text{satellite}}$, we find the following equation for the distance r of the Lagrange point L_1 to the center of the Earth

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where R denotes the distance between Earth and Moon, ω is the orbital frequency of the Moon and the satellite, and G is the gravitational constant.

Depending on your choice of programming language, either use `scipy` (python) or `GSL` (C) or `FGSL` (FORTRAN) to solve this equation and to obtain the location of L_1. Use the following parameters $M = 5.974 \times 10^{24}$ kg, $m = 7.348 \times 10^{22}$ kg, $R = 3.844 \times 10^{8}$ m, $\omega = 2.662 \times 10^{-6}\,\mathrm{s}^{-1}$.

(exercise is taken from Newman, Computational Physics†)

(0b10 points)

Useful links may be:
C: `https://www.gnu.org/software/gsl/doc/html/roots.html`
python: `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.root_scalar.html`
FORTRAN: `https://github.com/reinh-bader/fgsl/`
and `https://www.gnu.org/software/gsl/` for more wrappers for other languages
and sorry Dennis, I have not looked for the lib in julia.

in total (0b1010 points)

Deadline:
Tue, May 27th at midnight, via email (including code in attachment) to `ch.schaefer@uni-tuebingen.de`.

† ... which is by the way a really good book.