# **Computational Astrophysics Exercises** — Assignments Set 4

## Integration

The error of an integration scheme is defined as

$$E[f] = I[f] - Q[f] \tag{1}$$

where $I[f]$ denotes the exact solution of the integral and $Q[f]$ the numerical solution for the integral of function $f(x)$ in the interval $[a, b]$.
The scheme for Simpson's integration reads

$$Q_2[f] = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{2}$$

The error of an integration scheme with polynomials in the order of $n$ is given for even $n$ by

$$E_n[f] = \frac{K_n}{(n+2)!} f^{(n+2)}(\xi) \qquad \text{with} \qquad K_n = \int_a^b x W_n(x) dx, \tag{3}$$

where $\xi$ is a point in $[a, b]$. The error polynomial $W_n(x)$ is given by

$$W_n(x) = \prod_{i=0}^{n} (x - x_i), \tag{4}$$

where $x_i$ denote the nodes of the polynomial.

### 1. Simpson's rule — Error estimation

Show, that the integration error of Simpson's rule (n=2) is given by

$$E_2[f] = -\frac{f^{(4)}(\xi)}{90}\left(\frac{b-a}{2}\right)^5, \tag{5}$$

with the nodes $x_i$ $x_0 = a$, $x_1 = (b+a)/2$, $x_2 = b$.

**Hint:** Use the variable transformation

$$t = \frac{x-a}{b-a}, \tag{6}$$

that transforms the interval $[a, b]$ to $[0, 1]$.

(0b11 points)

### 2. Integration using Trapez, Simpson's and the adaptive Simpson's method

Develop a program to calculate the integral $I(f)$ of the function $f(x)$ over the interval $[a, b]$.

$$I(f) = \int_a^b f(x) dx. \tag{7}$$

Your program should implement the Trapez method, the Simpson's rule and the adaptive Simpson's method. The last method uses an estimate of the error using Simpson's rule. If the error exceeds a defined value, the algorithm subdivides the interval of integration in two and

the adaptive Simpson's method is applied to each subinterval in a recursive manner, see, e.g., `https://en.wikipedia.org/wiki/Adaptive_Simpson%27s_method`.
Calculate the values for the Si and Fresnel integrals $\mathsf{Si}(1)$ and $\mathsf{C}(5)$

$$\mathsf{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \qquad \text{and} \qquad \mathsf{C}(x) = \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt, \tag{8}$$

with an accuracy of $\varepsilon = 10^{-8}$ using the adaptive Simpson's rule*. Calculate the integral using the Trapez method and lower (slice the interval into 2) the step size until you have reached an accuracy of $\varepsilon = 10^{-8}$ using the following scheme: 1. Choose an initial number of steps $N_1 = 1$ and calculate the first approximation $I_1$ to the integral. 2. Double the number of steps and calculate an improved approximation $I_2$. 3. Continue until the error between the two last steps is less than your desired accuracy.

(0b111 points)

in total (≈$10^4$ masses of Jupiter in units of $M_\odot$ points)

**Deadline:**
Wed, 4 June at noon 🏛 via email (including code) to
`ch.schaefer@uni-tuebingen.de`.

*You might want to compare your implementation's results to the scipy implementation of Simpson's rule `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.simpson.html`