1. The vis-viva equation

We have shown in the lecture that the energy in the two body problem is conserved and it holds

$$\frac{1}{2}\dot{\mathbf{r}}^2 - \frac{\mu}{r} = \text{const.}, \tag{1}$$

where $\mu = G(m_1 + m_2)$ is the standard gravitational parameter. Show that the vis-viva equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{2}$$

follows from the conservation of energy and the conservation of angular momentum. Here, a denotes the semi-major axis of the orbit and r is the distance between the two bodies. (Hint: Consider the energy at periapsis and apoapsis to calculate const. with the help of angular momentum.)

(0b100 points)

2. Natural units

Show that if the two-body problem is analyzed using units in which the gravitational constant, the sum of the masses of the bodies and the distance between these bodies are equal to 1, the orbital period of the two bodies around the center of mass is 2π.

(0b000 points)

1 $\frac{1}{2}\dot{r}^2 - \frac{\mu}{r} = \text{const.}$ => spec. Energy is conserved. Same at periapsis and apoapsis

$$\frac{1}{2}\dot{r}_p^2 - \frac{\mu}{r_p} = \frac{1}{2}\dot{r}_a^2 - \frac{\mu}{r_a}$$

$$\Leftrightarrow \frac{1}{2}\left(\dot{r}_p^2 - \dot{r}_a^2\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right), \quad \vec{h} = \vec{r} \times \vec{v} = \vec{r}_p \times \dot{\vec{r}}_p = \vec{r}_a \times \dot{\vec{r}}_a$$

$$\vec{r}_a \perp \dot{\vec{r}}_a \Rightarrow |\vec{h}| = h = r_a v_a = r_p v_p$$

$$\Rightarrow v_p = \frac{r_a v_a}{r_p}$$

$$\Rightarrow \frac{1}{2}\left(v_p^2 - v_a^2\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$\Leftrightarrow \frac{v_a^2}{2}\left(\frac{r_a^2 - r_p^2}{r_p^2}\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$\Leftrightarrow \frac{v_a^2}{2} = \mu\left(\frac{r_p - r_a}{r_p r_a}\right)\cdot\left(\frac{r_p^2}{r_p^2 - r_a^2}\right) = \mu \frac{r_p}{r_a(r_p + r_a)}$$

$2a = r_a + r_p$

$$\Rightarrow \frac{v_a^2}{2} = \mu\frac{2a - r_a}{2a\, r_a} \Leftrightarrow \frac{v_a^2}{2} - \frac{\mu}{r_a} = -\frac{\mu}{2a}$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \Leftrightarrow \dot{r}^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

② $G = 1$, $m_1 + m_2 = 1$, $r = r_1 - r_2 = 1 \Rightarrow \mu = 1$
$a = 1$

Using : $T = \frac{2\pi}{\mu} a^{\frac{3}{2}} \Rightarrow T = 2\pi$