**ДЗ**: 1927, 1928, 1935 (см. указания), 1943, 1947, 1949.

# Интегрирование некоторых иррациональных функций

**Задача 1.** (**Д1927**)

$$\int \frac{dx}{x\left(1+2\sqrt{x}+\sqrt[3]{x}\right)}$$

□ Приведем подинтегральную функцию к рациональной:

$$t=\sqrt[6]{x}\Rightarrow t^6=x,\ dx=6t^5dt\Rightarrow$$

$$\Rightarrow\int \frac{dx}{x\left(1+2\sqrt{x}+\sqrt[3]{x}\right)}=\int\frac{6t^5dt}{t^6\left(1+2t^3+t^2\right)}=6\int\frac{dt}{t(1+2t^3+t^2)}$$

$$2t^3+t^2+1=0\Rightarrow t=-1\Rightarrow 2\cdot(-1)+1+1=0\Rightarrow 2t^3+t^2+1\div(t+1)\Rightarrow$$

$$\Rightarrow 2t^3+t^2+1-2t^2\cdot(t+1)\Rightarrow -t^2+1-(-t)\cdot(t+1)\Rightarrow t+1-1\cdot(t+1)\Rightarrow$$

$$\Rightarrow 2t^3+t^2+1=(t+1)\cdot(2t^2-t+1)$$

$$6\int\frac{dt}{t(1+2t^3+t^2)}=6\int\frac{dt}{t(t+1)(2t^2-t+1)}=6\int\frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{2t^2-t+1}dt$$

$$A(t+1)(2t^2-t+1)+Bt(2t^2-t+1)+(Ct+D)(t+1)t=1$$

$$2At^3-At^2+At+2At^2-At+A+2Bt^3-Bt^2+Bt+Ct^3+Ct^2+Dt^2+Dt=1$$

$$\begin{cases}1, & A=1\\ t, & A-A+B+D=0\\ t^2, & 2A-A-B+D+C=0\\ t^3, & 2A+2B+C=0\end{cases}\Rightarrow\begin{cases}A=1\\ B=-D\\ 2C=-3\\ 2B=-C-2A\end{cases}\Rightarrow\begin{cases}A=1\\ C=-\frac{3}{2}\\ B=\frac{3}{4}-1=-\frac{1}{4}\\ D=\frac{1}{4}\end{cases}$$

$$6\int\frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{2t^2-t+1}dt=6\int\frac{1}{t}dt-6\int\frac{1}{4(t+1)}dt-6\int\frac{6t-1}{4(2t^2-t+1)}dt=$$

$$=6\ln|t|-\frac{3}{2}\ln|t+1|-6\int\frac{6t-1}{4(2t^2-t+1)}dt$$

$$\frac{1}{4}\int\frac{6t-1}{2t^2-t+1}dt=\frac{3}{8}\int\frac{4t-1}{2t^2-t+1}dt+\frac{1}{4}\int\frac{\frac{1}{2}}{2t^2-t+1}dt=\frac{3}{8}\int\frac{d(2t^2-t+1)}{2t^2-t+1}+\frac{1}{16}\int\frac{dt}{t^2-\frac{t}{2}+\frac{1}{2}}=$$

$$=\frac{3}{8}\ln|2t^2-t+1|+\frac{1}{16}\int\frac{d\left(t-\frac{3}{8}\right)}{\left(t-\frac{1}{4}\right)^2+\frac{7}{16}}=\frac{1}{4}\ln|2t^2-t+1|+\frac{1}{4\sqrt{7}}\operatorname{arctg}\left(\frac{4t-1}{\sqrt{7}}\right)+C\Rightarrow$$

$$\Rightarrow\int\frac{dx}{x\left(1+2\sqrt{x}+\sqrt[3]{x}\right)}=6\ln\sqrt[6]{x}-\frac{3}{2}\ln\left(\sqrt[6]{x}+1\right)-\frac{9}{4}\ln\left(2\sqrt[3]{x}-\sqrt[6]{x}+1\right)-\frac{3}{2\sqrt{7}}\operatorname{arctg}\left(\frac{4\sqrt[6]{x}-1}{\sqrt{7}}\right)+C=$$

$$=\frac{3}{4}\cdot\ln\left(\frac{x\cdot\sqrt[3]{x}}{\left(1+\sqrt[6]{x}\right)^2\left(2\sqrt[3]{x}-\sqrt[6]{x}+1\right)^3}\right)-\frac{3}{2\sqrt{7}}\operatorname{arctg}\left(\frac{4\sqrt[6]{x}-1}{\sqrt{7}}\right)+C$$

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**Задача 2.** (**Д1928**)

$$\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx$$

$\square$ Приведем подинтегральную функцию к рациональной:

$$t=\sqrt[3]{2+x} \Rightarrow x=t^3-2,\ dx=3t^2dt \Rightarrow$$

$$\Rightarrow \int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx=\int \frac{(t^3-2)3t^3dt}{t^3-2+t}=3\int \frac{t^6-2t^3}{t^3+t-2}dt$$

$$t^6-2t^3 \div t^3+t-2 \Rightarrow t^6-2t^3-(t^3)(t^3+t-2) \Rightarrow -2t^3-t^4+2t^3-(-t)(t^3+t-2)=t^2-2t \Rightarrow$$

$$\Rightarrow 3\int \frac{t^6-2t^3}{t^3+t-2}dt=3\int t^3-t+\frac{t^2-2t}{t^3+t-2}dt=\frac{3}{4}t^4-\frac{3}{2}t^2+3\int \frac{t(t-2)}{t^3+t-2}dt$$

$$(1)^3+1-2=0 \Rightarrow t^3+t-2 \div (t-1) \Rightarrow$$

$$\Rightarrow t^3+t-2-t^2(t-1)=t^2+t-2 \Rightarrow t^2+t-2-t(t-1)=2t-2=2(t-1) \Rightarrow$$

$$t^3+t-2=(t-1)(t^2+t+2) \Rightarrow \frac{t(t-2)}{(t-1)(t^2+t+2)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+2} \Rightarrow$$

$$\Rightarrow At^2+At+2A+Bt^2+Ct-Bt-C=t^2-2t \Rightarrow$$

$$\Rightarrow \begin{cases} C=2A \\ A+B=1 \\ A+C-B=-2 \end{cases} \Rightarrow \begin{cases} C=2A \\ B=1-A \\ 2A+C=-1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=\frac{5}{4} \\ C=-\frac{1}{2} \end{cases}$$

$$3\int \frac{t(t-2)}{t^3+t-2}dt=-3\int \frac{1}{4(t-1)}dt+3\int \frac{\frac{5}{4}t-\frac{1}{2}}{t^2+t+2}dt=-\frac{3}{4}\ln|t-1|+\frac{3}{4}\int \frac{5t-2}{t^2+t+2}dt$$

$$\int \frac{5t-2}{t^2+t+2}dt=\frac{5}{2}\int \frac{2t+1}{t^2+t+2}dt-\frac{9}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{7}{4}}=\frac{5}{2}\ln|t^2+t+2|-\frac{9}{2}\cdot\frac{2}{\sqrt{7}}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{7}}\right)+C=$$

$$=\frac{5}{2}\ln(t^2+t+2)-\frac{9}{\sqrt{7}}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{7}}\right)+C \Rightarrow$$

$$\Rightarrow 3\int \frac{t(t-2)}{t^3+t-2}dt=-\frac{3}{4}\ln|t-1|+\frac{15}{8}\ln(t^2+t+2)-\frac{27}{4\sqrt{7}}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{7}}\right)+C \Rightarrow$$

$$\Rightarrow \int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx=\frac{3}{4}t^4-\frac{3}{2}t^2-\frac{3}{4}\ln|t-1|+\frac{15}{8}\ln(t^2+t+2)-\frac{27}{4\sqrt{7}}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{7}}\right)+C,\ t=\sqrt[3]{2+x}$$

$\blacksquare$

**Задача 3.** (**Д1935**)

$$\int \frac{dx}{1+\sqrt{x}+\sqrt{x+1}}$$

□ Предложим следующую замену (пусть $u>0$):

$$x=\left(\frac{u^2-1}{2u}\right)^2 \Rightarrow \sqrt{x}=\frac{u^2-1}{2u},\ \sqrt{x+1}=\sqrt{\frac{u^4-2u^2+1}{4u^2}+1}=\sqrt{\frac{(u^2+1)^2}{4u^2}}=\frac{u^2+1}{2u}\Rightarrow$$

$$dx=2\cdot\frac{u^2-1}{2u}\cdot\frac{2u2u-2(u^2-1)}{4u^2}du=\left(\frac{u^2-1}{u}\right)\cdot\frac{2u^2+2}{4u^2}du=\frac{u^4-1}{2u^3}du\Rightarrow$$

$$\int \frac{dx}{1+\sqrt{x}+\sqrt{x+1}}=\int \frac{u^4-1}{2u^3}\cdot\frac{2u}{2u+u^2-1+u^2+1}du=\int \frac{u^4-1}{2u^3\cdot(u+1)}du=\int \frac{(u-1)(u^2+1)}{2u^3}du=$$

$$=\int \frac{u^3-u^2+u-1}{2u^3}du=\int \frac{1}{2}-\frac{1}{2u}+\frac{1}{2u^2}-\frac{1}{2u^3}du=\frac{1}{2}\cdot\left(u-\ln u-\frac{1}{u}+\frac{1}{2u^2}\right)+C$$

Найдем выражение $u$ через $x$:

$$u^2-2\sqrt{x}u-1=0\Rightarrow D=4x+4=4(x+1),\ u_{1,2}=\frac{2\sqrt{x}\pm 2\sqrt{x+1}}{2}$$

$$u>0\Rightarrow u=\sqrt{x}+\sqrt{x+1}\Rightarrow \int \frac{dx}{1+\sqrt{x}+\sqrt{x+1}}=-\frac{1}{2}\ln\left(\sqrt{x}+\sqrt{x+1}\right)+$$

$$+\frac{\sqrt{x}+\sqrt{x+1}}{2}-\frac{1}{2\sqrt{x}+2\sqrt{x+1}}+\frac{1}{4\left(\sqrt{x}+\sqrt{x+1}\right)^2}+C$$

$$-\frac{1}{2\sqrt{x}+2\sqrt{x+1}}=-\frac{\sqrt{x}-\sqrt{x+1}}{2(x-x-1)}=\frac{\sqrt{x}-\sqrt{x+1}}{2}$$

$$\frac{1}{4\left(\sqrt{x}+\sqrt{x+1}\right)^2}=\frac{(\sqrt{x}-\sqrt{x+1})^2}{4}=\frac{x-2\sqrt{x(1+x)}+1+x}{4}=\frac{x}{2}-\frac{\sqrt{x(1+x)}}{2}+\frac{1}{2}\Rightarrow$$

$$\Rightarrow \int \frac{dx}{1+\sqrt{x}+\sqrt{x+1}}=-\frac{1}{2}\ln\left(\sqrt{x}+\sqrt{x+1}\right)+\frac{x}{2}-\frac{\sqrt{x(1+x)}}{2}+\sqrt{x}+C$$

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**Задача 4.** (**Д1943**)

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$$

□

$$y=\sqrt{1+2x-x^2},\ P_3(x)=x^3,\ \deg P_3=3$$

Тогда воспользуемся модифицированной формулой Остроградского:

$$\int \frac{P_n(x)}{y}dx=Q_{n-1}(x)\cdot y+\lambda\cdot\int \frac{dx}{y},\quad \deg P_n=n,\ \deg Q_{n-1}\le n-1,\ \lambda\in\mathbb{R}$$

$$\int \frac{x^3}{y}dx=(Ax^2+Bx+C)\cdot y+\lambda\cdot\int \frac{dx}{y}\Rightarrow$$

$$\Rightarrow \frac{x^3}{y} = (2Ax+B)\cdot y + (Ax^2+Bx+C)\cdot\frac{1}{2}\cdot\frac{-2x+2}{\sqrt{1+2x-x^2}} + \frac{\lambda}{y} \Rightarrow$$

$$\Rightarrow x^3 = (2Ax+B)(1+2x-x^2) + (Ax^2+Bx+C)\cdot(1-x) + \lambda$$

$$\begin{cases} x^3, & 1 = -2A - A \\ x^2, & 0 = 4A - B + A - B \\ x, & 0 = 2A + 2B + B - C \\ 1, & 0 = B + C + \lambda \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = \frac{5}{2}A = -\frac{5}{6} \\ C = 2A + 3B = -\frac{2}{3} - \frac{15}{6} = -\frac{19}{6} \\ \lambda = -B - C = \frac{5}{6} + \frac{19}{6} = 4 \end{cases} \Rightarrow$$

$$\Rightarrow \int \frac{x^3}{y}dx = -\frac{1}{6}\cdot\left(2x^2+5x+19\right)\cdot y + 4\cdot\int\frac{dx}{y}$$

$$\int \frac{1}{\sqrt{1+2x-x^2}}dx = \int\frac{dx}{\sqrt{2-(x^2-2x+1)}} = \int\frac{dx}{\sqrt{2-(x-1)^2}} = \arcsin\left(\frac{x-1}{\sqrt{2}}\right) + C$$

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx = -\frac{1}{6}\cdot\left(2x^2+5x+19\right)\cdot\sqrt{1+2x-x^2} + 4\arcsin\left(\frac{x-1}{\sqrt{2}}\right) + C$$

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**Задача 5.** (**Д1947**)

$$\int\frac{dx}{x^3\sqrt{x^2+1}}$$

□ Пусть $t = \frac{1}{x} > 0$, тогда:

$$x = \frac{1}{t},\ \sqrt{x^2+1} = \frac{\sqrt{t^2+1}}{t},\ dx = -\frac{1}{t^2}dt \Rightarrow$$

$$\Rightarrow \int\frac{dx}{x^3\sqrt{x^2+1}} = \int t^3\cdot\frac{t}{\sqrt{t^2+1}}\cdot\left(-\frac{1}{t^2}\right)dt = -\int\frac{t^2}{\sqrt{t^2+1}}dt \Rightarrow$$

$$\Rightarrow y = \sqrt{t^2+1},\ P_2(t) = t^2,\ \deg P_2 = 2 \Rightarrow$$

$$\int\frac{t^2}{y}dt = (At+B)\cdot y + \lambda\cdot\int\frac{dx}{y} \Rightarrow \frac{t^2}{y} = A\cdot y + (At+B)\cdot\frac{2t}{2\cdot y} + \frac{\lambda}{y} \Rightarrow$$

$$\Rightarrow t^2 = A(t^2+1) + (At+B)t + \lambda \Rightarrow 1 = 2A,\ 0 = B,\ 0 = A + \lambda \Rightarrow A = \frac{1}{2},\ B = 0,\ \lambda = -\frac{1}{2} \Rightarrow$$

$$\Rightarrow \int\frac{dx}{\sqrt{t^2+1}} = \ln\left|t + \sqrt{t^2+1}\right| + C = \ln\left|\frac{1}{x} + \sqrt{\frac{1+x^2}{x^2}}\right| + C = \ln\frac{1+\sqrt{1+x^2}}{|x|} + C \Rightarrow$$

$$\Rightarrow \int\frac{dx}{x^3\sqrt{x^2+1}} = -\frac{\sqrt{1+x^2}}{2x^2} + \frac{1}{2}\ln\frac{1+\sqrt{1+x^2}}{|x|} + C$$

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**Задача 6.** (**Д1949**)

$$\int \frac{1}{(x-1)^3\sqrt{x^2+3x+1}}dx$$

□ Воспользуемся модификацией метода Остроградского и сделаем замену:

$$x-1=\frac{1}{t},\ x=\frac{t+1}{t},\ dx=\frac{t-t-1}{t^2}dt=-\frac{1}{t^2}dt,\ (x-1)^3=\frac{1}{t^3}$$

$$x^2+3x+1=\frac{t^2+2t+1}{t^2}+3\frac{t+1}{t}+1=\frac{t^2+2t+1+3t^2+3t+t^2}{t^2}=\frac{5t^2+5t+1}{t^2}$$

$$\int \frac{1}{(x-1)^3\sqrt{x^2+3x+1}}dx=\int \frac{t^3\cdot t}{\sqrt{5t^2+5t+1}}\cdot\left(-\frac{1}{t^2}\right)dt=-\int\frac{t^2}{\sqrt{5t^2+5t+1}}dt$$

$$y=\sqrt{5t^2+5t+1},\ \int\frac{P_2(t)}{y}dt=Q_1(t)\cdot y+\lambda\cdot\int\frac{dt}{y}$$

$$\deg P_2=2,\ \deg Q_1\le 1,\ \lambda\in\mathbb{R}\Rightarrow Q_1(t)=At+B$$

$$-\int\frac{t^2}{\sqrt{5t^2+5t+1}}dt=(At+B)\sqrt{5t^2+5t+1}+\lambda\int\frac{dt}{\sqrt{5t^2+5t+1}}$$

Продифференцируем и умножим на $y$:

$$-t^2=A(5t^2+5t+1)+(At+B)\cdot\left(5t+\frac{5}{2}\right)+\lambda$$

$$\begin{cases} t^2, & -1 & = & 5A+5A \\ t, & 0 & = & 5A+\frac{5}{2}A+5B \\ 1, & 0 & = & A+\frac{5}{2}B+\lambda \end{cases}\Rightarrow\begin{cases} A & = & -\frac{1}{10} \\ B & = & \frac{3}{20} \\ \lambda & = & \frac{1}{10}\quad-\frac{15}{40}=-\frac{11}{40} \end{cases}$$

$$\int\frac{dt}{\sqrt{5t^2+5t+1}}=\int\frac{dt}{\sqrt{5\left(t+\frac{1}{2}\right)^2-\frac{1}{4}}}\Rightarrow 5\left(t+\frac{1}{2}\right)^2-\frac{1}{4}=\frac{1}{4}\cdot\left(20\left(t+\frac{1}{2}\right)^2-1\right)$$

$$2\sqrt{5}\left(t+\frac{1}{2}\right)=\operatorname{ch}u\Rightarrow 20\left(t+\frac{1}{2}\right)^2-1=\operatorname{sh}^2u,\ dt=\frac{1}{2\sqrt{5}}\operatorname{sh}u du$$

$$\int\frac{dt}{\sqrt{5\left(t+\frac{1}{2}\right)^2-\frac{3}{2}}}=\int\frac{1}{2\sqrt{5}}\frac{\operatorname{sh}u du}{\frac{1}{2}\operatorname{sh}u}=\frac{1}{\sqrt{5}}u+C=\frac{1}{\sqrt{5}}\ln\left(2\sqrt{5}\left(t+\frac{1}{2}\right)+\sqrt{20\left(t+\frac{1}{2}\right)^2-1}\right)$$

$$2\sqrt{5}\left(t+\frac{1}{2}\right)+\sqrt{20\left(t+\frac{1}{2}\right)^2-1}=2\sqrt{5}\frac{x+1}{2(x-1)}+\sqrt{\frac{5(x^2+2x+1)}{(x-1)^2}-1}=$$

$$=\sqrt{5}\frac{x+1}{x-1}+\sqrt{\frac{5(x^2+2x+1)-x^2+2x-1}{(x-1)^2}}=\sqrt{5}\frac{x+1}{x-1}+\frac{2}{x-1}\sqrt{x^2+3x+1}$$

$$\sqrt{5t^2+5t+1}=\frac{\sqrt{x^2+3x+1}}{(x-1)}\Rightarrow(At+B)\sqrt{5t^2+5t+1}=\left(-\frac{1}{10(x-1)}+\frac{3}{20}\right)\cdot\frac{\sqrt{x^2+3x+1}}{(x-1)}=$$

$$= \frac{3x-3-2}{20(x-1)^2} \cdot \sqrt{x^2+3x+1} = \frac{3x-5}{20(x-1)^2} \cdot \sqrt{x^2+3x+1}$$

Следовательно:

$$\int \frac{1}{(x-1)^3\sqrt{x^2+3x+1}}dx = \frac{3x-5}{20(x-1)^2} \cdot \sqrt{x^2+3x+1} - \frac{11}{40\sqrt{5}} \ln\left|\frac{\sqrt{5}(x+1)+2\sqrt{x^2+3x+1}}{x-1}\right| + C$$

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