CHAPTER 7

STABILITY, INERTIA, AND ROBUST STABILITY

Topics covered

- Lyapunov Stability Theory for Continuous-Time and Discrete-Time Systems
- Controllability and Observability Grammians via Lyapunov Equations
- Theory and Computation of the Inertia
- Distance to Instability
- Robust Stability
- Stability Radius

7.1 INTRODUCTION

This chapter deals with **stability of a linear time-invariant system** and the associated aspects such as the **inertia of a matrix, distance from an unstable system, robust stability, and stability radius** and computing the **H_2-norm of a stable transfer function**. A classical approach to determine the stability and inertia is to solve a **Lyapunov equation** or to find the characteristic polynomial of the state matrix A followed by application of the **Routh–Hurwitz criterion** in the continuous-time case and the **Schur–Cohn** criteria in the discrete-time case. These approaches are historically important and were developed at a time when numerically finding the eigenvalues of a matrix, even of a modest order, was a difficult problem. However, nowadays, with the availability of the QR iteration method for eigenvalue computation which is reliable, accurate, and fast, these approaches for stability and inertia, seem to have very little practical value. Furthermore, the Lyapunov equation approach is counterproductive in a practical computational setting in the sense that the most numerically viable method, currently available for solution of the Lyapunov equation, namely, the **Schur method (described in Chapter 8)**, is based on transformation of the matrix A to a real Schur form (RSF) and the latter either explicitly displays the eigenvalues of A or the eigenvalues can be trivially found once A is transformed into this form. Also, as mentioned before, **finding the characteristic polynomial of a matrix, in general, is a numerically unstable process.** In view of the above statements, it is clear that the best way to numerically check the stability and inertia is to explicitly compute all the eigenvalues. However, by computing the eigenvalues, one gets more than stability and inertia. Furthermore, if the eigenvalues of A are very ill-conditioned, determining the stability and inertia using eigenvalues may be misleading (**see Section 7.6**). The question, therefore, arises if an approach can be developed that does not require explicit computation of the eigenvalues of the state matrix A nor solution of a Lyapunov equation. Such an implicit method (**Algorithm 7.5.1**) is developed in **Section 7.5**. **This method is about three times faster than the eigenvalue method** and, **of course, many times faster than solving Lyapunov equation in a numerically effective way using the Schur method.**

An important practical problem "**How nearly unstable is a stable system** (or equivalently a stable matrix)?" is discussed in **Section 7.6**. A simple **bisection algorithm** (**Algorithm 7.6.1**) due to Byers (1988) to measure the distance of a stable matrix A from a set of unstable matrices is provided. A brief discussion of **robust stability** is the topic of **Section 7.7**.

The concept of **stability radius** in the context of robust stability is introduced in **Section 7.8** and a recent important formula for real stability radius due to Qiu *et al.* (1995) is stated. This concept will again be revisited in **Chapter 10**, where a connection of the complex stability radius with an algebraic Riccati equation (ARE) will be made.

The relationships between the controllability and observability Grammians and the H_2-norm of an asymptotically stable system with Lyapunov equations are discussed in **Sections 7.2.3, 7.2.4, and 7.3**, and a computational algorithm (**Algorithm 7.2.1**) for computing the H_2-norm of a stable continuous-time system is described in **Section 7.2.4**.

Reader's Guide to Chapter 7

Readers familiar with the basic concepts of stability and Lyapunov stability theory can skip Sections 7.2 and 7.3.

7.2 STABILITY OF A CONTINUOUS-TIME SYSTEM

The stability of a system is defined with respect to an equilibrium state.

Definition 7.2.1. *An* **equilibrium state** *of the unforced system*

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{7.2.1}$$

is the vector x_e satisfying

$$Ax_e = 0.$$

Clearly, $x_e = 0$ is an equilibrium state and it is the unique equilibrium state if and only if A is nonsingular.

Definition 7.2.2. *An equilibrium state x_e is* **asymptotically stable** *if for any initial state, the state vector $x(t)$ approaches x_e as time increases.*

The system (7.2.1) is asymptotically stable if and only if the equilibrium state $x_e = 0$ is asymptotically stable. Thus, the system (7.2.1) is asymptotically stable if and only if $x(t) \to 0$ as $t \to \infty$.

7.2.1 Eigenvalue Criterion of Continuous-Time Stability

Below we state a well-known criterion of asymptotic stability of a continuous-time system.

Theorem 7.2.1. *The system* (7.2.1) *is asymptotically stable if and only if all the eigenvalues of the matrix A have negative real parts.*

Proof. From Chapter 5, we know that the general solution of (7.2.1) is

$$x(t) = e^{At}x_0.$$

Thus, $x(t) \to 0$ if and only if $e^{At} \to 0$ as $t \to \infty$. We will now show that this happens if and only if all the eigenvalues of A have negative real parts.

Let $X^{-1}AX = \operatorname{diag}(J_1, J_2, \ldots, J_k)$ be the Jordan canonical form (JCF) of A. Then,

$$e^{At} = X \operatorname{diag}(e^{J_1 t}, e^{J_2 t}, \ldots, e^{J_k t})X^{-1}.$$

Let λ_i be the eigenvalue of A associated with J_i. Then $e^{J_i t} \to 0$ if and only if λ_i has a negative real part. Therefore, $e^{At} \to 0$ if and only if all the eigenvalues of A have negative real parts. ■

Definition 7.2.3. *A matrix A is called a* **stable matrix** *if all of the eigenvalues of A have negative real parts.*

A stable matrix is also known as a **Hurwitz** matrix in control literature. In analogy, an eigenvalue with negative real part is called a **stable eigenvalue**.

Since the asymptotic stability of (7.2.1) implies that its zero-input response approaches zero exponentially, the asymptotic stability is also referred to as **exponential stability.**

Definition 7.2.4. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then the distance* $\min\{-Re(\lambda_i)\colon i = 1, \ldots, n\}$ *to the imaginary axis is called the* **stability margin** *of A.*

In this book, the "stability" of a system means "**asymptotic stability**," and the associated matrix A will be referred to as "**stable matrix**," **not** asymptotically stable matrix.

Bounded-Input Bounded-Output Stability

The continuous-time linear system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned} \tag{7.2.2}$$

is said to be **bounded-input bounded-output** (BIBO) **stable** if for any bounded input, the output is also bounded.

The BIBO stability is governed by the poles of the transfer function $G(s) = C(sI - A)^{-1}B$. Specifically, the following result can be proved: (**Exercise 7.5**).

Theorem 7.2.2. *The system* (7.2.2) *is BIBO stable if and only if every pole of* $G(s)$ *has a negative real part.*
Since every pole of $G(s)$ *is also an eigenvalue of* A, **an asymptotically stable system is also BIBO stable.** *However,* **the converse is not true.** *The following simple example illustrates this.*

Example 7.2.1.

$$\dot{x} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}x + \begin{pmatrix}0\\1\end{pmatrix}u, \qquad y = (1, 1)x.$$

$$G(s) = C(sI - A)^{-1}B = (1, 1)\begin{pmatrix}s-1 & 0\\ 0 & s+1\end{pmatrix}^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{s+1}.$$

Thus, the system is BIBO (note that the pole of $G(s)$ is -1), but not asymptotically stable.

Bounded-Input Bounded-State (BIBS) Stability

Definition 7.2.5. *The system* (7.2.2) *is* ***BIBS*** *stable if, for any bounded input, the state response is also bounded.*

The following characterization of BIBS can be given in terms of eigenvalues of A and the controllability of the modes. For a proof of Theorem 7.2.2′, see DeCarlo (1989, pp. 416–417).

Theorem 7.2.2′. *BIBS. The system* (7.2.2) *is BIBS stable if and only if*

(i) *All the eigenvalues of A have nonnegative real parts.*

(ii) *If an eigenvalue λ_i has a zero real part, then the order of the associated factor in the minimal polynomial of A must be 1.*

(iii) *The mode associated with an eigenvalue with zero real part must be uncontrollable.*

7.2.2 Continuous-Time Lyapunov Stability Theory

In this section, we present the historical Lyapunov criterion of stability. Before the advent of computers, finding the eigenvalues of a matrix A was an extremely difficult task. The early research on stability, therefore, was directed toward finding the criteria that do not require explicit computation of the eigenvalues of a matrix. In 1892, the Russian mathematician A. Lyapunov (1857–1918) developed a historical stability criterion for nonlinear systems of equations. In the linear case, this criterion may be formulated in terms of the solution of a matrix equation.

Theorem 7.2.3. *Lyapunov Stability Theorem. The system:*

$$\dot{x}(t) = Ax(t),$$

is asymptotically stable if and only if, for any symmetric positive definite matrix M, there exists a unique symmetric positive definite matrix X satisfying the equation:

$$XA + A^{\mathrm{T}}X = -M. \tag{7.2.3}$$

Proof. Let's define a matrix X by

$$X = \int_0^\infty e^{A^{\mathrm{T}}t} M e^{At} dt. \tag{7.2.4}$$

Then, we show that when the system is asymptotic stable, X is a unique solution of the equation (7.2.3) and is symmetric positive definite.

Using the expression of X in (7.2.3), we obtain

$$\begin{aligned} XA + A^{\mathrm{T}}X &= \int_0^\infty e^{A^{\mathrm{T}}t} M e^{At} A\, dt + \int_0^\infty A^{\mathrm{T}} e^{A^{\mathrm{T}}t} M e^{At} dt \\ &= \int_0^\infty \frac{d}{dt}(e^{A^{\mathrm{T}}t} M e^{At})\, dt = \left[e^{A^{\mathrm{T}}t} M e^{At}\right]_0^\infty \end{aligned}$$

Since A is stable, $e^{A^{\mathrm{T}}t} \to 0$ as $t \to \infty$. Thus, $XA + A^{\mathrm{T}}X = -M$, showing that X defined by (7.2.4) satisfies the Eq. (7.2.3).

To show that X is positive definite, we have to show that $u^{\mathrm{T}}Xu > 0$ for any nonzero vector u. Using (7.2.4) we can write

$$u^{\mathrm{T}}Xu = \int_0^{\infty} u^{\mathrm{T}}e^{A^{\mathrm{T}}t}Me^{At}\, u\, dt.$$

Since the exponential matrices $e^{A^{\mathrm{T}}t}$ and e^{At} are both nonsingular and M is positive definite, we conclude that $u^{\mathrm{T}}Xu > 0$.

To prove that X is unique, assume that there are two solutions X_1 and X_2 of (7.2.3). Then,

$$A^{\mathrm{T}}(X_1 - X_2) + (X_1 - X_2)A = 0,$$

which implies that

$$e^{A^{\mathrm{T}}t}(A^{\mathrm{T}}(X_1 - X_2) + (X_1 - X_2)A)e^{At} = 0$$

or

$$\frac{d}{dt}\left[e^{A^{\mathrm{T}}t}(X_1 - X_2)e^{At}\right] = 0,$$

and hence $e^{A^{\mathrm{T}}t}(X_1 - X_2)e^{At}$ is a constant matrix for all t.

Evaluating at $t = 0$ and $t = \infty$ we conclude that $X_1 - X_2 = 0$.

We now prove the converse, that is, we prove that if X is a symmetric positive definite solution of the equation (7.2.3), then A is stable.

Let (λ, x) be an eigenpair of A. Then premultiplying the equation (7.2.3) by x^* and postmultiplying it by x, we obtain:

$$x^*XAx + x^*A^{\mathrm{T}}Xx = \lambda x^*Xx + \bar{\lambda}x^*Xx = (\lambda + \bar{\lambda})x^*Xx = -x^*Mx.$$

Since M and X are both symmetric positive definite, we have $\lambda + \bar{\lambda} < 0$ or $\mathrm{Re}(\lambda) < 0$. ■

Note: The matrix X defined by (7.2.4) satisfies the Eq. (7.2.3) even when M is not positive definite.

Definition 7.2.6. *The matrix equation:*

$$XA + A^{\mathrm{T}}X = -M$$

and its dual

$$AX + XA^{\mathrm{T}} = -M$$

are called the **Lyapunov equations.**

Remark (Lyapunov Function)

- The Lyapunov stability theory was originally developed by Lyapunov (Liapunov (1892)) in the context of stability of a nonlinear system. The stability of a nonlinear system is determined by **Lyapunov functions.** See Luenberger (1979) for details. For the linear system:

$$\dot{x}(t) = Ax(t),$$

the function $V(x) = x^{\mathrm{T}}Xx$, where X is symmetric is a **Lyapunov function** if the $\dot{V}(x)$, the derivative of $V(x)$, is negative definite. This fact yields an alternative proof of Theorem 7.2.3. This can be seen as follows:

$$\begin{aligned}\dot{V}(x) &= \dot{x}^{\mathrm{T}}Xx + x^{\mathrm{T}}X\dot{x}, \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}X + XA)\,x, \\ &= x^*(-M)x.\end{aligned}$$

Thus, $\dot{V}(x)$ is negative definite if and only if M is positive definite.

We note the following from the proof of Theorem 7.2.3.

Integral Representations of the Unique Solutions of Lyapunov Equations

Let A be a stable matrix and let M be symmetric, positive definite, or semidefinite. Then,

1. The unique solution X of the Lyapunov equation:

$$XA + A^{\mathrm{T}}X = -M$$

is given by

$$X = \int_0^\infty e^{A^{\mathrm{T}}t}Me^{At}\,dt. \tag{7.2.5}$$

2. The unique solution X of the Lyapunov equation

$$AX + XA^{\mathrm{T}} = -M$$

is given by

$$X = \int_0^\infty e^{At}Me^{A^{\mathrm{T}}t}\,dt. \tag{7.2.6}$$

As we will see later, the Lyapunov equations also arise in many other important control theoretic applications. In many of these applications, the right-hand side matrix M is positive semi-definite, rather than positive definite. The typical examples are $M = BB^{\mathrm{T}}$ or $M = C^{\mathrm{T}}C$, where B and C

are, respectively, the input and output matrices. The Lyapunov equations of the above types arise in finding **Grammians** of a stable system (see **Section 7.2.3**).

Theorem 7.2.4. *Let A be a stable matrix. Then the Lyapunov equation:*

$$XA + A^{\mathrm{T}}X = -C^{\mathrm{T}}C \tag{7.2.7}$$

has a unique symmetric positive definite solution X if and only if (A, C) is observable.

Proof. We first show that the observability of (A, C) and stability of A imply that X is positive definite.

Since A is stable, by (7.2.5) the unique solution X of the equation (7.2.7) is given by

$$X = \int_0^{\infty} e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} dt.$$

If X is not positive definite, then there exists a nonzero vector x such that $Xx = 0$. In that case

$$\int_0^{\infty} \|Ce^{At}x\|^2 \, dt = 0;$$

this means that $Ce^{At}x = 0$. Evaluating $Ce^{At}x = 0$ and its successive derivatives at $t = 0$, we obtain $CA^i x = 0, i = 0, 1, \ldots, n-1$. This gives $O_{\mathrm{M}}x = 0$, where O_{M} is the observability matrix. Since (C, A) is observable, O_{M} has full rank, and this implies that $x = 0$, which is a contradiction.

Hence $Ce^{At}x \neq 0$, for every t. So, X is positive definite.

Next, we prove the converse. That is, we prove that the stability of A and definiteness of X imply that (A, C) is observable. The proof is again by contradiction.

Suppose (A, C) is not observable. Then, according to criterion (v) of Theorem 6.3.1, there is an eigenvector x of A such that $Cx = 0$. Let λ be the eigenvalue corresponding to the eigenvector x. Then from the equation:

$$XA + A^{\mathrm{T}}X = -C^{\mathrm{T}}C,$$

we have $x^*XAx + x^*A^{\mathrm{T}}Xx = -x^*C^{\mathrm{T}}Cx$ or $(\lambda + \bar{\lambda})x^*Xx = -\ \|Cx\|^2$.

So, $(\lambda + \bar{\lambda})x^*Xx = 0$.

Since A is stable, $\lambda + \bar{\lambda} < 0$. Thus,

$$x^*Xx = 0.$$

But X is positive definite, so x must be a zero vector, which is a contradiction. ■

We next prove a necessary and sufficient condition of stability assuming that (A, C) is observable.

Theorem 7.2.5. *Let (A, C) be observable. Then A is stable if and only if there exists a unique symmetric positive definite solution matrix X satisfying the Lyapunov equation* (7.2.7).

Proof. We have already proved the theorem in one direction, that is, we have proved that if A is stable and (A, C) is observable, then the Lyapunov equation (7.2.7) has a unique symmetric positive definite solution X given by:

$$X = \int_0^{\infty} e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} dt.$$

We now prove the other direction. Let (λ, x) be an eigenpair of A. Then as before we have

$$(\lambda + \bar{\lambda}) x^* X x = - \|Cx\|^2 .$$

Since (A, C) is observable, $Cx \neq 0$, and since X is positive definite, $x^* X x > 0$. Hence $\lambda + \bar{\lambda} < 0$, which means that A is stable. ■

For the sake of convenience, we combine the results of Theorems 7.2.4 and 7.2.5 in Theorem 7.2.6.

In the rest of this chapter, for notational convenience, a symmetric positive definite (positive semidefinite) matrix X will be denoted by the symbol $X > 0$ (≥ 0).

Theorem 7.2.6. *Let X be a solution of the Lyapunov equation* (7.2.7). *Then the followings hold:*

(i) *If $X > 0$ and (A, C) is observable, then A is a stable matrix.*
(ii) *If A is a stable matrix and (A, C) is observable, then $X > 0$.*
(iii) *If A is a stable matrix and $X > 0$, then (A, C) is observable.*

Since observability is a dual concept of controllability, the following results can be immediately proved by duality of Theorems 7.2.4 and 7.2.5.

Theorem 7.2.7. *Let A be a stable matrix. Then the Lyapunov equation:*

$$AX + XA^{\mathrm{T}} = -BB^{\mathrm{T}} \tag{7.2.8}$$

has a unique symmetric positive definite solution X if and only if (A, B) is controllable.

Theorem 7.2.8. *Let (A, B) be controllable. Then A is stable if and only if there exists a unique symmetric positive definite X satisfying the Lyapunov equation* (7.2.8).

Theorems 7.2.7 and 7.2.8 can also be combined, in a similar manner, as in Theorem 7.2.6, to obtain the following:

Theorem 7.2.9. *Let X be a solution of the Lyapunov equation* (7.2.8). *Then the followings hold:*

(i) *If $X > 0$ and (A, B) is controllable, then A is a stable matrix.*
(ii) *If A is a stable matrix and (A, B) is controllable, then $X > 0$.*
(iii) *If A is a stable matrix and $X > 0$, then (A, B) is controllable.*

7.2.3 Lyapunov Equations and Controllability and Observability Grammians

Definition 7.2.7. *Let A be a stable matrix. Then the matrix:*

$$C_{\mathrm{G}} = \int_0^{\infty} e^{At} B B^{\mathrm{T}} e^{A^{\mathrm{T}} t} dt \tag{7.2.9}$$

*is called the **controllability Grammian**, and the matrix:*

$$O_{\mathrm{G}} = \int_0^{\infty} e^{A^{\mathrm{T}} t} C^{\mathrm{T}} C e^{At} dt \tag{7.2.10}$$

*is called the **observability Grammian**.*

In view of these definitions, Theorems 7.2.7 and 7.2.4 can be, respectively, restated as follows.

Theorem 7.2.10. *Controllability Grammian and the Lyapunov Equation. Let A be a stable matrix. Then the controllability Grammian C_{G} satisfies the Lyapunov equation*

$$AC_{\mathrm{G}} + C_{\mathrm{G}}A^{\mathrm{T}} = -BB^{\mathrm{T}} \tag{7.2.11}$$

and is symmetric positive definite if and only if (A, B) is controllable.

Theorem 7.2.11. *Observability Grammian and the Lyapunov Equation. Let A be a stable matrix. Then the observability Grammian O_{G} satisfies the Lyapunov equation*

$$O_{\mathrm{G}}A + A^{\mathrm{T}}O_{\mathrm{G}} = -C^{\mathrm{T}}C \tag{7.2.12}$$

and is symmetric positive definite if and only if (A, C) is observable.

Example 7.2.2. Let

$$A = \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The controllability Grammian C_G obtained by solving the Lyapunov equation (using MATLAB command **lyap**) $AX + XA^T = -BB^T$ is

$$C_G = \begin{pmatrix} 0.2917 & 0.0417 & 0.0417 \\ 0.0417 & 0.1667 & 0.1667 \\ 0.0417 & 0.1667 & 0.1667 \end{pmatrix},$$

which is clearly singular. So, (A, B) is not controllable.

Verify: The singular values of the controllability matrix C_M are 25.6766, 0.8425, and 0.

7.2.4 Lyapunov Equations and the H_2-Norm

In this section, we show how the H_2-norm of the transfer matrix of an asymptotically stable continuous-time system can be computed using Lyapunov equations.

Definition 7.2.8. *The H_2-norm of the transfer matrix $G(s)$ of an asymptotically stable continuous-time system:*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{7.2.13}$$

denoted by $\|G\|_2$, is defined by

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(G(j\omega)^* G(j\omega))\, d\omega \right)^{1/2}. \tag{7.2.14}$$

Thus, **the H_2-norm measures the steady-state covariance of the output response $y = Gv$ to the white noise inputs v.**

Computing the H_2-Norm

By Parseval's theorem in complex analysis (Rudin 1966, p. 191), (7.2.14) can be written as

$$\|G(s)\|_2 = \left(\int_0^{\infty} \text{Trace}\ (h^T(t) h(t))\, dt \right)^{1/2},$$

where $h(t)$ is the impulse response matrix:

$$h(t) = Ce^{At}B.$$

Thus,

$$\|G\|_2^2 = \text{Trace}\left(B^{\mathrm{T}}\left(\int_0^{\infty} e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}dt\right)B\right),$$
$$= \text{Trace}(B^{\mathrm{T}}O_{\mathrm{G}}B),$$

where O_{G} is the observability Grammian given by (7.2.10).

Similarly, we can show that

$$\|G\|_2^2 = \text{Trace}(CC_{\mathrm{G}}C^{\mathrm{T}}), \tag{7.2.15}$$

where C_{G} is the controllability Grammian given by (7.2.9).

Since A is stable, the controllability and observability Grammians satisfy, respectively, the Lyapunov equations (7.2.11) and (7.2.12).

Thus, a straightforward method for computing the H_2-norm is as follows:

Algorithm 7.2.1. *Computing the H_2-Norm*

Input. *The system matrices A, B, and C.*

Output. *The H_2-norm of the system (A, B, C).*

Assumption. *A is stable.*

Step 1. *Solve the Lyapunov equation* (7.2.11) *or* (7.2.12)

Step 2. *Compute either* $\text{Trace}(CC_{\mathrm{G}}C^{\mathrm{T}})$ *or* $\text{Trace}(B^{\mathrm{T}}O_{\mathrm{G}}B)$, *depending upon which of the two Lyapunov equations is solved, and take the square-root of either of these two values as the value of the H_2-norm.*

Example 7.2.3.

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Step 1. The solution of the Lyapunov equation (7.2.11), C_{G}, is

$$C_{\mathrm{G}} = \begin{pmatrix} 9.1833 & 2.5667 & 1.0167 \\ 2.5667 & 1.0333 & 0.5333 \\ 1.0167 & 0.5333 & 0.3333 \end{pmatrix},$$

$$C' = CC_{\mathrm{G}}C^{\mathrm{T}} = \begin{pmatrix} 18.7833 & 18.7833 \\ 18.7833 & 18.7833 \end{pmatrix}.$$

The solution of the Lyapunov equations (7.2.12), O_G, is

$$O_G = \begin{pmatrix} 1 & 1.3333 & 1.9167 \\ 1.3333 & 1.8333 & 2.7000 \\ 1.9167 & 2.7000 & 4.0500 \end{pmatrix},$$

$$B' = B^T O_G B = \begin{pmatrix} 18.7833 & 18.7833 \\ 18.7833 & 18.7833 \end{pmatrix}.$$

Step 2. H_2-norm $= \sqrt{\text{Trace}(B')} = \sqrt{\text{Trace}(C')} = \sqrt{37.5667} = 6.1292$.

MATCONTROL note: Algorithm 7.2.1 has been implemented in MATCONTROL function **h2nrmcg** and **h2nrmog**.

MATLAB Note. MATLAB function **norm(sys)** computes the H_2-norm of a system.

7.3 STABILITY OF A DISCRETE-TIME SYSTEM

7.3.1 Stability of a Homogeneous Discrete-Time System

Consider the discrete-time system:

$$x_{k+1} = Ax_k \tag{7.3.1}$$

with initial value x_0.

A well-known mathematical criterion for asymptotic stability of the homogeneous discrete-time system now follows. The proof is analogous to that of Theorem 7.2.1 and can be found in Datta (1995).

Theorem 7.3.1. *The system* (7.3.1) *is asymptotically stable if and only if all the eigenvalues of A are inside the unit circle.*

Definition 7.3.1. *A matrix A having all its eigenvalues inside the unit circle is called a* **discrete-stable matrix,** *or a* ***convergent matrix*** *or a* ***Schur matrix.***

We shall use the terminology ***discrete-stable*** *throughout the book.*

Discrete-Time Lyapunov Stability Theory

Each of the theorems in Section 7.2 has a discrete counterpart. In the discrete case, the continuous-time Lyapunov equations $XA + A^TX = -M$ and $AX + XA^T = -M$ are, respectively, replaced by their discrete-analogs $X - A^TXA = M$ and $X - AXA^T = M$.

These discrete counterparts of the continuous-time Lyapunov equations are called the **Stein equations.** The Stein equations are also known as **discrete-Lyapunov equations** in control literature.

In the following, we state and prove a discrete analog of Theorem 7.2.3. The statements and proofs of the discrete counterparts of Theorems 7.2.4 through 7.2.9 are analogous. In fact, the Lyapunov and Stein equations are related via the matrix analogs of the well-known bilinear transformation (known as the **Cayley transformation**):

$$z = \frac{1+s}{1-s}, \qquad s = \frac{z-1}{z+1} \tag{7.3.2}$$

Note that $|z| < 1 \Leftrightarrow \mathrm{Re}(s) < 0$ and $|z| = 1 \Leftrightarrow \mathrm{Re}(s) = 0$.

Theorem 7.3.2. *Discrete-Time Lyapunov Stability Theorem. The discrete-time system* (7.3.1) *is asymptotically stable if and only if, for any positive definite matrix M, there exists a unique positive definite matrix X satisfying the discrete Lyapunov equation:*

$$X - A^{\mathrm{T}}XA = M. \tag{7.3.3}$$

Proof. We prove the theorem in one direction, that is, we prove that if A is discrete-stable, then Eq. (7.3.3) has a unique symmetric positive definite solution X. The proof of the other direction is left as an **Exercise (7.10)**.

Define the matrix

$$X = \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k M A^k. \tag{7.3.4}$$

Since A is discrete-stable, the infinite series on the right-hand side converges. Furthermore, the matrix X is symmetric and positive definite.

We now show that X is the unique solution of the Eq. (7.3.3). Indeed,

$$X - A^{\mathrm{T}}XA = \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k M A^k - \sum_{k=1}^{\infty}(A^{\mathrm{T}})^k M A^k = M. \tag{7.3.5}$$

Thus, X defined by (7.3.4) satisfies the Eq. (7.3.3).

To prove that X is unique, let's assume that there is another symmetric positive definite solution X_1 of (7.3.3).

Then,

$$X_1 - A^{\mathrm{T}}X_1A = M,$$

and

$$\begin{aligned} X &= \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k M A^k = \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k (X_1 - A^{\mathrm{T}}X_1A)A^k, \\ &= \sum_{k=0}^{\infty}(A^{\mathrm{T}})^k X_1 A^k - \sum_{k=1}^{\infty}(A^{\mathrm{T}})^k X_1 A^k = X_1. \quad \blacksquare \end{aligned}$$

Remark (BIBO and BIBS Stability of a Discrete-Time System)

- Results on BIBO stability and BIBS stability, the discrete counter parts of Theorem 7.2.2 and Theorem 7.2.2′ can be obtained, for the discrete-time system:

$$x_{k+1} = Ax_k + Bu_k.$$

See **Exercises 7.7** and **7.8** and the book by DeCarlo (1989).

Definition 7.3.2. *Let A be discrete-stable. Then the matrices:*

$$C_{\mathrm{G}}^{\mathrm{D}} = \sum_{k=0}^{\infty} A^k B B^{\mathrm{T}} (A^{\mathrm{T}})^k \tag{7.3.6}$$

and

$$O_{\mathrm{G}}^{\mathrm{D}} = \sum_{k=0}^{\infty} (A^{\mathrm{T}})^k C^{\mathrm{T}} C A^k \tag{7.3.7}$$

are, respectively, called the ***discrete-time controllability Grammian*** *and* **discrete-time observability Grammians**.

The discrete counterparts of Theorems 7.2.10 and 7.2.11 are:

Theorem 7.3.3. *Discrete-Time Controllability Grammian and Lyapunov Equation. Let A be discrete-stable. Then the discrete-time controllability Grammian $C_{\mathrm{G}}^{\mathrm{D}}$ satisfies the discrete Lyapunov equation*

$$C_{\mathrm{G}}^{\mathrm{D}} - AC_{\mathrm{G}}^{\mathrm{D}}A^{\mathrm{T}} = BB^{\mathrm{T}} \tag{7.3.8}$$

and is symmetric positive definite if and only if (A, B) is controllable.

Theorem 7.3.4. *Discrete-Time Observability Grammian and Lyapunov Equation. Let A be discrete-stable. Then the discrete-time observability Grammian $O_{\mathrm{G}}^{\mathrm{D}}$ satisfies the discrete Lyapunov equation:*

$$O_{\mathrm{G}}^{\mathrm{D}} - A^{\mathrm{T}}O_{\mathrm{G}}^{\mathrm{D}}A = C^{\mathrm{T}}C \tag{7.3.9}$$

and is symmetric positive definite if and only if (A, C) is observable.

7.4 SOME INERTIA THEOREMS

Certain design specifications require that the eigenvalues lie in a certain region of the complex plane. Thus, finding if a matrix is stable is not enough in many practical instances. We consider the following problem, known as the inertia problem, which is concerned with counting the number of eigenvalues in a given region."

Definition 7.4.1. *The inertia of a matrix A of order n, denoted by* $\mathrm{In}(A)$*, is the triplet* $(\pi(A), \nu(A), \delta(A))$*, where* $\pi(A)$, $\nu(A)$, *and* $\delta(A)$ *are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.*

Note that $\pi(A) + \nu(A) + \delta(A) = n$ and A **is a stable matrix if and only if** $\mathrm{In}(A) = (0, n, 0)$.

The inertia, as defined above, is the **half-plane** or the **continuous-time inertia.**

The inertia with respect to the other regions of the complex plane can be defined similarly.

The **discrete-time inertia** or the **unit-circle inertia** is defined by the triplet $(\pi_O(A), \nu_O(A), \delta_O(A))$, where $\pi_O(A), \nu_O(A), \delta_O(A)$, are, respectively the number of eigenvalues of A outside, inside, and on the unit circle. It will be denoted by $\mathrm{In}_O(A)$.

Unless otherwise stated, by "inertia" we will mean the "half-plane inertia."

Much work has been done on the inertia theory of matrices. We will just give here a glimpse of the existing inertia theory and then present a computational algorithm for computing the inertia. For details, we refer the curious readers to the recent survey paper of the author (Datta 1999). This paper gives an overview of the state-of-the-art theory and applications of matrix inertia and stability. The applications include new matrix theoretic proofs of several classical stability tests, applications to D-stability and to continued functions, etc. (Datta 1978a, 1978b, 1979, 1980). For other control theoretic applications of the inertia of a matrix, see Glover (1984), and the book by Zhou *et al.* (1996).

7.4.1 The Sylvester Law of Inertia

A classical law on the inertia of a symmetric matrix A is the **Sylvester Law of Inertia,** stated as follows:

Let A be a symmetric matrix and P be a nonsingular matrix. Then,

$$\mathrm{In}(A) = \mathrm{In}(PAP^{\mathrm{T}}).$$

Proof. See Horn and Johnson (1985, pp. 223–229).

Computing the Inertia of a Symmetric Matrix

If A is symmetric, then Sylvester's law of inertia provides an inexpensive and numerically effective method for computing its inertia.

A symmetric matrix A admits a triangular factorization:

$$A = UDU^{\mathrm{T}},$$

where U is a product of elementary unit upper triangular and permutation matrices, and D is a symmetric block diagonal with blocks of order 1 or 2. This is known as **diagonal pivoting factorization**. Thus, by Sylvester's law of inertia $\mathrm{In}(A) = \mathrm{In}(D)$. Once this diagonal pivoting factorization is obtained, the inertia of the symmetric matrix A can be obtained from the entries of D as follows:

Let D have p blocks of order 1 and q blocks of order 2, with $p + 2q = n$. Assume that none of the 2×2 blocks of D is singular. Suppose that out of p blocks of order 1, p' of them are positive, p'' of them are negative, and p''' of them are zero (i.e., $p' + p'' + p''' = p$). Then,

$$\begin{aligned}
\pi(A) &= p' + q, \\
\nu(A) &= p'' + q, \\
\delta(A) &= p'''.
\end{aligned}$$

The **diagonal pivoting factorization can be achieved in a numerically stable way**. It requires only $n^3/3$ flops. For details of the diagonal pivoting factorization, see Bunch (1971), Bunch and Parlett (1971), and Bunch and Kaufman (1977).

LAPACK implementation: The diagonal pivoting method has been implemented in the LAPACK routine **SSYTRF**.

7.4.2 The Lyapunov Inertia Theorems

The Sylvester Law of Inertia and the matrix formulation of the Lyapunov criterion of stability seem to have made a significant impact on the development of nonsymmetric inertia theorems. Many inertia theorems for nonsymmetric matrices have been developed over the years. These theorems attempt to find a symmetric matrix X, given a nonsymmetric matrix A, as a solution of a certain matrix equation, in such a way that, under certain conditions, the inertia of the nonsymmetric matrix A becomes equal to the inertia of the symmetric matrix X. Once the symmetric matrix X is obtained, its inertia can be computed rather cheaply by application of the Sylvester Law of Inertia to the LDL^{T} decomposition of X.

Theorem 7.4.1 is the **Fundamental Theorem** on the inertia of a nonsymmetric matrix and is known as the **Main Inertia Theorem (MIT)** (Taussky (1961) , and Ostrowski and Schneider (1962)). This theroem is also known as **Ostrowski-Schneider-Taussky** (OST) **Theorem**.

Theorem 7.4.1. *The Main Inertia Theorem. (i) There exists a unique symmetric matrix X such that*

$$XA + A^{\mathrm{T}}X = M > 0 \tag{7.4.1}$$

if and only if $\delta(A) = 0$.
(ii) Whenever Eq. (7.4.1) *has a symmetric solution X,* $\mathrm{In}(A) = \mathrm{In}(X)$.

Recovery of the Lyapunov Stability Theorem

As an immediate corollary of Theorem 7.4.1, we obtain the following.

Corollary 7.4.1. *A necessary and sufficient condition for A to be stable is that there exists a symmetric positive definite matrix X such that*

$$XA + A^{\mathrm{T}}X = -M, \quad M > 0.$$

The Lyapunov Stability Theorem (**Theorem 7.2.3**) now follows from Corollary 7.4.1 by noting the fact that the Lyapunov equation for any given positive definite matrix M, has a unique solution if and only if $\Delta(A) = \prod_{i,j=1}^{n}(\lambda_i + \lambda_j) \neq 0$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, and $\Delta(A) \neq 0$ implies that $\delta(A) = 0$ (see Chapter 8).

Theorem 7.4.2. *Continuous-Time Semidefinite Inertia Theorem. Assume that $\delta(A) = 0$ and let X be a nonsingular symmetric matrix such that*

$$XA + A^{\mathrm{T}}X = M \geq 0.$$

Then $\mathrm{In}(A) = \mathrm{In}(X)$.

Remarks

- Theorem 7.4.2 is due to Carlson and Schneider (1963).
- For a discrete version of Theorem 7.4.1, see Wimmer (1973), and Taussky (1964).
- For a discrete version of Theorem 7.4.2, see Datta (1980).
- The condition $\delta(A) = 0$ in Theorem 7.4.2 can be shown to be equivalent to the controllability of the pair (A^{T}, M); see Chen (1973) and Wimmer (1974). For discrete analogue, see Wimmer and Ziebur (1975).

7.5 DETERMINING THE STABILITY AND INERTIA OF A NONSYMMETRIC MATRIX

From our discussions in the two previous sections, it is clear that the stability and inertia of a nonsymmetric matrix can be determined by solving an appropriate Lyapunov equation.

Unfortunately **this is computationally a counterproductive approach.** The reason is that the most numerically effective (and widely used) method for solving a Lyapunov equation, the Schur method **(see Chapter 8)**, is based on reduction of the matrix A to the RSF. The RSF either displays the eigenvalues of A or can be trivially obtained from there. Of course, once the eigenvalues are computed, the stability and inertia are immediately known.

An alternative classical approach (see Marden 1966) is to compute the characteristic polynomial of A, followed by application of the **Routh–Hurwitz criterion** in the continuous-time case and the **Schur–Cohn Criterion** in the discrete-time case. **This is, unfortunately, also not a numerically viable approach.** The reasons are that: (i) computing the characteristic polynomial may be a highly numerically unstable process and (ii) the coefficients of a polynomial may be extremely sensitive to small perturbations. See our discussions in Chapter 4 **(Section 4.1)**.

In view of the above considerations, the numerical analysts believe that the most numerically effective way to compute the inertia and stability of a matrix A is to explicitly compute the eigenvalues of A. However, by explicitly computing the eigenvalues of A, one gets more than what is needed, and furthermore, since the eigenvalues of a matrix can be sensitive to small perturbations, computing the inertia and stability this way may be quite misleading sometimes (see the example in *Section* 7.6 *which shows that a perfectly stable matrix may become unstable by a very small perturbation of a single entry of the matrix).*

It is, therefore, of interest to develop a method for inertia and stability that does not require solution of a Lyapunov equation, or explicit computation of the characteristic polynomial or the eigenvalues of A. We will now describe such a method.

Algorithm 7.5.1 is based on the **implicit solution** of a matrix equation. The algorithm constructs a symmetric matrix F which satisfies a Lyapunov matrix equation with a positive semidefinite matrix on the right-hand side, but **the Lyapunov matrix equation is not explicitly solved.** The algorithm was developed by Carlson and Datta (1979b).

Algorithm 7.5.1. *An Implicit Matrix Equation Method for Inertia and Stability*

Input. *An $n \times n$ real matrix A*

Output. *The inertia of A.*

Step 1. *Transform A to a lower Hessenberg matrix H using an orthogonal similarity. Assume that H is unreduced* (**Chapter 4**).

Step 2. *Construct a nonsingular lower triangular matrix L such that*

$$LH + HL = R = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

is a matrix whose first $(n-1)$ *rows are zero, starting with the first row* l_1 *of* L *as* $l_1 = (1, 0, \dots, 0)$.

Step 3. *Having constructed* L, *compute the last row* r *of* R.

Step 4. *Construct now a matrix* S *such that*

$$SH = H^{\mathrm{T}}S,$$

with the last row s_n *of* S *as the last row* r *of* R.

Step 5. *Compute* $F = L^{\mathrm{T}}S$.

Step 6. *If* F *is nonsingular, compute the inertia of the symmetric matrix* F, *using the Sylvester law of inertia, as described in Section* 7.4.1.

Step 7. *Obtain the inertia of* A: $\mathrm{In}(A) = \mathrm{In}(F)$.

Theorem 7.5.1. (i) *If* F *is nonsingular, then it is symmetric and* $\mathrm{In}(A) = \mathrm{In}(F)$. (ii) A *is stable if and only if* F *is negative definite.*

Proof. Proof of Part (i).

$$\begin{aligned} FH + H^{\mathrm{T}}F &= L^{\mathrm{T}}SH + H^{\mathrm{T}}L^{\mathrm{T}}S = L^{\mathrm{T}}H^{\mathrm{T}}S + H^{\mathrm{T}}L^{\mathrm{T}}S, \\ &= (L^{\mathrm{T}}H^{\mathrm{T}} + H^{\mathrm{T}}L^{\mathrm{T}})S = R^{\mathrm{T}}S = r^{\mathrm{T}}r \geq 0. \end{aligned}$$

The nonsingularity of F implies the nonsingularity of S, and it can be shown (see Datta and Datta (1987)) that S is nonsingular if and only if H and $-H$ do not have a common eigenvalue. Thus, F is a unique solution of the matrix equation (see Theorem 8.2.1):

$$FH + H^{\mathrm{T}}F = r^{\mathrm{T}}r \geq 0,$$

and is, therefore, necessarily symmetric. Furthermore, since H and $-H$ do not have a common eigenvalue, we have $\delta(H) = 0$. Theorem 7.4.2 now can be applied to the above matrix equation to obtain Part (i) of Theorem 7.5.1.

Proof of Part (ii). First suppose that A is stable, then we prove that F is negative definite. Since A is stable, so is H, and therefore, $\delta(H) = 0$. Again $\delta(H) = 0$ implies that H and $-H$ do not have an eigenvalue in common. Therefore, by Theorem 8.2.1 (see Chapter 8), the Lyapunov equation:

$$FH + H^{\mathrm{T}}F = r^{\mathrm{T}}r \geq 0$$

has a unique solution F and therefore, must be symmetric F. By Theorem 7.4.2, we then have

$$\mathrm{In}(F) = \mathrm{In}(A) = (0, n, 0).$$

Thus, F is negative definite. Conversely, let F be negative definite. Then F is nonsingular. By part (i), we then have that $\mathrm{In}(A) = \mathrm{In}(F) = (0, n, 0)$. So, A is stable. ∎

Computational remarks

- *Computation of L.* Once the first row of $L = (l_{ij})$ in step 2 is prescribed, the diagonal entries of L are immediately known. These are: $1, -1, 1, \ldots, (-1)^{n-1}$. Having known these diagonal entries, the $n(n-1)/2$ off-diagonal entries l_{ij} $(i > j)$ of L lying below the main diagonal can now be uniquely determined by solving a lower triangular system if these entries are computed in the following order: $l_{21}; l_{31}, l_{32}; \ldots, l_{n1}, l_{n2}, \ldots, l_{n,n-1}$.
- *Computation of S.* Similar remarks hold for computing S in Step 4. Knowing the last row of the matrix S, the rows s_{n-1} through s_1 of S can be computed directly from the relation $SH = H^{\mathrm{T}}S$.

Notes

1. The above algorithm has been modified and made more efficient by *Datta and Datta* (1987). The modified algorithm uses the matrix-adaptation of the well-known Hyman method for computing the characteristic polynomial of a Hessenberg matrix (see *Wilkinson* 1965), which is numerically effective with proper scaling.
2. The algorithm has been extended by *Datta and Datta (1986)* to obtain information on the number of eigenvalues of a matrix in several other regions of the complex plane including strips, ellipses, and parabolas.
3. A method of this type for finding distribution of eigenvalues of a matrix with respect to the unit circle has been reported by *L.Z. Lu* (an unpublished manuscript (1987)).
4. A comparison of various methods for inertia computation, and a computationally more effective version of the algorithm reported in this section appeared in the M.Sc. Thesis of *Daniel Pierce* (1983).

Flop-count of Algorithm 7.5.1 and comparisons with other methods: Algorithm 7.5.1 requires about n^3 flops once the matrix A has been transformed to the lower Hessenberg matrix H. Since it requires $\frac{10}{3}n^3$ flops to transform A to H, a total of about $\frac{13}{3}n^3$ flops is needed to determine the inertia and stability of A using Algorithm 7.5.1. This count compares very favorable with about $12n^3$ flops needed to compute the eigenvalues of A using the QR iteration algorithm described in Chapter 4. **Thus, Algorithm 7.5.1 seems to be about three times faster than the eigenvalue method.**

We have not included the Lyapunov equation approach and the characteristic polynomial approach in our comparisons here because of the numerical difficulties with the characteristic polynomial approach and the counterproductivity of the Lyapunov equation approach, as mentioned in the beginning of this section.

Example 7.5.1. We compute $\mathrm{In}(A)$ using Algorithm 7.5.1 with

$$A = \begin{pmatrix} 1.997 & -0.724 & 0.804 & -1.244 & -1.365 & -2.014 \\ 0.748 & 2.217 & -0.305 & 1.002 & -2.491 & -0.660 \\ -1.133 & -1.225 & -0.395 & -0.620 & 1.504 & 1.498 \\ -0.350 & 0.515 & -0.063 & 2.564 & 0.627 & 0.422 \\ -0.057 & -0.631 & 1.544 & 0.001 & 1.074 & -1.750 \\ -1.425 & -0.788 & 1.470 & -1.515 & 0.552 & -0.036 \end{pmatrix}.$$

Step 1. Reduction to Lower Hessenberg form:

$$H = \begin{pmatrix} 1.9970 & 2.9390 & & & & \\ 0.6570 & -1.0007 & 1.9519 & & & \\ 0.4272 & 1.5242 & 0.4502 & 0.8785 & & \\ -0.1321 & 1.2962 & 0.9555 & 1.4541 & 0.4940 & \\ -0.0391 & -1.5738 & 0.6601 & 0.2377 & 2.3530 & -0.4801 \\ -1.8348 & -0.5976 & 0.7595 & 0.1120 & -3.3993 & 2.1673 \end{pmatrix}.$$

Step 2. Construction of the lower triangular L such that $LH + HL = R$:

$$L = \begin{pmatrix} 1 & & & & & \\ -1.3590 & 1 & & & & \\ 0.6937 & 1.0209 & 1 & & & \\ -1.3105 & -1.6810 & -3.2933 & 1 & & \\ 16.4617 & 22.7729 & 19.3373 & 11.7433 & 1 & \\ 198.8687 & 258.5635 & 229.9657 & 128.4966 & 21.8842 & 1 \end{pmatrix}.$$

Step 3. Last row of the matrix R is

$$r = (1023.6330,\ 1293.0942,\ 1177.7393,\ 632.4162,\ 162.4031, -14.8420).$$

Step 4. Construction of S such that $SH = H^{\mathrm{T}}S$:

$$S = \begin{pmatrix} 2.1404 & 3.3084 & 2.7775 & 1.3224 & -1.0912 & 0.4808 \\ 3.3084 & 5.0426 & 4.1691 & 2.0997 & -1.2521 & 0.6073 \\ 2.7775 & 4.1691 & 3.6050 & 1.8169 & -1.1757 & 0.5531 \\ 1.3224 & 2.0996 & 1.8169 & 0.8899 & -0.6070 & 0.2970 \\ -1.0912 & -1.2521 & -1.1757 & -0.6070 & -0.3845 & 0.0763 \\ 0.4808 & 0.6073 & 0.5531 & 0.2970 & 0.0763 & -0.0070 \end{pmatrix}.$$

Step 5. Computation of $F = L^{\mathrm{T}}S$:

$$F = \begin{pmatrix} 75.4820 & 96.7596 & 87.8785 & 47.6385 & 9.4298 & -0.4808 \\ 96.7596 & 124.1984 & 112.7028 & 61.2339 & 12.0384 & -0.6073 \\ 87.8785 & 112.7028 & 102.0882 & 55.4523 & 10.9292 & -0.5531 \\ 47.6385 & 61.2339 & 55.4523 & 30.1476 & 5.8930 & -0.2970 \\ 9.4298 & 12.0384 & 10.9292 & 5.8930 & 1.2847 & -0.0763 \\ -0.4808 & -0.6073 & -0.5531 & -0.2970 & -0.0763 & 0.0070 \end{pmatrix}.$$

Step 6. Gaussian elimination with diagonal pivoting: $PFP^{\mathrm{T}} = WDW^{\mathrm{T}}$, gives

$$W = \begin{pmatrix} 1 & & & & & \\ 0.9074 & 1 & & & & \\ 0.7791 & -0.4084 & 1 & & & \\ 0.0969 & -0.0274 & 0.4082 & 1 & & \\ 0.4930 & 0.6227 & -0.8765 & 0.0102 & 1 & \\ -0.0049 & 0.0111 & -0.0651 & -0.1454 & 0.0502 & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 124.1984 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1831 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1298 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0964 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0715 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0016 \end{pmatrix}.$$

Step 7. $\mathrm{In}(A) = \mathrm{In}(F) = \mathrm{In}(D) = (4, 2, 0)$.
Verification: The eigenvalues of A are:

$$\{-2.1502,\ 0.8553,\ 3.6006,\ 2.0971,\ 3.1305,\ -0.1123\},$$

confirming that $\mathrm{In}(A) = (4, 2, 0)$.

MATCONTROL note: Algorithm 7.5.1 has been implemented in MATCONTROL function **inertia**.

7.6 DISTANCE TO AN UNSTABLE SYSTEM

Let A be an $n \times n$ **complex stable matrix.** A natural question arises:

How "nearly unstable" is the stable matrix A?

We consider the above question in this section.

Definition 7.6.1. *Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalue on the imaginary axis. Let $U \in \mathbb{C}^{n\times n}$ be the set of matrices having at least one eigenvalue on the imaginary axis. Then, with $\|\cdot\|$ as the 2-norm or the Frobenius norm, the distance from A to U is defined by*

$$\beta(A) = \min\{\|E\| \mid A + E \in U\}.$$

If A is stable, then $\beta(A)$ is the distance to the set of unstable matrices.

*The concept of "**distance to instability**" is an important practical concept. Note that a theoretically perfect stable matrix may be very close to an unstable matrix. For example, consider the following matrix (Petkov et al. 1991):*

$$A = \begin{pmatrix} -0.5 & 1 & 1 & 1 & 1 & 1 \\ 0 & -0.5 & 1 & 1 & 1 & 1 \\ 0 & 0 & -0.5 & 1 & 1 & 1 \\ 0 & 0 & 0 & -0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & -0.5 \end{pmatrix}.$$

Since its eigenvalues are all -0.5, *it is perfectly stable. However, if the* $(6, 1)$*th entry is perturbed to* $\epsilon = 1/324$ *from zero, then the eigenvalues of this slightly perturbed matrix become:*

$-0.8006, -0.7222 \pm 0.2485j, -0.3775 \pm 0.4120j, 0.000.$

Thus, the perturbed matrix is unstable, showing that the stable matrix A is very close to an unstable matrix.

We now introduce a measure of $\beta(A)$ in terms of singular values and describe a simple bisection algorithm to approximately measure it.

Let $\sigma_{\min}(A - j\omega I)$ be the smallest singular value of $A - j\omega I$. Then it can be shown **(Exercise 7.14)** that

$$\beta(A) = \min_{\omega \in \mathcal{R}} \sigma_{\min}(A - j\omega I). \tag{7.6.1}$$

So, for any real ω, $\sigma_{\min}(A - j\omega I)$ is an upper bound on $\beta(A)$, that is, $\beta(A) \leq \sigma_{\min}(A - j\omega I)$.

Based on this idea, Van Loan (1985) gave two estimates for $\beta(A)$. One of them is a **heuristic estimate**:

$$\beta(A) \approx \min \left\{ \sigma_{\min}(A - j\mathrm{Re}(\lambda)I) \middle| \lambda \in \Lambda(A) \right\}, \tag{7.6.2}$$

where $\Lambda(A)$ denotes the spectrum of A.

Thus, using this heuristic estimate, $\beta(A)$ may be estimated by finding the singular values of the matrix $(A - j\mathrm{Re}(\lambda)I)$, for every eigenvalue λ of A. This approach was thought to give an upper bound within an order of magnitude of $\beta(A)$. However, Demmel (1987) has provided examples to show this bound can be larger than $\beta(A)$ by an arbitrary amount.

The other approach of Van Loan requires application of a general nonlinear minimization algorithm to $f(\omega) = \sigma_{\min}(A - j\omega I)$. We will not pursue these approaches here. Rather, we will describe a simple bisection method to estimate

$\beta(A)$ due to Byers (1988). The bisection algorithm estimates $\beta(A)$ within a factor of 10 or indicates that $\beta(A)$ is less than a small tolerance. This is sufficient in practice. The algorithm makes use of the crude estimate of the upper bound $\beta(A) \leq \frac{1}{2}||A + A^*||_2$.

To describe the algorithm, let's define a $2n \times 2n$ **Hamiltonian matrix** $H(\sigma)$, given $\sigma \geq 0$, by

$$H(\sigma) = \begin{pmatrix} A & -\sigma I \\ \sigma I & -A^* \end{pmatrix}. \tag{7.6.3}$$

The bisection method is based on the following interesting spectral property of the matrix $H(\sigma)$. **For more on Hamiltonian matrices, see Chapters 10 and 13.**

Theorem 7.6.1. *$\sigma \geq \beta(A)$ if and only if $H(\sigma)$ defined by (7.6.3) has a purely imaginary eigenvalue.*

Proof. Let ω_i be a purely imaginary eigenvalue of $H(\sigma)$. Then there exist nonzero complex vectors u, v such that

$$\begin{pmatrix} A & -\sigma I \\ \sigma I & -A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \omega_i \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7.6.4}$$

This gives us

$$(A - \omega_i I)\, u = \sigma v \tag{7.6.5}$$

and

$$(A - \omega_i I)^*\, v = \sigma u \tag{7.6.6}$$

This means that σ is a singular value of the matrix $A - \omega_i I$. Also, since $\beta(A) \leq \sigma_{\min}(A - j\omega I)$ for any real ω, we obtain $\sigma \geq \beta(A)$.

Conversely, suppose that $\sigma \geq \beta(A)$. Define

$$f(\alpha) = \sigma_{\min}(A - j\alpha I).$$

The function f is continuous and $\lim_{\alpha\to\infty} f(\alpha) = \infty$. Therefore, f has a minimum value $f(\alpha) = \beta(A) \leq \sigma$, for some real α.

By the Intermediate Value Theorem of Calculus, we have $f(\omega) = \sigma$ for some real ω.

So, σ is a singular value of $A - j\omega I = A - \omega_i I$ and there exist unit complex vectors u and v satisfying (7.6.5) and (7.6.6). This means that ω_i is a purely imaginary eigenvalue of $H(\sigma)$. ∎

Algorithm 7.6.1. *The Bisection Algorithm for Estimating the Distance to an Unstable System*

Inputs. *A—An $n \times n$ stable complex matrix*

τ—Tolerance (> 0).

Outputs. *Real numbers α and ν such that either $\nu/10 \leq \alpha \leq \beta(A) \leq \nu$ or $0 = \alpha \leq \beta(A) \leq \nu \leq 10\tau$.*

Step 1. *Set* $\alpha \equiv 0,\ \nu = \frac{1}{2}\|(A + A^*)\|_2$

Step 2. *Do while* $\nu > 10\max(\tau, \alpha)$

$\sigma \equiv \sqrt{\nu \max(\tau, \alpha)}$

If $=$ $H(\sigma)$ *has a purely imaginary eigenvalue, then set* $\nu \equiv \sigma$*; else* $\alpha \equiv \sigma$

Example 7.6.1. Consider finding $\beta(A)$ for the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -0.0001 \end{pmatrix}.$$

$$\tau = 0.00100$$

Iteration 1.

Step 1. Initialization: $\alpha = 0$, $\nu = 1.2071$.

Step 2. $10 \times \max(\tau, \alpha) = 0.0100$.

$$\sigma = 0.0347$$

$$H(\sigma) = \begin{pmatrix} -1 & 1 & -0.0347 & 0 \\ 0 & -0.0001 & 0 & -0.0347 \\ 0.0347 & 0 & 1 & 0 \\ 0 & 0.0347 & -1 & 0.0001 \end{pmatrix}$$

The eigenvalues of $H(\sigma)$ are $\pm 1, \pm 0.0491j$. Since $H(\sigma)$ has an purely imaginary eigenvalue, we set

$$\nu = \sigma = 0.0347.$$

$$\nu = 0.0347 > 10 \ \max\ (\tau, \alpha) = 0.0100,$$

the iteration continues, until $\nu = 0.0059$ is reached, at which point, the iteration terminates with $\beta(A) \leq 0.0059 < 10\tau$.

Conclusion: $\beta(A) \leq 0.0059 < 10\tau$.

Computational remarks:

- The bulk of the work of the Algorithm 7.6.1 is in deciding whether $H(\sigma)$ has an imaginary eigenvalue.
- Also, the decision of whether $H(\sigma)$ has an imaginary eigenvalue in a computational setting (in the presence of round-off errors) is a tricky one. Some sort of threshold has to be used. However, if that decision is made in a numerically effective way, then in the worst case, "$\beta(A)$ **might lie outside the bound given by the algorithm by an amount proportional to the precision of the arithmetic**" (Byers 1988).

- Because of the significant computational cost involved in deciding if the matrix $H(\sigma)$ at each step has an imaginary eigenvalue, **the algorithm may not be computationally feasible for large problems.**

Convergence. If $\tau = \frac{1}{2}10^{-p}\|A + A^*\|$, then at most $\log_2 p$ bisection steps are required; for example, if $\tau = \frac{1}{2} \times 10^{-8}\|A + A^*\|$, then at most three bisection steps are required.

MATCONTROL note: Algorithm 7.6.1 has been implemented in MATCONTROL function **disstabc**.

Relation to Lyapunov Equation

Since Lyapunov equations play a vital role in stability analysis of a linear system, it is natural to think that the distance to a set of unstable matrices $\beta(A)$ is also related to a solution of a Lyapunov equation. Indeed, the following result can be proved (See Malyshev and Sadkane (1999) and also Hewer and Kenney (1988)).

Theorem 7.6.2. *Distance to an Unstable System and Lyapunov Equation. Let* ***A be complex stable*** *and let X be the unique positive Hermitian definite solution of the Lyapunov equation:*

$$XA + A^*X = -M, \tag{7.6.7}$$

where M is Hermitian positive definite. Then

$$\beta(A) \geq \frac{\lambda_{\min}(M)}{2\ \|X\|_2},$$

where $\lambda_{\min}(M)$ denotes the smallest eigenvalue of M.

Proof. Let $\omega \in \mathbb{R}$ and u be a unit vector such that

$$\frac{1}{\beta(A)} = \max_{Re(z)=0} \|(A - zI)^{-1}\| = \|(A - j\omega I)^{-1}u\| . \tag{7.6.8}$$

Let $x = (A - j\omega I)^{-1}u$. Then, $\|x\|_2 = 1/\beta(A)$.

Multiplying the Lyapunov equation (7.6.7) by x^* to the left and x to the right, we have

$$x^*(XA + A^*X)x = -x^*Mx,$$
$$x^*XAx + x^*A^*Xx = -x^*Mx.$$

Then,

$$x = (A - j\omega I)^{-1}u \Rightarrow (A - j\omega I)x = u \Rightarrow Ax = u + j\omega x$$

and

$$x = (A - j\omega I)^{-1} u \Rightarrow x^*(A - j\omega I)^* = u^* \Rightarrow x^* A^* + j\omega x^*$$
$$= u^* \Rightarrow x^* A^* = u^* - j\omega x^*.$$

Therefore,

$$|x^* X A x + x^* A^* X x| = |x^* X(u + j\omega x) + (u^* - j\omega x^*) X x|$$
$$= 2|u^* X x| \leq 2\|X\|_2 \|x\|_2. \qquad (7.6.9a)$$

Also, by the Rayleigh quotient (see Datta (1995) or Golub and Van Loan (1996)), we have, $\lambda_{\min}(M) \leq x^* M x / x^* x$, that is,

$$\lambda_{\min}(M)\|x\|_2^2 \leq x^* M x = |-x^* M x|. \qquad (7.6.9b)$$

Thus, combining (7.6.9a) and (7.6.9b) yields

$$\lambda_{\min}(M)\|x\|_2^2 \leq |x^* M x| = |x^*(XA + A^* X)x| \leq 2\|X\|_2 \|x\|_2$$

or

$$\lambda_{\min}(M)\|x\|_2 \leq 2\|X\|_2.$$

Since $\|x\|_2 = 1/\beta(A)$, this means that $\lambda_{\min}(M)(1/\beta(A)) \leq 2\|X\|_2$ or $\beta(A) \geq \lambda_{\min}/2\|X\|_2$. ■

Example 7.6.2. Consider Example 7.6.1 again.

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -0.0001 \end{pmatrix}.$$

Take $M = I_2$. Then

$$X = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 9999.5 \end{pmatrix}.$$

$$\beta(A) \approx 0.0059 \geq \frac{1}{2\|\mathrm{x}\|_2} = 5.0002 \times 10^{-5}.$$

Verify: The eigenvalues of $A + 5.0002 \times 10^{-5} I$ are -0.9999 and 0.

Distance to a Discrete Unstable System

The discrete analog of $\beta(A)$ is defined to be

$$\gamma(A) = \min\{\|E\| \mid \text{for some } \theta \in \mathbb{R};\ e^{i\theta} \in \Omega(A + E)\}. \qquad (7.6.10)$$

That is, $\gamma(A)$ **measures the distance from A to the nearest matrix with an eigenvalue on the unit circle.** If A is discrete-stable, then $\gamma(A)$ is a measure of how "**nearly discrete-unstable**" A is. In above, $\Omega(M)$ denotes the spectrum of M. A discrete-analog of Theorem 7.6.1 is:

Theorem 7.6.3. *Given an $n \times n$ complex matrix A, there exists a number $\Gamma(A) \in \mathbb{R}$ such that $\Gamma(A) \geq \gamma(A)$ and for $\Gamma(A) \geq \sigma \geq \gamma(A)$, the $2n \times 2n$ Hamiltonian matrix pencil*

$$H_{\mathrm{D}}(\sigma) = F(\sigma) - \lambda G(\sigma) = \begin{pmatrix} -\sigma I_n & A \\ I_n & 0 \end{pmatrix} - \lambda \begin{pmatrix} 0 & I_n \\ A^{\mathrm{T}} & -\sigma I_n \end{pmatrix}$$

has a generalized eigenvalue of magnitude 1. Furthermore, if $\sigma < \gamma(A)$, then the above pencil has no generalized eigenvalue of magnitude 1.

Proof. See Byers (1988). ■

Based on the above result, Byers (1988) described the following bisection algorithm to compute $\gamma(A)$, analogous to Algorithm 7.6.1.

The algorithm estimates $\gamma(A)$ within a factor of 10 or indicates that $\gamma(A)$ is less than a tolerance. The algorithm uses a crude bound $\Gamma(A) \geq \sigma_{\min}(A - I)$.

Algorithm 7.6.2. *The Bisection Algorithm for Estimating the Distance to a Discrete-Unstable System*

Inputs. *An $n \times n$ complex matrix A and a tolerance $\tau > 0$.*

Outputs. *Real numbers α and δ such that $\delta/10 \leq \alpha \leq \gamma(A) \leq \delta$ or $0 = \alpha \leq \gamma(A) \leq \delta \leq 10\tau$.*

Step 1. *Set $\alpha \equiv 0$; $\delta \equiv \sigma_{\min}(A - I)$.*

Step 2. *Do while $\delta > 10\max(\tau, \alpha)$*

$\sigma \equiv \sqrt{\delta \max(\tau, \alpha)}$.

If the pencil $F(\sigma) - \lambda G(\sigma)$, defined above, has a generalized eigenvalue of magnitude 1, then set $\delta \equiv \sigma$, else $\alpha \equiv \sigma$.

End.

Example 7.6.3. Let $A = \begin{pmatrix} 0.9999 & 1 \\ 0 & 0.5 \end{pmatrix}$, $\tau = 10^{-8}$. The matrix A is discrete-stable.

Iteration 1:

Step 1: $\alpha = 0$, $\delta = 4.4721 \times 10^{-5}$.

Step 2: $\delta > 10\max(\tau, \alpha)$ is verified, we compute $\sigma^{\boldsymbol{\cdot}} = 6.6874 \times 10^{-6}$. The eigenvalues of $H_{\mathrm{D}}(\sigma)$ are 2, 1.0001, 0.9999, and 0.5000. Thus, $\alpha \equiv 6.6874 \times 10^{-6}$.

Iteration 2: Since $\delta > 10\max(\tau, \alpha)$ is verified, we compute $\sigma = 5.4687 \times 10^{-6}$. The eigenvalues of $H_{\mathrm{D}}(\sigma)$ are 2, 1.001, 0.999, 0.5000; we set $\alpha = \sigma = 5.4687 \times 10^{-6}$.
Iteration 3: $\delta < 10\max(\tau, \alpha)$, the iteration stops, and on exit we obtain

$$\alpha = 5.4687 \times 10^{-6} \qquad \delta = 4.4721 \times 10^{-5}.$$

MATCONTROL note: Algorithm 7.6.2 has been implemented in MATCONTROL function **disstabd**.

7.7 ROBUST STABILITY

Even though a system is known to be stable, it is important to investigate if the system remains stable under certain perturbations. Note that in most physical systems, the system matrix A is not known exactly; what is known is $A + E$, where E is an $n \times n$ perturbation matrix. Thus, in this case the stability problem becomes the problem of finding if the system:

$$\dot{x}(t) = (A + E)x(t) \tag{7.7.1}$$

remains stable, given that A is stable.

The solution of the Lyapunov equations can be used again to obtain bounds on the perturbations that guarantee that the perturbed system (7.7.1) remains stable.

In Theorem 7.7.1, $\sigma_{\max}(M)$, as usual, stands for the largest singular value of M.

We next state a general result on robust stability due to Keel *et al.* (1988). The proof can be found in Bhattacharyya *et al.* (1995, pp. 519–520). The result there is proved in the context of feedback stabilization, and we will revisit the result later in that context. The other earlier results include those of Patel and Toda (1980) and Yedavalli (1985).

Theorem 7.7.1. *Let A be a stable matrix and let the perturbation matrix E be given by*

$$E = \sum_{i=1}^{r} p_i E_i, \tag{7.7.2}$$

where $E_i, i = 1, \ldots, r$ are matrices determined by structure of the perturbations.

Let Q be a symmetric positive definite matrix and X be a unique symmetric positive definite solution of the Lyapunov equation:

$$XA + A^{\mathrm{T}}X + Q = 0. \tag{7.7.3}$$

Then the system (7.7.1) remains stable for all p_i satisfying

$$\sum_{i=1}^{r} |p_i|^2 < \frac{\sigma_{\min}^2(Q)}{\sum_{i=1}^{r} \mu_i^2},$$

where $\sigma_{\min}(Q)$ denotes the minimum singular value of Q and μ_i is given by

$$\mu_i = \| E_i^{\mathrm{T}} X + X E_i \|_2 .$$

Example 7.7.1. Let $r = 1$, $p_1 = 1$. Take

$$E = E_1 = \begin{pmatrix} 0.0668 & 0.0120 & 0.0262 \\ 0.0935 & 0.0202 & 0.0298 \\ 0.0412 & 0.0103 & 0.0313 \end{pmatrix}.$$

Let $A = \begin{pmatrix} -4.1793 & 9.712 & 1.3649 \\ 0 & -1.0827 & 0.3796 \\ 0 & 0 & -9.4673 \end{pmatrix}$

Choose $Q = 2I$.
Then, $\mu_1 = \| E_1^{\mathrm{T}} X + X E_1 \|_2 = 0.7199$, and the right-hand side of (7.7.3) is 7.7185.
Since $|p_1^2| = 1 < 7.7185$, the matrix $A + E$ is stable.

A result similar to that stated in Theorem 7.7.1 was also proved by Zhou and Khargonekar (1987). We state the result below.

Theorem 7.7.2. *Let A be a stable matrix and let E be given by (7.7.2). Let X be the unique symmetric positive definite solution of (7.7.3). Define*

$$X_i = (E_i^{\mathrm{T}} X + X E_i)/2, \quad i = 1, 2, \ldots, r \tag{7.7.4}$$

and

$$X_e = (X_1, X_2, \ldots, X_r).$$

Then (7.7.1) remains stable if

$$\sum_{k=1}^{r} p_k^2 < \frac{1}{\sigma_{\max}^2(X_e)} \quad or \quad \sum_{i=1}^{r} |p_i| \sigma_{\max}(X_i) < 1$$

$$or \quad |p_i| < \frac{1}{\sigma_{\max}\left(\sum_{i=1}^{r} |X_i|\right)}, \quad i = 1, \ldots, r. \tag{7.7.5}$$

Remark

- It should be noted that Theorems 7.7.1 and 7.7.2 and others in Patel and Toda (1980) and Yedavalli (1985) all give only **sufficient conditions** for robust stability. A number of other sufficient conditions can be found in the book by Boyd *et al.* (1994).

7.8 THE STRUCTURED STABILITY RADIUS

In Section 7.6, we introduced the concept of the distance of a stable matrix from the set of unstable matrices. Here we specialize this concept to "**structured stability**," meaning that we are now interested in finding the distance from a stable matrix to the set of unstable matrices, **where the distance is measured by the size of the additive perturbations of the form $B\Delta C$, with B and C fixed, and Δ variable.**

Let A, B, and C be, respectively, $n \times n$, $n \times m$, and $r \times n$ matrices over the field $\mathbb{F}$ ($\mathbb{F}$ can be $\mathbb{C}$ or $\mathbb{R}$). Then the (structured) **stability radius** of the matrix triple (A, B, C) is defined as

$$r_{\mathbb{F}}(A, B, C) = \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times r} \text{ and } A + B\Delta C \text{ is unstable }\}, \qquad (7.8.1)$$

where $\bar{\sigma}(M)$ following the notation of Qiu *et al.* (1995), **denotes the largest singular value of** M (i.e., $\bar{\sigma}(M) = \sigma_{\max}(M)$). For real matrices (A, B, C), $r_{\mathbb{R}}(A, B, C)$ is called the **real stability radius** and, for complex matrices (A, B, C), $r_{\mathbb{C}}(A, B, C)$ is called the **complex stability radius. The stability radius, thus, determines the magnitude of the smallest perturbation needed to destroy the stability of the system.**

"Stability" here is referred to as either continuous-stability (with respect to the left half-plane) or discrete-stability (with respect to the unit circle).

Let $\partial\mathbb{C}_g$ denote the boundary of either the half plane or the unit circle. Let A be stable or discrete-stable.

Then,

$$r_{\mathbb{F}}(A, B, C) = \inf\{\bar{\sigma}(\Delta) | \Delta \in \mathbb{F}^{m\times r} \text{ and } A + B\Delta C \text{ has an eigenvalue on } \partial\mathbb{C}_g\}.$$

$$= \inf_{s\in\partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) | \Delta \in \mathbb{F}^{m\times r} \text{ and } \det(sI - A - B\Delta C) = 0\} \qquad (7.8.2)$$

$$= \inf_{s\in\partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) | \Delta \in \mathbb{F}^{m\times r} \text{ and } \det(I - \Delta G(s)) = 0\},$$

where $G(s) = C(sI - A)^{-1}B$.

Thus, given a complex $r \times m$ matrix M, the stability radius problem reduces to the problem of computing:

$$\mu_{\mathbb{F}}(M) = [\inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times r} \text{ and } \det(I - \Delta M) = 0\}]^{-1}.$$

The Complex Stability Radius

It is easy to see that

$$\mu_{\mathbb{C}}(M) = \bar{\sigma}(M).$$

Thus, we have the following formula for the complex stability radius.

Theorem 7.8.1. *The Complex Stability Radius Formula*

$$r_{\mathbb{C}}(A, B, C) = \left\{ \sup_{s \in \partial \mathbb{C}_g} \bar{\sigma}(G(s)) \right\}^{-1}. \tag{7.8.3}$$

The Real Stability Radius

If $\mathbb{F}$ is $\mathbb{R}$, then according to the above we have

$$r_{\mathbb{R}}(A, B, C) = \left\{ \sup_{s \in \partial \mathbb{C}_g} \mu_{\mathbb{R}}[C(sI - A)^{-1}B] \right\}^{-1}. \tag{7.8.4}$$

For the real stability radius, the major problem is then is the problem of computing $\mu_{\mathbb{R}}(M)$, given M.

The following important formula for computing $\mu_{\mathbb{R}}(M)$ has been recently obtained by Qiu *et al.* (1995). We quote the formula from this paper. The proof is involved and we refer the readers to the paper for the proof. Following the notation of this paper, **we denote the second largest singular value of M by** $\sigma_2(M)$, and so on.

Denote the real and imaginary parts of a complex matrix M by $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$, respectively. That is, $M = \mathrm{Re}(M) + j\mathrm{Im}(M)$.

Then the following result holds:

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \mathrm{Re}(M) & -\gamma \mathrm{Im}(M) \\ \frac{1}{\gamma}\mathrm{Im}(M) & \mathrm{Re}(M) \end{bmatrix} \right). \tag{7.8.5}$$

The function to be minimized is a unimodular function on (0, 1].

Furthermore, if $\mathrm{rank}(\mathrm{Im}(M)) = \lambda$, then

$$\mu_{\mathbb{R}}(M) = \max\{\bar{\sigma}(U_2^{\mathrm{T}}\mathrm{Re}(M)), \bar{\sigma}(\mathrm{Re}(M)V_2)\},$$

where U_2 and V_2 are defined by the SVD of $\mathrm{Im}(M)$, that is, they satisfy

$$\mathrm{Im}(M) = [U_1, U_2] \begin{bmatrix} \bar{\sigma}(\mathrm{Im}(M)) & 0 \\ 0 & 0 \end{bmatrix} [V_1, V_2]^{\mathrm{T}}.$$

Note that since the function to be minimized is unimodular, any local minimum is also a global minimum.

Notes

(i)
$$r_{\mathbb{R}}(A, B, C) \geq r_{\mathbb{C}}(A, B, C). \tag{7.8.6}$$

(ii) The ratio $r_{\mathbb{R}}(A, B, C)/r_{\mathbb{C}}(A, B, C)$ can be arbitrarily large.

The following example taken from Hinrichsen and Pritchard (1990) illustrates (ii).

Example 7.8.1. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\epsilon \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ -\epsilon \end{pmatrix},$$

and

$$C = (1, 0).$$

Then the transfer function:

$$G(s) = C(j\omega I - A)^{-1}B = \frac{-\epsilon}{1 - \omega^2 + j\omega\epsilon}.$$

By (7.8.4), the real stability radius:

$$r_{\mathbb{R}}(A, B, C) = 1/\epsilon.$$

Since

$$|G(j\omega)|^2 = \frac{\epsilon^2}{(1 - \omega^2)^2 + \epsilon^2\omega^2},$$

it is easy to see that $|G(j\omega)|^2$ is maximized when $\omega^2 = 1 - \epsilon^2/2$, if $\epsilon < \sqrt{2}$. So, by (7.8.3)

$$r_{\mathbb{C}}^2(A, B, C) = 1 - (\epsilon^2/4).$$

Thus, if ϵ is considered as a parameter, then $r_{\mathbb{C}}(A, B, C)$ is always bounded by 1 whereas $r_{\mathbb{R}}(A, B, C)$ can be made arbitrarily large by choosing ϵ small enough.

Specialization of the Stability Radius to the Distance from Unstable Matrices

From (7.8.3) we immediately have the following relation between the distance to an unstable system and the stability radius:

$$\beta = r_{\mathbb{C}}(A, I, I) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - j\omega I) = \beta(A).$$

Also, the following formula for $\beta(A)$, when A is a real stable matrix, can be proved.

Theorem 7.8.2. *Let A be a* **real stable matrix.** *Then*

$$\beta(A) \equiv r_{\mathbb{R}}(A, I, I) = \min_{s \in \partial\mathbb{C}_g} \max_{\gamma \in (0,1]} \sigma_{2n-1}\left(\begin{bmatrix} A - \mathrm{Re}(sI) & -\gamma \mathrm{Im}(sI) \\ \frac{1}{\gamma}\mathrm{Im}(sI) & A - \mathrm{Re}(sI) \end{bmatrix}\right). \tag{7.8.7}$$

Note: For each fixed s, the function in (7.8.7) to be maximized is quasiconcave.

7.9 SOME SELECTED SOFTWARE

7.9.1 MATLAB Control System Toolbox

norm—Computes the H_2-norm of the system.
bode—Computes the magnitude and phase of the frequency response, which are used to analyze stability and robust stability.
nyquist—Calculates the Nyquist frequency response. System properties such as **gain margin**, **phase margin**, and **stability** can be analyzed using Nyquist plots. (The gain margin and phase margin are widely used in classical control theory as measures of robust stability).
gram controllability and observability grammrians.

7.9.2 MATCONTROL

INERTIA	Determining the inertia and stability of a matrix without solving a matrix equation or computing eigenvalues
H2NRMCG	Finding H_2-norm using the controllability Grammians
H2NRMOG	Finding H_2-norm using the observability Grammian
DISSTABC	Determining the distance to the continuous-time stability
DISSTABD	Determining the distance to the discrete-time stability

7.9.3 SLICOT

AB13BD	H_2 or L_2 norm of a system
AB13ED	Complex stability radius using bisection
AB13FD	Complex stability radius using bisection and SVD

7.10 SUMMARY AND REVIEW

The stability of the system:

$$\dot{x}(t) = Ax(t)$$

or that of

$$x(k+1) = Ax(k)$$

is essentially governed by the eigenvalues of the matrix A.

Mathematical Criteria of Stability

The continuous-time system $\dot{x}(t) = Ax(t)$ is asymptotically stable if and only if the eigenvalues of A are all in the left half plane **(Theorem 7.2.1)**. Similarly, the discrete-time system $x(k+1) = Ax(k)$ is asymptotically stable if and only if all the eigenvalues of A are inside the unit circle. **(Theorem 7.3.1).** Various Lyapunov stability theorems (**Theorems 7.2.3–7.2.9**, and **Theorem 7.3.2**) have been stated and proved.

The Inertia of a Matrix

Two important inertia theorems (**Theorems 7.4.1 and 7.4.2**) and the classical **Sylvester Law of Inertia** have been stated. These inertia theorems generalize the Lyapunov stability results.

Methods for Determining Stability and Inertia

The **Characteristic Polynomial Approach** and the **Matrix Equation Approach** are two classical approaches for determining the stability of a system and the inertia of a matrix. Both these approaches have some computational drawbacks.

The zeros of a polynomial may be extremely sensitive to small perturbations. Furthermore, the numerical methods to compute the characteristic polynomial of a matrix are usually unstable.

The most numerically effective method (**The Schur method**, described in Chapter 8), for solving a Lyapunov matrix equation is based on reduction of the matrix A to RSF, and the RSF displays the eigenvalues of A or the eigenvalues can be trivially computed out of this form.

Thus, the characteristic equation approach is not numerically viable and the matrix equation approach for stability and inertia is counterproductive.

Hence, the most numerically effective approach for stability and inertia is the eigenvalue approach: compute all the eigenvalues of A.

By explicitly computing the eigenvalues, one, however, gets much more than what is needed for stability and inertia. Furthermore, since the eigenvalues of a matrix can be very sensitive to small perturbations, determining the inertia and stability by computing explicitly the eigenvalues can be misleading.

An implicit matrix equation approach **(Algorithm 7.5.1),** which does not require computation of eigenvalues nor explicit solution of any matrix equation has been described. **Algorithm 7.5.1 is about three times faster than the eigenvalue method** (According to the flop-count).

Distance to an Unstable System

Given a stable matrix A, the quantity $\beta(A)$ defined by

$$\beta(A) = \min \{\|E\|_F \text{ such that } A + E \in U\},$$

where U is the set of $n \times n$ matrices with at least one eigenvalue on the imaginary axis, is the distance to the set of unstable matrices.

A bisection algorithm (**Algorithm 7.6.1**) based on knowing if a certain Hamiltonian matrix (the matrix (7.6.3)) has a purely imaginary eigenvalue, is described. The algorithm is based on **Theorem 7.6.1**, which displays a relationship between a spectral property of the Hamiltonian matrix and the quantity $\beta(A)$.

The discrete-analog of $\beta(A)$ is defined to be

$$\gamma(A) = \min\{||E|| \text{ for some } \theta \in \mathbb{R};\, e^{i\theta} \in \Omega(A + E)\}.$$

An analog of Theorem 7.6.1 (**Theorem 7.6.3**) is stated and a bisection algorithm (**Algorithm 7.6.2**) based on this theorem is described.

Robust Stability

Given a stable matrix A, one naturally wonders if the matrix $A+E$ remains stable, where E is a certain perturbed matrix. Two bounds for E guaranteeing the stability of the perturbed matrix $(A+E)$ are given, in terms of solutions of certain Lyapunov equations (**Theorems 7.7.1** and **7.7.2**).

Stability Radius

Section 7.8 deals with the **structured stability radius**. If the perturbations are of the form $B\Delta C$, where Δ is an unknown perturbation matrix, then it is of interest to know the size of smallest Δ (measured using 2-norm) that will destabilize the perturbed matrix $A + B\Delta C$. In this context, the concept of **stability radius** is introduced, and formulas both for the complex stability radius (**Theorem 7.8.1**) and the real stability radius are stated.

H_2-Norm

The H_2-norm of a stable transfer, transfer function measures the steady-state covariance of the output response $y = Gv$ to the white noise inputs v. An algorithm (**Algorithm 7.2.1**) for computing the H_2-norm, based on computing the controllability or observability Grammian via Lyapunov equations is given.

7.11 CHAPTER NOTES AND FURTHER READING

A voluminous work has been published on Lyapunov stability theory since the historical monograph "**Problème de la stabilité du Mouvement**" was published by the Russian mathematician A.M. Liapunov in 1892. Some of the books that exclusively deal with Lyapunov stability are those by LaSalle and Lefschetz (1961), Lehnigk (1966), etc., and a good account of diagonal stability and diagonal-type

Lyapunov functions appears in the recent book by Kaszkurewicz and Bhaya (1999). For a good account of BIBO and BIBS stability, see the book by DeCarlo (1989).

In Section 7.2, we have just given a very brief account of the Lyapunov stability adapted to the linear case. The matrix equation version in the linear case seems to have first appeared in the book by Gantmacher (1959, Vol. II). There exist many proofs of Lyapunov stability theorem (**Theorem 7.2.3**). The proof given here is along the line of Bellman (1960). See also Hahn (1955). The proofs of the other theorems in this section can be found in most linear systems books, including the books by Chen (1984), Kailath (1980), Wonham (1986), etc.

The inertia theory has been mainly confined to the linear algebra literature. An excellent account of its control theoretic applications appear in Glover (1984) and in the book by Zhou *et al.* (1996).

There are also a few papers on the inertia theory with respect to more general regions in the complex plane other than the half-planes and the unit circle given in Section 7.4. Inertia theory has been applied to obtain elementary proofs of several classical root-location problems in Datta (1978a, 1978b, 1979). For an account of this work, see the recent survey paper of the author (Datta 1999). The inertia and stability algorithm is due to Carlson and Datta (1979b). The algorithm has been modified by Datta and Datta (1987) and extended to other regions in the complex plane in Datta and Datta (1986).

The concept of distance to instability was perhaps introduced by Van Loan (1985). The bisection algorithm (**Algorithm 7.6.1**) is due to Byers (1988).

There are now several good books on robust control. These include the books by Dorato and Yedavalli (1989), Hinrichsen and Martensson (1990), Barmish (1994), Bhattacharyya *et al.* (1995), Green and Limebeer (1995), Zhou *et al.* (1996). The concept of complex stability radius as robustness measures for stable matrices (in the form given here) was introduced by Hinrichsen and Pritchard (1986). There are several good papers on this subject in the book "**Control of uncertain systems**," edited by Hinrichsen and Martensson (1990). Discussion of Section 7.8 has been taken from Qiu *et al.* (1995).

Exercises

7.1 Verify that the spring-mass system of Example 5.2.3 is not asymptotically stable. What is the physical interpretation of the above statement?

7.2 Consider the problem of a cart with two sticks considered in Exercise 5.3 of Chapter 5. Take $M_1 = M_2 = M$.

(a) Show that at the equilibrium states, $\bar{x}_1$ and $\bar{x}_2$ are nonzero and $\bar{x}_3 = \bar{x}_4 = 0$. What is the physical significance of this?

(b) Show that the system is not asymptotically stable.

7.3 Consider the stick-balancing problem in Example 5.2.4. Give a mathematical explanation of the fact that without an input to the control system, if the stick is not upright with zero velocity, it will fall.

7.4 Give a proof of Theorem 7.3.2 from that of Theorem 7.2.3 using the matrix version of the Cayley transformation.

7.5 Prove that the system (7.2.2) is BIBO if and only if $G(s) = C(sI - A)^{-1}B$ has every pole with negative real part.

7.6 Prove that the discrete-time system:

$$x_{k+1} = Ax_k + Bu_k$$

is BIBO stable if and only if all the poles of the transfer functions lie inside the open unit circle of the z-plane.

7.7 Prove that the discrete-time system in Exercise 7.6 is BIBS if and only if (i) all the eigenvalues of A lie in the closed unit disc, (ii) the eigenvalues on the unit disc have multiplicity 1 in the minimal polynomial of A, and (iii) the unit circle modes are uncontrollable (consult DeCarlo (1989, p. 422)).

7.8 Let X and M be the symmetric positive definite matrices such that

$$XA + A^{\mathrm{T}}X + 2\lambda X = -M,$$

then prove that all eigenvalues of A have a real part that is less than $-\lambda$.

7.9 Prove that A is a stable matrix if and only if $\|e^{At}\| \leq k$, for some $k > 0$.

7.10 Prove that if M is positive definite and the discrete Lyapunov equation:

$$X - A^{\mathrm{T}}XA = M$$

has a symmetric positive definite solution X, then A is discrete-stable.

7.11 Prove the following results:

(a) Suppose that A is discrete-stable. Then (A, B) is controllable if and only if the discrete Lyapunov equation:

$$X - AXA^{\mathrm{T}} = BB^{\mathrm{T}}$$

has a unique positive definite solution.

(b) Suppose that (A, B) is controllable. Then A is discrete-stable if and only if the discrete Lyapunov equation:

$$X - AXA^{\mathrm{T}} = BB^{\mathrm{T}}$$

has a unique positive definite solution

(c) Suppose that (A, C) is observable. Then A is discrete-stable if and only if there exists a unique positive definite solution X of the discrete Lyapunov equation:

$$X - A^{\mathrm{T}}XA = C^{\mathrm{T}}C.$$

7.12 (Glover 1984).
Let

$$X = X^{\mathrm{T}} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

with $\delta(X) = 0$. Suppose that

$$AX + XA^{\mathrm{T}} = -BB^{\mathrm{T}},$$
$$A^{\mathrm{T}}X + XA = -C^{\mathrm{T}}C.$$

Partition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

and $C = (C_1, C_2)$, conformably with X. Then prove the following:

(a) If $\gamma(X_1) = 1$, then $\pi(A_{11}) = 0$.
(b) If $\delta(A) = 0$ and $\lambda_i(X_1^2) \neq \lambda_j(X_2^2)$ $\forall i, j$, then $\mathrm{In}(A_{11}) = \mathrm{In}(-X_1)$ and $\mathrm{In}(A_{22}) = \mathrm{In}(-X_2)$. (Here $\lambda_i(M)$ denotes the ith eigenvalue of M.)

7.13 Prove that in Theorem 7.4.2, the assumption that (A^{T}, M) is controllable implies that $\delta(X) = 0$.

7.14 Let A be a stable matrix. Prove that (i) $\beta(A) = \min_{\omega \in R} \sigma_{\min}(A - j\omega I)$, (ii) $\beta(A) \leq |\alpha(A)|$, where $\alpha(A) = \max\{Re(\lambda) | \lambda \text{ is an eigenvalue of } A\}$.

7.15 Give an example to show that the formula of $\beta(A)$ given by (7.6.2) can be arbitrary large (Consult the paper of Demmel (1987)).

7.16 Construct an example to show that a matrix A can be very near to an unstable matrix without $\alpha(A)$, defined in Exercise 7.14, being small.

7.17 Let $\mathrm{Arg}(z)$ represent the argument of the complex number z. Let $r > 0$ and $\rho \in \mathbb{C}$, then prove that $r^{-1}\gamma(r(A + \rho I))$ is the distance from A to the nearest matrix with an eigenvalue on the circle $\{z \in C \mid |z - \rho| = r^{-1}\}$, where $\gamma(M)$ denotes the distance of a discrete-stable matrix M to instability, defined by (7.6.10).
Use the result to develop an algorithm to estimate this quantity.

7.18 Give proofs of Theorems 7.7.1 and 7.7.2 (consult the associated papers, as necessary).

7.19 Consider the perturbed system:

$$\dot{x} = (A + BKC)x,$$

where

$$A = \mathrm{diag}(-1, -2, -3), \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} -1 + k_1 & 0 \\ 0 & -1 + k_2 \end{pmatrix},$$

and k_1 and k_2 are two uncertain parameters varying in the intervals around zero. Use each of the Theorems 7.7.1 and 7.7.2 to calculate and compare the allowable bounds on k_1 and k_2 that guarantee the stability of $A + BKC$.

7.20 Construct an example to verify each of the followings:

(a) The real stability radius is always greater than or equal to the complex stability radius.

(b) The ratio of the real stability radius to the complex stability radius can be made arbitrarily large.

7.21 Prove that the H_2-norm of the discrete-time transfer matrix

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

can be computed as

$$\|G(z)\|_2^2 = \mathrm{Trace}(CC_{\mathrm{G}}^{\mathrm{D}}C^{\mathrm{T}}) = B^{\mathrm{T}}O_{\mathrm{G}}^{\mathrm{D}}B,$$

where $C_{\mathrm{G}}^{\mathrm{D}}$ and $O_{\mathrm{G}}^{\mathrm{D}}$ are, respectively, the discrete-time controllability and observability Grammians given by (7.3.6) and (7.3.7), respectively. Write down a Lyapunov equation based algorithm to compute the H_2-norm of a discrete-time system based on the above formula.

7.22 Give a proof of Theorem 7.8.2.

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