CHAPTER 10

FEEDBACK STABILIZATION, EIGENVALUE ASSIGNMENT, AND OPTIMAL CONTROL

Topics covered

- State-Feedback Stabilization
- Eigenvalue Assignment (EVA)
- Linear Quadratic Regulator Problems
- H_∞ Control
- Stability Radius (Revisited)

10.1 INTRODUCTION

In this chapter, we first consider the problem of stabilizing a linear control system by choosing the control vector appropriately. Mathematically, the problem is to find a feedback matrix K such that $A - BK$ is stable in the continuous-time case or is discrete-stable in the discrete-time case. Necessary and sufficient conditions are established for the existence of stabilizing feedback matrices, and Lyapunov-style methods for constructing such matrices are described in **Section 10.2**.

A concept dual to stabilizability, called **detectability**, is then introduced and its connection with a Lyapunov matrix equation is established in **Section 10.3**.

In certain practical situations, stabilizing a system is not enough; a designer should be able to control the eigenvalues of $A - BK$ so that certain design constraints are met. This gives rise to the **eigenvalue assignment (EVA) problem** or the so-called **pole placement problem**. Mathematically, the problem is to find a feedback matrix K such that $A - BK$ has a preassigned spectrum. A well-known and a very important result on the solution of this problem is: Given a real pair of matrices (A, B) and Λ, an arbitrary set of n complex numbers, closed under complex conjugation, there exists a real matrix K such that the spectrum of $A - BK$ is the set Λ if and only if (A, B) is controllable. The matrix K is unique in the single-input case.

This important result is established in **Theorem 10.4.1**. The proof of this result is constructive, and leads to several well-known formulas, the most important of which is the **Ackermann formula**. However, these formulas do not yield numerically viable methods for pole placement. **Numerical methods for pole placement are presented in Chapter 11.**

Since there are no set guidelines as to where the poles (the eigenvalues) need to be placed, very often, in practice, a compromise is made in which a feedback matrix is constructed in such a way that not only the system is stabilized, but a certain performance criterion is satisfied. This leads to the well-known **Linear Quadratic Regulator** (LQR) problem. Both continuous-time and discrete-time LQR problems are discussed in **Section 10.5** of this chapter. The solutions of the LQR problems require the solutions of certain quadratic matrix equations, called the **algebraic Riccati equations** (AREs). **Numerical methods for the AREs are described in Chapter 13.**

The next topic in this chapter is the $\boldsymbol{H_\infty}$-control problems. Though a detailed discussion on the H_∞-control problems is beyond the scope of the book, some simplified versions of these problems are stated in **Section 10.6** in this chapter. The H_∞-control problems are concerned with stabilization of perturbed versions of a system, when certain bounds of perturbations are known. The solutions of the H_∞-control problems also require solutions of certain AREs. Two algorithms (**Algorithms 10.6.1 and 10.6.2**) are given in Section 10.6 for computing the H_∞-norm."

The concept of **stability radius** introduced in Chapter 7 is revisited in the final section of this chapter (**Section 10.7**), where a relationship between the complex stability radius and an ARE (**Theorem 10.7.3**) is established, and a bisection algorithm (**Algorithm 10.7.1**) for determining the complex stability radius is described.

Reader's Guilde for Chapter 10

The readers familiar with concepts and results of state-feedback stabilizations, pole-placement, LQR design, and H_∞ control can skip Sections 10.2–10.6. However, two algorithms for computing the H_∞-norm (**Algorithms 10.6.1 and 10.6.2**) and material on stability radius (**Section 10.7**) should be of interests to most readers.

10.2 STATE-FEEDBACK STABILIZATION

In this section, we consider the problem of stabilizing the linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{10.2.1}$$

Suppose that the state vector $x(t)$ is known and let's choose

$$u(t) = v(t) - Kx(t), \tag{10.2.2}$$

where K is a constant matrix, and $v(t)$ is a reference input vector.

Then feeding this input vector $u(t)$ back into the system, we obtain the system:

$$\begin{aligned} \dot{x}(t) &= (A - BK)x(t) + Bv(t), \\ y &= (C - DK)x(t) + Dv(t). \end{aligned} \tag{10.2.3}$$

The problem of stabilizing the system (10.2.1) then becomes the problem of finding K such that the system (10.2.3) becomes stable. The problem of state-feedback stabilization can, therefore, be stated as follows:

Given a pair of matrices (A, B), find a matrix K such that $A - BK$ is stable.

Graphically, the state-feedback problem can be represented as in Figure 10.1.

In the next subsection we will investigate the conditions under which such a matrix K exists. The matrix K, when it exists, is called a **stabilizing feedback matrix**; and in this case, the pair (A, B) is called a **stabilizable pair.** The system (10.2.3) is called the **closed-loop system** and the matrix $A - BK$ is called the **closed-loop matrix.**

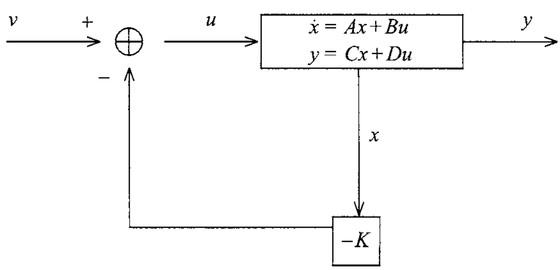

FIGURE 10.1: State feedback configuration.

Analogously, for the discrete-time system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

if there exists a matrix K such that $A - BK$ is discrete-stable, that is, if it has all its eigenvalues inside the unit circle, then the pair (A, B) will be called a **discrete-stabilizable** pair, and the matrix K will be called a **discrete-stabilizing feedback matrix.**

In what follows, we will present simple criteria of stabilizability and algorithms for constructing stabilizing feedback matrices via Lyapunov matrix equations.

10.2.1 Stabilizability and Controllability

In this section, we describe necessary and sufficient conditions for a given pair (A, B) to be a stabilizable pair. We start with the continuous-time case.

Theorem 10.2.1. *Characterization of Continuous-Time Stabilizability. The following, are equivalent:*

(i) (A, B) *is stabilizable.*

(ii) *Rank*$(A - \lambda I, B) = n$ *for all* $\mathrm{Re}(\lambda) \geq 0$. *In other words, the unstable modes of A are controllable.*

(iii) *For all* λ *and* $x \neq 0$ *such that* $x^*A = \lambda x^*$ *and* $\mathrm{Re}(\lambda) \geq 0$, *we have* $x^*B \neq 0$.

Proof. We prove the equivalence of (i) and (ii) and leave the equivalence of (i) and (iii) as an exercise **(Exercise 10.1)**.

Without any loss of generality we may assume (see Theorem 6.4.1) that the pair (A, B) is given in the form:

$$PAP^{-1} = \bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix}, \qquad PB = \bar{B} = \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix},$$

where $(\bar{A}_{11}, \bar{B}_1)$ is controllable.

Since $(\bar{A}_{11}, \bar{B}_1)$ is controllable, by the eigenvalue criterion of controllability (Theorem 6.2.1 (v)), we have $\mathrm{rank}(\lambda I - \bar{A}_{11}, \bar{B}_1) = p$, where p is the order of $\bar{A}_{11}$. Therefore,

$$\mathrm{rank}(\lambda I - \bar{A}, \bar{B}) = \mathrm{rank}\begin{pmatrix} \lambda I - \bar{A}_{11} & -\bar{A}_{12} & \bar{B}_1 \\ 0 & \lambda I - \bar{A}_{22} & 0 \end{pmatrix} < n,$$

if and only if $\mathrm{rank}(\lambda I - \bar{A}_{22}) < n - p$, that is, if and only if λ is an eigenvalue of $\bar{A}_{22}$.

The proof now follows from the fact that if (A, B) is a stabilizable pair, the matrix $\bar{A}_{22}$ must be a stable matrix. This can be seen as follows:

The stabilizability of the pair (A, B) implies the stabilizability of the pair $(\bar{A}, \bar{B})$. Since $(\bar{A}, \bar{B})$ is a stabilizable pair, there exists a matrix $\bar{K}$ such that $\bar{A} - \bar{B}\bar{K}$ is stable. This means that if $\bar{K} = (\bar{K}_1, \bar{K}_2)$, then the matrix

$$\begin{pmatrix} \bar{A}_{11} - \bar{B}_1\bar{K}_1 & \bar{A}_{12} - \bar{B}_1\bar{K}_2 \\ 0 & \bar{A}_{22} \end{pmatrix}$$

is a stable matrix, which implies that $\bar{A}_{22}$ must be stable. ■

Corollary 10.2.1. *If the pair (A, B) is controllable, then it must be stabilizable.*

Proof. If (A, B) is controllable, then again by the eigenvalue criterion of controllability, $\text{rank}(A - \lambda I, B) = n$ for every λ. In particular, $\text{rank}(A - \lambda I, B) = n$ for every λ for which $\text{Re}(\lambda) \geq 0$. Thus, (A, B) is stabilizable. ■

The above result tells us that **the controllability implies stabilizability**.

However, the converse is not true. The stabilizability is guaranteed as long as the unstable modes are controllable.

The following simple example illustrates the fact.

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(A, b) is not controllable; $\text{rank}(b, Ab, A^2b) = 2$.

However, the row vector $f^{\mathrm{T}} = (-126.5, -149.5, 0)$ is such that the eigenvalues of $A - bf^{\mathrm{T}}$ are $\{-10 \pm 11.4891j, -3\}$.

So, $A - bf^{\mathrm{T}}$ is stable, that is, (A, b) is stabilizable.

The Discrete Case

A theorem, analogous to Theorem 10.2.1, can be proved for the discrete-time system as well. We state the result without proof. The proof is left as an exercise **(Exercise 10.2)**.

Theorem 10.2.2. *Characterization of Discrete-Stabilizability. The following conditions are equivalent:*

(i) *The pair (A, B) is discrete-stabilizable.*
(ii) $\text{Rank}(A - \lambda I, B) = n$ *for every λ such that $|\lambda| \geq 1$.*
(iii) *For all λ and $x \neq 0$ such that $x^*A = \lambda x^*$ and $|\lambda| \geq 1$, we have $x^*B \neq 0$.*

10.2.2 Stabilization via Lyapunov Equations

From the discussions of the previous section, it is clear that for finding a feedback stabilizing matrix K for a given pair (A, B), we can assume that the pair (A, B) is controllable. For, if (A, B) is not controllable but stabilizable, then we can always put it in the form:

$$TAT^{-1} = \bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix}, \qquad TB = \bar{B} = \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix}, \tag{10.2.4}$$

where $(\bar{A}_{11}, \bar{B}_1)$ is controllable, and $\bar{A}_{22}$ is stable.

Once a stabilizing matrix $\bar{K}_1$ for the controllable pair $(\bar{A}_{11}, \bar{B}_1)$ is obtained, the stabilizing matrix K for the pair (A, B) can be obtained as:

$$K = \bar{K}T,$$

where

$$\bar{K} = (\bar{K}_1, \bar{K}_2) \tag{10.2.5}$$

and $\bar{K}_2$ is arbitrary.

We can therefore concentrate on stabilizing a controllable pair. Theorem 10.2.3 shows how to stabilize a controllable pair using a Lyapunov equation.

Theorem 10.2.3. *Let (A, B) be controllable and let β be a scalar such that*

$$\beta > |\lambda_{\max}(A)|,$$

where $\lambda_{\max}(A)$ is the eigenvalue of A with the largest real part. Let K be defined by

$$K = B^{\mathrm{T}}Z^{-1}, \tag{10.2.6}$$

where Z (necessarily symmetric positive definite) satisfies the Lyapunov equation:

$$-(A + \beta I)Z + Z[-(A + \beta I)]^{\mathrm{T}} = -2BB^{\mathrm{T}}, \tag{10.2.7}$$

then $A - BK$ is stable, that is, (A, B) is stabilizable.

Proof. Since $\beta > |\lambda_{\max}(A)|$, the matrix $-(A + \beta I)$ is stable.

Also, since (A, B) is controllable, the pair $(-(A+\beta I), B)$ is controllable. Thus, by Theorem 7.2.6, the Lyapunov equation (10.2.8) has a unique symmetric positive definite solution Z.

Again, Eq. (10.2.8) can be written as:

$$(A - BB^{\mathrm{T}}Z^{-1})Z + Z(A - BB^{\mathrm{T}}Z^{-1})^{\mathrm{T}} = -2\beta Z.$$

Then, from (10.2.7) we have:

$$(A - BK)Z + Z(A - BK)^{\mathrm{T}} = -2\beta Z. \tag{10.2.8}$$

Since Z is symmetric positive definite, $A - BK$ is stable by Theorem 7.2.3.

This can be seen as follows:

Let μ be an eigenvalue of $A - BK$ and y be the corresponding eigenvector.

Then multiplying both sides of Eq. (10.2.9) first by y^* to the left and then by y to the right, we have

$$2\,\mathrm{Re}(\mu)y^*Zy = -2\beta y^*Zy.$$

Since Z is positive definite, $y^*Zy > 0$. Thus, $\mathrm{Re}(\mu) < 0$. So, $A - BK$ is stable. ■

The above discussion leads to the following method for finding a stabilizing feedback matrix (see Armstrong 1975).

A Lyapunov Equation Method For Stabilization

Let (A, B) be a controllable pair. Then the following method computes a stabilizing feedback matrix K.

Step 1. Choose a number β such that $\beta > |\lambda_{\max}(A)|$, where $\lambda_{\max}(A)$ denotes the eigenvalue of A with the largest real part.

Step 2. Solve the Lyapunov equation (10.2.8) for Z:

$$-(A + \beta I)Z + Z[-(A + \beta I)]^{\mathrm{T}} = -2BB^{\mathrm{T}}.$$

Step 3. Obtain the stabilizing feedback matrix K:

$$K = B^{\mathrm{T}}Z^{-1}.$$

MATCONTROL note: The above method has been implemented in MATCONTROL function **stablyapc**.

A Remark on Numerical Effectiveness

The Lyapunov equation in Step 2 can be highly ill-conditioned, even when the pair (A, B) is robustly controllable. In this case, the entries of the stabilizing feedback matrix K are expected to be very large, giving rise to practical difficulties in implementation. See the example below.

Example 10.2.1 (*Stabilizing the Motion of the Inverted Pendulum Consider*). Example 5.2.5 (The **problem of a cart with inverted pendulum**) with the following data:

$$m = 1\,kg, \quad M = 2\,kg, \quad l = 0.5\,m, \quad \text{and } g = 9.18\,m/s^2.$$

Then,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -3.6720 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 22.0320 & 0 \end{pmatrix}.$$

The eigenvalues of A are 0, 0, ± 4.6938. Thus, with no control input, there is an instability in the motion and the pendulum will fall. We will now stabilize the motion by using the Lyapunov equation method with A as given above, and

$$B = \begin{pmatrix} 0 \\ 0.4 \\ 0 \\ -0.4 \end{pmatrix}.$$

Step 1. Let's choose $\beta = 5$. This will make $-(A + \beta I)$ stable.
Step 2.

$$Z = \begin{pmatrix} 0.0009 & -0.0044 & -0.0018 & 0.0098 \\ -0.0044 & 0.0378 & 0.0079 & -0.0593 \\ -0.0018 & 0.0079 & 0.0054 & -0.0270 \\ 0.0098 & -0.0593 & -0.0270 & 0.1508 \end{pmatrix}.$$

(The computed Z is symmetric positive definite but highly ill-conditioned).
Step 3. $K = B^{\mathrm{T}} Z^{-1} = 10^3(-0.5308, -0.2423, -1.2808, -0.2923)$.
Verify: The eigenvalues of $A - BK$ are $\{-5 \pm 11.2865j, -5 \pm 0.7632j\}$.

Note that the entries of K are large. The pair (A, B) is, however, robustly controllable, which is verified by the fact that the singular values of the controllability matrix are 8.9462, 8.9462, 0.3284, 0.3284.

Remark

- If the pair (A, B) is not controllable, but stabilizable, then after transforming the pair (A, B) to the form $(\bar{A}, \bar{B})$ given by

$$TAT^{-1} = \bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad TB = \bar{B} = \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix},$$

we will apply the above method to the pair $(\bar{A}_{11}, \bar{B}_1)$ (which is controllable) to find a stabilizing feedback matrix $\bar{K}_1$ for the pair $(\bar{A}_{11}, B_1)$ and then obtain K that stabilizes the pair (A, B) as

$$K = (\bar{K}_1, \bar{K}_2)\, T,$$

choosing $\bar{K}_2$ arbitrarily.

Example 10.2.2. Consider the uncontrollable, but the stabilizable pair (A, B):

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Step 1. $A = \bar{A},\ B = \bar{B}$. So, $T = I$.

$$\bar{A}_{11} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \bar{A}_{22} = -3, \qquad \bar{B}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Step 2. Choose $\beta_1 = 10$. The unique symmetric positive definite solution Z_1 of the Lyapunov equation:

$$-(\bar{A}_{11} + \beta_1 I)Z_1 + Z_1[-(\bar{A}_{11} + \beta_1 I)]^{\mathrm{T}} = -2\bar{B}_1\bar{B}_1^{\mathrm{T}}$$

is

$$Z_1 = \begin{pmatrix} 0.0991 & -0.0906 \\ -0.0906 & 0.0833 \end{pmatrix}.$$

Step 3. $\bar{K}_1 = \bar{B}_1^{\mathrm{T}} Z_1^{-1} = (-126.5, -149.5)$.
Step 4. Choose $\bar{K}_2 = 0$. Then $K = \bar{K} = (\bar{K}_1, \bar{K}_2) = (-126.5, -149.5, 0)$.
Verify: The eigenvalues of $A - BK$ are $-10 \pm 11.489j, -3$.

Discrete-Stabilization via Lyapunov Equation

The following is a discrete-analog of Theorem 10.2.3. We state the theorem without proof. The proof is left as an exercise **(Exercise 10.3)**.

Theorem 10.2.4. *Let the discrete-time system $x_{k+1} = Ax_k + Bu_k$ be controllable. Let $0 < \beta \leq 1$ be such that $|\lambda| \geq \beta$ for any eigenvalue λ of A.*
Define $K = B^{\mathrm{T}}(Z + BB^{\mathrm{T}})^{-1}A$, where Z satisfies the Lyapunov equation, $AZA^{\mathrm{T}} - \beta^2 Z = 2BB^{\mathrm{T}}$, then $A - BK$ is discrete-stable.

Theorem 10.2.4 leads to the following Lyapunov method of discrete-stabilization. The method is due to Armstrong and Rublein (1976).

A Lyapunov Equation Method for Discrete-Stabilization

Step 1. Find a number β such that $0 < \beta < \min(1, \min_i |\lambda_i|)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of (A).

Step 2. Solve the discrete Lyapunov equation for Z:

$$AZA^{\mathrm{T}} - \beta^2 Z = 2BB^{\mathrm{T}}.$$

Step 3. Compute the discrete-stabilizing feedback matrix K,

$$K = B^{\mathrm{T}}(Z + BB^{\mathrm{T}})^{-1}A.$$

Example 10.2.3. Consider the **cohort population model** in Luenberger (1979, pp. 170), with $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$, and

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of A are -1.9276, -0.7748, $-0.0764 \pm 0.8147j$. The matrix A is not discrete-stable.

Step 1. Choose $\beta = 0.5$

Step 2. The solution Z to the discrete Lyapunov equation $AZA^{\mathrm{T}} - \beta^2 Z = 2BB^{\mathrm{T}}$ is

$$Z = -\begin{pmatrix} -0.0398 & 0.0321 & -0.0003 & 0.0161 \\ 0.0321 & -0.1594 & 0.1294 & -0.0011 \\ -0.0003 & 0.1214 & -0.6376 & 6.5135 \\ 0.0161 & -0.0011 & 6.5135 & -2.5504 \end{pmatrix}.$$

Step 3. $K = (1.2167, 1.0342, 0.9886, 0.9696)$

Verify: The eigenvalues of $A - BK$ are $-0.0742 \pm 0.4259j$, -0.4390, and 0.3708. Thus, $A - BK$ is discrete-stable.

Note: If (A, B) is not a discrete-controllable pair, but is discrete-stabilizable, then we can proceed exactly in the same way as in the continuous-time case to stabilize the pair (A, B).

The following example illustrates how to do this.

MATCONTROL note: Discrete Lyapunov stabilization method as described above has been implemented in MATCONTROL function **stablyapd**.

Example 10.2.4. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & -0.9900 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The pair (A, B) is not discrete-controllable, but is discrete-stabilizable.

Using the notations of Section *10.2.2*, we have $\bar{A} = A,\ \bar{B} = B$. The eigenvalues of $\bar{A}$ are

$$\{1.7321, -1.7321, -0.9900\}.$$

$$\bar{A}_{11} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix},\ \bar{B}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The pair $(\bar{A}_{11}, \bar{B}_1)$ is controllable.

We now apply the Lyapunov method of discrete-stabilization to the pair $(\bar{A}_{11}, \bar{B}_1)$.

Step 1. Choose $\beta = 1$.

Step 2. The solution Z_1 of the discrete Lyapunov equation: $\bar{A}_{11} Z_1 \bar{A}_1^{\mathrm{T}} - Z_1 = 2\bar{B}_1 \bar{B}_1^{\mathrm{T}}$ is

$$Z_1 = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}.$$

Step 3.

$$\bar{K}_1 = (0, 2.4000).$$

Step 4. The matrix $\bar{A}_{11} - \bar{B}_1 \bar{K}_1$ is discrete-stable. To obtain $\bar{K}$ such that $\bar{A} - \bar{B}\bar{K}$ is discrete-stable, we choose $\bar{K} = (\bar{K}_1, 0)$. The eigenvalue of $\bar{A} - \bar{B}\bar{K}$ are $0.7746, -0.7746, -0.9900$, showing that $\bar{A} - \bar{B}\bar{K}$ is discrete-stable, that is, $A - B\bar{K}$ is discrete-stable.

Remark

- For an efficient implementation of the Lyapunov method for feedback stabilization using the Schur method, see Sima (1981).

10.3 DETECTABILITY

As observability is a dual concept of controllability, a concept dual to stabilizability is called **detectability.**

Definition 10.3.1. *The pair (A, C) is* **detectable** *if there exists a matrix L such that $A - LC$ is stable.*

By duality of Theorem 10.2.1, we can state the following result. The proof is left as an exercise **(Exercise 10.8)**.

Theorem 10.3.1. *Characterization of Continuous-Time Detectability. The following conditions are equivalent:*

(i) *(A, C) is detectable.*

(ii) *The matrix $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ has full column rank for all* $\mathrm{Re}(\lambda) \geq 0$.

(iii) *For all λ and $x \neq 0$ such that $Ax = \lambda x$ and* $\mathrm{Re}(\lambda) \geq 0$, *we have $Cx \neq 0$*

(iv) *$(A^{\mathrm{T}}, C^{\mathrm{T}})$ is stabilizable.*

We have seen in Chapter 7 that the controllability and observability play important role in the existence of positive definite and semidefinite solutions of Lyapunov equations.

Similar results, therefore, should be expected involving detectability. We prove one such result in the following.

Theorem 10.3.2. *Detectability and Stability. Let (A, C) be detectable and let the Lyapunov equation:*

$$XA + A^{\mathrm{T}}X = -C^{\mathrm{T}}C \tag{10.3.1}$$

have a positive semidefinite solution X. Then A is a stable matrix.

Proof. The proof is by contradiction. Suppose that A is unstable. Let λ be an eigenvalue of A with $\mathrm{Re}(\lambda) \geq 0$ and x be the corresponding eigenvector. Then premultiplying the equation (10.3.1) by x^* and postmultiplying it by x, we obtain $2\mathrm{Re}(\lambda)(x^*Xx) + x^*C^{\mathrm{T}}Cx = 0$. Since $X \geq 0$ and $Re(\lambda) \geq 0$, we must have that $Cx = 0$. This contradicts the fact that (A, C) is detectable. ■

Discrete-Detectability

Definition 10.3.2. *The pair (A, C) is discrete-detectable if there exists a matrix L such that $A - LC$ is discrete-stable.*

Theorems analogous to Theorems 10.3.1 and 10.3.2 also hold in the discrete case. We state the discrete counterpart of Theorem 10.3.1 in Theorem 10.3.3 and leave the proof as an exercise (**Exercise 10.10**).

Theorem 10.3.3. *The following are equivalent:*

(i) *(A, C) is discrete-detectable.*

(ii)
$$\mathrm{Rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n$$
for every λ such that $|\lambda| \geq 1$.

(iii) *For all λ and $x \neq 0$ such that $Ax = \lambda x$ and $|\lambda| \geq 1$, we have $Cx \neq 0$.*

(iv) *$(A^{\mathrm{T}}, C^{\mathrm{T}})$ is discrete-stabilizable.*

10.4 THE EIGENVALUE AND EIGENSTRUCTURE ASSIGNMENT PROBLEMS

We have just seen how an unstable system can be possibly stabilized using feedback control. However, in practical instances, stabilization alone may not be enough.

The stability of the system needs to be monitored and/or the system response needs to be altered. To meet certain design constraints, a designer should be able to choose the feedback matrix such that the closed-loop system has certain transient properties defined by the eigenvalues of the system. We illustrate this with the help of a second-order system.

Consider the second-order system:

$$\ddot{x}(t) + 2\zeta\omega_{\mathrm{n}}\dot{x}(t) + \omega_{\mathrm{n}}^2 x(t) = u(t).$$

The poles of this second-order system are of the form: $\lambda_{1,2} = -\zeta\omega_{\mathrm{n}} \pm j\omega_{\mathrm{n}}\sqrt{1-\zeta^2}$. The quantity ζ is called the **damping ratio** and ω_{n} is called the undamped **natural frequency**. The responses of the dynamical system depends upon ζ and ω_{n}. *In general, for a fixed value of ω_{n}, the larger the value of ζ ($\zeta \geq 1$), the smoother but slower the responses become; on the contrary, the smaller the value of ζ ($0 \leq \zeta < 1$), the faster but more oscillatory the response is.* Figures 10.2 and 10.3 illustrate the situations.

For Figure 10.2, $\omega_{\mathrm{n}} = 1$ and $\zeta = 3$. It takes about eight time units to reach the steady-state value 1.

For Figure 10.3, $\omega_{\mathrm{n}} = 1$ and $\zeta = 0.5$. The response is much faster as it reaches the steady-state value 1 in about three units time. However, it does not maintain that value; it oscillates before it settles down to 1.

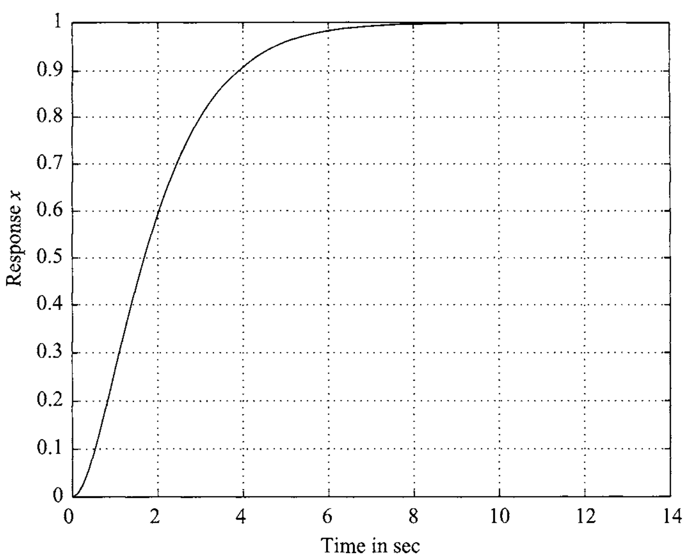

FIGURE 10.2: Unit step response when $\zeta = 3$ and $\omega_{\mathrm{n}} = 1$.

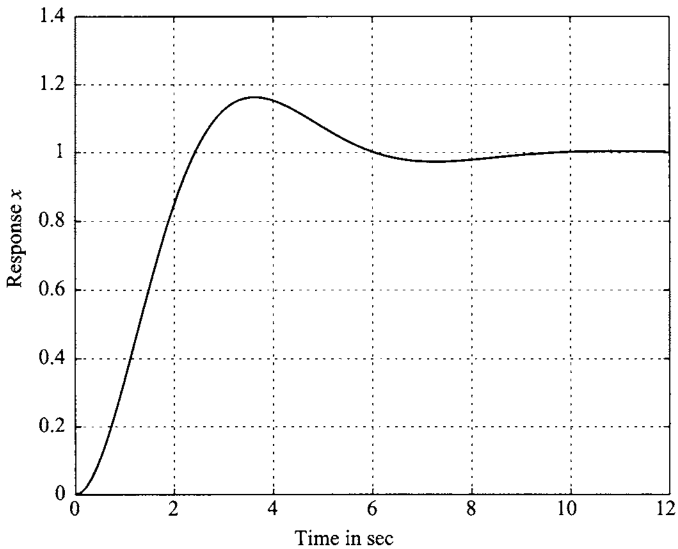

FIGURE 10.3: Unit step response when $\zeta = 0.5$ and $\omega_n = 1$.

These quantities thus need to be chosen according to a desired transient response. If the poles are close to $j\omega$-axis in the left half s-plane, then the transient responses decay relatively slowly. On the other hand, the poles far away from the $j\omega$-axis cause rapidly decaying time responses. Normally, "*The closed-loop poles for a system can be chosen as a desired pair of dominant second-order poles, with the rest of the poles selected to have real parts corresponding to sufficiently damped modes so that the system will mimic a second-order response with a reasonable control effort*" (Franklin *et al.* 1986). The dominant poles are the poles that have dominant effects on the transient response behavior. As far as transient response is concerned, the poles with magnitudes of real parts at least five times greater than the dominant poles may be considered as insignificant. We give below some illustrative examples.

Case 1. Suppose that it is desired that the closed-loop system response have the minimum decay rate $\alpha > 0$, that is, $\mathrm{Re}(\lambda) \leq -\alpha$ for all eigenvalues λ. Then the eigenvalues should lie in the shifted half plane as shown in Figure 10.4.

Case 2. Suppose that it is desired that the system have the minimal damping ratio $\zeta_{\min}$. Then the eigenvalues should lie in the sector as shown in Figure 10.5.

Case 3. Suppose that it is desired that the closed-loop system have a minimal undamped frequency $\omega_{\min}$. Then the eigenvalues of the closed-loop matrix should lie outside of the following half of the disk: $0 < \omega_{\min} \leq \omega_n$, as shown in Figure 10.6.

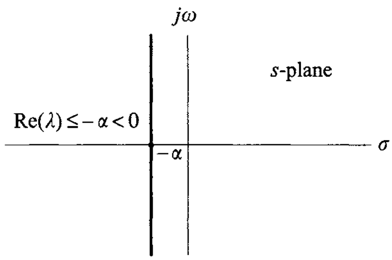

FIGURE 10.4: The minimum decay rate α of the closed-loop system.

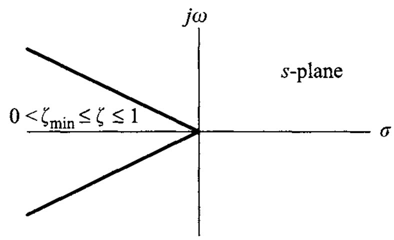

FIGURE 10.5: Minimal damping ratio ζ of the closed-loop system: the poles lie in the sector $\{\lambda \in \mathbb{C}\colon |\mathrm{Im}(\lambda)| \leq -\mathrm{Re}(\lambda)\sqrt{\zeta^{-2} - 1}\}$.

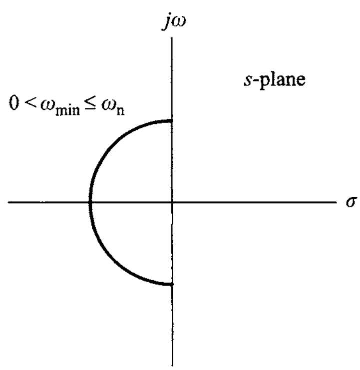

FIGURE 10.6: The minimal undamped frequency $\omega_{\min}$ of the closed-loop system: the poles lie in the region $\{\lambda \in \mathbb{C}\colon |\lambda| \geq \omega_{\min}\}$.

Knowing that to obtain certain transient responses, the eigenvalues of the closed-loop system should be placed in certain specified regions of the complex plane, the question arises: **where should these eigenvalues be placed?** An excellent discussion to this effect is given in the books by Friedland (1986, pp. 243–246) and Kailath (1980, chapter 3).

If the eigenvalues of the closed-loop system are moved far from those of the open-loop system, then from the explicit expression of the feedback vector (to be given later) in the single-input case, it is easily seen that a large feedback f will be required. From the control law,

$$u = v - f^{\mathrm{T}}x(t),$$

it then follows that this would require large control inputs, and there are practical limitations on how large control inputs can be.

Thus, although the eigenvalues have to be moved to stabilize a system, "**the designer should not attempt to alter the dynamic behavior of the open-loop process more than is required**" (Friedland 1986).

10.4.1 Eigenvalue Assignment by State Feedback

The problem of assigning the eigenvalues at certain desired locations in the complex plane using the control law (10.2.2) is called the **EVA Problem by state feedback**. In control theory literature, it is more commonly known as the **pole-placement problem.**

Here is the precise mathematical statement of the problem.

> Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ $(m \leq n)$, and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, where Λ is closed under complex conjugation, find $K \in \mathbb{R}^{m\times n}$ such that
>
> $$\Omega(A - BK) = \Lambda.$$
>
> Here, $\Omega(R)$ stands for the spectrum of R.

The matrix K is called the **state-feedback matrix**.

Theorem 10.4.1 gives the conditions of existence and uniqueness of K.

Theorem 10.4.1. *The State-Feedback EVA Theorem. The EVA problem is solvable for all Λ if and only if (A, B) is controllable. The solution is unique if and only if the system is a single-input system (i.e., if B is a vector). In the multi-input case, if the problem is solvable, there are infinitely many solutions.*

Proof. We first prove the **necessity.** The proof is by contradiction.

Suppose that the pair (A, B) is not controllable. Then according to the eigenvalue criteria of controllability, we have $\operatorname{rank}(A - \lambda I, B) < n$ for some λ. Thus there exists a vector $z \neq 0$ such that $z^{\mathrm{T}}(A - \lambda I) = 0$, $z^{\mathrm{T}}B = 0$. This means that for any K, we have $z^{\mathrm{T}}(A - \lambda I - BK) = 0$, which implies that λ is an eigenvalue of $A - BK$ for every K, and thus λ cannot be reassigned.

Next we prove the **sufficiency.**

Case 1. Let's consider first the **single-input case**. That is, we prove that if (A, b) is controllable, then there exists a unique vector f such that the matrix $A - bf^{\mathrm{T}}$ has the desired spectrum.

Consider the (lower) controller-companion form $(C, \tilde{b})$ of the controllable pair (A, b) (see **Chapter 6**):

$$TAT^{-1} = C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{pmatrix} \tag{10.4.1}$$

and

$$\tilde{b} = Tb = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{10.4.2}$$

We now show that there exists a row vector $\hat{f}^{\mathrm{T}}$ such that the closed-loop matrix $C - \tilde{b}\hat{f}^{\mathrm{T}}$ has the desired spectrum.

Let the characteristic polynomial of the desired closed-loop matrix be $d(\lambda) = \lambda^n + d_n\lambda^{n-1} + \cdots + d_1$. Let $\hat{f}^{\mathrm{T}} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$.

Then

$$C - \tilde{b}\hat{f}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 - \hat{f}_1 & -a_2 - \hat{f}_2 & \cdots & & -a_n - \hat{f}_n \end{pmatrix}. \tag{10.4.3}$$

The characteristic polynomial $c'(\lambda)$ of $C - \tilde{b}\hat{f}^{\mathrm{T}}$, then, is $\lambda^n + (a_n + \hat{f}_n)\lambda^{n-1} + \cdots + a_1 + \hat{f}_1$. Comparing the coefficients of $c'(\lambda)$ with those of $d(\lambda)$, we immediately have

$$\hat{f}_i = d_i - a_i, \quad i = 1, 2, \ldots, n. \tag{10.4.4}$$

Thus, the vector $\hat{f}$ is completely determined by the coefficients of the characteristic polynomial of the matrix C and the coefficients of the characteristic polynomial of the desired closed-loop matrix. Once the vector $\hat{f}$ is known, the vector f such that the original closed-loop matrix $A - bf^{\mathrm{T}}$ has the desired spectrum, can now be found from the relation:

$$f^{\mathrm{T}} = \hat{f}^{\mathrm{T}} T. \tag{10.4.5}$$

(Note that $\Omega(A - bf^{\mathrm{T}}) = \Omega(TAT^{-1} - Tbf^{\mathrm{T}}T^{-1}) = \Omega(C - \tilde{b}\hat{f}^{\mathrm{T}})$.)

Uniqueness: From the construction of $\hat{f}$, it is clear that $\hat{f}$ is unique. We now show that the uniqueness of $\hat{f}$ implies that of f. The proof is by contradiction.

Suppose there exists $g \neq f$ such that $\Omega(A - bg^{\mathrm{T}}) = \Omega(A - bf^{\mathrm{T}})$. Then $\Omega(C - \tilde{b}\hat{f}^{\mathrm{T}}) = \Omega(C - \tilde{b}\hat{g}^{\mathrm{T}})$, where $\hat{g}^{\mathrm{T}} = g^{\mathrm{T}}T^{-1} \neq \hat{f}^{\mathrm{T}}$, which contradicts the uniqueness of the vector $\hat{f}$.

Case 2. Now we turn to the **multi-input case**. Since (A, B) is controllable, there exists a matrix F and a vector g such that $(A - BF, Bg)$ is controllable (see Chen (1984, p. 344)). Thus, by Case 1, there exists a vector h such that the matrix $A - BF - Bgh^{\mathrm{T}}$ has the desired spectrum.

Then with $K = F + gh^{\mathrm{T}}$, we have that $A - BK$ has the desired spectrum.

Uniqueness: Since the choice of the pair (F, g) is not unique, there exist infinitely many feedback matrices K in the multi-input case. ■

The Bass–Gura Formula

Note that using the expression of T from Chapter 6, the above expression for f in the single-input case can be written as **(Exercise 10.13)**:

$$f = T^{\mathrm{T}}\hat{f} = [(C_{\mathrm{M}}W)^{\mathrm{T}}]^{-1}(d - a), \tag{10.4.6}$$

where d is the vector of the coefficients of the desired characteristic polynomial, a is the vector of the coefficients of the characteristic polynomial of A, C_{M} is the controllability matrix, and W is a certain Toeplitz matrix.

The above formula for f is known as the **Bass–Gura formula** (see Kailath (1980, p. 199)).

Ackermann's Formula (Ackermann 1972)

A closely related formula for the single-input feedback vector f is the well-known Ackermann formula:

$$f = e_{\mathrm{n}}^{\mathrm{T}} C_{\mathrm{M}}^{-1} d(A), \tag{10.4.7}$$

where C_{M} is the controllability matrix and $d(A)$ is the characteristic polynomial of the desired closed-loop matrix.

We also leave the derivation of Ackermann's formula as an exercise *(Exercise 10.14)*.

Notes: We remind the readers again that, since $T = C_{\mathrm{M}}^{-1}$ can be very ill-conditioned, **computing f using the constructive proof of Theorem 10.4.1 or by**

the Ackermann or by the Bass–Gura formula can be highly numerically unstable. We will give some numerical examples in **Chapter 11** to demonstrate this.

The MATLAB function **acker** has implemented Ackermann's formula and **comments have been made about the numerical difficulty with this formula in the MATLAB user's manual.**

10.4.2 Eigenvalue Assignment by Output Feedback

Solving the EVA problem using the feedback law (10.2.2) requires knowledge of the full state vector $x(t)$. Unfortunately, in certain situations, the full state is not measurable or it becomes expensive to feedback each state variable when the order of the system is large. In such situations, the feedback law using the output is more practical. Thus, if we define the output feedback law by

$$u(t) = -Ky(t), \qquad y(t) = Cx(t), \tag{10.4.8}$$

we have the closed-loop system

$$\dot{x}(t) = (A - BKC)x(t).$$

The **output feedback EVA problem** then can be defined as follows.

Given the system (10.2.1), find a feedback matrix K such that the matrix $A - BKC$ has a preassigned set of eigenvalues.

The following is a well-known result by Kimura (1975) on the solution of the output feedback problem.

Theorem 10.4.2. *The Output Feedback EVA Theorem. Let (A, B) be controllable and (A, C) be observable. Let* $\text{rank}(B) = m$ *and* $\text{rank}(C) = r$. *Assume that $n \leq r + m - 1$. Then an almost arbitrary set of distinct eigenvalues can be assigned by the output feedback law* (10.4.8).

10.4.3 Eigenstructure Assignment

So far, we have considered the problem of only assigning the eigenvalues. However, **if the system transient response needs to be altered, then the problem of assigning both eigenvalues and eigenvectors needs to be considered.** This can be seen as follows. We have taken the discussion here from Andry *et al.* (1983).

Suppose that the eigenvalues $\lambda_k, k = 1, \ldots, n$ of A are distinct. Let $M = (v_1, \ldots, v_n)$ be the matrix of eigenvectors, which is necessarily nonsingular.

Then every solution $x(t)$ of the system:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0,$$

representing a free response can be written as

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}} = M^{-1}x_0$.

Thus, from above, we see that the **eigenvalues determine the rate at which the system response decays or grows** and the **eigenvectors determine the shape of the response.**

The problem of assigning both eigenvalues and eigenvectors is called the **eigenstructure assignment problem.**

Formally, the problem is stated as follows:

> Given the sets $S = \{\mu_1, \ldots, \mu_n\}$ and $M = \{v_1, \ldots, v_n\}$ of scalars and vectors, respectively, both closed under complex conjugation, find a feedback matrix K such that the matrix $A + BK$ has the μ_is as the eigenvalues and the v_is as the corresponding eigenvectors.

The following result, due to Moore (1976), gives a necessary and sufficient condition for a solution of the eigenstructure assignment problem by state feedback (see Andry *et al.* (1983) for details and proof).

Define

$$R_\lambda = \begin{bmatrix} N_\lambda \\ M_\lambda \end{bmatrix},$$

where the columns of R_λ form a basis for the null space of the matrix $(\lambda I - A, B)$.

Theorem 10.4.3. *The State-Feedback Eigenstructure Assignment Theorem. Assume that the numbers $\{\mu_i\}$ in the set S are distinct and self-conjugate. Then there exists a matrix K such that $(A + BK)v_i = \mu_i v_i$, $i = 1, \ldots, n$ if and only if the following conditions are satisfied:*

(i) *The vectors $v_1, \ldots, v_n$ are linearly independent*
(ii) *$v_i = v_j^*$ whenever $\mu_i = \mu_j^*$, $i = 1, 2, \ldots, n$*
(iii) *$v_i \in \mathrm{span}\{N_{\mu_i}\}$, $i = 1, 2, \ldots, n$.*

If B has full rank and K exists, then it is unique. When μ_is are all real and distinct, an expression for K is

$$K = (-M_{\mu_1}z_1, -M_{\mu_2}z_2, \ldots, -M_{\mu_n}z_n)(v_1, v_2, \ldots, v_n)^{-1},$$

where the vector z_i is given by

$$v_i = N_{\mu_i}z_i, \quad i = 1, 2, \ldots, n.$$

The following result on the eigenstructure assignment by output feedback is due to Srinathkumar (1978).

Theorem 10.4.4. *The Output Feedback Eigenstructure Assignment Theorem. Let (A, B) be controllable and (A, C) be observable. Assume that* $\mathrm{rank}(B) = m$ *and* $\mathrm{rank}(C) = r$. *Then* $\max(m, r)$ *eigenvalues and* $\max(m, r)$ *eigenvectors with* $\min(m, r)$ *entries in each eigenvector can be assigned by the output feedback law* (10.4.8).

Note: Numerically effective algorithms for the output feedback problem are rare. *Perhaps, the first comprehensive work in this context is the paper by Misra and Patel (1989), where algorithms for both the single-input and the multi-output systems, using implicit shifts, have been given.* We refer the readers to the above paper for a description of this algorithm.

10.5 THE QUADRATIC OPTIMIZATION PROBLEMS

We have just seen that if a system is controllable, then the closed-loop eigenvalues can be placed at arbitrarily chosen locations of the complex plane. But, the lack of the existence of a definite guideline of where to place these eigenvalues makes the design procedure a rather difficult one in practice. *A designer has to use his or her own intuition of how to use the freedom of choosing the eigenvalues to achieve the design objective.*

It is, therefore, desirable to have a design method that can be used as an initial design process while the designer develops his or her insight.

A "compromise" is often made in practice to obtain such an initial design process. Instead of trying to place the eigenvalues at desired locations, the system is stabilized while satisfying certain performance criterion.

Specifically, the following problem, known as the **Linear Quadratic Optimization Problem**, is solved. The problem is also commonly known as the LQR problem.

10.5.1 The Continuous-Time Linear Quadratic Regulator (LQR) Problem

> Given matrices Q and R, find a control signal $u(t)$ such that the quadratic cost function $J_C(x) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]\,dt$ is minimized, subject to $\dot{x} = Ax + Bu, x(0) = x_0$.

The matrices Q and R represent, respectively, weights for the states and the control vectors.

The quadratic form $x^T Qx$ represents the deviation of the state x from the initial state, and the term $u^T Ru$ represents the "cost" of control. The matrices Q and R need to be chosen according to the requirements of a specific design. Note that the magnitude of the control signal u can be properly controlled by choosing R appropriately. In fact, by selecting large R, $u(t)$ can be made small (see the expression of the unique control law in Theorem 10.5.1), which is desirable. The choice of Q is related to which states are to be kept small.

Unfortunately, again it is hard to set a specific guideline of how to choose Q and R. "The choice of these quantities is again more of an art than a science" (Kailath (1980, p. 219). For a meaningful optimization problem, however, it is assumed that **Q is symmetric positive semidefinite and R is symmetric positive definite. Unless mentioned otherwise, we will make these assumptions throughout the rest of the chapter.**

The solution of the above problem can be obtained via the solution of a quadratic matrix equation called the ARE, as shown by the following result. See Anderson and Moore (1990) for details.

Theorem 10.5.1. *The Continuous-Time LQR Theorem. Suppose the pair (A, B) is stabilizable and the pair (A, Q) is detectable. Then there exists a unique optimal control $u^0(t)$ which minimizes $J_C(x)$. The vector $u^0(t)$ is given by $u^0(t) = -Kx(t)$, where $K = R^{-1}B^TX$, and X is the unique positive semidefinite solution of the ARE:*

$$XA + A^TX + Q - XBR^{-1}B^TX = 0. \tag{10.5.1}$$

Furthermore, the closed-loop matrix $A - BK$ is stable and the minimum value of $J_C(x)$ is equal to $x_0^TXx_0$, where $x_0 = x(0)$.

The proof of the existence and uniqueness of the stabilizing solution (under the conditions that (A, B) is stabilizable and (A, Q) is detectable) will be deferred until Chapter 13. Here we give a proof of the optimal control part, assuming that such a solution exists.

Proof. *Proof of the Optimal Control Part of Theorem 10.5.1*

$$
\begin{aligned}
\frac{d}{dt}\left(x^{\mathrm{T}}Xx\right) &= \dot{x}^{\mathrm{T}}Xx + x^{\mathrm{T}}X\dot{x},\\
&= (Ax+Bu)^{\mathrm{T}}Xx + x^{\mathrm{T}}X(Ax+Bu),\\
&= (u^{\mathrm{T}}B^{\mathrm{T}} + x^{\mathrm{T}}A^{\mathrm{T}})Xx + x^{\mathrm{T}}X(Ax+Bu),\\
&= x^{\mathrm{T}}(A^{\mathrm{T}}X + XA)x + u^{\mathrm{T}}B^{\mathrm{T}}Xx + x^{\mathrm{T}}XBu,\\
&= x^{\mathrm{T}}(XBR^{-1}B^{\mathrm{T}}X - Q)x + u^{\mathrm{T}}B^{\mathrm{T}}Xx\\
&\quad + x^{\mathrm{T}}XBu \text{ (using (10.5.1))}\\
&= x^{\mathrm{T}}XBR^{-1}B^{\mathrm{T}}Xx + u^{\mathrm{T}}B^{\mathrm{T}}Xx + x^{\mathrm{T}}XBu + u^{\mathrm{T}}Ru\\
&\quad - u^{\mathrm{T}}Ru - x^{\mathrm{T}}Qx,\\
&= (u^{\mathrm{T}} + x^{\mathrm{T}}XBR^{-1})R(u + R^{-1}B^{\mathrm{T}}Xx) - (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)
\end{aligned}
$$

or

$$
x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru = -\frac{d}{dt}(x^{\mathrm{T}}Xx) + (u^{\mathrm{T}} + x^{\mathrm{T}}XBR^{-1})R(u + R^{-1}B^{\mathrm{T}}Xx).
$$

Integrating with respect to t from 0 to T, we obtain

$$
\begin{aligned}
&\int_0^T (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)\,dt\\
&\quad = -x^{\mathrm{T}}(T)Xx(T) + x_0^{\mathrm{T}}Xx_0 + \int_0^T (u + R^{-1}B^{\mathrm{T}}Xx)^{\mathrm{T}}R(u + R^{-1}B^{\mathrm{T}}Xx)\,dt.
\end{aligned}
$$

(Note that $X = X^{\mathrm{T}} \geq 0$ and $R = R^{\mathrm{T}} > 0$.)

Letting $T \to \infty$ and noting that $x(T) \to 0$ as $T \to \infty$, we obtain

$$
J_{\mathrm{C}}(x) = x_0^{\mathrm{T}}Xx_0 + \int_0^{\infty} (u + R^{-1}B^{\mathrm{T}}Xx)^{\mathrm{T}}R(u + R^{-1}B^{\mathrm{T}}Xx)\,dt
$$

Since R is symmetric and positive definite, it follows that $J_{\mathrm{C}}(x) \geq x_0^{\mathrm{T}}Xx_0$ for all x_0 and for all controls u. Since the first term $x_0^{\mathrm{T}}Xx_0$ is independent of u, the minimum value of $J_{\mathrm{C}}(x)$ occurs at

$$
u^0(t) = -R^{-1}B^{\mathrm{T}}Xx(t) = -Kx(t).
$$

The minimum value of $J_{\mathrm{C}}(x)$ is therefore $x_0^{\mathrm{T}}Xx_0$. ■

Definition 10.5.1. *The ARE:*

$$
XA + A^{\mathrm{T}}X + Q - XSX = 0, \tag{10.5.2}
$$

*where $S = BR^{-1}B^{\mathrm{T}}$ is called the **Continuous-Time Algebraic Riccati Equation** or in short **CARE**.*

Definition 10.5.2. *The matrix H defined by*

$$H = \begin{pmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \tag{10.5.3}$$

is the Hamiltonian matrix associated with the CARE (10.5.2).

Definition 10.5.3. *A symmetric solution X of the CARE such that $A - SX$ is stable is called a* **stabilizing solution.**

Relationship between Hamiltonian Matrix and Riccati Equations

The following theorem shows that there exists a very important relationship between the Hamiltonian matrix (10.5.3) and the CARE (10.5.2). The proof will be deferred until Chapter 13.

Theorem 10.5.2. *Let (A, B) be stabilizable and (A, Q) be detectable. Then the Hamiltonian matrix H in (10.5.3) has n eigenvalues with negative real parts, no eigenvalues on the imaginary axis and n eigenvalues with positive real parts. In this case the CARE (10.5.2) has a unique stabilizing solution X. Furthermore, the closed-loop eigenvalues, that is, the eigenvalues of $A - BK$, are the stable eigenvalues of H.*

A note on the solution of the CARE: It will be shown in **Chapter 13** that the unique stabilizing solution to (10.5.2) can be obtained by constructing an invariant subspace associated with the stable eigenvalues of the Hamiltonian matrix H in (10.5.3). Specifically, if H does not have any imaginary eigenvalue and $\binom{X_1}{X_2}$ is the matrix with columns composed of the eigenvectors corresponding to the stable eigenvalues of H, then, assuming that X_1 is nonsingular, the matrix $X = X_2X_1^{-1}$ is a unique stabilizing solution of the CARE. For details, see **Chapter 13**.

The MATLAB function **care** solves the CARE. The matrix S in CARE is assumed to be nonnegative definite.

The Continuous-Time LQR Design Algorithm

From Theorem 10.5.1, we immediately have the following LQR design algorithm.

Algorithm 10.5.1. *The Continuous-Time LQR Design Algorithm.*

Inputs. *The matrices A, B, Q, R, and $x(0) = x_0$.*

Outputs. *X—The solution of the CARE.*

K—The LQR feedback gain matrix.

$J_{C\,\min}$—The minimum value of the cost function $J_C(x)$.

Assumptions.

1. *(A, B) is stabilizable and (A, Q) is detectable.*
2. *Q is symmetric positive semidefinite and R is symmetric positive definite.*

Step 1. *Compute the stabilizing solution X of the CARE:*

$$XA + A^{\mathrm{T}}X - XSX + Q = 0, \quad S = BR^{-1}B^{\mathrm{T}}.$$

Step 2. *Compute the LQR feedback gain matrix:*

$$K = R^{-1}B^{\mathrm{T}}X.$$

Step 3. *Compute the minimum value of* $J_{\mathrm{C}}(x)$: $J_{\mathrm{C}\,\min} = x_0^{\mathrm{T}}Xx_0$.

Example 10.5.1 (LQR Design for the Inverted Pendulum). We consider Example 10.2.1 again, with A and B, the same as there and $Q = I_4$, $R = 1$, and

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 1. The unique positive definite solution X of the CARE (obtained by using MATLAB function **care**) is

$$X = 10^3 \begin{pmatrix} 0.0031 & 0.0042 & 0.0288 & 0.0067 \\ 0.0042 & 0.0115 & 0.0818 & 0.0191 \\ 0.0288 & 0.0818 & 1.8856 & 0.4138 \\ 0.0067 & 0.0191 & 0.4138 & 0.0911 \end{pmatrix}.$$

Step 2. The feedback gain matrix K is

$$K = (-1, -3.0766, -132.7953, -28.7861).$$

Step 3. The minimum value of $J_{\mathrm{C}}(x)$ is 3100.3.

The eigenvalues of $A - BK$ are: -4.8994, -4.5020, $-0.4412 \pm 0.3718j$. Thus, X is the unique positive definite stabilizing solution of the CARE.

(Note that the entries of K in this case are smaller compared to those of K in Example 10.2.1.)

Comparison of Transient Responses with Lyapunov Stabilization

Figures 10.7a and b show the transient responses of the closed-loop systems with: (i) K from Example 10.2.1 and (ii) K as obtained above. The initial condition $x(0) = (5, 0, 0, 0)^{\mathrm{T}}$.

In Figure 10.7a, the transient solutions initially have large magnitudes and then they decay rapidly. In Figure 10.7b, the solutions have smaller magnitudes but the

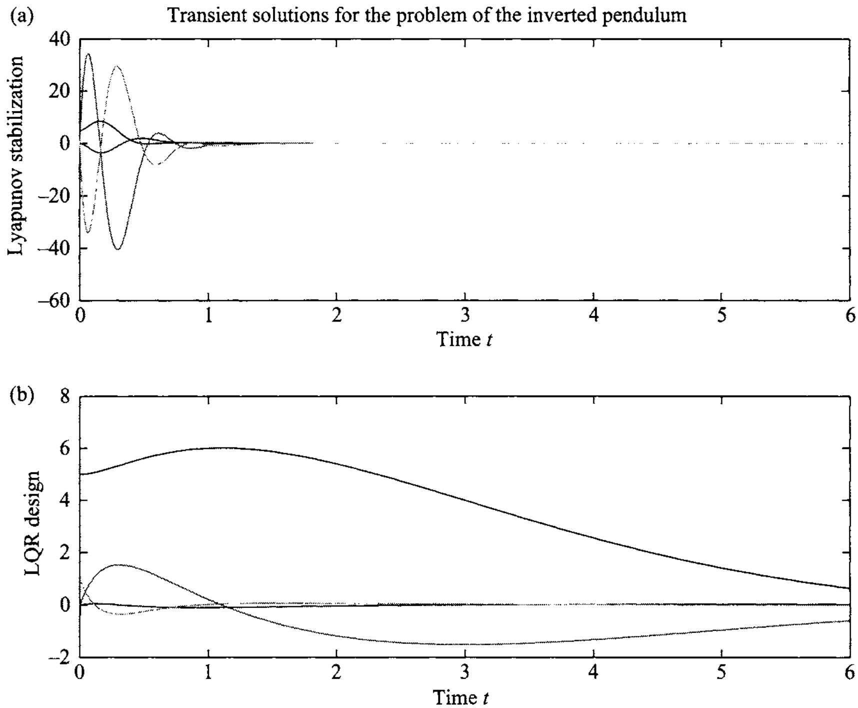

FIGURE 10.7: Transient responses: (a) Lyapunov Method and (b) LQR design.

decay rate is much slower. The largest magnitude in transient solution in (a) is roughly six times larger than the one in (b). In some dynamical systems, strong initial oscillations in the state components must be avoided, but sometimes a faster stabilization is desired; in other cases, a slow but smooth stabilization is required.

Note that the transient solutions in (a), however, depend upon β and in (b) depend upon Q and R (discussed in the following sections).

Stability and Robustness Properties of the LQR Design

The LQR design has some very favorable stability and robustness properties. We will list some important ones here.

Guaranteed Stability Properties

Clearly, the closed-loop eigenvalues of the LQR design depend upon the matrices Q and R. We will show here how the choice of R affects the closed-loop poles.

Suppose $R = \rho I$, where ρ is a positive scalar. Then, the associated Hamiltonian matrix:

$$H = \begin{pmatrix} A & -\frac{1}{\rho}BB^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{pmatrix}.$$

The closed-loop eigenvalues are the roots with negative real parts of the characteristic polynomial

$$d_{\mathrm{c}}(s) = \det(sI - H).$$

Let $Q = C^{\mathrm{T}}C$. It can be shown that

$$d_{\mathrm{c}}(s) = (-1)^n d(s)d(-s)\det\left[I + \frac{1}{\rho}G(s)G^{\mathrm{T}}(-s)\right],$$

where $d(s) = \det(sI - A)$, and $G(s) = C(sI - A)^{-1}B$.

Case 1. Low gain. When $\rho \to \infty$, $u(t) = -(1/\rho)B^{\mathrm{T}}Xx(t) \to 0$. Thus, the LQR controller has low gain. In this case, from the above expression of $d_{\mathrm{c}}(s)$, it follows that

$$(-1)^n d_{\mathrm{c}}(s) \to d(s)d(-s).$$

Since the roots with negative real parts of $d_{\mathrm{c}}(s)$; that is, the closed-loop eigenvalues, are stable, this means that **as ρ increases**:

- the stable open-loop eigenvalues remain stable.
- the unstable ones get reflected across the imaginary axis.
- if any open-loop eigenvalues are exactly on the $j\omega$-axis, the closed-loop eigenvalues start moving just left of them.

Case 2. *High gain.* If $\rho \to 0$, then $u(t)$ becomes large; thus, the LQR controller has high gain.

In this case, for finite s, the closed-loop eigenvalues approach the finite zeros of the system or their stable images.

As $s \to \infty$, the closed-loop eigenvalues will approach zeros at infinity in the so-called stable **Butterworth patterns**. (For a description of Butterworth patterns, see Friedland (1986).) An example is given in Figure 10.8.

These properties, provide good insight into the stability property of LQR controllers and, thus, can be used by a designer as a guideline of where to place the poles.

Robustness Properties of the LQR Design

As we have seen before, an important requirement of a control system design is that the closed-loop system be **robust** to uncertainties due to modeling errors, noise, and disturbances. It is shown below that the LQR design has some desirable robustness properties.

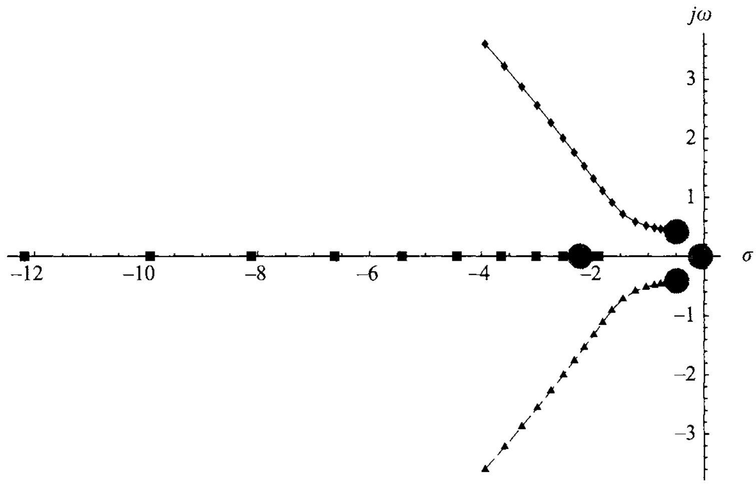

FIGURE 10.8: Illustration of Butterworth patterns.

The classical robust stability measures are **gain** and **phase margins**, defined with respect to the **Bode plot** (see **Chapter 5**) of a single-input single-output (SISO) system.

The **gain margin** is defined to be the amount of gain increase required to make the loop gain unity where the phase angle is 180°. That is, **it is the reciprocal of the gain at the frequency where the phase angle is** 180°. Thus, the *gain margin is a measure by which the system gain needs to be increased for the closed-loop system to become just unstable.*

Similarly, the difference between the phase of the response and –180° when the loop gain is unity is called the **phase margin**. *That is, the phase margin is the minimum amount of phase lag that is needed to make the system unstable.*

The robustness properties of the LQR design for a multi-input multi-output (MIMO) system can be studied by considering the **return difference matrix:** $I + G_{\mathrm{LQ}}(s)$, where $G_{\mathrm{LQ}}(s)$ is the LQR loop transfer function matrix given by

$$G_{\mathrm{LQ}}(s) = K(sI - A)^{-1}B.$$

The **optimal return difference identity** is:

$$[I+K(-sI-A)^{-1}B]^{\mathrm{T}}R[I+K(sI-A)^{-1}B] = R+B^{\mathrm{T}}(-sI-A)^{-\mathrm{T}}Q(sI-A)^{-1}B.$$

or

$$(I+G_{\mathrm{LQ}}^{\mathrm{T}}(-s))R(I+G_{\mathrm{LQ}}(s)) = R+G^{\mathrm{T}}(-s)QG(s), \text{ where } G(s) = (sI-A)^{-1}B.$$

From the above equation, we have

$$(I + G_{\text{LQ}}^*(j\omega))R(I + G_{\text{LQ}}(j\omega)) \geq R.$$

It has been shown in Safonov and Athans (1977) that if R is a diagonal matrix so that

$$(I + G_{\text{LQ}}^*(j\omega))(I + G_{\text{LQ}}(j\omega)) \geq I,$$

then there is at least 60° of phase margin in each input channel and the gain margin is infinity. *This means that a phase shift of up to 60° can be tolerated in each of the input channels simultaneously and the gain in each channel can be increased indefinitely without losing stability.* It also follows from above (**Exercise 10.20(b)**) that for all ω,

$$\sigma_{\min}(I + G_{\text{LQ}}(j\omega)) \geq 1.$$

This means that the LQ design always results in decreased sensitivity.

See Anderson and Moore (1990, pp. 122–135), the article "Optimal Control" by F.L. Lewis in the **Control Handbook** (1996, pp. 759–778) edited by William Levine, IEEE/CRC Press, and Lewis (1986, 1992), and Maciejowski (1989) for further details.

Example 10.5.2. Consider Example 10.5.1 again. For $\omega = 1$, $G_{\text{LQ}}(j\omega) = -1.9700 + 0.5345j$,

$$\sigma_{\min}(1 + G_{LQ}(j\omega)) = 1.1076,$$

$$\sigma_{\min}(1 + G_{LQ}^{-1}(j\omega)) = 0.5426,$$

$$\text{The gain margin} = 0.4907,$$

$$\text{The phase margin} = 60.0998.$$

LQR Stability with Multiplicative Uncertainty

The inequality

$$\sigma_{\min}(I + G_{\text{LQ}}(j\omega)) \geq 1$$

also implies (**Exercise 10.20(c)**) that

$$\sigma_{\min}(I + (G_{\text{LQ}}(j\omega))^{-1}) \geq \tfrac{1}{2},$$

which means that LQR design remains stable for all unmeasured multiplicative uncertainties Δ in the system for which $\sigma_{\min}(\Delta(j\omega)) \leq \frac{1}{2}$.

MATLAB notes: The MATLAB command $[K, S, E] = \boldsymbol{lqr}(A, B, Q, R)$ solves the LQR problem. Here, K—feedback matrix, S—steady-state solution of the CARE, E—the vector containing the closed-loop eigenvalues.

$$SA + A^{\mathrm{T}}S - SBR^{-1}B^{\mathrm{T}}S + Q = 0.$$

The CARE is solved using the generalized **Schur algorithm** to be described in Chapter 13.

MATLAB function **margin** can be used to compute the gain and phase margins of a system.

10.5.2 The Discrete-Time Linear Quadratic Regulator Problem

In the discrete-time case, the function to be minimized is:

$$J_{\mathrm{D}}(x) = \sum_{k=0}^{\infty}(x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k). \tag{10.5.4}$$

and the associated ARE is:

$$A^{\mathrm{T}}XA - X + Q - A^{\mathrm{T}}XB(R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA = 0. \tag{10.5.5}$$

The above equation is called the **Discrete-time Algebraic Riccati Equation** (DARE).

A theorem on the existence and uniqueness of the optimal control u_k^0, similar to Theorem 10.5.1, is stated next. For a proof, see Sage and White (1977).

Theorem 10.5.3. *The Discrete-Time LQR Theorem. Let (A, B) be discrete-stabilizable and (A, Q) be discrete-detectable. Then the optimal control u_k^0, $k = 0, 1, 2, \dots$, that minimizes $J_{\mathrm{D}}(x)$ is given by $u_k^0 = Kx_k$, where $K = (R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA$, and X is the unique positive semidefinite solution of the DARE* (10.5.5). *Furthermore, the closed-loop discrete system:*

$$x_{k+1} = (A - BK)x_k$$

is discrete-stable (i.e., all the eigenvalues are strictly inside the unit circle), and the minimum value of $J_{\mathrm{D}}(x)$ is $x_0^{\mathrm{T}}Xx_0$, where x_0 is the given initial state.

Definition 10.5.4. *A symmetric solution X of the DARE that makes the matrix $A - BK$, where $K = (R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA$, discrete-stable is called a* **discrete-stabilizing solution** *of the DARE.*

Example 10.5.3.

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = 1.$$

The solution X of the DARE (computed using MATLAB function ***dlqr***) is:

$$X = 10^3 \begin{pmatrix} 0.0051 & -0.0542 & 0.0421 \\ -0.542 & 1.0954 & -0.9344 \\ 0.0421 & -0.9344 & 0.8127 \end{pmatrix}.$$

The discrete LQR gain matrix:

$$K = (-0.0437, \;\; 2.5872, \;\; -3.4543).$$

The eigenvalues of $A - BK$ are: -0.4266, -0.2186, -0.1228. Thus, X is a discrete-stabilizing solution.

MATLAB note: The MATLAB function **lqrd** computes the discrete-time feedback-gain matrix given in Theorem 10.5.3.

10.6 H_∞-CONTROL PROBLEMS

So far we have considered the stabilization of a system ignoring any effect of disturbances in the system. But, we already know that in practice a system is always acted upon by some kind of disturbances. Thus, it is desirable to stabilize perturbed versions of a system, assuming certain bounds for perturbations. This situation gives rise to the well-known "**H_∞-control problem.**"

H_∞-control theory has been the subject of intensive study for the last twenty years or so, since its introduction by Zames (1981). There are now excellent literature in the area: the books by Francis (1987), Kimura (1996), Zhou *et al.* (1996), Green and Limebeer (1995), etc., and the original important papers by Francis and Doyle (1987), Doyle *et al.* (1989), etc.

Let $\sigma_{\max}(M)$ and $\sigma_{\min}(M)$ denote, respectively, the largest and smallest singular value of M.

Definition 10.6.1. *The H_∞-norm of the stable transfer function $G(s)$, denoted by $\|G\|_\infty$, is defined by*

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega)).$$

In the above definition, "sup" means the supremum or the least upper bound of the function $\sigma_{\max}(G(j\omega))$.

Physical Interpretation of the H_∞-norm

Consider the system:

$$y(s) = G(s)u(s).$$

When the system is driven with a sinusoidal input of unit magnitude at a specific frequency, $\sigma_{\max}(G(j\omega))$ is the largest possible size of the output for the corresponding sinusoidal input. Thus, the H_∞-norm gives the largest possible amplification over all frequencies of a unit sinusoidal input.

A detailed discussion of H_∞ control problems is beyond the scope of this book. The original formulation was in an input/output setting. However, due to its computational attractiveness, the recent state-space formulation has become more popular. We only state two **simplified versions** of the state-space formulations of H_∞-control problems, and mention their connections with AREs. First, we prove the following well-known result that shows how the H_∞-norm of a stable transfer function matrix is related to an ARE or equivalently, to the associated Hamiltonian matrix.

Define the **Hamiltonian matrix** out of the coefficients of the matrices of the system (10.2.1)

$$M_\gamma = \begin{pmatrix} A + BR^{-1}D^{\mathrm{T}}C & BR^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}(I + DR^{-1}D^{\mathrm{T}})C & -(A + BR^{-1}D^{\mathrm{T}}C)^{\mathrm{T}} \end{pmatrix}, \tag{10.6.1}$$

where $R = \gamma^2 I - D^{\mathrm{T}}D$.

Theorem 10.6.1. *Let $G(s)$ be a stable transfer function and let $\gamma > 0$. Then $\|G\|_\infty < \gamma$ if and only if $\sigma_{\max}(D) < \gamma$ and M_γ defined by* (10.6.1) *has no imaginary eigenvalues.*

Proof. We sketch the proof in case $D = 0$. This proof can easily be extended to the case when $D \neq 0$, and is left as an exercise (**Exercise 10.23**). Without any loss of generality, assume that $\gamma = 1$. Otherwise, we can scale G to $\gamma^{-1}G$ and B to $\gamma^{-1}B$ and work with scaled G and B (note that $\|G\|_\infty < \gamma$ if and only if $\|\gamma^{-1}G\|_\infty < 1$).

Since $\gamma = 1$ and $D = 0$, we have $R = I$. Using the notation:

$$G(s) = C(sI - A)^{-1}B \equiv \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

an easy computation shows that if

$$\Gamma(s) = I - G(-s)^{\mathrm{T}}G(s), \tag{10.6.2}$$

then

$$\Gamma^{-1}(s) = \left[\begin{array}{cc|c} A & BB^{\mathrm{T}} & B \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} & 0 \\ \hline 0 & B^{\mathrm{T}} & I \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{M_\gamma} & B \\ & & 0 \\ \hline 0 & B^{\mathrm{T}} & I \end{array}\right]. \tag{10.6.3}$$

Therefore, from above it follows that M_γ does not have an eigenvalue on the imaginary axis if and only if $\Gamma^{-1}(s)$ does not have any pole there. We now show that this is true if and only if $\|G\|_\infty$ is less than 1.

If $\|G\|_\infty < 1$, then $I - G(j\omega)^*G(j\omega) > 0$ for every ω, and hence $\Gamma^{-1}(s) = (I - G(-s)^{\mathrm{T}}G(s))^{-1}$ does not have any pole on the imaginary axis. On the other hand, if $\|G\|_\infty \geq 1$, then $\sigma_{\max}(G(j\omega)) = 1$ for some ω, which means that 1 is an eigenvalue of $G(j\omega)^*G(j\omega)$, implying that $I - G(j\omega)^*G(j\omega)$ is singular. ■

The following simple example from Kimura (1996, p. 41) illustrates Theorem 10.6.1.

Example 10.6.1. Let

$$G(s) = \frac{1}{s+\alpha}, \quad \alpha > 0.$$

The associated Hamiltonian matrix

$$H = \begin{pmatrix} -\alpha & 1 \\ -1 & \alpha \end{pmatrix}.$$

Then H does not have any imaginary eigenvalue if and only if $\alpha > 1$.

Since $\|G\|_\infty = \sup_\omega 1/\sqrt{\omega^2+\alpha^2} = 1/\alpha$, we have, for $\alpha > 1$, $\|G\|_\infty < 1$.

10.6.1 Computing the H_∞-Norm

A straightforward way to compute the H_∞-norm is:

1. Compute the matrix $G(j\omega)$ using the Hessenberg method described in Chapter 5.
2. Compute the largest singular value of $G(j\omega)$.
3. Repeat steps 1 and 2 for many values of ω.

Certainly the above approach is impractical and computationally infeasible.

However, Theorem 10.6.1 gives us a more practically feasible method for computing the H_∞-norm. The method, then, will be as follows:

1. Choose γ.
2. Test if $\|G\|_\infty < \gamma$, by computing the eigenvalues of M_γ and seeing if the matrix M_γ has an imaginary eigenvalue.
3. Repeat, by increasing or decreasing γ, accordingly as $\|G\|_\infty < \gamma$ or $\|G\|_\infty \geq \gamma$.

The following bisection method of Boyd *et al.* (1989) is an efficient and systematic implementation of the above idea.

Algorithm 10.6.1. *The Bisection Algorithm for Computing the* H_∞**-Norm**

Inputs. *The system matrices* A, B, C, *and* D, *of dimensions* $n \times n$, $n \times m$, $r \times n$, *and* $r \times m$, *respectively.*

γ_{lb}—*A lower bound of the* H_∞*-norm*
γ_{ub}—*An upper bound of the* H_∞*-norm*
$\epsilon (> 0)$—*Error tolerance.*

Output. *An approximation of the* H_∞*-norm with a relative accuracy of* ϵ.

Assumption. A *is stable.*

Step 1. *Set* $\gamma_{\mathrm{L}} \equiv \gamma_{\mathrm{lb}}$, *and* $\gamma_{\mathrm{U}} = \gamma_{\mathrm{ub}}$

Step 2. *Repeat until* $\gamma_{\mathrm{U}} - \gamma_{\mathrm{L}} \leq 2\epsilon\gamma_{\mathrm{L}}$
Compute $\gamma = (\gamma_{\mathrm{L}} + \gamma_{\mathrm{U}})/2$
Test if M_γ *defined by* (10.6.1) *has an imaginary eigenvalue*
If not, set $\gamma_{\mathrm{U}} = \gamma$
Else, set $\gamma_{\mathrm{L}} = \gamma$.

Remark

- After k iterations, we have $\gamma_{\mathrm{U}} - \gamma_{\mathrm{L}} = 2^{-k}(\gamma_{\mathrm{ub}} - \gamma_{\mathrm{lb}})$. Thus, on exit, the algorithm is guaranteed to give an approximation of the H_∞-norm with a relative accuracy of ϵ.

Convergence: The convergence of the algorithm is **linear** and is independent of the data matrices A, B, C, and D.

Note: An algorithm equivalent to the above bisection algorithm was also proposed by Robel (1989).

Remark

- The condition that M_γ does not have an imaginary eigenvalue (in Step 2) can also be expressed in terms of the associated Riccati equation:

$$XA + A^{\mathrm{T}}X + \gamma^{-1}XBR^{-1}B^{\mathrm{T}}X + \gamma^{-1}C^{\mathrm{T}}C = 0$$

(Assuming that $D = 0$.)

Example 10.6.2.

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = (1,\ 1,\ 1), \quad D = 0, \quad \epsilon = 0.0014.$$

Step 1. $\gamma_L = 0.2887, \qquad \gamma_U = 1.7321.$
Step 2.
Iteration 1

$$\gamma = 1.0104.$$

The eigenvalues of M_γ are $\{2, \ 4, \ -0.1429, \ 0.1429 \ -2.0000, \ -4, 0000\}$. Since M_γ does not have purely imaginary eigenvalues, we continue.

$$\gamma_L = 0.28867, \qquad \gamma_U = 1.0103.$$

Iteration 2

$$\gamma = 0.6495.$$

The eigenvalues of M_γ are $\{2, \ 4, \ -2, \ -4, \ -0 \pm 1.1706j\}$.
Since M_γ has a purely imaginary eigenvalue, we set

$$\gamma_L = 0.6495, \qquad \gamma_U = 1.0103,$$

Iteration 3

$$\gamma = 0.8299.$$

The eigenvalues of M_γ are $\{2, \ 4, \ -2, \ -4, \ 0 \pm 0.6722j\}$. Since M_γ has a purely imaginary eigenvalue, we set

$$\gamma_L = 0.8299, \qquad \gamma_U = 1.0103.$$

The values of γ at successive iterations are found to be 0.9202, 0.9653, 0.9878, 0.9991, 0.9998, and 1; and the iterations terminated at this point satisfying the stopping criterion. Thus, we take H_∞-norm $= 1$.

Computing γ_{lb} and γ_{ub}: For practical implementation of the above algorithm, we need to know how to compute γ_{lb} and γ_{ub}. We will discuss this aspect now.

Definition 10.6.2. *The* **Hankel singular values** *are the square roots of the eigenvalues of the matrix $C_G O_G$, where C_G and O_G are the controllability and observability Grammians, respectively.*

The bounds γ_{lb} and γ_{ub} may be computed using the *Hankel singular values* as follows:

$$\begin{aligned} \gamma_{\text{lb}} &= \max\{\sigma_{\max}(D), \sigma H_1\}, \\ \gamma_{\text{ub}} &= \sigma_{\max}(D) + 2\sum_{i=1}^{n} \sigma H_i, \end{aligned} \tag{10.6.4}$$

where σH_ks are the ordered **Hankel singular values,** that is, σH_i is the ith largest Hankel singular value. These bounds are due to Enns (1984) and Glover (1984) and the formula (10.6.4) is known as the **Enns–Glover formula**.

A scheme for computing γ_{lb} and γ_{ub} will then be as follows:

1. Solve the Lyapunov equations (7.2.11) and (7.2.12) to obtain C_{G} and O_{G}, respectively.
2. Compute the eigenvalues of $C_{\text{G}} O_{\text{G}}$.
3. Obtain the Hankel singular values by taking the square roots of the eigenvalues of $C_{\text{G}} O_{\text{G}}$.
4. Compute γ_{lb} and γ_{ub} using the above Enns–Glover formula.

As an alternative to eigenvalue computations, one can also use the following formulas:

$$\gamma_{\text{lb}} = \max\{\sigma_{\max}(D), \sqrt{\text{Trace}(C_{\text{G}} O_{\text{G}})/n}\},$$
$$\gamma_{\text{ub}} = \sigma_{\max}(D) + 2\sqrt{n\text{Trace}(\text{C}_{\text{G}} O_{\text{G}})}.$$

Remark

- Numerical difficulties can be expected in forming the product $C_{\text{G}} O_{\text{G}}$ explicitly.

MATCONTROL note: Algorithm 10.6.1 has been implemented in MATCONTROL function **hinfnrm.**

The Two-Step Algorithm

Recently, Bruinsma and Steinbuch (1990) have developed a **"two-step"** algorithm to compute H_∞-norm of $G(s)$. Their algorithm is believed to be faster than the bisection algorithm just stated. **The convergence is claimed to be quadratic.**

The algorithm is called a "two-step" algorithm, because, the algorithm starts with some lower bound of $\gamma < \|G\|_\infty$, as the first step and then in the second step, this lower bound is iteratively improved and the procedure is continued until some "tolerance" is satisfied.

A New Lower Bound of the H_∞-norm

The two-step algorithm, like the bisection algorithm, requires a starting value for γ_{lb}. The Enns–Glover formula can be used for this purpose. However, the authors have proposed that the following starting value for γ_{lb} be used:

$$\gamma_{\text{lb}} = \max\{\sigma_{\max}(G(0)), \sigma_{\max}(G(j\omega_{\text{p}})), \sigma_{\max}(D)\},$$

where $\omega_{\text{p}} = |\lambda_i|$, λ_i a pole of $G(s)$ with λ_i selected to maximize

$$\left|\frac{\text{Im}(\lambda_i)}{\text{Re}(\lambda_i)}\frac{1}{|\lambda_i|}\right|,$$

if $G(s)$ has poles with $\text{Im}(\lambda_i) \neq 0$ or to minimize $|\lambda_i|$, if $G(s)$ has only real poles.

Algorithm 10.6.2. *The Two-Step Algorithm for Computing the H_∞-norm*

Inputs. *The system matrices A, B, C, and D, respectively, of dimensions $n \times n$, $n \times m$, $r \times n$, and $r \times m$. ϵ–error tolerance.*

Output. *An approximate value of the H_∞-norm.*

Assumption. *A is stable.*

Step 1. *Compute a starting value for γ_{lb}, using the above criterion.*

Step 2. *Repeat*

2.1 *Compute $\gamma = (1 + 2\epsilon)\gamma_{\text{lb}}$.*

2.2 *Compute the eigenvalues of M_γ with the value of γ computed in Step* 2.1. *Label the purely imaginary eigenvalues of M_γ as $\omega_1, \ldots, \omega_k$.*

2.3 *If M_γ does not have a purely imaginary eigenvalue, set $\gamma_{\text{ub}} = \gamma$ and stop.*

2.4 *For $i = 1, \ldots, k - 1$ do*

(a) Compute $m_i = \frac{1}{2}(\omega_i + \omega_{i+1})$.

(b) Compute the singular values of $G(jm_i)$.

Update $\gamma_{\text{lb}} = \max\limits_i(\sigma_{\max}(G(jm_i))$.

End

Step 3. $\|G\|_\infty = \frac{1}{2}(\gamma_{\text{lb}} + \gamma_{\text{ub}})$.

MATLAB note: Algorithm 10.6.2 has been implemented in MATLAB Control System tool box. The usage is: **norm** (sys, inf).

In the above, "sys" stands for the linear time-invariant system in the matrices A, B, C, and D. "sys" can be generated as follows:

$A = [\ ]$, $B = [\ ]$, $C = [\ ]$, $D = [\ ]$, sys $= ss(A, B, C, D)$.

Remark

- Boyd and Balakrishnan (1990) have also proposed an algorithm similar to the above "two-step" algorithm. Their algorithm converges quadratically. Algorithm 10.6.2 is also believed to converge quadratically, but no proof has been given. See also the paper by Lin *et al.* (1999) for other recent reliable and efficient methods for computing the H_∞-norms for both the state and output feedback problems.

Connection between H_∞-norm and the Distance to Unstable Matrices

Here we point out a connection between the H_∞-norm of the resolvent of A and $\beta(A)$, the distance to the set of unstable matrices. The proof is easy and left as an [**Exercise 10.27**].

Theorem 10.6.2. *Let A be a complex stable matrix, then $\beta(A) = \|(sI - A)^{-1}\|_\infty^{-1}$.*

Computing the H_∞-Norm of a Discrete-Stable Transfer Function Matrix

Let $M(z) = C(zI - A)^{-1}B$ be the transfer function matrix of the **asymptotically stable discrete-time system:**

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k. \end{aligned}$$

Then

Definition 10.6.3. *The H_∞-norm of $M(z)$ is defined as*

$$\|M(z)\|_\infty = \sup_{|z|\geq 1} \sigma_{\max}(M(z)).$$

The following is a discrete-analog of Theorem 10.6.1. We state the result here without proof. For proof, we refer the readers to the book by Zhou *et al.* (1996, pp. 547–548).

Theorem 10.6.3. *Let*

$$S = \begin{pmatrix} A - BB^{\mathrm{T}}(A^{\mathrm{T}})^{-1}C^{\mathrm{T}}C & BB^{\mathrm{T}}(A^{\mathrm{T}})^{-1} \\ -(A^{\mathrm{T}})^{-1}C^{\mathrm{T}}C & (A^{\mathrm{T}})^{-1} \end{pmatrix}$$

be the symplectic matrix associated with the above stable discrete-time system. Assume that A is nonsingular and that the system does not have any uncontrollable and unobservable modes on the unit circle.

Then $\|M(z)\|_\infty < 1$ if and only if S has no eigenvalues on the unit circle and $\|C(I - A)^{-1}B\|_2 < 1$.

Computing H_∞-Norm of a Discrete-Stable System

The above theorem can now be used as a basis to develop a **bisection algorithm**, analogous to Algorithm 10.6.1, for computing the H_∞-norm of a discrete stable system. We leave this as an exercise (**Exercise 10.24**).

10.6.2 H_∞-Control Problem: A State-Feedback Case

Consider the following linear control system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \quad x(0) = 0 \\ z(t) &= Cx(t) + Du(t). \end{aligned} \tag{10.6.5}$$

Here $x(t)$, $u(t)$, and $z(t)$, denote the state, the control input, and controlled output vectors. The vector $d(t)$ is the disturbance vector. The matrices A, B, C, D, and E are matrices of appropriate dimensions. Suppose that a state-feedback control law

$$u(t) = Kx(t)$$

is applied to the system. Then the closed-loop system becomes:

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + Ed(t) \\ z(t) &= (C + DK)x(t). \end{aligned} \tag{10.6.6}$$

The transfer function from d to z is:

$$T_{zd}(s) = (C + DK)(sI - A - BK)^{-1}E. \tag{10.6.7}$$

Suppose that the influence of the disturbance vector $d(t)$ on the output $z(t)$ is measured by the H_∞-norm of $T_{zd}(s)$. **The goal of the state feedback H_∞ control problem is to find a constant feedback matrix K such that the closed-loop system is stable and the H_∞-norm of the transfer matrix $T_{zd}(s)$ is less than a prescribed tolerance.**

Specifically, the state feedback H_∞ problem is stated as follows:

Given a positive real number γ, find a real $m \times n$ matrix K such that the closed-loop system is stable and that $\|T_{zd}(s)\|_\infty < \gamma$.

Thus, by solving the above problem, one will stabilize perturbed versions of the original system, as long as the size of the perturbations does not exceed a certain given tolerance.

The following theorem due to Doyle *et al.* (1989) states a solution of the above problem in terms of the solution of an ARE.

Theorem 10.6.4. *A State-Feedback H_∞ Theorem. Let the pair (A, C) be observable and the pairs (A, B), and (A, E) be stabilizable. Assume that $D^T D = I$, and $D^T C = 0$. Then the H_∞ control problem (as stated above) has a solution if and only if there exists a positive semi-definite solution X of* *the ARE:*

$$A^{\mathrm{T}}X + XA - X\left(BB^{\mathrm{T}} - \frac{1}{\gamma^2}EE^{\mathrm{T}}\right)X + C^{\mathrm{T}}C = 0, \tag{10.6.8}$$

such that

$$A + \left(\frac{1}{\gamma^2}EE^{\mathrm{T}} - BB^{\mathrm{T}}\right)X$$

is stable. In this case one such state feedback matrix K is given by

$$K = -B^{\mathrm{T}}X. \tag{10.6.9}$$

Proof. The proof follows by noting the relationship between the ARE (10.6.8) and the associated Hamilton matrix:

$$H_\gamma = \begin{pmatrix} A & \frac{1}{\gamma^2}EE^{\mathrm{T}} - BB^{\mathrm{T}} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{pmatrix}, \tag{10.6.10}$$

as stated in Theorem 10.5.2, and then applying Theorem 10.6.1 to the transfer function matrix $T_{zd}(s)$. ■

Notes

1. The application of Theorem 10.6.1 to $T_{zd}(s)$ amounts to replacing A, B, C, and R of Theorem 10.6.1 as follows:

$$\begin{aligned} &A \to A + BK = A - BB^{\mathrm{T}}X, \\ &C \to C + DK = C - DB^{\mathrm{T}}X, \\ &B \to E, \\ &R \to \gamma^2 I - I = (\gamma^2 - 1)I, \end{aligned}$$

   and using the assumptions $D^{\mathrm{T}}D = I$ and $D^{\mathrm{T}}C = 0$.
2. The Riccati equation (10.6.8) is not the standard LQ Riccati equation, the CARE (Eq. (10.5.2)), because the term $(BB^{\mathrm{T}} - (1/\gamma^2)EE^{\mathrm{T}})$ may be indefinite for certain values of γ.

   However, when $\gamma \to \infty$, the Riccati equation (10.6.8) becomes the CARE with $R = I$:

$$XA + A^{\mathrm{T}}X - XBB^{\mathrm{T}}X + C^{\mathrm{T}}C = 0.$$

3. It can be shown **(Exercise 10.26)** that if H_γ has imaginary eigenvalues, then the H_∞ problem as defined above does not have a solution.

In a more realistic situation when a dynamic measurement feedback is used, rather than the state feedback as used here, the solution of the corresponding H_∞-control problem is provided by a **pair of AREs.** Details can be found in the pioneering paper of Doyle *et al.* (1989), and in the recent books by Kimura (1996), Green and Limebeer (1995), Zhou *et al.* (1996). We only state the result from the paper of Doyle *et al.* (1989).

10.6.3 The H_∞-Control Problem: Output Feedback Case

Consider a system described by the state-space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t), \end{aligned} \tag{10.6.11}$$

where $x(t)$—the state vector, $w(t)$—the disturbance signal, $u(t)$—the control input, $z(t)$— the controlled output, and $y(t)$—the measured output.

The transfer function from the inputs $\begin{bmatrix} w \\ u \end{bmatrix}$ to the outputs $\begin{bmatrix} z \\ y \end{bmatrix}$ is given by

$$G(s) = \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1}(B_1, B_2) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}.$$

Define a feedback controller $K(s)$ by $u = K(s)y$.

Then the closed-loop transfer function matrix $T_{zw}(s)$ from the disturbance w to the output z is given by

$$T_{zw}(s) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

Then the goal of the output feedback H_∞-control problem in this case is to find a controller $K(s)$ that $\|T_{zw}(s)\|_\infty < \gamma$, for a given positive number γ.

Figure 10.9, P is the linear system described by 10.6.11.

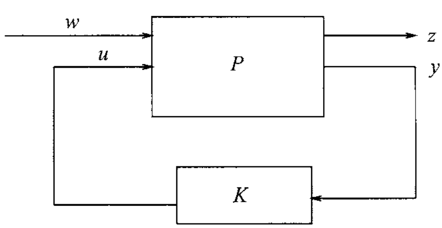

FIGURE 10.9: Output feedback H_∞ configuration.

A solution of the above H_∞-control problem is given in the following theorem. The following **assumptions** are made:

(i) (A, B_1) is stabilizable and (A, C_1) is detectable. (10.6.12)

(ii) (A, B_2) is stabilizable and (A, C_2) is detectable. (10.6.13)

(iii) $D_{12}^T(C_1, D_{12}) = (0, I)$ (10.6.14)

(iv) $\begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}$. (10.6.15)

Here I stands for an identity matrix of appropriate dimension.

Theorem 10.6.5. *An Output Feedback H_∞ Theorem. Under the assumptions (10.6.12–10.6.15), the output feedback H_∞-control problem as stated above has a solution if and only if there exist unique symmetric positive semidefinite stabilizing solutions X and Y, respectively, to the pair of AREs*

$$XA + A^TX - X\left(B_2B_2^T - \frac{1}{\gamma^2}B_1B_1^T\right)X + C_1^TC_1 = 0, \tag{10.6.16}$$

$$AY + YA^T - Y\left(C_2^TC_2 - \frac{1}{\gamma^2}C_1^TC_1\right)Y + B_1B_1^T = 0, \tag{10.6.17}$$

and $\rho(XY) < \gamma^2$, where $\rho(XY)$ is the spectral radius of the matrix XY.

Furthermore, in this case, one such controller is given by the transfer function

$$K(s) = -F(sI - \hat{A})^{-1}ZL, \tag{10.6.18}$$

where

$$\hat{A} = A + \frac{1}{\gamma^2}B_1B_1^TX + B_2F + ZLC_2 \tag{10.6.19}$$

and

$$F = -B_2^TX, \qquad L = -YC_2^T, \qquad Z = \left(I - \frac{1}{\gamma^2}YX\right)^{-1} \tag{10.6.20}$$

Proof. For a proof of Theorem 10.6.5, we refer the readers to the original paper of Doyle *et al.* (1989). ■

Notes

1. The second Riccati equation is dual to the first one and is of the type that arises in the Kalman filter (**Chapter 12**).
2. A general solution without the assumptions (10.6.14) and (10.6.15) can be found in Glover and Doyle (1988).

MATLAB Note: To solve the Riccati equations (10.6.16) and (10.6.17) using **care**, these equations have to be rewritten in the usual **care** format. For example, Eq. (10.6.16) can be rewritten as:

$$A^{\mathrm{T}}X + XA - X(B_1, B_2)\begin{pmatrix} -\gamma^{-2}I & 0 \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} B_1^{\mathrm{T}} \\ B_2^{\mathrm{T}} \end{pmatrix}X + C_1^{\mathrm{T}}C_1 = 0.$$

Example 10.6.3. Zhou *et al.* (1996, pp. 440–443). Let

$$A = a, \quad B_1 = (1, 0), \quad B_2 = b_2,$$

$$C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_2 = c_2, \quad D_{21} = (0, 1).$$

Then

$$D_{12}^{\mathrm{T}}(C_1, D_{12}) = (0, 1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 1)$$

and

$$\begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the conditions (10.6.14) and (10.6.15) are satisfied.

Let $a = -1$, $b_2 = c_2 = 1$. Let $\gamma = 2$.

Then it is easy to see that

$$\rho(XY) < \gamma^2,$$

$$T_{zw} = \begin{pmatrix} -1.7321 & 1 & -0.7321 \\ 1 & 0 & 0 \\ -0.7321 & 0 & -0.7321 \end{pmatrix},$$

and $\|T_{zw}\|_\infty = 0.7321 < \gamma = 2$.

Optimal H_∞ Control

The output H_∞-control problem in this section is usually known as the **Suboptimal H_∞-Control** problem.

Ideally, we should have considered **Optimal H_∞-Control problem:**

Find all admissible controllers $K(s)$ such that $\|T_{zw}\|_\infty$ is minimized.

Finding an optimal H_∞ controller is demanding both theoretically and computationally and, in practice, very often a suboptimal controller is enough, because suboptimal controllers are close to the optimal ones.

The behavior of the suboptimal H_∞ controller solution as γ approaches the infimal achievable norm, denoted by γ_{opt}, is discussed in the book by Zhou *et al.* (1996, pp. 438–443). It is shown there that for Example 10.6.3, $\gamma_{\text{opt}} = \|T_{zw}\|_\infty = 0.7321$.

10.7 THE COMPLEX STABILITY RADIUS AND RICCATI EQUATION

Assume in this section that A, B, and C are **complex matrices.** In Chapter 7, we introduced the concept of stability radius in the context of robust stabilization of the stable system $\dot{x} = Ax$ under structured perturbations of the form $B\Delta C$. The system:

$$\dot{x} = (A + B\Delta C)x \tag{10.7.1}$$

may be interpreted as a closed-loop system obtained by applying the output feedback (with unknown feedback matrix Δ) to the system (10.7.3) given below. Thus, the stability radius is related to the output feedback stabilization, as well.

In this section, we will discuss the role of the complex stability radius $r_{\mathbb{C}}(A, B, C)$ in certain parametric optimization problems.

Consider the following parametric optimization problem: Minimize

$$J_\rho(x) = \int_0^\infty \left[\|u(t)\|^2 - \rho\|y(t)\|^2\right] dt \tag{10.7.2}$$

subject to

$$\dot{x} = Ax + Bu, \qquad y = Cx. \tag{10.7.3}$$

If $\rho \leq 0$, then we have the usual LQR problem, which can be solved by solving the associated Riccati equation, as we have just seen. We will now show that for certain other values of $\rho > 0$, the above optimization problem is still solvable, by relating ρ to the stability radius. The key to that is to show the existence of a stabilizing solution of the associated Riccati equation for a given value of ρ.

Before we state the main result, we note the following, that shows that for certain values of ρ, the minimization cost is finite. **For simplicity, we will write $r_{\mathbb{C}}(A, B, C)$ as** r.

Theorem 10.7.1. *Let $J_\rho(x)$ be defined by* (10.7.2). *Then*

(i) *Inf $J_\rho(0) = 0$, if and only if $\rho \leq r^2$, if and only if $I - \rho G^*(i\omega)G(i\omega) \geq 0$, for all $\omega \in \mathbb{R}$.*
(ii) *Suppose A is stable and $r < \infty$. Then for all $\rho \in (-\infty, r^2)$, we have* $|\inf J_\rho(x_0)| <, \infty$.

Proof. See Hinrichsen and Pritchard (1986a). ■

The ARE associated with the above minimization problem is

$$XA + A^*X - \rho C^*C - XBB^*X = 0. \tag{10.7.4}$$

Since this equation is dependent on ρ, we denote this Riccati equation by ARE_ρ.
The parameterized Hamiltonian matrix associated with the ARE_ρ is

$$H_\rho = \begin{pmatrix} A & -BB^* \\ \rho C^*C & -A^* \end{pmatrix}. \tag{10.7.5}$$

The following theorems characterize $r_{\mathbb{C}}(= r)$ in terms of H_ρ.

Theorem 10.7.2. *Characterization of the Complex Stability Radius. Let H_ρ be defined by* (10.7.5). *Then $\rho < r$ if and only if H_{ρ^2} does not have an eigenvalue on the imaginary axis.*

Proof. See Hinrichsen and Pritchard (1986a). ■

Example 10.7.1. Consider Example 7.8.1.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad C = (1, 0).$$

From Example 7.8.1, we know that $r^2 = \frac{3}{4}$.
Case 1. Let $\rho = 0.5 < r_{\mathbb{C}} = r = 0.8660$. Then,

$$H_{\rho^2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 0.25 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues of H_{ρ^2} are $-0.4278 \pm 0.8264j, 0.4278 \pm 0.8264j$. Thus, H_{ρ^2} does not have a purely imaginary eigenvalue.
Case 2. Let $\rho = 1 > r_{\mathbb{C}} = r = 0.8660$. Then,

$$H_{\rho^2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues of H_{ρ^2} are $0.0000 \pm 1.0000j, 0, 0$, which are purely imaginary. Therefore we obtain an upper and a lower bound for r: $0.5 < r \leq 1$.

We have already mentioned the relationship between a Hamiltonian matrix and the associated ARE. In view of Theorem 10.7.2, therefore, the following result is not surprising. The proof of Theorem 10.7.3 has been taken from Hinrichsen and Pritchard (1986b).

Theorem 10.7.3. *Stability Radius and ARE. Let A be a complex stable matrix and let $r \equiv r_{\mathbb{C}}(A, B, C) < \infty$. Let $\rho \in (-\infty, r^2)$. Then there exists a unique Hermitian stabilizing solution X of the Riccati equation* (10.7.4)*:*

$$XA + A^*X - \rho C^*C - XBB^*X = 0.$$

Moreover, when $\rho = r^2$, there exists a unique solution X having the property that the matrix $A - BB^*X$ is unstable.

Conversely, if A is stable and if there exists a Hermitian solution X of the above ARE, then necessarily $\rho \leq r^2$.

Remark

- Note that if the Riccati equation (10.7.4) has a stabilizing solution X, then the control law

$$u(t) = -B^*Xx(t)$$

minimizes the performance index (10.7.2), and the minimum value of the performance index is $x_0^{\mathrm{T}}Xx_0$.

Note: **There is no controllability assumption here on the pair** (A, B).

Proof. Considering the orthogonal decomposition of $\mathbb{C}^n$ into the controllability subspace and its orthogonal complement, we can find a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad TB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

and

$$CT^{-1} = (C_1, C_2),$$

where (A_1, B_1) is controllable. Multiplying the Riccati equation (10.7.4) on the left by T^{-1*} and on the right by T^{-1} and setting

$$T^{-1*}XT^{-1} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

we have

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + \begin{pmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

$$-\rho\begin{pmatrix} C_1^*C_1 & C_1^*C_2 \\ C_2^*C_1 & C_2^*C_2 \end{pmatrix} - \begin{pmatrix} X_1B_1B_1^*X_1 & X_1B_1B_1^*X_2 \\ X_3B_1B_1^*X_1 & X_3B_1B_1^*X_2 \end{pmatrix} = 0 \qquad (10.7.6)$$

Eq. (10.7.6) breaks into the following four equations:

$$X_1A_1 + A_1^*X_1 - \rho C_1^*C_1 - X_1B_1B_1^*X_1 = 0, \qquad (10.7.7)$$
$$X_2A_3 + (A_1 - B_1B_1^*X_1^*)^*X_2 + X_1A_2 - \rho C_1^*C_2 = 0, \qquad (10.7.8)$$
$$X_3(A_1 - B_1B_1^*X_1) + A_3^*X_3 + A_2^*X_1 - \rho C_2^*C_1 = 0, \qquad (10.7.9)$$
$$X_4A_3 + A_3^*X_4 + X_3A_2 + A_2^*X_2 - \rho C_2^*C_2 - X_3B_1B_1^*X_2 = 0. \qquad (10.7.10)$$

Since (A_1, B_1) is controllable, there is a unique solution $X_{1\rho}$ of (10.7.7) with the property that $A_1 - B_1B_1^*X_{1\rho}$ is stable if $\rho \in (-\infty, r^2)$, and if $\rho = r^2$, then $A_1 - B_1B_1^*X_{1\rho}$ is not stable. (In fact it has one eigenvalue on the imaginary axis). Substituting this stabilizing solution $X_{1\rho}$ into the Sylvester equations (10.7.8) and (10.7.9), it can be shown that the solutions $X_{2\rho}$ and $X_{3\rho}$ of (10.7.8) and (10.7.9) are unique and moreover $X_{3\rho} = X_{2\rho}^*$ (note that the spectrum of A_3 is in the open left-half plane). Substituting these $X_{2\rho}$ and $X_{3\rho}$ in Eq. (10.7.10), similarly, we see that $X_{4\rho}$ is also unique and $X_{4\rho}^* = X_{4\rho}$. Finally, we note that the matrix $TAT^{-1} - (TB \cdot B^*T^*X_\rho)$, where

$$X_\rho = \begin{pmatrix} X_{1\rho} & X_{2\rho} \\ X_{3\rho} & X_{4\rho} \end{pmatrix},$$

is stable. Thus, $A - BB^*X_\rho$ is stable.

To prove the converse, we note that if $X = X^*$ satisfies the Riccati equation (10.7.6), then

$$(A - j\omega I)^*X + X(A - j\omega I) - \rho C^*C - XBB^*X = 0,$$

for all $\omega \in \mathbb{R}$. Thus,

$$\begin{aligned} 0 &\leq (B^*X(A - j\omega I)^{-1}B - I)^*(B^*X(A - j\omega I)^{-1}B - I), \\ &= I - \rho G^*(j\omega)G(j\omega), \qquad \text{for all } \omega \in \mathbb{R}. \end{aligned}$$

Thus, $\rho \leq r^2$ by the first part of Theorem 10.7.1. ■

Example 10.7.2.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad B = (0, -1)^{\mathrm{T}}, \qquad C = (1, 0).$$

Then we know from Example 7.8.1 that $r^2 = \frac{3}{4}$.

Choose $\rho = \frac{1}{2}$. Then the unique solution X_ρ to the Riccati equation:

$$XA + A^{\mathrm{T}}X - XBB^{\mathrm{T}}X - \rho C^{\mathrm{T}}C = 0$$

is

$$X_\rho = \begin{pmatrix} -0.5449 & -0.2929 \\ -0.2929 & -0.3564 \end{pmatrix},$$

which is symmetric.

The eigenvalues of $A - BB^{\mathrm{T}}X_\rho$ are $-0.3218 \pm 0.7769j$. So, the matrix $A - BB^{\mathrm{T}}X_\rho$ is stable. Thus, X_ρ is a stabilizing solution.

If ρ is chosen to be $\frac{3}{4}$, then the solution

$$X = \begin{pmatrix} -1 & -0.5 \\ -0.5 & -1 \end{pmatrix},$$

which is symmetric, but not stabilizing. The eigenvalues of $A - BB^{\mathrm{T}}X$ are $0 \pm 0.7071j$.

A Bisection Method for Computing the Complex Stability Radius

Theorem 10.7.2 suggests a bisection algorithm for computing the complex stability radius $r_{\mathbb{C}}$.

The idea is to determine $r_{\mathbb{C}}$ as that value of ρ for which the Hamiltonian matrix H_ρ given by (10.7.5), associated with the Riccati equation (10.7.4), has an eigenvalue on the imaginary axis for the first time.

If ρ_0^- and ρ_0^+ are some lower and upper estimates of $r_{\mathbb{C}}$, then the successive better estimates can be obtained by the following algorithm.

Algorithm 10.7.1. *A Bisection Method for the Complex Stability Radius*
Inputs.

1. *The system matrices A, B, and C.*
2. *Some upper and lower estimates ρ_0^+ and ρ_0^- of the complex stability radius ρ.*

Output. *An approximate value of the complex stability radius ρ.*
For $k = 0, 1, 2, \ldots,$ do until convergence

Step 1. *Take $\rho_k = \dfrac{\rho_k^- + \rho_k^+}{2}$ and compute $H_{\rho_k^2}$.*

Step 2. *If $H_{\rho_k^2}$ has eigenvalues on the imaginary axis, set $\rho_{k+1}^- \equiv \rho_k^-$ and $\rho_{k+1}^+ \equiv \rho_k$.*

Otherwise set $\rho_{k+1}^- \equiv \rho_k$ *and* $\rho_{k+1}^+ \equiv \rho_k^+$.
End

Example 10.7.3. Consider Example 10.7.1. Take $\rho_0^- = 0$, $\rho_0^+ = 1$.
$k = 0$. **Step 1.** $\rho_0 = \frac{1}{2}$. $H_{\rho_0^2}$ does not have an imaginary eigenvalue.
Step 2. $\rho_1^- = \frac{1}{2}$, $\rho_1^+ = 1$.
$k = 1$. **Step 1.** $\rho_1 = \frac{3}{4}$. $H_{\rho_1^2}$ does not have an imaginary eigenvalue.
Step 2. $\rho_2^- = \frac{3}{4}$, $\rho_2^+ = 1$.
$k = 2$. **Step 1.** $\rho_2 = \frac{7}{8}$. $H_{\rho_2^2}$ has an imaginary eigenvalue.
Step 2. $\rho_3^- = \frac{3}{4}$, $\rho_3^+ = \frac{7}{8}$.
$k = 3$. **Step 1.** $\rho_3 = \frac{13}{16}$. $H_{\rho_3^2}$ does not have an imaginary eigenvalue.
Step 2. $\rho_4^- = \frac{13}{16}$, $\rho_4^+ = \frac{7}{8}$.
$k = 4$. $\rho_4 = \frac{27}{32}$.
The iteration is converging toward $r = 0.8660$. The readers are asked to verify this by carrying out some more iterations.

MATCONTROL note: Algorithm 10.7.1 has been implemented in MATCONTROL function **stabradc**.

10.8 SOME SELECTED SOFTWARE

10.8.1 MATLAB Control System Toolbox

LQG design tools include:

lqr	LQ-optimal gain for continuous systems
dlqr	LQ-optimal gain for discrete systems
lqry	LQR with output weighting
lqrd	Discrete LQ regulator for continuous plant
care	Solve CARE
dare	Solve DARE
norm(sys, 2)	H_2-norm of the system
norm(sys, inf)	H_∞-norm of the system

10.8.2 MATCONTROL

STABLYAPC	Feedback stabilization of continuous-time system using Lyapunov equation
STABLYAPD	Feedback stabilization of discrete-time system using Lyupunov equation
STABRADC	Finding the complex stability radius using the bisection method
HINFNRM	Computing H_∞-norm using the bisection method.

10.8.3 CSP-ANM

Feedback stabilization

- Constrained feedback stabilization is computed by `StateFeedbackGains` [*system, region*], where the available regions are `DampingFactorRegion [`α`]`, `SettlingTimeRegion [`t_s, ϵ`]`, `DampingRatioRegion [`$\zeta_{\min}$`]` and `NaturalFrequencyRegion [`$\omega_{n\,\min}$`]`, and their intersections.
- The Lyapunov algorithms for the feedback stabilization is implemented as `StateFeedbackGains` [*system, region*, `Method` $\rightarrow$ `LyapunovShift`] and `StateFeedbackGains` [*system, region*, `Method` $\rightarrow$ `PartialLyapunovShift`].

10.8.4 SLICOT

Optimal regulator problems, system norms, and complex stability radius

SB10DD	H_∞ (sub)optimal state controller for a discrete-time system
SB10FD	H_∞ (sub)optimal state controller for a continuous-time system
AB13BD	H_2- or L_2-norm of a system
AB13CD	H_∞-norm of a continuous-time stable system
AB13ED	Complex stability radius using bisection
AB13FD	Complex stability radius using bisection and SVD.

10.8.5 MATRIX$_X$

Purpose: Computing L_∞-norm of the transfer matrix of a discrete-time system.

Syntax: [SIGMA, OMEGA] = DLINFNORM (S, NS, {TOL, { MAXITER}})

Purpose: Compute optimal state-feedback gain for discrete-time system.

Syntax: [EVAL, KR] = DREGULATOR (A, B, RXX, RUU, RXU) OR
[EVAL, KR, P] = DREGULATOR (A, B, RXX, RUU, RXU)

Purpose: Computing L_∞-norm of a transfer matrix .

Syntax: [SIGMA, OMEGA] = LINFNORM (S, NS, { TOL, { MAXITER} })

Purpose: Compute optimal state-feedback gain for continuous-time system.

Syntax: [EVAL, KR]=REGULATOR (A, B, RXX, RUU, RXU) OR
[EVAL, KR, P]=REGULATOR (A, B, RXX, RUU, RXU)

Purpose: Computes and plots the Singular Values of a continuous system.

Syntax: [OMEGA, SVALS]=SVPLOT (S, NS, WMIN, WMAX, { NPTS} , {OPTIONS }) OR
[SVALS]=SVPLOT (S, NS, OMEGA, { OPTIONS})

10.9 SUMMARY AND REVIEW

The following topics have been discussed in this chapter.

- State-feedback stabilization
- EVA and eigenstructure assignments by state and output feedback
- The LQR design
- H_∞-control problems
- Stability radius.

Feedback Stabilization

The problem of stabilizing the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

by using the state-feedback law $u(t) = -Kx(t)$ amounts to finding a feedback matrix K such that $A - BK$ is stable.

The state-feedback stabilization of a discrete-time system is similarly defined.

The **characterizations** of the continuous-time and discrete-time state-feedback stabilizations are, respectively, given in **Theorems 10.2.1** and **10.2.2**.

It is shown how a system can be stabilized using the solution of a Lyapunov equation. For the continuous-time system, the Lyapunov equation to be solved is

$$-(A + \beta I)X + X(-(A + \beta I))^{\mathrm{T}} = -2BB^{\mathrm{T}},$$

where β is chosen such that $\beta > |\lambda_{\max}(A)|$.

The stabilizing feedback matrix K is given by

$$K = B^{\mathrm{T}} X^{-1}.$$

In the discrete-time case, the Lyapunov equation to be solved is

$$AXA^{\mathrm{T}} - \beta^2 X = 2BB^{\mathrm{T}},$$

where β is chosen such that $0 < \beta \leq 1$ and $|\lambda| \geq \beta$ for any eigenvalue λ of A. The stabilizing feedback matrix in this case is

$$K = B^{\mathrm{T}}(X + BB^{\mathrm{T}})^{-1} A.$$

Detectability

The detectability of the pair (A, C) is a dual concept of the stabilizability of the pair (A, B). Characterizations of the continuous-time and discrete-time detectability are, respectively, stated in **Theorems 10.3.1** and **10.3.3.**

The Eigenvalue Assignment

For the transient responses to meet certain designer's constraints, it is required that the eigenvalues of the closed-loop matrix lie in certain specified regions of the complex plane. This consideration gives rise to the well-known EVA problem.

The EVA problem by state feedback is defined as follows:

Given the pair (A, B) and a set Λ of the complex numbers, closed under complex conjugations, find a feedback matrix K such that $A - BK$ has the spectrum Λ.

The conditions of solvability for the EVA problem and the uniqueness of the matrix K are:

> There exists a matrix K such that the matrix $A - BK$ has the spectrum Λ for every complex-conjugated set Λ if and only if (A, B) is controllable. The feedback matrix K, when it exists, is unique if and only if the system is a single-input system **(Theorem 10.4.1)**.

The constructive proof of Theorem 10.4.1 and several related well-known formulas such as the **Bass–Gura formula** and the **Ackermann formula** suggest computational methods for **single-input EVA problem.** Unfortunately, however, these methods are based on the reduction of the pair (A, b) to a controller–companion pair, and **are not numerically effective.** Numerically effective algorithms for EVA are based on the reduction of the pair (A, b) or the pair (A, B) (in the multi-output case) to the controller–Hessenberg pair. **These methods will be described in Chapter 11.**

The EVA problem by output feedback is discussed in Section 10.4.2 and a well-known result on this problem is stated in **Theorem 10.4.2**.

The Eigenstructure Assignment

The eigenvalues of the state matrix A determine the rate at which the system response decays or grows. On the other hand, the eigenvectors determine the shape of the response. Thus, in certain practical applications, it is important that both the eigenvalues and the eigenvectors are assigned. The problem is known as the **eigenstructure assignment problem.** The conditions under which eigenstructure assignment is possible are stated in Theorem **10.4.3** for the state-feedback law and in Theorem **10.4.4** for the output feedback law.

The Linear Quadratic Regulator (LQR) Problem

The continuous-time LQR problem is the problem of finding a control vector $u(t)$ that minimizes the performance index

$$J_{\mathrm{C}}(x) = \int_0^{\infty} \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] dt$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$
$$y(t) = Cx(t),$$

where Q and R are, respectively, the state-weight and the control-weight matrices.

It is shown in **Theorem 10.5.1** that the continuous-time LQR problem has a solution if (A, B) is stabilizable and (A, Q) is detectable.

The solution is obtained by solving the CARE:

$$XA + A^{\mathrm{T}}X - XSX + Q = 0,$$

where $S = BR^{-1}B^{\mathrm{T}}$.

The optimal control vector $u^0(t)$ is given by

$$u^0(t) = -R^{-1}B^{\mathrm{T}}Xx(t),$$

where X is the unique symmetric positive semidefinite solution of the CARE.

The matrix $K = -R^{-1}TB^{\mathrm{T}}X$ is such that $A - BK$ is stable, that is, X is a stabilizing solution.

The LQR design has the following **guaranteed stability and robustness properties:**

Stability property. The stable open-loop eigenvalues remain stable and the unstable eigenvalues get reflected across the imaginary axis (when $R = \rho I$ and $\rho \to \infty$).

Robustness properties. Using the optimal return difference identity, it can be shown that

$$\sigma_{\min}(I + G_{\mathrm{LQ}}(j\omega)) \geq 1$$

and $\sigma_{\min}(I + G_{\mathrm{LQ}}(j\omega)^{-1}) \geq \frac{1}{2}$, where $G_{\mathrm{LQ}}(s) = K(sI - A)^{-1}B$.

These relations show that the upward and downward gain margins are, respectively, ∞ and 0.5. The phase margin is at least $\pm 60°$.

The discrete-time LQR problem is similarly defined. In this case, the performance index is given by

$$J_{\mathrm{D}}(x) = \sum_{k=0}^{\infty} \left(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right).$$

The DARE is

$$A^{\mathrm{T}}XA - X + Q - A^{\mathrm{T}}XB(R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA = 0.$$

If (A, B) is discrete-stabilizable and (A, Q) is discrete-detectable, then the above Riccati equation (DARE) has a unique symmetric positive semidefinite solution X and the optimal control is $u_k^0 = Kx_k$, where

$$K = (R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA.$$

Furthermore, X is a discrete-stabilizing solution, that is, $(A - BK)$ is discrete-stable.

H_∞-Control Problems

The H_∞-control problems are concerned with robust stabilization for unstructured perturbations in the frequency domain. The goal of a H_∞ control is to determine a controller that guarantees a closed-loop system with an H_∞-norm bounded by a certain specified quantity γ when such a controller exists. Both the state feedback and the output feedback H_∞-control problems have been discussed briefly in Sections 10.6.2 and 10.6.3, respectively. Both problems require solutions of certain Riccati equations. Under the assumptions (10.6.12)–(10.6.15), the solution of the output feedback H_∞-control problem reduces to solving a pair of Riccati equations:

$$XA + A^{\mathrm{T}}X - X\left(B_2B_2^{\mathrm{T}} - \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}\right)X + C_1^{\mathrm{T}}C_1 = 0,$$

$$AY + YA^{\mathrm{T}} - Y\left(C_2^{\mathrm{T}}C_2 - \frac{1}{\gamma^2}C_1^{\mathrm{T}}C_1\right)Y + B_1B_1^{\mathrm{T}} = 0,$$

where A, B_1, B_2, C_1, and C_2 are defined by (10.6.11). The expression for a H_∞ controller is given in (10.6.18)–(10.6.20). Also, two computational algorithms: one, the well-known **bisection algorithm** by Boyd *et al.* and the other, the two-step algorithm by Bruinsma *et al.* (1990), for computing the H_∞-norm are given in Section 10.6.1. Algorithm 10.6.2 seems to be faster than Algorithm 10.6.1. but the latter is easier to implement.

Stability Radius

The concept of stability radius has been defined in Chapter 7. Here a connection of the **complex stability radius** r is made with the ARE:

$$XA + A^*X - \rho C^*C - XBB^*X = 0$$

via the parametrized Hamiltonian matrix

$$H_\rho = \begin{pmatrix} A & -BB^* \\ \rho C^*C & -A^* \end{pmatrix}.$$

It is shown in **Theorem 10.7.3** that if $\rho \in (-\infty, r^2)$, then the above Riccati equation has a unique stabilizing solution X. Conversely, if A is stable and if there exists a Hermitian solution X of the above equation, then $\rho \leq r^2$.

In terms of the eigenvalues of the Hamiltonian matrix H_ρ, it means that $\rho < r$ if and only if H_{ρ^2} does not have an eigenvalue on the imaginary axis (**Theorem 10.7.2**).

Based on the latter result, a **bisection algorithm (Algorithm 10.7.1)** for determining the complex stability radius is described.

10.10 CHAPTER NOTES AND FURTHER READING

Feedback stabilization and EVA (pole-placement) are two central problems in control theory. For detailed treatment of these problems, see Brockett (1970), Brogan (1991), Friedland (1986), Chen (1984), Kailath (1980), Wonham (1985), Kwakernaak and Sivan (1972), etc. Most of the books in linear systems theory, however, do not discuss feedback stabilization via Lyapunov equations. Discussions on feedback stabilization via Lyapunov equations in Section 10.2 have been taken from the papers of Armstrong (1975) and Armstrong and Rublein (1976). For a Schur method for feedback stabilization, see Sima (1981). For stabilization methods of descriptor systems, see Varga (1995). For more on the output feedback problem, see Kimura (1975), Porter (1977), Sridhar and Lindorff (1973), Srinathkumar (1978), and Misra and Patel (1989).

For a discussion on eigenstructure assignment problem, see Andry *et al.* (1983).

The authoritative book by Anderson and Moore (1990) is an excellent source for a study on the LQR problem. The other excellent books on the subject include Athans and Falb (1966), Lewis (1986), Mehrmann (1991), Sima (1996), Kučera (1979), etc. For a proof of the discrete-time LQR Theorem (**Theorem 10.5.3**), see Sage and White (1977). An excellent reference book on the theory of Riccati equations is the recent book by Lancaster and Rodman (1995). There are also several nice papers on Riccati equations in the books edited by Bittanti *et al.* (1991) and Bittanti (1989). $\boldsymbol{H}_\infty$-control problem has been dealt with in detail in the books by Kimura (1996), Zhou *et al.* (1996), Green and Limebeer (1995), Dorato *et al.* (1992, 1995). The original papers by Francis and Doyle (1987) and by Doyle *et al.* (1989) are worth reading. A recent book by Kirsten Morris (2001) contains an excellent coverage on feedback control, in particular, H_∞ feedback control. Algorithms 10.6.1 and 10.6.2 for computing the H_∞-norm have been taken, respectively, from the papers of Boyd *et al.* (1989) and Bruinsma and Steinbuch (1990). A gradient method for computing the optimal H_∞-norm has been proposed in Pandey *et al.* (1991). Recently, Lin *et al.* (2000) have proposed numerically reliable algorithms for computing H_∞-norms of the discrete-time systems, both for the state and the output feedback problems. The discussion on the complex stability radius and Riccati equation in Section 10.7 has been taken from Hinrichsen and Pritchard (1986b). For an iterative algorithm for H_∞-control by state feedback, see Scherer (1989). Theorem 10.7.3 is an extension of the work of Brockett (1970) and Willems (1971). For an application of the ARE to compute H_∞ optimization, see Zhou and Khargonekar (1988).

For more on return difference matrix, phase and gain margins of the multivariable LQR design, see Lewis (1986), Safonov *et al.* (1981), Safonov (1980), etc. For an excellent and very readable account of classical control design using H_∞ techniques, see Helton and Merino (1998).

Exercises

10.1 Prove the equivalence of (i) and (iii) in Theorem 10.2.1.

10.2 Prove Theorem 10.2.2.

10.3 Prove Theorem 10.2.4.

10.4 Construct a state-space continuous-time system that is stabilizable, but not controllable.
Apply the Lyapunov stabilization method (modify the method in the book as necessary) to stabilize this system.

10.5 Repeat Problem 10.4 with a discrete-time system.

10.6 Develop algorithms for feedback stabilization, both for the continuous-time and discrete-time systems, based on the reduction of A to the real Schur form (see Sima 1981).
Compare the efficiency of each of these Schur algorithms with the respective Lyapunov equation based algorithms given in the book.

10.7 Using the real Schur decomposition of A, develop partial stabilization algorithms, both for the continuous-time and discrete-time systems in which only the unstable eigenvalues of A are stabilized using feedback, leaving the stable eigenvalues unchanged.

10.8 Prove Theorem 10.3.1.

10.9 State and prove the discrete counterpart of Theorem 10.3.2.

10.10 Prove Theorem 10.3.3.

10.11 Give an alternative proof of the state-feedback EVA Theorem (Theorem 10.4.1).

10.12 Construct an example to verify that if the eigenvalues of the closed-loop system are moved far from those of the open-loop system, a large feedback will be required to place the closed-loop eigenvalues.

10.13 Using the expression of the transforming matrix T, which transforms the system (A, b) to a controller-companion form (10.4.1)–(10.4.2), and the expression for the feedback formula (10.4.5), derive the Bass–Gura formula (10.4.6).

10.14 Derive the Ackermann formula (10.4.7).

10.15 Work out an example to illustrate each of the following theorems: Theorems 10.5.1, 10.5.2, 10.5.3, 10.6.1, 10.6.2, 10.6.3, 10.6.4, and 10.6.5. (Use MATLAB function **care** to solve the associated Riccati equation, whenever needed.)

10.16 *Design of a regulator with prescribed degree of stability.* Consider the LQR problem of minimizing the cost function

$$J_\alpha = \int_0^\infty e^{2\alpha t}(u^{\mathrm{T}}Ru + x^{\mathrm{T}}Qx)\,dt.$$

(a) Show that the Riccati equation to be solved in this case is:

$$(A + \alpha I)^{\mathrm{T}}X + X(A + \alpha I) + Q - XBR^{-1}B^{\mathrm{T}}X = 0$$

and the optimal control is given by the same feedback matrix K as in Theorem 10.5.1.

(b) Give a physical interpretation of the problem.

10.17 *Cross-weighted LQR.* Consider the LQR problem with the quadratic cost function with a cross penalty on state and control:

$$J_{\mathrm{CW}} = \int_0^\infty \left[x^{\mathrm{T}}Qx + 2x^{\mathrm{T}}Nu + u^{\mathrm{T}}Ru\right]dt$$

subject to $\dot{x} = Ax + Bu, x(0) = x_0$, where Q, R, N are, respectively, the state-weighting matrix, the control-weighting matrix, and the cross-weighting matrix. Define $A_R = A - BR^{-1}N^{\mathrm{T}}$.

(a) Show that the Riccati equation to be solved in this case is:

$$XA_R + A_R^{\mathrm{T}}X + (Q - NR^{-1}N^{\mathrm{T}}) - XBR^{-1}B^{\mathrm{T}}X = 0,$$

and the optimal control law is given by $u(t) = -Kx(t)$, where $K = R^{-1}(N^{\mathrm{T}} + B^{\mathrm{T}}X)$.

(b) What are the assumptions needed for the existence of the unique, symmetric positive semidefinite solution X of the Riccati equation in (a)?

10.18 Consider the LQR problem of minimizing the cost

$$J = \int_0^\infty \left[x^2(t) + \rho^2u^2(t)\right]dt,$$

with $\rho > 0$, subject to

$$m\ddot{q} + kq(t) = u(t).$$

(a) Find an expression for the feedback matrix K in terms of ρ, by solving an appropriate ARE.

(b) Plot the closed-loop poles as ρ varies over $0 < \rho < \infty$.

(c) Write down your observations about the behavior of the closed-loop poles with respect to stability.

10.19 Using the MATLAB function ***lqr***, design an LQR experiment with a single-input system to study the behavior of the closed-loop poles and the feedback vector with varying ρ in $R = \rho I$ in the range $[1, 10^6]$, taking $\rho = 1, 10, 10^2, 10^3$, and 10^6. Plot the open loop poles, the closed loop poles, and the step responses. Make a table of the gain margins and phase margins with each feedback vector.

10.20 (a) Using the return-difference identity, show that the ith singular value σ_i of the return-difference matrix with $s = j\omega$ is:

$$\sigma_i(I + G_{\mathrm{LQ}}(j\omega)) = \left[1 + \frac{1}{\rho}\sigma_i^2(H(j\omega))\right]^{1/2},$$

where $H(s) = C(sI - A)^{-1}B,\ R = \rho I$, and $Q = C^{\mathrm{T}}C$.

(b) Using the result in (a), prove that

$$\sigma_{\min}(I + G_{\mathrm{LQ}}(j\omega)) \geq 1.$$

(c) Using (b), prove that

$$\sigma_{\min}(I + (G_{\mathrm{LQ}}(j\omega))^{-1}) \geq \tfrac{1}{2}.$$

10.21 In certain applications, the homogeneous ARE:

$$XA + A^{\mathrm{T}}X + XWX = 0$$

is important.

Prove that the above homogeneous Riccati equation has a stabilizing solution (i.e., $A + WX$ is stable) if A has no eigenvalues on the imaginary axis.

10.22 *Computing the H_∞-norm over an interval.* Define the H_∞-norm of $G(s)$ over an interval $0 \leq \alpha < \beta$ as:

$$\|G\|_{[\alpha,\beta]} = \sup \sigma_{\max}(C(j\omega)), \quad \alpha \leq \omega \leq \beta.$$

(a) Develop an algorithm to compute $\|G\|_{[\alpha,\beta]}$ by modifying the bisection algorithm (**Algorithm 10.6.1**) as follows:

1. Take $\gamma_{lb} = \max\{\sigma_{\max}(G(j\alpha)), \sigma_{\max}(G(j\beta))\}$
2. Modify the eigenvalue test in Step 2 as: if M_γ has no imaginary eigenvalues between $j\alpha$ and $j\beta$.

(b) Work out a numerical example to test your algorithm.

10.23 Give a linear algebraic proof of Theorem 10.6.1 (consult the paper by Boyd *et al.* (1989)).

10.24 Develop an algorithm to compute the H_∞-norm of a discrete-stable transfer function, based on Theorem 10.6.3.

10.25 (Kimura 1996). For the second-order system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= w_1 + u_1, \\ z_1 &= x_1, \\ z_2 &= \delta u_1, \\ y &= c_2 x_1 + d_2 u_2, \end{aligned}$$

find the conditions under which the output feedback problem has a solution. Find the transfer function for H_∞ controller.

10.26 Prove that if the Hamiltonian matrix H_γ defined by (10.6.10) has an imaginary eigenvalue, then the state feedback H_∞-control problem does not have a solution.

10.27 Prove Theorem 10.6.2: If A is a complex stable matrix, then the distance to instability

$$\beta(A) = \|(sI - A)^{-1}\|_\infty^{-1}.$$

10.28 (a) Let $G(s) = C(sI - A)^{-1}B$. Then prove

$$r = \begin{cases} \dfrac{1}{\max\limits_{\omega\in\mathbb{R}} \|G(j\omega)\|}, & \text{if } G \neq 0, \\ \infty, & \text{if } G = 0. \end{cases}$$

(Consult Hinrichsen and Pritchard (1986b)).

(b) Give an example to show that $r(A, I, I)$ can change substantially under similarity transformation on A.

References

Ackermann J. "Der entwurf linear regelung systeme im zustandsraum," *Regulungestechnik und prozessedatenverarbeitung*, Vol. 7, pp. 297–300, 1972.

Anderson B. D. O. and Moore J. B. *Optimal Control: Linear Quadratic Methods*, Prentice Hall, Englewood Cliffs, NJ, 1990.

Andry A. N., Jr., Shapiro E. Y., and Chung J. C. "Eigenstructure assignment for linear systems," *IEEE Trans. Aerospace and Electronic Syst.*, Vol. AES-19(5), pp. 711–729, 1983.

Armstrong E. S. "An extension of Bass' algorithm for stabilizing linear continuous constant systems," *IEEE Trans. Autom. Control*, Vol. AC-20, pp. 153–154, 1975.

Armstrong E. S. and Rublein G. T. "A stabilizing algorithm for linear discrete constant systems," *IEEE Trans. Autom. Control*, Vol. AC-21, pp. 629–631, 1976.

Athans M. and Falb P. *Optimal Control*, McGraw Hill, New York, 1966.

Bittanti S. (ed.). *The Riccati Equation in Control, Systems, and Signals*, Lecture Notes, Pitagora Editrice, Bologna, 1989.

Bittanti S., Laub A.J., and Willems J.C. (eds.), *The Riccati Equation*, Springer-Verlag, Berlin, 1991.

Boyd S., Balakrishnan V., and Kabamba P. "A bisection method for computing the H_∞-norm of a transfer function matrix and related problems," *Math. Control Signals Syst.*, Vol. 2, pp. 207–219, 1989.

Boyd S. and Balakrishnan V. "A regularity result for the singular values of a transfer function matrix and a quadratically convergenet algorithm for computing its L_∞-norm," *Syst. Control Lett.*, pp. 1–7, 1990.

Brockett R.W. *Finite Dimensional Linear Systems*, Wiley, New York, 1970.

Brogan W.L. *Modern Control Theory*, 3rd edn, Prentice Hall, Englewood Cliffs, NJ, 1991.

Bruinsma N.A. and Steinbuch M. "A fast algorithm to compute the H_∞-norm of a transfer function matrix," *Syst. Control Lett.*, Vol. 14, pp. 287–293, 1990.

Chen C.-T. *Linear System Theory and Design*, CBS College Publishing, New York, 1984.

Dorato P., Fortuna L., and Muscato G. *Robust Control for Unstructured Perturbations – An Introduction* (Thoma M. and Wyner A., eds.) Springer-Verlag, Berlin, 1992.

Dorato P., Abdallah C., and Cerone V. *Linear Quadratic Control, An Introduction*, Prentice Hall, Englewood Cliffs, NJ, 1995.

Doyle J., Glover K., Khargonekar P., and Francis B. "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. Autom. Control*, Vol. AC-34, pp. 831–847, 1989.

Enns D.F. "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," *Proc. IEEE Conf. Dec. Control*, Las Vegas, pp. 127–132, 1984.

Francis B.A. and Doyle J.C. "Linear control theory with an H_∞ optimality criterion," *SIAM J. Control Optimiz.*, Vol. 25, pp. 815–844, 1987.

Francis B.A. *A Course in H_∞ Control Theory*, Lecture Notes in Control and Information Sciences, Vol. 88, 1987.

Franklin G.F., Powell J.D., and Emami-Naeini A. *Feedback Control of Dynamic Systems*, Addison-Wesley, New York, 1986.

Friedland B. *Control System Design: An Introduction to State-Space Methods*, McGraw-Hill, New York, 1986.

Glover K. "All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds," *Int. J. Control*, Vol. 39, pp. 1115–1193, 1984.

Glover K. and Doyle J. "State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity," *Syst. Control Lett.*, Vol. 11, pp. 167–172, 1988.

Green M. and Limebeer D.J.N. *Linear Robust Control*, Prentice Hall Information and System Sciences Series (Thomas Kailath, ed.), Prentice Hall, Englewood Cliffs, NJ, 1995.

Helton J. W. and Merino O., *Classical control using H^∞ Methods: Theory, optimization and Design*, SIAM, Philadelphia, 1998.

Hinrichsen D. and Pritchard A.J. "Stability radii of linear systems," *Syst. Control Lett.*, Vol. 7, pp. 1–10, 1986a.

Hinrichsen D. and Pritchard A.J. "Stability radius for structured perturbations and the algebraic Riccati equation," *Syst. Control Lett.*, Vol. 8, pp. 105–113, 1986b.

Kailath T. *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

Kimura H. "Pole assignment by gain output feedback," *IEEE Trans. Autom. Control*, Vol. AC-20(4), pp. 509–516, 1975.

Kimura H. *Chain-Scattering to H^{∞} Control*, Birkhäuser, Boston, 1996.

Kucĕra V. *Discrete Linear Control*, John Wiley & Sons, New York, 1979.

Kwakernaak H. and Silvan R. *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

Lancaster P. and Rodman L. *The Algebraic Riccati Equation*, Oxford University Press, Oxford, UK, 1995.

William S. Levine (ed.). *The Control Handbook*, CRC Press and IEEE Press, Boca Raton, Fl., 1996.

Lewis F.L. *Optimal Control*, John Wiley & Sons, New York, 1986.

Lewis F.L. *Applied Optimal Control and Estimation—Digital Design and Implementation*, Prentice Hall and Texas Instruments Digital Signal Processing Series, Prentice Hall, Englewood Cliffs, NJ, 1992.

Lin W.-W., Wang J.-S., and Xu Q.-F. "On the Computation of the optimal H_{∞}-norms for two feedback control problems," *Lin. Alg. Appl.*, Vol. 287, pp. 233–255, 1999.

Lin W.-W., Wang J.-S., and Xu Q.-F. "Numerical computation of the minimal H_{∞}-norms of the discrete-time state and output feedback control problems," *SIAM. J. Num. Anal.*, Vol. 38(2), pp. 515–547, 2000.

Luenberger D. Introduction to Dynamic Systems: Theory, Model and Applications, John Wiley, New York, 1979.

Maciejowski J.M. *Multivariable Feedback Design*, Addison-Wesley, Wokingham, 1989.

Mehrmann V. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution*, Lecture Notes in Control and Information Sciences, Springer-Verlag, Heidelberg, 1991.

Morris K. *Introduction to Feedback Control*, Academic Press, Boston, 2001.

Misra P. and Patel R.V. "Numerical algorithms for eigenvalue assignment by constant and dynamic output feedback," *IEEE Trans. Autom. Control*, Vol. 34, pp. 579–588, 1989.

Moore B.C. "On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment," *IEEE Trans. Autom. Control*, Vol. AC-21, pp. 689–692, 1976.

Pandey P., Kenney C.S., Packard A., and Laub A.J. "A gradient method for computing the optimal H_{∞} norm," *IEEE Trans. Autom. Control*, Vol. 36, pp. 887–890, 1991.

Porter B. "Eigenvalue assignment in linear multivariable systems by output feedback," *Int. J. Control*, Vol. 25, pp. 483–490, 1977.

Robel G. "On computing the infinity norm," *IEEE Trans. Autom. Control*, Vol. 34, pp. 882–884, 1989.

Safonov M.G. and Athans M. "Gain and phase margins of multiloop LQG regulators," *IEEE Trans. Autom. Control*, Vol. AC-22, pp. 173–179, 1977.

Safonov M.G., Laub A.J., and Hartmann G.L. "Feedback properties of multivariable systems: The role and use of the return difference matrix," *IEEE Trans. Autom. Control*, Vol. AC-26, pp. 47–65, 1981.

Safonov M.G. *Stability and Robustness of Multivariable Feedback Systems*, MIT Press, Boston, 1980.

Sage A.P. and White C.C. *Optimal Systems Control*, 2nd edn, Prentice Hall, Englewood Cliffs, NJ, 1977.

Scherer C. "H_∞-control by state feedback: An iterative algorithm and characterization of high-gain occurence," *Syst. Control Lett.*, Vol. 12, pp. 383–391, 1989.

Sima V. "An efficient Schur method to solve the stabilizing problem," *IEEE Trans. Autom. Control*, Vol. AC-26, pp. 724–725, 1981.

Sima V. *Algorithms for Linear-Quadratic Optimization*, Marcel Dekker, New York, 1996.

Sridhar B. and Lindorff D.P. "Pole placement with constant gain output feedback," *Int. J. Control*, Vol. 18, pp. 993–1003, 1973.

Srinathkumar S. "Eigenvalue/eigenvector assignment using output feedback," *IEEE Trans. Autom. Control*, Vol. AC-21, pp. 79–81, 1978.

Varga A. "On stabilization methods of descriptor systems," *Syst. Control Lett.*, Vol. 24, pp. 133–138, 1995.

Willems J.C. "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Autom. Control*, Vol. AC-16, pp. 621–634, 1971.

Wonham W.H. *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, Berlin, 1985.

Zames G. "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Trans. Autom. Control*, Vol. AC-26, pp. 301–320, 1981.

Zhou K., Doyle J.C., and Glover K. *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.

Zhou K. and Khargonekar P. "An algebraic Riccati equation approach to H_∞ optimization," *Syst. Control Lett.*, Vol. 11(2), pp. 85–91, 1988.