CHAPTER 12

STATE ESTIMATION: OBSERVER AND THE KALMAN FILTER

Topics covered

- State Estimation via Eigenvalue Assignment (EVA)
- State Estimation via Sylvester-Observer Equation
- Characterization of the Unique Nonsingular Solution to the Sylvester Equation
- Numerical Methods for the Sylvester-Observer Equation
- A Numerical Method for the Constrained Sylvester-Observer Equation
- Kalman Filter
- Linear Quadratic Gaussian (LQG) Design

12.1 INTRODUCTION

We have seen in Chapter 10 that all the state-feedback problems, such as feedback stabilization, eigenvalue and eigenstructure assignment, the LQR and the state-feedback H_∞-control problems, etc., require that the state vector $x(t)$ should be explicitly available. However, in most practical situations, the states are not fully accessible and but, however, the designer knows the output $y(t)$ and the input $u(t)$. The unavailable states, somehow, need to be estimated accurately from the knowledge of the matrices A, B, and C, the output vector $y(t)$, and the input vector $u(t)$.

In this chapter, we discuss how the states of a continuous-time system can be estimated. **The discussions here apply equally to the discrete-time systems, possibly with some minor changes.** So we concentrate on the continuous-time case only.

We describe two common procedures for state estimation: **one, via eigenvalue assignment** (EVA) and **the other, via solution of the Sylvester-observer equation.**

The Hessenberg–Schur method for the Sylvester equation, described in Chapter 8, can be used for numerical solution of the Sylvester-observer equation. We, however, describe two other numerical methods (**Algorithms 12.7.1** and **12.7.2**), especially designed for this equation. Both are based on the reduction of the observable pair (A, C) to the observer-Hessenberg pair, described in Chapter 6 and are recursive in nature. *Algorithm* 12.7.2 *is a block-generalization of Algorithm* 12.7.1 *and seems to be a little more efficient than the later.* Algorithm 12.7.2 is also suitable for high performance computing. Both seem to have good numerical properties.

The chapter concludes with a well-known procedure developed by Kalman (**Kalman filtering**) for optimal estimation of the states of a **stochastic system**, followed by a brief discussion on the Linear Quadratic Gaussian (**LQG**) problem that deals with optimization of a performance measure for a stochastic system.

12.2 STATE ESTIMATION VIA EIGENVALUE ASSIGNMENT

Consider the linear time-invariant continuous-time system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{12.2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$.

Let $\hat{x}(t)$ be an estimate of the state vector $x(t)$. Obviously, we would like to construct the vector $\hat{x}(t)$ in such a way that the error $e(t) = x(t) - \hat{x}(t)$ approaches zero as fast as possible, for all initial states $x(0)$ and for every input $u(t)$). Suppose, we design a dynamical system using our available resources: the output variable $y(t)$, input variable $u(t)$, and the matrices A, B, C, satisfying

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Ky(t) + Bu(t), \tag{12.2.2}$$

where the matrix K is to be constructed. Then,

$$\begin{aligned} \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) &= Ax(t) + Bu(t) - A\hat{x}(t) + KC\hat{x}(t) - Ky(t) - Bu(t), \\ &= (A - KC)x(t) - (A - KC)\hat{x}(t) = (A - KC)e(t). \end{aligned}$$

The solution of this system of differential equations is $e(t) = e^{(A-KC)t}e(0)$, which shows that the rate at which the entries of the error vector $e(t)$ approach zero can be controlled by the eigenvalues of the matrix $A - KC$. For example, if all the eigenvalues of $A - KC$ have negative real parts less than $-\alpha$, then the error $e(t)$ will approach zero faster than $e^{-\alpha t}e(0)$.

The above discussion shows that the problem of state estimation can be solved by finding a matrix K such that the matrix $A - KC$ has a suitable desired spectrum.

Note that if (A, C) is observable, then such K always exists because, the observability of (A, C) implies the controllability of $(A^{\mathrm{T}}, C^{\mathrm{T}})$. Also, if $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable, then by the EVA Theorem (**Theorem 10.4.1**), there always exists a matrix L such that $(A^{\mathrm{T}} + C^{\mathrm{T}} L)$ has an arbitrary spectrum. We can therefore choose $K = -L^{\mathrm{T}}$ so that the eigenvalues of $A^{\mathrm{T}} - C^{\mathrm{T}} K^{\mathrm{T}}$ (which are the same as those of $A - KC$) will be arbitrarily assigned.

Theorem 12.2.1. *If (A, C) is observable, then the states $x(t)$ of the system* (12.2.1) *can be estimated by*

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Ky(t) + Bu(t), \tag{12.2.3}$$

where K is constructed such that $A - KC$ is a stable matrix. The error $e(t) = x(t) - \hat{x}(t)$ is governed by

$$\dot{e}(t) = (A - KC)e(t)$$

and $e(t) \to 0$ as $t \to \infty$.

12.3 STATE ESTIMATION VIA SYLVESTER EQUATION

We now present another approach for state estimation. Knowing A, B, C, $u(t)$ and $y(t)$, let's construct the system

$$\dot{z}(t) = Fz(t) + Gy(t) + Pu(t), \tag{12.3.1}$$

where F is $n \times n$, G is $n \times r$, and P is $n \times m$, in such a way that for some constant $n \times n$ nonsingular matrix X, the error vector $e(t) = z(t) - Xx(t) \to 0$ for all $x(0)$, $z(0)$, and for every input $u(t)$. The vector $z(t)$ will then be an estimate of $Xx(t)$. The system (12.3.1) is then said to be the **state observer** for the system (12.2.1). The idea originated with D. Luenberger (1964) and is hence referred to in control theory as the **Luenberger observer**.

We now show that the system (12.3.1) will be a state observer if the matrices X, F, G, and P satisfy certain requirements.

Theorem 12.3.1. *Observer Theorem. The system* (12.3.1) *is a state-observer of the system* (12.2.1), *that is, $z(t)$ is an estimate of $Xx(t)$ in the sense that the error $e(t) = z(t) - Xx(t) \to 0$ as $t \to \infty$ for any initial conditions $x(0)$, $z(0)$, and $u(t)$ if*

(i) $XA - FX = GC$,
(ii) $P = XB$,
(iii) *F is stable.*

Proof. We need to show that if the conditions (i)–(iii) are satisfied, then $e(t) \to 0$ as $t \to 0$.

From $e(t) = z(t) - Xx(t)$, we have

$$\begin{aligned}\dot{e}(t) &= \dot{z}(t) - X\dot{x}(t),\\ &= Fz(t) + Gy(t) + Pu(t) - X(Ax(t) + Bu(t)).\end{aligned} \tag{12.3.2}$$

Substituting $y(t) = Cx(t)$ while adding and subtracting $FXx(t)$ in Eq. (12.3.2), we get

$$\dot{e}(t) = Fe(t) + (FX - XA + GC)x(t) + (P - XB)u(t).$$

If the conditions (i) and (ii) are satisfied, then we obtain

$$\dot{e}(t) = Fe(t).$$

If, in addition, the condition (iii) is satisfied, then clearly $e(t) \to 0$ as $t \to \infty$, for any $x(0)$, $z(0)$, and $u(t)$.

Hence $z(t)$ is an estimate of $Xx(t)$. ■

The Sylvester-Observer Equation

Definition 12.3.1. *The matrix equation*

$$XA - FX = GC, \tag{12.3.3}$$

where A and C are given and X, F, and G are to be found will be called the **Sylvester-observer** *equation.*

The name **"Sylvester-observer equation"** is justified, because the equation arises in construction of an observer and it is a variation of the classical Sylvester equation (discussed in Chapter 8):

$$XA + TX = R,$$

where A, T, and R are given and X is the only unknown matrix.

Theorem 12.3.1 suggests the following method for the observer design.

Algorithm 12.3.1. *Full-Order Observer Design via Sylvester-Observer Equation*

Inputs. *The system matrices A, B, and C of order $n \times n$, $n \times m$, and $r \times n$, respectively.*

Output. *An estimate $\hat{x}(t)$ of the state vector $x(t)$.*

Assumptions. *(A, C) is observable.*

Step 1. *Find a* **nonsingular solution** *X of the Sylvester-observer equation (12.3.3) by choosing F as a stable matrix and choosing G in such a way that the resulting solution X is nonsingular.*

Step 2. *Compute $P = XB$.*

Step 3. *Construct the observer $z(t)$ by solving the system of differential equations:*

$$\dot{z}(t) = Fz(t) + Gy(t) + Pu(t), \quad z(0) = z_0.$$

Step 4. *Find an estimate $\hat{x}(t)$ of $x(t)$: $\hat{x}(t) = X^{-1}z(t)$.*

Example 12.3.1.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C = (1 \ \ 0).$$

(A, C) is observable.

Step 1. Choose $G = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\quad F = \mathrm{diag}(-1, -3)$.
Then a solution X of $XA - FX = GC$ is

$$X = \begin{pmatrix} 0.6667 & -0.3333 \\ 0.8000 & -0.2000 \end{pmatrix}$$

(computed by MATLAB function **lyap**). The matrix X is nonsingular.
Step 2.

$$P = XB = \begin{pmatrix} 0.6667 \\ 0.8000 \end{pmatrix}.$$

Step 3. An estimate $\hat{x}(t)$ of $x(t)$ is

$$\hat{x}(t) = X^{-1}z(t) = \begin{pmatrix} -1.5 & 2.5 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} -1.5z_1 + 2.5z_2 \\ -6z_1 + 5z_2 \end{pmatrix},$$

where

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

is given by

$$\dot{z}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 3 \end{pmatrix} y(t) + \begin{pmatrix} 0.6667 \\ 0.8000 \end{pmatrix} u(t), \quad z(0) = z_0.$$

Comparison of the state and estimate for Example 12.3.1: In Figure 12.1, we compare the estimate $\hat{x}(t)$, obtained by Algorithm 12.3.1, with the state $x(t)$, found by directly solving Eq. (12.2.1) with $u(t)$ as the *unit step function*, and $x(0) = (6, 0)^{\mathrm{T}}$. The differential equation in Step 3 was solved with $z(0) = 0$. The MATLAB function **ode23** was used to solve both the equations. The solid line corresponds to the exact states and the dotted line corresponds to the estimated state.

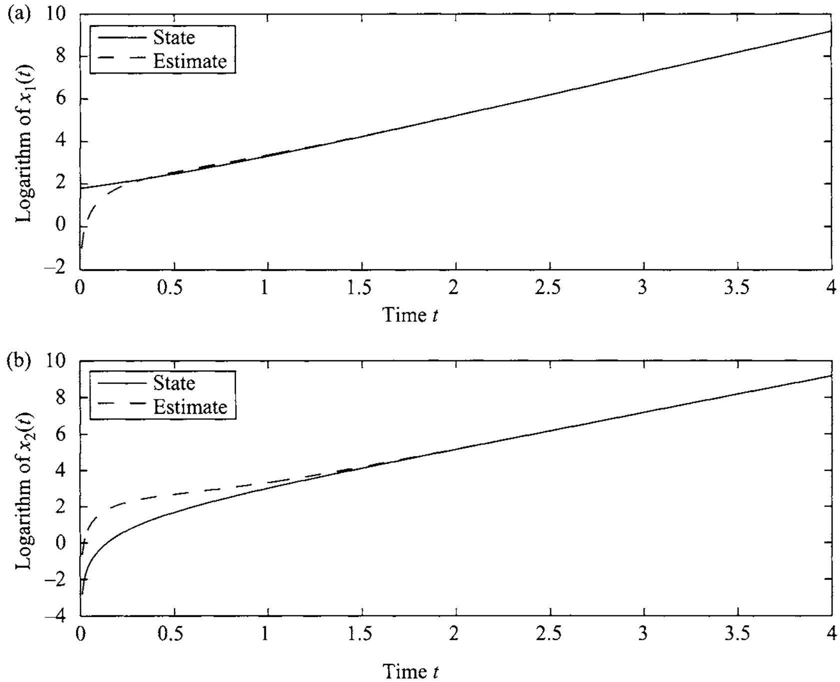

FIGURE 12.1: The (a) first and (b) second variables of the state $x(t)$ and estimate $\hat{x}(t)$ for Example 12.3.1.

12.4 REDUCED-ORDER STATE ESTIMATION

In this section, we show that if the $r \times n$ output matrix C has full rank r, then the problem of finding a full nth order state estimator for the system (12.2.1) can be reduced to the problem of finding an $(n - r)$th order estimator.

Such an estimator is known as a **reduced-order estimator**. Once a reduced-order estimator of order $n - r$ rather than n is constructed, the full states of the original system can be obtained from the $(n - r)$ state variable of this observer together with the r variables available from measurements. As in the full-dimensional case, we will describe two approaches for finding a reduced-order estimator. We start with the **EVA approach**.

For the sake of convenience, in the next two sections, we will denote the vector $x(t)$ and its derivative $\dot{x}(t)$ just by x and $\dot{x}$. Similarly, for the other vectors.

12.4.1 Reduced-Order State Estimation via Eigenvalue Assignment

Assume as usual that A is an $n \times n$ matrix, B is an $n \times m$ matrix $(m \leq n)$, C is an $r \times n$ matrix with full rank $(r < n)$, and (A, C) is observable.

Since C has full rank, we can choose an $(n - r) \times n$ matrix R such that the matrix $S = \begin{pmatrix} C \\ R \end{pmatrix}$ is nonsingular.

Introducing the new variable $\tilde{x} = Sx$, we can then transform the system (12.2.1) to

$$\begin{aligned} \dot{\tilde{x}} &= SAS^{-1}\tilde{x} + SBu, \\ y &= CS^{-1}\tilde{x} = (I_r, 0)\tilde{x}. \end{aligned} \tag{12.4.1}$$

Let's now partition

$$\bar{A} = SAS^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \qquad \bar{B} = SB = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \tag{12.4.2}$$

and $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, where $\bar{A}_{11}$ and $\tilde{x}_1$ are, respectively, $r \times r$ and $r \times 1$. Then we have

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} &= \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} u, \\ y &= (I_r, 0) \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \tilde{x}_1. \end{aligned}$$

That is,

$$\dot{y} = \dot{\tilde{x}}_1 = \bar{A}_{11}\tilde{x}_1 + \bar{A}_{12}\tilde{x}_2 + \bar{B}_1 u, \tag{12.4.3}$$

$$\dot{\tilde{x}}_2 = \bar{A}_{21}\tilde{x}_1 + \bar{A}_{22}\tilde{x}_2 + \bar{B}_2 u, \tag{12.4.4}$$

$$y = \tilde{x}_1. \tag{12.4.5}$$

Since $y = \tilde{x}_1$, we only need to find an estimator for the vector $\tilde{x}_2$ of this transformed system.

The transformed system is not in standard state-space form. However, the system can be easily put in standard form by introducing the new variables

$$\bar{u} = \bar{A}_{21}\tilde{x}_1 + \bar{B}_2 u = \bar{A}_{21} y + \bar{B}_2 u \tag{12.4.6}$$

and

$$v = \dot{y} - \bar{A}_{11} y - \bar{B}_1 u. \tag{12.4.7}$$

From (12.4.4)–(12.4.7), we then have

$$\dot{\tilde{x}}_2 = \bar{A}_{22}\tilde{x}_2 + \bar{u}, \qquad v = \bar{A}_{12}\tilde{x}_2, \tag{12.4.8}$$

which is in standard form.

Since (A, C) is observable, it can be shown (**Exercise 12.3**) that $(\bar{A}_{22}, \bar{A}_{12})$ is also observable. Since $\tilde{x}_2$ has $(n - r)$ elements, we have thus reduced the full n-dimensional estimation problem to an $(n-r)$-dimensional problem. We, therefore, now concentrate on finding an estimate of $\tilde{x}_2$.

By (12.2.2) an $(n - r)$ dimensional estimate $\hat{\tilde{x}}_2$ of $\tilde{x}_2$ defined by (12.4.8) is of the form:

$$\dot{\hat{\tilde{x}}}_2 = (\bar{A}_{22} - L\bar{A}_{12})\hat{\tilde{x}}_2 + Lv + \bar{u},$$

for any matrix L chosen such that $\bar{A}_{22} - L\bar{A}_{12}$ is stable.

Substituting the expressions for $\bar{u}$ and v from (12.4.6) and (12.4.7) into the last equation, we have

$$\dot{\hat{\tilde{x}}}_2 = (\bar{A}_{22} - L\bar{A}_{12})\hat{\tilde{x}}_2 + L(\dot{y} - \bar{A}_{11}y - \bar{B}_1u) + (\bar{A}_{21}y + \bar{B}_2u).$$

Defining another new variable

$$z = \hat{\tilde{x}}_2 - Ly,$$

we can then write

$$\begin{aligned}\dot{z} &= (\bar{A}_{22} - L\bar{A}_{12})(z + Ly) + (\bar{A}_{21} - L\bar{A}_{11})y + (\bar{B}_2 - L\bar{B}_1)u \\ &= (\bar{A}_{22} - L\bar{A}_{12})z + [(\bar{A}_{22} - L\bar{A}_{12})L + (\bar{A}_{21} - L\bar{A}_{11})]y + (\bar{B}_2 - L\bar{B}_1)u\end{aligned} \tag{12.4.9}$$

Comparing Eq. (12.4.9) with (12.2.2) and noting that $\bar{A}_{22} - L\bar{A}_{12}$ is a stable matrix, we see that $z + Ly$ is also an estimate of $\tilde{x}_2$.

Once an estimate of $\tilde{x}_2$ is found, an estimate of the original n-dimensional state vector x from the estimate of $\tilde{x}_2$ can be easily constructed, as shown below.

Since $y = \tilde{x}_1$ and $\hat{\tilde{x}}_2 = z + Ly$, we immediately have

$$\hat{\tilde{x}} = \begin{pmatrix} \hat{\tilde{x}}_1 \\ \hat{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} y \\ Ly + z \end{pmatrix} \tag{12.4.10}$$

as an estimate of $\tilde{x}$.

Finally, since $\tilde{x} = Sx$, an estimate $\hat{x}$ of x can be constructed from an estimate of $\tilde{x}$ as:

$$\hat{x} = S^{-1}\hat{\tilde{x}} = \begin{pmatrix} C \\ R \end{pmatrix}^{-1} \begin{pmatrix} y \\ Ly + z \end{pmatrix}.$$

The above discussion can be summarized in the following algorithm:

Algorithm 12.4.1. *Reduced-Order Observer Design via EVA*

Inputs. *The system matrices A, B, C, respectively, of order $n \times n, n \times m$, and $r \times n$.*

Output. *An estimate $\hat{x}$ of the state vector x.*

Assumptions. (i) (A, C) *is observable.* (ii) C *is of full rank.*

Step 1. *Find an* $(n-r) \times n$ *matrix* R *such that* $S = \begin{pmatrix} C \\ R \end{pmatrix}$ *is nonsingular.*

Step 2. *Compute* $\bar{A} = SAS^{-1}$, $\bar{B} = SB$, *and partition them as*

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \tag{12.4.11}$$

where $\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{21}, \bar{A}_{22}$ *are, respectively,* $r \times r$, $r \times (n-r)$, $(n-r) \times r$, *and* $(n-r) \times (n-r)$ *matrices.*

Step 3. *Find a matrix L such that* $\bar{A}_{22} - L\bar{A}_{12}$ *is stable.*

Step 4. *Construct a reduced-order observer by solving the systems of differential equations:*

$$\begin{aligned} \dot{z} = &(\bar{A}_{22} - L\bar{A}_{12})z + [(\bar{A}_{22} - L\bar{A}_{12})L + (\bar{A}_{21} - L\bar{A}_{11})]y \\ &+ (\bar{B}_2 - L\bar{B}_1)u, z(0) = z_0. \end{aligned} \tag{12.4.12}$$

Step 5. *Find* $\hat{x}$, *an estimate of* x:

$$\hat{x} = \begin{pmatrix} C \\ R \end{pmatrix}^{-1} \begin{pmatrix} y \\ Ly + z \end{pmatrix}. \tag{12.4.13}$$

Example 12.4.1. Consider the design of a reduced-order observer for the linearalized model of the Helicopter problem discussed in Doyle and Stein (1981), and also considered in Dorato *et al.* (1995), with the following data:

$$A = \begin{pmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{pmatrix}.$$

Since $\text{rank}(C) = 2$, $r = 2$.

Step 1. Choose $R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. The matrix $S = \begin{pmatrix} C \\ R \end{pmatrix}$ is nonsingular.

Step 2. $\bar{A} = SAS^{-1} = \begin{pmatrix} 1.7400 & -0.5009 & -0.1400 & -1.1600 \\ -57.3006 & -1 & 0 & 57.3000 \\ -0.0370 & -1.0698 & -0.1600 & 0.6600 \\ 2.3580 & -0.4695 & -0.1400 & -1.7600 \end{pmatrix}$,

$$\bar{B} = SB = \begin{pmatrix} 0.3600 & -8.6000 \\ 0 & 0 \\ 0.8500 & -8.7191 \\ 0.7100 & -8.5991 \end{pmatrix}.$$

$$\bar{A}_{11} = \begin{pmatrix} 1.7400 & -0.5009 \\ -57.3006 & -1 \end{pmatrix}, \qquad \bar{A}_{12} = \begin{pmatrix} -0.14 & -1.16 \\ 0 & 57.3 \end{pmatrix},$$

$$\bar{A}_{21} = \begin{pmatrix} -0.0370 & -1.0698 \\ 2.3580 & -0.4695 \end{pmatrix}, \qquad \bar{A}_{22} = \begin{pmatrix} -0.1600 & 0.6600 \\ -0.1400 & -1.7600 \end{pmatrix},$$

$$\bar{B}_1 = \begin{pmatrix} 0.3600 & -8.6000 \\ 0 & 0 \end{pmatrix}, \qquad \bar{B}_2 = \begin{pmatrix} 0.85 & -8.7191 \\ 0.7100 & -8.5991 \end{pmatrix}.$$

Step 3. The matrix

$$L = \begin{pmatrix} -6 & -0.1099 \\ 1 & 0.0244 \end{pmatrix}$$

is such that the eigenvalues of $\bar{A}_{22} - L\bar{A}_{12}$ (the observer eigenvalues) are $\{-1, -2\}$ (L is obtained using MATLAB function **place**).

Step 4. The reduced-order 2-dimensional observer is given by:

$$\dot{z} = (\bar{A}_{22} - L\bar{A}_{12})z + [(\bar{A}_{22} - L\bar{A}_{12})L + (\bar{A}_{21} - L\bar{A}_{11})]y + (\bar{B}_2 - L\bar{B}_1)u,\ Z(0) = Z_0.$$

with $\bar{A}_{11}$, $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_1$, and $\bar{B}_2$ as computed above.

An estimate $\hat{x}$ of the state vector x is then

$$\hat{x} = \begin{pmatrix} C \\ R \end{pmatrix}^{-1} \begin{pmatrix} y \\ Ly + z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & -0.0175 & 0 & 1 \\ 0 & 0.0175 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ Ly + z \end{pmatrix},$$

where z is determined from (12.4.12).

Remark

- The explicit inversion of the matrix $S = \begin{pmatrix} C \\ R \end{pmatrix}$, which could be a source of large round-off errors in case this matrix is ill-conditioned, can be avoided by taking the QR decomposition of the matrix C: $C = RQ_1$ and then choosing an orthogonal matrix Q_2 such that the matrix $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ is orthogonal. The matrix Q can then be used in place of S. We leave the details for the readers as an exercise **(Exercise 12.18)**.

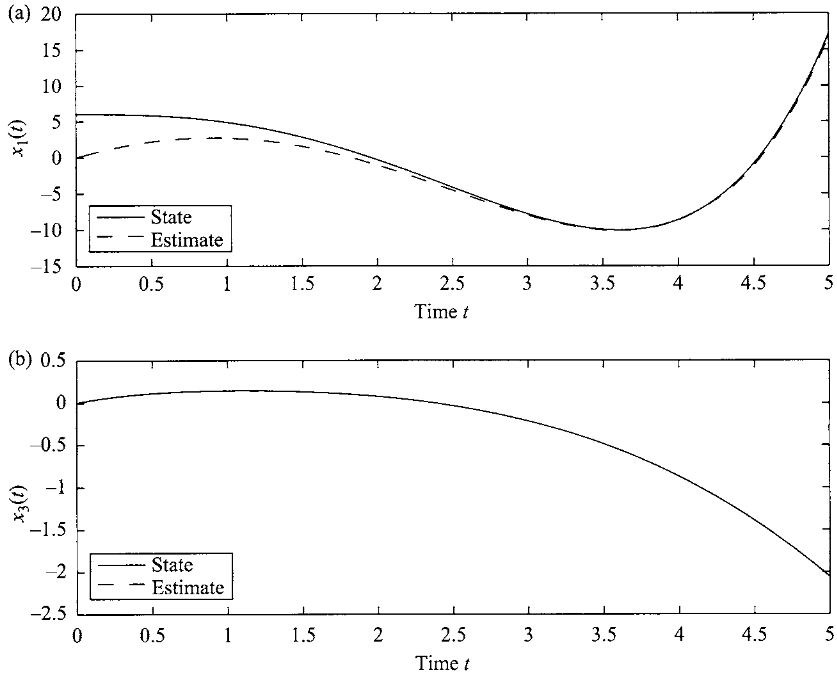

FIGURE 12.2: The (a) first and (b) third variables of the state $x(t)$ and estimate $\hat{x}(t)$, for Example 12.4.1.

Comparison of the state and estimate for example 12.4.1: In Figure 12.2, we compare the estimate $\hat{x}(t)$, obtained by Algorithm 12.4.1 with the state $x(t)$, found by directly solving Eq. (12.2.1) with $u(t) = H(t)[1\ 1]^T$, $H(t)$ is the unit step function and $x(0) = (6, 0, 0, 0)^T$. To solve Eqs. (12.2.1) and (12.4.12), **MATLAB** function **ode23** was used. For Eq. (12.4.12), the initial condition was $z(0) = 0$. The first and the third components of the solutions are compared. The solid line corresponds to the exact state and the dotted line corresponds to the estimated state.

12.4.2 Reduced-Order State Estimation via Sylvester-Observer Equation

As in the case of a full-dimensional observer, a reduced-order observer can also be constructed via solution of a Sylvester-observer equation. The procedure is as follows:

Algorithm 12.4.2. *Reduced-order Observer Design via Sylvester-Observer Equation*

Inputs. *The matrices A, B, and C of order $n \times n$, $n \times m$, and $r \times n$, respectively.*

Output. *An estimate $\hat{x}$ of the state vector x.*

Assumptions. *(i) (A, C) is observable. (ii) C is of full rank.*

Step 1. *Choose an $(n-r) \times (n-r)$* ***stable*** *matrix F.*

Step 2. *Solve the reduced-order Sylvester-observer equation for a full rank $(n-r) \times n$ solution matrix X:*

$$XA - FX = GC,$$

*choosing the $(n-r) \times r$ matrix G appropriately. (**Numerical methods for solving the Sylvester-observer equation will be described in Section 12.7**).*

Step 3. *Compute $P = XB$.*

Step 4. *Find the $(n-r)$ dimensional reduced-order observer z by solving the system of differential equations:*

$$\dot{z} = Fz + Gy + Pu, \qquad z(0) = z_0. \tag{12.4.14}$$

Step 5. *Find an estimate $\hat{x}$ of x:*

$$\hat{x} = \begin{pmatrix} C \\ X \end{pmatrix}^{-1} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Note: If we write $\begin{pmatrix} C \\ X \end{pmatrix}^{-1} = (\bar{S}_1, \bar{S}_2)$, then $\hat{x}$ can be written in the compact form:

$$\hat{x} = \bar{S}_1 y + \bar{S}_2 z. \tag{12.4.15}$$

Example 12.4.2. Consider Example 12.4.1 again.

Step 1. Choose $F = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

Step 2. Choose $G = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

The solution X of the Sylvester-observer equation $XA - FX = GC$ is

$$X = \begin{pmatrix} -0.117 & -0.0822 & 62.1322 & 37.2007 \\ -0.1364 & -1.9296 & 428.2711 & -173.4895 \end{pmatrix}.$$

Step 3. $P = XB = \begin{pmatrix} 21.7151 & 1.2672 \\ 149.1811 & 20.4653 \end{pmatrix}$.

Step 4. The two-dimensional reduced-order observer is given by $\dot{z} = Fz + Gy + Pu$, where F, G, and P are the matrices found in Step 1, Step 2, and Step 3, respectively.

An estimate $\hat{x}$ of x is

$$\hat{x} = \begin{pmatrix} C \\ X \end{pmatrix}^{-1} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -24.5513 & -135.1240 & 124.1400 & -18.0098 \\ 1 & 0 & 0 & 0 \\ -0.0033 & -0.0360 & 0.0395 & -0.0034 \\ 0 & 0.0175 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

(Note that if

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

then $\hat{x}_2 = y_1$, $\hat{x}_4 = 0.0175 y_2$, same as was obtained in Example 12.4.1 using the EVA method).

Comparison of the states and estimates for Example 12.4.2: In Figure 12.3, we compare the actual state vector with the estimated one obtained by Algorithm 12.4.2 on the data of Example 12.4.1. The solid line corresponds to

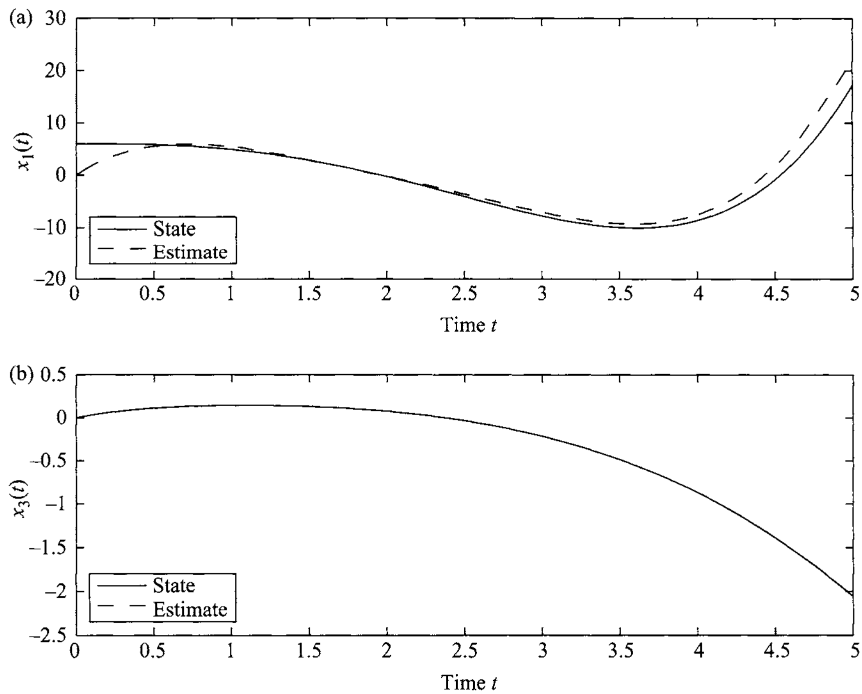

FIGURE 12.3: The (a) first and (b) third variables of the state $x(t)$ and the estimate $\hat{x}(t)$, for Example 12.4.2.

the actual state and the dotted line corresponds to the estimated state. MATLAB function **ode23** was used to solve the underlying differential equations with the same initial conditions as in Example 12.4.1. *The third components are indistinguishable.*

12.5 COMBINED STATE FEEDBACK AND OBSERVER DESIGN

When an estimate $\hat{x}$ of x is used in the feedback control law

$$u = s - K\hat{x} \tag{12.5.1}$$

in place of x, one naturally wonders: **what effect will there be on the EVA?** We consider only the reduced-order case, here. The same conclusion, of course, is true for a full-order observer.

Using (12.5.1) in (12.2.1), we obtain

$$\begin{aligned}
\dot{x} &= Ax + B(s - K\hat{x}),\\
&= Ax + B(s - K\bar{S}_1 y - K\bar{S}_2 z), \qquad \text{(using (12.4.15))}\\
&= Ax + B(s - K\bar{S}_1 Cx - K\bar{S}_2 z),\\
&= (A - BK\bar{S}_1 C)x - BK\bar{S}_2 z + Bs.
\end{aligned}$$

Also, Eq. (12.4.14), can be written as

$$\begin{aligned}
\dot{z} &= Fz + Gy + Pu = Fz + GCx + P(s - K\bar{S}_1 y - K\bar{S}_2 z),\\
&= (GC - PK\bar{S}_1 C)x + (F - PK\bar{S}_2)z + Ps
\end{aligned}$$

(using (12.5.1) and (12.4.15)).

Thus, the combined (feedback and observer) system (Figure 12.4) is given by

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A - BK\bar{S}_1 C & -BK\bar{S}_2 \\ GC - PK\bar{S}_1 C & F - PK\bar{S}_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ P \end{pmatrix} s,\\
y &= (C, 0)\begin{pmatrix} x \\ z \end{pmatrix}.
\end{aligned} \tag{12.5.2}$$

Applying to this system the equivalence transformation, given by the nonsingular matrix

$$\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}$$

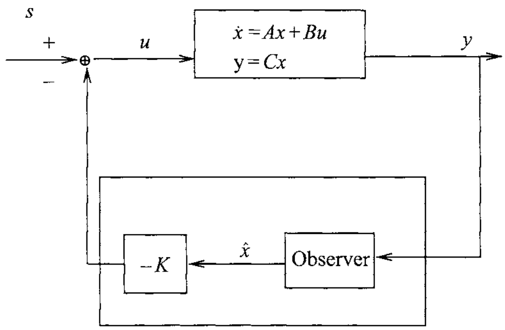

FIGURE 12.4: Observer-based state feedback.

and noting that $e = z - Xx$, $XA - FX = GC$, and $P = XB$, we have, after some algebraic manipulations,

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} &= \begin{pmatrix} A - BK & -BK\bar{S}_2 \\ 0 & F \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} s, \\ y &= (C, 0) \begin{pmatrix} x \\ e \end{pmatrix}. \end{aligned} \tag{12.5.3}$$

Thus, the eigenvalues of the combined system are the union of the eigenvalues of the closed-loop matrix $A - BK$ and of the observer matrix F.

Therefore, the observer design and feedback design can be carried out independently, and the calculation of the feedback gain is not affected whether the true state x or the estimated state $\hat{x}$ is used.

This property is known as the **separation property**.

12.6 CHARACTERIZATION OF NONSINGULAR SOLUTIONS OF THE SYLVESTER EQUATION

We have just seen that the design of an observer via the Sylvester-observer equation requires a nonsingular solution X for the full-order design (**Algorithm 12.3.1**) or a full rank solution X for the reduced-order design (**Algorithm 12.4.2**). In this section, we describe some necessary conditions for a unique solution of the Sylvester equation to have such properties. For the sake of convenience, we consider the full-order case (i.e., A and F are $n \times n$) only. The results, however, hold for the reduced-order case also and the proofs given here can be easily modified to deal with the latter and are left as an exercise (**Exercise 12.7**).

The following theorem was proved by Bhattacharyya and DeSouza (1981). T· proof here has been taken from Chen (1984).

Theorem 12.6.1. *Necessary Conditions for Nonsingularity of the Sylvester Equation Solution. Let A, F, G, and C, respectively, be of order $n \times n$, $n \times n$, $n \times r$, and $r \times n$. Let X be a unique solution of the Sylvester-observer equation*

$$XA - FX = GC. \tag{12.6.1}$$

Then, **necessary conditions** *for X to be nonsingular are that (A, C) is observable and (F, G) is controllable.*

Proof. From the given Eq. (12.6.1), we have

$$\begin{aligned}
XA^0 - F^0X &= 0, \quad (\text{Noting that } A^0 = I_{n\times n} \text{ and } F^0 = I_{n\times n}.)\\
XA - FX &= GC,\\
XA^2 - F^2X &= GCA + FGC,\\
&\vdots\\
XA^n - F^nX &= GCA^{n-1} + FGCA^{n-2} + \cdots + F^{n-1}GC\ .
\end{aligned}$$

Let $a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of A, and let's denote the controllability matrix of the pair (F, G) by C_{FG}, and the observability matrix of the pair (A, C) by O_{AC}.

First of all, we note that the uniqueness of X implies that the matrix $a(F)$ is nonsingular and vice versa. This is seen as follows: By Theorem 8.2.1, X is a unique solution of (12.6.1) if and only if A and F do not have a common eigenvalue. Again, A and F do not have a common eigenvalue if and only if the matrix $a(F)$ is non-singular because the eigenvalues of $a(F)$ are the n numbers $\prod_{j=1}^{n}(\mu_i - \lambda_j)$, $i = 1, \ldots, n$; where, λ_is are the eigenvalues of A and μ_is are the eigenvalues of F. Thus, $a(F)$ is nonsingular if and only if X is a unique solution of (12.6.1).

Now, multiplying the above equations, respectively, by $a_n, a_{n-1}, \ldots, 1$, and using the Cayley–Hamilton theorem, we obtain after some algebraic manipulations:

$$X = -[a(F)]^{-1} C_{\mathrm{FG}}\, R\, O_{\mathrm{AC}}, \tag{12.6.2}$$

where

$$R = \begin{pmatrix} a_{n-1}I & a_{n-2}I & \cdots & a_1 I & I\\ a_{n-2}I & a_{n-3}I & \cdots & I & 0\\ \vdots & \vdots & & \vdots & \vdots\\ a_1 I & I & \cdots & 0 & 0\\ I & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

From (12.6.2), it then immediately follows that for X to be nonsingular, the rectangular matrices C_{FG} and O_{AC} must have full rank; or, in other words, the pair (F, G) must be controllable and the pair (A, C) must be observable. ■

Corollary 12.6.1. *If G is $n \times 1$ and C is $1 \times n$, then* **necessary and sufficient conditions** *for the unique solution X of* (12.6.1) *to be nonsingular are that (F, G) is controllable and (A, C) is observable.*

Proof. In this case, both the matrices C_{FG} and O_{AC} are square matrices. Thus, from (12.6.2), it immediately follows that X is nonsingular if and only if (F, G) is controllable and (A, C) is observable. ∎

Theorem 12.6.1 has recently been generalized by Datta *et al.* (1997) giving a necessary and sufficient condition for nonsingularity of X. We state the result below and refer the readers to the paper for the proof.

Theorem 12.6.2. *Characterization of the Nonsingularity of the Sylvester Equation Solution. Let A, F, and R be $n \times n$ matrices. Let $a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of A.*

Define

$$S = (F^{n-1} + a_1 F^{n-2} + \cdots + a_{n-1} I)R + (F^{n-2} + a_1 F^{n-3} + \cdots + a_{n-2} I) \times RA + \cdots + (F + a_1 I)RA^{n-2} + RA^{n-1}.$$

Then a unique solution X of the Sylvester equation

$$FX - XA = R$$

is nonsingular if and only if S is nonsingular. Furthermore, the unique solution X is given by

$$X = (a(F))^{-1} S.$$

(Note again that the uniqueness of X implies that $a(F)$ is nonsingular).

Remark

- The results of Theorems 12.6.1 and 12.6.2 also hold in case the matrix X is not necessarily a square matrix. In fact, this general case has been dealt with in the papers by Bhattacharyya and DeSouza (1981), and Datta *et al.* (1997), and conditions for the unique solution to have full rank have been derived there.

12.7 NUMERICAL SOLUTIONS OF THE SYLVESTER-OBSERVER EQUATION

In this section, we discuss numerical methods for solving the Sylvester-observer equation. These methods are based on the reduction of the observable pair (A, C) to the observer-Hessenberg form $(H, \bar{C})$, described in Chapter 6.

The methods use the following template.

Step 1. *Reduction of the problem.* The pair (A, C) is transformed to observer-Hessenberg form by orthogonal similarity, that is, an orthogonal matrix O is constructed such that

$$OAO^{\mathrm{T}} = H, \text{ an unreduced block upper-Hessenberg matrix,}$$
$$CO^{\mathrm{T}} = \bar{C} = (0, C_1).$$

The equation $XA - FX = GC$ is then transformed to $XO^{\mathrm{T}}OAO^{\mathrm{T}} - FXO^{\mathrm{T}} = GCO^{\mathrm{T}}$

or

$$YH - FY = G\bar{C}, \tag{12.7.1}$$

where $Y = XO^{\mathrm{T}}$.

Step 2. *Solution of the reduced problem.* The reduced Hessenberg Sylvester-observer equation (12.7.1) is solved.

Step 3. *Recovery of the Solution X of the Original Problem.* The solution X of the original problem is recovered from the solution of the reduced problem:

$$X = YO. \tag{12.7.2}$$

We now discuss the implementation of Step 2. Step 3 is straightforward. Implementation of Step 1 has been described in Chapter 6.

The simplest way to solve Eq. (12.7.1) is to choose the matrices F and G completely satisfying the controllability requirement of the pair (F, G). In that case, the Sylvester-observer equation reduces to an ordinary Sylvester equation, and, therefore, can be solved using the **Hessenberg–Schur method**, described in Chapter 8.

Indeed, F can be chosen in the lower real Schur form (RSF), as required by the method. Therefore, computations will be greatly reduced. We will not repeat the procedure here. Instead, we will present below two simple recursive procedures, designed specifically for solution of the reduced-order Sylvester-observer equation (12.7.1).

12.7.1 A Recursive Method for the Hessenberg Sylvester-Observer Equation

In the following, we describe a recursive procedure for solving the reduced multi-output Sylvester-observer equation

$$YH - FY = G\bar{C}. \tag{12.7.3}$$

The procedure is due to Van Dooren (1984). The procedure computes simultaneously the matrices F, Y, and G, assuming that $(H, \bar{C})$ is observable.

Set $q = n - r$ and assume that Y has the form:

$$Y = \begin{pmatrix} 1 & y_{12} & \cdots & & \cdots & y_{1,n} \\ & \ddots & \ddots & & & \vdots \\ 0 & & 1 & y_{q,q+1} & \cdots & y_{q,n} \end{pmatrix} \tag{12.7.4}$$

and choose F in lower triangular form (for simplicity):

$$F = \begin{pmatrix} f_{11} & 0 & \cdots & \cdots & 0 \\ f_{21} & f_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ f_{q1} & \cdots & \cdots & \cdots & f_{qq} \end{pmatrix}, \qquad G = \begin{pmatrix} g_1^{\mathrm{T}} \\ g_2^{\mathrm{T}} \\ \cdots \\ g_9^{\mathrm{T}} \end{pmatrix}, \tag{12.7.5}$$

where the diagonal entries $f_{ii}, i = 1, \ldots, q$ are known and the remaining entries of F are to be found. It has been shown in (Van Dooren 1984) that a solution Y in the above form always exists. The reduced Sylvester-observer equation can now be solved recursively for Y, F, and G, as follows.

Let g_i^{T} denote the ith row of G. Comparing the first row of Eq. (12.7.3), we obtain

$$(1, y_1)H - f_{11}(1, y_1) = g_1^{\mathrm{T}}\bar{C}. \tag{12.7.6}$$

Similarly, comparing the ith row of that equation, we have

$$(0, 0, \ldots, 0, 1, y_i)H - (f_i, f_{ii}, 0, \ldots, 0)Y = g_i^{\mathrm{T}}\bar{C}, \quad i = 2, 3, \ldots, q \tag{12.7.7}$$

In the above, $y_i = (y_{i,i+1}, \ldots, y_{i,n})$ and $f_i = (f_{i1}, f_{i2}, \ldots, f_{i,i-1})$.

The Eqs. (12.7.6) and (12.7.7) can be, respectively, written as

$$(y_1, g_1^{\mathrm{T}}) \begin{bmatrix} (H - f_{11}I)_{\text{bottom}(n-1)} \\ -\bar{C} \end{bmatrix} = -[\text{1st row } of \ (H - f_{11}I)], \tag{12.7.8}$$

and

$$(f_i, y_i, g_i^{\mathrm{T}}) \begin{bmatrix} -Y_{\text{top}(i-1)} \\ (H - f_{ii}I)_{\text{bottom}(n-i)} \\ -\bar{C} \end{bmatrix} = -\left[i\text{th row of } (H - f_{ii}I)\right], \tag{12.7.9}$$

where $Y_{\text{top}(i-1)}$ and $(H - f_{ii}I)_{\text{bottom}(n-i)}$ denote, respectively, the top $i - 1$ rows of Y and the bottom $n - i$ rows of $H - f_{ii}I$. Because of the structure of the observer-Hessenberg form $(H, \bar{C})$, the above systems are consistent and these systems can be solved recursively to compute the unknown entries of the matrices Y, F, and G.

We illustrate how to solve these equations in the special case with $n = 3, r = 1$. The reduced equation to be solved in this case is:

$$\begin{pmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & h_{32} & h_{33} \end{pmatrix} - \begin{pmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} g_{11} \\ g_{21} \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} 0 & 0 & c_1 \end{pmatrix}}_{\tilde{C}}.$$

Comparing the first row of the last equation, we have

$$\begin{cases} y_{12}h_{21} = f_{11} - h_{11}, \\ y_{12}(h_{22} - f_{11}) + y_{13}h_{32} = -h_{12}, \\ y_{13}(h_{33} - f_{11}) + y_{12}h_{23} - g_{11}c_1 = -h_{13}. \end{cases} \tag{12.7.10}$$

Similarly, comparing the second row, we have

$$\begin{cases} -f_{21} = -h_{21}, \\ y_{23}h_{32} - f_{21}y_{12} - f_{22} = -h_{22}, \\ y_{23}h_{33} - f_{21}y_{13} - f_{22}y_{23} - g_{21}c_1 = -h_{23}. \end{cases} \tag{12.7.11}$$

The system (12.7.10) can be written as

$$(y_{12}, y_{13}, g_{11}) \begin{pmatrix} h_{21} & h_{22} - f_{11} & h_{23} \\ 0 & h_{32} & h_{33} - f_{11} \\ 0 & 0 & -c_1 \end{pmatrix} = \begin{pmatrix} f_{11} - h_{11} \\ -h_{12} \\ -h_{13} \end{pmatrix}^{\mathrm{T}}$$

Similarly, the system (12.7.11) can be written as

$$(f_{21}, y_{23}, g_{21}) \begin{pmatrix} -1 & -y_{12} & -y_{13} \\ 0 & h_{32} & h_{33} - f_{22} \\ 0 & 0 & -c_1 \end{pmatrix} = \begin{pmatrix} -h_{21} \\ f_{22} - h_{22} \\ -h_{23} \end{pmatrix}^{\mathrm{T}}.$$

Note that since the pair (A, C) is observable, h_{21}, h_{32}, and c_1 are different from zero and, therefore, the matrices of the above two systems are nonsingular.

Algorithm 12.7.1. *A Recursive Algorithm for the Multi-Output Sylvester-Observer Equation*

Inputs. *The matrices* $A_{n\times n}$, *and* $C_{r\times n}$.

Output. *A full-rank solution* X *of the reduced-order Sylvester-observer equation:*

$$XA - FX = GC.$$

Assumption. (A, C) *is observable.*

Step 0. *Set* $n - r = q$.

Step 1. *Transform the pair* (A, C) *to the observer-Hessenberg pair* $(H, \bar{C})$*:*

$$OAO^{\mathrm{T}} = H, \; CO^{\mathrm{T}} = \bar{C}.$$

Step 2. *Choose* $F = (f_{ij})$ *as a* $q \times q$ *lower triangular matrix, where the diagonal entries* f_{ii}, $i = 1, \ldots, q$ *are arbitrarily given numbers, and the off-diagonal entries are to be computed.*

Step 3. *Solve for* Y *satisfying*

$$YH - FY = G\bar{C},$$

where Y *has the form* (12.7.4), *as follows:*

Compute the first row of Y *and the first row of* G *by solving the system* (12.7.8). *Compute the second through* qth *rows of* Y, *the second through* qth *rows of* F, *and the second through* qth *rows of* G *simultaneously, by solving the system* (12.7.9).

Step 4. *Recover* X *from* Y*:*

$$X = YO.$$

Example 12.7.1. Consider Example 12.4.1 again.

Here $n = 4$, $r = 2$.

Step 1. The observer-Hessenberg pair of (A, C) is given by:

$$H = \begin{pmatrix} -0.0200 & 2.4000 & 0.0050 & -32.0000 \\ 0 & -1.6000 & 0.0180 & 1.2000 \\ -0.1400 & -1.3000 & 0.4400 & -30.0000 \\ 0 & 1.000 & 0 & 0 \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 57.3 \end{pmatrix}.$$

The transforming matrix

$$O = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Step 2. Let's choose $f_{11} = -1, \; f_{22} = -2$

Step 3. The solution of the system (12.7.8) is $(0, 7, 6.7, 10.085, -4.1065)$.

Thus, $y_1 = (0, 7, 6.7), \; g_1 = (10.085, -4.1065)$. The first row of $Y = (1, 0, 7, 6.7)$.

The solution of the system (12.7.9) is $(0.0007, -0.0053, -0.4068, 0, 0.0094)$.

Thus, $f_{21} = 0.0007$ and $y_2 = (-0.0053, -0.4068), \; g_2 = (0, 0.0094)$. So,

$$F = \begin{pmatrix} -1 & 0 \\ 0.0007 & -2 \end{pmatrix}, G = \begin{pmatrix} 10.085 & -4.1065 \\ 0 & 0.0094 \end{pmatrix}.$$

The second row of $Y = (0, 1, -0.0053, -0.4068)$.

Therefore,

$$Y = \begin{pmatrix} 1 & 0 & 7 & 6.7 \\ 0 & 1 & -0.0053 & -0.4068 \end{pmatrix}.$$

Step 4. Recover X from Y:

$$X = YO = \begin{pmatrix} 1 & 7 & 0 & 6.7 \\ 0 & -0.0053 & 1 & -0.4068 \end{pmatrix}.$$

Flop-count: Solving for F, G, and Y (using the special structures of these matrices): $2(n-r)rn^2$ flops.

Obtaining the observer-Hessenberg form: $2(3n+r)n^2$ flops (including the construction of O).

Recovering X from Y: $2(n-r)n$

Total: (**About**) $(6+2r)n^3$.

MATCONTROL note: Algorithm 12.7.1 has been implemented in MATCONTROL function **sylvobsm**.

12.7.2 A Recursive Block-Triangular Algorithm for the Hessenberg Sylvester-Observer Equation

A block version of Algorithm 12.7.1 has recently been obtained by Carvalho and Datta (2001). This block algorithm seems to be computationally slightly more efficient than Algorithm 12.7.1 and is suitable for high-performance computing. We describe this new block algorithm below.

As in Algorithm 12.7.1, assume that the observable pair (A, C) has been transformed to an observer-Hessenberg pair $(H, \bar{C})$, that is, an orthogonal matrix O has been computed such that

$$OAO^{\mathrm{T}} = H \quad \text{and} \quad \bar{C} = C\,O^{\mathrm{T}} = \begin{bmatrix} 0 & \ldots & 0, & C_1 \end{bmatrix},$$

where $H = (H_{ij})$ is block upper Hessenberg with diagonal blocks $H_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i = 1, 2, \ldots, p$ and $n_1 + \cdots + n_p = n$.

Given the Observer-Hessenberg pair $(H, \bar{C})$, we now show how to compute the matrices Y, F, and G in **blocks** such that

$$YH - FY = G\bar{C}. \qquad (12.7.12)$$

Partitioning the matrices F, Y, and G conformably with H allows us to write the above equation as

$$\begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1p} \\ & Y_{22} & \dots & Y_{2p} \\ & & Y_{qq} & Y_{qp} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1p} \\ H_{21} & H_{22} & \dots & H_{2p} \\ & H_{32} & \dots & H_{3p} \\ & & H_{p-1,p} & H_{pp} \end{bmatrix}$$

$$- \begin{bmatrix} F_{11} & & \\ F_{21} & F_{22} & \\ F_{q1} & \dots & F_{qq} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1p} \\ & Y_{22} & \dots & Y_{2p} \\ & & Y_{qq} & Y_{qp} \end{bmatrix}$$

$$= \begin{bmatrix} G_1 \\ \dots \\ G_q \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 & C_1 \end{bmatrix}.$$

We set $Y_{ii} = I_{r \times r}, i = 1, 2, \dots, q$ for simplicity. Since matrix F is required to have a preassigned spectrum $\mathcal{S}$, we distribute the elements of $\mathcal{S}$ among the diagonal blocks of F in such a way that $\Omega(F) = \mathcal{S}$, where $\Omega(M)$ denotes the spectrum of M. A complex conjugate pair is distributed as a 2×2 matrix and a real one as a 1×1 scalar on the diagonal of F. Note that, some compatibility between the structure of $\mathcal{S}$ and the parameters n_i, $i = 1, \dots, p$ is required to exist for this to be possible.

Equating now the corresponding blocks on left- and right-hand sides, we obtain:

$$\sum_{k=i}^{j+1} Y_{ik}H_{kj} - \sum_{k=1}^{\min(i,j)} F_{ik}Y_{kj} = 0, \quad j = 1, 2, \dots, p-1. \tag{12.7.13}$$

$$\sum_{k=i}^{p} Y_{ik}H_{kp} - \sum_{k=1}^{i} F_{ik}Y_{kp} = G_iC_1. \tag{12.7.14}$$

From (12.7.13) and (12.7.14), we conclude $F_{ij} = 0$ for $j = 1, 2, \dots, i-2$, and $F_{ij} = H_{ij}$ for $j = i-1$.

Thus, Eqs. (12.7.13) and (12.7.14) are reduced to

$$\sum_{k=i}^{j+1} Y_{ik}H_{kj} - \sum_{k=\max(i-1,1)}^{i} F_{ik}Y_{kj} = 0, \qquad j = i,\ i+1, \dots, p-1. \tag{12.7.15}$$

$$\sum_{k=i}^{p} Y_{ik}H_{kp} - \sum_{k=\max(i-1,1)}^{i} F_{ik}Y_{kp} = G_iC_1, \quad \text{for } i = 1, 2, \dots, q. \tag{12.7.16}$$

For a computational purpose we rewrite Eq. (12.7.15) as

$$\sum_{k=i}^{j} Y_{ik}H_{kj}+Y_{i,j+1}H_{j+1,j}-\sum_{k=\max(i-1,1)}^{i} F_{ik}Y_{kj}=0, \qquad j=i,\ i+1,\ldots,p-1,$$

that is, for $j = i, i+1, \ldots, p-1$,

$$Y_{i,j+1}H_{j+1,j} = -\sum_{k=i}^{j} Y_{ik}H_{kj} + \sum_{k=\max(i-1,1)}^{i} F_{ik}Y_{kj}. \tag{12.7.17}$$

Equations (12.7.16) and (12.7.17) allow us to compute the off-diagonal blocks Y_{ij} of Y and the blocks G_i of G recursively.

This is illustrated in the following, in the special case when $p = 4, q = 3$:

First row: $i = 1$

$H_{11} + Y_{12}H_{21} - F_{11} = 0$ (solve for Y_{12})
$H_{12} + Y_{12}H_{22} + Y_{13}H_{32} - F_{11}Y_{12} = 0$ (solve for Y_{13})
$H_{13} + Y_{12}H_{23} + Y_{13}H_{33} + Y_{14}H_{43} - F_{11}Y_{13} = 0$ (solve for Y_{14})
$H_{14} + Y_{12}H_{24} + Y_{13}H_{34} + Y_{14}H_{44} - F_{11}Y_{14} = G_1C_1$ (solve for G_1).

Second row: $i = 2$

$H_{22} + Y_{23}H_{32} - F_{21}Y_{12} - F_{22} = 0$ (solve for Y_{23})
$H_{23} + Y_{23}H_{33} + Y_{24}H_{43} - F_{21}Y_{13} - F_{22}Y_{23} = 0$ (solve for Y_{24})
$H_{24} + Y_{23}H_{34} + Y_{24}H_{44} - F_{21}Y_{14} - F_{22}Y_{24} = G_2C_1$ (solve for G_2)

Third row: $i = 3$

$H_{33} + Y_{34}H_{43} - F_{32}Y_{23} - F_{33} = 0$ (solve for Y_{34})
$H_{34} + Y_{34}H_{44} - F_{32}Y_{24} - F_{33}Y_{34} = G_3C_1$ (solve for G_3)

The above discussion leads to the following algorithm:

Algorithm 12.7.2. *A Recursive Block Triangular Algorithm for the Multi-Output Sylvester Observer Equation*

Input. *Matrices* $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{r\times n}$ *of full-rank and the self-conjugate set* $\mathcal{S} \in C^{n-r}$.

Output. *Block matrices* X, F, *and* G, *such that* $\Omega(F) = \mathcal{S}$ *and* $XA - FX = GC$.

Step 1. Reduce (A, C) to observer-Hessenberg form $(H, \bar{C})$. Let $n_i, i = 1, \ldots, p$ be the dimension of the diagonal blocks H_{ii} of the matrix H.

Step 2. Partition matrices Y, F, and G in blocks according to the block structure of H. Let $q = p - 1$.

Step 3. Distribute the elements of $\mathcal{S}$ along the diagonal blocks $F_{ii}, i = 1, 2, \ldots, q$ such that $\Omega(F) = \mathcal{S}$; the complex conjugate pairs as 2×2 blocks and the real ones as 1×1 scalars along the diagonal of the matrix F.

Step 4. Set $Y_{11} = I_{n_1 \times n_1}$.
Step 5. For $i = 2, 3, \ldots, q$, set

$$F_{i,i-1} = H_{i,i-1}, \qquad Y_{ii} = I_{n_i \times n_i}.$$

Step 6. For $i = 1, 2, \ldots, q$ do
6.1. For $j = i, i+1, \ldots, p-1$, solve the upper triangular system for $Y_{i,j+1}$:

$$Y_{i,j+1} H_{j+1,j} = -\sum_{k=i}^{j} Y_{ik} H_{kj} + \sum_{k=\max(i-1,1)}^{i} F_{ik} Y_{kj}.$$

6.2. Solve the triangular system for G_i:

$$G_i C_1 = \sum_{k=i}^{p} Y_{ik} H_{kp} - \sum_{k=\max(i-1,1)}^{i} F_{ik} Y_{kp}.$$

Step 7. Form the matrices Y, F, and G from their computed blocks.
Step 8. Recover $X = Y\ O$.

Return

Remark

- Recall that once the matrix X is obtained, the estimated state-vector $\hat{x}(t)$ can be computed from

$$\begin{bmatrix} C \\ X \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}.$$

It is interesting to note that the matrix X does not need to be computed explicitly for this purpose because the above system is equivalent to:

$$\begin{bmatrix} \bar{C} \\ Y \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} O^{\mathrm{T}}.$$

The matrix $\begin{pmatrix} \bar{C} \\ Y \end{pmatrix}$ is a nonsingular block upper Hessenberg by the construction of Y. This structure is very important from the computational point of view since it can possibly be exploited in high-performance computations.

Flop-count and comparison of efficiency

Flop-count of Algorithm 12.7.2

1. Reduction to observer-Hessenberg form using the staircase algorithm:

$$6n^3 + 2rn^2 \text{ flops}$$

2. Computation of Y using Steps 4–8 of the algorithm:

$$\sum_{i=1}^{p-1}\sum_{j=i}^{p}\left[(j-i+1)(2r^3)+2(2r^3)+r^2\right]$$
$$=\sum_{i=1}^{p-1}\sum_{j=i}^{p}\left\{[2(j-i)+7]r^3+r^2\right\} \approx \sum_{i=1}^{p-1}\left[(p-i)^2+(p-i)\right]r^3$$
$$\approx \left[\frac{(p-1)p(2p-1)}{6}+\frac{(p-1)p}{2}\right]r^3 \approx \frac{n^3}{3}+\frac{rn^2}{2}\text{flops.}$$

3. Computation of X from Y: n^3 flops (note that the matrix Y is a unit block triangular matrix).

Thus, total count is $(19n^3/3)+(5r/2)n^2$ flops.

Comparison of Efficiency. Algorithm 12.7.1 requires about $(6+2r)n^3$ flops. [*Note*: the flop count given in Van Dooren (1984) is nearly one half of that given here; this is because a "flop" is counted there as a multiplication/division coupled with an addition/subtraction.]

Also, it can be shown that a recent block algorithm of Datta and Sarkissian (2000) requires about $52n^3/3$ flops.

Thus Algorithm 12.7.2 *is much faster than both Van Dooren's (***Algorithm** 12.7.1*) and the Datta–Sarkissian algorithms.*

Besides, this algorithm is suitable for implementations using the recently developed and widely used scientific computing software package LAPACK (Anderson *et al.* (1999)), since it is composed of BLAS-3 (Basic Linear Algebra Subroutines Level 3) operations such as matrix–matrix multiplications, QR factorizations, and solutions of triangular systems with multiple right hand sides.

Example 12.7.2. We consider Example 12.7.1 again,

Step 1. The matrices H, $\bar{C}$, and O are given by:

$$H=\begin{bmatrix}-0.0200 & 2.4000 & 0.0050 & -32.0000\\ 0 & -1.6000 & 0.0180 & 1.2000\\ -0.1400 & -1.3000 & 0.4400 & -30.0000\\ 0 & 1.0000 & 0 & 0\end{bmatrix},$$

$$C_1=\begin{bmatrix}1.0000 & 0\\ 0 & 57.3000\end{bmatrix},\quad O=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix},\quad \text{and}\quad \bar{C}=(0,C_1).$$

Step 2. $q=1$.

Steps 3 and 4.

$$F = F_{11} = \begin{bmatrix} -1.00 & 0 \\ 0 & -2.00 \end{bmatrix}, \qquad Y_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 5. Skipped ($q = 1$).
Step 6. $i = 1$.
6.1. $j = 1$. Solve the triangular system $Y_{12}H_{21} = -Y_{11}H_{11} + F_{11}Y_{11}$ for Y_{12}:

$$Y_{12} = \begin{bmatrix} 7.0000 & 6.7000 \\ 0 & -0.4000 \end{bmatrix}.$$

6.2: Solve triangular system $G_1C_1 = Y_{11}H_{12} + Y_{12}H_{22} - F_{11}Y_{12}$ for G_1:

$$G_1 = \begin{bmatrix} 10.0850 & -4.1065 \\ 0.0180 & 0.0070 \end{bmatrix}.$$

Step 7. Form matrices Y, F, and G from the computed blocks:

$$Y = \begin{bmatrix} 1 & 0 & 7.0000 & 6.7000 \\ 0 & 1 & 0 & -0.4000 \end{bmatrix},$$

$$F = \begin{bmatrix} -1.000 & 0 \\ 0 & -2.000 \end{bmatrix}, \qquad G = \begin{bmatrix} 10.0850 & -4.1065 \\ 0.0180 & 0.0070 \end{bmatrix}$$

Step 8. Recover $X = YO$:

$$X = \begin{bmatrix} 1 & 7.0000 & 0 & 6.7000 \\ 0 & 0 & 1 & -0.4000 \end{bmatrix}.$$

Verify: $\|XA - FX - GC\|_2 = O(10^{-13})$ and $\Omega(F) = \{-2.0000, -1.0000\}$. Thus, the residue is small and the spectrum of F has been assigned accurately.

MATCONTROL note: Algorithm 12.7.2 has been implemented in MATCONTROL function **sylvobsmb**.

Comparison of the state and estimate for Example 12.7.2: Figure 12.5 shows the relative error between the exact state $x(t)$ and the estimate $\hat{x}(t)$ satisfying

$$\begin{bmatrix} \overline{C} \\ Y \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} O^{\mathrm{T}}$$

with the data above and $u(t)$ as the unit step function. The underlying systems of ordinary differential equations were solved by using **MATLAB** procedure **ode45** with zero initial conditions. The relative error is defined by

$$\frac{\|x(t) - \hat{x}(t)\|_2}{\|x(t)\|_2}.$$

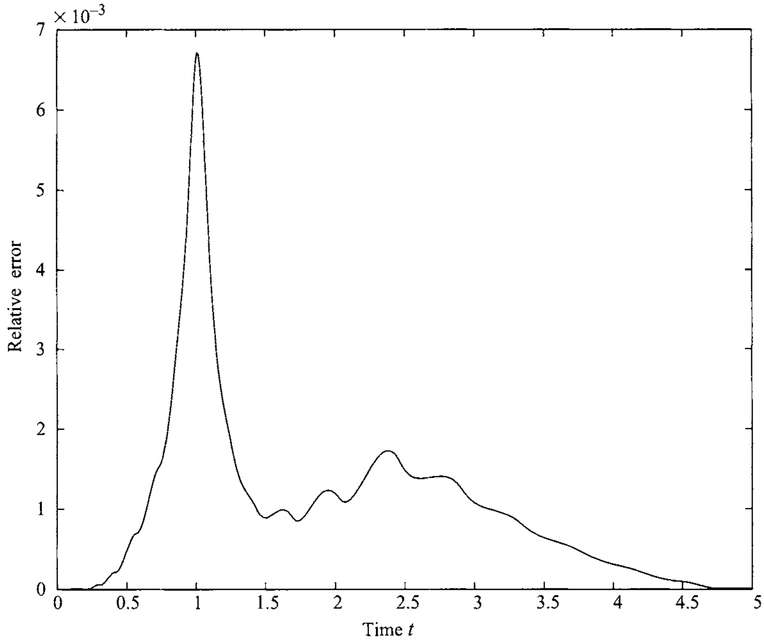

FIGURE 12.5: Relative error between the state and estimate.

12.8 NUMERICAL SOLUTION OF A CONSTRAINED SYLVESTER-OBSERVER EQUATION

In this section, we consider the problem of solving a constrained reduced-order Sylvester-observer equation. Specifically, the following problem is considered:

Solve the reduced-order Sylvester-observer equation

$$XA - FX = GC \tag{12.8.1}$$

such that

$$XB = 0 \tag{12.8.2}$$

and

$$\begin{bmatrix} X \\ C \end{bmatrix} \tag{12.8.3}$$

has full rank.

The importance of solving the constrained Sylvester equation lies in the fact that if the constraint (12.8.2) is satisfied, then the feedback system with the reduced-order observer has the same robustness properties as that of the direct feedback system (see Tsui (1988)).

We state a recent method of Barlow *et al.* (1992) to solve the above problem.

A basic idea behind the method is to transform the given equation to a reduced-order unconstrained equation and then recover the solution of the constrained equation from that of the reduced unconstrained equation. We skip the details and present below just the algorithm. For details of the development of the algorithm, see the above paper by Barlow *et al.* (1992).

Algorithm 12.8.1. *An Algorithm for Constrained Sylvester-observer Equation*

Inputs.

(i) *The system matrices* A, B, *and* C *of order* $n \times n, n \times m$, *and* $r \times n$, *respectively.*

(ii) *A matrix* F *of order* $(n - r)$.

Output. *An* $(n - r) \times n$ *matrix* X *and an* $(n - r) \times r$ *matrix* G *satisfying* (12.8.1) *such that* $\begin{pmatrix} X \\ C \end{pmatrix}$ *is nonsingular and* $XB = 0$.

Assumptions. (A, C) *is observable,* $n > r > m$, *and* $\text{rank}(CB) = \text{rank}(B) = m$.

Step 1. *Find the* QR *factorization of* B:

$$B = W \begin{pmatrix} S \\ 0 \end{pmatrix},$$

where S *is* $m \times m$, *upper triangular and has full rank, and* W *is* $n \times n$ *and orthogonal.*

Partition $W = (W_1, W_2)$, *where* W_1 *is* $n \times m$ *and* W_2 *is* $n \times (n - m)$.

Step 2. *Set*

$$A_1 = W_2^{\mathrm{T}} A W_1, \qquad A_2 = W_2^{\mathrm{T}} A W_2, \qquad C_1 = C W_1, \qquad C_2 = C W_2$$

Step 3. *Find a* QR *factorization of* C_1:

$$C_1 = (Q_1, Q_2) \begin{pmatrix} R \\ 0 \end{pmatrix},$$

where Q_1 *is* $r \times m$, Q_2 *is* $r \times (r - m)$, *and* R *is an* $m \times m$ *upper triangular matrix with full rank.*

Step 4. *Define* E *by*

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = Q^{\mathrm{T}} C_2,$$

where E_1 *is* $m \times (n - m)$, E_2 *is* $(r - m) \times (n - m)$, *and* $Q = (Q_1, Q_2)$.

Step 5. *Form* $\hat{A} = A_2 - A_1 R^{-1} E_1$. *Solve the Sylvester equation:*

$$Z\hat{A} - FZ = G_2 E_2,$$

choosing G_2 *randomly.* (*Use* **Algorithm 8.5.1.**)

Step 6. *Set* $G_1 = ZA_1R^{-1} = ZJ,\ G = (G_1, G_2)Q^{\mathrm{T}}$, *and* $X = ZW_2^{\mathrm{T}}$.

(*Note that* Z *is of order* $(n-r) \times (n-m)$ *and* $J = A_1R^{-1}$ *is computed by solving the upper triangular system* $JR = A_1$).

MATHCONTROL note: Algorithm 12.8.1 has been implemented in MATCONTROL functions **sylvobsc**.

Example 12.8.1. Consider solving the Eq. (12.8.1) using Algorithm 12.8.1 with

$$A = \begin{pmatrix} -0.02 & 0.005 & 2.4 & -3.2 \\ -0.14 & 0.44 & -1.3 & -3 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}.$$

Then, $n = 4,\ r = 2,\ m = 1$.

Step 1. $W_1 = (-0.5, -0.5, -0.5, -0.5)^{\mathrm{T}}$.

$$W_2 = \begin{pmatrix} -0.5 & -0.5 & -0.5 \\ 0.8333 & -0.1667 & -0.1667 \\ -0.1667 & 0.8333 & -0.1667 \\ -0.1667 & -0.1667 & 0.8333 \end{pmatrix}, \qquad S = -2.$$

Step 2. $A_1 = W_2^{\mathrm{T}} A W_1 = \begin{pmatrix} 1.5144 \\ -0.2946 \\ -0.9856 \end{pmatrix},$

$$A_2 = W_2^{\mathrm{T}} A W_2 = \begin{pmatrix} 0.9015 & -1.6430 & 0.5596 \\ 0.1701 & -2.5976 & 2.9907 \\ -0.4185 & -0.2230 & 1.5604 \end{pmatrix},$$

$$C_1 = CW_1 = \begin{pmatrix} -0.5 \\ -28.65 \end{pmatrix},$$

$$C_2 = CW_2 = \begin{pmatrix} 0.8333 & -0.1667 & -0.1667 \\ -9.55 & -9.55 & 47.75 \end{pmatrix}.$$

Step 3. $Q = \begin{pmatrix} -0.0174 & -0.9998 \\ -0.9998 & 0.0174 \end{pmatrix}, \qquad R = 28.6544.$

Step 4. $E = Q^{\mathrm{T}} C_2 = \begin{pmatrix} 9.5340 & 9.5515 & -47.7398 \\ -0.9998 & 0 & 0.9998 \end{pmatrix}$,

$$E_2 = \begin{pmatrix} -0.9998 & 0 & 0.9998 \end{pmatrix}$$

Step 5. $\hat{A} = A_2 - A_1 R^{-1} E_1 = \begin{pmatrix} 0.3976 & -2.1478 & 1.96 \\ -0.0721 & -2.4944 & 2.4999 \\ -0.0905 & 0.1056 & -0.0817 \end{pmatrix}$.

Choose

$$G_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution Z of the Sylvester equation: $Z\hat{A} - FZ = G_2 E_2$

$$Z = \begin{pmatrix} -0.6715 & 0.9589 & -0.0860 \\ -0.2603 & -0.5957 & 0.9979 \end{pmatrix}.$$

Step 6. $G_1 = Z A_1 R^{-1} = \begin{pmatrix} -0.0424 \\ -0.0420 \end{pmatrix}$.

$$G = (G_1, G_2) Q^{\mathrm{T}} = \begin{pmatrix} -0.9991 & 0.0598 \\ 0.0007 & 0.0419 \end{pmatrix},$$

$$X = \begin{pmatrix} -0.1007 & -0.7050 & 0.9254 & -0.1196 \\ -0.0709 & -0.2839 & -0.6199 & 0.9743 \end{pmatrix}.$$

Verify:

(i) $\|XA - FX - GC\| = O(10^{-3})$,

(ii) $XB = 10^{-3} \begin{pmatrix} 0.1000 \\ -0.4000 \end{pmatrix}$, and

(iii) $\operatorname{rank} \begin{pmatrix} X \\ C \end{pmatrix} = 4$.

Note: If G_2 were chosen as $G_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the solution X would be rank-deficient and consequently $\begin{pmatrix} X \\ C \end{pmatrix}$ would be also rank-deficient. Indeed, in this case,

$$X = \begin{pmatrix} -0.1006 & -0.7044 & 0.9246 & -0.1195 \\ -0.1006 & -0.7044 & 0.9246 & -0.1195 \end{pmatrix},$$

which has rank 1.

12.9 OPTIMAL STATE ESTIMATION: THE KALMAN FILTER

So far we have discussed the design of an observer ignoring the "noise" in the system, that is, we assumed that all the inputs were given exactly and all the outputs were measured exactly without any errors. But in a practical situation, the measurements are always corrupted with noise. Therefore, it is more practical to consider a system with noise. In this section, we consider the problem of finding the optimal steady-state estimation of the states of a **stochastic system**. Specifically, the following problem is addressed.

Consider the stochastic system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{12.9.1}$$

where $w(t)$ and $v(t)$ represent "noise" in the input and the output, respectively. The problem is to find the linear estimate $\hat{x}(t)$ of $x(t)$ from all past and current output $\{y(s), s \leq t\}$ that minimizes the mean square error:

$$E[\|x(t) - \hat{x}(t)\|^2], \text{ as } t \to \infty, \tag{12.9.2}$$

where $E[z]$ is the expected value of a vector z.

The following assumptions are made:

1. The system is **controllable** and **observable**. (12.9.3)

Note that the controllability assumption implies that the noise $w(t)$ excites all modes of the system and the observability implies that the noiseless output $y(t) = Cx(t)$ contains information about all states.

2. Both w and v are white noise, **zero-mean** stochastic processes.

   That is, for all t and s,

$$E[w(t)] = 0, \qquad E[v(t)] = 0, \tag{12.9.4}$$

$$E[w(t)w^{\mathrm{T}}(s)] = W\delta(t-s), \tag{12.9.5}$$

$$E[v(t)v^{\mathrm{T}}(s)] = V\delta(t-s), \tag{12.9.6}$$

   where W and V are **symmetric and positive semidefinite and positive definite covariance matrices,** respectively, and $\delta(t-s)$ is the Dirac delta function.

3. The noise processes w and v are **uncorrelated** with one another, that is,

$$E[w(t)v^{\mathrm{T}}(s)] = 0. \tag{12.9.7}$$

4. The initial state x_0 is a **Gaussian zero-mean** random variable with known covariance matrix, and uncorrelated with w and v. That is,

$$\begin{gathered} E[x_0] = 0, \\ E[x_0x_0^{\mathrm{T}}] = S, \qquad E[x_0w^{\mathrm{T}}(t)] = 0, \qquad E[x_0v^{\mathrm{T}}(t)] = 0, \end{gathered} \tag{12.9.8}$$

   where S is the positive semidefinite covariance matrix.

The following is a well-known (almost classical) result on the solution of the above problem using an algebraic Riccati equation (ARE). For a proof, see Kalman and Bucy (1961). For more details on this topic, see Kailath *et al.* (2000).

Theorem 12.9.1. *Under the assumptions* (12.9.3)–(12.9.8), *the best estimate* $\hat{x}(t)$ *(in the linear least-mean-square sense) can be generated by the* **Kalman filter** *(also known as the* **Kalman-Bucy filter**).

$$\dot{\hat{x}}(t) = (A - K_{\mathrm{f}}C)\hat{x}(t) + Bu(t) + K_{\mathrm{f}}y(t), \qquad (12.9.9)$$

where $K_{\mathrm{f}} = X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}$, *and* X_{f} *is the symmetric positive definite solution of the ARE:*

$$AX + XA^{\mathrm{T}} - XC^{\mathrm{T}}V^{-1}CX + FWF^{\mathrm{T}} = 0. \qquad (12.9.10)$$

Definition 12.9.1. *The matrix* $K_{\mathrm{f}} = X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}$ *is called the* ***filter gain matrix.***

Note: The output estimate $\hat{y}(t)$ is given by $\hat{y}(t) = C\hat{x}(t)$.

The error between the measured output $y(t)$ and the predicted output $C\hat{x}(t)$ is given by the residual $r(t)$:

$$r(t) = y(t) - C\hat{x}(t).$$

where $\hat{x}$ is generated by (12.9.9).

Algorithm 12.9.1. *The State Estimation of the Stochastic System Using Kalman Filter*

Inputs.

1. *The matrices* A, B, C, *and* F *defining the system* (12.9.1)
2. *The covariance matrices* V *and* W *(both symmetric and positive definite).*

Output. *An estimate* $\hat{x}(t)$ *of* $x(t)$ *such that* $E[\|x(t) - \hat{x}(t)\|^2]$ *is minimized, as* $t \to \infty$.

Assumptions. (12.9.3)–(12.9.8).

Step 1. *Obtain the unique symmetric positive definite solution* X_{f} *of the ARE:*

$$AX_{\mathrm{f}} + X_{\mathrm{f}}A^{\mathrm{T}} - X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}CX_{\mathrm{f}} + FWF^{\mathrm{T}} = 0.$$

Step 2. *Form the filter gain matrix* $K_{\mathrm{f}} = X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}$.

Step 3. *Obtain the estimate* $\hat{x}(t)$ *by solving* (12.9.9).

Duality Between Kalman Filter and the LQR Problems

The ARE (12.9.10) in Theorem 12.9.1 is dual to the Continuous-time Algebraic Riccati Equation (CARE) that arises in the solution of the LQR problem. To distinguish it from the CARE, it will be referred to as the **Continuous-time Filter Algebraic Riccati Equation** (CFARE).

Using this duality, the following important properties of the Kalman filter, *dual to those of the LQR problem described in Chapter 10*, can be established **(Exercise 12.15)**.

1. *Guaranteed stability.* The filter matrix $A - K_f C$ is stable, that is, $\text{Re}\lambda_i(A - K_f C) < 0$; $i = 1, 2, \dots, n$, where $\lambda_i, i = 1, \dots, n$, are the eigenvalues of $A - K_f C$.
2. *Guaranteed robustness.* Let V be a diagonal matrix and let $W = I$. Let $G_{\text{KF}}(s)$ and $G_{\text{FOL}}(s)$ denote, respectively, the Kalman-filter loop-transfer matrix and the filter open-loop transfer matrix (from $w(t)$ to $y(t)$), that is,

$$G_{\text{KF}}(s) \equiv C(sI - A)^{-1}K_f \tag{12.9.11}$$

and

$$G_{\text{FOL}}(s) \equiv C(sI - A)^{-1}F. \tag{12.9.12}$$

Then the following equality holds:

$$(I + G_{\text{KF}}(s))V(I + G_{\text{KF}}(s))^* = V + G_{\text{FOL}}(s)G^*_{\text{FOL}}(s). \tag{12.9.13}$$

Using the above equality, one obtains

$$(I + G_{\text{KF}}(s))(I + G_{\text{KF}}(s))^* \geq I. \tag{12.9.14}$$

In terms of singular values, one can then deduce that

$$\sigma_{\min}(I + G_{\text{KF}}(s)) \geq 1 \tag{12.9.15}$$

or

$$\sigma_{\max}(I + G_{\text{KF}}(s))^{-1} \leq 1$$

and

$$\sigma_{\min}(I + G^{-1}_{\text{KF}}(s)) \geq \tfrac{1}{2}. \tag{12.9.16}$$

See the article by Athans on "Kalman filtering" in the *Control Handbook* (1996, pp. 589–594), edited by W.S. Levine, IEEE Press/CRC Press.

Example 12.9.1. Consider the stochastic system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + w(t),\\ y(t) &= Cx(t) + v(t)\end{aligned}$$

with A, B, and C as in Example 12.4.1.

Take

$$W = BB^{\mathrm{T}}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F = I_{4\times 4}.$$

Step 1. The symmetric positive definite solution X_f of the CFARE

$$AX + XA^{\mathrm{T}} - XC^{\mathrm{T}}V^{-1}CX + FWF^{\mathrm{T}} = 0$$

is

$$X_{\mathrm{f}} = \begin{pmatrix} 8.3615 & 0.0158 & 0.0187 & -0.0042 \\ 0.0158 & 9.0660 & 0.0091 & -0.0031 \\ 0.0187 & 0.0091 & 0.0250 & 0.0040 \\ -0.0042 & -0.0031 & 0.0040 & 0.0016 \end{pmatrix}.$$

Step 2. The filter gain matrix $K_{\mathrm{f}} = X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}$ is

$$K_{\mathrm{f}} = \begin{pmatrix} 0.0158 & -0.2405 \\ 9.0660 & -0.1761 \\ 0.0091 & 0.2289 \\ -0.0031 & 0.0893 \end{pmatrix}.$$

The optimal state estimator of $\hat{x}(t)$ is given by

$$\dot{\hat{x}}(t) = (A - K_{\mathrm{f}}C)\hat{x}(t) + Bu(t) + K_{\mathrm{f}}y(t).$$

The filter eigenvalues, that is, the eigenvalues of $A - K_{\mathrm{f}}C$, are $\{-0.0196, -8.6168, -3.3643 \pm j2.9742\}$.

MATLAB note: The MATLAB function **kalman** designs a Kalman state estimator given the state-space model and the process and noise covariance data. **kalman** is available in MATLAB **Control System Toolbox**.

Comparison of the state and the estimate for Example 12.9.1: In Figure 12.6 we compare the actual state with the estimated state obtained in Example 12.9.1 with $x(0) = \hat{x}(0) = (-6 \quad -1 \quad 1 \quad 2)^{\mathrm{T}}$ and $u(t) = H(t)(1 \quad 1 \quad 1 \quad 1)^{\mathrm{T}}$, where $H(t)$ is the unit step function. Only the first and second variables are compared. The solid line corresponds to the exact state and the dotted line corresponds to the estimated state. *The graphs of the second variables are indistinguishable.*

The Kalman Filter for the Discrete-Time System

Consider now the discrete stochastic system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Fw_k, \\ y_k &= Cx_k + v_k, \end{aligned} \tag{12.9.17}$$

where w and v are the process and measurement noise. Then, under the same assumptions as was made in the continuous-time case, it can be shown that the

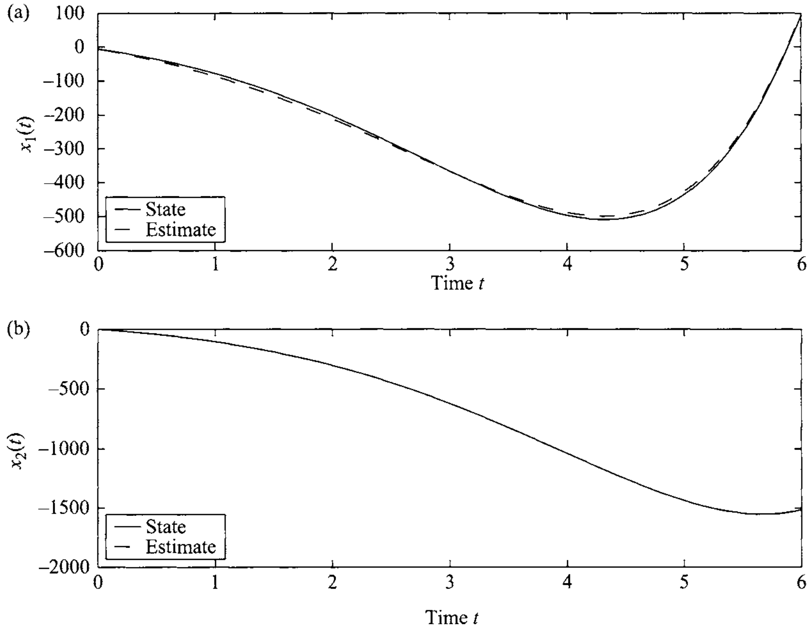

FIGURE 12.6: The (a) first and (b) second variables of the state $x(t)$ and estimate $\hat{x}(t)$, obtained by Kalman filter.

state error covariance is minimized in steady-state when the filter gain is given by

$$K_{\mathrm{d}} = AX_{\mathrm{d}}C^{\mathrm{T}}(CX_{\mathrm{d}}C^{\mathrm{T}} + V)^{-1}, \tag{12.9.18}$$

where X_{d} is the symmetric positive semidefinite solution of the Riccati equation:

$$X = A(X - XC^{\mathrm{T}}(CXC^{\mathrm{T}} + V)^{-1}CX)A^{\mathrm{T}} + FWF^{\mathrm{T}}, \tag{12.9.19}$$

and V and W are the symmetric positive definite and positive semidefinite covariance matrices, that is,

$$E\left[v_k v_j^{\mathrm{T}}\right] = V\delta_{kj}, \quad E\left[w_k w_j^{\mathrm{T}}\right] = W\delta_{kj}; \tag{12.9.20}$$

and

$$\delta_{kj} = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j. \end{cases} \tag{12.9.21}$$

For details, see Glad and Ljung (2000, pp. 137–138).

Definition 12.9.2. *In analogy with the continuous-time case, the discrete-time algebraic Riccati equation* (*DARE*) (12.9.19), *arising in discrete Kalman filter will be called the discrete filter algebraic Riccati equation or DFARE, for short.*

12.10 THE LINEAR QUADRATIC GAUSSIAN PROBLEM

The linear quadratic regulator (LQR) problems deal with optimization of a performance measure for a deterministic system. The **Linear Quadratic Gaussian** (LQG) problems deal with optimization of a performance measure for a stochastic system.

Specifically, the **continuous-time** LQG problem is defined as follows:

> Consider the controllable and observable stochastic system (12.9.1) and the quadratic objective function
>
> $$J_{QG} = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} (x^T Q x + u^T R u) dt\right],$$
>
> where the weighting matrices Q and R are, respectively, symmetric positive semidefinite and positive definite. Suppose that the noise $w(t)$ and $v(t)$ are both Gaussian, white, zero-mean, and stationary processes with positive semidefinite and positive definite covariance matrices W and V. The problem is to find the optimal control $u(t)$ that minimizes the average cost.

Solution of the LQG Problem via Kalman Filter

The solution of the LQG problem is obtained by combining the solutions of the deterministic LQR problem and the optimal state estimation problem using the Kalman filter (see the next subsection on the separation property of the LQG design).

The control vector $u(t)$ for the LQG problem is given by

$$u(t) = -K_c \hat{x}(t), \tag{12.10.1}$$

where

(i) the matrix K_c is the feedback matrix of the associated LQR problem, that is,

$$K_c = R^{-1} B^T X_c, \tag{12.10.2}$$

X_c satisfying the CARE: $X_c A + A^T X_c + Q - X_c B R^{-1} B^T X_c = 0.$ (12.10.3)

(ii) the vector $\hat{x}(t)$ is generated by the Kalman filter:

$$\dot{\hat{x}}(t) = (A - K_f C)\hat{x}(t) + Bu(t) + K_f y(t). \tag{12.10.4}$$

The filter gain matrix $K_f = X_f C^T V^{-1}$ and X_f satisfies the CFARE

$$A X_f + X_f A^T - X_f C^T V^{-1} C X_f + F W F^T = 0. \tag{12.10.5}$$

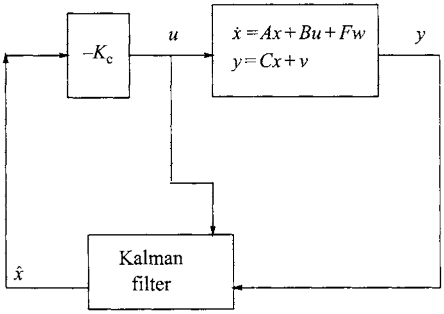

FIGURE 12.7: The LQG design via Kalman filter.

For a proof of the above, see Dorato *et al.* (1995).

The LQG design via Kalman filter is illustrated in Figure 12.7.

The LQG Separation Property

In this section, we establish the LQG separation property. For the sake of convenience, we assume that $F = I$. By substituting (12.10.1) into (12.10.3), we obtain the compensator:

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - BK_c - K_f C)\hat{x}(t) + K_f y(t), \\ u(t) &= -K_c \hat{x}(t). \end{aligned} \tag{12.10.6}$$

The transfer function $M(s)$ of this compensator (from $y(t)$ to $u(t)$) can be easily written down:

$$M(s) = -K_c(sI - A + BK_c + K_f C)^{-1} K_f. \tag{12.10.7}$$

From (12.10.6) and (12.9.1), it is easy to see that the closed-loop matrix satisfies the differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} = \begin{pmatrix} A & -BK_c \\ K_f C & A - BK_c - K_f C \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} + \begin{pmatrix} I & O \\ O & K_f \end{pmatrix} \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}. \tag{12.10.8}$$

Define the error vector

$$e(t) = x(t) - \hat{x}(t). \tag{12.10.9}$$

Then from (12.10.8) and (12.10.9), we obtain

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A - BK_c & BK_c \\ O & A - K_f C \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} + \begin{pmatrix} I & O \\ I & -K_f \end{pmatrix} \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}.$$

Thus, the $2n$ closed-loop eigenvalues are the union of the n eigenvalues of $A - BK_c$ and the n eigenvalues of $A - K_f C$.

Furthermore, if (A, B) is controllable and (A, C) is observable, then both the matrices $A - BK_c$ and $A - K_f C$ are stable. However, the matrix $A - BK_c - K_f C$ is not necessarily stable.

Algorithm 12.10.1. *The Continuous-time LQG Design Method*
Inputs.

(i) *The matrices A, B, C, and F defining the system* (12.9.1).
(ii) *The covariance matrices V and W.*

Output. *The control vector $u(t)$ generated by the LQG regulator.*
Assumptions. (12.9.3)–(12.9.8).
Step 1. *Obtain the symmetric positive definite stabilizing solution X_c of the CARE:*

$$XA + A^T X - XBR^{-1}B^T X + Q = 0. \tag{12.10.10}$$

Step 2. *Compute $K_c = R^{-1}B^T X_c$*
Step 3.
3.1. *Solve the CFARE:*

$$AX + XA^T - XC^T V^{-1} CX + FWF^T = 0 \tag{12.10.11}$$

to obtain the symmetric positive definite stabilizing solution X_f.
3.2. *Compute filter gain matrix*

$$K_f = X_f C^T V^{-1}. \tag{12.10.12}$$

Step 4. *Solve for $\hat{x}(t)$:*

$$\dot{\hat{x}}(t) = (A - BK_c - K_f C)\hat{x}(t) + K_f y(t), \quad \hat{x}(0) = \hat{x}_0. \tag{12.10.13}$$

Step 5. *Determine the control law:*

$$u(t) = -K_c \hat{x}(t). \tag{12.10.14}$$

Remarks

- Though the optimal closed-loop system will be asymptotically stable, the LQG design method described above does not have the same properties as the LQR design method; in fact, most of the nice properties of the LQR design are lost by the introduction of the Kalman filter. See Doyle (1978) and Zhou *et al.* (1996, pp. 398–399).

- **Overall, the LQG design has lower stability margins than the LQR design and its sensitivity properties are not as good as those of the LQR design.**
- It might be possible to recover some of the desirable properties of the LQR design by choosing the weights appropriately. This is known as the **Loop Transfer Recovery** (LTR). The details are beyond the scope of this book. See Doyle and Stein (1979, 1981) and the book by Anderson and Moore (1990).

Example 12.10.1. We consider the LQG design for the helicopter problem of Example 12.9.1, with

$$Q = C^{\mathrm{T}}C \quad \text{and} \quad R = I_{2\times 2},$$

and the same W and V.

Step 1. The stabilizing solution X_{c} of the CARE (computed by MATLAB function **care**) is

$$X_{\mathrm{c}} = \begin{pmatrix} 0.0071 & -0.0021 & -0.0102 & -0.0788 \\ -0.0021 & 0.1223 & 0.0099 & -0.1941 \\ -0.0102 & 0.0099 & 41.8284 & 174.2 \\ -0.0788 & -0.1941 & 174.2 & 1120.9 \end{pmatrix}.$$

Step 2. The control gain matrix K_{c} is

$$K_{\mathrm{c}} = R^{-1}B^{\mathrm{T}}X_{\mathrm{c}} = \begin{pmatrix} -0.0033 & 0.0472 & 14.6421 & 60.8894 \\ 0.0171 & -1.0515 & 0.2927 & 3.2469 \end{pmatrix}.$$

Step 3. The filter gain matrix K_{f} computed in Example 12.9.1 is

$$K_{\mathrm{f}} = \begin{pmatrix} 0.0158 & -0.2405 \\ 9.0660 & -0.1761 \\ 0.0091 & 0.2289 \\ -0.0031 & 0.0893 \end{pmatrix}.$$

The closed-loop eigenvalues: The closed-loop eigenvalues are the union of the eigenvalues of $A - BK_{\mathrm{c}}$ (the controller eigenvalues) and those of $A - K_{\mathrm{f}}C$ (the filter eigenvalues):

$$\begin{aligned} &\{-3.3643 \pm 2.9742j, -0.0196, -8.6168\} \\ &\quad \cup \{-0.0196, -8.6168, -3.3643 \pm 2.9742j\}. \end{aligned}$$

MATLAB note: The MATLAB function (from the **control system toolbox**) **lqgreg** forms the LQG regulator by combining the Kalman estimator designed with **Kalman** and the optimal state feedback gain designed with **lqr.** In case of a discrete-time system, the command **dlqr** is used in place of **lqr.**

12.11 SOME SELECTED SOFTWARE

12.11.1 MATLAB Control System Toolbox

LQG design tools

kalman	Kalman estimator
kalmd	Discrete Kalman estimator for continuous plant
lqgreg	Form LQG regulator given LQ gain and Kalman estimator.

12.11.2 MATCONTROL

SYLVOBSC	Solving the constrained multi-output Sylvester-observer equation
SYLVOBSM	Solving the multi-output Sylvester-observer equation
SYLVOBSMB	Block triangular algorithm for the multi-output Sylvester-observer equation

12.11.3 CSP-ANM

Design of reduced-order state estimator (observer)

- The reduced-order state estimator using pole assignment approach is computed by `ReducedOrderEstimator` [*system, poles*].
- The reduced-order state estimator via solution of the Sylvester-observer equation using recursive bidiagonal scheme (a variation of the triangular scheme of van Dooren (1984)) is computed by `ReducedOrderEstimator` [*system, poles*, `Method` → `RecursiveBidiagonal`] and `ReducedOrderEstimator` [*system, poles*, `Method` → `RecursiveBlockBidiagonal`] (block version of the recursive bidiagonal scheme).
- The reduced-order state estimator via solution of the Sylvester-observer equation using recursive triangular scheme is computed by `ReducedOrderEstimator` [*system, poles*, `Method` → `RecursiveTriangular`] and `ReducedOrderEstimator` [*system, poles*, `Method` → `RecursiveBlockTriangular`] (block version of the recursive triangular scheme).

12.11.4 SLICOT

FB01RD	Time-invariant square root covariance filter (Hessenberg form)
FB01TD	Time-invariant square root information filter (Hessenberg form)
FB01VD	One recursion of the conventional Kalman filter
FD01AD	Fast recursive least-squares filter.

12.11.5 MATRIX$_X$

Purpose: Calculate optimal state estimator gain matrix for a discrete time system.

Syntax: [EVAL, KE]=DESTIMATOR (A, C, QXX, QYY, QXY) OR
[EVAL, KE, P]=DESTIMATOR (A, C, QXX, QYY, QXY)

Purpose: Calculate optimal state estimator gain matrix for a continuous time system.

Syntax: [EVAL, KE]=ESTIMATOR (A, C, QXX, QYY, QXY)
[EVAL, KE, P]=ESTIMATOR (A, C, QXX, QYY, QXY)

Purpose: Given a plant and optimal regulator, this function designs an estimator which recovers loop transfer robustness via the design parameter RHO. Plots of singular value loop transfer response are made for the (regulator) and (estimator + regulator) systems.

Syntax:
[SC, NSC, EVE, KE, SLTF, NSLTF]=LQELTR (S, NS, QXX, QYY, KR, RHO, WMIN, WMAX,
{NPTS} , {OPTION}); OR
[SC, NSC, EVE, KR, SLTF, NSLTF]=LQRLTR (S, NS, RXX, RUU, KE, RHO, OMEGA,
{OPTION});

Purpose: Given a plant and optimal estimator, this function designs a regulator which recovers loop transfer robustness via the design parameter RHO. Plots of singular value loop transfer response are made for the (estimator) and (regulator + estimator) systems.

Syntax:
[SC, NSC, EVR, KR, SLTF, NSLTF]=LQRLTR (S, NS, RXX, RUU, KE, RHO, WMIN, WMAX,
{NPTS} , {OPTION}); OR
[SC, NSC, EVR, KR, SLTF, NSLTF]=LQRLTR (S, NS, RXX, RUU, KE, RHO, OMEGA,
{OPTION});

12.12 SUMMARY AND REVIEW

In Chapters 10 and 11 we have discussed feedback stabilization, EVA and related problems. Solutions of these problems require that the states are available for measurements. Unfortunately, in many practical situations, all the states are not accessible. One therefore needs to estimate the states by knowing only input and output. This gives rise to **state estimation** problem, which is the subject matter of this chapter.

Full State Estimation

The states can be estimated using

- EVA approach **(Theorem 12.2.1)**
- Solving the associated Sylvester-like matrix equation, called the **Sylvester-observer equation (Algorithm 12.3.1)**.

In "**the eigenvalue assignment approach**," the states x can be estimated by constructing the observer

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Ky(t) + Bu(t),$$

where the matrix K is constructed such that $A - KC$ is a stable matrix, so that the error $e(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$.

Using "**the Sylvester equation approach**," the states are estimated by solving the Sylvester-observer equation

$$XA - FX = GC,$$

where the matrix F is chosen to be a stable matrix and G is chosen such that the solution X is nonsingular. The estimate $\hat{x}(t)$ is given by $\hat{x}(t) = X^{-1}z(t)$, where $z(t)$ satisfies $\dot{z}(t) = Fz(t) + Gy(t) + XBu(t)$.

Reduced-Order State Estimation

If the matrix C has full rank r, then the full state estimation problem can be reduced to the problem of estimating only the $n - r$ states.

Again, two approaches: **the EVA approach** and **the Sylvester-observer matrix equation** can be used for reduced-order state estimation.

Reduced-order state estimation via EVA (**Algorithm 12.4.1**) is discussed in **Section 12.4.1**. Here the EVA problem to be solved is of order $n - r$.

In the Sylvester equation approach for reduced-order state estimation, one solves a reduced-order equation

$$XA - FX = GC$$

by choosing F as an $(n-r) \times (n-r)$ stable matrix and choosing G as an $(n-r) \times r$ matrix such that the solution matrix X has full rank. The procedure is described in **Algorithm 12.4.2.**

Two numerical methods for the multi-output equation, both based on reduction of the pair (A, C) to the observer-Hessenberg pair $(H, \bar{C})$, are proposed to solve the above reduced-order Sylvester-observer equation. These methods are described in Section 12.7 (**Algorithms 12.7.1** and **12.7.2**).

Optimal State Estimation: The Kalman Filter

If there is "noise" in the system, then one has to consider the state estimation problem for a **stochastic system**. The optimal steady-state estimation of a stochastic system is traditionally done by constructing the **Kalman filter.**

For the continuous-time stochastic system (12.9.1), the Kalman filter is given by

$$\dot{\hat{x}}(t) = (A - K_{\mathrm{f}}C)\hat{x}(t) + Bu(t) + K_{\mathrm{f}}y(t),$$

where $K_{\mathrm{f}} = X_{\mathrm{f}}C^{\mathrm{T}}V^{-1}$, and X_{f} the symmetric positive definite solution of the CFARE: $AX + XA^{\mathrm{T}} - XC^{\mathrm{T}}V^{-1}CX + FWF^{\mathrm{T}} = 0$.

The matrices V and W are the covariance matrices associated with "noise" in the output and input, respectively. The matrix K_{f} is called the **Kalman filter gain.**

It can be shown that under the assumptions (12.9.3)–(12.9.8), the above Riccati equation has a symmetric positive definite solution and the estimate $\hat{x}(t)$ is such that

$$E[\|x(t) - \hat{x}(t)\|^2]$$

is minimized as $t \to \infty$.

Like the LQR design, the Kalman filter also possesses **guaranteed stability** and **robustness properties**:

- The matrix $A - K_{\mathrm{f}}C$ is stable.
- $\sigma_{\min}(I + G_{\mathrm{KF}}(s)) \geq 1$
- $\sigma_{\min}(I + G_{\mathrm{KF}}^{-1}(s)) \geq \frac{1}{2}$,

where $G_{\mathrm{KF}}(s) = C(sI - A)^{-1}K$.

For the discrete-time system, the DFARE to be solved is

$$X = A(X - XC^{\mathrm{T}}(CXC^{\mathrm{T}} + V)^{-1}CX)A^{\mathrm{T}} + FWF^{\mathrm{T}}$$

and the discrete **Kalman filter** gain is given by

$$K_{\mathrm{d}} = X_{\mathrm{d}}C^{\mathrm{T}}(CX_{\mathrm{d}}C^{\mathrm{T}} + V)^{-1},$$

where X_{d} is the stabilizing solution of the above discrete Riccati equation (DFARE).

The Linear Quadratic Gaussian (LQG) Problem

The LQG problem is the problem of finding an optimal control that minimizes a performance measure given a **stochastic** system. **Thus, it is the counterpart of the deterministic LQR problem for a stochastic system.**

Given the stochastic system (12.9.1) and the performance measure J_{QG} defined in Section 12.10, the optimal control $u(t)$ for the LQG problem can be computed as

$$u(t) = -K_c \hat{x}(t),$$

where $K_c = R^{-1}B^T X_c$, X_c being the solution of the CARE arising in the solution of the deterministic LQR problem. The estimate $\hat{x}(t)$ is determined by using the Kalman filter. Specifically, $\hat{x}(t)$ satisfies

$$\dot{\hat{x}}(t) = (A - K_f C)\hat{x}(t) + Bu(t) + K_f y(t),$$

where K_f is the Kalman filter gain computed using the stabilizing solution of the CFARE.

Thus, the LQG problem is solved by first solving the LQR problems followed by constructing a Kalman filter.

Unfortunately, the LQG design described as above does not have some of the nice properties of the LQR problem that we have seen before in Chapter 10. They are lost by the introduction of the Kalman filter.

12.13 CHAPTER NOTES AND FURTHER READING

State estimation is one of the central topics in control systems design and has been discussed in many books (Kailath 1980; Chen 1984; Anderson and Moore 1990; etc.). The idea of reduced-order observers is well-known (Luenberger (1964, 1966, 1971, 1979)). The treatment of Section 12.4 on the reduced-order estimation has been taken from Chen (1984).

The term "**Sylvester-observer equation**" was first introduced by the author (Datta 1994). Algorithm 12.7.1 was developed by Van Dooren (1984), while Algorithm 12.7.2 was by Carvalho and Datta (2001). For large-scale solution of this equation, see Datta and Saad (1991); for computing an orthogonal solution to the Sylvester-observer equation, see Datta and Hetti (1997). For a discussion of the numerical properties of the method in Datta and Saad (1991), see Calvetti *et al.* (2001). A parallel algorithm for the multi-output Sylvester-observer equation appears in Bischof *et al.* (1996). For numerical solution of the Sylvester-observer equation with F as the JCF see Tsui (1993) and the references therein. For other algorithms for this problem see Datta (1989) and Datta and Sarkissian (2000). The last paper contains an algorithm for designing a "**functional observer**," which can be used to compute the feedback control law $y = K\hat{x}(t)$ without any matrix inversion.

The method for the constrained Sylvester-observer equation presented in Section 12.8 was developed by Barlow *et al.* (1992). For numerical methods dealing with nearly singular constrained Sylvester-observer equation, see Ghavimi and Laub (1996).

The topic of Kalman filter is now a classical topic. Since the appearance of the pioneering papers by Kalman (1960), Kalman and Bucy (1961), and Kalman (1964), many books and papers have been written on the subject (see, e.g., Kwakernaak and Sivan 1972; Anderson and Moore 1979; Maybeck 1979; Lewis 1986, 1992; etc.).

A special issue of *IEEE Transactions on Automatic Control*, edited by Athans (1971b) was published on the topic of LQG design, which contains many important earlier papers in this area and an extensive bibliography on this subject until 1971. See Dorato *et al.* (1995) for up-to-date references. For aerospace applications of LQG design see McLean (1990). Gangsaas (1986), Bernstein and Haddad (1989) have discussed LQG control with H_∞ performance bound.

We have not discussed in detail the stability and robustness properties of the LQG design. See the papers of Safonov and Athans (1977) and Doyle (1978) in this context.

For discussions on the LQG loop transfer recovery, see the original paper of Doyle and Stein (1979) and the survey of Stein and Athans (1987), and Section 7.2 of the recent book by Dorato *et al.* (1995).

Exercises

12.1 Consider Example 5.2.5 with the following data: $M = 2, m = 1, g = 0.18$, and $l = 1$. Take $C = (1, 1, 1, 1)$.
(i) Find a feedback matrix K such that the closed-loop matrix $A - BK$ has the eigenvalues $-1, -2, -3, -4$.
(ii) Assuming now that the state x is not available for feedback, construct a full-dimensional observer using (a) the eigenvalue assignment method and (b) the Sylvester-observer equation. Compare the results by plotting the error between the true and observed states.
(iii) Construct a three-dimensional reduced-order observer using (a) the eigenvalue assignment method and (b) the Sylvester-observer equation. Compare the results by plotting the error between the true and observed states.
In each case (ii) and (iii), choose the observer eigenvalues to be three times as those of the matrix $A - BK$.

12.2 Are the conditions of Theorem 12.3.1 also necessary? Give reasons for your answer.

12.3 Prove that the pair (A, C) is observable if and only if the pair $(\bar{A}_{22}, \bar{A}_{12})$ is observable, where $\bar{A}_{12}$ and $\bar{A}_{22}$ are given by (12.4.2).

12.4 Establish the "**separation property**" stated in Section 12.5 for a full-dimensional observer.

12.5 Prove that the transfer function matrix of the combined system (12.5.2) of the state feedback and observer can be computed from

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

and the transfer function matrix is

$$\hat{G}(s) = C(sI - A + BK)^{-1}B.$$

How do you interpret this result?

12.6 Construct an example to show that the necessary conditions for the unique solution X of the Sylvester equation stated in Theorem 12.6.1 are not sufficient, unless $r = 1$.

12.7 Using the ideas from the proof of Theorem 12.6.1 prove that necessary conditions for the existence of a unique full rank solution X in $XA - FX = GC$ such that $T = \begin{pmatrix} C \\ X \end{pmatrix}$ is nonsingular are that (A, C) is observable and (F, G) is controllable. Prove further that for the single-output case ($r = 1$), the conditions are sufficient as well.

12.8 Deduce Theorem 12.6.1 from Theorem 12.6.2.

12.9 Workout a proof of Algorithm 12.8.1 (consult Barlow *et al.* (1992)).

12.10 Prove that the EVA approach and the Sylvester-observer equation approach, both for full-dimensional and reduced-order state-estimation, are mathematically equivalent.

12.11 Compare flop-count of Algorithm 12.4.1 with that of Algorithm 12.4.2. (To implement Step 3 of Algorithm 12.4.1, assume that Algorithm 11.3.1 has been used, and to implement Step 2 of Algorithm 12.4.2, assume that Algorithm 12.7.2 has been used).

12.12 *Functional estimator* (Chen (1984, p. 369)). Consider the problem of finding an estimator of the form:

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t) + Hu(t), \\ w(t) &= Mz(t) + Ny(t), \end{aligned}$$

where M and N are row vectors, so that $w(t)$ will approach $kx(t)$ for a constant row vector k, as $t \to \infty$.

(a) Show that if M and N are chosen so as to satisfy the equation:

$$MT + N\bar{C} = \bar{k},$$

with T given by

$$\begin{aligned} T\bar{A} - FT &= G\bar{C}, \\ H &= T\bar{B}, \end{aligned}$$

where $\bar{A}$ and $\bar{B}$ are the same as in (12.4.2), and $\bar{C} = CS^{-1}$, $\bar{k} = kS^{-1}$, and F is a stable matrix, then $w(t)$ will approach $kx(t)$ as $t \to \infty$.

(b) Based on the result in (a), formulate an algorithm for designing such an estimator and apply your algorithm to Example 12.4.1.

12.13 Prove that if (A, C) is observable, then a state-estimator for the discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k \end{aligned}$$

may be constructed as

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k),$$

where L is such that the eigenvalues of $A - LC$ have moduli less than 1.

12.14 Show that for a "deadbeat" observer, that is, for an observer with the "observer eigenvalues" equal to zero, the observer state equals the original state.

12.15 Establish the **"Guaranteed stability"** and **"Guaranteed robustness"** properties of the Kalman Filter, stated in Section 12.9.

12.16 Design an experiment for the Kalman filter estimation of the linearized state-space model of the motion of a satelite in Example 5.2.6, choosing the initial values of the variables and the covariance matrices appropriately. Show the error behavior by means of a graph.

12.17 Design an experiment to show that the LQG design has lower stability margins than the LQR design.

12.18 Rework Algorithm 12.4.1 using the QR decomposition of the matrix C, so that the explicit inversion of the matrix S can be avoided.

References

Alfriend K.T. "Special section on robust control design for a benchmark problem," *AIAA J. Guidance, Control and Dynam.*, Vol. 15, pp. 1060–1149, 1992.

Anderson B.D.O. and Moore J.B. *Optimal Control: Linear Quadratic Methods*, Prentice Hall, Englewood Cliffs, NJ, 1990.

Anderson B.D.O. and Moore J.B. *Optimal Filtering*, Prentice Hall, Englewood Cliffs, NJ, 1979.

Anderson E., Bai Z., Bischof C., Blackford S., Demmel J., Dongarra J., Du Croz J., Greenbaum, A. Hammarling S., McKenney A., and Sorensen D. *LAPACK Users' Guide*, 3rd edn, SIAM, Philadelphia, 1999.

Athans M. "The role and use of the stochastic linear quadratic Gaussian problem in control system design," *IEEE Trans. Autom. Control*, Vol. AC-16, pp. 529–552, 1971a.

Athans M. (ed.), "Special issue on linear-quadratic-Gaussian problem," *IEEE Trans. Autom. Control*, Vol. AC-16, 1971b.

Barlow J.B., Monahemi M.M., and O'Leary D.P. "Constrained matrix Sylvester equations," *SIAM J. Matrix Anal. Appl.*, Vol. 13, pp. 1–9, 1992.

Bernstein D.S. and Haddad W.M. "LQG Control with $\mathcal{H}^\infty$ performances bound: A Riccati equation approach," *IEEE Trans. Autom. Control*, Vol. AC-34, pp. 293–305, 1989.

Bhattacharyya S.P. and DeSouza E. "Controllability, observability and the solution of $AX - XB = C$," *Lin. Alg. Appl.*, Vol. 39, pp. 167–188, 1981.

Birdwell J.P. and Laub A.J. "Balanced singular values for LQG/LTR design," *Int. J. Control*, Vol. 45, pp. 939–950, 1986.

Bischof C., Datta B.N., and Purkayastha A. "A parallel algorithm for the Sylvester-observer equation," *SIAM J. Sci. Comput.*, Vol. 17, no. 3, pp. 686–698, 1996.

Calvetti D., Lewis B., and Reichel L. "On the solution of large Sylvester-observer equation," *Num. Lin. Alg. Appl.*, Vol. 8, pp. 435–451, 2001.

Carvalho J. and Datta B.N. "A block algorithm for the Sylvester-Observer equation arising in state-estimation," *Proc. IEEE Conf. Dec. Control*, Orlando, Florida, 2001.

Chen C.-T. *Linear System Theory and Design*, CBS College Publishing, New York, 1984.

The Control Handbook, edited by William S. Levine, CRC Press and IEEE Press, Boca Raton, Florida, 1995.

Chiang R.Y. and Safonov M.G. *Robust-Control Toolbox for Use with MATLAB*, Math Works, Natick, MA, 1988.

Datta B.N. "Parallel and large-scale matrix computations in control; some ideas," *Lin. Alg. Appl.*, Vol. 12, pp. 243–264, 1989.

Datta B.N. "Linear and numerical linear algebra in control theory: some research problems," *Lin. Alg. Appl.*, Vol. 197/198, pp. 755–790, 1994.

Datta B.N. and Saad Y. "Arnoldi methods for large Sylvester-like matrix equations and an associated algorithm for partial spectrum assignment," *Lin. Alg. Appl.*, Vol. 156, pp. 225–244, 1991.

Datta B.N. and Sarkissian D. Block algorithms for state estimation and functional observers, Proc. *IEEE Joint Conf. Control Appl. Comput. aided Control Syst. Des.*, pp. 19–23, 2000.

Datta K., Hong Y.P., and Lee R.B. "Applications of linear transformations to matrix equations," *Lin. Alg. Appl.*, Vol. 267, pp. 221–240, 1997.

Datta B.N. and Hetti C. "Generalized Arnoldi methods for the Sylvester-observer equation and the multi-input pole placement problem," *Proc. IEEE Conf. Dec. Control*, pp. 4379–4383, 1997.

Dorato P., Abdallah C., and Cerone V. *Linear Quadratic Control: An Introduction*, Prentice Hall, Englewood Cliffs, NJ, 1995.

Doyle J.C. and Stein G. "Multivariable feedback design: Concepts for a classical/modern synthesis," *IEEE Trans. Autom. Control*, Vol. AC-26, pp. 4–16, 1981.

Doyle J.C. "Guaranteed margins for LQG regulators," *IEEE Trans. Autom. Control*, Vol. AC-23, pp. 756–757, 1978.

Doyle J.C. and Stein G. "Robustness with observers," *IEEE Trans. Autom. Control*, Vol. AC-24, pp. 607–611, 1979.

Friedland B. *Control System Design*, McGraw-Hill, New York, 1986.

Gangsaas D. "Application of modern synthesis to aircraft control: three case studies," *IEEE Trans. Autom. Control*, Vol. AC-31, pp. 995–1104, 1986.

Ghavimi A. and Laub A.J. "Numerical methods for nearly constrained matrix Sylvester equations," *SIAM J. Matrix Anal. Appl.*, Vol. 17, no. 1, pp. 212–221, 1996.

Glad T. and Ljung L. *Control Theory: Multivariable and Nonlinear Methods,* Taylor and Francis, London, 2000.

Kailath T., Sayed A.H., and Hassibi B. *Linear Estimation*, Prentice Hall, Englewood Cliffs, NJ, 2000.

Kailath T. *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

Kalman R.E. "Contribution to the theory of optimal control," *Bol. Soc. Matem. Mex.*, Vol. 5, pp. 102–119, 1960.

Kalman R.E. and Bucy R.S. "New results in linear filtering and prediction theory," *ASME Trans. Ser. D: J. Basic Engr.*, Vol. 83, pp. 95–107, 1961.

Kalman R.E. "When is a linear control system optimal?" *ASME Trans. Ser. D: J. Basic Engr.*, Vol. 86, pp. 51–60, 1964.

Kučera V. *Discrete Linear Control*, John Wiley & Sons, New York, 1979.

Kushner H.J. *Introduction to Stochastic Control*, Holt, Rinehart, and Winston, New York, 1971.

Kwakernaak H. and Sivan R. *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

Lewis F.L. *Optimal Control*, John Wiley & Sons, New York, 1986.

Lewis F.L. *Applied Optimal Control and Estimation*, Prentice Hall, Englewood Cliffs, NJ, 1992.

Levine W.S. (ed.) *The Control Handbook*, CRC Press and IEEE Press, Boca Raton, Florida, 1996.

Luenberger D.G. "Observing the state of a linear system," *IEEE Trans. Mil. Electr.*, Vol. 8, 74–80, 1964.

Luenberger, D.G. "Observers for multivariable systems," *IEEE Trans. Autom. Control*, Vol. 11, pp. 190–197, 1966.

Luenberger D.G. "An introduction to observers," *IEEE Trans. Autom. Control*, Vol. AC-16, pp. 596–602, 1971.

Luenberger D.G. *Introduction to Dynamic Systems; Theory, Models, and Applications*, John Wiley & Sons, New York, 1979.

Mahmoud M.S. "Structural properties of discrete systems with slow and fast modes," *Large Scale Systems*, Vol. 3, pp. 227–336, 1982.

MATLAB User's Guide, The Math Works, Inc., Natick, MA, 1992.

Maybeck P.S. *Stochastic Models, Estimation, and Control*, Vols. 1–3, Academic Press, New York, 1979.

McLean D. *Automatic Flight Control Systems*, International Series in Systems and Control Engineering, Prentice Hall, London, 1990.

Safonov M.G. and Athans M. "Gain and phase margins of multiloop LQG regulators," *IEEE Trans. Autom. Control*, Vol. AC-22, pp. 173–179, 1977.

DeSouza C.E. and Fragoso M.D. "On the existence of maximal solution for generalized algebraic Riccati equations arising in stochastic control," *Syst. Contr. Lett.* Vol. 14, pp. 233–239, (1990).

Stein G. and Athans M. "The LQG/LTR procedure for multi-variable feedback control design," *IEEE Trans. Autom. Control*, Vol. AC-32, pp. 105–114, 1987.

Tsui C.-C. "An algorithm for the design of multi-functional observers," *IEEE Trans. Autom. Control*, Vol. AC-30, pp. 89–93, 1985.

Tsui C.-C. "A new approach to robust observer design," *Int. J. Control*, Vol. 47, pp. 745–751, 1988.

Tsui C.-C. "On the solution to matrix equation $TA - FT = LC$," *SIAM J. Matrix Anal. Appl.*, Vol. 14, pp. 33–44, 1993.

Van Dooren P. "Reduced order observers: a new algorithm and proof," *Syst. Contr. Lett.*, Vol. 4, pp. 243–251, 1984.

Zhou K., Doyle J.C. and Glover K. *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.