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Random walk methods and wave diffraction

B.V. Budaev, D.B. Bogy *

Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

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Abstract

This paper develops an approach to problems of wave diffraction that combines the physical clarity of the ray method with the versatility of direct numerical methods. First it addresses scalar problems with general linear first-order boundary conditions, and then it considers problems formulated in adjacent domains with imposed interface conditions. We start by following closely the scheme of the ray method, but instead of looking for approximate expressions for the amplitudes of the Liouville decomposition we obtain their exact representations as the mathematical expectations of some functionals on the space of Brownian trajectories. The obtained solutions provide direct improvements of the ray method approximations to the exact solutions, and they are shown to admit efficient numerical evaluations.

Keywords: Wave propagation; Ray method; Brownian motion; Stochastic process

1. Introduction

Many problems of continuum mechanics are effectively approached today by direct numerical methods that one way or another reduce the problems to systems of algebraic equations. There are however areas such as wave propagation where such techniques remain ineffective. The nature of wave propagation phenomena reduces the efficiency of most conventional direct numerical techniques but it is often possible to use simple and physically meaningful asymptotic methods.

One of the most efficient tools for analysis of wave propagation is the 'geometrical theory of diffraction' or the 'ray method', whose adaptations to specific areas are also known as 'geometrical optics', 'geometrical acoustics', 'geometrical seismology', etc. These methods are based on representations of wave fields in the Liouville form $\sum \phi_n(x)e^{iS_n(x)}$, where the eikonals $S_n(x)$ are associated with the rays and the amplitudes $\phi_n(x)$ admit physical interpretations of quantities transported along the rays. The success of the geometrical theory of diffraction is secured by the existence of the canonical procedure to determine the eikonals $S_n(x)$ and by the fact that in many practically important cases there exist simple asymptotic approximations for the amplitudes $\phi_n(x)$. In general, however, these amplitudes are defined by partial differential equations which have neither exact nor approximate analytic solutions and are difficult for direct numerical analysis.

* Corresponding author.
E-mail address: dbogy@bogy.berkeley.edu (D.B. Bogy).

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Here we develop an approach to problems of wave propagation combining the physical clarity of the ray method with the versatility of direct numerical methods. We start our analysis following closely the scheme of the ray method, but instead of looking for approximate expressions for the amplitudes of the Liouville representations we obtain their exact values as the mathematical expectations of some functionals on the space of Brownian trajectories. The obtained solutions provide direct improvements of the ray method approximations, and they are shown to admit efficient numerical evaluations.

The first probabilistic solutions of partial differential equations were obtained in the 1920s (Philips and Wiener, 1923) as a result of the analysis of the Laplace equation by the finite difference scheme. Rapid progress in the development of probabilistic methods in partial differential equations was made in the 1950s, after the publication of landmark papers of Feynman (1942, 1948) and Kac (1949).

These theories have a long record of successful applications to numerical simulation of evolutions of quantum systems, and quite recently attempts have been made to apply path integral methods to acoustics (Schlottmann, 1999) and electromagnetics (Nevels et al., 2000). Probabilistic methods have also been applied to steady flow computations (Hunt et al., 1995) and these methods are increasingly used for analysis of geophysical wave propagation (Bal et al., 1999, 2000). In Bal et al. (2000) and Papanicolaou (1998) the competitiveness of probabilistic methods in wave propagation is discussed, and in Bal et al. (1999, 2000) such methods are used to study the transport of energy by waves propagating in random media. Such transport is described by first-order differential equations and a probabilistic method is developed in those papers for their analysis. Here, instead, we deal with problems in non-random media but employ probabilistic methods for the analysis of the second-order auxiliary equations whose first-order components are also known in ray theory as transport equations.

The basic ideas of our approach to problems of wave propagation are outlined in Budaev and Bogy, 2001, 2002) where the model problems with Dirichlet boundary conditions were discussed. Here we also take into consideration problems with general linear first-order boundary conditions and problems formulated in adjacent domains with imposed interface conditions.

Section 2 focuses on the fundamental notions underlying the application of random walk methods to partial differential equations. Results presented in the first part of the section are rather standard but are included to make the paper self-contained and to provide the necessary background for understanding what follows. The rest of this section addresses the application of random walk methods to boundary value problems. We first discuss the notion of Brownian motion with reflections and then represent solutions of problems with linear first-order boundary conditions by a probabilistic formula averaging random walks with reflections. This material is also not new, although it is presented in a non-standard form which, we believe, better suits our needs. In particular, in addition to a concept of 'local time' at the boundary, widely used in the literature, we introduce a 'local time in the domain', which makes some important formulas more transparent. Finally, we introduce random walks in domains that comprise two adjacent domains and employ the developed technique to solve problems with imposed interface conditions. We are not aware of other works reporting applications of probabilistic methods to problems of this type, although, from the point of view employed here, such problems are just special cases of problems on a single domain. The discussed material is illustrated by two numerical examples.

Section 3 focuses on the application of the random walk method to the Helmholtz equation. First, this equation is reduced to a complete transport equation defined through the eikonal which can always be computed by the canonical Hamilton–Jacobi technique. Then the transport equation is treated by the methods discussed in the previous sections and the obtained probabilistic solution is converted to a form that may be considered a direct improvement of the well-known ray method approximation to the exact solution of the Helmholtz equation. To provide an indication of the efficiency of the probabilistic solutions of the Helmholtz equation we consider three illustrative examples. First we recover the Hankel function by considering it as a solution of the Dirichlet problem for the Helmholtz equation outside a circle of radius as small as 3% of the wavelength. Next we simulate the solution of the Sommerfeld diffraction problem in a

wedge with Dirichlet boundary conditions. Finally, we compute the reflection and transmission coefficients of plane waves incident on the interface separating two half-spaces with different wave speeds.

It should be emphasized that all of the numerical examples considered in the paper are selected solely for illustrative purposes. We attempted to choose the simplest problems whose probabilistic solutions employ the discussed techniques, and we do not claim that the probabilistic solutions of these problems are preferable to solutions delivered by any other method. It is expected that the demonstrations provide convincing evidence that the methods employed will be applicable to more complex problems for which other methods fail. We will address these problems in future papers.

2. Probabilistic solutions of differential equations

2.1. Random motions in the entire space

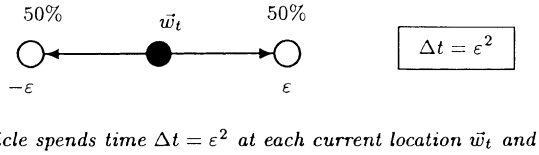
Let a particle start a random walk on the real axis $-\infty < x < \infty$ from the point $x = 0$ and jump at the instants $t_1 < t_2 < t_3 < \cdots$, the distance ε in either of two equally probable directions, as shown on the left diagram of Fig. 1. The particle's position x_n on the time interval $[t_n, t_{n+1})$ prior to the $(n+1)$-th jump is represented as a sum $x_n = \sum_{\nu=1}^{n} \Delta x_\nu$ of independent random variables $\Delta x_n = \pm\varepsilon$ with two equally possible values.

Assume that the instants t_n are equally spaced and that $t_n = n\Delta t$. Then the sequences x_n and t_n determine a piecewise constant function $\widetilde{w}_t = x_{\widetilde{t}/\Delta t}$, where $\widetilde{t} = t/\Delta t$, is the last instant of the series t_n preceding or coinciding with t. It is well known (Dynkin, 1965; Wiener, 1923) that if the time and space meshes decrease together such that $\Delta t = \varepsilon^2 \to 0$, then the jump-motion $\widetilde{w}_t$ converges in some sense to a continuous random motion w_t which is usually referred to as the one-dimensional Brownian motion or, equivalently, as the one-dimensional Wiener process. The N-dimensional Brownian motion in $\mathbb{R}^N$ is defined as a superposition of one-dimensional Brownian motions (see the right diagram of Fig. 1). Let $\vec{w}_t = (w_t^1, w_t^2, \ldots, w_t^N)$ be a path in $\mathbb{R}^N$ whose Cartesian coordinates are independent one dimensional Brownian motions. Then $\vec{w}_t$ is said to be the Brownian motion, or the Wiener process in $\mathbb{R}^N$.

A striking property of the Brownian motion is that it is closely related with partial differential equations. For instance, an N-dimensional elliptic equation

$$\tfrac{1}{2}\nabla^2\phi - B\phi + F = 0, \tag{2.1}$$

can be explicitly solved by the expectation



A particle spends time $\Delta t = \varepsilon^2$ at each current location $\vec{w}_t$ and then instantly jumps on the vector $\Delta\vec{w} = (\pm\varepsilon, \pm\varepsilon, \ldots, \pm\varepsilon)$.

The average single displacement vanishes, i. e. $\mathbf{E}(\Delta\vec{w}) = 0$, but $\mathbf{E}(\|\Delta\vec{w}\|^2) = \varepsilon\sqrt{N}$, where N is the space's dimension.

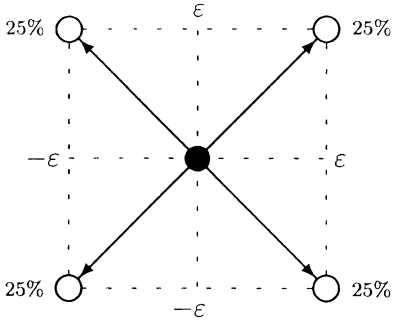


Fig. 1. Discrete Brownian motion.

$$\phi(x) = \mathbf{E}\left\{ \int_0^\infty F(\vec{\xi}_t^x) \mathrm{e}^{-\int_0^t B(\vec{\xi}_t^x)\mathrm{d}s}\, \mathrm{d}t \right\}, \tag{2.2}$$

where the averaging is extended over all the Brownian motions

$$\vec{\xi}_t^x = x + \vec{w}_t, \tag{2.3}$$

launched from the observation point. Solution (2.2) is widely known in the literature as the Feynman–Kac formula, and in Simon (1979) one may find a long list of papers providing different proofs of this formula employing different ideas and applying different hypotheses on the coefficients B, F. For our purposes it is enough to mention that if B and F are smooth and $B > 0$, then the mathematical expectation in (2.2) exists and presents a solution of the Eq. (2.1).

Eq. (2.1) is not the only one that can be explicitly solved by averaging over trajectories of random motions. In particular, a more general equation

$$\tfrac{1}{2}\nabla^2\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F = 0, \tag{2.4}$$

can also be solved by the formula (2.2), but the averaging in this case has to be extended over trajectories of Brownian motions with a drift, which is discussed below.

Let $\vec{A}(x)$ be a vector field on $\mathbb{R}^N$, and let $\vec{\xi}_t^x$ be a random motion (stochastic process) in $\mathbb{R}^N$ launched from x and defined as a superposition

$$\vec{\xi}_t^x = \vec{\zeta}_t^x + \vec{w}_t, \qquad \vec{\xi}_0^x = x, \tag{2.5}$$

of the N-dimensional Brownian motion $\vec{w}_t$, $\vec{w}_0 = 0$, and of the motion $\vec{\zeta}_t^x$ controlled by the ordinary differential equation

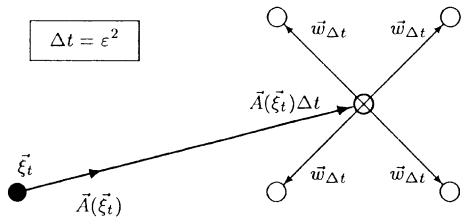
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\zeta}_t^x = \vec{A}\left(\vec{\xi}_t^x\right) \equiv \vec{A}\left(\vec{\zeta}_t^x + \vec{w}_t\right), \qquad \vec{\zeta}_0^x = x, \tag{2.6}$$

whose right-hand side depends on the Brownian motion $\vec{w}_t$. It should be emphasized that although both motions $\vec{w}_t$ and $\vec{\zeta}_t^x$ are random, they are not independent, because any particular Brownian path $\vec{w}_t$ completely determines the component $\vec{\zeta}_t^x$.

The geometrical meaning of the motion $\vec{\xi}_t^x$ becomes clear from the estimate

$$\Delta\vec{\xi}_t = \vec{A}\left(\vec{\xi}_t\right)\Delta t + \vec{w}_{\Delta t} + o(\Delta t), \qquad \Delta\vec{\xi}_t \equiv \vec{\xi}_{t+\Delta t} - \vec{\xi}_t, \tag{2.7}$$

which follows from (2.5) and (2.6) and states that on a short time interval from t to $t + \Delta t$ the increment $\Delta\vec{\xi}_t$ consists of a random move $\vec{w}_{\Delta t}$ and of a deterministic move $\Delta\vec{\zeta}_t = \vec{A}(\vec{\xi}_t^x)\Delta t$. Due to this interpretation, illustrated on Fig. 2, random walks described by (2.5) and (2.6) are usually referred to as Brownian motions with a drift or, equivalently, as Wiener processes with a drift. Passing in (2.7) to the limit $\Delta t \to 0$ we obtain a stochastic differential equation



A particle spends time $\Delta t = \varepsilon^2$ at each current location $\vec{\xi}_t$ and then instantly jumps on the vector $\Delta\vec{\xi} = \vec{A}(\vec{\xi}_t)\Delta t + \vec{w}_{\Delta t}$, whose first component $\vec{A}(\vec{\xi}_t)\Delta t$ is deterministic, while the second component $\vec{w}_{\Delta t} = (\pm\varepsilon, \pm\varepsilon)$ is a random Brownian jump.

As $\varepsilon \ll 1$, the random component $\vec{w}_{\Delta t}$ dominates in the total single displacement estimated as $\Delta\vec{\xi} = (\pm\varepsilon, \pm\varepsilon) + O(\varepsilon^2)$. However, the mathematical expectation of the random component vanishes and the average displacement in time Δt is estimated as $\mathbf{E}(\vec{\xi}_{\Delta t}) = \vec{A}(\vec{\xi}_t)\Delta t = \vec{A}(\vec{\xi}_t)\,\varepsilon^2$.

Fig. 2. Discrete Brownian motion with a drift.

$$\mathrm{d}\vec{\xi}_t^x = \mathrm{d}\vec{w}_t + \vec{A}\mathrm{d}t, \qquad \vec{\xi}_0 = x, \tag{2.8}$$

which provides a well known way (Dynkin, 1965; Ito and McKean, 1974) to analyze random motions with a drift.

Since any positive-definite second order differential equation can be represented in the form (2.4) any of these equations can be solved by the probabilistic formulas (2.2). This solution, however, can be extended to more general equations, such as degenerate elliptic equations including parabolic equations.

Let $\widehat{D}(x)$ be an $N \times N$ matrix function in $\mathbb{R}^N$ and let $\vec{A}(x)$ be a vector field in $\mathbb{R}^N$. Then, the random motion $\vec{\xi}_t^x$ determined by the equations

$$\vec{\xi}_t^x = \vec{\zeta}_t^x + \widehat{D}(x)\cdot\vec{w}_t, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\zeta}_t^x = \vec{A}\left(\vec{\xi}_t^x\right), \quad \vec{\zeta}_0^x = x, \tag{2.9}$$

may be viewed (see Fig. 3) as an asymmetric Brownian motion with a drift (Dynkin, 1965; Ito and McKean, 1974). The asymmetric motion defined by (2.9) is related to the partial differential equation

$$\mathfrak{D}\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F = 0, \tag{2.10}$$

in which the second order operator $\mathfrak{D}$ is defined as

$$\mathfrak{D}\phi = \frac{1}{2}\sum_{m,n=1}^{N} C_{mn}\frac{\partial^2\phi}{\partial x_m \partial x_n}, \qquad C_{mn} = \sum_{j=1}^{N} D_{mj}D_{nj}, \tag{2.11}$$

where D_{mn} are the elements of the matrix $\widehat{D}$ from (2.9) and, therefore, C_{mn} are the elements of the symmetric matrix $\widehat{C} = \widehat{D}\cdot\widehat{D}^{\top}$. Equations of the type (2.10) and (2.11) admit explicit solutions by formula (2.2) with the averaging over random walks satisfying (2.9), and this solution remains valid independently of the rank of the matrix $\widehat{D}$.

Consider, for example, the case $\widehat{D} = 0$. Then, (2.9) degenerates to a deterministic dynamical system $\mathrm{d}\xi_t^x = \vec{A}\mathrm{d}t$ which results in paths $\mathrm{d}\xi_t^x$ that do not have random components. Therefore, the sign of the mathematical expectation in (2.2) can be dropped and the solution has the form

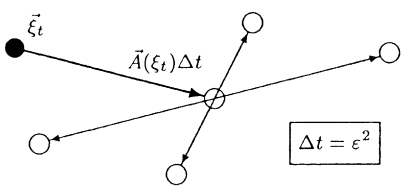
$$\phi(x) = \int_0^{\infty} F(\xi_t^x)\mathrm{e}^{-\int_0^t B(\xi_s^x)\mathrm{d}s}\mathrm{d}t, \qquad \mathrm{d}\xi_t^x = \vec{A}\mathrm{d}t, \quad \xi_0^x = x,$$

which is an obvious solution of the first order equation $\vec{A}\cdot\vec{\nabla}\phi - B\phi + F = 0$.

The above considerations can be straightforwardly extended to equations with complex coefficients. For instance, consider the complex equation

$$\frac{\mathrm{i}}{2}\nabla^2\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F = 0, \tag{2.12}$$

with N independent real variables $x_1, x_2, \ldots, x_N$, considered as the Cartesian components of the N-dimensional real vector $\vec{x}$. Let $\phi(z)$ be an analytic function of a complex N-dimensional argument $\vec{z} = \vec{x} + \mathrm{i}\vec{y}$.



A particle spends time $\Delta t = \varepsilon^2$ at each current location $\vec{\xi}_t$ and then instantly jumps on the vector $\Delta\vec{\xi} = \vec{A}(\vec{\xi}_t)\Delta t + \widehat{D}(\vec{\xi}_t)\cdot\vec{w}_{\Delta t}$, whose first component $\vec{A}(\vec{\xi}_t)\Delta t$ is deterministic, and the second component is a Brownian jump $\vec{w}_{\Delta t}$ transformed by the matrix $\widehat{D}(\vec{\xi}_t)$.

If $\widehat{D} = 0$ this is the deterministic motion along $\vec{A}$. If $\widehat{D} = \mathrm{diag}[1, 0, \ldots, 0]$ this is the deterministic drift along $\vec{A}$ perturbed by random fluctuations in the direction along the axis x_1.

Fig. 3. Discrete asymmetric Brownian motion with a drift.

Then, $\phi(\vec{z})$ can be treated as a complex-valued function $\phi(\vec{x},\vec{y})$ of two real vectors $\vec{x}$ and $\vec{y}$ satisfying Cauchy–Riemann conditions

$$\mathrm{i}\frac{\partial\phi}{\partial x_n}=\frac{\partial\phi}{\partial y_n},\qquad z_n=x_n+\mathrm{i}y_n,\quad n=1,2,\ldots,N, \tag{2.13}$$

where $x_n\in\mathbb{R}$ and $y_n\in\mathbb{R}$ are the components of $\vec{x}$ and $\vec{y}$. From (2.13) we derive identities

$$\frac{\partial^2\phi}{\partial x_n^2}=-\frac{\partial^2\phi}{\partial y_n^2},\qquad \frac{\partial^2\phi}{\partial x_n\partial y_n}=\mathrm{i}\frac{\partial^2\phi}{\partial x_n^2},\qquad \mathrm{i}\sum_{n=1}^{N}\frac{\partial^2\phi}{\partial x_n^2}=\sum_{n=1}^{N}\frac{\partial^2\phi}{\partial x_n\partial y_n}, \tag{2.14}$$

and

$$A_n\frac{\partial\phi}{\partial x_n}=\mathrm{Re}(A_n)\frac{\partial\phi}{\partial x_n}+\mathrm{i}\mathrm{Im}(A_n)\frac{\partial\phi}{\partial x_n}=\mathrm{Re}(A_n)\frac{\partial\phi}{\partial x_n}+\mathrm{Im}(A_n)\frac{\partial\phi}{\partial y_n} \tag{2.15}$$

which result in the representation of the Eq. (2.12) in the form

$$\frac{1}{2}\sum_{n=1}^{N}\frac{\partial^2\phi}{\partial x_n\partial y_n}+\sum_{n=1}^{N}\left[\mathrm{Re}(A_n)\frac{\partial\phi}{\partial x_n}+\mathrm{Im}(A_n)\frac{\partial\phi}{\partial y_n}\right]-B\phi+F=0, \tag{2.16}$$

i.e., the standard second-order differential equation with $2N$ independent variables.

Eq. (2.16) matches the structure of the Eqs. (2.10) and (2.11) with $2N$ real variables and, therefore, its solution can be obtained as the mathematical expectation (2.2) averaging $2N$-dimensional random walks of the type (2.9) with the matrix D determined by the second order component of the Eq. (2.16) as defined by (2.11). An elementary analysis shows that D has the rank N and that the above mentioned random walks are described by the equations

$$\mathrm{Re}(\mathrm{d}\vec{\xi}_t)=\frac{1}{\sqrt{2}}\vec{w}_t+\mathrm{Re}(\vec{A})\mathrm{d}t,\qquad \mathrm{Im}(\mathrm{d}\vec{\xi}_t)=\frac{1}{\sqrt{2}}\vec{w}_t+\mathrm{Im}(\vec{A})\mathrm{d}t,\quad \vec{w}_t\in\mathbb{R}^N, \tag{2.17}$$

where $\vec{w}_t$ is the standard Brownian motion in the N-dimensional real space $\mathbb{R}^N$. In complex notation, solutions of stochastic equations (2.17) take the form

$$\xi_t^x=\sqrt{\mathrm{i}}\,\vec{w}_t+\vec{\zeta}_t^x,\qquad \text{where}\quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\zeta}_t^x=\vec{A}(\vec{\zeta}_t^x),\quad \vec{\zeta}_0^x=x, \tag{2.18}$$

and this leads to the conclusion that the analytic solution ϕ of (2.12) can be represented by the formula (2.2) with the paths (2.18) running across the complex space $\mathbb{C}^N$ even while their Brownian component $\vec{w}_t$ remains real-valued. This conclusion is based on the hypothesis that the solution ϕ of (2.12) is analytic, while the analyticity of the coefficients $\vec{A}$, B and F is not formally required. However, if we assume that $\vec{A}$, B and F are analytic, then we may derive the analyticity of ϕ defined by (2.2) and (2.18), and, therefore, justify that ϕ solves Eq. (2.12).

Eq. (2.12) may also be considered (Simon, 1979) as a time independent Schrödinger equation in the presence of a magnetic potential, which leads to the solution of (2.12) represented by means of Feynman's path integrals. This solution does not require analyticity of the coefficients $\vec{A}$, B and F, but it is more difficult for numerical evaluation than the probabilistic solution (2.2) averaging complex random walks (2.18). However, if $\vec{A}$, B and F are analytic, then the contour deformation technique discussed in Chang and Miller (1987) makes it possible to convert Feynman's integrals solving (2.12) to probabilistic formulas (2.2) and (2.18).

2.2. Random motions in domains with boundaries

It is remarkable that the probabilistic approach can be extended from partial differential equations like (2.2) in the entire space to boundary value problems

$$\tfrac{1}{2}\nabla^2\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F\Big|_G = 0, \qquad \vec{a}\cdot\vec{\nabla}\phi - b\ \phi + f\Big|_{\partial G} = 0, \tag{2.19}$$

formulated in a domain $G \subset \mathbb{R}^N$ with the boundary ∂G. Coefficients $\vec{A}$, B and F are assumed here to be defined inside G, while the coefficients $\vec{a}$, b and f are defined on ∂G. Additionally, for definiteness, we assume that vectors $\vec{a}$ are oriented inwards toward G.

Applications of random walk methods to problems (2.19) are based on the idea of defining random walks (stochastic processes) on the closure $G \cup \partial G$ whose behavior inside G corresponds to the operator $L_G = \frac{1}{2}\nabla^2 + \vec{A}\cdot\vec{\nabla} - B$, and whose behavior on the boundary ∂G corresponds to the first-order operator $L_{\partial G} = \vec{a}\cdot\vec{\nabla} - b$. Since both operators L_G and $L_{\partial G}$ are particular cases of the general second-order operator discussed in the previous section, it is natural to expect that inside G the random walk should look like a Brownian motion with a drift associated with the vector fields $\vec{A}$, and on the boundary ∂G it should look like a deterministic motion along the vector $\vec{a}$. This idea can indeed be carried out but attention is needed to the continuity of the total motion, which may be violated if the time scales of the motions in G and ∂G are not properly coordinated.

To see the origin of the problem with the speed coordination consider first the motion inside G. It is composed of the standard Brownian motion $\vec{w}_t$ and of the deterministic drift $\vec{\zeta}(t)$ whose speed-vector $\vec{A}$ is uniquely determined because it comes from Eq. (2.19) normalized to the coefficient $\frac{1}{2}$ in front of the Laplacian. On the contrary, the boundary condition in (2.19) does not have a preferred normalization the result of which is that the vector $\vec{a}$ determines only the direction of the motion at the boundary ∂G but applies no restriction on its speed.

Stochastic processes $\vec{\xi}_t^x$ corresponding to the problem (2.19) are known as reflecting random motions and they can be introduced as continuous solutions of the stochastic differential equation

$$\mathrm{d}\vec{\xi}_t^x = \mathrm{d}\vec{w}_t + \vec{A}\ \mathrm{d}\Lambda_t + \vec{a}\ \mathrm{d}\lambda_t, \qquad \vec{\xi}_0^x = x, \quad \lambda_0 = 0, \tag{2.20}$$

with an additional unknown λ_t, which is required to be a continuous non-decreasing stochastic process increasing only on the 'visiting' set $\mathfrak{g} = \{t : \vec{\xi}_t \in \partial G\}$ of instants when the path $\vec{\xi}_t$ touches the boundary ∂G. The process λ_t is called the 'local time at ∂G' because it admits interpretation as the measure of the time spent by the path $\vec{\xi}_t$ on the boundary ∂G. As for Λ_t it is defined here as the Lebesgue measure of the set $\mathfrak{G} = \{s : 0 < s < t, \vec{\xi}_s \in G\}$ and, correspondingly, we call Λ_t the 'local time in G'.

Stochastic differential equations of the type similar to (2.20) were introduced by Skorokhod (1961), and Watanabe (1971) used such equations for analysis of general boundary value problems. The 'local time' λ_t on the boundary is extensively discussed in Ito and McKean (1974), where, in particular, it is proven that if $\vec{a} \neq 0$ then the Lebesgue measure of the visiting set $\mathfrak{g} = \{t : \vec{\xi}_t \in \partial G\}$ is equal to zero. This observation makes it possible to re-write Skorokhod's equation in the form (2.20), which is more convenient because it treats similarly the domain G and its boundary ∂G.

Continuous random motions corresponding to the boundary value problem (2.19) can be approximated by discrete random walks

$$\vec{\xi}_0 \to \vec{\xi}_1 \to \vec{\xi}_2 \to \cdots \to \vec{\xi}_{n-1} \to \vec{\xi}_n \to \cdots \tag{2.21}$$

whose individual steps are defined differently depending on the location of their initial positions with respect to the boundary ∂G. If $\vec{\xi}_{n-1}$ is located inside G and its distance from ∂G exceeds ε then the walk spends at this point the time $\Delta t_n = \varepsilon^2$ and then moves to the next random point

$$\vec{\xi}_n = \vec{\xi}_{n-1} + \vec{A}(\vec{\xi}_{n-1})\Delta t_n + \vec{w}_n, \quad \text{where} \quad \vec{w}_n = (\pm\varepsilon, \pm\varepsilon, \ldots, \pm\varepsilon), \quad \Delta t_n = \varepsilon^2, \tag{2.22}$$

which is defined by a deterministic move $\vec{A}(\vec{\xi}_{n-1})\Delta t$ and a random jump along a vector $\vec{w}_n$ with $2N$ equally probable values. Otherwise, the walk spends at $\vec{\xi}_{n-1}$ the time

$$\Delta t_n = \varepsilon \Big/ \left\| \vec{a}\left(\breve{\xi}_{n-1}\right) \right\|, \qquad \vec{\xi}_{n-1} \approx \breve{\xi}_{n-1} \in \partial G, \tag{2.23}$$

where $\breve{\xi}_{n-1}$ is the point on ∂G closest to $\vec{\xi}_{n-1}$, and then moves to the non-random point

$$\vec{\xi}_n = \vec{\xi}_{n-1} + \varepsilon \vec{a}(\breve{\xi}_{n-1}) \Big/ \left\| \vec{a}\left(\breve{\xi}_{n-1}\right) \right\|. \tag{2.24}$$

As $\varepsilon \to 0$ the lengths of the jumps defined by (2.24) and by (2.22) have the same order and this provides convergence of the process to a continuous random motion. Furthermore, the local times λ_t and Λ_t can be approximated as the limits

$$\Lambda_t = \lim_{\varepsilon \to 0} \left(\varepsilon^2 \, N_G^{\varepsilon}\right), \qquad \lambda_t = \lim_{\varepsilon \to 0} \left(\varepsilon \, N_{\partial G}^{\varepsilon} \Big/ \left\| \vec{a}\left(\breve{\xi}_{n-1}\right) \right\| \right),$$

where $N_{\partial G}^{\varepsilon}$ and N_G^{ε} are the total numbers of steps of the discrete walk (2.21) determined by the rules (2.24) or (2.22), respectively.

It is important to emphasize that the described motion, illustrated by the Fig. 4 is random only inside G, but on the boundary ∂G the motion is completely deterministic and determines a shift along the vector field $\vec{a}$ defined on ∂G.

Since trajectories $\vec{\xi}_t^x$ cannot visit simultaneously both the domain G and its boundary ∂G, it is natural to introduce global coefficients

$$\vec{\mathscr{A}} = \begin{cases} \vec{A}, & \text{in } G \\ \vec{a}, & \text{on } \partial G, \end{cases} \qquad \mathscr{B} = \begin{cases} B, & \text{in } G \\ b, & \text{on } \partial G, \end{cases} \qquad \mathscr{F} = \begin{cases} F, & \text{in } G \\ f, & \text{on } \partial G, \end{cases} \tag{2.25}$$

which unify the coefficients of the equation and of the boundary condition from (2.19) to functions defined on $G \cup \partial G$. Introducing also the global time measure

$$\mathrm{d}t_t = \begin{cases} \mathrm{d}\lambda_t, & \text{if } \vec{\xi}_t \in \partial G, \\ \mathrm{d}\Lambda_t, & \text{if } \vec{\xi}_t \in G, \end{cases} \tag{2.26}$$

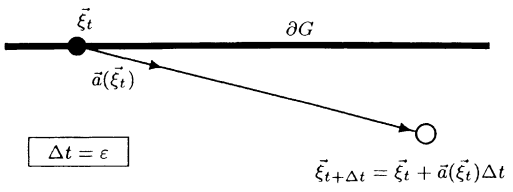
we re-arrange (2.20) into a stochastic equation

$$\mathrm{d}\vec{\xi}_t^x = \mathrm{d}\vec{w}_t + \vec{\mathscr{A}}\mathrm{d}t_t, \quad \vec{\xi}_0^x = x, \tag{2.27}$$

similar to (2.8) and then the solution of the problem (2.19) can be expressed in the form

$$\phi(x) = \mathbf{E} \int_0^{\infty} \mathscr{F}(\vec{\xi}_t^x) \mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{d}t_s} \mathrm{d}t_t, \tag{2.28}$$

which may be considered as a direct extension of (2.2). This solution may, certainly, not be valid in all circumstances, and some restrictions must be applied to the domain G and to other parameters of the



Inside the domain G a particle performs the Brownian motion with a drift, as shown on Fig 2. A particle spends time $\Delta t = \varepsilon^2$ at each location and then jumps on a random vector $\Delta\vec{\xi} = (\pm\varepsilon, \pm\varepsilon) + O(\varepsilon^2)$.

If the path $\vec{\xi}_t$ hits the boundary ∂G it spends there time $\Delta t = \varepsilon$, and then is reflected back to G jumping on the deterministic vector $\Delta\vec{\xi} = \vec{a}(\vec{\xi}_t)\Delta t$.

Fig. 4. Discrete Brownian motion with reflections.

problem. A comprehensive discussion of this topic may be found in Freidlin (1985), so we do not go into more detail here, assuming that the correctness of the solution (2.28) must be questioned in each particular case when this formula is used.

It is well-known that the problem (2.19) is not necessarily always solvable and some conditions of solvability must be imposed. Thus, the Neuman problem in an interior domain G with the boundary condition $\partial\phi/\partial\vec{n}|_{\partial G} = f$ does not have solutions unless $\int_{\partial G} f = 0$. Such conditions of solvability are naturally included in the probabilistic solution (2.28) as conditions providing convergence of the involved mathematical expectation.

2.3. Probabilistic solution of the Dirichlet problem

As an instructive example illustrating the solution (2.28) of the general boundary value problem (2.19) we consider here a simpler Dirichlet problem

$$\tfrac{1}{2}\nabla^2\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F\Big|_G = 0, \qquad \phi|_{\partial G} = f, \tag{2.29}$$

which is a particular case of (2.19) corresponding to the values $\vec{a} = 0$, $b = 1$.

Since the vector field $\vec{a}$ from (2.19) determines the speed of the trajectories $\vec{\xi}_t^x$ on the boundary ∂G, the vanishing of $\vec{a}$ means that the trajectories stop as soon as they reach the boundary, i.e. $\vec{\xi}_t^x = \vec{\xi}_\tau^x$, for any $t \geqslant \tau$, where the 'exit time' (Dynkin, 1965; Ito and McKean, 1974)

$$\tau \equiv \inf\{t : \vec{\xi}_t^x \notin G\}, \tag{2.30}$$

may be viewed as the first instant when the path $\vec{\xi}_t^x$ hits the boundary. Then, the local times Λ_t and λ_t in the domain and at the boundary are described as

$$\Lambda_t = \begin{cases} t, & t < \tau, \\ \tau, & t \geqslant \tau, \end{cases} \qquad \lambda_t = \begin{cases} 0, & t < \tau, \\ t-\tau, & t \geqslant \tau, \end{cases}$$

and, correspondingly

$$\mathrm{d}\Lambda_t = \begin{cases} \mathrm{d}t, & t < \tau, \\ 0, & t \geqslant \tau, \end{cases} \qquad \mathrm{d}\lambda_t = \begin{cases} 0, & t < \tau, \\ \mathrm{d}t, & t \geqslant \tau. \end{cases}$$

Taking into account the definition (2.26) of the global time measure t_t we re-write solution (2.28) in the form

$$\phi(x) = \mathbf{E}\big(I_F + I_f\big),$$

where

$$I_F = \int_0^\infty F(\vec{\xi}_t^x)\mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{dt}_s}\mathrm{d}\Lambda_t = \int_0^\tau F(\vec{\xi}_t^x)\mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{dt}_s}\mathrm{d}t,$$
$$I_f = \int_0^\infty f(\vec{\xi}_t^x)\mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{dt}_s}\mathrm{d}\Lambda_t = f(\vec{\xi}_\tau^x)\int_\tau^\infty \mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{dt}_s}\mathrm{d}t.$$

Then, assuming that $b = 1$, we similarly evaluate the integral from the exponents:

$$\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{dt}_s = \int_0^t \mathrm{d}\lambda_s + \int_0^t B(\vec{\xi}_s^x)\mathrm{d}\Lambda_s = (t-\tau)\chi_{t-\tau} + \int_0^{\min(t,\tau)} B(\vec{\xi}_s^x)\mathrm{d}s,$$

where χ_t is the Heaviside step-function equal to zero for $t < 0$ and to unity otherwise. Substituting this result into the expressions for I_f and I_F we have

$$I_F = \int_0^\tau F(\vec{\xi}_t^x)\mathrm{e}^{-\int_0^t \mathscr{B}(\vec{\xi}_s^x)\mathrm{d}t_s}\mathrm{d}t, \quad I_f = f(\vec{\xi}_\tau^x)\int_\tau^\infty \mathrm{e}^{\tau-t+\int_0^\tau B(\vec{\xi}_s^x)\mathrm{d}s}\mathrm{d}t = f(\vec{\xi}_\tau^x)\mathrm{e}^{-\int_0^\tau B(\vec{\xi}_s^x)\mathrm{d}s},$$

and the solution of the Dirichlet problem (2.29) arrives at its final form

$$\phi(x) = \mathbf{E}\left\{ f(\vec{\xi}_\tau^x)\mathrm{e}^{-\int_0^\tau B(\vec{\xi}_s^x)\mathrm{d}s} + \int_0^\tau F(\vec{\xi}_t^x)\mathrm{e}^{-\int_0^t B(\vec{\xi}_s^x)\mathrm{d}s}\mathrm{d}t \right\}, \tag{2.31}$$

which is well known in the literature (Dynkin, 1965; Freidlin, 1985), where the precise hypotheses providing convergence and correctness of (2.31) are also discussed.

The early versions of the solution (2.31) of the Dirichlet problem were presented long ago, as early as the 1920s. Philips and Wiener (1923) and Courant et al. (1928) considered random walks on plane nets, introduced 'exit times' from closed domains, and actually obtained the solution of the Dirichlet problem for the Laplace equation as the limit of the corresponding discrete problem when the net's mesh passes to zero. In the 1930s these ideas were further developed by Petrovsky (1934) and Khinchine (1933) who considered more general random walks and more general differential operators. The solution (2.31) was obtained by Doob (1956) for the case when $F = D = 0$, and later it was extended to problems

$$\mathfrak{D}\phi + \vec{A}\cdot\vec{\nabla}\phi - B\phi + F = 0, \qquad \phi|_{\partial G} = -f, \tag{2.32}$$

involving a general second-order operator $\mathfrak{D}$ from (2.11). It is remarkable that the solution of the general problem (2.32) may still be expressed by the probabilistic formula (2.31) averaging random walks by (2.9).

The advantage of the probabilistic solution (2.31) of the Dirichlet problem (2.29) is that it can be easily estimated by a simple statistical procedure.

Let $x^1(t), x^2(t), \ldots, x^M(t)$, be M independent Brownian paths launched from x, and let $\tau_1, \tau_2, \ldots, \tau_M$, be the exit times of these sample paths from the domain G. Then the mathematical expectation in (2.31) can be approximated by statistical averaging over the sample paths, which leads to the following approximation

$$\phi(x) \approx \frac{1}{M}\sum_{m=1}^{M}\left\{ f(x^m(\tau_m))\mathrm{e}^{-\int_0^\tau B(x^m(s))\mathrm{d}s} + \int_0^{\tau_m} F(x^m(t))\mathrm{e}^{-\int_0^t B(x^m(s))\mathrm{d}s}\mathrm{d}t \right\},$$

of the solution $\phi(x)$ of (2.29). Next, continuous paths $x^m(t)$ can be approximated by discrete paths

$$x_0^m \to x_1^m \to x_2^m \to \cdots \to x_{K_m}^m \to \ldots, \tag{2.33}$$

with random jumps

$$\Delta x_n^m = x_n^m - x_{n-1}^m - \Delta(x_{n-1}^m)\Delta t = \overbrace{(\pm\varepsilon, \pm\varepsilon, \ldots, \pm\varepsilon)}^{N \text{ positions}}, \quad \varepsilon > 0,$$

occurring with the time interval $\Delta t = \varepsilon^2$. Then, (2.31) is approximated by the finite expression

$$\phi(x) \approx \frac{1}{M}\sum_{m=1}^{M}\left\{ \widetilde{f}\left(x_{K_m}^m\right)\prod_{v=1}^{K_m} E_v^m + \varepsilon^2\sum_{n=1}^{K_m} F\left(x_n^m\right)\prod_{v=1}^{K_m} E_v^m \right\}, \quad E_v^m = \mathrm{e}^{-B(x_v^m)\Delta x_v^m}, \tag{2.34}$$

where $x_{K_m}^m$ is the last point of the discrete path (2.33) located inside G, and $\widetilde{f}\left(x_{K_m}^m\right)$ is the value of the boundary function $f(x)$ at the point $x \in \partial G$ closest to $x_{K_m}^m$.

Fig. 5 displays some numerical results demonstrating the efficiency of the probabilistic solution (2.31) of Dirichlet problems. In particular, we consider the problem

$$\nabla^2\phi - 4(r\cos\theta + 1) = 0, \qquad \phi(1,\theta) = \cos\theta\left(2\cos\theta + \sin\theta + 2\sin^2\theta\right), \tag{2.35}$$

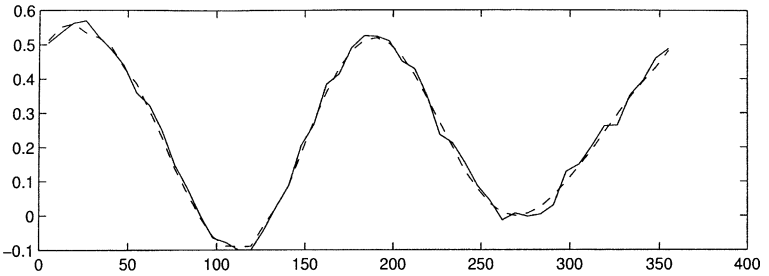


Fig. 5. Probabilistic solution of the Dirichlet problem (2.35).

formulated in the disk $r \leqslant 1$, and recover its obvious solution

$$\phi(r,\theta) = r^2 \cos\theta\big(2\cos\theta + \sin\theta + 2\sin^2\theta\big), \tag{2.36}$$

by the probabilistic formula (2.31). Simulated values of $\phi(\frac{1}{2},\theta)$ along the circle $r = \frac{1}{2}$ are plotted by a solid line while the exact values are plotted by a dashed line. The mathematical expectation from (2.31) is estimated by (2.34) by averaging over 4000 discrete random walks with the space increment $\varepsilon = 0.025$.

2.4. Random walks in adjacent domains

Let G be an N-dimensional domain with the boundary ∂G and let G be subdivided on two subdomains G_1 and G_2 with the boundaries ∂G_1 and ∂G_2. Let ∂G_1^e, ∂G_2^e be the external boundaries of G_1, G_2, and let ∂G_0 be the interface between G_1 and G_2 (Fig. 6).

Then, one may formulate the problem with two unknown functions ϕ_n, $n = 1,2$, defined in G_n and satisfying differential equations

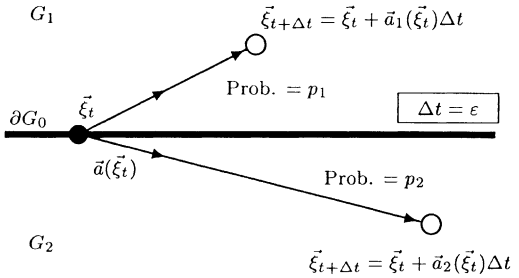
$$\tfrac{1}{2}\nabla^2\phi_n + \vec{A}_n \cdot \vec{\nabla}\phi_n - B_n\phi_n + F_n\Big|_{G_n} = 0, \qquad n = 1,2, \tag{2.37}$$

boundary conditions

$$\overrightarrow{a_n} \cdot \vec{\nabla}\phi_n - b_n\phi_n + f_n\Big|_{\partial G_n^0} = 0, \qquad n = 1,2, \tag{2.38}$$

and interface conditions

$$\phi_1 - \phi_2|_{\partial G_0} = 0, \tag{2.39}$$



Inside domains G_1, G_2 a particle performs the Brownian motion with the drift as shown on Fig 2. A particle spends time $\Delta t = \varepsilon^2$ at each location and then instantly jumps on a random vector $\Delta\vec{\xi} = (\pm\varepsilon, \pm\varepsilon) + O(\varepsilon^2)$.

If the path $\vec{\xi}_t$ hits the interface ∂G, it spends there time $\Delta t = \varepsilon^2$ and then – with the probability p_1 it moves to G_1 jumping on the deterministic vector $d\vec{\xi}_{\Delta t} = \vec{a}_1(\vec{\xi}_t)\Delta t$, or, with the probability p_2 it moves to G_2 jumping on the deterministic vector $\Delta\vec{\xi} = \vec{a}_2(\vec{\xi}_t)\Delta t$.

Fig. 6. Discrete Brownian motion in adjacent domains.

$$p_1\vec{e}_1\cdot\vec{\nabla}\phi_1+p_2\vec{e}_2\cdot\vec{\nabla}\phi_2-b_0\phi_1+f_0\Big|_{\partial G_0}=0,\qquad p_{1,2}\geqslant 0,\quad p_1+p_2=1, \tag{2.40}$$

with the coefficients $\vec{A}_n$, $B_n>0$, and F_n defined inside domains G_n; with the coefficients $\vec{a}_n$, $b_n>0$, and f_n defined on the external boundaries ∂G_n^e; and with the coefficients $\vec{e}_n$, p_n, $b_0>0$, f_0, defined on the interface ∂G_0. For definiteness vectors $\vec{e}_n$ are assumed to be oriented inward in the corresponding domains and normalized as $\|\vec{e}_n\|=1$. The coefficients $p_{1,2}\geqslant 0$ are assumed to be non-negative and normalized by $p_1+p_2=1$. Additionally we assume that all involved boundaries and coefficients are smooth and bounded.

Previous considerations suggest that to approach this problem by the random walk method we need, first, to define a random motion $\vec{\xi}_t$ in $G\cup\partial G$ corresponding to the operator $\mathfrak{L}_\xi$ defined as

$$\mathfrak{L}_\xi\phi=\begin{cases}\frac{1}{2}\nabla^2\phi_n+\vec{A}_n\cdot\vec{\nabla}\phi_n, & \text{inside domains } G_n\\ \vec{a}_n\cdot\vec{\nabla}\phi_n, & \text{on the external boundaries } \partial G_n^e,\\ p_1\vec{e}_1\cdot\vec{\nabla}\phi_1+p_2\vec{e}_2\cdot\vec{\nabla}\phi_2, & \text{on the interface } \partial G_0.\end{cases} \tag{2.41}$$

The required random motion can be defined in a manner similar to that in the previous section. Namely, at every instant the future direction of the path $\vec{\xi}_t$ is determined by its current position x_0. From a position inside either of the domains G_n, $n=1,2$, the motion continues as the Brownian walk with a drift corresponding to the operator $\frac{1}{2}\nabla^2+\vec{A}_n(x_0)\cdot\vec{\nabla}$. From a position on the external boundaries ∂G_n^e, the motion continues as a deterministic drift into G_n along the vector $\vec{a}_n(x_0)$ with a time scale corresponding to the reflection from the boundary discussed above. From a position on the interface ∂G_0 the motion continues with the probability p_1 as a deterministic drift into G_1 along the vector $\vec{e}_1(x_0)$, or it continues with the probability p_2 as a deterministic drift into G_2 along the vector $\vec{e}_2(x_0)$. The time scale of the drift into the domains G_n, corresponds to the problem inside G_n with the boundary conditions $p_n\vec{e}_n\cdot\vec{\nabla}\phi_n-b_0\phi_n+f_0=0$, formulated as if the other domain did not exist.

The motion corresponding to the problem (2.37)–(2.40) can also be introduced as a continuous solution $\vec{\xi}_t^x$ of the stochastic differential equation

$$\mathrm{d}\vec{\xi}_t^x=\mathrm{d}\vec{w}_t+\sum_{n=1}^{2}\left(\vec{A}_n\mathrm{d}\Lambda_t^n+\vec{a}_n\mathrm{d}\lambda_t^n\right)+p_{\nu_t}\vec{e}_{\nu_t}\mathrm{d}\lambda_t^0, \tag{2.42}$$

with initial conditions

$$\vec{\xi}_0^x=x,\qquad \Lambda_0^{1,2}=\lambda_0^{0,1,2}=0,\qquad \text{and}\quad \nu_0=n,\quad \text{if } x\in G_n. \tag{2.43}$$

where Λ_t^n, $n=1,2$, are the local times in the domains G_n, λ_t^n are the local times on the exterior boundaries ∂G_n^e, and λ_t^0 is the local time on the interface ∂G_0. This equation has five unknown stochastic processes (random functions): the path $\vec{\xi}_t^x$, local times λ_t^0, λ_t^1, λ_t^2, and a random index ν_t, which changes its value at the instants when the path $\vec{\xi}_t^x$ touches the interface ∂G_0 taking the values $\nu=1$ or $\nu=2$ with probabilities p_1 and p_2 that are functions defined on the interface ∂G_0. Solutions of the stochastic differential equation (2.42) can be approximated by discrete random walks similar to that discussed in the previous section as approximations of the Brownian motion with reflections described by the stochastic equation (2.20).

When stochastic functions satisfying (2.42) and, consequently, (2.41), are found, then the solution of the problem (2.37)–(2.40) can be represented by the formula (2.2) with F and B replaced by $\mathscr{F}$ and $\mathscr{B}$, defined, together with the global time t_t, by the formulas

$$\mathscr{B}=\begin{cases}B_n,\\ b_t,\\ b_0,\end{cases}\qquad \mathscr{F}=\begin{cases}F_n,\\ F_t,\\ f_0,\end{cases}\qquad \mathrm{t}_t=\begin{cases}\Lambda_t^n, & \text{if } \vec{\xi}_t^x\in G_n,\\ \lambda_t^n, & \text{if } \vec{\xi}_t^x\in\partial G_n^e,\\ \lambda_t^0, & \text{if } \vec{\xi}_t^x\in\partial G_0,\end{cases} \tag{2.44}$$

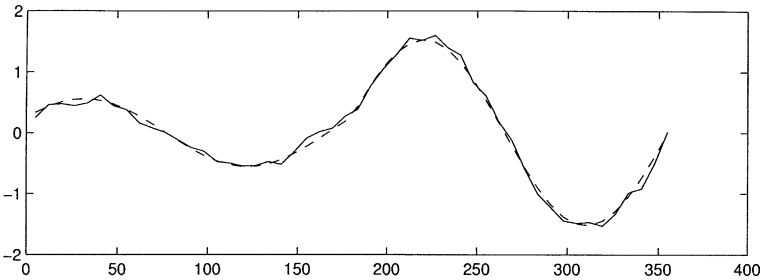


Fig. 7. Probabilistic solution of the problem (2.37)–(2.40).

depending on the current location of the path $\vec{\xi}_t^x$ with respect to the domain $G_{1,2}$, their external boundaries $\partial G_{1,2}^e$, and the interface ∂G_0.

Fig. 7 displays some numerical results demonstrating the efficiency of the probabilistic solution of the problem (2.37)–(2.40). We recover function

$$\phi(x,y)=x^2-y^2+4C(y)xy, \qquad \text{where} \quad C(y)=\begin{cases}2 & \text{if } y>0,\\ 1 & \text{if } y<0,\end{cases} \tag{2.45}$$

in the disk $x^2+y^2\leqslant 1$ considering it as the solution of the equation $\nabla^2\phi=0$ in domains $y>0$ and $y<0$, satisfying the interface condition

$$3\left.\frac{\partial\phi}{\partial y}\right|_{y=0+}-\left.\frac{\partial\phi}{\partial y}\right|_{y=0-}=4x[3C(1)-C(-1)]\equiv 20x,$$

on the line $y=0$, and meeting predefined values on the circle $x^2+y^2=1$. The probabilistic solution $\phi(x,y)$ along the circle $x^2+y^2=\frac{1}{4}$ is plotted by a solid line while the dashed line presents exact values of this function. The results were obtained by averaging over 4000 discrete random walks with the space increment $\varepsilon=0.025$.

3. Probabilistic solutions of the Helmholtz equation

3.1. The Dirichlet problem for the Helmholtz equation

Consider the boundary value problem

$$\nabla^2\psi+k^2\varkappa^2(x)\psi=0, \quad \psi|_{\partial G}=\psi_0, \qquad x\in G\subset\mathbb{R}^N, \tag{3.1}$$

for the N-dimensional Helmholtz equation with a positive wave number $\kappa\equiv k\varkappa(x)>0$ subdivided for future convenience into two multiplicative components: a variable parameter $\varkappa(x)>0$ and a constant $k>0$ which is often considered to be large. A typical problem of wave radiation consists of finding the solution of (3.1) that is defined in an exterior domain $G\subset\mathbb{R}^N$ and satisfies some additional conditions at infinity, most commonly, the Sommerfeld radiation condition.

Although (3.1) is an elliptic boundary value problem similar to (2.1) its solution may not be straightforwardly expressed by probabilistic formulas like (2.31) because the inequality $\kappa^2>0$ leads to divergent integration in (2.31). There is, however, a less straightforward way to obtain probabilistic solutions of (3.1).

Let us seek a solution of (3.1) in the Liouville product form

$$\psi(x) = \phi(x)\mathrm{e}^{-\mathrm{i}kS(x)}, \tag{3.2}$$

which has been used since the early 1800s as an anzatz for exact and approximate solutions of partial and ordinary differential equations. Then, the Helmholtz equation from (3.1) can be split into two equations

$$(\vec{\nabla}S)^2 = \varkappa^2, \tag{3.3}$$

$$\frac{\mathrm{i}}{2}\nabla^2\phi + k\vec{\nabla}\phi\cdot\vec{\nabla}S + \frac{k}{2}(\nabla^2 S)\,\phi = 0, \qquad \phi|_{\partial G} = \psi_0\mathrm{e}^{\mathrm{i}kS}\big|_{\partial G}. \tag{3.4}$$

Eq. (3.3) is a well-known eikonal equation from ray optics (Keller, 1958; Maslov and Fedoriuk, 1981) and it is a particular case of the Hamilton–Jacobi equation of classical mechanics (Arnold, 1978). These equations have been exhaustively studied in the literature so we may take as granted that the eikonal $S(x)$ corresponding to the imposed radiation conditions at infinity is already defined either on the domain G or on a multi-sheeted Lagrangian manifold constructed over G similar to Riemann surfaces in the theory of analytic functions.

After the eikonal equation (3.3) is solved, either in phase space or in a more general Lagrangian manifold, Eq. (3.4) may be considered as a second order partial differential equation with the Dirichlet boundary condition depending only on the boundary values ϕ_0 and the eikonal $S(x)$. Eqs. (3.3) and (3.4) together are equivalent to the Helmholtz equation (3.1), but Eq. (3.4) considered with an already defined eikonal $S(x)$ has been widely used as a starting point of different approximate approaches to the general wave radiation problem.

If $k \gg 1$, one may neglect in (3.4) the first term and arrive at the 'transport' equation $2\vec{\nabla}S\cdot\nabla\phi + (\nabla^2 S)\,\phi = 0$, widely used in the geometrical theory of diffraction (Keller, 1958; Maslov and Fedoriuk, 1981) for derivation of short-wave asymptotic approximations of wave fields. Another approximate approach to Eq. (3.4) arises if instead of neglecting all of the first term in (3.4) we neglect only part of it. A broad spectrum of 'parabolic-equation' methods in the theory of high-frequency wave propagation is based on this idea, originating from the contributions of Fock and Leontovich (1965).

3.2. A probabilistic solution of the complete transport equation

Let $\vec{A}$ be a vector field in an N-dimensional domain $G \subset \mathbb{R}^N$ with the boundary ∂G and let f be a function on ∂G. Then, the Dirichlet problem

$$\frac{\mathrm{i}}{2}\nabla^2\phi + k\vec{A}\cdot\vec{\nabla}\phi + \frac{k}{2}\mathrm{div}(\vec{A})\phi = 0; \qquad \phi|_{\partial G} = f, \tag{3.5}$$

with a constant parameter k may be considered as a generalization of the complete transport equations (3.4) which, in turn, may be treated as a particular case of (3.5) with $\vec{A} = \vec{\nabla}S$, where S is an eikonal determined from an independent eikonal equation (3.3).

Our approach to the problem (3.5) with a complex-valued differential operator $\frac{\mathrm{i}}{2}\nabla^2 + \vec{A}\cdot\vec{\nabla}$ employs a scheme to replace (3.5) by another Dirichlet problem with a real-valued differential operator which makes it possible to obtain a solution in a probabilistic form similar to (2.31). This idea was partially developed above in the analysis (2.13)–(2.16) of the Eq. (2.12) similar to (3.5), so all that is needed here is to extend the analysis to accommodate the presence of the boundary conditions in (3.5).

Let $\phi(\vec{z})$ be an analytic function of a complex argument $\vec{z} = \vec{x} + \mathrm{i}\vec{y}$, and consider it as a function $\phi(\vec{x},\vec{y})$ of two real N-dimensional vectors $\vec{x}$, $\vec{y}$, satisfying Cauchy–Riemann conditions (2.13). Then, following the reasoning employed in (2.12)–(2.16) we convert the equation from (3.5) with N independent variables to the equation

$$\frac{1}{2}\sum_{n=1}^{N}\frac{\partial^2\phi}{\partial x_n\partial y_n}+k\sum_{n=1}^{N}\left[\mathrm{Re}(A_n)\frac{\partial\phi}{\partial x_n}+\mathrm{Im}(A_n)\frac{\partial\phi}{\partial y_n}\right]+\frac{k}{2}\left(\sum_{n=1}^{N}\frac{\partial A_n}{\partial x_n}\right)\phi=0, \tag{3.6}$$

with a real-valued differential operator but with $2N$ independent variables.

The next step is to augment Eq. (3.6) with some Dirichlet boundary conditions in order to formulate a boundary value problem that may be considered, in an appropriate sense, as an extension of the problem (3.5).

Let G^c and ∂G^c be a domain and its boundary in a complex space $\mathbb{C}^N$ satisfying

$$G^c\cap\mathbb{R}^N=G,\qquad \partial G^c\cap\mathbb{R}^N=\partial G, \tag{3.7}$$

so that the intersections of G^c and ∂G^c with the real space $\mathbb{R}^N$ coincide with the domain G and its boundary ∂G of the Dirichlet problem (3.5). Next, assume $f^c(\vec{z})$ are the values of $\phi(\vec{z})$ on the boundary ∂G^c. Such $f^c(\vec{z})$ may be considered as the extension of the boundary values $f(\vec{z})$ from the real $(N-1)$-dimensional boundary $\partial G\in\mathbb{R}^N$ onto the $(2N-1)$-dimensional surface $\partial G^c\subset\mathbb{C}^N\equiv\mathbb{R}^{2N}$. Then, identifying the complex variable $\vec{z}$ with the pair $(\vec{x},\vec{y})$ of real variables we arrive at the boundary value problem consisting of the Eq. (3.6) and the Dirichlet condition

$$\phi(\vec{x},\vec{y})|_{\vec{z}\in\partial G^c}=f^c(\vec{x},\vec{y}),\qquad \text{where } \vec{z}=\vec{x}+\mathrm{i}\vec{y}. \tag{3.8}$$

Since the problem (3.6)–(3.8), has the structure of the Dirichlet problem (2.32) with $2N$ independent variables x_n, y_n, $n=1,2,\ldots,N$, its solution may be obtained from the formula (2.31) which, in the considered particular case, can be conveniently arranged into the form

$$\phi(\vec{x},\vec{y})=\mathbf{E}\left\{f^c(\vec{\xi}^{x,y}_\tau)\mathrm{e}^{-\int_0^\tau B(\vec{\xi}^{x,y}_t)\mathrm{d}t}\right\},\qquad B=\frac{k}{2}\sum_{n=1}^{N}\frac{\partial A_n}{\partial x_n}, \tag{3.9}$$

averaging random walks $\vec{\xi}^{x,y}_t$ launching from the point $(\vec{x},\vec{y})$ and running across the space $\mathbb{R}^{2N}\equiv\mathbb{C}^N$ according to the real stochastic equations

$$\mathrm{Re}(\mathrm{d}\vec{\xi}_t)=\frac{1}{\sqrt{2}}\vec{w}_t+k\mathrm{Re}(\vec{A})\mathrm{d}t,\qquad \mathrm{Im}(\mathrm{d}\vec{\xi}_t)=\frac{1}{\sqrt{2}}\vec{w}_t+k\mathrm{Im}(\vec{A})\mathrm{d}t,\qquad \vec{w}_t\in\mathbb{R}^N, \tag{3.10}$$

where $\vec{w}_t$ is the standard Brownian motion in the real space $\mathbb{R}^N$. As for the exit time τ, it is defined as the first instant when the path $\vec{\xi}^{z}_t=\vec{\xi}^{x,y}_t$ touches the boundary ∂G^c, which is equivalent to the condition that the projection $\mathrm{Re}(\vec{\xi}^{z}_t)$ of the path $\vec{\xi}^{z}_t$ to the real space touches the boundary ∂G.

Since $\phi(\vec{x},\vec{y})$ defined by (3.9) obeys boundary conditions (3.8) it is clear that the restriction $\phi(\vec{x},0)$ of this function to the real space obeys the boundary condition from the original Dirichlet problem (3.5). Therefore, to insure that $\phi(\vec{x},0)$ solves the problem (3.5) it is sufficient to show that the equation from (3.5) is equivalent to (3.6). But equation (3.6) was derived from (3.5) under the hypothesis that $\phi(\vec{x},\vec{y})$ satisfies the Cauchy–Riemann conditions (2.13), which is equivalent to the requirement that $\phi(\vec{z},0)$ is an analytic function of the complex variable $\vec{z}=\vec{x}+\mathrm{i}\vec{y}$. On the other hand, if we assume that $\vec{A}(x)$ and $f(x)$ from (3.5) are analytic and that f^c from (3.8) analytically continues f, then the mathematical expectation (3.9), if it converges, determines an analytic function $\phi(\vec{z})=\phi(\vec{x}+\mathrm{i}\vec{y})$ which, consequently, obeys the complex equation from (3.5).

So, we finally conclude that if functions $\vec{A}$ and f from the problem (3.5) are analytic, and $f^c(z)$ is the analytic continuation of $f(x)$, then the solution of the Dirichlet problem (3.5) can be represented by the probabilistic formula

$$\phi(x)=\mathbf{E}\left\{f(\vec{\xi}^{x}_\tau)\mathrm{e}^{\frac{k}{2}\int_0^\tau\mathrm{div}(\vec{A}(\vec{\xi}^{x}_t))\mathrm{d}t}\right\}, \tag{3.11}$$

where $\vec{\xi}_s^x$ is the random motion defined by the stochastic differential equation

$$\mathrm{d}\vec{\xi}_t^x = \sqrt{\mathrm{i}}\mathrm{d}\vec{w}_t + k\vec{A}\mathrm{d}t, \qquad \vec{\xi}_0^x = x, \tag{3.12}$$

which is equivalent to (3.10), and τ is the exit time of the projection $\mathrm{Re}\left(\vec{\xi}_t^x\right)$ of the trajectory $\vec{\xi}_t^x$ from the domain G.

It is worth noting that formulas (3.11) remain valid if the paths $\vec{\xi}_t^x$ are governed by the equations

$$\mathrm{d}\vec{\xi}_t^x = \mathrm{e}^{\mathrm{i}(\frac{\pi}{4}+\frac{\vec{\alpha}}{2})}\mathrm{d}\vec{w}_t + k\mathrm{e}^{\mathrm{i}\vec{\alpha}}\vec{A}\mathrm{d}t, \qquad \vec{\xi}_0^x = x, \tag{3.13}$$

slightly generalizing (3.12) by the presence of an arbitrary function $\vec{\alpha}(z)$ of the complex space argument. To justify (3.12) it suffices to multiply (3.5) by $\mathrm{e}^{\mathrm{i}\vec{\alpha}}$ and to re-apply, with obvious modifications, all the reasoning embodied in (3.6)–(3.11). In particular, it may be convenient to introduce $\vec{\alpha}(z)$ by the formula

$$\vec{\alpha}(z) = \frac{\pi}{2} - 2\arg(\vec{A}(z)), \tag{3.14}$$

which guarantees that the differentials of the Brownian and of the drift components of the path $\vec{\xi}_t^x$ belong to the same N-dimensional subspace $\mathrm{e}^{\mathrm{i}(\frac{\pi}{4}+\frac{\vec{\alpha}}{2})}\mathbb{R}^N$ of $\mathbb{C}^N$.

Representation (3.11) is valid for any domain G^c satisfying (3.7), but the right-hand sides of these formulae depend on values of $f^c(\vec{z})$ on the surface ∂G^c, while the boundary conditions from (3.5) specify this function only on its intersection $\partial G = \partial G^c \cap \mathbb{R}^N$ with the real space. However, (3.11) may still be used for solution of the problem (3.5), because: (a) in many cases in problems of wave propagation, such as the numerical examples presented later on, the correct extension of the boundary values $f(\vec{z})$ from ∂G to ∂G^c is known a priori; (b) in many cases there is a unique continuation of the analytic boundary values $f(\vec{z})$ of the problem (3.5), from the $(N-1)$-dimensional boundary ∂G to the specific $(2N-1)$-dimensional surface ∂G^c.

To construct such an extension of the $(N-1)$-dimensional boundary ∂G and of the boundary values $g(\vec{z})$ defined on ∂G, assume that $f(\vec{z})$ and ∂G are both analytic, which provides a unique analytic continuation of $f(\vec{z})$ onto the $(2N-2)$-dimensional analytic continuation $\partial G_{\mathbb{C}}$ of ∂G. The integral lines of the vector field $\mathrm{i}\vec{A}\mathrm{e}^{-2\mathrm{i}\arg(\vec{A})}$ originating from $\partial G_{\mathbb{C}}$ form the $(2N-1)$-dimensional surface ∂G^c which has a property that a path started on ∂G^c and defined by (3.13) and (3.14) never leaves ∂G^c. The last property makes it possible to compute the function $\phi(z)$ on ∂G^c by the formula (3.11) with the paths $\vec{\xi}_t^x$ from (3.13) and (3.14), running on the $(2N-1)$-dimensional ∂G^c and stopping on the $(2N-2)$-dimensional analytical continuation $\partial G_{\mathbb{C}}$ of the real boundary ∂G.

In order to illustrate the mechanism of continuation of boundary conditions, consider the one-dimensional problem

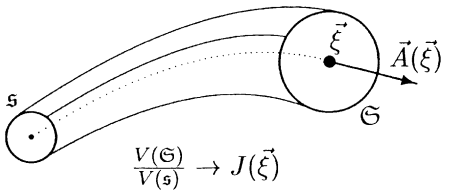
$$\frac{\mathrm{i}}{2}\phi_{rr} + \left(\frac{\mathrm{i}}{2r} - 1\right)\phi_r - \frac{1}{2r}\phi = 0, \quad \text{on } r > 1, \text{ with } \phi(1) = 1, \quad \text{and } \phi(\infty) = 0,$$

with obvious solution $\phi = \mathrm{e}^{-\mathrm{i}r}H_0^1(r)/H_0^1(1)$. It is clear that the paths (3.13) and (3.14), with $A = \frac{\mathrm{i}}{2r} - 1$, do not leave the line ∂G^c, defined on the complex plane $\mathbb{C}$ by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}A\mathrm{e}^{-2\mathrm{i}\arg(A)} \equiv \mathrm{i}\bar{A}(r), \qquad r(0) = 1.$$

Consequently, formula (3.11) with paths from (3.13) and (3.14) determines the solution of this problem on the line ∂G^c transversally crossing the real axis at $r = 1$. After that, the same formula (3.11), but with the paths from (3.12), determines $\phi(r)$ outside ∂G^c including the real semi-axis $r > 1$.

If the problem (3.5) stands alone, then restriction of its solution (3.11) and (3.12) to cases with analytic vector fields $\vec{A}$ and analytic boundary values f may be considered as too severe. Here, however, our interests are limited to a particular case (3.4) of the problem (3.5) arising from the analysis of the Helmholtz



A small surface element $\mathfrak{S}$ with center at $\vec{\xi}$ and transversal to $\vec{A}(\vec{\xi})$ is projected by the dynamical system $\frac{d\vec{\xi}}{dt} = \vec{A}(\vec{\xi})$ to the piece $\mathfrak{S}_0$ of a fixed reference surface S_0.

The ratio $V(\mathfrak{S})/V(\mathfrak{s})$ of volumes of $\mathfrak{S}$ and $\mathfrak{s}$ converges to the geometric divergence $J(\vec{\xi})$ as the diameter of $\mathfrak{S}$ tends to zero.

Fig. 8. Geometrical divergence of the vector field.

equation (3.1) describing propagation of time-harmonic waves. In this case it is rather common to deal with analytic eikonals S determining analytic vector fields $\vec{A} = \vec{\nabla}S$, and with analytic boundary conditions f generated by analytic incident waves.

It should also be mentioned that in order to guarantee that the mathematical expectation (3.11) exists the vector field $\vec{A}$ and the domain G must satisfy certain conditions, which are discussed in detail in the literature (Freidlin, 1985).

A specific feature of the solution (3.11) is that its exponent can be converted to an alternative form which provides a convenient way for the asymptotic analysis of $\phi(x)$ in the limit $k \gg 1$.

Let $\mathscr{S}$ be a $(N-1)$-dimensional surface (see Fig. 8) non-tangent to the vector field $\vec{A}$ and let $\vec{\xi}_0(t,\sigma)$ be a solution of the initial value problem

$$\frac{d\vec{\xi}_0}{dt} = \vec{A}(\vec{\xi}_0), \qquad \vec{\xi}_0(0) = \vec{\sigma} \in \mathscr{S},$$

parameterized by a vector $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_{N-1})$. Then, the set $(t, \vec{\sigma})$ may be considered as coordinates of the point $\vec{\xi}_0(t,\vec{\sigma})$, and one may introduce the function

$$J(\vec{\xi}) = \det\left[V(\vec{\xi})\right], \quad \vec{\xi} = \vec{\xi}_0(t,\vec{\sigma}), \quad V(\vec{\xi}) = \frac{\partial\vec{\xi}_0(t,\sigma_1,\sigma_2,\ldots,\sigma_{N-1})}{\partial(t,\sigma_1,\sigma_2,\ldots,\sigma_{N-1})}, \tag{3.15}$$

where $V(\vec{\xi})$ is a Jacobi matrix of the transformation $(t,\vec{\sigma}) \to \vec{\xi}_0(t,\vec{\sigma})$. The function $J(\vec{\xi})$ from (3.15) is widely known in the geometrical theory of diffraction (ray theory) as the 'geometrical divergence' of the vector field $\vec{A}$, and the Liouville theorem establishes the relationship

$$\operatorname{div}\left[\vec{A}(\vec{\xi})\right] = \vec{\nabla}_{\vec{A}}\mathfrak{J}(\vec{\xi}) = \frac{d}{ds}\bigg|_{s=0}\mathfrak{J}\left(\vec{\xi} + s\vec{A}(\vec{\xi})\right), \qquad \text{where } \mathfrak{J}(\vec{\xi}) = \ln\left[J(\vec{\xi})\right], \tag{3.16}$$

between the divergence of $\vec{A}$ and the derivative of $\ln\left[J(\vec{\xi})\right]$ along $\vec{A}$.

Taking into account (3.16) and applying some elementary stochastic calculus, we can re-arrange solution (3.11) of the boundary value problem (3.5) to the form

$$\phi(x) = \mathbf{E}\left\{ f(\vec{\xi}^x_\tau)\sqrt{\frac{J(\vec{\xi}^x_\tau)}{J(\vec{\xi}^x_0)}}\, e^{\int_0^\tau \frac{i}{4}\vec{\nabla}^2(\ln J)\,dt - \frac{\sqrt{i}}{2}\vec{\nabla}(\ln J)\,d\vec{w}} \right\}, \tag{3.17}$$

given in terms of a geometric characteristic $J(\vec{\xi})$.

It should be noticed that although solution (3.17) of the problem (3.5) does not explicitly depend on the parameter k, this parameter, nevertheless, heavily affects the solution through the structure (3.12) of the trajectories $\vec{\xi}^x_t$. Consider, for example, the case when $k \gg 1$. Then the speed of the drift along the vector field $\vec{A}$ is much higher than the average speed of the Brownian motion and the trajectory $\vec{\xi}^x_t$ follows closely along the 'rays' $\vec{\zeta}^x_t$ described by the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\zeta}_t^x = \vec{A}(\vec{\zeta}_t^x), \qquad \vec{\zeta}_0^x = x.$$

Therefore, assuming that the ray $\vec{\zeta}_t^x$ hits the boundary ∂G at the point x_τ, we get an approximation

$$\phi(x) \approx \phi_0(x) = f(x_\tau)\sqrt{\frac{J(x_\tau)}{J(x)}}, \qquad k \gg 1, \tag{3.18}$$

which, indeed, can be rigorously justified when the boundary ∂G, the boundary values f and the vector field $\vec{A}$ satisfy some conditions of regularity.

This approximation may also be obtained as an exact solution of the equation

$$2\vec{A}\cdot\vec{\nabla}\phi_0 + \mathrm{div}(\vec{A})\ \phi_0 = 0; \qquad \phi_0|_{\partial G} = f, \tag{3.19}$$

which may be regarded as an approximation to the problem (3.5). Equation (3.19) is well known in the geometrical theory of diffraction (ray method) as a 'transport equation' because it describes the transport of a quantity ϕ_0 along the vector field $\vec{A}$.

3.3. Numerical examples

To get an indication of the numerical efficiency of the probabilistic formulas (3.17) we consider a particular two dimensional case of (3.5) with $k = 1$ and with the vector field $\vec{A} = \vec{x}/\|\vec{x}\|$ generated by the eikonal $S(x) = \|x\| = r$.

Eq. (3.5) corresponding to this case takes the form $\frac{1}{2}\nabla^2\phi + \mathrm{i}(\partial\phi)/(\partial r) + \mathrm{i}(\phi/2r) = 0$, which has an obvious solution $\phi(r,\theta) = \mathrm{e}^{-\mathrm{i}r}H_0^{(1)}(r)$, defined in any exterior domain $r \geqslant R > 0$. Since the vector field $\vec{A}(r,\theta) = \vec{x}/\|\vec{x}\|$ and its geometric divergence $J(r,\theta) = r$ are both analytic in the domain $\mathrm{Re}(r) \geqslant R$, the probabilistic formula (3.17) can be used for simulation of the function $\mathrm{e}^{-\mathrm{i}r}H_0^{(1)}(r)$ considered as the solution of the Dirichlet problem with boundary conditions $\phi(R,\theta) = \mathrm{e}^{-\mathrm{i}R}H_0^{(1)}(R)$ in the domain $r \geqslant R$.

Fig. 9 shows the results of the numerical simulation of the function $\mathrm{e}^{-2\pi\mathrm{i}r}H_0^{(1)}(2\pi r)$. This function is computed twice: as a Hankel function and as the solution of the discussed Dirichlet problem given by the probabilistic formula (3.17). The amplitudes of the computed functions are displayed on the first diagram,

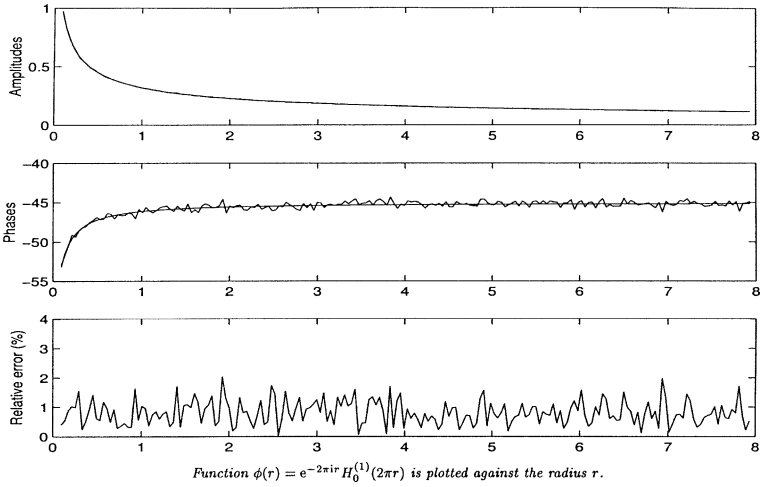


Fig. 9. Simulation of a Hankel function by random walks.

and the second diagram displays their phases. The last diagram presents the relative error of the probabilistic solution.

The presented results correspond to the radius $2\pi R = \lambda/30$, where $\lambda = 2\pi$ is the wave length. The mathematical expectations from (3.17) were estimated by statistical sums of the type (2.34) where the averaging included 2000 independent discrete Brownian walks with Cartesian increments $\Delta x = \Delta y \approx 0.07$. The computations were very stable and despite the use of a rather rough discretization and the simplest algorithms, the relative error was maintained below the low 2% level over the entire trial interval from $r = 0.05$ to $r \approx 8$ wave lengths.

Next we consider a classical Sommerfeld problem of diffraction in a wedge, which consists of finding the solution of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$, defined in a wedge $|\theta| \leqslant \alpha$ with the Dirichlet boundary conditions $\psi(r, \pm\alpha) = 0$ imposed on its faces. The excitation is generated by the incident plane wave

$$\psi_i(r, \theta) = \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\theta_0)}, \qquad |\theta_0| < \alpha,$$

arriving from infinity along the ray $\theta = \theta_0$, and except for this incident wave there should be no other waves arriving from infinity. The last condition can be formalized as a requirement that the solution $\psi(r, \theta)$ admits the decomposition

$$\psi(r, \theta) = \psi_{\mathrm{g}}(r, \theta) + \psi_{\mathrm{d}}(r, \theta), \tag{3.20}$$

into a 'geometrical' component $\psi_{\mathrm{g}}(r, \theta)$, which includes the incident wave as well as a finite number of plane reflected waves generated according to the laws of geometrical optics, and a diffracted component $\psi_{\mathrm{d}}(r, \theta)$ satisfying the Sommerfeld radiation condition at infinity.

The geometrical field $\psi_{\mathrm{g}}(r, \theta)$ can be defined without any prior knowledge of the diffracted field by the Sommerfeld integral

$$\psi_{\mathrm{g}}(r, \theta) = \frac{1}{2\pi\mathrm{i}} \int_C \left[\Psi_{\mathrm{g}}(\theta - \omega + \pi) - \Psi_{\mathrm{g}}(\theta + \omega - \pi)\right] \mathrm{e}^{\mathrm{i}kr\cos\omega} \mathrm{d}\omega, \tag{3.21}$$

taken along the standard $\bigcup$-like contour of integration C passing from $\frac{\pi}{2} + \mathrm{i}\infty$ to $\frac{3\pi}{2} + \mathrm{i}\infty$. The amplitude function $\Psi_g(\omega)$ can be defined as a finite sum of rational functions:

$$\Psi_{\mathrm{g}}(\omega) = \sum_{m=M_1}^{M_2} \left\{ \frac{1}{\omega - \theta_0 - 4\alpha m} - \frac{1}{\omega - \theta_0 - 4\alpha m - 2\alpha} \right\}, \tag{3.22}$$

where M_1 and M_2 are any integers satisfying the inequalities $M_2 > \frac{\pi - \theta_0}{4\alpha}$ and $M_1 < -\frac{2\alpha + \theta_0}{4\alpha}$.

The diffracted field has the structure $\psi_{\mathrm{d}}(r, \theta) = \mathrm{e}^{\mathrm{i}kr}\phi_{\mathrm{d}}(r, \theta)$, with the eikonal $S = r$ and with the factor $\phi_{\mathrm{d}}(r, \theta)$ satisfying in the wedge $|\theta| < \alpha$ the complete transport equations (3.5) with the boundary conditions

$$\phi_{\mathrm{d}}(r, \pm\alpha) = g(r, \pm\alpha) \equiv -\phi_{\mathrm{g}}(r, \pm\alpha)\mathrm{e}^{-\mathrm{i}kr}. \tag{3.23}$$

Since the field $\phi_{\mathrm{d}}(r, \theta)$ is defined as the solution of the boundary value problem of the type (3.5) it can be computed by a probabilistic formula (3.17) which leads to the solution of the Sommerfeld problem of diffraction computed by the random walk method. On the other hand, the exact solution of this problem can be represented by the Sommerfeld integral

$$\psi(r, \theta) = \frac{1}{2\pi\mathrm{i}} \int_C [\Psi(\theta - \omega + \pi) - \Psi(\theta + \omega - \pi)], \mathrm{e}^{\mathrm{i}kr\cos\omega} \mathrm{d}\omega, \tag{3.24}$$

similar to (3.22) but with a different amplitude function:

$$\Psi(\omega) = \frac{\pi}{4\alpha} \left\{ \cot\left[\frac{\pi}{4\alpha}(\omega - \theta_0)\right] + \tan\left[\frac{\pi}{4\alpha}(\omega + \theta_0)\right] \right\}. \tag{3.25}$$

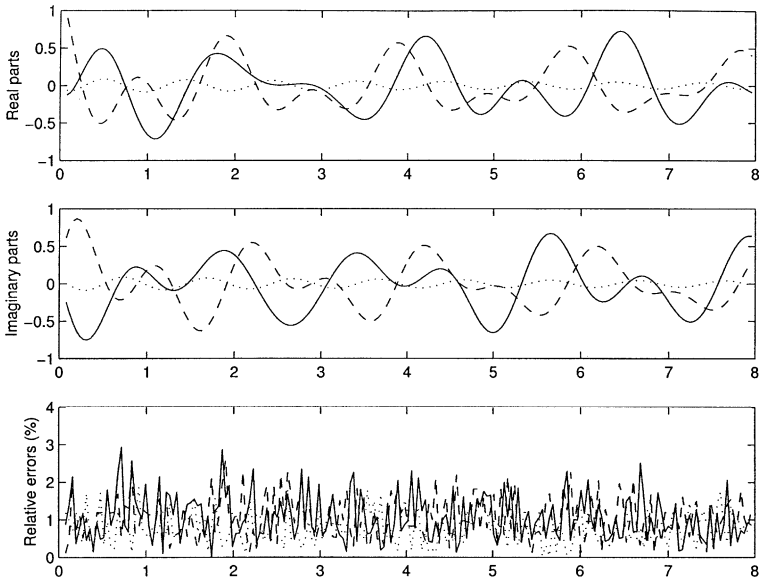


Fig. 10. Probabilistic solution of the Sommerfeld diffraction problem.

So, the Sommerfeld problem of diffraction in a wedge provides another opportunity to compare solutions obtained by the random walk method with the exact solutions represented by conventional quadratures.

Fig. 10 shows the results of the numerical simulation of the wave field $\psi(2\pi r,\theta)$ from (3.24) and (3.25) generated in a wedge by an incident plane wave. This field is computed twice: by direct numerical integration in (3.24) and (3.25), and through the probabilistic simulation of the diffracted component $\phi_d(r,\theta)$ of the total wave field.

The presented results correspond to the wedge $|\theta| < 135°$ with the incident wave arriving along the direction $\theta_0 = 90°$. Solid lines display the wave field along the ray $\theta = 30°$ which is exposed to the incident, reflected and diffracted waves. Dashed lines correspond to the ray $\theta = -30°$ which is exposed to the incident, and diffracted waves. Dotted lines correspond to the ray $\theta = -120°$ exposed only to the diffracted waves. Computations using random walks with the same parameters as in the previous examples were stable and the relative difference between the diffracted fields computed by the different methods was below the 3% level on the entire wavelength interval of computation (0.05, 8).

3.4. Wave propagation in two media

In the above we discussed the application of the random walk method to the computation of solutions of the Helmholtz equation in a single medium, and in Section 2.4 we discussed the application of this method to differential equations formulated in two adjacent media with some interface boundary conditions imposed. Comparing the presented material it is easy to see that these two cases may be combined, which results in the possibility of employing probabilistic methods for the analysis of wave propagation through several media. Here we consider a rather simple but, nevertheless, representative case of wave propagation in a space which comprises two half-spaces with different wave speeds.

Let a plane (x,y) be subdivided by the line $y=0$ on two half-spaces $G_1=\{x,y: y>0\}$ and $G_2=\{x,y: y<0\}$ occupied by different wave conducting media. Consider a plane wave

$$\psi_1^i = \mathrm{e}^{\mathrm{i}k_1(x\cos\theta_0+y\sin\theta_0)}, \qquad 0<\theta_0<\pi, \tag{3.26}$$

propagating in the half-space G_1 in the direction from infinity towards the interface $y=0$ separating G_1 from G_2. The interaction of this wave with the interface causes the excitation of secondary waves ψ_1 and ψ_2, which are radiated to the domains G_1 and G_2, respectively. The wave ψ_1 radiated to G_1 is usually referred to as the reflected wave, while the wave ψ_2 radiated to G_2 is referred to as the transmitted or refracted wave. The goal of the present section is to compute the reflected and transmitted waves by the probabilistic method.

More precisely, the problem considered here is to compute functions $\psi_{1,2}$ which are defined in the domains $G_{1,2}$, and satisfy the Helmholtz equations

$$\begin{aligned} &\nabla^2\psi_1 + k_1^2\psi_1 = 0, \quad \text{in } y>0,\\ &\nabla^2\psi_2 + k_2^2\psi_2 = 0, \quad \text{in } y<0 \end{aligned} \tag{3.27}$$

with different wave numbers $k_{1,2}$. Additionally, functions $\phi_{1,2}$ must satisfy the interface conditions

$$(\psi_1^i+\psi_1)\big|_{y=0} = \psi_2|_{y=0}, \tag{3.28}$$

$$\frac{\partial}{\partial y}\bigg|_{y=0}(\psi_1^i+\psi_1) = \frac{\partial}{\partial y}\bigg|_{y=0}\psi_2, \tag{3.29}$$

and must obey the radiation conditions in the corresponding domains $G_{1,2}$.

We seek solutions $\psi_{1,2}$ in the form

$$\psi_1 = \phi_1\ \mathrm{e}^{-\mathrm{i}k_1S_1}, \qquad \psi_2 = (1+\phi_2)\ \mathrm{e}^{-\mathrm{i}k_2S_2}, \tag{3.30}$$

with the eikonals

$$S_n = x\cos\theta_n + y\sin\theta_n, \tag{3.31}$$

whose angles

$$\theta_1 = -\theta_0, \qquad \theta_2 = \arccos\left(\frac{k_1}{k_2}\cos\theta_0\right) \tag{3.32}$$

are determined by the direction θ_0 of the incident wave from (3.26). Then, comparing (3.27) with (3.30)–(3.32) we find that the new unknown functions $\phi_{1,2}$ must satisfy equations

$$\frac{\mathrm{i}}{2}\nabla^2\phi_n + k_n\vec{e}_n\cdot\vec{\nabla}\phi_n + \frac{1}{2}\ k_n\phi_n = 0, \tag{3.33}$$

with the unit vectors

$$\vec{e}_1 = (\cos\theta_1,\ \sin\theta_1), \qquad \vec{e}_2 = (\cos\theta_2,\ \sin\theta_2), \tag{3.34}$$

and these functions are coupled through the interface conditions

$$\phi_1 - \phi_2 = 0, \tag{3.35}$$

$$\mathrm{i}\left(\frac{\partial}{\partial y}\phi_1 - \frac{\partial}{\partial y}\phi_2\right) + k_1\sin\theta_1\phi_1 - k_2\sin\theta_2\phi_2 = k_1\sin\theta_1 + k_2\sin\theta_2, \tag{3.36}$$

imposed on the line $y=0$.

Problem (3.33)–(3.36) has an obvious solution

$$\phi_1 = \phi_2 = \frac{k_1 \sin\theta_1 + k_2 \sin\theta_2}{k_1 \sin\theta_1 - k_2 \sin\theta_2} = \frac{k_1 \sin\theta_0 - \sqrt{k_2^2 - k_1^2 \cos^2\theta_2}}{k_1 \sin\theta_0 + \sqrt{k_2^2 - k_1^2 \cos^2\theta_2}},$$

from which it follows that the solution $\psi_{1,2}$ of the original problem (3.27)–(3.29) has the structure

$$\psi_n = K_n \, \mathrm{e}^{-\mathrm{i}k_n S_n}, \tag{3.37}$$

where the reflection coefficient K_1 and the transmission coefficient K_2 are explicitly determined as

$$K_1 = \frac{\sin\theta_0 - \sqrt{\gamma^2 - \cos^2\theta_2}}{\sin\theta_0 + \sqrt{\gamma^2 - \cos^2\theta_2}}, \qquad K_2 = \frac{-2\sin\theta_0}{\sin\theta_0 + \sqrt{\gamma^2 - \cos^2\theta_2}}, \qquad \gamma = \frac{k_1}{k_2}. \tag{3.38}$$

On the other hand, problem (3.33)–(3.36) is of the type (2.37)–(2.40) discussed in Section 2.4 as an example of problems admitting explicit probabilistic solutions. Therefore, applying the scheme of the random walk method described in Section 2.4 we can compute the reflection and transmission coefficients K_1, K_2 satisfying (3.37), and comparing the simulated results with exact values known from (3.38) we get another opportunity to justify the efficiency of the random walk approach to problems of wave propagation.

Fig. 11 shows results of the probabilistic computations of the reflection coefficients K_1 for different plane incident waves. These coefficients are computed twice: by direct evaluation of (3.38) and through the probabilistic solution of the problem (3.33)–(3.36). The presented results correspond to the case when the upper half space has the wave number $k_1 = 1.5$ and the wave number of the lower half space is $k_2 = 1$, which means that waves propagate in the lower domain 1.5 time faster than in the upper domain. The reflected field was computed at the point $x = (0, 0.5)$ by the probabilistic formula where the mathematical

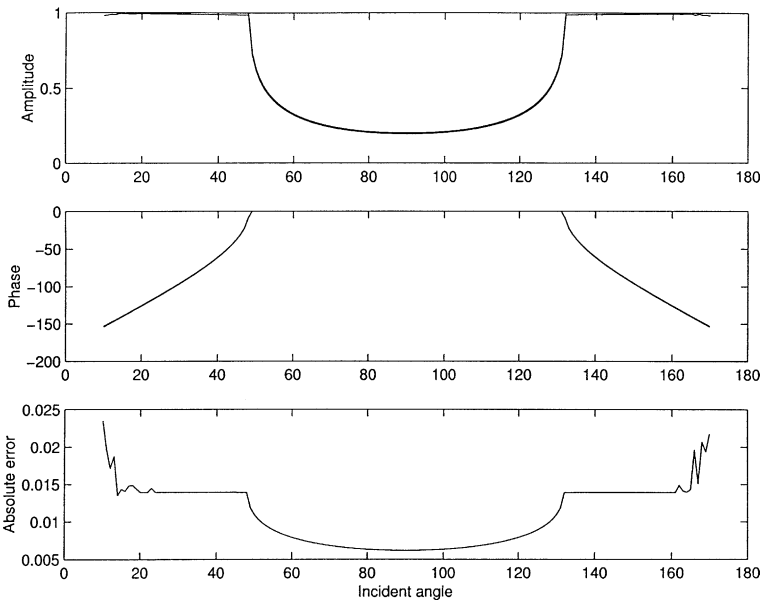


Fig. 11. Simulation of the reflection coefficient K_1.

expectations were computed by averaging 1000 independent discrete random walks with the spatial increment $\varepsilon = 0.05$. The computations were stable and the absolute error was in the range 0.01–0.02 for most of the incident angles.

4. Conclusion

The results presented here show that the synthesis of the ray method and the probabilistic methods provides a promising approach to problems of wave propagation which may be used both for effective numerical evaluations and for asymptotic analysis. The advantages of this combination include, but are not limited to: physical meaningfulness is retained from the ray theory; versatility and minimal requirements of the problem's data; numerical implementations may employ simple and scalable parallel algorithms with minimal use of computer memory.

Here we explored basic ideas of the developing method and considered examples illustrating its applications to scalar problems with general first-order boundary conditions and to scalar problems formulated in adjacent domains. In future papers we will extend the approach to vector problems of elastodynamics and electromagnetics and apply the method to diffraction problems as yet unsolved by other methods.

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D. Bogy received his B.S. degrees in Geology and Mechanical Engineering in 1959, and his M.S. degree in Mechanical Engineering from the Rice University in 1961. He received a Ph.D. degree in Applied Mathematics from Brown University in 1966 and spent a year as Postdoctoral Fellow at the California Institute of Technology. In 1967, D. Bogy joined the faculty of the University of California at Berkeley. In 1991–1999, D. Bogy served as a Chairman of the Department of Mechanical Engineering at University of California at Berkeley. D. Bogy is a member of the National Academy of Engineering, and he serves on the US National Committee on Theoretical and Applied Mechanics. He has served as Chair of the Executive Committees of the Applied Mechanics and the Tribology Divisions of the American Society of Mechanical Engineers.

B. Budaev received his M.S. (Diploma) and Ph.D. (Candidate of Sciences) in Applied Mathematics and Mechanics in 1976 and 1979, both from the Department of Mathematics and Mechanics of the Leningrad State University. In 1995, B. Budaev was awarded a degree 'Doctor of Sciences'. In 1984, B. Budaev joint the staff of the St. Petersburg Branch of the Steklov Mathematical Institute. Since 1995, B. Budaev continues research at the University of California, Berkeley.