Two-Dimensional Diffraction by a Wedge With Impedance Boundary Conditions

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***Abstract*—The two dimensional problem of diffraction by a wedge with impedance boundary conditions on its faces is explicitly solved in a form that admits effective numerical simulation by simple perfectly scalable algorithms with unlimited capability for parallel processing. The solution is represented as a superposition of the geometric field that is completely determined by elementary ray analysis and of the waves diffracted by the tip of the wedge. The diffracted field is explicitly represented as a mathematical expectation of a specified functional on trajectories of a random motion determined by the configurations of the problem and by the boundary conditions. The numerical results confirm the efficiency of this approach.**

I. Introduction

PROBLEMS of wave propagation in wedges have a long, but at the same time, short history. As early as 1892 and 1897 Poincaré [16], [17] published two papers on acoustic wave diffraction by a wedge with Dirichlet or Neumann boundary conditions. Using the technique of separation of variables he derived explicit solutions in the form of series of Bessel functions multiplied by the corresponding trigonometric expressions. Almost at the same time Sommerfeld analyzed the two-dimensional (2-D) problem of diffraction by a semi-infinite screen, and in [20] he obtained the exact solution in the form of the so-called Sommerfeld integrals representing the wave field as expansions in terms of plane waves.

It was not until the 1950s that the next fundamental step was taken in the study of wave propagation in wedge-shaped regions. Namely, the 2-D problem of plane wave diffraction by a wedge with impedance boundary conditions was independently solved by Maliuzhinets (Malyuzhinets, Maliuzhinetz) [12]–[14], Senior [19] and Williams [21]. These authors used techniques, different in detail but similar in essence, employing the Sommerfeld representation of wave fields that reduced the diffraction problem to a scalar Hilbert problem of conjugation, or one of its equivalents such as Wiener-Hopf or difference equations, which admit conventional closed-form analytic solutions.

The above mentioned approaches provide explicit analytic solutions to a class of scalar problems of electromagnetic wave diffraction by wedge-shaped scatterers with impedance boundary conditions. This class includes problems of diffraction with an incident wave propagating in the direction orthogonal to the edge of the wedge and it also includes some particular problems of diffraction of skew incident waves by a wedge with the impedance tensor satisfying specific conditions discussed in detail in [2], [10], [11]. As for generic problems of diffraction of arbitrary plane incident waves by a wedge with anisotropic impedance boundary conditions, it appears that they do not have conventional closed-form solutions represented by arithmetic operations and quadratures applied to admissible special functions as found in sources such as [1], [18] for example. The latter statement, however, does not eliminate the possibility that the considered problems may admit other kinds of closed-form solutions, such as probabilistic solutions introduced in [4]–[6] for the analysis of similar problems of wave propagation.

Here we explicitly solve the 2-D problem of diffraction by a wedge with impedance boundary conditions on its faces using the random walk method developed in [4] for the Dirichlet boundary conditions. The solution is represented as a superposition of the geometric field that is completely determined by an elementary ray analysis and of the waves diffracted by the tip of the wedge. To compute the diffracted field we express it in the form $U_d = e^{ikr}u$, where r is the polar radius. Then the amplitude $u(r,\theta,\phi)$ is determined by the complete transport equation $(1/ik)\nabla^2 u + 2\vec{\nabla} r \cdot \vec{\nabla} u + (\nabla^2 r)u = 0$ which is solved by the Feynman-Kac formula in terms of a mathematical expectation of a specified functional on trajectories of a random motion determined by the geometry of the problem and by the boundary conditions. Then, we discuss the convergence of the obtained expressions and observe that it is guaranteed in all cases when the considered problem of diffraction has a unique bounded solution. Moreover, it is shown that the analysis of convergence provides the "probabilistic" explanation of the surface waves which may be generated in certain conditions. Also, we present numerical results which demonstrate the feasibility of the approach and show agreement with published results obtained using conventional methods.

It should finally be emphasized that the goal of the present paper is restricted to the adaptation of the random walk approach to problems of diffraction with impedance boundary conditions. For this purpose we consider only 2-D problems which gives an opportunity to expose the main ideas of handling the impedance boundary conditions without going into nonessential detail. As for the extension of the approach to physically important three-dimensional (3-D) vector problems of diffraction by a wedge with anisotropic impedances this will be the subject of a separate paper [7]. Since the underlying ideas of the extensions of the random walk method to impedance boundary conditions and to vector problems are different, we choose not to present both of these extensions in a single paper.

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II. Formulation of the Problem

Let (r, θ, z) be cylindrical coordinates in the physical space, so that the inequalities $0 < \theta < \alpha$ define a wedge-shaped domain $\Gamma^3(\alpha)$ with the edge coinciding with the z-axis. It is assumed that this wedge is filled by a medium supporting propagation of electromagnetic waves and that the faces $\theta = 0$ and $\theta = \alpha$ of the wedge have constant impedances which may be complex and different from each other.

We are interested in a problem of diffraction of a plane electromagnetic wave arriving to the wedge $\Gamma^3(\alpha)$ in a direction orthogonal to its edge. It is well known [3] that such a problem can always be split on two independent problems, usually referred to as the TE and TM problems, which admit reductions to a 2-D scalar problem of diffraction as formulated below.

Let (r, θ) be polar coordinates and let

$$\Gamma: \quad r > 0 \quad 0 < \theta < \alpha \tag{2.1}$$

be a wedge, which may be considered as a projection of the 3-D wedge $\Gamma^3(\alpha)$ to the plane $z = 0$. Then, the plane wave arriving to $\Gamma(\alpha)$ is described as

$$U_0(r, \theta) = e^{-ikr\cos(\theta - \theta_0)} \quad 0 < \theta_0 < \alpha \tag{2.2}$$

and the problem of its diffraction by the screen $G = \partial\Gamma$ with impedance boundary conditions can be formulated as the problem of finding the solution of the Helmholtz equation

$$\nabla^2 U + k^2 U = 0 \tag{2.3}$$

which is bounded anywhere and does not contain any waves arriving from infinity, except for the incident wave $U_0(r, \theta)$, and satisfies the impedance boundary conditions

$$\begin{aligned} \left[\frac{1}{r}\frac{\partial U}{\partial \theta} + ikB_1 U\right]_{\theta=0} &= 0 \\ \left[-\frac{1}{r}\frac{\partial U}{\partial \theta} + ikB_2 U\right]_{\theta=\alpha} &= 0 \end{aligned} \tag{2.4}$$

with the generally complex coefficients B_1 and B_2 restrained by the inequalities

$$\mathrm{Re}(B_1) \geq 0 \quad \mathrm{Re}(B_2) \geq 0 \tag{2.5}$$

which guarantee the existence and uniqueness of a bounded solution of the considered problem.

Elementary geometric-optical analysis suggests that the solution of this diffraction problem can be represented as a superposition

$$U(r, \theta) = U_g(r, \theta) + U_d(r, \theta) \tag{2.6}$$

of the discontinuous wave fields $U_g(r, \theta)$ and $U_d(r, \theta)$, which are referred to as the geometric wave and the diffracted wave, respectively.

The geometric field $U_g(r, \theta)$ consists of the incident wave and a finite number of reflected waves, which can be computed *a priori* by the recursive application of the laws of geometrical optics. This field can always be represented as a sum

$$U_g(r, \theta) = \sum_{m=3}^{M} K_m e^{ikr\cos(\theta - \widetilde{\theta}_m)} + U_g^1(r, \theta) + U_g^2(r, \theta) \tag{2.7}$$

of a finite number of continuous plane waves propagating in the uniquely determined directions $\theta = \widetilde{\theta}_m$ and of at most two discontinuous plane waves

$$\begin{aligned} U_g^n(r, \theta) &= \chi(\theta_n - \theta) K_n e^{ikr\cos(\theta - \theta_n)} \\ 0 &< \theta_n < \alpha \quad n = 1, 2 \end{aligned} \tag{2.8}$$

where

$$\chi(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \tag{2.9}$$

is the standard step-function, and $\theta = \theta_1$, $\theta = \theta_2$ are rays which are contained inside the wedge $\Gamma(\alpha)$. As for the coefficients $K_1, K_2, \ldots, K_m$ from (2.7), (2.8) they are completely determined by the wedge angle, direction of the incident wave and the impedances of the wedge faces.

To avoid too much emphasis on material that is not essential for our purposes here we do not present general formulas for all angles α and θ_0, but only mention that if the following conditions (see Fig. 1) are satisfied

$$0 < \theta_0 < \alpha - \pi \tag{2.10}$$

then the geometric field is defined by

$$\begin{aligned} U_g(r, \theta) = \chi(\theta_1 - \theta) K_1 e^{ikr\cos(\theta - \theta_1)} \\ + \chi(\theta_2 - \theta) e^{ikr\cos(\theta - \theta_2)} \end{aligned} \tag{2.11}$$

where

$$\theta_1 = \pi - \theta_0 \quad \theta_2 = \pi + \theta_0. \tag{2.12}$$

and

$$K_1 = \frac{\sin\theta_0 - B_1}{\sin\theta_0 + B_1} \tag{2.13}$$

is the reflection coefficient of the face $\theta = 0$ illuminated by the incident wave.

The geometric field $U_g(r, \theta)$ can be viewed as the first approximation to the exact solution which obeys all conditions of the problem with the exception that the Helmholtz equation is violated on the rays $\theta = \theta_1$ and $\theta = \theta_2$ where $U_g(r, \theta)$ is discontinuous.

As for the diffracted field $U_d(r, \theta)$ it has to obey the Helmholtz equation everywhere except on the rays $\theta = \theta_{1,2}$ along which it must obey the jump conditions

$$U_d(r, \theta_1 + 0) - U_d(r, \theta_1 - 0) = K_1 e^{ikr} \tag{2.14}$$

$$U_d(r, \theta_2 + 0) - U_d(r, \theta_2 - 0) = e^{ikr} \tag{2.15}$$

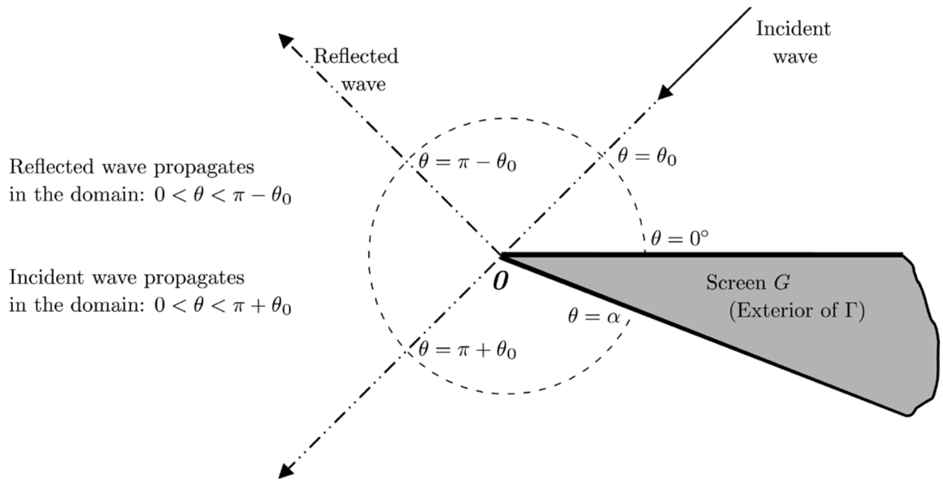

Fig. 1. Geometry of the problem.

and

$$\left.\frac{\partial U_d}{\partial \theta}\right|_{\theta=\theta_m+0} = \left.\frac{\partial U_d}{\partial \theta}\right|_{\theta=\theta_m-0} \quad m = 1, 2. \tag{2.16}$$

Additionally, $U_d(r, \theta)$ has to satisfy the radiation condition

$$e^{-ikr} U_d(r, \theta) = o(1) \quad r \to \infty \quad \theta \neq \pi \pm \theta_0. \tag{2.17}$$

and the boundary conditions (2.4).

We seek the diffracted field in the product form

$$U_d(r, \theta) = u(r, \theta) e^{ikr} \tag{2.18}$$

whose amplitude $u(r, \theta)$ should be a bounded solution of the equation

$$\frac{r^2}{2} \frac{\partial^2 u}{\partial r^2} + r\left(\frac{1}{2} + ikr\right)\frac{\partial u}{\partial r} + \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{ikr}{2} u = 0 \tag{2.19}$$

accompanied by the interface conditions

$$u(r, \theta_1 + 0) - u(r, \theta_1 - 0) = K_1 \tag{2.20}$$
$$u(r, \theta_2 + 0) - u(r, \theta_2 - 0) = 1 \tag{2.21}$$
$$u'_\theta(r, \theta_m + 0) - u'_\theta(\theta_m - 0) = 0 \quad m = 1, 2 \tag{2.22}$$

together with the condition at infinity

$$u(r, \theta) = o(1) \quad r \to \infty \quad \theta \neq \theta_{1,2} \tag{2.23}$$

and the boundary conditions

$$\begin{aligned} u'_\theta(r, 0) + ikrB_1 u(r, 0) &= 0 \\ -u'_\theta(r, \alpha) + ikrB_2 u(r, \alpha) &= 0. \end{aligned} \tag{2.24}$$

It should be mentioned that the technique employed here is closely related to the parabolic equation method widely used in the theory of diffraction as a source of practical approximations to the diffracted wave fields. To observe this relationship it suffices to divide (2.19) by $(kr)^2$ to get an equivalent expression

$$\frac{1}{2}\frac{\partial^2 u}{\partial (kr)^2} + \left(\frac{1}{2kr} + i\right)\frac{\partial u}{\partial (kr)} + \frac{1}{2k^2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{iu}{2kr} = 0. \tag{2.25}$$

Then, accepting the typical restrictions for the parabolic equation method estimates

$$kr \gg 1 \quad \left|\frac{\partial^2 u}{\partial r^2}\right| \ll \left|\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right| \tag{2.26}$$

we approximate (2.25) by the Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial (kr)} - \frac{1}{2(kr)^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2.27}$$

which is used in the parabolic equation method as a substitute for (2.19).

III. Probabilistic Representation of the Diffracted Field

Equation (2.19) in a wedge $0 < \theta < \alpha$ with homogeneous Dirichlet boundary conditions has been studied in [4] where its exact solution is represented as a mathematical expectation computed over the trajectories of a specified random motion that is stopped as soon as it hits one of the faces $\theta = 0$ or $\theta = \alpha$ of the wedge. To use the analysis from [4] here, we first assume that the values of $u(r, \theta)$ are already known on the faces $\theta = 0$ and $\theta = \alpha$ of the wedge. Then, a straightforward combination of the general theory described, for example, in [9] with the specifics of the problem (2.19)–(2.23) discussed in [4] makes it possible to represent $u(r, \theta)$ in the form

$$u(r, \theta) = \mathbf{E}\left\{ u(\xi_\theta, \eta_\tau) e^{(i/2)k\int_0^\tau \xi_t dt} + \sum_{\nu=1}^{\tau_\nu < \tau} \delta_\nu Q_\nu e^{(i/2)k\int_0^{\tau_\nu} \xi_s ds} \right\} \tag{3.1}$$

the exact meaning of which will be further explained.

The mathematical expectation $\mathbf{E}$ is computed over the trajectories of the independent random motions ξ_t and η_t referred to hereafter as the radial and the angular motions, respectively. The radial motion is launched at the time $t = 0$ from the position $\xi_0 = r$ and is controlled by the stochastic differential equation

$$\xi_0 = r \quad d\xi_t = \xi_t dw_t^1 + \xi_t\left(\frac{1}{2} + ik\xi_t\right) dt \tag{3.2}$$

where w_t^1 is the standard one-dimensional Brownian motion (Wiener process). As shown in [4] this motion is confined to the first quadrant $\mathrm{Re}(\xi_t) > 0, \mathrm{Im}(\xi_t) \geq 0$ and has a drift toward the unreachable point $\xi = i/2k$. The angular motion η_t runs inside the interval $0 < \eta < \alpha$ as a standard Brownian motion which is launched from the initial position $\eta_0 = \theta$ and is stopped at the exit time $t = \tau$ defined as the time when it reaches one of the boundary points $\eta = 0$ or $\eta = \alpha$. The angular motion running inside the interval $0 \leq \eta \leq \alpha$ crosses the interior points $\eta = \pi \pm \theta_0$ (2.12) at the times $t = \tau_\nu$ enumerated by the index $\nu \geq 1$ which determines the factors δ_ν and Q_ν of (3.1) by the following rules:

$$\delta_\nu = \begin{cases} 1, & \text{if } \eta_{\tau_\nu} < \eta_{\tau_\nu - 0}, \text{ and } \eta_{\tau_\nu + 0} < \eta_{\tau_\nu} \\ -1, & \text{if } \eta_{\tau_\nu} > \eta_{\tau_\nu - 0}, \text{ and } \eta_{\tau_\nu + 0} > \eta_{\tau_\nu} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

and

$$Q_\nu = \begin{cases} K_1, & \text{if } \eta_{\tau_\nu} = \theta_1 \\ 1, & \text{if } \eta_{\tau_\nu} = \theta_2. \end{cases} \tag{3.4}$$

It is clear that $\delta_\nu = 1$ if at the time τ_ν the interface $\eta = \theta_1$ or $\eta = \theta_2$ is crossed from left to right. Similarly, the value $\delta_\nu = -1$ corresponds to the crossing from right to left, and $\delta_\nu = 0$ corresponds to the case when the interface is touched but not intersected. As for Q_ν, its value is determined by the particular interface $\eta = \theta_1$ or $\eta = \theta_2$ that is touched at the time $t = \tau_\nu$.

Expression (3.1) can not be accepted as a solution of the problem (2.19)–(2.24) because its right-hand side involves the yet unknown boundary value $u(\xi_\tau, \eta_\tau)$. However, boundary conditions (2.24) make it possible to estimate this value as

$$u(\xi_\tau, \eta_\tau) = \begin{cases} u(\xi_\tau, \eta_\tau + \Delta t) e^{ikB_1 \xi_\tau \Delta t} + o(\Delta t), & \text{if } \eta_\tau = 0 \\ u(\xi_\tau, \eta_\tau - \Delta t) e^{ikB_2 \xi_\tau \Delta t} + o(\Delta t), & \text{if } \eta_\tau = \alpha. \end{cases} \tag{3.5}$$

It is clear that if $\Delta t > 0$, then both of the angles $\eta = \eta_\tau + \Delta t$ and $\eta = \eta_\tau - \Delta t$ belong to the interval $0 < \eta < \alpha$ which means that the corresponding function values $u(\xi_\tau, \eta_\tau + \Delta t)$ and $u(\xi_\tau, \eta_\tau - \Delta t)$ from the right-hand part of (3.5) can be represented by the (3.1). This iterative process can obviously be continued indefinitely and it results in the representation of the amplitude $u(r, \theta)$ of the diffracted field in the form

$$u(r,\theta) = \mathbf{E}\left\{ \sum_{\nu=1}^{\infty} \delta_\nu Q_\nu \times \exp\left(\int_0^{\tau_\nu} ik\xi_s \left[\frac{1}{2} ds + B_1 d\lambda_s^1 + B_2 d\lambda_s^2 \right] \right) \right\} \tag{3.6}$$

which retains all of the notation from (3.1) except that the random motions ξ_t and η_t in (3.6) are governed by the equations

$$\xi_0 = r \quad d\xi_t = \xi_t dw_t^1 + \xi_t \left(\frac{1}{2} + ik\xi_t \right) dt \tag{3.7}$$

$$\eta_0 = \theta \quad d\eta_t = \begin{cases} dw_t^2, & \text{if } \eta_t \neq 0, \alpha \\ -dt, & \text{if } \eta_t = 0 \\ dt, & \text{if } \eta_t = \alpha \end{cases} \tag{3.8}$$

which show that inside the segment $0 \leq \eta \leq \alpha$ the motion η_t runs as a standard Brownian motion, but when it reaches the segment's borders it is deterministically reflected back. Additionally, expression (3.6) involves the so-called local times λ_t^1 and λ_t^2, which may be considered as measures of the time spent by the angular motion η_t on the interfaces $\theta = \theta_1$ and $\theta = \theta_2$, respectively.

A rigorous discussion of stochastic differential equations, stochastic integrals, and of local times can be found in the literature on stochastic processes [8], [9], but for our purposes it suffices to view the random motions ξ_t, η_t and the integrals from (3.6) as the limits as $\Delta t \to 0$ of discrete processes as described below.

The radial motion ξ_t that is controlled by the stochastic equation (3.2) can be considered as a sequence of random jumps

$$\xi_t \longrightarrow \xi_{t+\Delta t} = \xi_t \pm \xi_t \sqrt{\Delta t} + \xi_t \left(\frac{1}{2} + ik\xi_t \right) \Delta t \tag{3.9}$$

following each other with an infinitesimally small time increment $\Delta t \to 0$. Similarly, the angular motion η_t may be approximated by discrete jumps determined by the rule

$$\eta_t \longrightarrow \eta_{t+\Delta t} = \begin{cases} \eta_t \pm \sqrt{\Delta t}, & \text{if } \eta_t \neq 0, \alpha \\ \eta_t - \Delta t, & \text{if } \eta_t = 0 \\ \eta_t + \Delta t, & \text{if } \eta_t = \alpha \end{cases} \tag{3.10}$$

depending on the current position of the moving point. These discrete approximations of the radial and angular random motions are closely related to the possibility of approximating the integrals from (3.6) by the Riemann sums

$$\int_0^t \xi_s ds \approx \Delta t \sum_{\nu=0}^{\nu\Delta t \leq t} \xi_{\nu\Delta t}$$

$$\int_0^t \xi_s d\lambda_s^{1,2} \approx \Delta t \sum_{\nu=0}^{\nu\Delta t \leq t} \Omega_\nu^{1,2} \xi_{\nu\Delta t} \tag{3.11}$$

where the factors

$$\Omega_\nu^1 = \begin{cases} 1, & \text{if } \nu\Delta t = 0 \\ 0, & \text{otherwise,} \end{cases} \text{ and } \Omega_\nu^2 = \begin{cases} 1, & \text{if } \nu\Delta t = \alpha \\ 0, & \text{otherwise} \end{cases} \tag{3.12}$$

indicate the times when the angular motion η_t is reflected by the boundaries $\eta = 0$ and $\eta = \alpha$, respectively. Correspondingly, the local times λ_t^1 and λ_t^2 can be approximated as

$$\lambda_t^n = \int_0^t d\lambda_t^n \approx \Delta t \sum_{\nu=0}^{\nu\Delta t \leq t} \Omega_\nu^n \quad n = 1, 2. \tag{3.13}$$

It is important to emphasize that the discrete motions (3.9) and (3.10) can be regarded as the Euler approximations to the solutions of the stochastic differential equations (3.2) and (3.7), but these approximations should by no means be considered as the sole methods of numerical simulation of the random motions ξ_t and η_t defined by (3.2), (3.7). Similarly, expressions (3.11) should not be regarded as the sole methods of computing the integrals from (3.6) which they indeed approximate. The situation here is reminiscent of the classical theory of integration, where the Riemann sums provide a handy interpretation of the integrals while the actual integration is more efficiently done either analytically on the basis of the Newton-Leibnitz formula or numerically by more sophisticated schemes, such as Gauss-Legendre or high-order Newton-Cotes formulas, for example.

Finally, to demonstrate the versatility of the probabilistic approach it is instructive to observe that the solution (3.6) of the problem (2.19)–(2.24) can be easily adjusted to the solution of the approximate problem obtained by stripping the complete (2.19) down to the Schrödinger (2.27) used in the parabolic equation method. More precisely, to do this the solution (3.6) itself does not require any modifications except that the radial random motion ξ_t should be replaced by the deterministic motion governed by the ordinary differential equation

$$\xi_0 = r \quad d\xi_t = ik(\xi_t)^2 dt \tag{3.14}$$

which may be considered as an approximation of the stochastic (3.2) in the case $k \gg 1$ assumed by the parabolic equation method. This approach may be used for further simplifications of the solution (3.6). Indeed, observing that the solution of (3.14) has the structure $\xi_t = r/(1 - ikrt)$, we evaluate the first integral from (3.6) as

$$\frac{ik}{2}\int_0^{\tau_\nu} \xi_s ds = \frac{1}{2}\int_0^{\tau_\nu} \frac{ikrds}{1 - ikrs} = -\frac{1}{2}\log(1 - ikr\tau_\nu) \tag{3.15}$$

and transform (3.6) to a form

$$u(r,\theta) = \mathbf{E}\left\{\sum_{\nu=1}^{\infty} \frac{\delta_\nu Q_\nu}{\sqrt{1 - ikr\tau_\nu}} \times \exp\left(ik\int_0^{\tau_\nu} \left[B_1 d\lambda_s^1 + B_2 d\lambda_s^2\right]\right)\right\} \tag{3.16}$$

which approximates (3.6).

IV. Dependence of the Probabilistic Solution on the Boundary Impedances

It is well known that the qualitative and quantitative properties of the solution of the diffraction problem (2.2)–(2.4) may significantly depend on the boundary conditions (2.4). Therefore, since the exact solution of this problem is delivered by the probabilistic expression (3.6), it is instructive to examine how the coefficients B_1 and B_2 of the boundary conditions affect the convergence of (3.6).

In the simplest case of Neumann boundary conditions on both faces of the wedge the coefficients in (2.4) vanish ($B_1 = B_2 = 0$) and the general solution (3.6) reduces to

$$u(r,\theta) = \mathbf{E}\left\{\sum_{\nu=1}^{\infty} \delta_\nu Q_\nu \exp\left(\frac{1}{2}\int_0^{\tau_\nu} ik\xi_t dt\right)\right\} \tag{4.1}$$

where ξ_t is the radial random motion controlled by (3.7) and τ_ν are the times when the angular random motion η_t controlled by (3.8) intersects one of the interfaces $\eta = \theta \pm \theta_0$.

As shown in [4] the radial random motion ξ_t is contained in the first quarter $0 \le \arg(\xi_t) < \pi/2$ and it drifts along the deterministic lines

$$\tilde{\xi}_t = \frac{i}{2k} + \frac{(1 + 2ikr)e^{-t/2}}{2kr + i(1 + 2ikr)e^{-t/2}} \tag{4.2}$$

toward the unreachable imaginary point $\xi_* = i/2k$, so that after an initial time the motion ξ_t eventually arrives nearby the point ξ_* which does not depend on the initial position $\xi_0 = r$. The described behavior of the motion ξ_t results in the estimate

$$\frac{1}{2}\int_0^{\tau_\nu} ik\xi_t dt \approx -\frac{\tau_\nu}{4} \quad \tau_\nu \to \infty \tag{4.3}$$

which insures the convergence of the probabilistic expression (4.1). Moreover, a more detailed analysis performed, for example, in [4] makes it possible to show that as $r \to \infty$, the left side of (4.3) is estimated as $O(1/\sqrt{r})$ which agrees with the expected rate of decay of the solution (4.1).

To estimate the convergence of the expression (3.6) in the case of nonvanishing coefficients B_1, B_2 we observe that due to the described behavior of the radial motion ξ_t the integral in the exponent in (3.6) is estimated as

$$\int_0^t ik\xi_s \left[\frac{1}{2}ds + B_1 d\lambda_s^1 + B_2 d\lambda_s^2\right] \approx -\frac{t}{2} - B_1\frac{\lambda_t^1}{2} - B_2\frac{\lambda_t^2}{2} \quad \tau_\nu \to \infty \tag{4.4}$$

where λ_t^1 and λ_t^2 are the local times defined by (3.13). Then, combining (4.4) with (3.6) we find that the convergence of (3.6) is guaranteed if and only if the impedances B_1 and B_2 obey the inequalities $\mathrm{Re}(B_1) \ge 0$ and $\mathrm{Re}(B_2) \ge 0$ which coincide with the conditions of solvability (2.5) of the considered problem of diffraction.

It is also instructive to observe that although the convergence of (3.6) is guaranteed by the inequalities $\mathrm{Re}(B_1) \ge 0$ and $\mathrm{Re}(B_2) \ge 0$, the rate of convergence and the magnitude of the resulting value of $u(r,\theta)$ significantly depend on the imaginary parts of the impedances and on the distance of the observation point from the faces $\theta = 0$ and $\theta = \alpha$ of the wedge.

To illustrate the last statement, consider first the case of a positive impedance $B_1 = |B_1|$. Then since the motion ξ_t is contained in the quarter plane $0 \le \arg(\xi_t) < \pi/2$ we obtain the bound $\pi/2 \le \arg(ik\xi_t B_1) < \pi$ which means that the presence of the factor $\exp\left(\int ikB_1\xi_t d\lambda_t^1\right)$ does not adversely affect the rate of convergence in (3.6) compared to the Neumann case when $B_1 = 0$. Similarly, in the case when B_1 is positive imaginary such that $B_1 = i|B_1|$, we have the bound $\pi \le \arg(ik\xi_t B_1) < 3\pi/2$ which also does not diminish the rate of convergence in (3.6). However, if $B_1 = -i|B_1|$, then $0 \le \arg(ik\xi_t B_1) < \pi/2$ which may reduce the rate of convergence in (3.6) and may significantly increase the magnitude of $u(r,\theta)$ represented by (3.6). Indeed, if $\theta \approx 0$, then the angular motion η_t starts bouncing from the face $\theta = 0$ very soon after

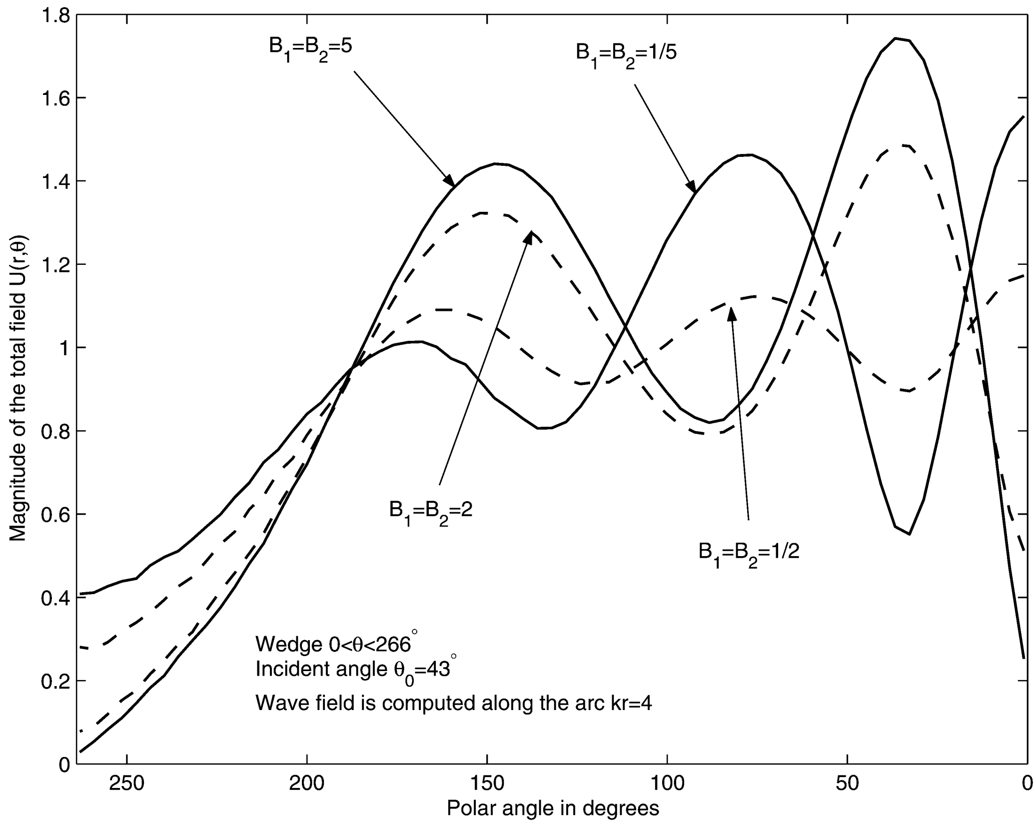

Fig. 2. Total fields in the wedge with positive impedances.

launching from $\eta_0 = \theta$, while the radial motion ξ_t is still located near the initial point $\xi_0 = r > 0$. As a result we obtain the estimate

$$\mathrm{Re}\left(\int_0^t ikB_1\xi_s d\lambda_s^1\right) \geq 0, \text{ if } t \ll 1 \tag{4.5}$$

which increases the magnitude of $u(r,\theta)$ compared to the reference case when $B_1 = 0$. Then, as time increases, ξ_t approaches the vicinity of the attracting point $\xi_* = i/2k$ and the estimate

$$\arg(ikB_1\xi_t) \approx \frac{\pi}{2} \quad t \gg 1 \tag{4.6}$$

secures convergence in (3.6). Similarly, if $\theta \not\approx 0$ then the estimate (4.6) becomes valid before the angular motion η_t hits the boundary $\theta = 0$, and in this case, the magnitude of $u(r,\theta)$ is not considerably affected in comparison to the Neumann case $B_1 = 0$.

It is clear that the above reasoning can also be applied to the second face $\theta = \alpha$ and that these arguments provide a "probabilistic" explanation of such phenomena as surface waves which are known to be generated on either of the faces $\theta = 0$ or $\theta = \alpha$ if they have an imaginary negative impedance.

V. Example

To illustrate the suitability of the obtained probabilistic solution for calculations of the problem of diffraction by a wedge with different impedances on its faces we conducted a series of numerical simulations for the wedge $\Gamma(266°)$ exposed to the incident plane wave $U_0(r,\theta) = e^{-ir\cos(\theta-\theta_0)}$ arriving along the ray $\theta_0 = 43°$. In this configuration, which was selected for comparability with [15], the shadow domain $223° < \theta < 266°$ is illuminated only by the diffracted waves, the sector $137° < \theta < 223°$ is open to the incident and diffracted waves, and the domain $0 < \theta < 137°$ is exposed to the incident, reflected and diffracted waves.

Fig. 2 shows the magnitudes of the total wave fields in the wedge with real positive impedance $B_1 = B_2 = B$ ranging from $B = 1/5$ to $B = 5$. The dashed lines correspond to the impedances $B = 1/2$ and $B = 2$, while the solid lines correspond to the impedances $B = 1/5$ and $B = 5$ which were considered in [15] by two conventional methods, including the Maliuzhinets' closed-form solution. Since the computations reported in [15, Fig. 1] were made along the arc $kr = 4$, we set $k = 1$ and $r = 4$, which allows us to show that the numerical results obtained by the probabilistic method agree with those delivered by other more traditional techniques. The results provided by both methods appear to be identical.

Fig. 3 shows results of numerical simulations in the cases when the face $\theta = 266°$ has a fixed positive impedance $B_2 = 5$ while the face $\theta = 0°$ has a purely imaginary impedance B_1 which has three different values $B_1 = -i/3$, $B_1 = 0$ and $B_1 = i/3$. The solid line corresponds to the Neumann boundary condition of $B_1 = 0$ and it appears to decay with the expected rate $O(1/\sqrt{kr})$. The dashed line corresponds to the imaginary negative coefficient $B_1 = -i/3$ which supports the generation of surface waves, and wherefore, as expected, this line does not have a visible decay as kr increases. Finally, the dash-dotted line corresponds to the imaginary positive coefficient $B_1 = i/3$

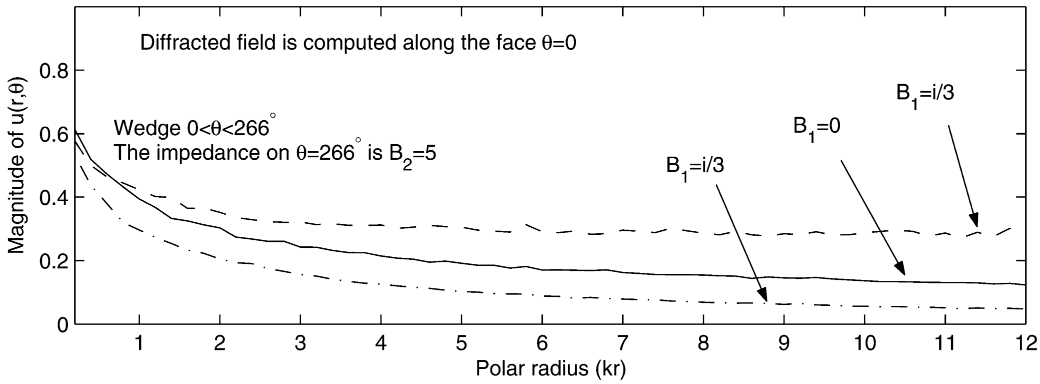

Fig. 3. Diffracted fields in the wedge with an imaginary impedance on $\theta = 0°$.

which does not support the generation of surface waves. Fig. 3 clearly confirms the quantitative analysis from the previous section predicting the mechanism of formation of surface waves.

All of the numerical results presented here were obtained by the averaging of 2000 discrete random walks (3.9)–(3.11) with the time increment $\Delta t = 0.01$. The computations which took a few seconds per point were carried out on a 900 Mhz Notebook PC using a simple MATLAB[1] code. We include this code in the Appendix just to illustrate its amazingly short length and also to enable the reader to easily compute other cases of interest.

VI. Conclusion

The version of the random walk method developed in [4]–[6] and employed here results in a simple exact solution of the canonical problem of diffraction of a plane incident wave by a wedge with different impedance boundary conditions on its faces. The obtained solution (3.6) is exact in the same rigorous sense as $f = \sin(x)$ is an exact solution of the equation $f''+f = 0$. Of course, in most cases the mathematical expectation (3.6) can not be computed without numerical error, but computation of $\sin(x)$ is also impossible without numerical error.

As with analytic or asymptotic methods, the probabilistic solutions provided by the random walk method are local in the sense that they make it possible to compute functions of interest at individual points without computing them on dense meshes. Moreover, such solutions admit simple perfectly scalable implementations with practically unlimited capability for parallel processing, and such solutions admit meaningful physical interpretations which do not contradict, but compliment, elementary models of wave propagation employed by ray theory. All of these features together make the random walk method attractive for the analysis of wave propagation, and we hope that the present paper will stimulate its further development and practical use.

Appendix
Matlab Code

```
function[u] = point(r0, f0, a0, M1, M2, alpha, k, ep, N)
 ik = i.*k;
 Cr = (sin(a0) - M1)./(sin(a0) + M1);
 f = repmat(f0, N, 1);
 r = repmat(r0, N, 1);
Q = ones(N, 1);
[u, J1, J2] = deal(0, (f > pi - a0), (f > pi + a0));
while isempty(r)
 ds = ep./abs(r);
 wr = ds.*sign(rand(length(r), 1) - 0.5);
 wf = ds.*sign(rand(length(r), 1) - 0.5);
 f = f + wf;
 [Ja, Jb] = deal(f > pi - a0, f > pi + a0);
 u = u + sum(Q.*(Cr.*(J1 - Ja) + (J2 - Jb)))./N;
 [I1, I2] = deal(f < 0.02, f > alpha + 0.02);
 ds(ds > ep) = ep;
 f(I1) = ds(I1);
 f(I2) = alpha - ds(I2);
 Q(I1) = Q(I1).*exp(ik.*ds(I1).*M1.*r(I1));
 Q(I2) = Q(I2).*exp(ik.*ds(I2).*M2.*r(I2));
 Q = Q.*exp(0.25.*ik.*r.*wr.^2);
 r = r.*(1 + wr + (0.5 + ik.*r).*wr.^2);
 Q = Q.*exp(0.25.*ik.*r.*wr.^2);
 I = abs(Q) > 1e - 3;
 [f, r, Q, J1, J2] = deal(f(I), r(I), Q(I), Ja(I), Jb(I));
end
```

Input parameters `r0`, `f0` are the polar coordinates of the observation point; `a0` is the incidence angle; `B1` $= B_1$ and `B2` $= B_2$ are the impedances on $\theta = 0$ and $\theta = \alpha$; `alpha` is the wedge angle; `k` $= k$ is the wave number; `ep` $= \sqrt{\Delta t}$ is the spatial step; and `N` is the number of averaged random walks. The output parameter `u` $= u(r, \theta)$ is the amplitude of the diffracted field, and `U` $= U(r, \theta)$ is the total field.

[1]MATLAB is a registered trademark of The MathWorks, Natick, MA.

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