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Novel solutions of the Helmholtz equation and their application to diffraction

By Bair V. Budaev* and David B. Bogy

Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

This paper presents a series of novel representations for the solutions of the Helmholtz equation in a broad class of wedge-like domains including those with curvilinear, non-flat faces. These representations are obtained by an original method which combines ray theory with the probabilistic approach to partial differential equations and uses a specific technique to deal with a need for analytical continuation of the specified boundary function. The main results are reminiscent of the standard Feynman–Kac formula but differ in that the averaging over solutions of stochastic differential equations is replaced by averaging over the trajectories of a new two-scale random motion introduced here.

The paper focuses on the development of the solutions and for this reason it includes only a brief outline of numerous applications, consequences, extensions and variations of the method, which include, but are not limited to, problems of diffraction and scattering, problems in three-dimensional domains and problems of wave propagation in non-homogeneous media.

Keywords: wave propagation; diffraction; Brownian motion; stochastic process

1. Introduction

It is well known (Dynkin 1965; Simon 1979; Freidlin 1985) that the solutions of the boundary value problem

$$\nabla^2 U(\boldsymbol{x}) - k^2 U = 0, \qquad U|_{\partial G} = f, \tag{1.1}$$

in an N-dimensional domain G can be represented as the mathematical expectation

$$U(\boldsymbol{x}) = \boldsymbol{E}\left\{ f(\boldsymbol{\xi}_\tau)\mathrm{e}^{-k^2\tau} \right\}, \tag{1.2}$$

computed over the trajectories of the N-dimensional Brownian motion $\boldsymbol{\xi}_t$ which starts at time $t=0$ from the observation point $\boldsymbol{\xi}_0 = \boldsymbol{x}$ and stops at the exit time $t=\tau$, defined as the first time when $\boldsymbol{\xi}_t$ hits the boundary ∂G. This simple expression, widely known as the Feynman–Kac formula (Feynman 1942; Kac 1949), admits generalizations to the problem (1.1) with more general boundary conditions $aU + b\partial U/\partial n|_{\partial G} = f$, which makes it possible to obtain explicit solutions corresponding to virtually arbitrary non-constant coefficients $k=k(\boldsymbol{x})$, $a=a(\boldsymbol{x})$ and $b=b(\boldsymbol{x})$, as well as to domains of virtually arbitrary shape. Such solutions have

* Author for correspondence (budaev@berkeley.edu).

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been known theoretically for decades (Philips & Wiener 1923; Courant *et al.* 1928; Petrovsky 1934; Kakutani 1944; Khasminskii 1960; Daletsky 1962; Keller & McLaughlin 1975), and in recent years they have attracted increasing practical interest (Pobedrya & Chistyakov 1988; Hunt *et al.* 1995; Asmussen 1998; Busnello 1999; Shia & Hui 2000; Chati *et al.* 2001; Jourdain & Méléard 2004) owing to their suitability for parallel processing and for the analysis of problems with uncertain data.

The advantages of the probabilistic approach might be even more beneficial for problems of wave propagation, which are customarily formulated in unbounded domains and, therefore, are not perfectly suited for direct numerical methods. However, straightforward application of the Feynman–Kac formula to the Helmholtz equation $\nabla^2 U + k^2 U + F = 0$ is not possible, because it leads to a representation like equation (1.2) but with the decreasing exponent $e^{-k^2\tau}$ replaced by an increasing exponent $e^{k^2\tau}$ in equation (1.2), which may cause divergence of the resulting expression. Although the Helmholtz equation $\nabla^2 U + k^2 U = 0$ cannot be addressed by the probabilistic method directly, it is nevertheless possible to apply this technique indirectly (Buslaev 1967; Bal *et al.* 1999, 2000; Schlottmann 1999; Galdi *et al.* 2000; Nevels *et al.* 2000; Fishman 2002; Budaev & Bogy 2004, 2005*a*,*c*, 2006). One of the indirect probabilistic approaches to the Helmholtz equation follows closely the ray method (Keller 1958) and seeks the solution in the product form $U(\boldsymbol{x}) = u(\boldsymbol{x})e^{ikS(\boldsymbol{x})}$. Then, the Helmholtz equation splits into the eikonal equation $(\boldsymbol{\nabla} S)^2 = k^2$, which can be solved by the Hamilton–Jacobi method, and the transport equation

$$\frac{1}{2}\nabla^2 u + ik\boldsymbol{\nabla} S \cdot \boldsymbol{\nabla} U + \frac{ik}{2}(\nabla^2 S)u = 0, \tag{1.3}$$

which can often be solved on the basis of the Feynman–Kac formula.

The outlined method has already been successfully applied to the notoriously difficult problems of diffraction by a plane angular sector (Budaev & Bogy 2004) and the problem of diffraction by an infinite wedge with anisotropic face impedances (Budaev & Bogy 2005*a*). Most recently, the random walk method was applied to the even more complex two-dimensional problem of diffraction by an arbitrary convex polygon with face-wise constant impedances (Budaev & Bogy 2005*c*, 2006). In all of these cases, the probabilistic solutions are transparent, and they admit numerical simulation by simple and short algorithms.

Here, we complete the development started in Budaev & Bogy (2005*c*, 2006) and find a new class of explicit probabilistic representations for the solutions of the Helmholtz equation in the wedge-like domains described as

$$\mathfrak{G}: \quad r > 0, \quad \alpha_1(r) < \theta < \alpha_2(r), \tag{1.4}$$

where (r, θ) are standard polar coordinates. The obtained representations look similar to the standard Feynman–Kac formula, but they involve averaging over trajectories of random motions of a specific type, which is introduced here.

The paper presents a practically self-contained derivation of the main results, and it outlines possible ways to apply them to selected typical problems of diffraction. However, we do not attempt to discuss all of the consequences, extensions and variations of the approach, which are so numerous that their exploration may overshadow the presentation of the fundamental developments and, as a result, undermine the foundation for their future development.

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The main technical tool of the paper is developed in §2 where the Helmholtz equation is considered in a wedge $\mathfrak{G}_+ = \{r, \theta : r > 0, 0 < \theta < \alpha \le 3\pi/2\}$ with the face $\theta = \alpha$ grounded in the sense that the Dirichlet boundary condition $U(r, \alpha) = 0$ must be valid for any wave field in $\mathfrak{G}_+$. In this idealized setting, we derive an important formula (2.31) which opens the way to the probabilistic representation (2.32) for the wave fields in the domain $\mathfrak{G}_+$ with arbitrary boundary values. This representation is a considerable leap forward compared to the similar representations employed in Budaev & Bogy (2004, 2005*a*,*c*), which were severely restricted by the requirement there of analytic boundary functions.

In §3, the auxiliary results of §2 are gradually extended to several representations of wave fields in the wedge defined by the inequalities $r > 0$, $\alpha_1 < \theta < \alpha_2$. The first representation (3.4)–(3.7) is just a straightforward generalization of the expression (2.32) to a two-face wedge. Then, recursively applying formulae (3.4)–(3.7), we transform them into the much more versatile representation (3.24)–(3.27), which is the main result of the paper.

In §4, we demonstrate that the representation (3.24)–(3.27) remains valid in a broad class of domains of the type (1.4) with curvilinear boundaries of virtually arbitrary shape. This observation immediately lifts the random walk method from a conventional method suitable to solve problems with relatively simple geometries to a universal approach to problems of wave propagation.

Finally, §5 illustrates how the obtained representations of wave fields can be applied to a typical problem of diffraction, and §6 discusses a few of the most obvious generalizations of the obtained results and outlines directions for future work.

2. Representations of wave fields in a wedge with a 'grounded' face

Here, we obtain specific representations of the wave field $U(r, \theta)$ which is defined in a wedge

$$\mathfrak{G}_+ = \{r, \theta : r > 0, 0 < \theta < \alpha\}, \quad (\alpha \le 3\pi/2), \tag{2.1}$$

which has the face $\theta = \alpha$ grounded in the sense that the boundary conditions $U(r, \alpha) = 0$ must be valid. Besides that, the field $U(r, \theta)$ is supposed to obey the Helmholtz equation $\nabla^2 U + k^2 U = 0$, the boundary condition $U(r, 0) = F(r)$ and the radiation condition $U(r, \theta)\mathrm{e}^{-\mathrm{i}kr} = o(1)$ as $r \to \infty$.

We are looking for representations of $U(r, \theta)$ in the product form

$$U(r, \theta) = u(r, \theta)\mathrm{e}^{\mathrm{i}kr}, \tag{2.2}$$

where the amplitude $u(r, \theta)$ vanishes at infinity and satisfies the transport equation

$$\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2} + r\left(\frac{1}{2} + \mathrm{i}kr\right)\frac{\partial u}{\partial r} + \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\mathrm{i}kr}{2}u = 0, \tag{2.3}$$

accompanied by the boundary conditions

$$u(r, 0) = f(r, 0), \qquad u(r, \alpha) = f(r, \alpha), \tag{2.4}$$

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where

$$f(r,\theta) = \begin{cases} F(r)\mathrm{e}^{-\mathrm{i}kr}, & \text{if} \quad \theta = 0, \\ 0, & \text{if} \quad \theta = \alpha, \end{cases} \tag{2.5}$$

is a 'boundary function' defined only on the faces $\theta=0$ and α of the wedge $\mathfrak{G}_+$. For definiteness, we assume that both $f(r, 0)$ and $f(r, \alpha)$ are piece-wise continuous functions of the radius r.

The problem (2.3) and (2.4) is well posed and has a unique solution determined by the boundary function $f(r, \theta)$ restrained by rather general conditions of regularity. However, particular representations of the solution may require different additional assumptions on the properties of $f(r, \theta)$.

(*a*) *The first representation of* $U(r,\theta)$ *in* $\mathfrak{G}_+$

Let the boundary function $f(\xi, \eta)$ admit analytic continuation to the first quarter $0 \le \arg \xi < \pi/2$ of the complex plane. Then, applying the random walk method similar to that from Budaev & Bogy (2004, 2005*c*), we get the representation

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\{f(\xi_\tau,\eta_\tau))\mathrm{e}^{S(t)}\} \qquad S(t) = \frac{1}{2}\int_0^t \mathrm{i}k\xi_t \mathrm{d}t, \tag{2.6}$$

where $\mathbb{E}$ denotes the mathematical expectation computed over the trajectories of the radial and angular motions ξ_t and η_t that are controlled by the stochastic equations

$$\mathrm{d}\xi_t = \xi_t \mathrm{d}w_t^1 + \xi_t\left(\frac{1}{2} + \mathrm{i}k\xi_t\right)\mathrm{d}t, \qquad \mathrm{d}\eta_t = \mathrm{d}w_t^2, \tag{2.7}$$

driven by standard one-dimensional Brownian motions w_t^1 and w_t^2. The process $P_t=(\xi_t, \eta_t)$ starts from the initial position $P_0=(r, \theta)$ and stops at the exit time $t=\tau$, defined as the first time when P_t eventually hits one of the faces $\eta_t=0$ or α.

It is easy to see that for any $t>0$, the angular motion η_t is contained in the segment $[0,\alpha]$ and the radial motion ξ_t runs in the first quarter $0<\arg(\xi)<\pi/2$ of the complex plane drifting to an unreachable point $\xi=\mathrm{i}/2k$. Indeed, if $\arg(\xi_t)=0$, then $\mathrm{Im}(\mathrm{d}\xi_t)>0$ which implies that the motion ξ_t cannot penetrate to the fourth quarter. Similarly, if $\arg(\xi_t)=\pi/2$, then $\mathrm{Re}(\mathrm{d}\xi_t)=0$ which implies that the motion cannot cross the imaginary axis, and therefore it can only stay in the first quarter. Such localization of ξ_t ensures that $S(t)$ has a positive imaginary component which increases as t increases and which, therefore, improves the convergence in equation (2.6). On the other hand, the fact that ξ_t runs in the complex space implies that the boundary function $f(\xi, \eta)$ must be analytic in the domain $0<\arg(\xi)<\pi/2$ and that it should not grow too fast there along the random walk.

Equations (2.7) belong to the well-studied class of Ito's stochastic differential equations (Ito & McKean 1965), and the theory of such equations implies that the two-dimensional motion (ξ_t, η_t) can be considered as the discrete random walk

$$(r,\theta) \equiv (\xi_0,\eta_0) \to (\xi_{\mathrm{d}t},\eta_{\mathrm{d}t}) \to \cdots \to (\xi_t,\eta_t) \to (\xi_{t+\mathrm{d}t},\eta_{t+\mathrm{d}t}) \to \cdots, \tag{2.8}$$

with infinitesimally small steps determined by the stochastic rules

$$\xi_{t+\mathrm{d}t} = \xi_t \mathrm{e}^{\pm\sqrt{\mathrm{d}t}+\mathrm{i}k\xi_t\mathrm{d}t}, \qquad \eta_{t+\mathrm{d}t} = \eta_t \pm \sqrt{\mathrm{d}t}, \tag{2.9}$$

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where all four combinations of signs are equally probable. This interpretation of the motion (ξ_t, η_t) is convenient because it treats the components ξ_t and η_t as one-dimensional random motions independent of each other. However, in some cases, it is beneficial to consider (ξ_t, η_t) as a random walk (2.8) with the steps

$$(\xi_{t+\mathrm{d}t}, \eta_{t+\mathrm{d}t}) = \begin{cases} \left(\xi_t \mathrm{e}^{\pm\sqrt{2\,\mathrm{d}t}+\mathrm{i}k\xi_t\mathrm{d}t}, \eta_t\right), & \text{with probability} \quad 1/2, \\ (\xi_t, \eta_t \pm \sqrt{2\,\mathrm{d}t}), & \text{with probability} \quad 1/2, \end{cases} \tag{2.10}$$

which are either purely radial or purely angular.

Although both of the schemes (2.9) and (2.10) define the same random motion, there are technical differences between them. Thus, in equation (2.9), every step consists of the simultaneous jumps in both the radial and the angular directions, while in equation (2.10) the jumps in these directions are separated from each other.

Properties of stochastic differential equations and their interplay with partial differential equations are thoroughly discussed in an immense literature. For example, theoretical aspects of these topics are discussed in Dynkin (1965), Ito & McKean (1965), Dynkin & Yushkevich (1969), Friedman (1971), Hida (1975), Simon (1979), Freidlin (1985) and Schulman (1996), while Kloeden & Platen (1992), Higham (2001) and Milstein & Tretyakov (2004) are focused on numerical aspects.

(*b*) *Representation of U(r,θ) in $\mathfrak{G}_+$ with a two-piece constant boundary condition*

Let U(r,θ) be a wave field in the domain $\mathfrak{G}_+$ with the boundary function

$$f(r,\theta) = \begin{cases} f_0, & \text{if} \quad \theta = 0, \quad r \le L, \\ f_1 & \text{if} \quad \theta = 0, \quad r > L, \\ 0 & \text{if} \quad \theta = \alpha, \end{cases} \tag{2.11}$$

where f_0, f_1 and $L>0$ are given constants.

It is clear that the segment $I_0(L) \equiv \{\theta=0,\ 0<r\le L)$ and the half-line $I_1(L) \equiv \{\theta=0,\ r>L)$, which comprise the face $\theta=0$ of $\mathfrak{G}_+$, can be characterized by the analytic equations

$$I_0 \equiv I_0(L): \qquad \mathcal{P}(r,\theta,p) = p - r, \quad \forall\ p \ge L, \tag{2.12}$$

$$I_1 \equiv I_1(L): \qquad \mathcal{P}(r,\theta,p) = r - p, \quad \forall\ p < L, \tag{2.13}$$

where

$$\mathcal{P}(r,\theta,p) = \sqrt{r^2 + p^2 - 2rp\cos\theta} \equiv \sqrt{(r\mathrm{e}^{\mathrm{i}\theta} - p)(r\mathrm{e}^{-\mathrm{i}\theta} - p)}. \tag{2.14}$$

Indeed, applying the cosine theorem to the triangle ΔOPQ shown in figure 1, we see that $r_p = \mathcal{P}(r,\theta,p)$ and that this triangle collapses to the segment $(OP]$ in the case when $r_p = p - r$. We conclude that if the vertex Q belongs to the segment $(0, L]$, then the equality $r_p = p - r$ must remain valid for all $p \ge L$.

It is worth mentioning that equations (2.12) and (2.13) provide analytic continuation of the real one-dimensional intervals I_0 and I_1 to two-dimensional surfaces in the complex space formed by the pairs (r, θ), where both of the components are considered as complex numbers.

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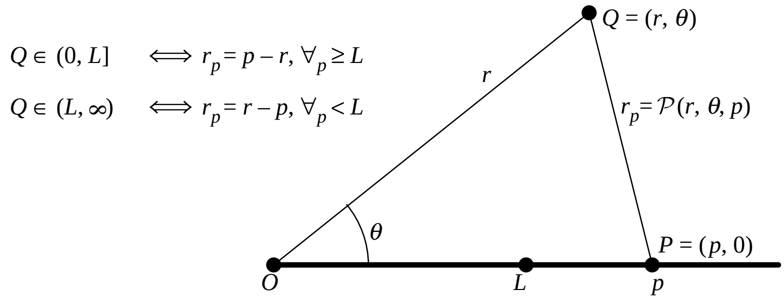

Figure 1. Analytic description of the segment $(0,\ L]$.

Applying the same variation of the random walk method as employed above, we obtain the representation

$$U(r,\theta)=\mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{f(\rho_\tau,\eta_\tau)\exp\left(\frac{1}{2}\int_0^t \mathrm{i}k\xi_t\mathrm{d}t\right)\right\},\qquad \rho_\tau=\begin{cases}L/2 & \text{if}\quad \nu(\tau)=0,\\ 2L, & \text{if}\quad \nu(\tau)=1,\end{cases}\tag{2.15}$$

where the mathematical expectation is computed over the trajectories of the stochastic processes ξ_t and η_t which are described by (2.7) and are stopped at the exit time τ defined as the first time when $\eta_t=0$ or α. It is obvious that if $\eta_\tau=0$ then the exit point $P_\tau=(\xi_\tau,\ \eta_\tau)$ belongs either to I_0 or to I_1, and elementary analysis shows that it belongs to $I_{\nu(\tau)}$ with the index

$$\nu(\tau)=\begin{cases}0, & \text{if } \mathcal{P}(\xi_\tau,\eta_\tau,p)=p-\xi_\tau,\quad \forall\ p\geq L,\\ 1, & \text{if } \mathcal{P}(\xi_\tau,\eta_\tau,p)=\xi_\tau-p,\quad \forall\ p< L,\end{cases}\tag{2.16}$$

which depends on the entire trajectory of the motion $(\xi_t,\ \eta_t)$, rather than on its final position.

To compute $\nu(\tau)$ by the formula (2.16), it is necessary to trace the value of the radical

$$\Lambda_t\equiv\mathcal{P}(\xi_t,\eta_t,p)=\sqrt{(\xi_t\mathrm{e}^{\mathrm{i}\eta_t}-p)(\xi_t\mathrm{e}^{-\mathrm{i}\eta_t}-p)},\tag{2.17}$$

along the trajectory of the motion P_t. This procedure provides a clear and unambiguous definition of $\nu(\tau)$, but it hides a relationship between $\nu(\tau)$ and the global structure of the trajectory of the motion P_t, which becomes transparent from a closer look at equation (2.17).

Consider first the radical $\Xi(\zeta)=\sqrt{\zeta-p}$, where $p>0$ is a positive constant and ζ is a complex variable. It is well-known that $\Xi(\zeta)$ is an analytic function defined on a two-sheet Riemann surface $\tilde{\mathbb{C}}_p$ consisting of two identical copies $\mathbb{C}_p^1$ and $\mathbb{C}_p^2$ of the complex plane that are slit along the ray $[p,\infty)$ and are attached to each other in such a way that the upper side of the slit of $\mathbb{C}_p^1$ is identified with the lower side of the slit of $\mathbb{C}_p^2$ and vice versa. The values of $\Xi(\zeta)$ on the sheets $\mathbb{C}_p^1$ and $\mathbb{C}_p^2$ are defined as

$$\Xi(\zeta)=\begin{cases}\sqrt{|\zeta-p|}\quad \mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(\zeta-p)/2}, & \text{if } \zeta\in\mathbb{C}_p^1,\\ \sqrt{|\zeta-p|}\quad \mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(\zeta-p)/2+\mathrm{i}\pi}, & \text{if } \zeta\in\mathbb{C}_p^2,\end{cases}\tag{2.18}$$

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where $\sqrt{|\zeta-p|}\geq 0$ is the arithmetic value of the square root of the non-negative number $|\zeta-p|\geq 0$, while $\mathrm{Arg}(\zeta-p)\in[0,2\pi)$ is the principle value of the argument of the complex number $(\zeta-p)$. It is obvious that $\Xi(\zeta)$ is a continuous function on the two-sheet manifold $\tilde{\mathbb{C}}_p$ and that it admits analytical continuation along any path ζ_t, $t\geq 0$, which is continuous on $\tilde{\mathbb{C}}_p$.

Let ζ_t be a continuous path on the two-sheet surface $\tilde{\mathbb{C}}_p$ originating from the point ζ_0 located on the first sheet $\mathbb{C}_p^1$. Then, the value of $\Xi(\zeta_0)$ is represented by the first option in equation (2.18) and the values of $\Xi(\zeta_0)$ with $t>0$ can be computed either by the analytic continuation or by the formula (2.18), but in the second case it is necessary to identify which of the sheets $\mathbb{C}_p^1$ or $\mathbb{C}_p^2$ contains ζ_t. The answer to the last question can be obtained by a continuation along the trajectory of ζ_t, and elementary analysis shows that if ζ_t (more precisely, if its projection to the standard complex plane $\mathbb{C}$) intersects the ray $[p,\infty)$ an even number of times, then $\zeta_t\in\mathbb{C}_p^1$, and $\zeta_t\in\mathbb{C}_p^2$ if the number of such intersections is odd.

The outlined method of analytical continuation of an elementary radical Ξ_t from equation (2.18) can now be used for the evaluation of a more complex radical Λ_t from equation (2.17), which can be represented as a product

$$\Lambda_t=\Xi_t(\zeta_t^+)\Xi(\zeta_t^-), \tag{2.19}$$

where ζ_t^+ and ζ_t^- are complex-valued motions

$$\zeta_t^+=\xi_t \mathrm{e}^{\mathrm{i}\eta_t}, \qquad \zeta_t^-=\xi_t \mathrm{e}^{-\mathrm{i}\eta_t}, \tag{2.20}$$

determined through the random motions ξ_t and η_t controlled by stochastic equations (2.7).

Assume, for definiteness, that $\zeta_0^{\pm}\in\mathbb{C}_p^1$, so that $\Xi(\zeta_0^{\pm})$ have the values

$$\Xi(\zeta_0^{\pm})=\sqrt{|r\mathrm{e}^{\pm\mathrm{i}\theta}-p|}\exp\left[\frac{\mathrm{i}}{2}\mathrm{Arg}(r^{\pm\mathrm{i}\theta}-p)\right]. \tag{2.21}$$

Then, for any $p>0$ and $t>0$, the radical $\Xi(\zeta_t^+)$ can be computed by the similar formula

$$\Xi(\zeta_t^+)=\sqrt{|\zeta_t^+-p|}\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(\zeta_t^+-p)}, \quad \forall\, p>0, \tag{2.22}$$

which means that ζ_t^+ is always located on the first sheet $\mathbb{C}_p^1$ of the surface $\tilde{\mathbb{C}}_p$. Indeed, since the motions ξ_t and η_t start from the real positive values $\xi_0=r$ and $\eta_0=\theta$, and since these motions are contained in the domains $0\leq\arg(\xi_{\mathrm{t}})<\pi/2$ and $0\leq\eta_{\mathrm{t}}<\alpha\leq 3\pi/2$, the complex variable $\zeta_t^+=\xi_t\mathrm{e}^{\mathrm{i}\eta_t}$ is contained in the domain $0\leq\arg(\zeta_t)<2\pi$ which coincides with the sheet $\mathbb{C}_p^1$. Therefore, $\Xi(\zeta_t^+)$ should be computed by the first line in equation (2.18) which gives exactly equation (2.22).

To compute the second factor $\Xi(\zeta_t^-)$ from equation (2.17), we need to trace the trajectory of the point $\zeta_t^-=\xi_t\mathrm{e}^{-i\eta_t}$, which starts from the position $\zeta_0^-=r\mathrm{e}^{-\mathrm{i}\theta}\in\mathbb{C}_p^1$ and stops at the point $\zeta_\tau^-=\xi_\tau$ located in the first quarter $0<\arg(\zeta_\tau^-)<\pi/2$. Taking into account the restraints $0\leq\arg(\xi_{\mathrm{t}})<\pi/2$ and $0\leq\eta_{\mathrm{t}}<3\pi/2$, we see that the trajectory of ζ_t^- is contained in the domain $-3\pi/2<\arg(\zeta_t^-)<\pi/2$ and, therefore, it never intersects the positive part of the imaginary axis. This information about the continuous motion ζ_t^- guarantees that it crosses the ray $\arg(\zeta)=0$ an odd number of times, as illustrated in figure 2.

Let $\mathfrak{Z}_\tau=\{p_n\}$ be the set of all positive points $p_{\mathrm{n}}>0$ where the trajectory of ζ_t^- on the time-interval $0\leq t\leq\tau$ intersects the ray $\arg(\zeta)=0$, and let $N(p,\mathfrak{Z}_\tau)$ be the number of points of the set $\mathfrak{Z}_\tau$ located to the left of p. Then, applying the above-discussed rule of analytical continuation of the radical $\Xi(\xi_t^-)$, we readily derive

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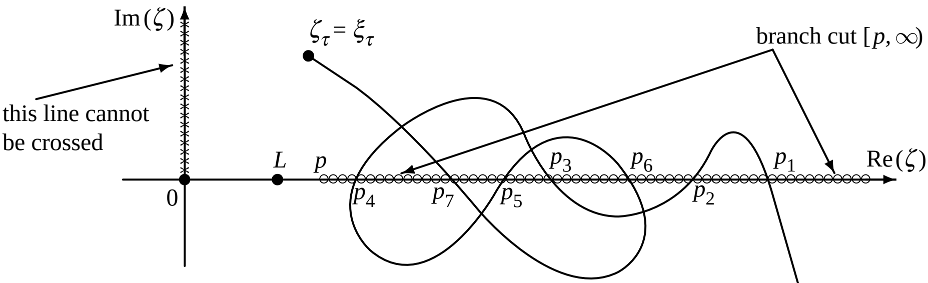

$\mathfrak{Z}_\tau = \{p_1, p_2, \ldots, p_7\}, \quad \inf\{\mathfrak{Z}_\tau\} = p_4 > L \;\Rightarrow\; (\xi_\tau, \eta_\tau) \in I_0$

Figure 2. Trajectory of $\zeta_t = \xi_t \mathrm{e}^{-\mathrm{i}\eta_t}$ and determination of $\rho_\tau = \inf\{p_n\}$.

that the location of ζ_τ^- is defined by the formulae

$$\zeta_\tau^- \in \begin{cases} \mathbb{C}_p^1, & \text{if} \quad N(p, \mathfrak{Z}_\tau) \quad \text{is even}, \\ \mathbb{C}_p^2, & \text{if} \quad N(p, \mathfrak{Z}_\tau) \quad \text{is odd}, \end{cases} \tag{2.23}$$

and, therefore, the radical $\Xi(\xi_t^-)$ has the value

$$\Xi(\zeta_\tau^-) \equiv \Xi(\xi_\tau) = \begin{cases} \sqrt{|\xi_\tau - p|}\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(\xi_\tau - p)/2}, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is even}, \\ \sqrt{|\xi_\tau - p|}\mathrm{e}^{\mathrm{i}\,\mathrm{Arg}(\xi_\tau - p)/2 + \mathrm{i}\pi}, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is odd}, \end{cases} \tag{2.24}$$

which is completely determined by the disposition of the parameter p with respect to the set $\mathfrak{Z}_\tau$ and does not depend on other details of trajectory of ζ_t^-. Indeed, if $N(p, \mathfrak{Z}_\tau)$ is even, then ζ_t^- intersects the branch cut $[p, \infty)$ an odd number of times (because the number of intersections with the ray $\theta = 0$ is odd), and this leads to the first line of equation (2.23). Otherwise, if $N(p, \mathfrak{Z}_\tau)$ is odd, then the trajectory of ζ_t has an even number of intersections with the cut $[p, \infty)$ and this leads to the second option of equation (2.23).

Finally, combining equation (2.17) with equations (2.22) and (2.24), we get

$$\mathcal{P}(\xi_\tau, \eta_\tau, p) = \begin{cases} \xi_\tau - p, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is even}, \\ p - \xi_\tau, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is odd}, \end{cases} \tag{2.25}$$

and combining equation (2.25) with equations (2.12) and (2.13), we arrive at the remarkable formula

$$\nu(\tau) = \begin{cases} 0, \text{ if } \inf\{\mathfrak{Z}_\tau\} \le L, \\ 1, \text{ if } \inf\{\mathfrak{Z}_\tau\} > L, \end{cases} \tag{2.26}$$

which makes it possible to evaluate equation (2.15) by tracing the intersections of the trajectory of $\zeta_t = \xi_t \mathrm{e}^{-\mathrm{i}\eta_t}$ with the positive semi-axis, but without tracing the radical (2.17).

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(*c*) *Representation of $U(r, \theta)$ in $\mathfrak{G}_+$ with arbitrary boundary values*

Formulae (2.6) and (2.15) represent the field $U(r, \theta)$ in the special cases when the boundary function $f(r, \theta)$ is either analytic in the first quarter or when it is a piece-wise constant on $\theta=0$ with only one jump. To obtain the representation of $U(r, \theta)$ corresponding to an arbitrary boundary function $f(r, \theta)$, we first assume that $f(r, \theta)$ has a piece-wise constant structure

$$f(r,\theta) = \begin{cases} f_n, & \text{if } \theta = 0, \quad r \in I_n = (L_n, L_{n+1}], \quad n \geq 0, \\ 0, & \text{if } \theta = \alpha, \end{cases} \tag{2.27}$$

where $\{L_n\}$ is a monotonically increasing sequence with the first element L_0 and f_n are some constants.

To employ the technique from the previous subsection, we observe that the intervals I_n can be represented as $I_n = I(L_{n+1}) \backslash I(L_n)$, where $I(L)$ is the domain in the space (r, θ) defined as

$$I(L): \quad \mathcal{P}(r, \theta, p) = p - r, \quad \forall\, p \geq L. \tag{2.28}$$

Then, the problem (2.3) and (2.4) can be considered in the complex domain bounded by the junction $\bigcup I_n$, and the random walk method leads to the representation

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(L_{\nu(\tau)+1}, \eta_\tau)\exp\left(\frac{1}{2}\int_0^t \mathrm{i}k\xi_t \mathrm{d}t\right)\right\}, \tag{2.29}$$

where the mathematical expectation is computed over the trajectories of the stochastic processes ξ_t and η_t, which are described by equation (2.7), and are stopped at the exit time τ defined as the first time when $\eta_t=0$ or α. It is easy to see that if $\eta_\tau=0$ then the exit point $P_\tau = (\xi_\tau, \eta_\tau)$ belongs to the interval $I_{\nu(\tau)}$ with the index

$$\nu(\tau) = n, \quad \text{if} \quad L_n < \inf\{\mathfrak{Z}_\tau\} \leq L_{n+1}, \tag{2.30}$$

where $\mathfrak{Z}_\tau$ is the full intersection of the trajectory of $\zeta_t = \xi_t \mathrm{e}^{-\mathrm{i}\eta_t}$ with the semi-axis $\arg(\zeta)=0$. Comparison of the last formula with equation (2.27) shows that if the point

$$\rho_\tau = \inf\{\mathfrak{Z}_\tau\} \tag{2.31}$$

belongs to the interval I_n, then $\nu_\tau=n$ and $f_{\nu(\tau)} = f(\rho_\tau, 0)$. As a result, the solution (2.29) can be converted to the form

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\rho_\tau, \eta_\tau)\exp\left(\frac{1}{2}\int_0^t \mathrm{i}k\xi_t \mathrm{d}t\right)\right\}, \tag{2.32}$$

which does not rely on the piece-wise structure of $f(r, 0)$, and which, therefore, can be straightforwardly extended to the case when $f(r, 0)$ has a virtually arbitrary structure, in particular, to the cases when it is not analytic at all or when it is analytic but grows excessively fast in the first quarter of the complex plane, where the radial motion ξ_t runs.

It is worth mentioning that if we redefine the exit position ξ_τ of the radial motion as $\xi_\tau \equiv \rho_\tau$, then the representation (2.32) can be considered as a generalization of equation (2.6) to the case where ξ_t, instead of being continuous on the entire lifespan $0 \leq t \leq \tau$, has a jump at its end.

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3. Representations of wave fields in an arbitrary wedge

Probabilistic representations (2.6) and (2.32) of the wave field in a wedge $\mathfrak{G}_+$ with the grounded face $\theta=0$ can be easily generalized to describe the field $U(r, \theta)$ in the angular domain

$$\mathfrak{G} \equiv \mathfrak{G}(\alpha_1, \alpha_2): \quad r>0, \quad \alpha_1<\theta<\alpha_2, \qquad (\alpha_2-\alpha_1 \leq 3\pi/2), \tag{3.1}$$

with the boundary conditions

$$U(r, \alpha_n) = f(r, \alpha_n)\mathrm{e}^{\mathrm{i}kr}, \quad (n=1,2), \tag{3.2}$$

where $f(r, \alpha_1)$ and $f(r, \alpha_2)$ are given functions of the radius r. As usual, we assume that $U(r, \theta)$ obeys the Helmholtz equation and the radiation condition $U(r,\theta)\mathrm{e}^{-\mathrm{i}kr}=o(1)$, as $r\rightarrow\infty$.

(a) Basic representations

If the boundary functions $f(\xi, \alpha_n)$ are analytic in the first quarter $0<\arg(\xi)<\pi/2$ of the complex plane, then the field $U(r, \theta)$ in $\mathfrak{G}(\alpha_1, \alpha_2)$ can be represented by the formula

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\xi_\tau, \eta_\tau)\exp\left(\frac{1}{2}\int_0^t \mathrm{i}k\xi_t \mathrm{d}t\right)\right\}, \tag{3.3}$$

where the averaging is computed over the random motion (ξ_t, η_t) which is controlled by the equation (2.7) and stopped at the exit time $t=\tau$ when the angular component η_t hits either of the faces $\eta_\tau=\alpha_1$ or $\eta_\tau=\alpha_2$.

Alternatively, the field $U(r, \theta)$ can be represented by a less restrictive formula

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\hat{\xi}_\tau, \eta_\tau)\exp\left(\frac{1}{2}\int_0^t \mathrm{i}k\hat{\xi}_t \mathrm{d}t\right)\right\}, \tag{3.4}$$

where the radial and the angular random motions $\hat{\xi}_t$ and η_t start from the initial positions

$$\hat{\xi}_0 = r, \qquad \eta_0 = \theta, \tag{3.5}$$

and are controlled thereafter by the stochastic rules

$$\begin{aligned} \hat{\xi}_{t+\mathrm{d}t} &= \hat{\xi}_t \mathrm{e}^{\mathrm{d}w_t^1+\mathrm{i}k\hat{\xi}_t\mathrm{d}t}, \qquad \mathrm{d}\eta_{t+\mathrm{d}t} = \eta_t + \mathrm{d}w_t^2, \quad \text{if } 0\leq t<\tau, \\ \hat{\xi}_\tau &= \inf\{\mathfrak{Z}_\tau(r, \alpha_1, \alpha_2; \eta_\tau)\}, \quad \eta_\tau = \eta_{\tau-0}, \end{aligned} \tag{3.6}$$

where τ is the exit time of the angular motion η_t from the segment (α_1, α_2) and $\mathfrak{Z}_\tau(r, \alpha_1, \alpha_2; \eta_\tau)$ is the intersection of the trajectory of the motion

$$\zeta_t = \begin{cases} \xi_t \mathrm{e}^{-\mathrm{i}(\eta_t-\alpha_1)}, & \text{if } \eta_\tau = \alpha_1, \\ \xi_t \mathrm{e}^{-\mathrm{i}(\alpha_2-\eta_t)}, & \text{if } \eta_\tau = \alpha_2, \end{cases} \tag{3.7}$$

with the semi-axis $\arg(\zeta)=0$.

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It is clear that the radial motion $\hat{\xi}_t$ runs in the complex space exactly as the continuous motion ξ_t from equation (3.3), but at the exit time $t=\tau$ it jumps to a positive real point defined by the entire trajectory of both components of $(\hat{\xi}_t, \eta_t)$.

(*b*) *Representation using a 'two-scale' random motion in complex space*

The representation (3.4) employs the random motion $(\hat{\xi}_t, \eta_t)$ running in complex space, but this formula can be used as an intermediate step for the derivation of another representation of $U(r, \theta)$ which does not involve random motions in the complex space.

Let $(2N+1)$ points β_n be distributed in the interval $\mathfrak{G}=[\alpha_1,\alpha_2]$ as

$$\alpha_1 \equiv \beta_0 < \beta_1 < \beta_2 < \beta_3 < \ldots < \beta_{2N-1} < \beta_{2N} \equiv \alpha_2, \tag{3.8}$$

and let $\mathfrak{G}_0$, $\mathfrak{G}_1$, …, $\mathfrak{G}_{2N-2}$ be the overlapping intervals (see figure 3) defined by the rule

$$\mathfrak{G}_n: \quad \beta_n \le \theta < \beta_{n+2}, \qquad (\beta_{n+2} - \beta_n \le 3\pi/2), \tag{3.9}$$

which implies that the interval $[\alpha_1,\alpha_2)$ is covered by the segments $\mathfrak{G}_{2m}$ with even indices and that the left and the right ends of the interval $\mathfrak{G}_{2m}$ belong the intervals $\mathfrak{G}_{2m-1}$ and $\mathfrak{G}_{2m+1}$, respectively.

To compute $U(r, \theta)$ inside the wedge $\alpha_1 < \theta < \alpha_2$, we first find an even number n_0 such that $\theta \in \mathfrak{G}_{n_0} = [\beta_{n_0}, \beta_{n_0+2})$. Then, applying formula (3.4) to the wedge $\mathfrak{G}_{n_0}$ we find that

$$U(r,\theta) = \mathbb{E}\left\{ U(\hat{\xi}_{\tau_1}, \eta_{\tau_1}) \exp\left(\mathrm{i}k\left[r - \hat{\xi}_{\tau_1} + \frac{1}{2}\int_0^{\tau_1} \hat{\xi}_t \mathrm{d}t \right]\right)\right\}, \tag{3.10}$$

where the averaging includes the random motion $(\hat{\xi}_t, \eta_t)$ which starts from (r, θ) and runs under the control of equations (3.5) and (3.6) until η_t reaches the boundary of $\mathfrak{G}_{n_0}$. Then, $U(\hat{\xi}_{\tau_1}, \eta_{\tau_1})$ can be computed by a similar formula applied to the wedge $\mathfrak{G}_{n_1}$ which contains the exit point $(\hat{\xi}_{\tau_1}, \eta_{\tau_1})$ from equation (3.10). This leads to the expression

$$U(\hat{\xi}_{\tau_1}, \eta_{\tau_1}) = \mathbb{E}\left\{ U(\hat{\xi}_{\tau_2}, \eta_{\tau_2}) \exp\left(\mathrm{i}k\left[\hat{\xi}_{\tau_1} - \hat{\xi}_{\tau_2} + \frac{1}{2}\int_{\tau_1}^{\tau_2} \mathrm{i}k\hat{\xi}_t \mathrm{d}t \right]\right)\right\}, \tag{3.11}$$

which can be combined with equation (3.10) to deliver the following result

$$U(r,\theta) = \mathbb{E}\left\{ U(\hat{\xi}_{\tau_2}, \eta_{\tau_2}) \exp\left(\mathrm{i}k\left[r - \hat{\xi}_{\tau_2} + \frac{1}{2}\int_0^{\tau_2} \mathrm{i}k\hat{\xi}_t \mathrm{d}t \right]\right)\right\}. \tag{3.12}$$

Next, we compute $U(\hat{\xi}_{\tau_2}, \eta_{\tau_2})$ by a formula similar to equation (3.12), and continuing the iterations until the angular motion η_t hits the boundary of the original interval (α_1, α_2), we get the final representation

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\hat{\xi}_\tau, \eta_\tau) \exp\left(\frac{1}{2}\int_0^{\tau} \mathrm{i}k\hat{\xi}_t \mathrm{d}t, \right)\right\}, \tag{3.13}$$

where the averaging is extended over the stochastic processes $\hat{\xi}_t$, η_t and n_t described below.

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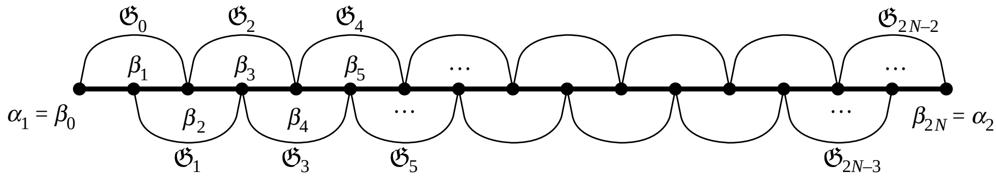

Figure 3. Overlapping intervals $\mathfrak{G}_m$.

The radial and the angular motions $\hat{\xi}_t$ and η_t are the stochastic processes driven by the Cartesian components w_t^1 and w_t^2 of the standard two-dimensional Brownian motion $\boldsymbol{w}_t = (w_t^1, w_t^2)$.

The angular motion η_t starts at $t = \tau_0 \equiv 0$ from the initial position $\eta_0 = \theta$ and runs as the Brownian motion $\eta_t = \theta + w_t^2$ which passes points from the set $\{\beta_m\}$ at the crossing times $\tau_1, \tau_2, \ldots$. Finally, η_t stops at the exit time $t = \tau_M \equiv \tau$ when it eventually hits one of the boundary points $\beta_0 \equiv \alpha_1$ or $\beta_{2N} \equiv \alpha_2$. The crossing times τ_m subdivide the 'lifespan' $\tau_0 \leq t \leq \tau_M$ of the continuous angular motion η_t on the semi-open intervals

$$T_m: \quad \tau_{m-1} \leq t < \tau_m, \quad m = 1, 2, \ldots, M, \tag{3.14}$$

on which the processes n_t and $\hat{\xi}_t$ are also continuous.

The integer-valued process n_t starts from the even number n_0 defined above, and then it changes its value at every crossing time τ_m, when it acquires the index of the interval $\mathfrak{G}_\mu$ which contains η_{τ_m} in its interior. It is obvious that

$$n(\tau_m) = \begin{cases} n(\tau_{m-1}) + 1, & \text{if } \eta(\tau_m) > \eta(\tau_{m-1}), \\ n(\tau_{m-1}) - 1, & \text{if } \eta(\tau_m) < \eta(\tau_{m-1}), \end{cases} \tag{3.15}$$

where the symbols $n(t)$ and $\eta(t)$ are used as synonyms for n_t and η_t.

The radial motion $\hat{\xi}_t$ is a piece-wise continuous stochastic process which starts from $\xi_0 = r$ and has jumps at the crossing times $t = \tau_m$. In the intervals T_m, this motion is controlled by the differential equation

$$\mathrm{d}\hat{\xi}_t = \hat{\xi}_t \mathrm{d}w_t^1 + \hat{\xi}_t\left(\frac{1}{2} + \mathrm{i}k\hat{\xi}_t\right)\mathrm{d}t, \quad t \in [\tau_{m-1}, \tau_m), \tag{3.16}$$

and at the crossing time $t = \tau_m$, it jumps to the positive real point $\hat{\xi}(\tau_m)$, defined by the formula

$$\hat{\xi}(\tau_m) = \inf\{\mathfrak{Z}_{\tau_m}(a_m, b_m, \eta_{\tau_m})\}, \tag{3.17}$$

where

$$a_m = \min(\mathfrak{G}_{n_t}), \qquad b_m = \max(\mathfrak{G}_{n_t}), \tag{3.18}$$

are the ends of the interval $\mathfrak{G}_{n_t}$, where the motion (ξ_t, η_t) is currently located, and $\mathfrak{Z}_{\tau_m}(a_m, b_m, \eta_{\tau_m})$ is the intersection of the trajectory of the point

$$\zeta_t = \begin{cases} \zeta_t \mathrm{e}^{-\mathrm{i}(\eta_t - a_m)}, & \text{if } \eta(\tau_m) < \eta(\tau_{m-1}), \\ \zeta_t \mathrm{e}^{-\mathrm{i}(b_m - \eta_t)}, & \text{if } \eta(\tau_m) > \eta(\tau_{m-1}), \end{cases} \tag{3.19}$$

with the semi-axis $\arg(\zeta) = 0$.

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(*c*) *Representation of $U(r,\theta)$ using a motion with micro structure in real space*

The motion $(\hat{\xi}_t, \eta_t)$ involved in expression (3.13) can be considered as a two-scale process which admits the decomposition

$$(\hat{\xi}_t, \eta_t) = (\rho_t, \vartheta_t) + (\mathfrak{x}_t, \mathfrak{y}_t), \tag{3.20}$$

into the 'macro' component (ρ_t, ϑ_t) and the 'micro' component $(\mathfrak{x}_t, \mathfrak{y}_t)$. The macro motion (ρ_t, ϑ_t) runs in real space as a sequence of the jumps

$$(r, \theta) \equiv (\hat{\xi}_{\tau_0}, \vartheta_{\tau_0}) \to (\hat{\xi}_{\tau_1}, \vartheta_{\tau_1}) \to \cdots \to (\hat{\xi}_{\tau_M}, \vartheta_{\tau_M}) \equiv (\hat{\xi}_{\tau}, \vartheta_{\tau}), \tag{3.21}$$

which occur at the crossing times $t = \tau_m$. The micro component $\mathfrak{x}_t$ starts afresh from the initial position $(\mathfrak{x}_{\tau_m}, \mathfrak{y}_{\tau_m}) = (0, 0)$, at every crossing time $t = \tau_m$, and it runs in complex space under the control of the stochastic differential equations

$$\mathrm{d}\mathfrak{x}_t = \mathfrak{x}_t \mathrm{d}w_t^1 + \mathfrak{x}_t\left(\frac{1}{2} + \mathrm{i}k\mathfrak{x}_t\right)\mathrm{d}t, \qquad \mathfrak{y}_{s+\mathrm{d}s} = \mathfrak{y}_s + \mathrm{d}w_{\mathrm{d}s}^2, \tag{3.22}$$

where w_s^1 and w_s^2 are standard Brownian motions, independent of each other. The macro and micro motions are not independent but affect each other through the mechanism described below under the assumption that the points β_n from equation (3.8) form a uniform mesh with the infinitesimally short distance $\mathrm{d}\theta$ between nodes.

The interval $\mathrm{d}t = \tau_{m+1} - \tau_m$ between crossing times is a random number

$$\mathrm{d}t = \text{exit time of } \mathfrak{y}_s \text{ from the segment } (-\mathrm{d}\theta, \mathrm{d}\theta), \tag{3.23}$$

which is determined by the angular component of the micro motion. Correspondingly, the angular component $\mathrm{d}\vartheta = \vartheta_{\tau_{m+1}} - \vartheta_{\tau_m}$ of the macro motion is defined by a simple formula

$$\mathrm{d}\vartheta = \mathfrak{y}_{\mathrm{d}t} = \pm \mathrm{d}\theta, \tag{3.24}$$

and the radial component $\mathrm{d}\rho_{\tau_m} = \rho_{\tau_{m+1}} - \rho_{\tau_m}$ of this motion is defined as

$$\mathrm{d}\rho_t = \inf\{\mathfrak{Z}_{\mathrm{d}t}(\rho_t, -\mathrm{d}\theta, \mathrm{d}\theta; \mathfrak{y}_{\mathrm{d}t})\}, \tag{3.25}$$

where $\mathfrak{Z}_{\mathrm{d}\tau}(\rho, -\mathrm{d}\theta, \mathrm{d}\theta; \mathfrak{y}_{\mathrm{d}\tau})$ is the full intersection of the trajectory of the point

$$\zeta_s = \begin{cases} \mathfrak{x}_s \mathrm{e}^{-\mathrm{i}(\mathfrak{y}_s + \mathrm{d}\theta)}, & \text{if } \mathfrak{y}_{\mathrm{d}t} = -\mathrm{d}\theta, \\ \mathfrak{x}_s \mathrm{e}^{-\mathrm{i}(\mathrm{d}\theta - \mathfrak{y}_s)}, & \text{if } \mathfrak{y}_{\mathrm{d}t} = \mathrm{d}\theta, \end{cases} \tag{3.26}$$

with the positive semi-axis $\arg(\zeta) = 0$.

As a result, we arrive at the representation

$$U(r, \theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\rho_\tau, \vartheta_\tau)\exp\left(\frac{1}{2}\int_0^\tau \mathrm{i}k\rho_t \mathrm{d}t\right)\right\}, \tag{3.27}$$

where the averaging is extended over the trajectories of the two-dimensional random motion (ρ_t, ϑ_t), defined by equations (3.23)–(3.26), which starts from the observation point $(\rho_0, \vartheta_0) = (r, \theta)$ and stops at the exit time τ when it hits the boundary of the domain $\mathfrak{G}$.

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It is instructive to compare the structure of the probabilistic representations (3.3), (3.4), (3.13) and (3.27) of the same wave field. All of these formulae employ averaging over trajectories of random motions which have the same angular component but have different radial components. In equation (3.3), the radial motion ξ_t from equation (3.6) is continuous, and at any time $t>0$ it is located inside the first quarter of the complex plane. In equation (3.4), the radial motion runs exactly as the radial motion in equation (3.3), but at the exit point $t=\tau$, it jumps to the uniquely defined point $\hat{\xi}_\tau$ located on the real axis. In equation (3.13), the radial motion $\hat{\xi}_t$ returns to the positive semi-axis at every crossing time $t=\tau_m$, when the angular motion η_t touches any of the points $\{\beta_n\}$ from equation (3.8). Finally, as shown below, the representation (3.27) makes it possible to compute wave fields in domains of various shapes, which are no longer restricted to infinite wedges. The price for this versatility is the use of the non-traditional random motion (ρ_t, ϑ_t) which may not be described by Ito's stochastic differential equations, but which, nevertheless, has a clear meaning, defined by the invisible 'micro' structure.

4. Representation of wave fields in a wedge with non-flat faces

(a) Representation using random motions in real space

Looking back through the reasoning of the previous section, it is easy to see that the probabilistic expression (3.27) can be straightforwardly extended to represent wave fields in an arbitrary wedge $\mathfrak{G}(\alpha_1, \alpha_2)$ with additional 'boundary' conditions imposed on an arbitrary number of two-sided radially oriented straight segments.

For example, consider an infinite wedge $\mathfrak{G}(-\infty, \infty)$ with cuts along the line segments

$$\mathfrak{I}_n: \quad \theta = \alpha_n, \quad r_n \le r < r_{n+1}, \tag{4.1}$$

where α_n are given angles in the range $-\infty < \alpha_n < \infty$ and r_n are the elements of a monotonically increasing sequence of non-negative numbers (figure 4). Let $U(r, \theta)$ be the continuous wave field in this infinite domain, which satisfies the Helmholtz equation $\nabla^2 U + k^2 U = 0$, the radiation condition $U\mathrm{e}^{-\mathrm{i}kr} = o(1)$ and the boundary conditions

$$U(r, \alpha_n) = f(r, \alpha_n)\mathrm{e}^{\mathrm{i}kr}, \quad \text{if } r_n \le r < r_{n+1}, \tag{4.2}$$

where $f(r, \theta)$ is a given function defined only on the cuts $\mathfrak{I}_n$. In general, $f(r, \theta)$ may take different values on the different sides of the cuts, but for our purposes it suffices to assume that these values coincide. Then, straightforward application of the method from the previous section implies that the field $U(r, \theta)$ can be represented by the formula (3.27) where the exit time is defined as the first time when the motion (ρ_t, ϑ_t) hits either of the sides of one of the intervals $\mathfrak{I}_n$ from equation (4.1).

It is clear that as the radial and the angular increments $\triangle r_n = r_{n+1} - r_n$ and $\triangle \alpha_n = \alpha_{n+1} - \alpha_n$ simultaneously tend to zero in such a way that the ratio $|\triangle \alpha_n / \triangle r_n|$ remains bounded, then the correspondences $r_n \to \alpha_n$ form a continuous function $\theta = \alpha(r)$. The graph of this function is formed by the

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segments $\mathfrak{I}_n$, which link without overlapping and subdivide the infinite domain in the angular direction $-\infty < \theta < \infty$ into two half-infinite domains $\theta > \alpha(r)$ and $\theta < \alpha(r)$, inside each of which the wave field can be represented by the formula (3.27). More generally, if $\alpha_1(r)$ and $\alpha_2(r)$ are two continuous functions related by $\alpha_1(r) < \alpha_2(r)$, then the wave field $U(r, \theta)$ in the domain

$$\mathfrak{G}: \quad r > 0, \quad \alpha_1(r) < \theta < \alpha_2(r), \tag{4.3}$$

with the boundary conditions

$$U(r, \alpha_n(r)) = f_n(r)\mathrm{e}^{\mathrm{i}kr}, \quad n = 1, 2, \tag{4.4}$$

can be represented by the mathematical expectation

$$U(r, \theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\Big\{f_1(\rho_{\tau_1}, \vartheta_{\tau_1})\mathrm{e}^{\mathrm{i}kS(\tau_1)} + f_2(\rho_{\tau_2}, \vartheta_{\tau_2})\mathrm{e}^{\mathrm{i}kS(\tau_2)}\Big\}, \tag{4.5}$$

$$S(t) = \frac{1}{2}\int_0^t \rho_t \mathrm{d}t, \tag{4.6}$$

where the averaging is extended over the trajectories of the two-dimensional random motion $P_t = (\rho_t, \vartheta_t)$, defined by (3.23)–(3.26), which starts from the observation point $(\rho_0, \vartheta_0) = (r, \theta)$ and stops at the earliest of exit times τ_1 or τ_2 defined as the first time when P_t hits the line $\vartheta = \alpha_1(\rho)$ or $\vartheta = \alpha_2(\rho)$, respectively.

(b) *Representation using complex- and real-valued random motions together*

In the above, we obtained probabilistic representations of wave fields using two types of random motions. In the first one, formulae from §3 employed averaging over the trajectories of the random motion $(\hat{\xi}_t, \eta_t)$ from equation (3.6), which runs in the complex space but jumps, at the exit time, to a certain real point on one of the wedge's faces. This random motion is simple to implement but it cannot be used in domains other than wedges with straight faces. The second one, introduced in §4, uses a more difficult to implement real-value random motion (ρ_t, ϑ_t), which made it possible to represent wave fields in domains of rather general shape. These two different types of random motion, however, can be easily combined to use the advantages of both of them.

To illustrate how the methods from §§3 and 4 can be combined, we consider a typical case where the boundaries of the domain $\mathfrak{G}$ of type (4.3) are restrained by the inequalities

$$\alpha_1(r) < \beta_1 < \beta_2 < \alpha_2(r), \tag{4.7}$$

which imply that $\mathfrak{G}$ can be decomposed, as shown in figure 5, into three subdomains

$$\begin{aligned} &\mathfrak{G}_1: \quad r > 0, \quad \alpha_1(r) < \theta < \beta_1, \\ &\mathfrak{G}_2: \quad r > 0, \quad \beta_2 < \theta < \alpha_2(r), \\ &\mathfrak{G}_0: \quad r > 0, \quad \beta_1 \le \theta \le \beta_2, \end{aligned} \tag{4.8}$$

separated by the straight rays $\theta = \beta_1$ and β_2. In this case, the field $U(r, \theta)$ in the entire domain $\mathfrak{G}$ with the boundary conditions (4.4) can be considered as three separate fields defined in the domains from equation (4.8) and connected through

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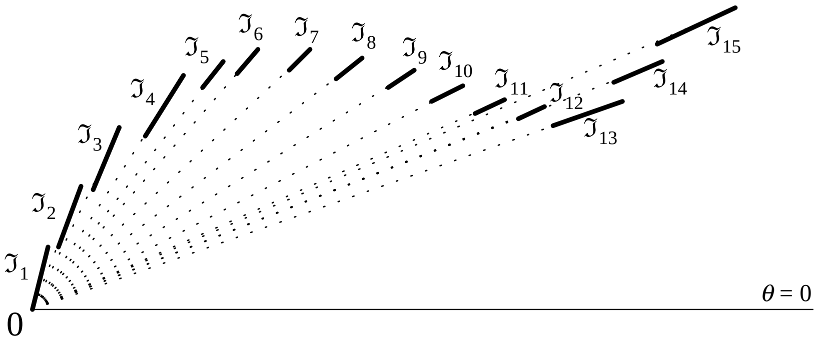

Figure 4. A system of radially oriented segments.

the interface conditions

$$U(r,\beta_n-0) = U(r,\beta_n+0), \qquad \frac{\partial U(r,\beta_n-0)}{\partial\theta} = \frac{\partial U(r,\beta_n+0)}{\partial\theta}, \qquad n=1,2, \tag{4.9}$$

which guarantee the smoothness of $U(r, \theta)$ inside $\mathfrak{G}$.

The approach to this problem is rather trivial: in the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ we represent the field $U(r, \theta)$ by the formula (3.27), which employs the real-valued random motion (ρ_t, ϑ_t) described in §3, but in the wedge $\mathfrak{G}_0$ we represent $U(r, \theta)$ by the formula (3.4), which employs the random motion $(\hat{\xi}_t, \eta_t)$ running in the complex space. The interface conditions (4.9) lead to the identities

$$U(r,\beta_n) = \frac{1}{2}[U(r,\beta+\mathrm{d}t) + U(r,\beta-\mathrm{d}t)] \equiv \mathbb{E}\{U(r,\beta\pm\mathrm{d}t)\}, \tag{4.10}$$

which express the values of $U(r, \theta)$ on the interfaces through the values of this field inside one of the domains $\mathfrak{G}_1$, $\mathfrak{G}_2$ or $\mathfrak{G}_0$.

It is easy to see that straightforward implementation of the outlined idea leads to the unified representation

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{f(\hat{\rho}_\tau,\vartheta_\tau)\exp\left(\frac{1}{2}\int_0^\tau \mathrm{i}k\hat{\rho}_t\mathrm{d}t\right)\right\}, \tag{4.11}$$

which differs from equation (4.11) only by the structure of the radial motion,

$$\hat{\rho}_{t+\mathrm{d}t} = \begin{cases} \inf\{\mathfrak{Z}_{\mathrm{d}t}(\hat{\rho}_t,-\mathrm{d}\theta,\mathrm{d}\theta;\mathfrak{y}_{\mathrm{d}t})\}, & \text{if } \beta_1 \le \vartheta_t \le \beta_2, \\ \hat{\rho}_t\mathrm{e}^{w_{\mathrm{d}t}+\mathrm{i}k\hat{\rho}_t\mathrm{d}t}, & \text{otherwise.} \end{cases} \tag{4.12}$$

It is worth reiterating that although solutions (3.27) and (4.11) are equivalent, the latter may be implemented more efficiently because the motion $\hat{\rho}_t$ employed there has a simplified micro structure in, at least, part of its running time. Moreover, during the time when $\hat{\rho}_t$ is complex, it has a positive imaginary part which depresses the value of the exponent in equation (4.11) and accelerates the convergence of this representation.

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5. Applications

(*a*) *Wave field in a curvilinear wedge with a discontinuity along a half-line $\theta=\phi$*

Probabilistic expression (3.27) represents wave fields that are smooth and obey the Helmholtz equation $\nabla^2 U + k^2 U = 0$ everywhere inside the curvilinear wedge $\mathfrak{G}$ of the type (4.3). However, to address some problems of diffraction, it is convenient to have similar representations for the field $U(r, \theta)$ which has a discontinuity on a half-line $\theta=\phi$ described by the interface conditions

$$U(r, \phi+0) - U(r, \phi-0) = g(r)\mathrm{e}^{\mathrm{i}kr}, \qquad \frac{\partial U(r, \phi+0)}{\partial \theta} = \frac{\partial U(r, \phi-0)}{\partial \theta}, \tag{5.1}$$

where the jump function $g(r)$ is bounded, but does not necessarily vanish at infinity. Additionally, we assume that $U(r, \theta)$ obeys the boundary conditions (4.2), and the radiation condition

$$U(r, \theta)\mathrm{e}^{-\mathrm{i}kr} = o(1), \quad r \to \infty, \quad \theta \neq \phi, \tag{5.2}$$

which is relaxed on the half-line $\theta=\phi$ because otherwise the jump function $g(r)$ from the first interface condition would have to vanish as $r \to \infty$.

To solve this problem, we start from the assumption that the field $U(r, \theta)$ is already known on both sides of the interface $\theta=\phi\pm 0$. Then, the value of $U(r, \theta)$ with $\theta \neq \phi$ can be evaluated by the formulae (3.27) applied to that domain $\mathfrak{G}(\phi, \alpha_2)$ or $\mathfrak{G}(\alpha_1, \phi)$ which contains the observation point (r, θ). These formulae lead to the expression

$$U(r, \theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\Big\{f_1(\rho_{\tau_1}, \vartheta_{\tau_1})\mathrm{e}^{\mathrm{i}kS(\tau_1)} + f_2(\rho_{\tau_2}, \vartheta_{\tau_2})\mathrm{e}^{\mathrm{i}kS(\tau_2)} + U(\rho_\iota, \vartheta_\iota)\mathrm{e}^{\mathrm{i}k[S(\iota)-\rho_\iota]}\Big\}, \tag{5.3}$$

with the averaging over the trajectories of the motion (ρ_t, ϑ_t) which stops at the earliest of the times $t=\tau_1$, $t=\tau_2$ or $t=\iota$, where τ_1 and τ_2 are the exit times through the first and the second sides of the original wedge $\mathfrak{G}$, and ι is the first time when the angular motion hits the interface $\eta=\phi$.

The auxiliary exit point η_ι takes one of the two values $\eta_\iota=\phi\pm 0$, which is determined by the location of the observation point (r, θ). Therefore, the value of $U(\hat{\xi}_\iota, \eta_\iota)$ can be represented by the corresponding expression

$$\begin{aligned} U(\rho_\iota, \vartheta_\iota) &= \frac{U(\rho_\iota, \phi+\mathrm{d}t) + [U(\rho_\iota, \phi-\mathrm{d}t) + g(\rho_\iota)\mathrm{e}^{\mathrm{i}k\hat{\xi}_\iota}]}{2}, \quad \text{if } \vartheta_\iota = \phi+0, \\ U(\rho_\iota, \vartheta_\iota) &= \frac{[U(\rho_\iota, \phi+\mathrm{d}t) - g(\rho_\iota)\mathrm{e}^{\mathrm{i}k\rho_\iota}] + U(\rho_\iota, \phi-\mathrm{d}t)}{2}, \quad \text{if } \vartheta_\iota = \phi-0, \end{aligned} \tag{5.4}$$

which immediately follow from the interface conditions (5.1), and which can be rewritten in the probabilistic form

$$U(\rho_\iota, \vartheta_\iota) = \mathbb{E}\{U(\rho_\iota, \vartheta_1) + \delta g(\rho_\iota)\mathrm{e}^{\mathrm{i}k\rho_\iota}\}, \tag{5.5}$$

where ϑ_1 is a random number with two equally possible values $\vartheta_1=\phi\pm\mathrm{d}t$, and δ depends on η_ι and η_1 by the formula

$$\delta = \begin{cases} 1, & \text{if } \vartheta_1 < \phi < \vartheta_\iota, \\ -1, & \text{if } \vartheta_1 > \phi > \vartheta_\iota. \end{cases} \tag{5.6}$$

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Since $(\rho_\iota, \vartheta_1)$ is located the interface $\theta=\phi$, the value of $U(\rho_\iota, \vartheta_1)$ can be evaluated by formulae like equations (5.4)–(5.6), and repeating the iterations we eventually arrive at the final expression

$$U(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{\sum_{\nu=1}^{\infty}\delta(t_\nu,\phi)g(\rho_{t_\nu})\mathrm{e}^{\mathrm{i}kS(\tau_\nu)} + f_1(\rho_{\tau_1},\vartheta_{\tau_1})\mathrm{e}^{\mathrm{i}k[S(\tau_1)-\rho_{\tau_1}]}\right. \\ \left. +f_2(\rho_{\tau_2},\vartheta_{\tau_2})\mathrm{e}^{\mathrm{i}k[S(\tau_2)-\rho_{\tau_1}]}\right\}, \tag{5.7}$$

where ρ_t, ϑ_t and $S(t)$ retain their meaning from equation (3.27), while the factor $\delta(t,\phi)$ is the stochastic process described below.

The angular motion running inside the interval $[\alpha_1,\alpha_2]$ touches the fixed points $\vartheta=\phi$, α_1 and α_2 at the times $t=t_\nu$ enumerated by the index $\nu\geq 1$ which determines the value of $\delta(t_\nu,\phi)$ by the rule

$$\delta(t,\phi) = \begin{cases} 1, & \text{if } \vartheta_{t-0} > \phi, \quad \vartheta_{t+0} < \phi, \\ -1, & \text{if } \vartheta_{t-0} < \phi, \quad \vartheta_{t+0} > \phi, \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

(b) *Diffraction by a wedge-like scatterer with non-flat faces*

Here, we apply the developed technique to the problem of diffraction of the plane incident wave

$$U_*(r,\theta) = \mathrm{e}^{-\mathrm{i}kr\,\cos(\theta-\theta_*)}, \tag{5.9}$$

by the boundaries of the domain

$$\mathfrak{G}: \quad r>0, \quad \alpha_1 < \theta < \alpha_2(r), \tag{5.10}$$

where $\alpha_1(r)$ and $\alpha_2(r)$ are continuous functions restrained by the inequalities

$$|r\alpha_1(r)| < \Delta, |r\alpha_2(r)-\alpha| < \Delta, \tag{5.11}$$

which makes it possible to treat $\mathfrak{G}$ as a wedge with a perturbed face, as shown in figure 5.

To reduce insignificant detail, we limit ourselves to the analysis of the configuration characterized by the inequalities $\alpha>\pi$ and $\theta_*<\alpha-\pi$, which guarantee that the face $\theta=\alpha_1(r)$ is illuminated by the incident wave, while the face $\theta=\alpha_2(r)$ is located in the shadow, as shown in figure 6. Additionally, we limit ourselves here to the Dirichlet boundary conditions

$$U(r,0) = 0, \qquad U(r,\alpha_2(r)) = 0, \tag{5.12}$$

so that the problem is reduced to the computation of the solution of the Helmholtz equation $\nabla^2 U + k^2 U = 0$, complimented by the boundary conditions (5.12) and the standard conditions at infinity, which require that the total field does not contain any waves arriving from infinity, except for the incident plane wave $U_*(r, \theta)$.

A comparison of the domain $\mathfrak{G}$ to the wedge $\mathfrak{G}(0, \alpha)$ justifies the representation of the field $U(r, \theta)$ as the superposition

$$U = U_i + U_r + U_1 + U_2, \tag{5.13}$$

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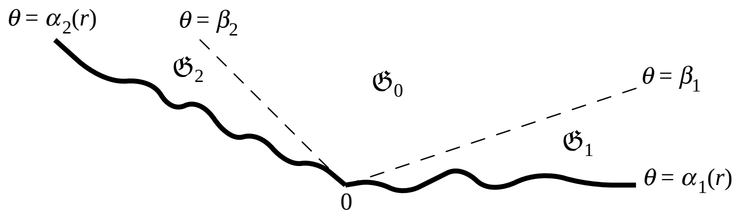

Figure 5. Wedge-like domain with non-flat faces.

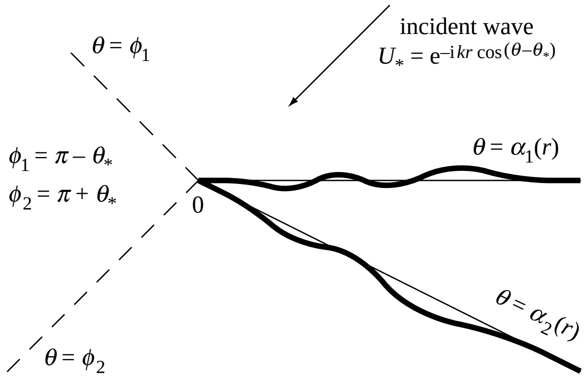

Figure 6. Diffraction in a domain with non-flat faces.

of the yet unknown scattered fields U_1 and U_2 together with two pre-defined components U_i and U_r defined by the formulae

$$U_i = \begin{cases} e^{-ikr\cos(\theta-\theta_*)}, & \text{if } \theta < \phi_2, \\ 0, & \text{if } \theta > \phi_2 \end{cases} \qquad U_r = \begin{cases} -e^{-ikr\cos(\theta+\theta_*)}, & \text{if } \theta < \phi_1, \\ 0, & \text{if } \theta > \phi_1, \end{cases} \tag{5.14}$$

where

$$\phi_1 = \pi - \theta_*, \qquad \phi_2 = \pi + \theta_*. \tag{5.15}$$

The scattered fields $U_n(r, \theta)$ must be solutions of the Helmholtz equations $\nabla^2 U_n + k^2 U_n = 0$ complimented by the conditions at infinity

$$U_n e^{-ikr} = o(1), \quad r \to \infty, \quad \theta \neq \phi_n, \tag{5.16}$$

interface conditions

$$U_1(r, \phi_1 + 0) - U_1(r, \phi_1 - 0) = e^{ikr}, \quad \frac{\partial U_1(r, \phi_1 + 0)}{\partial \theta} = \frac{\partial U_1(r, \phi_1 - 0)}{\partial \theta}, \tag{5.17}$$

$$U_2(r, \phi_2 + 0) - U_2(r, \phi_2 - 0) = -e^{ikr}, \quad \frac{\partial U_2(r, \phi_2 + 0)}{\partial \theta} = \frac{\partial U_2(r, \phi_2 - 0)}{\partial \theta}, \tag{5.18}$$

and the boundary conditions

$$U_n(r, \alpha_1(r)) = f(r) e^{ikr}, \qquad U_n(r, \alpha_2(r)) = 0, \tag{5.19}$$

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with the right-hand side

$$f(r) = 2\mathrm{i}\,\sin(kr\,\sin(\alpha_1(r))\sin\theta_*)\,e^{-\mathrm{i}kr[1+\cos\theta_*\cos(\alpha_1(r))]}. \tag{5.20}$$

It is obvious that the problem (5.16)–(5.19) coincides with the problem discussed in §5.1, and, therefore, the scattered fields $U_n(r, \theta)$ can be computed by the formula (4.5), or by (4.11), which result in the probabilistic expression

$$U_n(r,\theta) = \mathrm{e}^{\mathrm{i}kr}\mathbb{E}\left\{ f(\rho_\tau)e^{\mathrm{i}kS(\tau)} + (-1)^{n+1}\sum_{\nu=1}^{\infty}\delta(t_\nu,\phi_n)\mathrm{e}^{\mathrm{i}k[S(t_\nu)-\rho_{t_\nu}]}\right\}, \tag{5.21}$$

where the averaging is extended over the trajectories of the real-valued random motion $P_t = (\hat{\rho}_t, \vartheta_t)$, which stops at the earliest of the exit times $t=\tau$ or $t=\tilde{\tau}$, when P_t hits the line $\theta=\alpha_1(r)$ or $\theta=\alpha_2(r)$, respectively.

6. Conclusion

The combination of the probabilistic approach to partial differential equations using the product representation (2.2) together with the properties of the function $\mathcal{P}(r, \theta, L)$ from equation (2.17) made it possible to obtain an explicit representation (4.5) for the solutions of the Helmholtz equation with given boundary values in domains of the type $r>0$ and $\alpha_1(r)<\theta<\alpha_2(r)$. This representation opens a way to address numerous problems of wave propagation and diffraction, but this representation itself also has much room for further generalizations and extensions. It is impractical at this point to provide a comprehensive overview of all of the opportunities for future work, but it is also useful to outline a few of the most straightforward extensions of the obtained results.

First, it is worth mentioning that the representation (4.5) can be used to compute the solution of the Helmholtz equation in the domain (4.3) with the general impedance boundary conditions

$$aU + b\frac{\partial U}{\partial n}\Big|_{\partial G} = f,$$

where $a(r, \theta)$, $b(r, \theta)$ and $f(r, \theta)$ are defined on the boundary ∂G, and $\boldsymbol{n}(r, \theta)$ is the unit normal to ∂G. Indeed, the straightforward combination of equation (4.5) with the technique from Budaev & Bogy (2004, 2005*b*) leads to the probabilistic representation which employs angular random motion with reflections.

It should also be mentioned that by using several polar coordinate systems simultaneously, as in Budaev & Bogy (2006), we can extend the developed method from the wedge-like domains of the type $r>0$, $\alpha_1(r)<\theta<\alpha_2(r)$ to virtually arbitrary unbounded domains.

Another important feature of the obtained results is that they admit straightforward extension to multidimensional problems. Indeed, seeking solutions of the three-dimensional Helmholtz equation in the form $U(r,\theta,\phi)=\mathrm{e}^{\mathrm{i}kr}u(r,\theta,\phi)$, where (r, θ, ϕ) are the usual spherical coordinates, we get the transport equation

$$\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2} + r(1+\mathrm{i}kr)\frac{\partial u}{\partial r} + \frac{1}{2}\nabla^2_{\boldsymbol{\theta}}u + \mathrm{i}kru = 0,$$

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where $\nabla^2_{\theta,\phi}$ is the Laplace–Beltrami operator on the unit sphere $\mathbb{S}_2$. Then, a simple generalization of the technique from §3 makes it possible to compute $U(r, \theta, \phi)$ in the cone $r>0$, $(\theta, \phi)\in\mathfrak{S}$, where $\mathfrak{S}$ is a fixed domain on $\mathbb{S}_2$. After that, the scheme from §4 leads to the representation of $U(r, \theta, \phi)$ in an arbitrary domain described as $r>0$, $(\theta, \phi)\in\mathfrak{S}_r$, where $\mathfrak{S}_r$ is a domain on $\mathbb{S}_2$, which may depend on the radius r.

In the above, we mentioned only polar or spherical coordinates, but a careful analysis shows that the developed method may be used with any coordinate system that allows the representation of the unknown wave field in the form $U(\boldsymbol{x})=u(\boldsymbol{x})e^{iS(\boldsymbol{x})}$, where the eikonal $S(\boldsymbol{x})$ depends on only one of the coordinates of the used system and the amplitude $u(\boldsymbol{x})$ vanishes as this coordinate increases to infinity. In particular, it is possible to use any 'ray' coordinate system $(S, \boldsymbol{\theta})$, where S is an eikonal and $\boldsymbol{\theta}$ is the coordinate vector of the transverse coordinates, which reduces the Helmholtz equation to a transport equation of the type

$$a(S,\boldsymbol{\theta})\frac{\partial^2 u}{\partial S^2} + b(S,\boldsymbol{\theta})\frac{\partial u}{\partial S} + \nabla^2_{\boldsymbol{\theta}} u + c(S,\boldsymbol{\theta})u = 0,$$

where $\nabla^2_{\boldsymbol{\theta}}$ is the Laplace–Beltrami operator on the surface $S=\text{const}$. Since the ray coordinates can be defined for the Helmholtz equation with a non-constant wavenumber, this observation opens the way for the application of the random walk method to problems of wave propagation in non-homogeneous media.

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