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On the electromagnetic field singularities near the vertex of a dielectric wedge

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[1] The problem of electromagnetic scattering by a dielectric wedge has attracted attention recently because of reported disagreement between the theory and numerical experiments concerning a certain behavior of the fields near the wedge vertex. The goal of this paper is to show that what is reported has been not a disagreement but a misinterpretation of the existing theory, which predicts all of the numerically observed effects.

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1. Introduction

[2] The problem of electromagnetic scattering by a dielectric wedge has attracted considerable attention because of numerous reports [*Marx*, 1993; *Anderson and Solodukhov*, 1978; *Beker*, 1991] of disagreement between numerical experiments and the theory [*Meixner*, 1972; *Makarov and Osipov*, 1986; *Budaev*, 1991]. The theory implies that the asymptote of the electromagnetic field near the vertex of a dielectric wedge is described by a generalized Meixner series [*Makarov and Osipov*, 1986; *Budaev*, 1991] which starts from the same leading term as the asymptote of the electrostatic field in a similar configuration. However, numerical results obtained by *Marx* [1993], *Anderson and Solodukhov* [1978] and *Beker* [1991] show only partial agreement with the prediction that an electromagnetic field near the edge of a dielectric wedge behaves like a static field, 'and numerical experiments indicate that the discrepancy is not due to imprecision of the calculations [*Marx*, 1993]. Such a situation is not acceptable because it undermines the credibility of both the numerical and analytical approaches to the equations that model electromagnetic fields in dielectric bodies, and more generally, because it undermines the credibility of the mathematical models themselves.

[3] To clarify the issue, we analyzed the theoretical asymptotes of electromagnetic fields near the apex of dielectric wedges, and this analysis revealed that the asymptotes have 'hidden' second terms which are normally dominated by the leading terms, but, in certain circumstances, they become dominant. Here we revisit the problem considered by *Marx* [1993] and demonstrate that the discrepancies between the numerical simulations and the theory reported by *Marx* [1993] do not contradict the theory but, on the contrary, confirm the theory which, if applied correctly, predicts all of the observed effects.

2. Analysis

[4] Let (ρ, ϕ, z) be the standard cylindrical polar coordinates and let V_1 and V_2 be the wedges defined as

$$V_1: \quad \frac{\beta-\pi}{2} < \phi < \frac{3\pi-\beta}{2}, \qquad V_2: \quad \frac{3\pi-\beta}{2} < \phi < \frac{3\pi+\beta}{2}, \tag{1}$$

where β is an angle of the wedge V_2. The wedge V_1 is occupied by a homogeneous dielectric with permittivity ϵ_1 and permeability μ_1. Similarly, the wedge V_2 has permittivity ϵ_2 and permeability μ_2. This notation coincides with that by *Marx* [1993], but for our purposes it is convenient to introduce an additional coordinate system (ρ, θ, z) related with the system (ρ, ϕ, z) by the formula

$$\theta = \phi - \frac{\pi}{2}, \tag{2}$$

which makes it possible to define the wedges V_1 and V_2 by

$$V_1: \quad |\theta| < \pi - \frac{\beta}{2}, \qquad V_2: \quad \pi - \frac{\beta}{2} < |\theta| < \pi, \tag{3}$$

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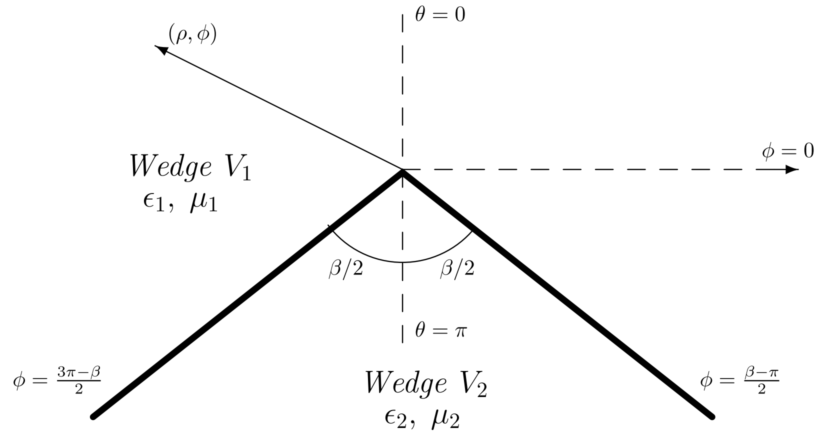

Figure 1. Geometry of the problem.

which imply that $-\pi < \theta < \pi$. The described configuration is illustrated in Figure 1 in which both of the coordinate axes $\phi = 0$ and $\theta = 0$ are shown.

[5] To simplify the analysis, we restrict ourselves to the transverse magnetic mode (TM) where the disagreement reported by *Marx* [1993] between the theory and numerical simulation is especially noticeable. In this mode the electromagnetic field can be defined in terms of a scalar solution H_z of the Helmholtz equation

$$\left(\nabla^2 + \lambda^2 k_m^2\right) H_z = 0, \qquad \text{if} \quad (\rho, \phi) \in V_m, \quad m = 1, 2, \tag{4}$$

where λ is an arbitrary parameter determining the scaling of the polar radius ρ. The field H_z and its derivative $(1/\epsilon)(\partial H_z/\partial\phi)$ have to be continuous everywhere except possibly at the vertex $\rho = 0$. Other components of the electromagnetic fields $\vec{E}$ and $\vec{H}$ are defined by the formulas

$$E_\rho = \frac{i}{\epsilon\mu\rho}\frac{\partial H_z}{\partial\phi}, \qquad E_\phi = \frac{i}{\epsilon\mu}\frac{\partial H_z}{\partial\rho},$$
$$E_z = H_\rho = H_\phi = 0. \tag{5}$$

[6] It is convenient to set the scaling parameter λ by $\lambda k_1 = 1$ which means that distances are measured in wavelength units. Then the structure of the field H_z near the vertex $\rho = 0$ is described in the coordinates (ρ, θ) by the generalized Meixner series [*Budaev*, 1991]

$$H_z \approx \sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\Big\{A_{n\nu}^{+}\rho^{t_n^{+}+\nu}\cos\left[\left(t_n^{+}+\nu\right)\theta\right] + A_{n\nu}^{-}\rho^{t_n^{-}+\nu}\sin\left[\left(t_n^{-}+\nu\right)\theta\right]\Big\}, \text{as} \quad \rho \to 0, \tag{6}$$

determined by the roots $0 \equiv t_0^{\pm} \leq t_1^{\pm}\ldots$ of the transcendental equations

$$\sin\left[\pi t_n^{\pm}\right] \pm p\sin\left[(\pi-\beta)t_n^{\pm}\right] = 0, \tag{7}$$

where

$$p = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}. \tag{8}$$

[7] It is worth observing that since (7) has a fixed root $t_{00}^{\pm} = 0$, the series always contains the "subseries" $A_{00}^{\pm} + A_{01}\rho + A_{02}\rho^2 + \ldots$. In certain particular circumstances the generic series (6) has to be modified. Namely, if for some $n > 0$ a root $t_n^{\pm}$ can be represented as $t_n^{\pm} = t_m^{\pm} + q$, where m, n and q are integers satisfying inequalities $m < n$ and $q > 0$, then, instead of terms with $\rho^{t_n^{\pm}+\nu}$ and $\rho^{t_m^{\pm}+q+\nu}$, the series should involve independent terms with $\rho^{t_n^{\pm}+\nu}(\ln\rho)^l$, $l = 0, 1, \ldots, L$, where L is the number of all possible different representations $t_n^{\pm} = t_m^{\pm} + q$ of the root $t_n^{\pm}$.

[8] The last modification, however, never affects the first few terms of (6) which are given by the expression

$$H_z \approx A_{00}^{+} + A_{10}^{-}\sin\left(t_1^{-}\theta\right)\rho^{t_1^{-}} + \left[A_{01}^{-}\sin\theta + A_{01}^{+}\cos\theta\right]\rho + A_{10}^{+}\cos\left(t_1^{+}\theta\right)\rho^{t_1^{+}}, \tag{9}$$

from which it follows that the fields E_ϕ and E_ρ have the local structure

$$E_\phi \approx B_{10}^{-}\sin\left(t_1^{-}\theta\right)\rho^{t_1^{-}-1} + \left[B_{01}^{-}\sin\theta + B_{01}^{+}\cos\theta\right] + B_{10}^{+}\cos\left(t_1^{+}\theta\right)\rho^{t_1^{+}-1},$$
$$E_\rho \approx -B_{10}^{-}\cos\left(t_1^{-}\theta\right)\rho^{t_1^{-}-1} - \left[B_{01}^{-}\cos\theta + B_{01}^{+}\sin\theta\right] - B_{10}^{+}\sin\left(t_1^{+}\theta\right)\rho^{t_1^{+}-1}, \tag{10}$$

with constants $B_{10}^{\pm}$ and $B_{01}^{\pm}$ proportional to $A_{10}^{\pm}$ and $A_{01}^{\pm}$, respectively.

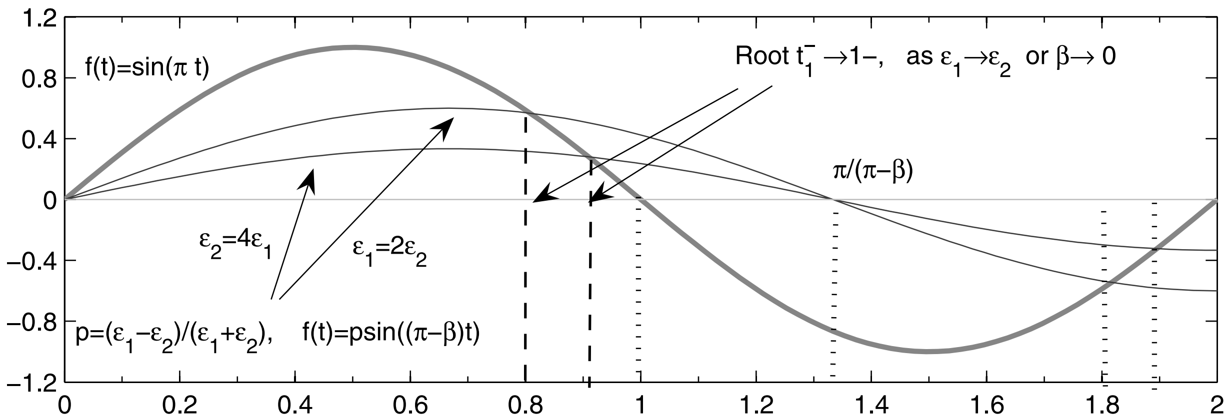

Figure 2. Graphical estimate of the root t_1^- in the case $\epsilon_2 > \epsilon_1$.

[9] It is instructive to observe that the middle terms in (10), which do not depend on the radius ρ, appear in electrodynamics but do not appear in electrostatics. Electrostatic fields near the vertex of the dielectric wedge are described by the Meixner series [*Meixner*, 1972] which may be considered as a particular case of the series (10) with $A_{n\nu}^{\pm} = 0$ for all $\nu \geq 1$. In this case the asymptotes corresponding to those in (10) degenerate to the expressions

$$\begin{aligned} E_{\phi}^{\text{stat}} &\approx B_{10}^{-}\sin\left(t_1^-\theta\right)\rho^{t_1^- -1} + B_{10}^{+}\cos\left(t_1^+\theta\right)\rho^{t_1^+ -1}, \\ E_{\rho}^{\text{stat}} &\approx -B_{10}^{-}\cos\left(t_1^-\theta\right)\rho^{t_1^- -1} - B_{10}^{+}\sin\left(t_1^+\theta\right)\rho^{t_1^+ -1}, \end{aligned} \tag{11}$$

which are, in general, similar to (10) but may diverge from (6) as $\cos(t_1^{\pm}\theta) \to 0$ or $\sin(t_1^{\pm}\theta) \to 0$.

[10] From elementary graphics shown in Figures 2 and 3 it is easy to see that

$$0 < t_1^- \leq 1 \leq t_1^+, \qquad \text{if} \quad \epsilon_2 > \epsilon_1, \tag{12}$$

and that

$$t_1^{\pm} \approx 1 \pm \frac{p\sin\beta}{p(\pi-\beta)-\pi} \longrightarrow 1, \tag{13}$$

as either $(\epsilon_1 - \epsilon_2) \to 0$ or as $\beta \to 0$.

[11] From (9) and (10) it follows that near the vertex H_z approaches a bounded constant and that the fields E_ρ, E_ϕ have singularities of the type $\rho^{t_1^- -1}$ or $\rho^{t_1^+ -1}$ depending on which of the inequalities $t_1^- < 1$ or $t_1^+ < 1$ is valid.

[12] To analyze the fields E_ρ, E_ϕ in detail, we assume for definiteness and for compatibility with the examples from *Marx* [1993] that $\epsilon_2 > \epsilon_1$ which means, in view of (12), that $t_1^- < 1 < t_1^+$.

[13] Consider first the field E_ϕ. From the inequality $t_1^- < 1 < t_1^+$ we see that, in general, this field becomes unbounded as $\rho \to 0$, but if the vertex is approached along the bisector $\theta = 0$, ($\phi = 90°$, in the original coordinates) then $B_{10}^{-}\sin(t_1^-\theta)\rho^{t_1^-}-1 = 0$ and E_ϕ remains

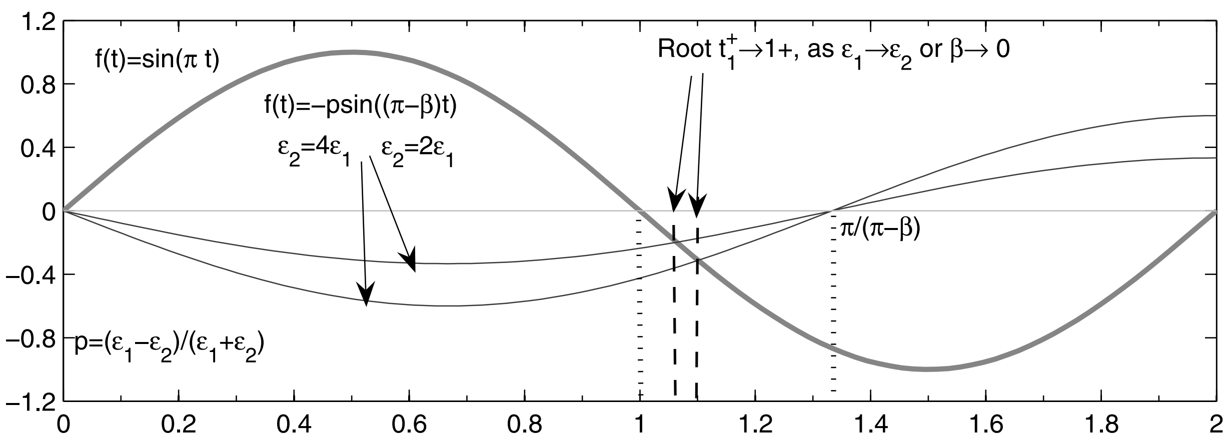

Figure 3. Graphical estimate of the root t_1^+ in the case $\epsilon_2 > \epsilon_1$.

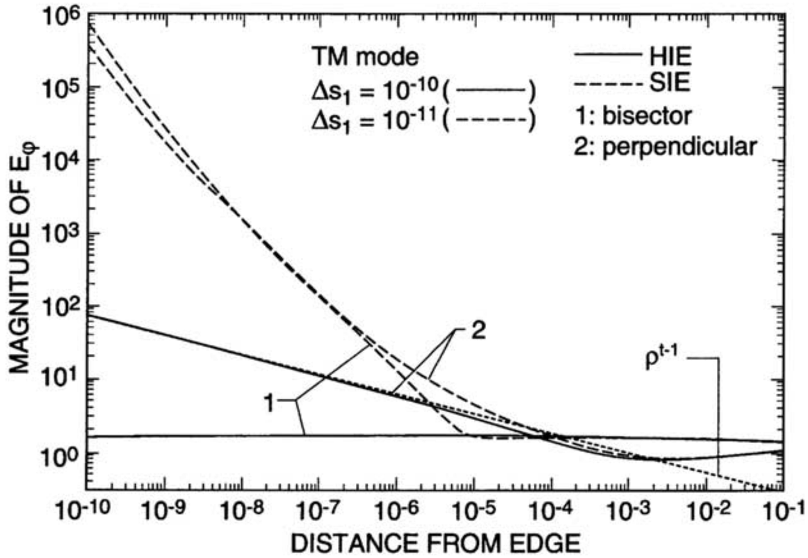

Figure 4. Comparison of E_ϕ in (TM) mode (reproduction of Figure 4 from *Marx* [1993]).

bounded approaching the finite value $E_\phi \approx B_{01}^{+}$, which explains the results in Figure 4.

[14] Figure 4 reproduces Figure 4 from *Marx* [1993], which compares the electromagnetic fields near the vertex of the dielectric wedge as predicted by the Meixner asymptotes [*Meixner*, 1972] with the numerical results obtained by *Marx* [1993] using a hypersingular equation (HIE) and with the results obtained by *Marx* [1990] using a singular integral equation (SIE). All of the reported results correspond to the configuration in Figure 1 with $\beta = 90°$ and $\epsilon_2 = 10\epsilon_1$. Both of the axes in Figure 4 have logarithmic scales, and they correspond to the distance from the vertex and to the magnitude of the electric field E_ϕ. The dotted line corresponds to the straight line $\log(E_\phi) = (t_1^- - 1)\log(\rho)$ + const, which represents the Meixner asymptote. The solid and dashed lines correspond to the results obtained by the methods using hypersingular and singular integral equations, respectively. Since the first method is more accurate, we do not discuss any dashed lines below.

[15] The solid line labeled "1" shows the electric field E_ϕ computed along the bisectrix $\theta = 0$, and this is essentially a horizontal line, which agrees well with the prediction that $E_\phi \approx B_{01}^{+}$ = const as $\rho \ll 1$. Similarly, the solid line labeled "2" shows that $E_\phi(\rho, \theta)$ along the perpendicular direction $\theta = -90°$ approaches an inclined line, which agrees well with the prediction

$$\log\bigl(E_\phi\bigr) \approx \bigl(t_1^- - 1\bigr)\log(\rho) + \log\bigl(B_{10}^-\bigr) + \log\bigl(\sin\bigl(\pi t_1^-/2\bigr)\bigr) = \bigl(t_1^- - 1\bigr)\log(\rho) + \text{const}, \tag{14}$$

based on the asymptote (10).

[16] Next we observe that (10) implies that for any $\theta \neq 0$, the length of the interval $0 < \rho < \rho_*(\theta, \delta)$ where the relative error of the asymptotic approximation of $E_\phi(\rho, \theta)$ by the first term of (10) does not exceed $\delta > 0$ can be estimated as

$$\rho_*(\theta, \delta) = O\Bigl(\bigl[\delta/\sin\bigl(t_1^-\theta\bigr)\bigr]^{1/\left(t_1^- - 1\right)}\Bigr). \tag{15}$$

[17] Then, applying this formula to the configuration in Figure 1 with the permittivities related by $\epsilon_2 = 5\epsilon_1$, we get the estimates

$$\rho_*(90°, 0.05) \approx \begin{cases} 10^{-14} & \text{if} \quad t_1^- - 1 \approx -0.1, \\ 10^{-10} & \text{if} \quad t_1^- - 1 \approx -0.15, \\ 10^{-7} & \text{if} \quad t_1^- - 1 \approx -0.2, \end{cases} \tag{16}$$

which explains why some of the curves in Figure 5 fit the theoretical predictions while others do not. Indeed, if $\beta > 30°$, then the power $(t_1^- - 1)$ varies from -0.15 to -0.2, and therefore the principal terms of the asymptotes in (10) should dominate starting from distances of 10^{-7} to 10^{-10}, included in Figure 5. However, if $\beta = 10°$, then $(t_1^- - 1) \approx -0.1$ and the principal term of (10) is expected to dominate at a distances smaller than 10^{-14} which are far beyond the range studied by *Marx* [1993].

[18] Finally, we consider the field E_ρ. It is clear from (10) that in general, $E_\rho(\rho, \theta)$ grows as $O(\rho^{t_1^-} - 1)$ when $\rho \to 0$, but if the vertex is approached along the rays $\theta = \pm\pi/2t_1^-$ where the coefficient $\cos(t_1^-\theta)$ of the singular

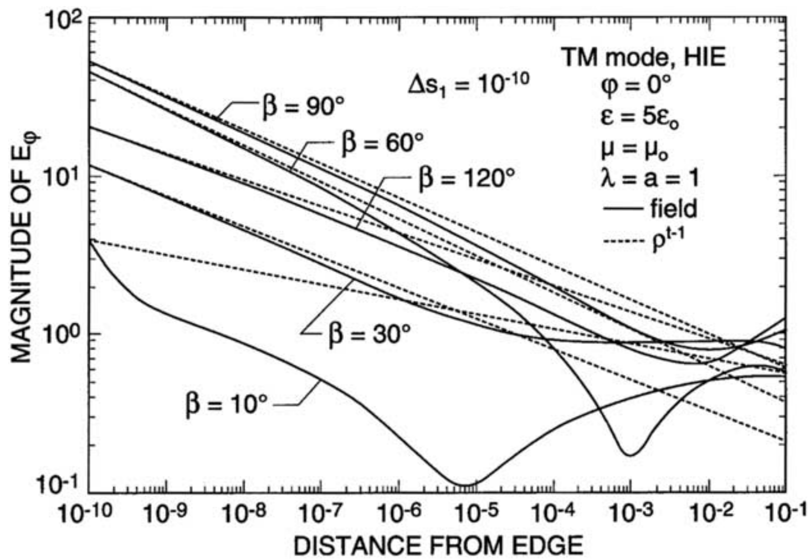

Figure 5. Dependence of E_ϕ on the wedge angle in the TM mode (Figure 5 from *Marx* [1993]).

term in (10) vanishes, then E_ρ does not grow and approaches a bounded constant. Unlike the case for the field E_ϕ these directions are not fixed but depend on the particular configurations. However, if the root t_1^- is close to unity ($t_1^- \approx 1$), then the rays $\theta = \pm\pi/2t_1^-$ are close to the rays $\theta = \pm 90°$ referred to by *Marx* [1993] as the perpendicular directions. Therefore, if $t_1^- \approx 1$ the field E_ρ is expected to approach a constant along the perpendicular direction $\theta = \pm 90°$ which agrees with the results in Figure 6. Indeed, the curve of Figure 6 labeled "2" and "$\epsilon = 2\epsilon_0$" corresponds to $t_1^- \approx 0.9$ and it is close to a horizontal line as predicted by the theory.

[19] As for the behaviors of the field E_ρ along other directions, they are similar to the behavior of the field E_ϕ, and (16) estimates the region where the field may be approximated by the first term of (10) with the relative

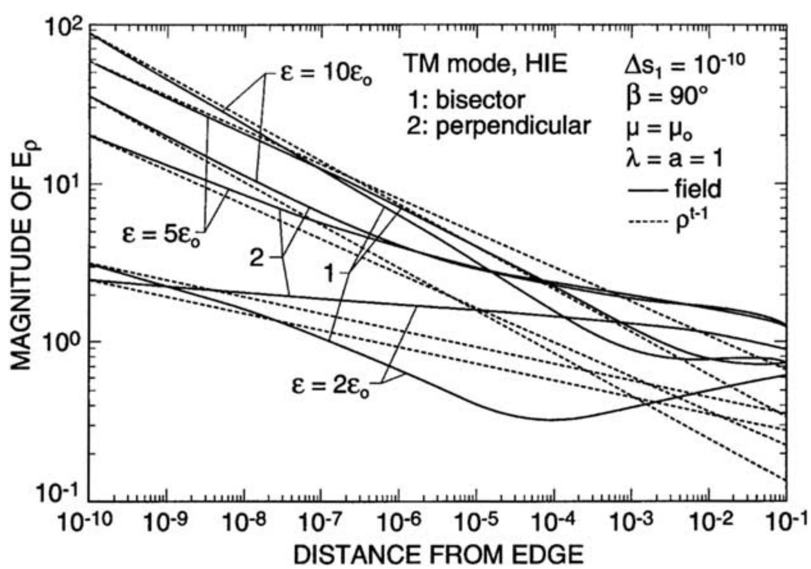

Figure 6. Dependence of E_ρ on the permittivity in the TM mode (Figure 6 from *Marx* [1993]).

error of 5%. These estimates obviously agree with the numerical results in Figure 6.

3. Conclusion

[20] Thus we see that the numerical results presented by *Marx* [1993] do not contradict the theory but, instead, confirm the theory: theoretical curves fit the numerical data in areas where they are expected to fit; they fail to fit in the areas where they should not fit; and the differences between the asymptotic and numerical results have the trends predicted by the analysis. This validates the computations by *Marx* [1993] and, at the same time, reconfirms that the generalized Meixner series (6) correctly describes electromagnetic fields near the vertex of the dielectric wedge. However, our analysis reveals that the leading term of the generalized Meixner series, which describes electrodynamic fields, may not always coincide with the leading term of the standard Meixner series [*Meixner*, 1972], which describes static fields. The discrepancy may occur in narrow domains where the leading term of the generalized Meixner series either completely annihilates or becomes comparable with lower-order terms, as happens in the examples considered above. Therefore the observations [*Marx*, 1993; *Anderson and Solodukhov*, 1978; *Beker*, 1991] that the electrodynamic fields near the vertex of the wedge do not behave like static fields are correct, but the theory gives no reason to expect overwise.

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