A probabilistic approach to wave propagation and scattering

Bair V. Budaev and David B. Bogy

Department of Mechanical Engineering, University of California, Berkeley, California, USA

Received 21 September 2004; revised 3 February 2005; accepted 23 March 2005; published 14 September 2005.

[1] The probabilistic approach to wave propagation starts in a way that is similar to ray theory, from the representation of the wave field as a product of the amplitude and of the exponent of the eikonal, which is computed by a canonical technique of analytical mechanics. However, an important difference is that the amplitude is not approximated but is represented by exact probabilistic formulas that admit efficient numerical evaluation, and that is a direct improvement of many asymptotic solutions. This approach is shown to be an effective tool for the analysis of numerous wave propagation problems, including those of wave diffraction by a screen occupying a plane angular sector and of electromagnetic diffraction by a wedge with anisotropic impedance boundary conditions.

Citation: Budaev, B. V., and D. B. Bogy (2005), A probabilistic approach to wave propagation and scattering, *Radio Sci.*, *40*, RS6S07, doi:10.1029/2004RS003176.

1. Introduction

[2] In the theory of wave propagation solutions of the Helmholtz equation

$$\nabla^2\Phi(x) + k^2\kappa^2(\vec{x})\Phi(\vec{x}) = \Upsilon(\vec{x}), \qquad k = \text{const} \quad (1)$$

are customarily sought in the product form $\Phi(\vec{x}) = u(\vec{x})\,\mathrm{e}^{\mathrm{i}kS(\vec{x})}$, where the phase $S(\vec{x})$ satisfies the eikonal equation $[\vec{\nabla}\ S(\vec{x})]^2 = \kappa^2(\vec{x})$, and the amplitude $u(\vec{x})$ satisfies the complete transport equation

$$\frac{\mathrm{i}}{2k}\nabla^2 u - \vec{\nabla}S\cdot\vec{\nabla}u - \frac{1}{2}\left(\nabla^2 S\right)u + F = 0, \qquad F = -\mathrm{i}\Upsilon\mathrm{e}^{-\mathrm{i}kS}/k. \quad (2)$$

[3] There are at least three reasons justifying the use of such a representation of the wave field: (1) The eikonal equation admits a constructive solution by use of the canonical Hamilton-Jacobi method of analytical mechanics; (2) the structure of the eikonal is closely connected with the intuitively clear idea of propagation along rays; and (3) in many cases the amplitude $u(\vec{x})$ can be approximated sufficiently well by the ray theory solution u_0 of the equation obtained from (2) by dropping its first term.

[4] If the first term in (2) is dropped then the resulting first-order equation has the solution

$$u_0(\vec{x}) = \int_0^\infty F\left(\vec{\xi}_t\right)\mathrm{e}^{-\frac{1}{2}\int_0^t \nabla^2 S\left(\vec{\xi}_s\right)\mathrm{d}s}\mathrm{d}t \equiv \int_0^\infty \sqrt{\frac{J\left(\vec{\xi}_t\right)}{J(x)}}\,F\left(\vec{\xi}_t\right)\mathrm{d}t, \quad (3)$$

where $J(\vec{x})$ is a characteristic of the vector field $\vec{\nabla}(S)$ widely known in the literature as its "geometrical divergence," and $\vec{\xi}_t = \vec{x} - \vec{\nabla}\ S(\vec{\xi}_t)t$ is the solution of the ordinary differential equation $\mathrm{d}\vec{\xi}_t = -\vec{\nabla}\ S(\vec{\xi}_t)\mathrm{d}t$, $\vec{\xi}_0 = \vec{x}$, so that the trajectory of $\vec{\xi}_t$ is the ray along which the wave arrives at $\vec{x}$.

[5] It is clear that the approximation $u \approx u_0$ is accurate only when $k \gg 1$ and when the geometrical divergence $J(\vec{x})$ has no singularities in the vicinity of the ray passing through the observation point, which is not the case in many important situations arising, for example, in problems of propagation of low-frequency waves, problems of diffraction, and problems of wave propagation in nonhomogeneous media. Such limitations naturally have generated numerous attempts to improve the elementary approximation $\phi \approx u_0$ either by constructing a series $\phi \approx u_0 + u_1 + \ldots$, or by a more complicated choice of the initial approximation u_0, or by a combination of both of these ideas.

[6] Although much progress has been achieved in finding asymptotic or approximate solutions of the complete transport equation (2), it is nevertheless instructive and useful to observe that the exact solution $u(\vec{x})$ can be represented by explicit probabilistic formulas

0048-6604/05/2004RS003176$11.00

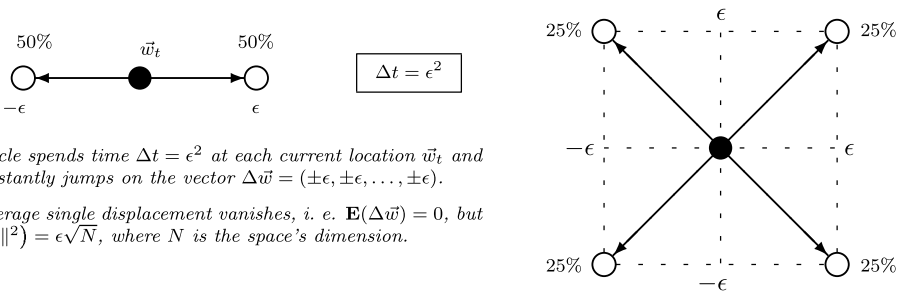


Figure 1. Discrete Brownian motions in $\mathbb{R}$ and in $\mathbb{R}^2$.

which are exact in the same rigorous sense as $f(x) = \sin(x)$ is an exact solution of $f'' + f = 0$. Correspondingly, these solutions do not fail anywhere including at caustics, and they do not loose any information which may be used for the analysis of the physical phenomena.

[7] The basic ideas of the probabilistic approach to wave propagation are traced back to the 1920s–1930s, when the link between partial differential equations and Brownian motion was first observed [*Philips and Wiener*, 1923; *Courant et al.*, 1928; *Petrovsky*, 1934], but the rapid development in this area was made only after the publications of the landmark papers of Feynman [*Feynman*, 1942, 1948] and *Kac* [1949] which presented similar but at the same time very different results: *Feynman* [1942, 1948] represented solutions of the Schrödinger equation by heuristically introduced path integrals which did not admit probabilistic interpretation, and *Kac* [1949] adapted Feynman's formula to the heat conduction equation which was solved by means of the rigourously justified Wiener integration in a functional space which had a clear probabilistic sense.

[8] Since the Schrödinger equation is closely related to the Helmholtz equation, it is not surprising that there have been attempts to employ Feynman's path integral for the analysis of wave propagation. In the first papers exploring this direction [*Buslaev*, 1967; *Keller and KcLaughlin*, 1975] the ray approximation of the wave field was derived from the path integral solution of the Helmholtz equation. Later, the path integrals were used for numerical simulations of acoustical [*Schlottmann*, 1999] and electromagnetic [*Nevels et al.*, 2000] waves, but as mentioned in the survey [*Galdi et al.*, 2000], the perspectives of broader application of the path integrals to wave propagation were limited, presumably because of the notorious difficulty of computation of the Feynman path integrals.

[9] It is well known [*Feynman*, 1998] that the probabilistic formulas employed in Kac's solution [*Kac*, 1949] of the heat conduction equation rest on a rigorous mathematical foundation and admit efficient numerical simulation, but this equation is not directly connected to the Helmholtz equation describing wave propagation. Nevertheless, it has been recently found that there is a natural way of solve the Helmholtz equation by a probabilistic "random walk" method which is based on Kac's formula and provides a direct improvement of the simple ray approximation of the theoretically exact solution of the Helmholtz equation.

[10] In the next section we briefly discuss the principles of random walk and of its relationship with differential equations. Then we derive solutions of the Helmholtz equation which directly improve the approximation provided by ray theory. Finally, to illustrate an application of the random walk method to problems of diffraction we derive a probabilistic solution of the two dimensional problem of diffraction by a wedge with impedance boundary conditions.

2. Probabilistic Solutions of Differential Equations

[11] To expose the relationship between differential equations and random motions we need first to introduce the notions of Brownian motion, of Brownian motion with a drift, and of reflected Brownian motion.

[12] Suppose a particle moves along the real axis $-\infty < x < \infty$ starting at the time $t = 0$ from $x = 0$ and jumping at the instants $t_n = n\Delta t$ the distance ϵ in either of two equally probable directions (see the left diagram of Figure 1). Then, the particle's position x_n in the time interval $[t_n, t_{n+1})$ prior to the $(n + 1)$th jump is represented by the sum $x_n = \sum_{\nu=1}^{n} \Delta x_\nu$ of independent random variables $\Delta x_n = \pm\epsilon$ with two equally probable values. The sequences x_n and t_n determine a piecewise constant function $\widetilde{w}_t = x_{\tilde{t}}$, where $\tilde{t} = [\frac{t}{\Delta t}]\,\Delta t$, is the last instant of the series t_n preceding or coinciding with t. It is well known [*Dynkin*, 1965; *Wiener*, 1923] that if the time and space meshes decrease together as $\Delta t = \epsilon^2 \to 0$, then the jump motion $\widetilde{w}_t$ converges in some sense to a continuous random motion w_t which is usually referred to as a one-dimensional Brownian motion or, equivalently, as a one-dimensional Wiener process. As for the

Brownian motion in $\mathbb{R}^N$ it is defined as a superposition $\vec{w}_t = (w_t^1, w_t^2, \ldots, w_t^N)$ of one-dimensional Brownian motions along each of the Cartesian axes (see the right diagram of Figure 1).

[13] We are now in a position to derive a probabilistic solution of the equation

$$\frac{1}{2}\nabla^2\phi(x) + B\phi(x) + F(x) = 0, \qquad B = \text{const} < 0, \tag{4}$$

considered, for transparency, on the line $-\infty < x < \infty$.

[14] It is clear that by using the approximation

$$\nabla^2\phi(x) = \frac{1}{\varepsilon^2}[\phi(x+\varepsilon) + \phi(x-\varepsilon) - 2\phi(x)] + o(\varepsilon^2), \tag{5}$$

we can represent the solution of (4) as

$$\phi(x) = \frac{[\phi(x+\varepsilon) + \varepsilon^2 F(x)]e^{B\varepsilon^2} + [\phi(x-\varepsilon) + \varepsilon^2 F(x)]e^{B\varepsilon^2}}{2} + o(\varepsilon^2), \tag{6}$$

which can also be written in the form

$$\phi(x) = \mathbf{E}\left\{\phi(x \pm \varepsilon)e^{B\varepsilon^2} + \varepsilon^2 F(x)e^{B\varepsilon^2}\right\} + o(\varepsilon^2), \tag{7}$$

where $\mathbf{E}$ denotes the average computed over the random choices of equally possible signs in $\phi(x \pm \varepsilon)$. Then, the values $\phi(x \pm \varepsilon)$ involved in (7) can themselves be computed by the formula (7) resulting in the representation

$$\phi(x) = \mathbf{E}\left\{\phi(x \pm \varepsilon \pm \varepsilon)e^{2B\varepsilon^2} + \left[F(x)e^{B\varepsilon^2} + F(x \pm \varepsilon)e^{2B\varepsilon^2}\right]\varepsilon^2\right\} + o(\varepsilon^2). \tag{8}$$

Obviously, the described iteration of (7) can be computed as many times as desired, and after n iterations we arrive at the representation

$$\phi(x) = \mathbf{E}\left\{\phi(x_n)e^{nB\varepsilon^2} + \varepsilon^2\sum_{\nu=0}^{n-1} F(x_\nu)e^{(\nu+1)B\varepsilon^2}\right\} + o(\varepsilon^2), \tag{9}$$

where

$$x_n = x\underbrace{\pm\varepsilon \pm \varepsilon \ldots \pm \varepsilon}_{n-\text{times}} \tag{10}$$

is a position on the n-legged discrete random walk with the spatial step $\Delta x = \varepsilon$ corresponding to the chronological step $\Delta t = \varepsilon^2$. Then, passing to the limit $\varepsilon \to 0$ we convert (9) to the expression

$$\phi(x) = \mathbf{E}\left\{\phi(\xi_t)e^{Bt} + \int_0^t F(\xi_s)e^{Bs}ds\right\}, \tag{11}$$

where the mathematical expectation is computed over all possible trajectories of the Brownian motion $\xi_t = x + w_t$ launched from the observation point $\xi_0 = x$.

[15] If the solution $\phi(x)$ of equation (4) is bounded and if $B < 0$, then in passing to the limit $t \to \infty$ we arrive at the Feynman-Kac formula

$$\phi(x) = \mathbf{E}\left\{\int_0^\infty F(x + w_t)e^{Bt}dt\right\}, \tag{12}$$

where w_t is the standard Brownian motion.

[16] Equation (4) is not the only one that can be explicitly solved by averaging over trajectories of random motions. In particular, the more general equation with variable coefficients

$$\frac{\sigma^2}{2}\nabla^2\phi(\vec{x}) + \vec{A}(\vec{x})\cdot\vec{\nabla}\phi(\vec{x}) + B(x)\phi(\vec{x}) + F(\vec{x}) = 0 \tag{13}$$

can be solved by a formula that is similar to but more general than (12),

$$\phi(x) = \mathbf{E}\left\{\int_0^\infty F(\xi_t)e^{\int_0^t B(\xi_s)ds}dt\right\}, \tag{14}$$

which has the averaging over a Brownian motion with a drift, which is described below.

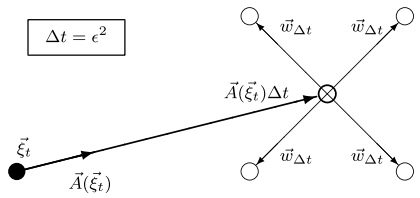
[17] Let $\vec{A}(\vec{x})$ be a vector field in $\mathbb{R}^N$, and let $\vec{\xi}_t$ be a random motion (stochastic process) in $\mathbb{R}^N$ launched from $\vec{x}$ and consisting of the jumps

$$\vec{\xi}_t \longrightarrow \vec{\xi}_{t+\Delta t} = \vec{\xi}_t + \sigma\left(\vec{\xi}_t\right)\Delta\vec{\xi}_t, \qquad \Delta\vec{\xi}_t = \vec{A}\left(\vec{\xi}_t\right)\Delta t + \vec{w}_{\Delta t}, \tag{15}$$

where $\vec{w}_{\Delta t}$ is the Brownian displacement on the time interval Δt. Then, passing in (15) to the limit $\Delta t \to 0$ we obtain a stochastic differential equation

$$d\vec{\xi}_t = \sigma d\vec{w}_t + \vec{A}dt, \qquad \vec{\xi}_0 = \vec{x}, \tag{16}$$

which is usually referred to as Ito's stochastic differential equation [*Dynkin*, 1965; *Ito and McKean*, 1965]. It is clear from (15) that the jump $\Delta\vec{\xi}_t$ can be considered as a superposition of the deterministic move $\Delta\vec{\zeta}_t = \vec{A}(\vec{\xi}_t)\,\Delta t$ and of the Brownian displacement $\vec{w}_{\Delta t}$. Because of this interpretation, illustrated in Figure 2, the random motion described by (16) is referred to as a Brownian motion with a drift.



A particle spends time $\Delta t = \epsilon^2$ at each current location $\vec{\xi}_t$ and then instantly jumps on the vector $\Delta\vec{\xi} = \vec{A}(\vec{\xi}_t)\Delta t + \vec{w}_{\Delta t}$, whose first component $\vec{A}(\vec{\xi}_t)\Delta t$ is deterministic, while the second component $\vec{w}_{\Delta t} = (\pm\epsilon, \pm\epsilon)$ is a random Brownian jump. As $\epsilon \ll 1$, the random component $\vec{w}_{\Delta t}$ dominates in the total single displacement estimated as $\Delta\vec{\xi} = (\pm\epsilon, \pm\epsilon) + O(\epsilon^2)$. However, the mathematical expectation of the random component vanishes and the average displacement in time Δt is estimated as $\mathbf{E}(\vec{\xi}_{\Delta t}) = \vec{A}(\vec{\xi}_t)\Delta t = \vec{A}(\vec{\xi}_t)\,\epsilon^2$.

Figure 2. Discrete Brownian motion with a drift.

[18] It is remarkable that the probabilistic approach can be extended from partial differential equations like (13) in unbounded domains to boundary value problems

$$\begin{aligned} &\frac{\sigma^2}{2}\nabla^2\phi + \vec{A}\cdot\vec{\nabla}\phi + B\phi + F\Big|_G = 0, \\ &\vec{a}\cdot\vec{\nabla}\phi + b\,\phi + f\,\Big|_{\partial G} = 0, \end{aligned} \tag{17}$$

formulated in a domain $G \in \mathbb{R}^N$ with the boundary ∂G. Coefficients $\vec{A}$, B and F are assumed to be defined inside G, while the coefficients $\vec{a}$, b and f are defined on ∂G. Additionally, for definiteness, we assume that vectors $\vec{a}$ are oriented inward toward G.

[19] Application of the random walk method to the problem (17) is based on the idea of defining random motions on the closure $G \cup \partial G$ whose behavior inside G corresponds to the operator $L_G = \frac{1}{2}\,\sigma^2\nabla^2 + \vec{A}\cdot\vec{\nabla}$, and whose behavior on the boundary ∂G corresponds to the first-order operator $L_{\partial G} = \vec{a}\cdot\vec{\nabla}$. Since both operators L_G and $L_{\partial G}$ are particular cases of the general second-order operator discussed in the previous section, it is natural to expect that inside G the random walk should be a Brownian motion with a drift associated with the vector fields $\vec{A}$, and on the boundary ∂G it should be a deterministic motion along the vector $\vec{a}$. More precisely, the random motion $\vec{\xi}_t$ can be defined by the equations

$$\mathrm{d}\vec{\xi}_t = \begin{cases} \sigma\left(\vec{\xi}_t\right)\mathrm{d}\vec{w}_t + \vec{A}\left(\vec{\xi}_t\right)\mathrm{d}t, & \text{if} \quad \vec{\xi}_t \in G, \\ \vec{a}\left(\vec{\xi}_t\right)\mathrm{d}t, & \text{if} \quad \vec{\xi}_t \in \partial G, \end{cases} \tag{18}$$

as illustrated in Figure 3.

[20] Stochastic processes $\vec{\xi}_t^{\,x}$ defined by (18) are known as reflecting random motions and they can also be introduced as continuous solutions of the stochastic differential equation

$$\begin{aligned} &\mathrm{d}\vec{\xi}_t = \sigma\left(\vec{\xi}_t\right)\mathrm{d}\vec{w}_t + \vec{A}\left(\vec{\xi}_t\right)\mathrm{d}t + \vec{a}\left(\vec{\xi}_t\right)\mathrm{d}\lambda_t, \\ &\vec{\xi}_0 = x, \quad \lambda_0 = 0, \end{aligned} \tag{19}$$

with an additional unknown λ_t, which is required to be a continuous nondecreasing stochastic process increasing only on the "visiting" set $\mathfrak{g} = \{t\colon \vec{\xi}_t \in \partial G\}$ of instants when the path $\vec{\xi}_t$ touches the boundary ∂G.

[21] The process λ_t is called the "local time at ∂G" because it admits interpretation as the measure of the time spent by the path $\vec{\xi}_t$ on the boundary ∂G. It follows from (18) that the local time λ_t can be approximated as $\lambda_t = \lim_{\epsilon\to 0}(\epsilon\, N^{\epsilon}_{\partial G})$, where $N^{\epsilon}_{\partial G}$ is the number of times when the corresponding discrete random motion visited the boundary.

[22] Random motions with reflections are discussed in detail by *Skorokhod* [1961], *Watanabe* [1971], *Ito and McKean* [1965], and *Freidlin* [1985], and it is shown by *Freidlin* [1985] that the solution of the problem (13), (17) can be represented by the formula

$$\begin{aligned} \phi(\vec{x}) = \mathbf{E}\Bigg\{ &\int_0^\infty F\left(\vec{\xi}_t\right)\mathrm{e}^{\int_0^t B\left(\vec{\xi}_s\right)\mathrm{d}s + b\left(\vec{\xi}_s\right)\mathrm{d}\lambda_s}\mathrm{d}t \\ &+ \int_0^\infty f\left(\vec{\xi}_t\right)\mathrm{e}^{\int_0^t B\left(\vec{\xi}_s\right)\mathrm{d}s + b\left(\vec{\xi}_s\right)\mathrm{d}\lambda_s}\mathrm{d}\lambda_t\Bigg\}, \end{aligned} \tag{20}$$

which may be regarded as an extension of the Feynman-Kac formula (14).

[23] It is worth emphasizing that the expression (20) remains valid in a quite general setting: it represents the solution of the problem (13), (17) in arbitrary domains, with arbitrary coefficients in the equation (13), and with arbitrary coefficients in the boundary conditions (17). In particular, this solution may be used in the case of the Dirichlet boundary conditions $\phi|_{\partial G} = f$, which correspond to the coefficients $a = 0$ and $b = -1$ in (17). In this case, the random motion $\vec{\xi}_t$ stops as soon as it hits the boundary ∂G and the Feynman-Kac formula reduces to a simpler form

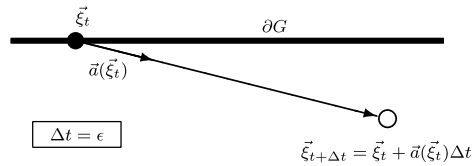
$$\phi(\vec{x}) = \mathbf{E}\left\{\int_0^\tau F\left(\vec{\xi}_t\right)\mathrm{e}^{\int_0^t B\left(\vec{\xi}_s\right)\mathrm{d}s}\mathrm{d}t + f\left(\vec{\xi}_\tau\right)\mathrm{e}^{\int_0^\tau B\left(\vec{\xi}_s\right)\mathrm{d}s}\right\}, \tag{21}$$

where τ is the "exit time" defined as the first time when $\vec{\xi}_t$ touches ∂G.

3. Probabilistic Solutions of the Helmholtz Equation

[24] A typical problem of wave propagation can be reduced to computation of the wave field $\Phi(\vec{\xi})$ which solves the Helmholtz equation

$$\nabla^2\Phi(\vec{x}) + k^2\kappa^2(\vec{x})\Phi(\vec{x}) = \Upsilon(\vec{x}), \tag{22}$$



Inside the domain G a particle performs the Brownian motion with a drift, as shown on Fig 2. A particle spends time $\Delta t = \epsilon^2$ at each location and then jumps on a random vector $\Delta\vec{\xi} = (\pm\epsilon, \pm\epsilon) + O(\epsilon^2)$.

If the path $\vec{\xi}_t$ hits the boundary ∂G it spends time $\Delta t = \epsilon$ there, and then is reflected back to G, jumping on the deterministic vector $\Delta\vec{\xi} = \vec{a}(\vec{\xi}_t)\Delta t$.

Figure 3. Discrete Brownian motion with reflections.

with the wave number subdivided for convenience into two components: a variable coefficient $\kappa(x)$ related to the material properties of the medium, and a constant k related to the frequency.

[25] Although equation (22) matches the structure of (13), its solution cannot be straightforwardly computed by the Feynman-Kac formula (14) because the positiveness of the coefficient $B(x) = k^2\kappa^2(x) > 0$ leads to a divergent integral in (14). This difficulty, however, can be overcome by a standard idea of seeking the solution of (22) in the product form

$$\Phi(\vec{x}) = u(\vec{x})\mathrm{e}^{\mathrm{i}kS(\vec{x})}, \tag{23}$$

which has been known since the early 1800s as a convenient anzatz for exact and approximate solutions of the Helmholtz equation.

[26] Direct substitution of (23) into (22) makes it possible to split the Helmholtz equation into the eikonal equation

$$|\vec{\nabla}S|^2 = \kappa^2(\vec{x}) \tag{24}$$

and the complete transport equation

$$\frac{\mathrm{i}}{2k}\nabla^2\phi - \vec{\nabla}S\cdot\vec{\nabla}\phi - \frac{1}{2}\left(\nabla^2 S\right)\phi + F = 0, \qquad F = \Upsilon\mathrm{e}^{-\mathrm{i}kS(\vec{x})}, \tag{25}$$

with the coefficients determined through $S(x)$.

[27] Equation (24) is a well-known eikonal equation from ray optics [*Keller*, 1958; *Maslov and Fedoriuk*, 1981] and it is a particular case of the Hamilton-Jacobi equation of classical mechanics [*Arnold*, 1989]. These equations have been exhaustively studied in the literature so it may be taken as granted that the eikonal $S(x)$ is already known either on the domain G or on a multisheeted Lagrangian manifold constructed over G similar to Riemann surfaces in the theory of analytic functions. After the eikonal is computed, equation (25) may be considered as a second-order partial differential equation with predefined coefficients.

[28] Equations (24) and (25) together are equivalent to the Helmholtz equation (22), and equation (25) has been widely used as the starting point of different approximate approaches to the general wave radiation problem.

[29] If $k \gg 1$, one may neglect the first term in (25) and arrive at the "transport" equation $2\vec{\nabla}\,S\cdot\nabla\phi + (\nabla^2 S)\,\phi = 0$, widely used in the geometrical theory of diffraction [*Keller*, 1958; *Maslov and Fedoriuk*, 1981] for derivation of short-wave asymptotic approximations of wave fields.

[30] Another approximate approach to equation (25) arises if instead of neglecting all of the first term in (25) we neglect only part of it. A broad spectrum of "parabolic equation" methods in the theory of high-frequency wave propagation is based on this idea originating from the contributions of Fock and Leontovich [*Fock and Leontovich*, 1946; *Fock*, 1965].

[31] It is remarkable that although much attention has been focused on the reduction of the complete transport equation (25) to simpler equations which admit efficient solutions, the probabilistic approach briefly described above makes it possible to solve the complete transport equation (25) directly, without any approximation.

[32] Indeed, observing that equation (22) matches the structure of (13) it is easy to conclude that its solution can be represented by the mathematical expectation

$$u(x) = \mathbf{E}\left\{\int_0^\infty F\left(\vec{\xi}_t\right)\exp\left(-\frac{1}{2}\int_0^t \nabla^2 S\left(\vec{\xi}_s\right)\mathrm{d}s\right)\right\}, \tag{26}$$

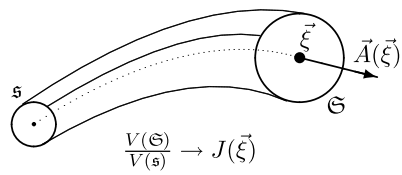
computed over the trajectories of the random motion $\vec{\xi}_t$ governed by the stochastic equation

$$\mathrm{d}\vec{\xi}_t = \sqrt{\frac{\mathrm{i}}{k}}\,\mathrm{d}\vec{w}_t - \vec{\nabla}S\left(\vec{\xi}_t\right)\mathrm{d}t, \qquad \vec{\xi}_t = 0, \tag{27}$$

driven by the standard Brownian motion $\vec{w}_t$.

[33] A specific feature of the complete transport equation (25) is that its coefficient $B = -\frac{1}{2}\nabla^2 S$, which appears in the exponent in (26), is related to the coefficient $\vec{A} = -\vec{\nabla}S$, which determines the drift of the random motion $\vec{\xi}_t$. This relationship makes it possible to convert the solution (26) to an alternative form which establishes the bridge between the exact solution of the problem and its approximations provided by the ray theory and the geometrical theory of diffraction.

[34] To transform the expression (26) we first observe that the Laplacian $\nabla^2 S$ can be represented by the Liou-



A small surface element $\mathfrak{S}$ with center at $\vec{\xi}$ and normal to $\vec{A}(\vec{\xi})$ is projected by the dynamical system $\frac{d\vec{\xi}}{dt} = \vec{A}(\vec{\xi})$ to the piece $\mathfrak{S}_0$ of a fixed reference surface S_0.

When the the diameter of $\mathfrak{S}$ tends to zero, the ratio $V(\mathfrak{S})/V(\mathfrak{s})$ of volumes of $\mathfrak{S}$ and $\mathfrak{s}$ converges to the geometric divergence $J(\vec{\xi})$ of the vector field $\vec{A}(\vec{\xi})$.

Figure 4. Geometrical divergence of the vector field.

ville formula [*Arnold*, 1989; *Maslov and Fedoriuk*, 1981] which states that

$$\nabla^2 S\left(\vec{\xi}\right) = \vec{\nabla} S\left(\vec{\xi}\right) \cdot \vec{\nabla}\left[\ln J\left(\vec{\xi}\right)\right], \tag{28}$$

where $J(\vec{\xi})$ is the "geometrical divergence" (see Figure 4) of the vector field $\vec{A} = -\vec{\nabla}S$. Then, we recall the chain rule of the stochastic calculus which states that if $\vec{\xi}_t$ is a random motion controlled by the stochastic equation $d\vec{\xi}_t = \sigma d\vec{w}_t + \vec{A}dt$, then the differential of $f(\vec{\xi}_t)$ is delivered by the formula

$$df\left(\vec{\xi}_t\right) = \left(\vec{A} \cdot \vec{\nabla} f + \frac{\sigma^2}{2}\nabla^2 f\right)dt + \sigma \vec{\nabla} f \cdot \vec{w}_t, \tag{29}$$

usually referred to as Ito's formula [*Dynkin*, 1965; *Ito and McKean*, 1965]. In particular, applying Ito's formula (29) to the function $f(\vec{\xi}_t) = \ln[J(\vec{\xi}_t)]$, we obtain the identity

$$d\ln\left[J\left(\vec{\xi}_t\right)\right] = -\vec{\nabla} S \cdot \vec{\nabla}\ln[J]dt - \frac{i}{2k}\nabla^2 \ln[J]dt + \sqrt{\frac{i}{k}}\vec{\nabla}\ln[J]d\vec{w}_t, \tag{30}$$

which may be combined with (28), resulting in the representation

$$-\frac{1}{2}\nabla^2 S = d\ln\sqrt{J} + \frac{i}{4k}\nabla^2 \ln[J]dt - \sqrt{\frac{i}{4k}}\vec{\nabla}\ln[J]d\vec{w}_t. \tag{31}$$

Finally, substituting (31) into (26) and taking into account the initial condition $\vec{\xi}_0 = \vec{x}$, we arrive at the solution of equation (25) in the form

$$u(\vec{x}) = \mathbf{E}\left\{\int_0^\infty F\left(\vec{\xi}_t\right)\sqrt{\frac{J\left(\vec{\xi}_t\right)}{J(x)}}\exp\left(\int_0^t \frac{i}{4k}\nabla^2 \ln J\left(\vec{\xi}_s\right)ds - \sqrt{\frac{i}{4k}}\vec{\nabla}\ln J\left(\vec{\xi}_s\right)d\vec{w}_s\right)\right\}, \tag{32}$$

given in terms of the geometrical divergence $J(x)$ of the ray field $\vec{A} = -\vec{\nabla}S$.

[35] It should be noticed that the solution (32) of equation (25) heavily depends on the wave number k. Consider, for example, the case when $k \gg 1$. Then, as follows from (27), the random component of the motion $\vec{\xi}_t$ becomes negligible and this motion may be approximated by the deterministic motion along the ray defined as the integral line of the ordinary differential equation $\dot{\vec{\xi}}_t = -\vec{\nabla}\, S(\vec{\xi}_t)dt$. Moreover, in cases when the geometrical divergence $J(\vec{\xi})$ is bounded, the integrals in the exponent in (32) tend to zero, and (32) can be approximated by the deterministic formula

$$u(\vec{x}) \approx u_0(x) = \int_0^\infty F\left(\vec{\xi}_t\right)\sqrt{\frac{J\left(\vec{\xi}_t\right)}{J(x)}}dt, \qquad d\vec{\xi}_t = -\vec{\nabla}S\left(\vec{\xi}_t\right), \quad \vec{\xi}_0 = \vec{x}, \tag{33}$$

which is well known from ray theory.

[36] One of the most serious drawbacks of the ray method is that (33) fails to approximate the solution of the complete transport equation (25) near caustics where $J(x) = 0$. Since the function $J(x)$ appears also in the denominator of the exact solution (32) it is important to notice that (32) does not fail on caustics, because as $J(x) \to 0$ the exponent in (32) becomes highly oscillatory which compensates for the vanishing of $J(x)$ in the denominator. On the other hand, to compute wave fields on caustics there is no need to use (32) at all, because the equivalent expression (26) with no vanishing denominator may be used instead.

4. Two-Dimensional Problem of Diffraction by an Impedance Wedge

[37] In the previous section we outlined the probabilistic approach to wave propagation in inhomogeneous media over the entire space. However, this method can also be employed for the analysis of wave radiation and diffraction. As an example illustrating the potential of the probabilistic approach we apply it here to the two dimensional problem of diffraction by a wedge with impedance boundary conditions.

[38] Let G be an infinite wedge $r > 0$, $0 < \theta < \alpha$ given in the polar coordinates (r, θ). Then, a classic problem of

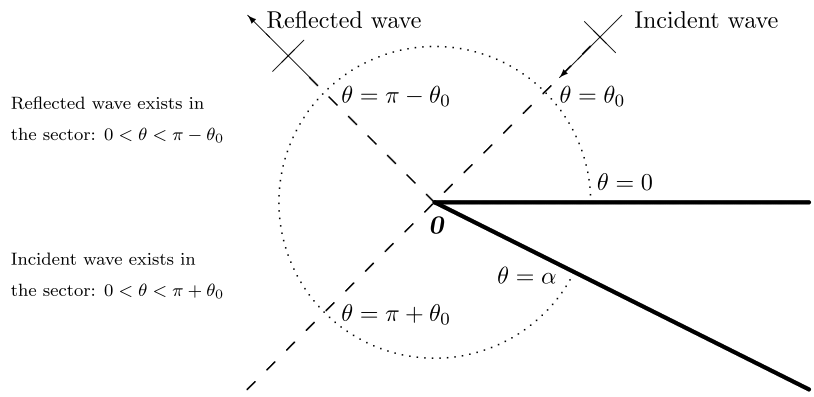


Figure 5. Geometry of the problem on the (r, θ) plane.

diffraction of a plane incident wave $U_0 = e^{-ikr\cos(\theta-\theta_0)}$ by a wedge G with impedance boundary conditions can be reduced to the Helmholtz equation

$$\nabla^2 U + k^2 U = 0, \tag{34}$$

complemented by condition at the vertex $U(r, \theta) = 0(1)$ as $r \to 0$, by the boundary conditions

$$\frac{A_1}{r}\frac{\partial U}{\partial \theta} + ikB_1 U\Big|_{\theta=0} = -\frac{A_2}{r}\frac{\partial U}{\partial \theta} + ikB_2 U\Big|_{\theta=\alpha} = 0, \tag{35}$$

and by the condition at infinity which requires that the solution $U(r, \theta)$ does not contain any arriving component except for the incident wave $U_0(r, \theta)$. The ratios ikB_n/A_n, $n = 1, 2$, are usually referred to as the surface impedances and it is well known that the inequalities

$$\mathrm{Re}\left(\frac{ikB_1}{A_1}\right) \geq 0, \qquad \mathrm{Re}\left(\frac{ikB_2}{A_2}\right) \geq 0 \tag{36}$$

guarantee the existence of the unique solution of this problem.

[39] Elementary geometric-optical analysis suggests that the solution of this diffraction problem can be represented as a superposition

$$U(r, \theta) = U_g(r, \theta) + u_d(r, \theta)e^{ikr} \tag{37}$$

of the discontinuous wave fields $U_g(r, \theta)$ and $U_d(r, \theta) = u_d(r, \theta)e^{ikr}$, which are referred to as the geometric wave and the diffracted wave, respectively.

[40] The geometric field $U_g(r, \theta)$ consists of the incident wave and a finite number of reflected waves, which can be computed a priori by the laws of geometrical optics. To avoid too much emphasis on material that is not essential for our purposes here we do not present general formulas for all angles α and θ_0, but only mention that if the following conditions (see Figure 5) are satisfied

$$\alpha > \pi, \qquad \pi + \theta_0 < \alpha, \tag{38}$$

then the geometric field is defined by

$$U_g(r, \theta) = \chi(\theta_1 - \theta)K_1 e^{ikr\cos(\theta-\theta_1)} + \chi(\theta_2 - \theta)e^{ikr\cos(\theta-\theta_2)}, \tag{39}$$

where $\chi(\theta)$ is the Heaviside step function, $\theta_n = \pi + (-1)^n\theta_0$, and

$$K_1 = \frac{A_1 \sin\theta_0 - B_1}{A_1 \sin\theta_0 + B_1}, \tag{40}$$

is the reflection coefficient of the face $\theta = 0$ illuminated by the incident wave.

[41] Unlike the geometric field $U_g(r, \theta)$, the amplitude $u_d(r, \theta)$ of the diffracted field is an unknown function which has to obey the equation

$$\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2} + r\left(\frac{1}{2} + ikr\right)\frac{\partial u}{\partial r} + \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{ikr}{2}u = 0, \tag{41}$$

accompanied by the interface conditions

$$u(r, \theta_1 + 0) - u(r, \theta_1 - 0) = K_1 \tag{42}$$

$$u(r, \theta_2 + 0) - u(r, \theta_2 - 0) = 1 \tag{43}$$

$$u'_\theta(r, \theta_m + 0) - u'_\theta(\theta_m - 0) = 0, \qquad m = 1, 2, \tag{44}$$

together with the condition at infinity

$$u(r, \theta) = o(1), \qquad r \to \infty, \quad \theta \neq \theta_{1,2} \tag{45}$$

and the impedance boundary conditions

$$A_1 u'_\theta(r,0) + \mathrm{i}krB_1 u(r,0) = 0,$$
$$-A_2 u'_\theta(r,\alpha) + \mathrm{i}krB_2 u(r,\alpha) = 0. \tag{46}$$

[42] Equation (41) in a wedge $0 < \theta < \alpha$ with homogeneous Dirichlet boundary conditions has been studied by *Budaev and Bogy* [2003] where its exact solution is represented as a mathematical expectation computed over the trajectories of a specified random motion that is stopped as soon as it hits one of the faces $\theta = 0$ or $\theta = \alpha$ of the wedge. However, as discussed in Section 2, the probabilistic solution of equation (41) with interface boundary conditions (42)–(43) can be extended to problems with impedance boundary conditions (46) by a simple modification of the behavior of random motion at the boundary, which should reflect the motion back to the domain instead of stopping it.

[43] More precisely, using a straightforward combination of the general theory [*Freidlin*, 1985] outlined in the end of Section 2 with the specifics of equation (41) and interface conditions (42)–(43), it is easy to arrive at the representation of the amplitude $u(r, \theta)$ in the form

$$u(r,\theta) = \mathbf{E}\left\{\sum_{\nu=1}^{\infty} \delta_\nu Q_\nu \exp\left(\frac{\mathrm{i}k}{2}\int_0^{\tau_\nu} \xi_s \mathrm{d}s + \mathrm{i}kB_1 \int_0^{\tau_\nu} \xi_s \mathrm{d}\lambda_s^1 + \mathrm{i}kB_2 \int_0^{\tau_\nu} \xi_s \mathrm{d}\lambda_s^2\right)\right\}, \tag{47}$$

the exact meaning of which is explained below.

[44] The mathematical expectation $\mathbf{E}$ is computed over the trajectories of the independent random motions ξ_t and η_t referred to hereafter as the radial and the angular motions, respectively. The radial motion is launched at the time $t = 0$ from the position $\xi_0 = r$ and is controlled by the stochastic differential equation

$$\xi_0 = r, \qquad \mathrm{d}\xi_t = \xi_t \mathrm{d}w_t^1 + \xi_t\left(\frac{1}{2} + \mathrm{i}k\xi_t\right)\mathrm{d}t, \tag{48}$$

where w_t^1 is the standard one-dimensional Brownian motion. As shown by *Budaev and Bogy* [2003] this motion is confined to the first quadrant $\mathrm{Re}(\xi_t) > 0$, $\mathrm{Im}(\xi_t) \geq 0$ and has a drift toward the unreachable point $\xi = \mathrm{i}/2k$. The angular motion η_t is launched from the position $\xi_0^2 = \theta$ and is governed by the equations

$$\eta_0 = \theta, \qquad \mathrm{d}\eta_t = \begin{cases} \mathrm{d}w_t^2, & \text{if} \quad \eta_t \neq 0,\ \alpha, \\ -A_1 \mathrm{d}t, & \text{if} \quad \eta_t = 0, \\ A_2 \mathrm{d}t, & \text{if} \quad \eta_t = \alpha, \end{cases} \tag{49}$$

which show that inside the segment $0 \leq \eta \leq \alpha$ the motion η_t runs as a standard Brownian motion, but when it reaches the segment's boundaries it is deterministically reflected back.

[45] The angular motion running inside the interval $0 \leq \eta \leq \alpha$ crosses interior points $\eta = \pi \pm \theta_0$ at the times $t = \tau_\nu$ enumerated by the index $\nu \geq 1$ which determines the factors δ_ν and Q_ν of (47) by the following rules:

$$\delta_\nu = \begin{cases} 1, & \text{if} \quad \eta_{\tau_\nu} < \eta_{\tau_\nu - 0}, \quad \text{and} \quad \eta_{\tau_\nu + 0} < \eta_{\tau_\nu}, \\ -1, & \text{if} \quad \eta_{\tau_\nu} > \eta_{\tau_\nu - 0}, \quad \text{and} \quad \eta_{\tau_\nu + 0} > \eta_{\tau_\nu}, \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

and

$$Q_\nu = K_1, \text{if} \quad \eta_{\tau_\nu} = \theta_1, 1, \text{if} \quad \eta_{\tau_\nu} = \theta_2. \tag{51}$$

It is clear that $\delta_\nu = 1$ if at the time τ_ν the interface $\eta = \theta_1$ or $\eta = \theta_2$ is crossed from left to right. Similarly, the value $\delta_\nu = -1$ corresponds to the crossing from right to left, and $\delta_\nu = 0$ corresponds to the case when the interface is touched but not intersected. The value of Q_ν, is determined by the particular interface $\eta = \theta_1$ or $\eta = \theta_2$ that is touched at the time $t = \tau_\nu$. Finally, λ_t^1 and λ_t^2 in the integrals from (47) represent the local times of the angular motion η_t on the interfaces $\eta = \theta_1$ and $\eta = \theta_2$.

[46] A rigorous discussion of stochastic differential equations, stochastic integrals, and of local times can be found in the literature on stochastic processes [*Dynkin*, 1965; *Freidlin*, 1985], but for our purposes it suffices to view the random motions ξ_t, η_t and the integrals from (47) as the limits as $\Delta t \to 0$ of discrete processes as described below.

[47] The radial motion ξ_t that is controlled by the stochastic equation (48) can be considered as a sequence of random jumps

$$\xi_t \longrightarrow \xi_{t+\Delta t} = \xi_t \pm \xi_t\sqrt{\Delta t} + \xi_t\left(\frac{1}{2} + \mathrm{i}k\xi_t\right)\Delta t, \tag{52}$$

following each other with an infinitesimally small time increment $\Delta t \to 0$. Similarly, the angular motion η_t may be approximated by discrete jumps determined by the rule

$$\eta_t \longrightarrow \eta_{t+\Delta t} = \begin{cases} \eta_t \pm \sqrt{\Delta t}, & \text{if} \quad \eta_t \neq 0,\ \alpha, \\ \eta_t - A_1 \Delta t, & \text{if} \quad \eta_t = 0, \\ \eta_t + A_2 \Delta t, & \text{if} \quad \eta_t = \alpha, \end{cases} \tag{53}$$

depending on the current position of the moving point. These discrete approximations of the radial and angular random motions are closely related to the possibility of approximating the integrals from (47) by the Riemann sums

$$\int_0^t \xi_s \mathrm{d}s \approx \Delta t \sum_{\nu=0}^{\nu\Delta t \leq t} \xi_{\nu\Delta t}\ ,$$
$$\int_0^t \xi_s \mathrm{d}\lambda_s^{1,2} \approx \Delta t \sum_{\nu=0}^{\nu\Delta t \leq t} \Omega_\nu^{1,2} \xi_{\nu\Delta t}, \tag{54}$$

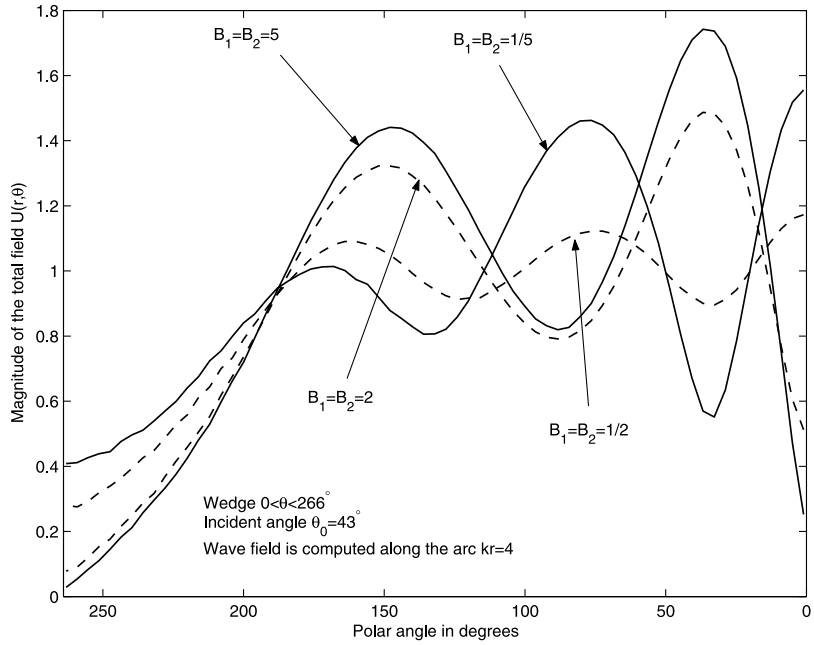


Figure 6. Simulated total wave fields.

where the factors

$$\Omega_\nu^1 = \begin{cases} 1, & \text{if} \quad \nu\Delta t = 0, \\ 0, & \text{otherwise} \end{cases}$$

and (55)

$$\Omega_\nu^2 = \begin{cases} 1, & \text{if} \quad \nu\Delta t = \alpha, \\ 0, & \text{otherwise} \end{cases}$$

indicate the times when the angular motion η_t is reflected by the boundaries $\eta = 0$ and $\eta = \alpha$, respectively.

[48] To illustrate the feasibility of the obtained probabilistic solution for calculations of the problem of diffraction by a wedge with different impedances on its faces we conducted a series of numerical simulations for the wedge $\Gamma(266°)$ exposed to the incident plane wave $U_0(r, \theta) = e^{-ir\cos(\theta-\theta_0)}$ arriving along the ray $\theta_0 = 43°$. In this configuration, which was selected for comparability with *Osipov* [2004], the shadow domain $223° < \theta < 266°$ is illuminated only by the diffracted waves, the sector $137° < \theta < 223°$ is open to the incident and diffracted waves, and the domain $0 < \theta < 137°$ is exposed to the incident, reflected and diffracted waves.

[49] Figure 6 shows the magnitudes of the total wave fields in the wedge with the coefficients $A_1 = A_2 = 1$ and $B_1 = B_2 = ikB$, with the impedance B ranging from $B = 1/5$ to $B = 5$. The dashed lines correspond to the impedances $B = 1/2$ and $B = 2$, while the solid lines correspond to the impedances $B = 1/5$ and $B = 5$ which were considered by *Osipov* [2004] by two conventional methods, including the Maliuzhinets' closed-form solution. Since the computations reported by *Osipov* [2004] (Figure 1) were made along the arc $kr = 4$, we set $k = 1$ and $r = 4$, which allows us to show that the numerical results obtained by the probabilistic method agree with those delivered by other more traditional techniques. The results provided by the two methods appear to be identical.

[50] All numerical results were obtained by the averaging of 2000 discrete random walks (52)–(54) with the time increment $\Delta t = 0.01$. The computations were carried out on a laptop PC using the simple MATLAB code presented below. We include this code just to illustrate its amazingly short length and transparency.

```
function [u, U] = point(r0, f0, a0, M1, M2, alpha, k, ep, N)
  ik = i.*k;
  Cr = (sin (a0) − M1)./(sin (a0) + M1);
  [f, r, Q] = deal (repmat (f0, N, 1), repmat (r0, N, 1), ones
(N, 1));
  [u, J1, J2] = deal (0, (f > pi − a0), (f > pi + a0));
    while isempty (r)
    ds = ep./abs (r);
    wr = ds.*sign (rand (length (r), 1) − 0.5);
    wf = ds.*sign (rand (length (r), 1) − 0.5);
    f = f + wf;
    [Ja, Jb] = deal (f > pi − a0, f > pi + a0);
    u = u + sum (Q.*(Cr.*(J1 − Ja) + (J2 − Jb))./N;
    [I1, I2] = deal (f < 0.02, f > alpha + 0.02);
    ds(ds > ep) = ep;
```

```
    f (I1) = ds (I1);
    f (I2) = alpha − ds (I2);
    Q (I1) = Q (I1).*exp (ik.*ds (I1).*M1.*r (I1));
    Q (I2) = Q (I2).*exp(ik.*ds (I2).*M2.*r (I2));
    Q = Q.*exp (0.25.*ik.*r.*wr.^2);
    r = r.*(1 + wr + (0.5 + ik.*r).*wr.^2);
    Q = Q.*exp(0.25.*ik.*r.*wr.^2);
    I = abs (Q) > 1e − 3;
    [f, r, Q, J1, J2] = deal (f (I), r (I), Q (I), Ja (I), Jb (I));
end
U = u.*exp(ik.*r0);
if f0 < pi − a0 U = U + Cr.*exp(−ik.*r0.*cos(f0 + a0));
end
if f0 < pi + a0 U = U + exp(−ik.*r0.*cos(f0 − a0)); end
```

[51] Input parameters r0, f0 are the polar coordinates of the observation point; a0 is the incidence angle; M1 = ikB_1/A_1 and M2 = ikB_2/A_2 are the impedances on $\theta = 0$ and $\theta = \alpha$; alpha is the wedge angle; k = k is the wave number; ep = $\sqrt{\Delta t}$ is the spatial step; and N is the number of averaged random walks. The output parameter u = $u(r, \theta)$ is the amplitude of the diffracted field, and U = $U(r, \theta)$ is the total field.

[52] Finally, it should be mentioned that we made no attempts to find the most efficient numerical scheme for simulation of the solution (47) or even to optimize the employed scheme. Although we admit the importance of development of efficient algorithms, we believe that this should be a subject of separate research.

5. Conclusion

[53] The results presented here show that the synthesis of the ray method with the probabilistic technique provides a promising approach to problems of wave propagation and diffraction which may be used both for effective numerical evaluation and for asymptotic analysis. The advantages of this combination include, but are not limited to: physical meaningfulness is retained from ray theory; solutions can be computed at individual points instead of on massive meshes; versatility; numerical implementations may employ simple and scalable parallel algorithms with minimal requirement of computer memory.

[54] The basic ideas of the probabilistic approach to wave propagation were developed by *Budaev and Bogy* [2001, 2002b]. *Budaev and Bogy* [2002a] showed that the random walk method makes it possible to describe such phenomena as backscattering which is predicted neither by the ray theory nor by a more general method of parabolic equations. More recently, the probabilistic method was successfully applied to a formidable three dimensional problem of diffraction by a plane angular sector. *Budaev and Bogy* [2004] considered this problem with Dirichlet boundary conditions, and *Budaev and Bogy* [2005a] extended the analysis to Neumann boundary conditions. Finally, *Budaev and Bogy* [2005b] extended the random walk method from scalar problems to the three dimensional vector problem of electromagnetic wave diffraction by a wedge with anisotropic impedance boundary conditions.

[55] All of these together make the random walk method attractive for the analysis of wave propagation, and we anticipate that it will evolve into a practical and broadly used tool.

[56] **Acknowledgments.** This research was supported by NSF grants CMS-0098418 and CMS-0408381 and by the William S. Floyd Jr. Distinguished Professorship in Engineering held by D. Bogy.

References

Arnold, V. I. (1989), *Mathematical Methods in Classical Mechanics*, *Grad. Texts Math.*, vol. 60, 2nd ed., Springer, New York.

Budaev, B. V., and D. B. Bogy (2001), Probabilistic solutions of the Helmholtz equations, *J. Acoust. Soc. Am.*, *109*(5), 2260–2262.

Budaev, B. V., and D. B. Bogy (2002a), Analysis of one-dimensional wave scattering by the random walk method, *J. Acoust. Soc. Am.*, *111*(6), 2555–2560.

Budaev, B. V., and D. B. Bogy (2002b), Application of random walk methods to wave propagation, *Q. J. Mech. Appl. Math.*, *55*(2), 209–226.

Budaev, B. V., and D. B. Bogy (2003), Random walk approach to wave propagation in wedges and cones, *J. Acoust. Soc. Am.*, *114*(4), 1733–1741.

Budaev, B. V., and D. B. Bogy (2004), Diffraction by a plane sector, *Proc. R. Soc. London, Ser. A*, *460*(2052), 3529–3546.

Budaev, B. V., and D. B. Bogy (2005a), Diffraction of a plane wave by a sector with Dirichlet or Neumann boundary conditions, *IEEE Trans. Antennas Propag.*, *53*(2), 711–718.

Budaev, B. V., and D. B. Bogy (2005b), Two dimensional diffraction by a wedge with impedance boundary conditions, *IEEE Trans. Antennas Propag.*, *53*(6), 2073–2080.

Buslaev, V. S. (1967), Continuun integrals and the asymptotic behavor of the solutions of parabolc equations as $t \rightarrow 0$: Applications to diffraction, in *Topics in Mathematical Physics*, edited by S. Berman, pp. 67–86, Consult. Bur., New York.

Courant, R., K. Friedrichs, and K. H. Lewy (1928), Uber die partiellen Differenzengleichungen der mathematischen Physik, *Math. Annal.*, *100*, 32–74.

Dynkin, E. B. (1965), *Markov Processes*, *Grundlehren Math Wiss. Einzeldarstellungen*, vol. 121–122, Springer, New York.

Feynman, R. P. (1942), The principle of least action in quantum mechanics, Ph.D. thesis, Princeton Univ., Princeton, N. J.

Feynman, R. P. (1948), Space-time approach to non-relativistic quantum mechanics, *Rev. Mod. Phys.*, *20*, 367–387.

Feynman, R. P. (1998), *Statistical Mechanics: A Set of Lectures*, Addison-Wesley, Boston, Mass.

Fock, V. A. (1965), *Electromagnetic Diffraction and Propagation Problems*, *Int. Ser. Monogr. Electromagn. Waves*, vol. 1, Elsevier, New York.

Fock, V. A., and M. A. Leontovich (1946), Solution of the problems of propagation of electromagnetic waves along the Earth's surface by the method of parabolic equation, *J. Phys. Moscow*, *10*, 1–13.

Freidlin, M. (1985), *Functional Integration and Partial Differential Equations*, *Annals Math. Stud.*, vol. 109, Princeton Univ. Press, Princeton, N. J.

Galdi, V., L. B. Felsen, and I. M. Pinto (2000), Path integrals for electromagnetics: Are they useful?, paper presented at XIII Italian National Conference on Electromagnetics, Ital. Soc. Electromagn., Como, Italy.

Ito, K., and H. P. McKean Jr. (1965), *Diffusion Processes and Their Sample Paths*, Springer, New York.

Kac, M. (1949), On distribution of certain Wiener functionals, *Trans. Am. Math. Soc.*, *65*(1), 1–13.

Keller, J. B. (1958), A geometric theory of diffraction, in *Calculus of Variations and Its Applications*, *Symp. Appl. Math.*, vol. 8, pp. 27–52, McGraw-Hill, New York.

Keller, J. B., and D. W. McLaughlin (1975), The Feynman integral, *Am. Math. Mon.*, *82*(5), 451–465.

Maslov, V. P., and M. V. Fedoriuk (1981), *Semi-Classical Approximation in Quantum Mechanics*, Springer, New York.

Nevels, R. D., J. A. Miller, and R. E. Miller (2000), A path integral time-domain method for electromagnetic scattering, *IEEE Trans. Antennas Propag.*, *48*(4), 565–573.

Osipov, A. V. (2004), A hybrid technique for the analysis of scattering by impedance wedges, paper presented at 2004 International Symposium on Electromagnetic Theory, Union Radio Sci. Int., Pisa, Italy.

Petrovsky, I. G. (1934), Über das Irrahrtproblem, *Math. Annal.*, *109*, 425–444.

Philips, H. P., and N. Wiener (1923), Nets and Dirichlet problem, *J. Math. Phys.*, *2*, 105–124.

Schlottmann, R. B. (1999), A path intergal formulation of acoustic wave propagation, *Geophys. J. Int.*, *137*(2), 353–363.

Skorokhod, A. V. (1961), Stochastic equations for diffusion processes in a bounded domain, *Theor. Probab. Appl.*, *6*, 264–274.

Watanabe, S. (1971), On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions, *J. Math. Kyoto Univ.*, *11*(1), 169–180.

Wiener, N. (1923), Differential space, *J. Math. Phys.*, *2*, 131–174.

D. B. Bogy and B. V. Budaev, Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA. (dbogy@cml.me.berkeley.edu; budaev@berkeley.edu)