Diffraction by a wedge with a face of variable impedance

Bair V. Budaev[1] and David B. Bogy[1]

Received 29 November 2006; revised 7 March 2007; accepted 9 April 2007; published 27 October 2007.

[1] The two-dimensional problem of diffraction by a wedge with a face of variable impedance is explicitly solved by a probabilistic random walk method. The solution admits numerical simulation based on simple scalable algorithms with unlimited capability for parallel processing. The diffracted field is represented as a mathematical expectation of a specified functional on trajectories of random motions determined by the configuration of the problem. The solution is not significantly more difficult than a similar probabilistic solution of the problem with constant impedance faces.

Citation: Budaev, B. V., and D. B. Bogy (2007), Diffraction by a wedge with a face of variable impedance, *Radio Sci.*, *42*, RS6S02, doi:10.1029/2006RS003597.

1. Introduction

[2] For more than a century problems of wave diffraction by an infinite wedge have been at the center of the mathematical theory of diffraction starting from the landmark papers of *Poincaré* [1892, 1897] and *Sommerfeld* [1896], who presented explicit descriptions of wavefields generated by a plane incident wave in an infinite wedge with ideally reflecting faces. However, they used very different methods. The solution of Poincaré was based on the method of separation of variables, and it led to a solution represented by an infinite series of Bessel functions multiplied by associated trigonometric polynomials. The solution of Sommerfeld was in terms of the so-called Sommerfeld integral representing wavefields as superpositions of plane waves. Although these two solutions look very different, they are equivalent and can be easily converted from one to the other. It took about fifty years before a problem of diffraction by a wedge with impedance boundary conditions was solved. In the 1950s this problem was independently solved by *Maliuzhinets* [1951, 1955], *Senior* [1959] and *Williams* [1959]. They used techniques that are similar in general but different in detail that employed Sommerfeld's representation of wavefields to reduce the problem to certain functional equations in the complex plane. In the following decades the Maliuzhinets method was applied to a number of two- and three-dimensional problems of diffraction by infinite wedges. However, all of these problems either had boundary conditions with constant coefficients or even simpler boundary conditions with the coefficients proportional to the distance from the vertex, as considered by *Felsen and Marcuvitz* [1972].

[3] Here we consider a two-dimensional problem of diffraction by a wedge with a variable impedance at its face. We employ a novel probabilistic approach to wave propagation that is not restricted to problems with simple canonical geometries or to problems with special (constant) boundary conditions. The probabilistic approach has already been successfully applied by the authors to a standard two-dimensional problem of diffraction by a wedge with constant impedances, as well as to more challenging three-dimensional problems of diffraction by a plane angular sector [*Budaev and Bogy*, 2004] and by an infinite wedge with anisotropic face impedances [*Budaev and Bogy*, 2006b]. Most recently, the probabilistic method was applied to diffraction by an arbitrary convex polygon with side-wise constant impedances [*Budaev and Bogy*, 2006a]. This problem with nontrivial geometry does not have known conventional closed-form solutions, but its probabilistic solution is not considerably more complex than the solutions of the other problems mentioned above. Here we take the next step and extend the probabilistic method in a way which makes it possible to handle boundary conditions with variable coefficients. This is the centerpiece of the paper, and it is based on further development of the idea of analytical continuation introduced by *Budaev and Bogy* [2005c] and *Budaev and Bogy* [2006a], who applied it to other problems of interest.

[4] The paper is organized as follows. In section 2 we formulate the problem and convert it to a form suitable for the application of the random walk method. Section 3

[1]Department of Mechanical Engineering, University of California, Berkeley, California, USA.

0048-6604/07/2006RS003597$11.00

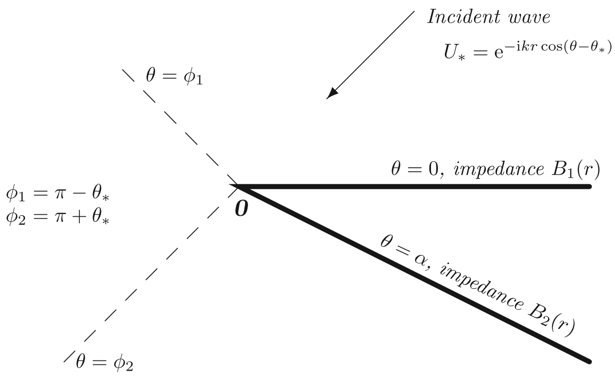

Figure 1. Diffraction by a wedge with faces of nonconstant impedance.

deals with the problem with constant coefficients. Most of this material is published by *Budaev and Bogy* [2005b], but it is included here also to make the presentation of section 4 transparent and self-consistent. The main material of the paper is concentrated in section 4, and section 5 presents numerical experiments based on the obtained solution.

2. Formulation of the Problem

[5] Let (r, θ) be standard polar coordinates and let $\mathfrak{G} = \mathfrak{G}(0, \alpha)$ be an angular domain

$$\mathfrak{G}(0,\alpha): \qquad r > 0, \quad 0 < \theta < \alpha. \tag{1}$$

Then the problem of diffraction of the plane incident wave

$$U_*(r,\theta) = e^{-ikr\cos(\theta-\theta_*)}, \qquad 0 < \theta_* < \alpha, \tag{2}$$

in the wedge $\mathfrak{G}$ consists of the computation of the wavefield $U(r, \theta)$, which obeys the Helmholtz equation $\nabla^2 U + k^2 U = 0$, accompanied by the standard conditions at infinity and by the boundary conditions

$$\begin{aligned} \left.\frac{1}{r}\frac{\partial U}{\partial \theta} + ikB_1(r)U\right|_{\theta=0} &= 0, \\ \left.-\frac{1}{r}\frac{\partial U}{\partial \theta} + ikB_2(r)U\right|_{\theta=\alpha} &= 0, \end{aligned} \tag{3}$$

where $B_1(r)$ and $B_2(r)$ are impedances assumed to satisfy the inequalities

$$\mathrm{Re}(B_n) \geq 0, \quad n = 1, 2, \tag{4}$$

which guarantee that the problem has a unique bounded solution.

[6] To reduce insignificant detail, we limit ourselves to the analysis of the configuration characterized by the inequalities $\alpha > \pi$ and $\theta_* < \alpha - \pi$, which guarantee that the face $\theta = 0$ is illuminated by the incident wave, while the face $\theta = \alpha$ is located in the shadow zone, as shown in Figure 1. Additionally, to focus only on diffraction we assume that the impedance of the illuminated face is constant, so that $B_1(r)$ = const. Then elementary analysis suggests that we seek the total field $U(r, \theta)$ in the form of the superposition

$$U = U_i + KU_r + KU_1 + U_2 \tag{5}$$

of the yet unknown scattered fields $U_1(r, \theta)$ and $U_2(r, \theta)$ together with the two predefined components

$$\begin{aligned} U_i &= \begin{cases} e^{-ikr\cos(\theta-\theta_*)}, & \text{if} \quad \theta < \phi_2, \\ 0, & \text{if} \quad \theta > \phi_2, \end{cases} \\ U_r &= \begin{cases} e^{-ikr\cos(\theta+\theta_*)}, & \text{if} \quad \theta < \phi_1, \\ 0, & \text{if} \quad \theta > \phi_1, \end{cases} \end{aligned} \tag{6}$$

where

$$\phi_1 = \pi - \theta_*, \qquad \phi_2 = \pi + \theta_*, \tag{7}$$

and

$$K = \frac{\sin\theta_* - B_1}{\sin\theta_* + B_1} \tag{8}$$

is the reflection coefficient of the surface with a constant impedance B_1.

[7] From (5) and (6) it follows that the scattered fields $U_n(r, \theta)$, where n = 1,2, must be solutions of the Helmholtz equation that satisfy the conditions at infinity

$$U_n e^{-ikr} = o(1), \qquad r \to \infty, \quad \theta \neq \phi_n, \tag{9}$$

the interface conditions

$$\begin{aligned} U_1(r, \phi_1 + 0) - U_1(r, \phi_1 - 0) &= e^{ikr}, \\ \frac{\partial U_1(r, \phi_1 + 0)}{\partial \theta} &= \frac{\partial U_1(r, \phi_1 - 0)}{\partial \theta}, \end{aligned} \tag{10}$$

$$\begin{aligned} U_2(r, \phi_2 + 0) - U_2(r, \phi_2 - 0) &= e^{ikr}, \\ \frac{\partial U_2(r, \phi_2 + 0)}{\partial \theta} &= \frac{\partial U_2(r, \phi_2 - 0)}{\partial \theta}, \end{aligned} \tag{11}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial U_n}{\partial \theta} + ikrB(r,0)U_n = 0, &\quad \text{if} \quad \theta = 0, \\ -\frac{\partial U_n}{\partial \theta} + ikrB(r,\alpha)U_n = 0, &\quad \text{if} \quad \theta = \alpha, \end{aligned} \tag{12}$$

where

$$B(r,\theta)=\begin{cases}B_1(r), & \text{if} \quad \theta=0,\\ B_2(r), & \text{if} \quad \theta=\alpha,\end{cases} \tag{13}$$

is a function of two variables, but it is actually defined only on the faces $\theta = 0$ and $\theta = \alpha$ of the wedge.

[8] It should be emphasized that the boundary value problem (9)–(13) is formulated and will be solved without the assumption $B_1(r)$ = const which has been mentioned above. This assumption makes it possible to define the geometrically reflected wave by an elementary formula $U_{\text{reflected}} = KU_r$, where U_r is a plane from (6) and K is the constant reflection coefficients from (8). As a result, the reduction of the original problem of diffraction to the boundary value problem is not complicated by insignificant detail, so that the paper may be focused on its principal innovation which is the procedure for the solution of the boundary value problem (9)–(13), which remains valid in most general case when both impedances $B_1(r)$ and $B_2(r)$ are arbitrary nonconstant functions.

3. Solution for the Case of Constant Impedances

[9] Formulas (9)–(12) determine two similar boundary value problems for the unknown wavefields U_1 (r, θ) and U_2 (r, θ), and these problems differ only by the position of the auxiliary interface, which is located either at the half line $\theta = \phi_1$ or at the half line $\theta = \phi_2$. Therefore we need only to find a method of computation of the wavefield $U(r, \theta; \phi) \equiv U(r, \theta)$ that is smooth everywhere inside the wedge $\mathfrak{G}(0, \alpha)$ except at the ray $\theta = \phi$ where it has a discontinuity on the half line $\theta = \phi$ described by the interface conditions

$$\begin{gathered}U(r,\phi+0;\phi)-U(r,\phi-0;\phi)=e^{\mathrm{i}kr},\\ \frac{\partial U(r,\phi+0;\phi)}{\partial\theta}=\frac{\partial U(r,\phi-0;\phi)}{\partial\theta}.\end{gathered} \tag{14}$$

The field $U(r, \theta)$ must also satisfy the condition at infinity

$$U(r,\theta;\phi)e^{-\mathrm{i}kr}=o(1), \qquad r\to\infty, \quad \theta\neq\phi, \tag{15}$$

and the boundary conditions (12) and (13).

[10] In the particular case of constant impedances B_1 and B_2 this problem was explicitly solved by *Budaev and Bogy* [2005b] by a probabilistic random walk method. Even though the solution obtained by *Budaev and Bogy* [2005b] was restricted to constant impedances, it is important for our current purposes and therefore is worth reproducing here.

[11] Assume first that the values of $U(r, \theta)$ are known on the rays $\theta = \alpha_1$ and $\theta = \alpha_2$ which are selected in such a way that either $0 < \alpha_1 < \alpha_2 < \phi$ or $\phi < \alpha_1 < \alpha_2 < \alpha$. Then, seeking $U(r, \theta; \phi)$ in the product form

$$U(r,\theta)=u(r,\theta)e^{\mathrm{i}kr}, \tag{16}$$

we come to the problem of finding the amplitude $u(r, \theta)$ that vanishes at infinity and satisfies the transport equation

$$\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2}+r\left(\frac{1}{2}+\mathrm{i}kr\right)\frac{\partial u}{\partial r}+\frac{1}{2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\mathrm{i}kr}{2}u=0, \tag{17}$$

accompanied by the boundary conditions

$$u(r,\alpha_1)=f_1(r), \qquad u(r,\alpha_2)=f_2(r), \tag{18}$$

where $f_1(r)$ and $f_2(r)$ are predefined functions. In the case when $f_1(r)$ and $f_2(r)$ are analytic and bounded in the first quarter $0 < \arg(r) < \pi/2$ of the complex plane, the auxiliary field $u(r, \theta)$ can be represented by the Feynman-Kac formula

$$u(r,\theta)=\mathbf{E}\left\{f(\xi_\tau,\eta_\tau)\right)e^{\mathrm{i}kS(t)}\Big\}, \qquad S(t)=\tfrac{1}{2}\int_0^t \mathrm{i}k\xi_t\mathrm{d}t, \tag{19}$$

where $\mathbf{E}$ denotes the mathematical expectation computed over the trajectories of the radial and angular motions ξ_t and η_t that are controlled by the stochastic equations

$$\mathrm{d}\xi_t=\xi_t\mathrm{d}w_t^1+\xi_t\left(\frac{1}{2}+\mathrm{i}k\xi_t\right)\mathrm{d}t, \qquad \mathrm{d}\eta_t=\mathrm{d}w_t^2, \tag{20}$$

driven by standard one-dimensional Brownian motions w_t^1 and w_t^2. The process $P_t = (\xi_t, \eta_t)$ starts from the initial position $P_0 = (r, \theta)$ and stops at the exit time $t = \tau$, defined as the first time when P_t eventually hits one of the faces $\eta_t = 0$ or $\eta_t = \alpha$.

[12] It is easy to see that for any $t > 0$ the angular motion η_t is contained in the segment $[0, \alpha]$ and the radial motion ξ_t runs in the first quarter $0 < \arg(\xi) < \pi/2$ of the complex plane, drifting to an unreachable point $\xi = \mathrm{i}/2k$. Such localization of ξ_t ensures that $S(t)$ has a positive imaginary component which improves the convergence in (19). On the other hand, the fact that ξ_t runs in the complex plane implies that the boundary function $f(\xi, \eta)$ must be analytic in the domain $0 < \arg(\xi) < \pi/2$ and that it should not grow too fast there along the random walk.

[13] Next we employ the formula (19) to derive the expression for the field $U(r, \theta)$ defined in the entire wedge $\mathfrak{G}(0, \alpha)$ with the interface condition (14) and with the boundary conditions (12) and (13) where $B_1(r)$ and $B_2(r)$ are constants.

[14] Let $U(r, \theta)$ be already known on the boundaries $\theta = 0$ and $\theta = \alpha$ of the wedge $\mathfrak{G}(0, \alpha)$ as well as on both sides of the interface $\theta = \phi \pm 0$. Then, the value of $U(r, \theta)$ with $\theta \neq \phi$ can be evaluated by the formula (19) applied to that wedge $\phi < \theta < \alpha$ or $0 < \theta < \phi$ which contains the observation point (r, θ). It is easy to see that this formula leads to the the expression

$$U(r,\theta) = e^{ikr}\mathbf{E}\Big\{U\big(\xi_{\tau_1},\eta_{\tau_1}\big)\big)e^{ik\left[S(\tau_1)-\xi_{\tau_1}\right]} + U\big(\xi_{t_1},\eta_{t_1}\big)\big)e^{ik\left[S(t_1)-\xi_{t_1}\right]}\Big\}, \tag{21}$$

where $S(t)$ has the same meaning as in (19). The mathematical expectation in (19) is computed over the trajectories of the random motion (ξ_t, η_t) which is similar to the motion from (19) and which stops at the earliest of the times $t = \tau_1$ or $t = t_1$, where τ_1 is the exit time through the side of the original wedge $\mathfrak{G}(0, \alpha)$ and t_1 is the first time when η_t hits the interface $\eta = \phi$.

[15] There are two unknown quantities $U(\xi_{\tau_1}, \eta_{\tau_1})$ and $U(\xi_{t_1}, \eta_{t_1})$ in the right-hand side of (19), but both of them can be evaluated using the boundary or interface conditions (12) or (14).

[16] To compute $U(\xi_{t1}, \eta_{t1})$, we observe that the auxiliary exit point η_{t1} takes one of two values $\eta = \phi \pm 0$ which is determined by the location of the observation point (r, θ). Therefore the value of $U(\xi_{t1}, \eta_{t1})$ can be represented by the expression

$$U\big(\xi_{t_1},\eta_{t_1}\big) = \begin{cases} \dfrac{U\big(\xi_{t_1},\phi+\mathrm{d}t\big) + \big[U\big(\xi_{t_1},\phi-\mathrm{d}t\big) + e^{ik\xi_{t_1}}\big]}{2}, & \text{if} \quad \eta_{t_1} = \phi + 0, \\ \dfrac{\big[U\big(\xi_{t_1},\phi+\mathrm{d}t\big) - e^{ik\xi_{t_1}}\big] + U\big(\xi_{t_1},\phi-\mathrm{d}t\big)}{2}, & \text{if} \quad \eta_{t_1} = \phi - 0, \end{cases} \tag{22}$$

which is an immediate consequence of the interface condition (14), and which can be rewritten in the probabilistic form

$$U\big(\xi_{t_1},\eta_{t_1}\big) = \mathbf{E}\big\{U\big(\xi_{t_1},\phi+\mathrm{d}\eta\big) + \delta e^{ik\xi_{t_1}}\big\}, \tag{23}$$

where $\mathrm{d}\eta = \pm\mathrm{d}t$ is a random number with two equally possible values and

$$\delta = \begin{cases} 1, & \text{if} \quad \phi + \mathrm{d}\eta < \phi < \eta_{t_1}, \\ -1, & \text{if} \quad \phi + \mathrm{d}\eta > \phi > \eta_{t_1}, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases} \tag{24}$$

The last formula implies that $\delta = 1$ in the case when the angular motion η_t crosses the interface $\eta = \phi$ from the right to the left, $\delta = -1$ if the interface is crossed from the left to the right and $\delta = 0$ in the case when the interface is touched but is not intersected.

[17] To compute the value $U(\xi_{\tau 1}, \eta_{\tau_1})$ of the field $U(r, \theta)$ on the boundary of the wedge $\mathfrak{G}(0, \alpha)$ where the impedance boundary conditions (12) are imposed, we rewrite these conditions in the form

$$U\big(\xi_{\tau_1},\eta_{\tau_1} + \mathrm{d}\eta\big) - U\big(\xi_{\tau_1},\eta_{\tau_1}\big) + ik\xi_{\tau_1}B\big(\xi_{\tau_1},\eta_{\tau_1}\big)U\big(\xi_{\tau_1},\eta_{\tau_1}\big)\mathrm{d}t = 0, \tag{25}$$

where two possible values of

$$\mathrm{d}\eta = \begin{cases} \mathrm{d}t, & \text{if} \quad \eta_{\tau_1} = 0, \\ -\mathrm{d}t, & \text{if} \quad \eta_{\tau_1} = \alpha, \end{cases} \tag{26}$$

correspond to the two different faces of the wedge $\mathfrak{G}(0, \alpha)$. Then, taking into account the identity $e^{-p\mathrm{d}t} = 1 - p\mathrm{d}t + o(\mathrm{d}t)$, we convert (25) to the expression

$$U\big(\xi_{\tau_1},\eta_{\tau_1}\big) = U\big(\xi_{\tau_1},\eta_{\tau_1} + \mathrm{d}\eta\big)e^{ik\xi_{\tau_1}B\left(\xi_{\tau_1},\eta_{\tau_1}\right)} + o(\mathrm{d}t), \tag{27}$$

which represents $U(\xi_{\tau_1}, \eta_{\tau_1})$ through the value of the field $U(r, \theta)$ at the point $(\xi_{\tau_1}, \eta_{\tau_1} + \mathrm{d}\eta)$ located in the interior of the wedge $\mathfrak{G}(0, \alpha)$.

[18] Since both of the points $(\xi_{\tau_1}, \phi + \mathrm{d}\eta)$ and $(\xi_{\tau_1}, \eta_{\tau_1} + \mathrm{d}\eta)$ that appear in (23) and (27) are located inside $\mathfrak{G}(0, \alpha)$ but not on the interface $\theta = \phi$, the values of $U(\xi_{t_1}, \phi + \mathrm{d}\eta)$ and $U(\xi_{\tau_1}, \eta_{\tau_1} + \mathrm{d}\eta)$ can be evaluated by formulas like (21), and repeating obvious iterations we eventually arrive at the final expressions

$$U(r,\theta;\phi) = e^{ikr}\mathbf{E}\left\{\sum_{\nu=1}^{\infty}\delta(t_\nu,\phi)e^{ikS(t_\nu)}\right\}, \tag{28}$$

$$S(t) = \frac{1}{2}\int_0^\infty \xi_t\mathrm{d}t + \int_0^\infty \xi_t B(\xi_t,\eta_t)\mathrm{d}\lambda_t, \tag{29}$$

where the mathematical expectation is computed over the trajectories of the stochastic processes ξ_t, η_t, $\delta(t_\nu, \phi)$ and λ_t described below.

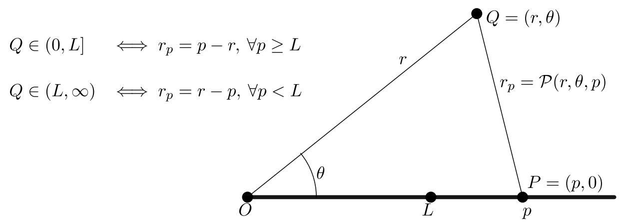

Figure 2. Analytic description of the segment (0, L].

[19] The radial motion ξ_t retains its meaning from (20), while the angular motion η_t is controlled by the stochastic equations

$$\eta_0 = \theta, \qquad d\eta_t = \begin{cases} dt, & \text{if} \quad \eta_t = \alpha_1, \\ -dt, & \text{if} \quad \eta_t = \alpha_2, \\ dw_t^2, & \text{if} \quad \alpha_1 < \eta_t < \alpha_2, \end{cases} \tag{30}$$

which imply that η_t is the so-called Brownian motion with reflections at the times $t = \tau_1$, $t = \tau_2$, ..., when it touches the boundary of the interval (0, α). Inside this interval, η_t runs as a standard Brownian motion but every time when it hits the boundary it is deterministically reflected back into the interval.

[20] The angular motion η_t completely determines the other stochastic processes λ_t and $\delta(t_\nu, \phi)$, which are controlled by the equations

$$\lambda_0 = 0, \qquad d\lambda_t = \begin{cases} dt, & \text{if} \quad \eta_t = \alpha_1, \\ -dt, & \text{if} \quad \eta_t = \alpha_2, \\ 0, & \text{if} \quad \alpha_1 < \eta_t < \alpha_2, \end{cases} \tag{31}$$

and

$$\delta(t_\nu, \phi) = \begin{cases} 1, & \text{if} \quad \eta_{t_\nu - 0} > \phi, \quad \eta_{t_\nu + 0} < \phi, \\ -1, & \text{if} \quad \eta_{t_\nu - 0} < \phi, \quad \eta_{t_\nu + 0} > \phi, \\ 0, & \text{otherwise}, \end{cases} \tag{32}$$

where $\{t_\nu\}$ is the sequence of times when the angular motion η_t touches the fixed point $\eta = \phi$. The process λ_t may be considered as a measure of the time that the angular motion η_t spends on the boundary of the interval (0, α), and correspondingly, this process is known as the 'local time' of the reflected Brownian motion η_t on the boundary.

4. Solution for the General Case of Nonconstant Impedances

[21] By analyzing the above described approach to the problem with constant impedances it is easy to see that in order to extend the approach to problems with arbitrary impedances it is necessary to find analytical continuation of the boundary impedances to the complex space (r, θ) with complex radial and angular coordinates.

[22] Consider first the simplest nontrivial case when B_2 is still constant but $B_1(r)$ is a piecewise constant function

$$B_1(r) = \begin{cases} b_0, & \text{if} \quad r \leq L, \\ b_1, & \text{if} \quad r > L, \end{cases} \tag{33}$$

where b_0, b_1 and $L > 0$ are given numbers.

[23] It is clear that the pieces $I_0(L) = \{\theta = 0, 0 < r \leq L\}$ and $I_1(L) = \{\theta = 0,\ r > L\}$ of the face $\theta = 0$ can be characterized by the analytic equations

$$I_0 \equiv I_0(L): \qquad \mathcal{P}(r, \theta, p) = p - r, \quad \forall p \geq L, \tag{34}$$

$$I_1 \equiv I_1(L): \qquad \mathcal{P}(r, \theta, p) = r - p, \quad \forall p \geq L, \tag{35}$$

where

$$\begin{aligned} \mathcal{P}(r, \theta, p) &= \sqrt{r^2 + p^2 - 2rp\cos\theta} \\ &\equiv \sqrt{(re^{i\theta} - p)(re^{-i\theta} - p)}. \end{aligned} \tag{36}$$

Indeed, applying the cosine theorem to the triangle ΔOPQ shown in Figure 2, we see that $r_p = \mathcal{P}(r, \theta, p)$ and that this triangle collapses to the segment (OP] in the case when $r_p = p - r$. We conclude that if the vertex Q belongs to the segment (0, L], then the equality $r_p = p - r$ must remain valid for all $p \geq$ L.

[24] The last result implies that if the point $P_t = (\xi_t, \eta_t)$ moves in the complex space, then at any time $t = t_*$ when $\eta_{t_*} = 0$, the point P_t belongs to the interval $I_{\nu(t_*)}$ where the value $\nu(t_*) = 0$ or $\nu(t_*) = 1$ of the index ν_t is determined by the entire preceding trajectory of the point P_t rather than by its position at the time $t = t_*$. In other words, equations (34) and (35) provide analytic continuation of the real one-dimensional intervals I_0 and I_1 to two-dimensional surfaces in the complex space formed by the pairs (r, θ), where both of the components are considered as complex variables. Correspondingly, these formulas define the analytical continuation of the boundary piecewise function $B_1(r)$ to the complex space (r, θ).

[25] Since $B_1(r)$ is analytically continued to the complex space, the field $U(r, \theta)$ can be computed by the same variation of the random walk method as employed in the previous section and the resulting solution has the form (28) with the phase $S(t)$ computed by a more complex rule than that in (29). More precisely, $S(t)$ is a stochastic process that starts from the initial position $S(0) = 0$ and evolves thereafter as

$$S(t + dt) = S(t) + \begin{cases} \frac{1}{2}\xi_t dt, & \text{if} \quad 0 < \eta_t < \alpha, \\ \xi_t B_2 dt, & \text{if} \quad \eta_t = \alpha, \\ \xi_t b_0 dt, & \text{if} \quad \eta_0 = 0, \quad \nu(t) = 0, \\ \xi_t b_1 dt, & \text{if} \quad \eta_0 = 0, \quad \nu(t) = 1, \end{cases} \tag{37}$$

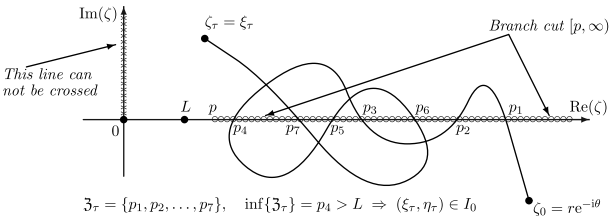

Figure 3. Trajectory of $\zeta_t = \xi_t e^{-i\eta_t}$ and determination of $\rho\tau = \inf\{p_n\}$.

where $\nu(t)$ is an index determining which part of the face $\theta = 0$ is touched.

[26] The value of $\nu(t)$ is explicitly defined by the formula

$$\nu(\tau) = \begin{cases} 0, & \text{if} \quad \mathcal{P}(\xi_\tau, \eta_\tau, p) = p - \xi_\tau, \quad \forall p \geq L, \\ 1, & \text{if} \quad \mathcal{P}(\xi_\tau, \eta_\tau, p) = \xi_\tau - p, \quad \forall p < L, \end{cases} \tag{38}$$

which requires tracing the value of the radical

$$\mathcal{P}(\xi_t, \eta_t, p) = \sqrt{(\xi_t e^{i\eta_t} - p)(\xi_t e^{-i\eta_t} - p)} \tag{39}$$

along the trajectory of the motion P_t. This procedure provides a clear and unambiguous definition of $\nu(\tau)$, but it hides the relationship between $\nu(\tau)$ and the global structure of the trajectory of the motion P_t, which becomes transparent from a closer look at (39).

[27] It is obvious that for any $p > 0$ the argument of the first radical in (39) has the value

$$\arg\left(\sqrt{\xi_t e^{i\eta_t} - p}\right) = \frac{1}{2}\arg\left(\xi_t e^{i\eta_t} - p\right), \quad \forall p > 0, \tag{40}$$

where the branches of the multivalued functions $\sqrt{\zeta - p}$ and $\arg(\zeta - p)$ are fixed by the cut along the ray $[p, \infty)$. Indeed, since ξ_t and η_t start from the real positive values $\xi_0 = r$ and $\eta_0 = \theta$, and since these motions are contained in the domains $0 \leq \arg(\xi_t) < \pi/2$ and $0 \leq \eta_t < \alpha \leq 3\pi/2$, the complex number $\zeta_t = \xi_t e^{i\eta_t}$ is contained in the domain $0 \leq \arg(\zeta_t) < 2\pi$. This guarantees that $\xi_t\, e^{i\eta_t}$ never crosses the cut $[p, \infty)$ where $p \geq 0$ and therefore that (40) holds for any possible trajectory of P_t.

[28] To compute the second radical in (39), we need to trace the trajectory of the point $\zeta_t = \xi_t e^{-i\eta_t}$, which starts from the position $\zeta_0 = re^{-i\theta}$ in the domain $\arg(\zeta) < 0$ and stops at the point $\zeta_\tau = \xi_\tau$ located in the quarter $0 < \arg(\zeta_\tau) < \pi/2$. Taking into account the restraints $0 \leq \arg(\xi_t) < \pi/2$ and $0 \leq \eta_t < 3\pi/2$ we conclude that ζ_t is contained in the domain $-3\pi/2 < \arg(\zeta) < \pi/2$. Such information about the continuous motion ζ_t guarantees that it crosses the ray $\arg(\zeta) = 0$ an odd number of times, as illustrated in Figure 3.

[29] Let $\mathfrak{Z}\tau = \{p_n\}$ be the set of all positive points $p_n > 0$ where the trajectory of ζ_t intersects the ray $\arg(\zeta) = 0$, and let $N(p, \mathfrak{Z}\tau)$ be the number of points of the set $\mathfrak{Z}\tau$ located to the left of p. Then, the argument of $\sqrt{\zeta - p}$ takes the values

$$\arg\left(\sqrt{\zeta_\tau - p}\right) = \begin{cases} \frac{1}{2}\arg(\zeta_\tau - p) + 2\pi m, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is even,} \\ \frac{1}{2}\arg(\zeta_\tau - p) + 2\pi m + \pi, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is odd,} \end{cases} \tag{41}$$

which is completely determined by the disposition of the parameter p with respect to the set $\mathfrak{Z}\tau$ and does not depend on other details of the trajectory of ζ_t. Indeed, if $N(p, \mathfrak{Z}\tau)$ is even, then ζ_t intersects the branch cut $[p, \infty)$ an odd number of times and this leads to the first line of (41). Otherwise, if $N(p, \mathfrak{Z}\tau)$ is odd, then the trajectory of ζ_t has an even number of intersections with the cut $[p, \infty)$ and this leads to the second option of (41).

[30] Finally, combining (39) with (40) and (41), we get

$$\mathcal{P}(\xi_\tau, \eta_\tau, p) = \begin{cases} \xi_\tau - p, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is even} \\ p - \xi_\tau, & \text{if } N(p, \mathfrak{Z}_\tau) \text{ is odd,} \end{cases} \tag{42}$$

and combining (42) with (34) and (35), we arrive at the remarkable formula

$$\nu(\tau) = \begin{cases} 0, & \text{if} \quad \inf\{\mathfrak{Z}_\tau\} \leq L, \\ 1, & \text{if} \quad \inf\{\mathfrak{Z}_\tau\} > L, \end{cases} \tag{43}$$

which makes it possible to evaluate $\nu(t)$ by tracing the intersections of the trajectory of $\zeta_t = \xi_t e^{-i\eta_t}$ with the positive semiaxis, but without tracing the radical (39).

[31] The previous result can be straightforwardly generalized to the representation of the field $U(r, \theta)$ in the special case when the impedance $B_1(r)$ has a piecewise constant structure

$$B_1(r) = b_n, \quad \text{if} \quad r \in I_n = (L_n, L_{n+1}], \qquad n \geq 0, \tag{44}$$

where $\{L_n\}$ is a monotonically increasing sequence with $L_0 = 0$, and b_n are some constants. Indeed, observing that the intervals I_n can be represented as $I_n = I(L_{n+1}) \backslash I(L_n)$, where $I(L)$ is the domain in the space (r, θ) defined as

$$I(L): \qquad \mathcal{P}(r, \theta, p) = p - r, \qquad \forall p \geq L, \tag{45}$$

we readily come to the representation of the field $U(r, \theta)$ in the form (28) with the phase $S(t)$ defined by the stochastic equation

$$S(t+\mathrm{d}t) = S(t) + \begin{cases} \frac{1}{2}\xi_t \mathrm{d}t, & \text{if} \quad 0 < \eta_t < \alpha, \\ \xi_t B_2 \mathrm{d}t, & \text{if} \quad \eta_t = \alpha, \\ \xi_t b_{\nu(t)\mathrm{d}t}, & \text{if} \quad \eta_0 = 0, \end{cases} \tag{46}$$

with the index $\nu(t)$ defined as

$$\nu(t) = n, \quad \text{if} \quad L_n < \inf\{\mathfrak{Z}_t\} \le L_{n+1}, \tag{47}$$

where $\mathfrak{Z}\tau$ is the full intersection of the trajectory of $\zeta_t = \xi_t e^{-\mathrm{i}\eta_t}$ with the semiaxis arg(ζ) = 0. Comparison of (47) with (44) shows that if the point

$$\rho_t = \inf\{\mathfrak{Z}_t\} \tag{48}$$

belongs to the interval I_n, then $\nu_\tau = n$ and $b_{\nu(t)} = B_1(\rho_t, 0)$. As a result, the solution (46) can be written in the form

$$S(t+\mathrm{d}t) = S(t) + \begin{cases} \frac{1}{2}\xi_t \mathrm{d}t, & \text{if} \quad 0 < \eta_t < \alpha, \\ \xi_t B_2 \mathrm{d}t, & \text{if} \quad \eta_t = \alpha, \\ \xi_t B_1(\rho_t)\mathrm{d}t, & \text{if} \quad \eta_0 = 0, \end{cases} \tag{49}$$

which does not rely on the piecewise structure of $B_1(r)$, and which therefore can be extended to the case when $B_1(r)$ is a virtually arbitrary function of the real variable $r > 0$.

[32] The probabilistic representation (28), (46) of the wavefield $U(r, \theta)$ in the wedge $\mathfrak{G}(0, \alpha)$ with the variable impedance $B_1(r)$ and constant impedance $B_2(r)$ can be easily generalized to describe this field in the case when both of the impedances are arbitrary functions. Indeed, straightforward reiteration of the above reasonings leads to the representation of the field $U(r, \theta)$ by the mathematical expectation

$$U(r, \theta; \phi) = e^{\mathrm{i}kr}\mathbf{E}\left\{\sum_{\nu=1}^{\infty} \delta(t_\nu, \phi) e^{\mathrm{i}} k S(t_\nu)\right\}, \tag{50}$$

$$S(t) = \frac{1}{2}\int_0^\infty \xi_t \mathrm{d}t + \int_0^\infty \xi_t B(\rho_t, \eta_t)\mathrm{d}\lambda_t, \tag{51}$$

where ξ_t, η_t, $\delta(t_\nu, \phi)$ and λ_t retain their meaning from (28)–(32) and ρ_t is the random process controlled by the stochastic equations

$$\rho_0 = r, \qquad \rho_{t+\mathrm{d}t} = \begin{cases} \inf\{\mathfrak{Z}_\tau^0(r; \eta_t)\}, & \text{if} \quad \eta_t = 0, \\ \inf\{\mathfrak{Z}_\tau^\alpha(r; \eta_t)\}, & \text{if} \quad \eta_t = \alpha, \\ \rho_t, & \text{otherwise}, \end{cases} \tag{52}$$

where $\mathfrak{Z}_t^0(r; \eta_t)$ and $\mathfrak{Z}_t^\alpha(r; \eta_t)$ are the intersection of the trajectory of the motions

$$\zeta_t^0 = \xi_t e^{-\mathrm{i}\eta_t}, \qquad \text{and} \qquad \zeta_t^\alpha = \xi_t e^{-\mathrm{i}(\alpha - \eta_t)}, \tag{53}$$

with the semiaxis arg(ζ) = 0.

5. Numerical Examples

[33] To illustrate the suitability of the obtained solution of the problem of diffraction by a wedge with a face of variable impedance, we conducted numerical simulations for the configuration considered by *Osipov* [2004] and *Budaev and Bogy* [2005b]. In this configuration the wave number is fixed as $k = 1$, and a plane incident wave $U_*(r, \theta) = e^{-\mathrm{i}r\cos(\theta - \theta_* st)}$ with $\theta_* = 43°$ propagates in the wedge $0 < \theta < 266°$ with impedance boundary conditions on its faces.

[34] In Figure 4 we consider four cases with the face $\theta = 0$ having a fixed constant impedance $B_1 = 5$ and with the face $\theta = 266°$ having four different impedances. In the first and the second cases the second impedance has constant values $B_2 = 5$ and $B_2 = 1/5$, respectively. The total wavefield corresponding to $B_2 = 5$ is shown by a thick solid line, and it obviously agrees with the results of *Osipov* [2004] and *Budaev and Bogy* [2005b]. The field corresponding to $B_2 = 1/5$ is shown by a thin line additionally marked by dots. In the third and the forth cases the impedance of the face $\theta = 266°$ is variable and defined as $B_2(r) = -i/(r^2 + 1)$ and $B_2(r) = 5(1 + \cos(r\pi/4))$, respectively. The total fields in these cases are shown by $\times$ and $°$ marks. As expected the total fields corresponding to different impedances of the face $\theta = 266°$ are practically undistinguishable in the domain $0° < \theta < 150°$ but they differ from each other considerably in the domain $200° < \theta < 266°$.

[35] Results of four other numerical simulations are shown in Figure 5. In all cases presented in Figure 5 the face $\theta = 0$ has a constant impedance $B_1 = 1/5$ while the impedance of the face $\theta = 266°$ are either constant or piecewise constant functions. The bold solid line shows the total field in the case with $B_2 = 1/5$, which has also been considered by *Osipov* [2004] and *Budaev and Bogy* [2005b]. The field corresponding to $B_2 = 5$ is shown by a thin line additionally marked by dots. In the two other cases the impedance B_2 is defined by

$$B_2(r) = \begin{cases} 1/5, & \text{if} \quad r > 3, \\ -\mathrm{i}/3, & \text{if} \quad r < 3, \end{cases} \qquad \text{or}$$

$$B_2(r) = \begin{cases} 1/5, & \text{if} \quad r > 3, \\ 5, & \text{if} \quad r < 3. \end{cases}$$

The total wavefields corresponding to these cases are marked by crosses and circles, respectively.

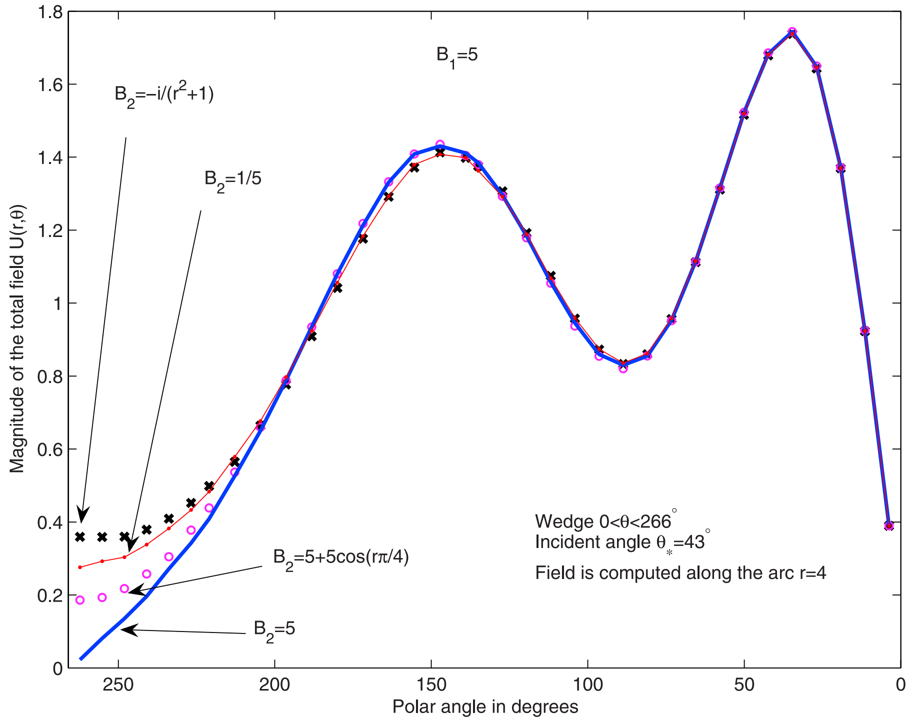

Figure 4. Total fields in a wedge with $B_1 = 5$.

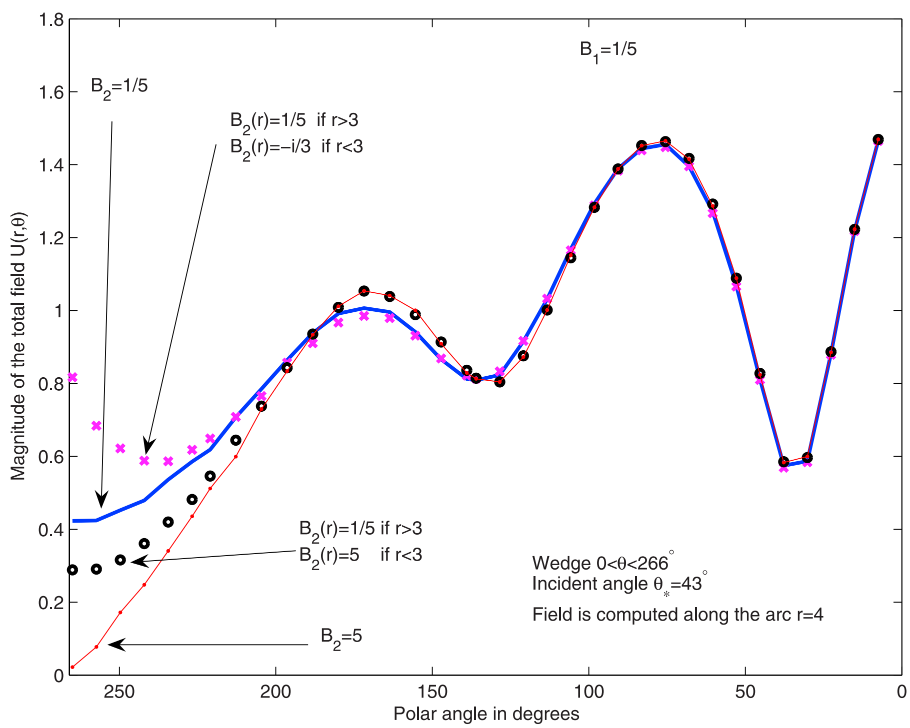

Figure 5. Total fields in a wedge with $B_1 = 1/5$.

[36] All of the computations presented here were obtained by the averaging of 1500 discrete random walks with the time increment dt = 0.01. The computations were carried out on a 900 Mhz notebook PC using MATLAB code, which was only a few lines longer than the code published by *Budaev and Bogy* [2005b] for the similar problem with constant impedances.

6. Conclusion

[37] The version of the probabilistic random walk method surveyed by *Budaev and Bogy* [2005a] made it possible to obtain a solution of the problem of diffraction by a wedge with a nonconstant impedance faces. The obtained solution is not considerably more complex than the similar solution of the standard problems of diffraction by a wedge with constant impedances, and it admits numerical simulation based on a simple very short algorithm. These features confirm that the probabilistic approach is well suited for further applications to problems of diffraction with complex shapes and boundary conditions.

[38] **Acknowledgment.** This research was supported by NSF grant CMS-0408381.

References

Budaev, B. V., and D. B. Bogy (2004), Diffraction by a plane sector, *Proc. R. Soc. London, Ser. A*, *460*(2052), 3529–3546.

Budaev, B. V., and D. B. Bogy (2005a), A probabilistic approach to wave propagation and scattering, *Radio Sci.*, *40*, RS6S07, doi:10.1029/2004RS003176.

Budaev, B. V., and D. B. Bogy (2005b), Two-dimensional diffraction by a wedge with impedance boundary conditions, *IEEE Trans. Antennas Propag.*, *53*(6), 2073–2080.

Budaev, B. V., and D. B. Bogy (2005c), Two-dimensional problems of diffraction by finite collinear structures, *J. Acoust. Soc. Am.*, *119*(2), 741–750, doi:10.1121/1.2159292.

Budaev, B. V., and D. B. Bogy (2006a), Diffraction by a convex polygon with side-wise constant impedance, *Wave Motion*, *43*(8), 631–645.

Budaev, B. V., and D. B. Bogy (2006b), Diffraction of a plane electromagnetic wave by a wedge with anisotropic impedance faces, *IEEE Trans. Antennas Propag.*, *54*(5), 1559–1567.

Felsen, L. B., and N. Marcuvitz (1972), *Radiation and Scattering of Waves*, Prentice-Hall, Englewood Cliffs, N. J.

Maliuzhinets, G. D. (1951), Some generalizations of the method of reflections in the theory of sinusoidal wave diffraction (in Russian), Doktoral dissertation, Fiz. Inst. Lebedev, Acad. Nauk SSR, Moscow, USSR.

Maliuzhinets, G. D. (1955), Radiation of sound by oscillating faces of an arbitrary wedge, *Acoust. J.*, *1*(2), 144–164.

Osipov, A. V. (2004), A hybrid technique for the analysis of scattering by impedance wedges, paper presented at URSI 2004 International Symposium on Electromagnetic Theory, Int. Union of Radio Sci., Pisa, Italy.

Poincaré, J. H. (1892), Sur la polarisation par diffraction, *Acta Math.*, *16*, 297–339.

Poincaré, J. H. (1897), Sur la polarisation par diffraction, *Acta Math.*, *20*, 313–355.

Senior, T. B. A. (1959), Diffraction by an imperfectly conducting wedge, *Commun. Pure Appl. Math.*, *12*, 337–372.

Sommerfeld, A. (1896), Mathematische Theorie der Diffraction, *Math. Ann.*, *47*, 317–374.

Williams, W. E. (1959), Diffraction of an E-polarized plane wave by an imperfectly conductiong wedge, *Proc. R. Soc. London, Ser. A*, *252*, 376–393.

D. B. Bogy and B. V. Budaev, Department of Mechanical Engineering, University of California, Etcheverry Hall, MC 1740, Berkeley, CA 94720, USA. (dbogy@cml.me.berkeley.edu; budaev@berkeley.edu)