Girsanov change of measure applied to Boundary value problems

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Theory

Theorem 1: (Girsanov) Let $W(t) \in \mathbb{R}^n$ be a Brownian motion with respect to the measure P, and $X(t) \in \mathbb{R}^n$ an Itô process given by

$$dX(t) = a(t, \omega)dt + dW(t), \quad 0 \leq t \leq T \leq \infty.$$

Where $a : [0, T] \times \Omega \to \mathbb{R}^n$ is the drift. Then with

$$M(t, \omega) = \exp\left(-\int_0^t a(s, \omega) \cdot dW(s) - \frac{1}{2}\int_0^t a^2(s, \omega)\, ds\right), \quad 0 \leq t \leq T$$

define the measure Q by

$$dQ(\omega) = M(T, \omega)dP(\omega),$$

then $X(t)$ is an N-dimensional Brownian motion wirh respect to Q.

Theorem 2: Let $D \subset \mathbb{R}^n$, open and connected, ∂D its boundary, and $f : \partial D \to \mathbb{R}$. Let $W(t)$, $X(t)$, $M(t)$, as in the last theorem. Define

$$\tau_x(\omega) = \inf\left(\{0 \leq t \leq T; X(t, \omega) \notin D, \}\right),$$

where $X(0) = x$. Then

$$u(x) = \int_\Omega f(X(\tau_x(\omega), \omega))\, M(\tau_x(\omega), \omega)\, dP(\omega) = E_{Q_x}[f(X(\tau_x))],$$

where $dQ_x(\omega) = M(\tau_x(\omega), \omega)dP(\omega)$, solves the equation

$$\nabla^2 u(x) = \begin{cases} 0, & x \in D \\ f(x), & x \in \partial D. \end{cases}$$

Idea: The drift, $a(t, \omega)$, may be chosen to increase the rate of convergence in the above integral.

Experiment

Verifying the hitting distribution in $\mathbb{R}^2$:

Let D be the open disc $\{x \in \mathbb{R}^2; ||x||_2 < 1\}$, so that $\partial D = \{x \in \mathbb{R}^2; ||x||_2 = 1\}$, and take $f : \partial D \to (-\pi, \pi]$, such that $x = (\cos(f(x)), \sin(f(x)))$ for all $x \in \partial D$, i.e. f is the function atan2 (see Wikipedia). Then let $Y_{x_0} = f(X(\tau_{x_0}))$, which is the hitting position of X, where $X(0) = x_0 \in D$; we'll examine the distributions $E \mapsto P(Y_{x_0}^{-1}(E))$, and $E \mapsto Q(Y_{x_0}^{-1}(E))$ for measurable sets $E \subset (-\pi, \pi]$. $Y_{x_0}^{-1}(E) = \{f(X(\tau_{x_0})) \in E_k\}$.

If $a \equiv 0$, then $X(t) = W(t)$, the standard Brownian motion, and in this case $P = Q$. So, for a given x_0, we expect that the distribution $E \mapsto P(Y_{x_0}^{-1}(E))$ with $a \equiv 0$ equals the distribution $E \mapsto Q(Y_{x_0}^{-1}(E))$ for an arbitrary a, which is the design of the measure Q. This is what we verify.

To sample $X(\tau_{x_0})$, the random hitting position, we discretize time into even steps of length dt, then take each $dW(t) = \sqrt{dt}N(0, 1)$, where $N(0, 1)$ is a vector of normally distributed random varaibles, with 0 mean and standard deviation 1. X is started as $X = x_0$, and incremented as $X(t+dt) = dW(t) + a(t)dt$. Similarly, $\log(M)$ computed as the sum of $-a(t) \cdot dW(t) - \frac{1}{2}a(t) \cdot a(t)dt$. This process is continued until X no longer is in the region D; in this case, untill $||X(t)||_2 > 1$. The final value of $X(t)$ is then $X(\tau_{x_0})$, and the final value of the sum of $\log(M(t))$ is then exponentiated to give $M(\tau_{x_0})$.

We partition the interval $(-\pi, \pi]$ into a number of disjoint, evenly spaced sub intervals, $\{E_k\}$. For each sample $f(X(\tau_{x_0}))$, we check in which of the invervals, $\{E_k\}$, it falls in. Two sums are associated to each E_k, and given that $f(X(\tau_{x_0}))$ fell in E_k, the first associated sum is incremented by $M(\tau_{x_0})$, and the second by 1. Finally, each sum is divided by the total number of samples, so that for each E_k, the first sum approximates $Q(Y_{x_0}^{-1}(E_k) = Q(\{f(X(\tau_{x_0})) \in E_k\})$, and the second sum approximates $P(Y_{x_0}^{-1}(E_k)) = P(\{f(X(\tau_{x_0})) \in E_k\})$, which are the desired ditributions.

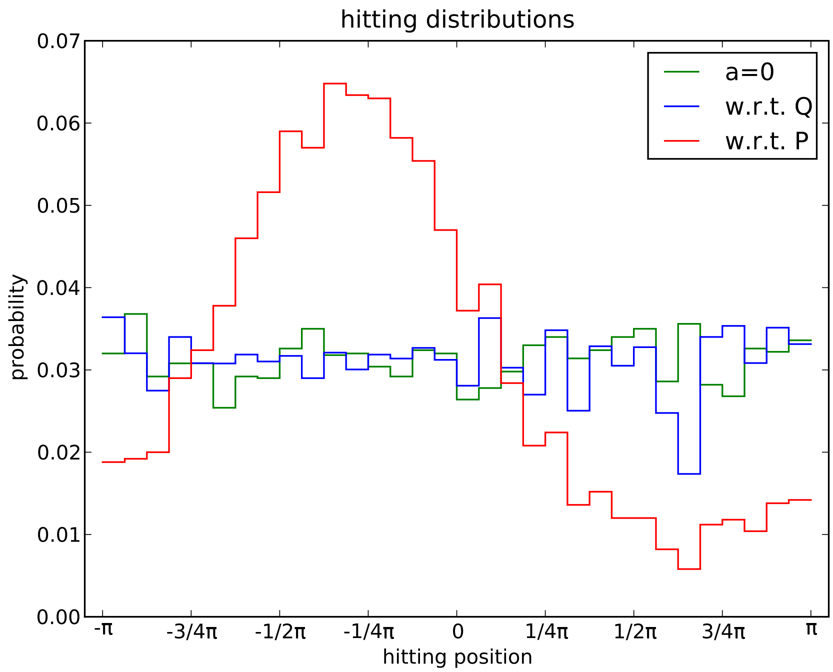

Figure 1: Results, with $a(t) = (-0.7, 0.5)$, $x_0 = (0.0, 0.0)$, $dt = 0.02$, 32 partitions of $(-\pi, \pi]$, and 5000 samples.

Verifying the hitting distribution in $\mathbb{R}^n$:

Let D be the open sphere $\{x \in \mathbb{R}^n; ||x||_2 < 1\}$, so that $\partial D = \{x \in \mathbb{R}^n; ||x||_2 = 1\}$. Then we define an enumeration of the quadrants of $\mathbb{R}^n$ as folows. For some $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, if $x_k \geq 0$, let $s_k = 2^k$, otherwise let $s_k = 0$, then take $f(x) = \sum_{k=0}^{n-1} s_k$. Then we'll follow the same steps as in the $\mathbb{R}^2$ case, now with $E_k = \{k\}$, for $0 \leq k < 2^n$.

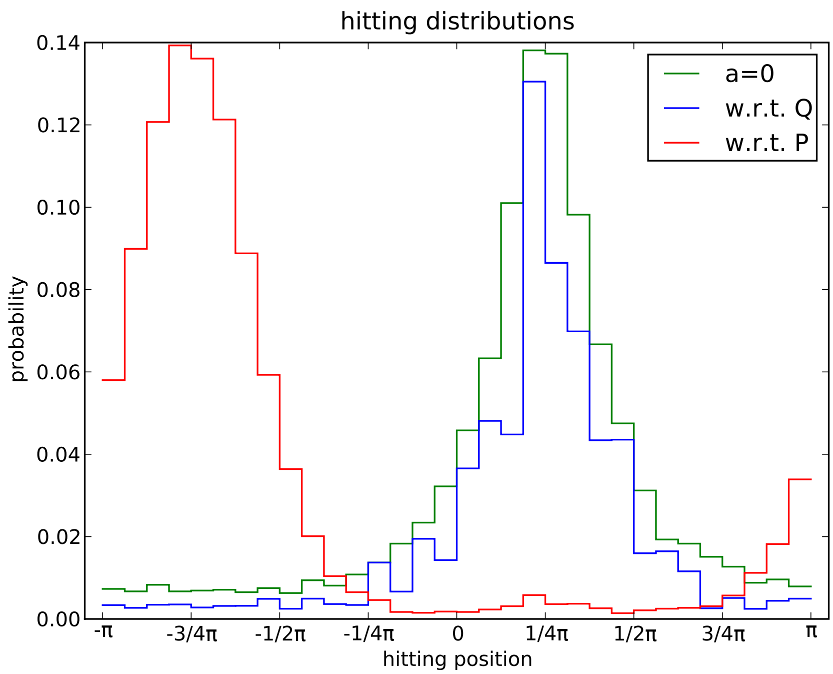

Figure 2: Results, with $a(t) = (-4.0, -4.0) + \xi(t+1)^2$, where ξ is a uniformly distributed random variable on $[-\frac{1}{2}, \frac{1}{2}]$, $x_0 = (0.5, 0.5)$, $dt = 0.02$, 32 partitions of $(-\pi, \pi]$, and 10,000 samples.

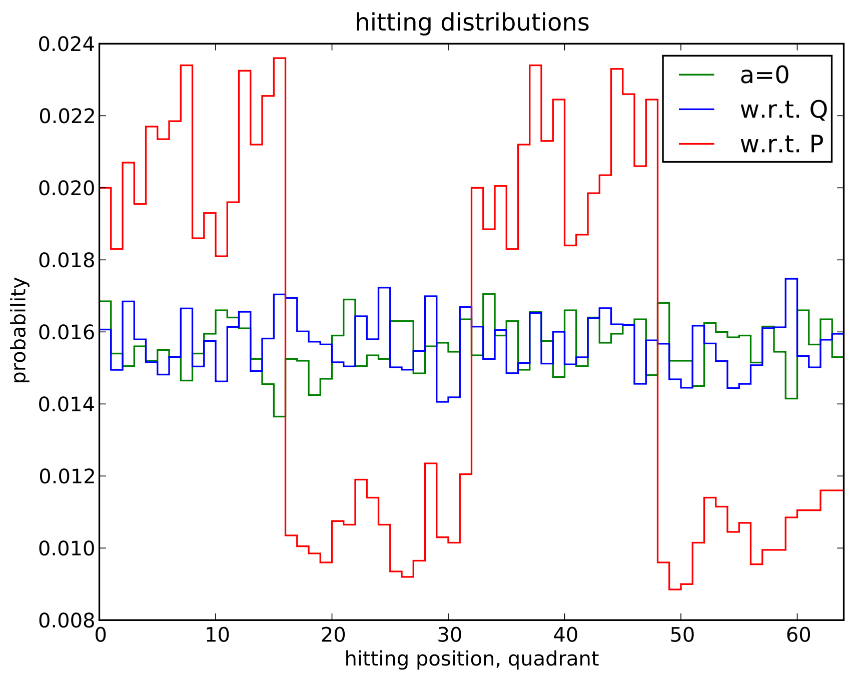

Figure 3: Results, in 6 dimensions, with $a(t) = (0, 0, 0.2, 0, -0.9, 0)$, $x_0 = 0$, $dt = 0.03$, and 20,000 samples.