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Random walk approach to wave radiation in cylindrical and spherical domains

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Abstract

Two- and three-dimensional Helmholtz equations in the exterior cylindrical and spherical domains are addressed by the random walk method. The solutions of the Dirichlet problems in such domains are represented as mathematical expectations of specified functionals on trajectories of random motions running in designated domains of a multi-dimensional complex space. The numerical examples confirm the efficiency of the random walk approach to the analysis of wave radiation.

Keywords: Wave propagation; Brownian motion; Stochastic process

1. Introduction

In a recent series of papers [2–7] problems of wave propagation have been addressed by a novel random walk method which combines the advantages of exact analytical methods with the versatility of direct numerical methods, and which has its roots in different fields spanning a range from ray theory and classical mechanics to the theory of finite differences and the stochastic calculus.

The fundamentals of the random walk approach to wave propagation were outlined in Refs. [2,4,5], where it was shown that the solution of the Helmholtz equation $\nabla^2 U(x) + k^2\kappa^2(x)U(x) = 0$, where $k = \text{const}$, can be represented in the Liouville form $U(x) = e^{ikS(x)}u(x)$ with both the eikonal $S(x)$ and the amplitude $u(x)$ constructively determined by theoretically exact formulas. Thus, $S(x)$ can be computed by the Hamilton–Jacobi method [1,14] as the solution of the eikonal equation $(\nabla S)^2 = \kappa^2$, and the amplitude $u(x)$ can be computed by the probabilistic Feynman–Kac formula [9,13] applied to the complete transport equation $\nabla^2 u + 2ik\vec{\nabla}S\cdot\vec{\nabla}u + 2ik(\nabla^2 S)u = 0$. It was also shown that in certain circumstances, the exact solutions of the Helmholtz equation provided by the random walk method degenerate to the well-known asymptotes of the ray method.

Later, in Ref. [3] it was shown that the random walk method, as an exact approach, makes it possible to describe such typical phenomena of wave propagation as back-scattering, which is predicted neither by the ray theory nor by a more general method of parabolic equations [11,12].

In the papers [6–8] the focus was shifted from general considerations to the analysis of particular problems of wave propagation and diffraction. In Ref. [7] the random walk method was applied to problems of wave propagation in wedge-shaped and conical domains. Then these results were used in Ref. [6] to obtain the explicit probabilistic solution of the important three-dimensional problem of diffraction by a plane sectorial screen, and in Ref. [8] the random walk method was applied to problems of wave scattering by surface-breaking cracks and cavities.

Here we continue our development of the random walk approach to wave propagation and consider problems of wave radiation into exterior cylindrical and spherical domains. The governing equations for these problems are similar to those employed in Refs. [6,7] for the analysis of waves in wedge-shaped and conical domains, but in order to accommodate boundary conditions formulated on non-plane surfaces we have to invoke a specific technique which makes it possible to ‘steer’ the random walks into a preferable, albeit still random, direction. This steering technique constitutes an important step towards the extensions of the random walk method to problems of wave

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propagation in domains with non-trivial geometry such as deformed spherical or cylindrical domains with surface-breaking cracks or cavities.

2. Waves radiating into the exterior of a circular cylinder

Here we focus on the computation of two-dimensional waves radiating into an exterior cylindrical domain

$$G_R = \{r, \theta : r \geq R > 0, 0 \leq \theta < 2\pi\}, \tag{2.1}$$

given in polar coordinates (r, θ). Such waves are described by the solution of the Helmholtz equation

$$\nabla^2 U + k^2 U = 0, \qquad k = \text{const}, \tag{2.2}$$

satisfying the Dirichlet boundary conditions

$$U(R, \theta) = F(\theta), \qquad 0 \leq \theta < 2\pi, \tag{2.3}$$

and the radiation condition at infinity

$$U(r, \theta) = C(\theta)\frac{e^{ikr}}{\sqrt{kr}}(1 + o(1)), \qquad r \to \infty, \tag{2.4}$$

where $C(\theta)$ is an unspecified function of θ usually referred to as the diffraction coefficient.

We seek the solution of the Helmholtz equation in the product form

$$U(r, \theta) = u(r, \theta)e^{ikr}, \tag{2.5}$$

where $u(r, \theta)$ is a new unknown function. Then, substituting Eq. (2.5) into Eq. (2.2) and multiplying the resulting expression by $Q = (1/2)r^2q^2(r, \theta)$, where $q(r, \theta)$ is an indefinite function to be specified later, we find that $u(r, \theta)$ has to obey the equation

$$\frac{q^2r^2}{2}\frac{\partial^2 u}{\partial r^2} + q^2 r\left(\frac{1}{2} + ikr\right)\frac{\partial u}{\partial r} + \frac{q^2}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{ikrq^2}{2}u = 0, \tag{2.6}$$

accompanied by the boundary condition

$$u(R, \theta) = f(\theta), \qquad f(\theta) = F(\theta)e^{-ikR}, \tag{2.7}$$

equivalent to Eq. (2.3), and by the condition

$$u(r, \theta) = O(1/\sqrt{r}), \qquad r \to \infty, \tag{2.8}$$

equivalent to Eq. (2.4).

Equations of the type (2.6) were discussed in Refs. [4–7] where, in particular, it was shown that in certain circumstances the explicit solution of the problem (2.6)–(2.8) can be obtained represented by the Feynman–Kac formula [13,15]

$$u(r, \theta) = \mathbf{E}\left\{u(\vec{\xi}_\tau)\exp\left(\frac{1}{2}\int_0^\tau ik\xi_t^1 q^2(\vec{\xi}_t)dt\right)\right\}, \qquad \vec{\xi}_0 = (r, \theta), \tag{2.9}$$

with the mathematical expectation $\mathbf{E}$ corresponding to the Wiener measure on the space of all possible trajectories of the two-component random motion

$$\vec{\xi}_t = (\xi_t^1, \xi_t^2), \tag{2.10}$$

running in the complex space $\mathbb{C}^2$ under the control of the stochastic differential equations

$$d\xi_t^1 = q(\vec{\xi}_t)\xi_t^1\, dw_t^1 + q^2(\vec{\xi}_t)\xi_t^1\left(\frac{1}{2} + ik\xi_t^1\right)dt, \qquad \xi_0^1 = r, \tag{2.11}$$

$$d\xi_t^2 = q(\vec{\xi}_t)dw_t^2, \qquad \xi_0^2 = \theta, \tag{2.12}$$

driven by the standard one-dimensional Wiener processes w_t^1 and w_t^2. As for the exit time τ, it is defined as the time when $\vec{\xi}_t$ first touches a surface $\mathfrak{g}$ where the function $u(r, \theta)$ is pre-assigned.

To make the solution (2.9)–(2.12) useful we should select the indefinite factor $q(\vec{\xi})$ in a way that guarantees the finiteness of the time τ required for the motion $\vec{\xi}_t$ to hit the surface $\mathfrak{g}$ where $u(r, \theta)$ is known. Such selection does not pose serious difficulties in cases when $\mathfrak{g}$ is a three-dimensional object in the four-dimensional space $\mathbb{C}^2$ where the random motion $\vec{\xi}_t$ runs. However, in the case under consideration here of a two-dimensional $\mathfrak{g}$, the right selection of $q(\vec{\xi})$ becomes a less trivial, although still solvable problem.

We assume that the boundary function $f(\theta)$ from condition (2.7) is analytic and can be continued from the circle $\mathbb{S}_R = \{r, \theta : r = R, 0 \leq \theta < 2\pi\}$ to the plane $\mathfrak{s}_R = \{r, \theta : r = R, \theta \in \mathbb{C}\}$. Then, in order to find a factor $q(\vec{\xi})$ which ensures that any motion $\vec{\xi}_t$ controlled by Eqs. (2.10)–(2.12) reaches $\mathfrak{s}_R$, we first solve a limited problem of 'steering' to $\mathfrak{s}_R$ the motions $\vec{\xi}_t$ launched from a specially designated three-dimensional manifold $\mathfrak{S}_R \supset \mathfrak{s}_R$. After that, we find how to steer to $\mathfrak{S}_R$ any motion $\vec{\xi}_t$ launched from the physical domain G_R introduced in Eq. (2.1).

Consider the motion $\vec{\xi}_t = (\xi_t^1, \xi_t^2)$ corresponding to the factor

$$q(\vec{\xi})\frac{1}{\frac{1}{2} + ik\xi^1}, \qquad \vec{\xi} \equiv (\xi^1, \xi^2), \tag{2.13}$$

which turns Eqs. (2.11) and (2.12) into the form

$$d\xi_t^1 = \frac{2\xi_t^1(dw_t^1 + dt)}{1 + 2ik\xi_t^1}, \tag{2.14}$$

$$d\xi_t^2 = \frac{2\, dw_t^2}{1 + 2ik\xi_t^1}. \tag{2.15}$$

Since the differential $ds = dw_t^1 + dt$ is real-valued, Eq. (2.14) implies that the complex point ξ_t^1 moves along one-dimensional integral lines of the ordinary differential equation

$$\frac{d\xi_s}{ds} = \frac{2\xi_s}{1 + 2ik\xi_s}, \tag{2.16}$$

whose general solution ξ_s satisfies the algebraic equation

$$\xi_s\, e^{2ik\xi_s} = \xi_0\, e^{2ik\xi_0}\, e^{2s}. \tag{2.17}$$

Let $\mathfrak{l}_R \subset \mathbb{C}$ be the integral line of Eq. (2.16) passing through the real point $\xi = R > 0$ involved in the boundary conditions (2.7). It follows from Eq. (2.17) that points $\xi = x + iy$ located on $\mathfrak{l}_R$ satisfy the condition

$$\mathrm{Im}(\xi\, e^{2ik(\xi - R)}) = 0, \qquad R \in \mathbb{R}, \tag{2.18}$$

which leads to the explicit definition

$$\mathfrak{l}_R = \left\{ x + iy : y = -x \tan[2k(x - R)],\, |x - R| \le \frac{\pi}{4k} \right\}. \tag{2.19}$$

Then an elementary analysis of Eq. (2.19) shows that $\mathfrak{l}_R$ may have one of three distinct shapes depending on the values of k and R. Thus, if $\pi/2 < kR$, then $\mathfrak{l}_R$ is a curve running between infinite points $\xi_- = (R - (\pi/4k)) + i\infty$ and $\xi_+ = (R + (\pi/4k)) - i\infty$. If $\pi/2 = kR$, then $\mathfrak{l}_R$ is a semi-infinite curve between $\xi = i/2k$ and $\xi = \xi_+$. Finally, if $\pi/2 > kR$, then $\mathfrak{l}_R$ is also a semi-infinite line connecting $\xi = 0$ and $\xi = \xi_+$. All of these options are shown in Fig. 1 which shows the lines $\mathfrak{l} = \mathfrak{l}_R$ corresponding to the fixed wave number $k = 1$ and to different values of the radius R.

Since $\mathfrak{l}_R$ is an integral line of Eq. (2.16) the motion ξ_t^1 controlled by the stochastic equation (2.14) and launched from a point $\xi_0^1 \in \mathfrak{l}_R$ remains on $\mathfrak{l}_R$ at all times $t > 0$, so that ξ_t^1 can be considered as a random motion on $\mathfrak{l}_R$. To determine the trend of this motion it suffices to use the representation $\xi_t^1 = x_t + iy_t$ and to re-arrange (2.14) into a system of real-valued equations

$$\begin{aligned} dx_t &= \frac{2x_t(dw_t^1 + dt)}{(1 - 2ky_t)^2 + 4k^2x_t^2}, \\ dy_t &= \frac{2(y_t - 2kx_t^2 - 2ky_t^2)(dw_t^1 + dt)}{(1 - 2ky_t)^2 + 4k^2x_t^2}, \end{aligned} \tag{2.20}$$

the first of which implies that in the domain $x_t \equiv \mathrm{Re}(\xi_t^1) > 0$ the motion ξ_t^1 drifts to the right. Then, taking into account the possible structures of the lines $\mathfrak{l}_R$ shown in Fig. 1 we conclude that ξ_t^1 randomly moves along $\mathfrak{l}_R$ drifting towards the end $\xi_+ = (R + (\pi/4k)) - i\infty$, which ensures that any motion ξ_t^1 launched from the point $\xi_0^1 \in \mathfrak{l}_R$ with $\mathrm{Im}(\xi_0^1) > 0$ eventually reaches the point $\xi_\tau^1 = R$ at the finite time τ.

After the line $\mathfrak{l}_R$ is specified by Eq. (2.19) we introduce the three-dimensional object $\mathfrak{S}_R$ consisting of the vectors $\vec{\xi} = (\xi^1, \xi^2)$ whose components are restricted by the conditions

$$\mathfrak{S}_R: \quad \xi^1 \in \mathfrak{l}_R, \qquad \mathrm{Im}(\xi^1) \ge 0, \quad \xi^2 \in \mathbb{C}, \tag{2.21}$$

which guarantee that any motion $\vec{\xi}_t = (\xi_t^1, \xi_t^2)$ controlled by Eqs. (2.14) and (2.15) and launched from $\mathfrak{S}_R$ never leaves $\mathfrak{S}_R$ and eventually reaches its boundary $\mathfrak{s}_R = \{\vec{\xi} : \xi^1 = R, \xi^2 \in \mathbb{C}\}$, where the solution $u(\xi^1, \xi^2)$ of the problem (2.6)–(2.8) is defined by the analytical continuation of the boundary conditions (2.7). As a result, formulas (2.9) and (2.13) together with (2.14) and (2.15) provide the extension of $u(r, \theta)$ from the two-dimensional plane $\mathfrak{s}_R = \{r, \theta : r = R, \theta \in \mathbb{C}\}$ to the three-dimensional domain $\mathfrak{S}_R$, which, however, does not include the important physical domain $G_R = \{r, \theta : r > R, 0 \le \theta < 2\pi\}$.

To make the expression (2.9) usable for the extension of $u(r, \theta)$ from $\mathfrak{S}_R$ to the physical domain G_R we consider the stochastic differential equations

$$d\xi_t^1 = \xi_t^1\, dw_t^1 + \xi_t^1\left(\tfrac{1}{2} + ik\xi_t^1\right)dt, \qquad \xi_0^1 = r, \tag{2.22}$$

$$d\xi_t^2 = dw_t^2, \qquad \xi_0^2 = \theta, \tag{2.23}$$

which are the particular case of Eqs. (2.11) and (2.12) corresponding to the factor $q(\vec{\xi}_t) \equiv 1$.

Eq. (2.23) implies that ξ_t^2 is the standard one-dimensional Brownian motion on the real axis, while Eq. (2.22) determines a more complicated random motion inside the quarter-plane $\mathrm{Re}(\xi_t^1) > 0$, $\mathrm{Im}(\xi_t^1) > 0$ of the complex plane. To verify the last statement we notice that ξ_t^1 can neither leave nor reach the axis $\mathrm{Re}(\xi_t^1) = 0$ because, due to Eq. (2.22), the differential $d\xi_t^1$ has on this axis purely

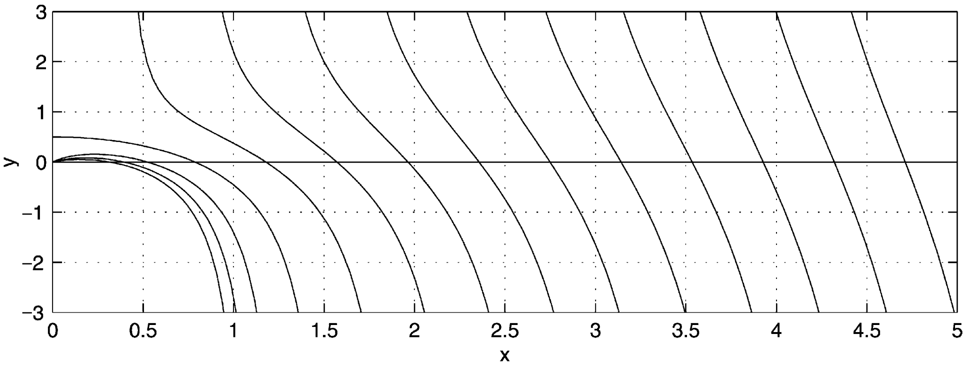

Fig. 1. Lines $\mathfrak{l}_R = \{x + iy : y = -x \tan[2k(x - R)], |x - R| \le \pi/4k\}$.

imaginary values. Similarly, ξ_t^1 cannot leave the quarter-plane $\mathrm{Re}(\xi_t^1) > 0$, $\mathrm{Im}(\xi_t^1) > 0$ through the horizontal semi-axis $\xi_t^1 > 0$ where $\mathrm{d}\xi_t^1$ has a positive imaginary part. Additional information about the motion ξ_t^1 comes from the observation [7] that it has a drift along the lines

$$\xi_t^1 = \frac{\mathrm{i}\xi_t^1}{2k\xi_t^1 - (2k\xi_t^1 - \mathrm{i})\mathrm{e}^{-t/2}}, \qquad t \geq 0, \tag{2.24}$$

which run from different initial points $\xi_0^1 = r > 0$ toward the single end-point $\xi_\infty^1 = \mathrm{i}/2k$. Combining all the above mentioned details about ξ_t^1 we conclude that this motion is localized in the quarter-plane $\mathrm{Re}(\xi_t^1) > 0$, $\mathrm{Im}(\xi_t^1) > 0$, and that the drift drags ξ_t^1 towards the point $\xi_\infty^1 = \mathrm{i}/2k$, so that as $t \to \infty$ the motion ξ_t^1 appears as a random walk in the vicinity of ξ_∞^1. Then, taking into account the structure of the lines I_R defined by Eq. (2.19) and shown in Fig. 1, it is easy to show that any of these lines is intersected by the trajectory of ξ_t^1 launched from any initial position $\xi_0^1 = r > 0$ and controlled thereafter by Eq. (2.22).

The last property of the motion ξ_t^1 controlled by Eq. (2.22) makes it possible to use the Feynman–Kac formula (2.9) with $q(\vec{\xi}) = 1$ for the extension of $u(r, \theta)$ from the auxiliary domain $\mathfrak{S}_R$ described by Eq. (2.21) to the physical domain G_R. At the same time, the continuation of $u(r, \theta)$ to $\mathfrak{S}_R$ from its boundary $\mathfrak{s}_R = \{r, \theta : r = R, \theta \in \mathbb{C}\}$ is also provided by the formula (2.9), but with the factor $q(\vec{\xi})$ defined on $\mathfrak{S}_R$ by Eq. (2.13). Comparing these results we readily conclude that the solution $u(r, \theta)$ of the Dirichlet problem (2.6)–(2.8) in the physical domain G_R can be obtained by the single application of the formula (2.9) with the factor $q(\vec{\xi})$ defined as

$$q(\vec{\xi}) \equiv q(\xi^1, \xi^2) = \begin{cases} 1, & \text{if } \xi^1 \notin \mathrm{I}_R, \\ 2/(1 + 2\mathrm{i}k\xi^1), & \text{if } \xi^1 \in \mathrm{I}_R, \end{cases} \tag{2.25}$$

where $\mathrm{I}_R \subset \mathbb{C}$ is the line parameterized by Eq. (2.19).

For better understanding of the obtained solution of the problem (2.6)–(2.8) it is instructive to discuss the structure of the random motion $\vec{\xi}_t = (\xi_t^1, \xi_t^2)$ launched from the point $\vec{\xi}_0 = (r, \theta)$ with $r > R$, and controlled by the stochastic differential equations (2.11) and (2.12) corresponding to the factor $q(\vec{\xi})$ introduced by Eq. (2.25).

Since the initial position $\xi_0^1 = r$ of the radial motion ξ_t^1 does not belong the line $\mathrm{I}_R \subset \mathbb{C}$, it is clear that immediately after launching the motion $\vec{\xi}_t$ is controlled by Eqs. (2.22) and (2.23), corresponding to $q = 1$, which means that the angular coordinate ξ_t^2 chaotically moves along the real axis, while the radial coordinate ξ_t^1 moves in the quarter-plane $0 < \arg(\xi_t^1) < \pi/2$ until it eventually hits at the time $t = \tau_1$ the line I_R involved in Eq. (2.25) and specified by Eq. (2.19). After that, at any time $t > \tau_1$ the motion $\vec{\xi}_t$ is controlled by Eqs. (2.14) and (2.15), which let the angular coordinate ξ_t^2 leave the real axis but force the radial coordinate ξ_t^1 to move along the line I_R until it hits, at the exit time $t = \tau$, the point $\xi_\tau^1 = R$.

The described behavior of the motions ξ_t^1 and ξ_t^2 is shown in Fig. 2 corresponding to the wave number $k = 1$ and radius $R = 1.5$ which is approximately 25% of the wave length $\lambda = 2\pi/k \approx 6.28$.

The radial motion ξ_t^1 is shown on the top diagram. The dotted curves on the top diagram with a more horizontal

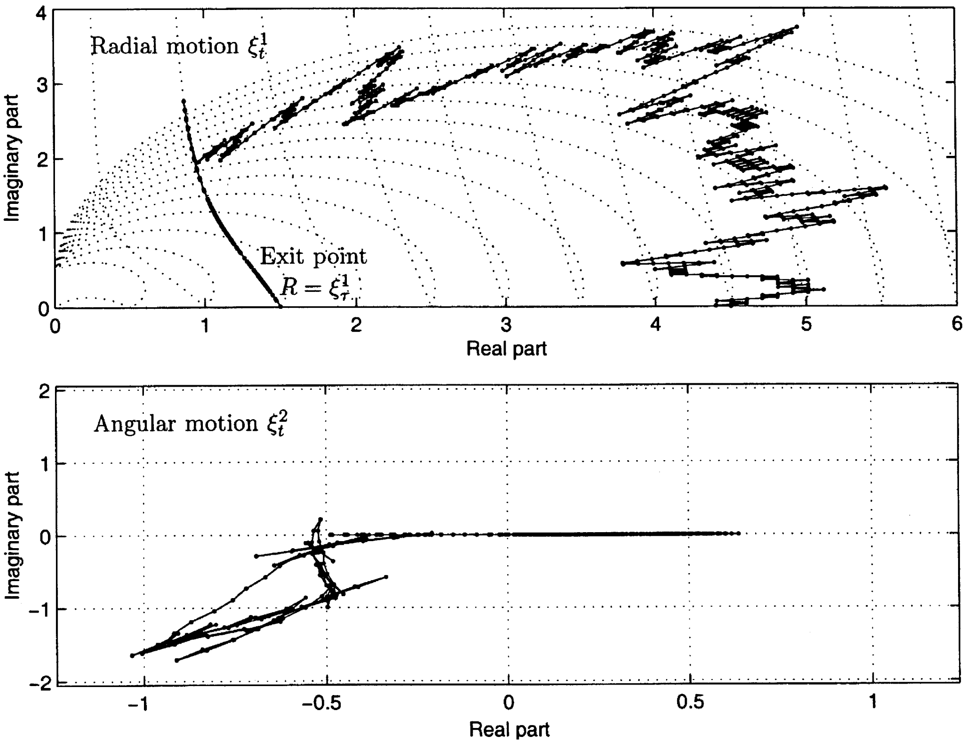

Fig. 2. Sample path of the radial and angular motions ξ_t^1 and ξ_t^2.

orientation show the lines (2.24) along which the motion ξ_t^1 has a drift. The dotted curves with a more vertical orientation show the lines $\mathfrak{l}_r$ defined by Eq. (2.19). The shown trajectory launches from the point $\xi_0^1 = 4.5$. It makes chaotic two-dimensional movements until it meets the line $\mathfrak{l}_R$, after which it stochastically runs along $\mathfrak{l}_R$ with some drift towards the real axis, which is hit at the point $\xi_t^1 = 1.5$, a point on the boundary.

The angular motion ξ_t^2 is shown on the bottom diagram of Fig. 2. This motion starts from the point $\xi_0^2 = 0$ and, in the beginning, it stochastically moves along the real axis. Then, after the radial motion hits the line $\mathfrak{l}_R$, it leaves the real axis and moves across the complex plane until it stops at a complex, in general, point at the time when the radial motion ξ_t^1 reaches the boundary $\xi_t^1 = R$.

In conclusion of the analysis of the problem (2.6)–(2.8) we convert its solution (2.9) and (2.25) to alternative forms, more convenient for use and more descriptive of the underlying phenomena.

First we observe that the solution (2.9) admits the representation

$$u(r,\theta) = \sqrt{\frac{R}{r}}\mathrm{e}^{-\mathrm{i}kR}\mathbf{E}\left\{F(\xi_\tau^2)\exp\left(-\frac{1}{2}\int_0^\tau q(\vec{\xi}_t)\mathrm{d}w_t^1\right)\right\}, \tag{2.26}$$

$$\vec{\xi}_0 = u(r,\theta),$$

which directly involves the boundary function $F(\theta)$ from the original boundary conditions (2.3), and which emphasizes the structure of $u(r,\theta)$ as $r \to \infty$, suggested by the radiation condition (2.4). To show this we note that Eq. (2.11) yields

$$\mathrm{i}k\xi_t^1 q^2(\vec{\xi}_t)\mathrm{d}t = \frac{\mathrm{d}\xi_t^1}{\xi_t^1} - q(\vec{\xi}_t)\mathrm{d}w_t^1 - \frac{q^2(\vec{\xi}_t)}{2}\mathrm{d}t, \tag{2.27}$$

and Ito's formula of stochastic differentiation [9,10] generates the identity

$$\mathrm{d}\ln(\xi_t^1) = \frac{\mathrm{d}\xi_t^1}{\xi_t^1} - \frac{1}{2}\left(\frac{\mathrm{d}\xi_t^1}{\xi_t^1}\right)^2 \equiv \frac{\mathrm{d}\xi_t^1}{\xi_t^1} - \frac{q^2(\vec{\xi}_t)}{2}\mathrm{d}t, \tag{2.28}$$

whose combination with Eq. (2.27) leads to the expression

$$\mathrm{i}k\xi_t^1 q^2(\vec{\xi}_t)\mathrm{d}t = \mathrm{d}\ln(\xi_t^1) - q(\vec{\xi}_t)\mathrm{d}w_t^1. \tag{2.29}$$

Then, integrating Eq. (2.29) and taking into account the initial and exit conditions $\xi_0^1 = r$ and $\xi_\tau^1 = R$ we get the expression

$$\int_0^\tau \mathrm{i}k\xi_t^1 q^2(\vec{\xi}_t)\mathrm{d}t = \ln(R) - \ln(r) - \int_0^\tau q(\vec{\xi}_t)\mathrm{d}w_t^1, \tag{2.30}$$

whose substitution into Eq. (2.9) generates Eq. (2.26).

Finally, taking advantage of the explicit definition (2.25) of the factor $q(\vec{\xi}_t)$ it is easy to re-write the expression (2.26) as

$$u(r,\theta) = \sqrt{\frac{R}{r}}\mathrm{e}^{-\mathrm{i}kR} \times\mathbf{E}\left\{F(\xi_\tau^2)\exp\left(-\frac{w_{\tau_1}^1}{2} - \int_{\tau_1}^{\tau}\frac{\mathrm{d}w_t^1}{1+2\mathrm{i}k\xi_t^1}\right)\right\},$$

$$\vec{\xi}_0 = (r,\theta), \tag{2.31}$$

where τ_1 is the turning time when the radial motion ξ_t^1 hits the line $\mathfrak{l}_R$ defined in Eq. (2.19), and τ is the exit time, i.e. the time when ξ_t^1, moving along $\mathfrak{l}_R$, hits the point $\xi_\tau^1 = R$. As part of the obtained solution it should be recalled that the mathematical expectation in Eq. (2.31) is computed over the trajectories of the random motion $\vec{\xi}_t = (\xi_t^1, \xi_t^2)$ which launches at the time $t = 0$ from the observation point $\vec{\xi}_0 = (r,\theta)$ and runs thereafter under the control of the stochastic differential equations (2.22) and (2.23) on the initial time interval $0 \le t < \tau_1$, and of Eqs. (2.14) and (2.15) on the final time interval $\tau_1 \le t < \tau$, ended at the exit time τ when the radial component ξ_t^1 of $\vec{\xi}_t$ hits the point $\xi_R^1 = R$.

As an example illustrating the obtained probabilistic solution of wave radiation problems we consider the two-dimensional problem (2.4)–(2.6) with the boundary function

$$F(\theta) = \frac{1}{1-\sigma\,\mathrm{e}^{\mathrm{i}\theta}} \equiv \sum_{n=0}^{\infty}\sigma^n\,\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.32}$$

corresponding to the exact solution $u(r,\theta)$ delivered by the series

$$u(r,\theta) = \sum_{n=0}^{\infty}\sigma^n\frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)}\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.33}$$

which converges fairly fast in the domain $r \ge R$ with moderate values of the radius $R > 1$.

Fig. 3 presents the results of numerical simulation of the solution of the problem (2.4)–(2.6) with the wave number $k = 1$, with the radius $R = 1.5$ selected close to 25% of the wave length $\lambda = 2\pi/k$, and with the boundary conditions (2.32) corresponding to the parameter $\sigma = 0.25$. The solution was computed along the spiral $r = r_0 + (\theta/2\pi)\times(r_1 - r_0)$ with the polar angle θ varying from 0° to 360° and with the radius r varying from $r_0 = 2$ at $\theta = 0°$ to $r_1 = 13 \approx 2\lambda$ at $\theta = 360°$.

The continuous lines in Fig. 3 show the results of direct summation of the series (2.33), while the results obtained by the random walk method are shown by the dotted lines. The probabilistic computations were based on the solution (2.31) with the factor $q(\xi)$ defined by Eq. (2.25). The mathematical expectation was simulated by the averaging of 2000 random motions approximated by the discrete random walks [6,7] with the spatial step $\varepsilon = 0.05$ corresponding to the time increment $\Delta t = \varepsilon^2 = 0.0025$. The computations were stable and the difference between the results was under 0.015 at all considered observation points.

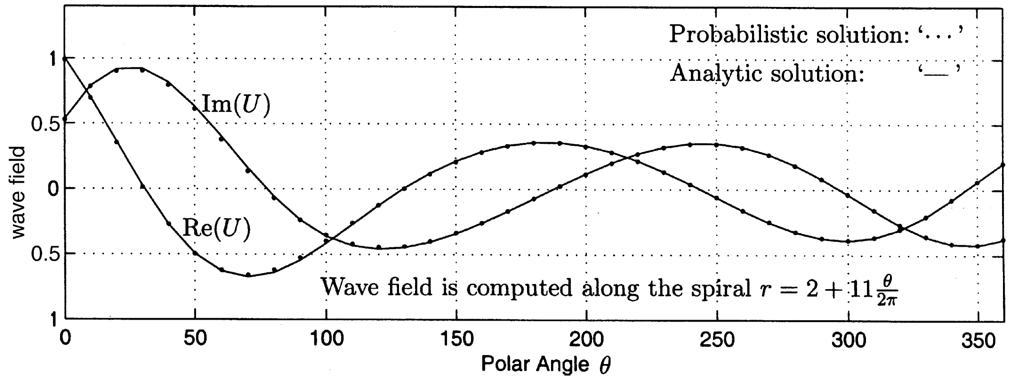

Fig. 3. Simulation of $u(r,\theta)=\sum_{n=0}^{\infty}(1/4^n)(H_n^{(1)}(kr)/H_n^{(1)}(kR))e^{in\theta}$.

3. Waves radiating from a sphere

It is remarkable that the technique developed above for the analysis of two-dimensional wave propagation in exterior cylindrical domains can be readily applied to three-dimensional problems of wave propagation in exterior spherical domains.

An exterior spherical domain $G_R \subset \mathbb{R}^3$ is specified by the condition

$$G_R:\ \{r,\theta,\phi : r>R, -\frac{\pi}{2}\le\theta<\frac{\pi}{2}, 0\le\phi<2\pi\}, \tag{3.1}$$

where (r,θ,ϕ) are the standard spherical coordinates in $\mathbb{R}^3$. We are interested in the solution of the Helmholtz equation

$$\nabla^2 U + k^2 U = 0, \quad k = \text{const}, \tag{3.2}$$

which is defined in G_R and satisfies the Dirichlet boundary conditions

$$U(R,\theta,\phi) = F(\theta,\phi), \tag{3.3}$$

with the pre-defined function $F(\theta,\phi)$, and the radiation condition

$$U(r,\theta,\phi) = C(\theta,\phi)\frac{e^{ikr}}{kr}(1+o(1)), \quad r\to\infty, \tag{3.4}$$

where $C(\theta,\phi)$ is an indefinite function known as the diffraction coefficient.

We seek the solution of the Helmholtz equation in the Liouville form

$$U(r,\theta,\phi) = u(r,\theta,\phi)e^{ikr}, \tag{3.5}$$

where $u(r,\theta,\phi)$ is a new unknown function satisfying the equation

$$\left[\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2} + r(1+ikr)\frac{\partial u}{\partial r}\right] + \left[\frac{1}{2}\frac{\partial^2 u}{\partial\theta^2} - \frac{\tan\theta}{2}\frac{\partial u}{\partial\theta}\right] + \frac{1}{2\cos^2\theta}\frac{\partial^2 u}{\partial\phi^2} + ikru = 0, \tag{3.6}$$

which can be deduced from Eqs. (3.2) and (3.5) by straightforward computations [7], similar to that used for derivation of Eq. (2.6). Additionally, $u(r,\theta,\phi)$ has to obey the boundary conditions

$$u(R,\theta,\phi) = f(\theta,\phi), \qquad f(\theta,\phi)\equiv F(\theta,\phi)e^{-ikR}, \tag{3.7}$$

and the condition at infinity

$$u(r,\theta,\phi) = O(1/r), \qquad r\to\infty, \tag{3.8}$$

following from Eqs. (3.3) and (3.5), and from Eqs. (3.4) and (3.5), respectively.

If the boundary function $F(\theta,\phi)$ is analytic with respect to both of the angular coordinates θ and ϕ, then the solution of the problem (3.6)–(3.8) can be represented by the formula

$$u(r,\theta,\phi) = e^{-ikR}\mathbf{E}\left\{F(\xi_\tau^2,\xi_\tau^3)\exp\left(\int_0^\tau ik\xi_t^1 q^2(\vec{\xi}_t)dt\right)\right\}, \tag{3.9}$$

$$\xi_\tau^1 = R,$$

with the mathematical expectation computed over trajectories of the three-dimensional random motion $\vec{\xi}_t = (\xi_t^1,\xi_t^2,\xi_t^3)$ launched at time $t=0$ from the observation point $\vec{\xi}_0 = (r,\theta,\phi)$, governed thereafter by the stochastic differential equations

$$d\xi_t^1 = \xi_t^1 q(\vec{\xi}_t)dw_t^1 + \xi_t^1(1+ik\xi_t^1)q^2(\vec{\xi})dt, \quad \xi_0^1 = r, \tag{3.10}$$

$$d\xi_t^2 = q(\vec{\xi}_t)dw_t^2 - \tfrac{1}{2}\tan(\xi_t^2)q^2(\vec{\xi})dt, \quad \xi_0^2 = \theta, \tag{3.11}$$

$$d\xi_t^3 = q(\vec{\xi})dw_t^3/\cos(\xi_t^2), \quad \xi_0^3 = \phi, \tag{3.12}$$

driven by independent one-dimensional Brownian motions w_t^1, w_t^2, w_t^3, and stopped at the exit time τ, when the radial component of $\vec{\xi}_t$ touches the point $\xi_\tau^1 = R$.

To make the solution (3.9)–(3.12) meaningful, the indefinite factor $q(\vec{\xi}_t)$ has to be selected in a way which guarantees that the radial random motion ξ_t^1 running, in general, in the complex plane does not miss the isolated single point $\xi^1 = R$.

Following the analogy between Eq. (3.10) and Eq. (2.11) studied in Section 2, we notice that the choice

$$q(\vec{\xi}) = \frac{1}{1+ik\xi^1}, \qquad \vec{\xi} = (\xi^1,\xi^2,\xi^3), \tag{3.13}$$

turns Eq. (3.10) into the form

$$d\xi_t^1 = \frac{\xi_t^1(dw_t^1 + dt)}{1 + ik\xi_t^1}, \qquad \xi_0^1 = r, \tag{3.14}$$

which makes it clear that the solution ξ_t^1 of Eq. (3.14) represents x random motion along a one-dimensional line $\mathrm{l}_R \subset \mathbb{C}$, defined as the integral line of the ordinary differential equation $(1 + ik\zeta)d\zeta = \zeta\, dt$ passing through the point $\zeta = R$. Then, an elementary analysis similar to that described in relation to Eqs. (2.16)–(2.19) shows that the line l_R is described by the condition

$$\mathrm{l}_R: \ \mathrm{Im}\{\zeta\, e^{ik(\zeta-R)}\} = 0, \qquad R \in \mathbb{R}, \tag{3.15}$$

which leads to the explicit definition

$$\mathrm{l}_R = \left\{x + iy : y = -x\tan[k(x-R)], |x - R| \le \frac{\pi}{2k}\right\}, \tag{3.16}$$

only slightly different from the formula (2.19) appearing in the analysis of the two-dimensional configuration of Section 2.

The definition (3.16) of the line l_R guarantees that any random motion $\vec{\xi}_t$ governed by Eqs. (3.10)–(3.12) with $q(\vec{\xi})$ from Eq. (3.13) and launched from a point $\vec{\xi}_0 = (\xi_0^1, \xi_0^2, \xi_0^3)$ where $\xi_0^1 \in \mathrm{l}_R$ and $\mathrm{Im}(\xi_0^1) \ge 0$ inevitably hits the plane $\vec{\xi}_\tau = (\xi_\tau^1, \xi_\tau^2, \xi_\tau^3)$ with $\xi_\tau^1 = R \in \mathbb{R}$, where the solution $u(r, \theta, \phi)$ of the problem (3.6)–(3.8) is assigned by the analytical continuation of the boundary condition (3.7). This observation shows that the Feynman–Kac formula (3.9) can be used for computation of the solution of the problem (3.6)–(3.8) on the five-dimensional manifold $\mathfrak{S}_R$ consisting of the vectors $\vec{\xi} == (\xi^1, \xi^2, \xi^3)$ whose components are restricted by the conditions

$$\mathfrak{S}_R: \ \xi^1 \in \mathrm{l}_R, \ \mathrm{Im}(\xi^1) \ge 0, \ \xi^2 \in \mathbb{C}, \ \xi^3 \in \mathbb{C}. \tag{3.17}$$

Next we extend the solution $u(r, \theta, \phi)$ of the problem (3.6)–(3.8) from $\mathfrak{S}_R$ defined as Eq. (3.17) to a physical domain G_R introduced by Eq. (3.1). This extension is also achieved by the Feynman–Kac formula (3.9) with the factor $q(\vec{\xi}) = 1$ which turns the stochastic equation (3.10) into the form

$$d\xi_t^1 = \xi_t^1\, dw_t^1 + \xi_t^1(1 + ik\xi_t^1)dt, \qquad \xi_0^1 = r, \tag{3.18}$$

only slightly differed from Eq. (2.22). Indeed, an analysis similar to that performed for Eq. (2.22) shows that any motion of ξ_t^1 controlled by Eq. (3.18) and launched from a real point $\xi_0^1 = r$ eventually reaches the manifold $\mathfrak{S}_R$ from Eq. (3.17), where the solution $u(r, \theta, \phi)$ is already known.

All of the above result in the conclusion that if

$$q(\vec{\xi}) \equiv q(\xi^1, \xi^2, \xi^3) = \begin{cases} 1, & \text{if } \xi^1 \notin \mathrm{l}_R, \\ 1/(1 + ik\xi^1), & \text{if } \xi^1 \in \mathrm{l}_R, \end{cases} \tag{3.19}$$

where $\mathrm{l}_R \subset \mathbb{C}$ is the line parameterized by Eq. (3.16), then the formulas (3.9)–(3.12) effectively solve the problem (3.6)–(3.8) anywhere in the physical domain $r \ge R$. Furthermore, it is easy to see that this solution admits transformation to the alternative form

$$u(r, \theta, \phi) = \frac{R}{r}e^{-ikR} \times \mathbf{E}\left\{F(\xi_\tau^2, \xi_\tau^3)\exp\left(-\int_0^\tau [q(\xi_t^1)dw_t^1 + \tfrac{1}{2}q^2(\xi_t^1)dt]\right)\right\}, \tag{3.20}$$

where τ_1 is the turning time when the radial motion ξ_t^1 hits the line l_R defined in Eq. (3.16), and τ is the exit time defined as the time when ξ_t^1 moving along l_R hits the point $\xi_\tau^1 = R$.

To derive Eq. (3.20) we note that Eq. (3.10) yields the relationship

$$ik\xi_t^1 q^2(\vec{\xi}_t)dt = \frac{d\xi_t^1}{\xi_t^1} - q(\vec{\xi}_t)dw_t^1 - q^2(\vec{\xi}_t)dt, \tag{3.21}$$

whose combination with Ito's formula (2.28) leads to the expressions

$$ik\xi_t^1 q^2(\vec{\xi}_t)dt = d\ln(\xi_t^1) - q(\vec{\xi}_t)dw_t^1 - \tfrac{1}{2}q^2(\vec{\xi}_t)dt, \tag{3.22}$$

and

$$\int_0^\tau ik\xi_t^1 q^2(\vec{\xi}_t)dt = \ln(R) - \ln(r) - \int_0^\tau [q(\vec{\xi}_t)dw_t^1 + \tfrac{1}{2}q^2(\vec{\xi}_t)dt], \tag{3.23}$$

the latter of which is obtained by the integration of Eq. (3.22) with the conditions $\xi_0^1 = r$ and $\xi_\tau^1 = R$ taken into account. Finally, substituting Eq. (3.23) into Eq. (3.9) we arrive at Eq. (3.20).

4. Variations of the approach

Since the probabilistic solutions (2.31) and (3.20) of the two- and three-dimensional problems considered above are essentially similar to each other, we restrict the following discussion to the simpler two-dimensional problems.

An attractive feature of the probabilistic approach to wave propagation is its versatility, which makes it possible to adjust practically every step of the method to the specifics of the problem under investigation. To illustrate such versatility we suggest here possible modifications of three crucial points of the approach, which are the product representation (2.5) of the solution, the choice of the auxiliary factor $q(\xi)$ determining through Eqs. (2.11) and (2.12) the random motion $\vec{\xi}_t$ involved in the Feynman–Kac solution (2.9), and the assumption of analyticity of the function $F(\theta)$ from the boundary condition (2.3).

Consider the representation of the solution of the Helmholtz equation (2.2) in the form

$$U(r, \theta) = u(r, \theta)e^{i[kr+\mu(\theta)]}, \tag{4.1}$$

generalizing the standard Liouville decomposition (2.5) by the presence of the additional indefinite function $\mu(\theta)$. Then, assuming that $\mu(\theta)$ is pre-defined we reduce the wave radiation problem (2.2)–(2.4) to the problem of finding the amplitude $u(r,\theta)$ which vanishes as $r\to\infty$ and satisfies the equation

$$\left[\frac{r^2}{2}\frac{\partial^2 u}{\partial r^2}+r\left(\frac{1}{2}+\mathrm{i}kr\right)\frac{\partial u}{\partial r}\right] +\left[\frac{1}{2}\frac{\partial^2 u}{\partial\theta^2}+\mathrm{i}\mu'\frac{\partial u}{\partial\theta}\right]+\frac{1}{2}[\mathrm{i}kr-(\mu')^2+i\mu'']u=0, \tag{4.2}$$

accompanied by the boundary conditions

$$u(R,\theta)=F(\theta)\mathrm{e}^{-\mathrm{i}kR-\mathrm{i}\mu(\theta)}, \tag{4.3}$$

depending on the indefinite function $\mu(\theta)$.

Eq. (4.2) is of the same type as Eq. (2.6) and its solution can be represented by the generalization of the Feynman–Kac formula (2.9) to the form

$$u(r,\theta)=\mathrm{e}^{-\mathrm{i}kR} \mathbf{E}\left\{F(\xi_\tau^2)\mathrm{e}^{-\mathrm{i}\mu(\xi_\tau^2)}\exp\left(\frac{1}{2}\int_0^\tau[\mathrm{i}k\xi_t^1-(\mu')^2+\mathrm{i}\mu'']q^2\,\mathrm{d}t\right)\right\}, \tag{4.4}$$

where the mathematical expectation is computed over the trajectories of the two-component random walk $\vec{\xi}_t=(\xi_t^1,\xi_t^2)$ governed by the stochastic differential equations

$$\mathrm{d}\xi_t^1=\xi_t^1q(\xi_t^1)\mathrm{d}w_t^1+\xi_t^1\left(\frac{1}{2}+\mathrm{i}k\xi_t^1\right)q^2(\xi_t^1)\mathrm{d}t, \tag{4.5}$$

$$\mathrm{d}\xi_t^2=q(\xi_t^1)\mathrm{d}w_t^2+\mathrm{i}\mu'(\xi_t^2)q^2(\xi_t^1)\mathrm{d}t, \tag{4.6}$$

with the auxiliary factor $q(\xi)$ defined by Eq. (2.25). The motion $\vec{\xi}_t$ starts at $t=0$ from the observation point $\vec{\xi}_0=(r,\theta)$ and stops at the exit time $t=\tau$ when the radial component ξ_t^1 of the motion $\vec{\xi}_t$ reaches the point $\xi_\tau^1=R$, which is guaranteed by the selection of the factor $q(\xi)$. Next, applying the identity (2.30) we re-arrange Eq. (4.4) to the alternative form

$$u(r,\theta)=\sqrt{\frac{R}{r}}\mathrm{e}^{-\mathrm{i}kR} \times\mathbf{E}\left\{F(\xi_\tau^2)\mathrm{e}^{-\mathrm{i}\mu(\xi_\tau^2)}\exp\left(\frac{1}{2}\int_0^\tau[\mathrm{i}\mu''-(\mu')^2]q^2\mathrm{d}t-q\mathrm{d}w_t^1\right)\right\}, \tag{4.7}$$

emphasizing the structure of $u(r,\theta)$ in the remote region $r\gg 1$.

It is clear that the expressions (4.4)–(4.7) directly generalize the similar expressions (2.11), (2.12), (2.26) and (2.31) by the inclusion of the indefinite function $\mu(\theta)$, whose appropriate selection may be beneficial, for example, by reducing the variance of the probabilistic simulation of the mathematical expectations (4.4) or (4.7).

Next we examine alternate possibilities of the definition (2.25) of the auxiliary factor $q(\xi)$ determining the structure of the random motion $\vec{\xi}_t$ governed by the stochastic equations (2.11) and (2.12). As explained in Section 2, the selection of $q(\xi)$ should guarantee that any random motion ξ_t^1 launched from the real point $\xi=r>R$ and controlled thereafter by Eq. (2.11) eventually reaches the point $\xi=R$, also located on the real axis. To achieve this goal we introduced by Eq. (2.19) the special line I_R crossing the real axis at the point $\xi=R$, and then we defined $q(\xi)$ differently according to whether it is on or away from the line I_R. The definition $q(\xi)=1$ outside I_R insures that the motion ξ_t^1 eventually hits the I_R in the upper half-plane $\mathrm{Im}(\xi)>0$, and definition (2.13) of $q(\xi)$ on I_R forces the ξ_t^1 to move along I_R, after this line is reached.

It is important to note that while the choice $q(\xi)=2/(1+2ik\xi)$ on the line I_R is essentially unique, the choice $q(\xi)=1$ outside I_R is not unique, and it may be replaced by any other complex-valued factor which guarantees that the motion ξ_t^1 does not leave the half-plane $\mathrm{Im}(\xi)>0$ and drifts towards the line I_R. For example, outside I_R the factor $q(\xi)$ may be defined by the formula

$$q^2(\xi)=1-\chi(\xi)-\frac{2\chi(\xi)}{1+2ik\xi},\quad \chi(\xi)=\tanh(a|\mathrm{Im}(\xi)|), \tag{4.8}$$

$$a>0,\ \xi\notin\mathrm{I}_R,$$

which provides a faster drift of ξ_t^1 towards I_R. Fig. 4 shows a sample path of the motion ξ_t^1 governed by the stochastic equation (2.11) with the factor $q(\xi)$ defined outside I_R by Eq. (4.8) with $a=10$. Comparing Fig. 4 with Fig. 2 corresponding to the simpler choice (2.25) of $q(\xi)$ one can observe that the drift of the random motion shown in Fig. 4 is aimed directly towards I_R which speeds up the random motion. As a result, the computations shown in Fig. 3 are 2–3 times faster if outside I_R the factor $q(\xi)$ is defined by Eq. (4.8) with $a=10$ instead of $q(\xi)=1$.

Finally, we discuss the simple but important modification of the method which makes it possible to address problems of wave radiation into domains with non-analytic boundary conditions.

Consider again the problem (2.6)–(2.8), but this time assuming only that the boundary data $f(\theta)$ is an integrable, but not necessarily analytic, function of the angle $\theta\in[0,2\pi]$. Since $f(\theta)$ is integrable the following Poisson integral

$$u_0(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-R^2}{r^2+R^2-2Rr\cos(\theta-\vartheta)}f(\vartheta)\mathrm{d}\vartheta, \tag{4.9}$$

correctly defines an auxiliary function $u_0(r,\theta)$ which is analytic with respect to both variables and satisfies the Laplace equation

$$\frac{\partial^2u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_0}{\partial\theta^2}=0, \tag{4.10}$$

together with the boundary conditions (2.7). Then the solution $u(r,\theta)$ of the problem (2.6)–(2.8) can be represented as the superposition

$$u=u_0+\tilde{u}, \tag{4.11}$$

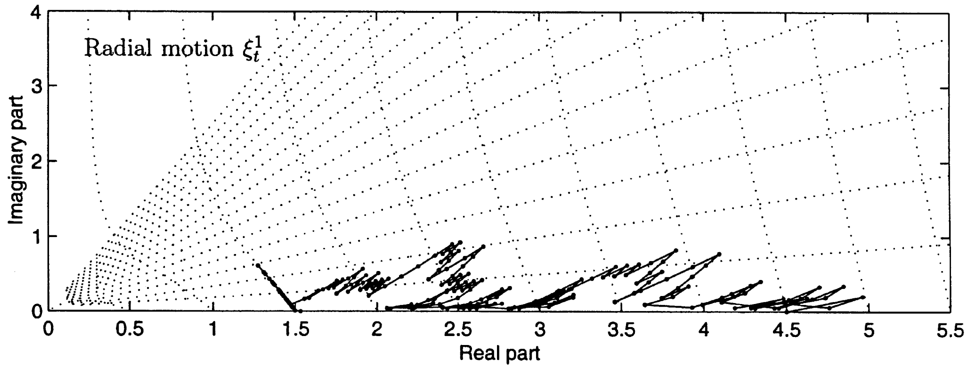

Fig. 4. Sample path of the motion ξ_t^1 corresponding to $q(\xi)$ from Eq. (4.8).

of the Poisson integral (4.9) and the new unknown function $\tilde{u}(r,\theta)$ which has to satisfy the analytic boundary conditions

$$\tilde{u}(R,\theta)=0, \tag{4.12}$$

and the equation

$$\frac{r^2}{2}\frac{\partial^2\tilde{u}}{\partial r^2}+r\left(\frac{1}{2}+\mathrm{i}kr\right)\frac{\partial\tilde{u}}{\partial r}+\frac{1}{2}\frac{\partial^2\tilde{u}}{\partial\theta^2}+\frac{\mathrm{i}kr}{2}\tilde{u}+\mathscr{F}=0, \tag{4.13}$$

with the pre-defined term

$$\mathscr{F}(r,\theta)=\mathrm{i}kr\left[r\frac{\partial u_0(r,\theta)}{\partial r}+\frac{u_0(r,\theta)}{2}\right], \tag{4.14}$$

which is analytic with respect to both variables r and θ.

Eq. (4.13) is of similar type as Eq. (2.6), and combining the technique from Section 2 with the well-known [9,13] Feynman–Kac formula for inhomogeneous equations, we straightforwardly arrive at the solution

$$\tilde{u}(r,\theta)=\mathbf{E}\left\{\int_0^\tau \mathscr{F}(\vec{\xi}_t)\mathrm{e}^{1/2\int_0^\tau \mathrm{i}k\xi_s^1 q^2(\vec{\xi}_t)\mathrm{d}s}\mathrm{d}t\right\}, \tag{4.15}$$

where the mathematical expectation is computed over the trajectories of the random motion $\vec{\xi}_t$ controlled by Eqs. (2.11) and (2.12) with the coefficient $q(\xi)$ defined either by Eq. (2.25) or by more complicated formulas like Eq. (4.8).

5. Conclusion

The random walk approach to wave propagation developed in Refs. [2–7] has been applied to problems of wave radiation into exterior cylindrical and spherical domains. The probabilistic formulas derived in Refs. [2–5] determine solutions of the Helmholtz equation as mathematical expectations of values of specified functionals computed over trajectories of the random motion composed of a deterministic drift along pre-defined complex rays and of random complex-valued fluctuations. However, as discussed in Ref. [7], such solutions become practical only when it is guaranteed that the random walks included in the averaging eventually reach a domain where the solution is pre-defined by the boundary conditions. It was also explained that the stochastic equations controlling these random walks involve an indefinite function whose appropriate selection steers the trajectories in the desired direction. In Refs. [6,7] it was shown how to select this indefinite function to solve problems of wave propagation in wedges and cones. Similarly, in the present paper it is described how to define random walks which provide convergence of the probabilistic representations of the wave fields in cylindrical and spherical domains.

As with analytic or asymptotic methods, probabilistic solutions provided by the random walk method are local in the sense that they make it possible to compute functions of interest at individual points without computing them on dense meshes. Moreover, such solutions admit simple perfectly scalable implementations with practically unlimited capability for parallel processing, and such solutions admit meaningful physical interpretation which do not contradict but compliment elementary models of wave propagation employed by the ray theory. All of these features together make the random walk method attractive both for qualitative and numerical analysis, and the examples discussed in the present paper and in Refs. [6,7] suggest that this method might be a powerful tool for the analysis of general wave propagation phenomena.

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References

[1] Arnold IV. Mathematical methods in classical mechanics, 2nd ed. Graduate texts in mathematics, New York: Springer; 1989.

[2] Budaev BV, Bogy DB. Probabilistic solutions of the Helmholtz equations. J Acoust Soc Am 2001;109:2260–2.
[3] Budaev BV, Bogy DB. Analysis of one-dimensional wave scattering by the random walk method. J Acoust Soc Am 2002;111:2555–60.
[4] Budaev BV, Bogy DB. Application of random walk methods to wave propagation. Q J Mech Appl Math 2002;55:209–26.
[5] Budaev BV, Bogy DB. Random walk methods and wave diffraction. Int J Solids Struct 2002;39:5547–70.
[6] Budaev BV, Bogy DB. Diffraction by a plane sector. Submitted for publication.
[7] Budaev BV, Bogy DB. Random walk approach to wave propagation in wedges and cones. J Acoust Soc Am 2003. Accepted for publication.
[8] Budaev BV, Bogy DB. Wave scattering by surface-breaking cracks and cavities. Submitted for publication.
[9] Dynkin EB. Markov processes. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Berlin: Springer; 1965.
[10] Feller W. An introduction to probability theory and its applications. Wiley series in probability and mathematical statistics, New York: Wiley; 1967.
[11] Fock VA. Electromagnetic diffraction and propagation problems. International series of monographs on electromagnetic waves, vol. 1. Oxford: Pergamon Press; 1965.
[12] Fock VA, Leontovich MA. Solution of the problems of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation. J Phys USSR 1946;10:1–13.
[13] Freidlin M. Functional integration and partial differential equations. The annals of mathematics studies, Princeton, New Jersey: Princeton University Press; 1985.
[14] Maslov VP, Fedoriuk MV. Semi-classical approximation in quantum mechanics. Dordrecht: Reidel; 1981.
[15] Simon B. Functional integration and quantum physics. Pure and applied mathematics; a series of monographs and textbooks, New York: Academic Press; 1979.