Notes on Integration and differentiation on $\mathbb{R}^n$, etc.
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A measure defined on a Borel σ-algebra is called a Borel measure. Write $\mathcal{M}(\mathbb{R}^n)$ for the Borel measures on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$, then $\mathcal{M}(E)$ are the borel measures on E, but $\mathcal{M}(E) \subset \mathcal{M}(\mathbb{R}^n)$, by zero padding.

Definition: a positive finite measure μ on $\mathbb{R}^n$ is regular if

1. $\mu(E) = \inf\{\mu(U); U \text{ open}, U \supset E\}$
2. $\mu(E) = \sup\{\mu(K); K \text{ compact}, K \subset E\}$

Every positive finite measure on $\mathbb{R}^n$ is regular. For every measure $\nu \in \mathcal{M}(\mathbb{R}^n)$, for all $E \in \mathcal{B}\ell(\mathbb{R}^n)$, there exists a sequence of open sets $U_k \supset E$, and compact sets $K_n \subset E$ such that $\nu(U_n) \to \nu(E)$ and $\nu(K_n) \to \nu(E)$.

Proof: ADD

Definition: we say a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular if each positive ν_k, in the Jordan decomposition $\nu = \sum_{k=0}^{3} i^k \nu_k$, is regular. By the last result, every $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular and then the condition about sequences of sets holds.

If $\nu \in \mathcal{M}(\mathbb{R})$, we define its distribution function by $F_\nu(x) = \nu((\infty, x])$. $\nu \mapsto F_\nu$ is injective and linear on $\mathcal{M}(\mathbb{R})$.

Proof: Linearity is easy, to see that $\nu \mapsto F_\nu$ is surjective, suppose $F_\nu = 0$, this means that $F_\nu(x) = 0$ for all $x \in \mathbb{R}$, so $\nu((-\infty, x]) = 0$ for all x, so $\nu((a,b]) = \nu((-\infty, b] \setminus (-\infty, a]) = 0 - 0 = 0$. Then because every open set in $\mathbb{R}$ is a countable union of disjoint open invervals, and open intervals can written as a countable union of disjoint half open intervals, every open set in $\mathbb{R}$ has ν-measure zero, and then by Jordan decomposition, ν can be written as a sum of regular measures, which are then all zero measures, so ν is the zero measure. Thus $\mathrm{Ker}(\nu \mapsto F_\nu) = 0$, and so $\nu \mapsto F_\nu$ is surjective.

$F : \mathbb{R} \to \mathbb{R}$ is of *bounded variation*, BV, or say $F \in BV$, if $\mathrm{Var}(F) < \infty$, where $\mathrm{Var}(F) := \sup\{V_F(x); x \in \mathbb{R}\}$, and $V_F(x)$ is the *total variation function* of F,

$$V_F(x) := \sup\left\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|;\ x_0 < x_1 < ... < x_n = x, (x_k) \in \mathbb{R}, n \in \mathbb{N}\right\}.$$

Def: $F : \mathbb{R} \to \mathbb{R}$ is in NBV if $F \in BV$, F is right continuous at all $x \in \mathbb{R}$, and $\lim_{x \to -\infty} F(x) = 0$; normalized BV.

If $\nu \in \mathcal{M}(\mathbb{R}, \mathcal{B}\ell(\mathbb{R}))$, then $F_\nu \in NBV$.

Proof: Let $x_0 < x_1 < ... < x_n = x \in \mathbb{R}$, then $\sum_{k=1}^{n} |F_\nu(x_k) - F_\nu(x_{k-1})| = \sum_{k=1}^{n} |\nu([x_{k-1}, x_k))| \leq \sum_{k=1}^{n} |\nu|([x_{k-1}, x_k)) = |\nu|([x_0, x_n)) \leq |\nu|(\mathbb{R}) = ||\nu|| < \infty$.

$$V_{F_\nu}(x) = \sup\left\{\sum_{k=1}^{n} |\nu((x_{k-1}, x_k])|;\ x_0 < x_1 < ... < x_n = x, (x_k) \in \mathbb{R}, n \in \mathbb{N}\right\} \leq$$

$$\sup\left\{\sum_{k=1}^{n}|\nu|((x_{k-1},x_k]);\ x_0<x_1<...<x_n=x,(x_k)\in\mathbb{R},n\in\mathbb{N}\right\}=V_{F_{|\nu|}}(x)\leq$$

$$\sup\left\{|\nu((-\infty,x_0])|+\sum_{k=1}^{n}|\nu|((x_{k-1},x_k]);\ x_0<x_1<...<x_n=x,(x_k)\in\mathbb{R},n\in\mathbb{N}\right\}\leq|\nu|((-\infty,x])$$

so $V_{F_\nu}(x)\leq V_{F_{|\nu|}}(x)\leq|\nu|((-\infty,x])\leq||\nu||<\infty$. So $V_{F_\nu}(x)$ is bounded by a constant in x, $||\nu||$, so $\sup(\{V_{F_\nu}(x);x\in\mathbb{R}\})\leq||\nu||$, so $\mathrm{Var}F_\nu|\leq||\nu||$, so $F_\nu\in BV$.

For $x\in[-\infty,\infty)$, pick $x_n\searrow x$, take Jordan decomposition of ν, to get $\nu=\nu_+-\nu_-$, with $\nu_\pm$ positive finite measures, then $\nu_\pm((-\infty,x_n])\to\nu_\pm(\cap_{k\in\mathbb{N}}(-\infty,x_k])=\nu_\pm((-\infty,x])$, by continuity from above, so $\nu_\pm((-\infty,x_n])\to\nu((-\infty,x_n])$. Thus $\lim_n F_\nu(x_n)=F_\nu(x)$, so $F_\nu(-\infty)=0$, and F_ν is right continuous by definition.

(Folland 3.28) If $F\in BV$ then $\lim_{x\to-\infty}V_F(x)=0$ and $F\in BV\Rightarrow V_F\in NBV$.
Proof: ADD

Properties of BV,

1) If $F,G:\mathbb{R}\to\mathbb{R}$, $c\in\mathbb{R}$, then $V_{F+G}(x)\leq V_F(x)+V_F(x)$ and $V_{cF}(x)=|c|V_F(x)$. Hence BV is a vector space and if $F,G\in BV$, then $\mathrm{Var}(F+G)\leq\mathrm{Var}(F)+\mathrm{Var}(G)$ and $\mathrm{Var}(cF)=|c|\mathrm{Var}(F)$. NBV is a subspace of BV.
2) If $F\in BV$, then $V_F(x)$ is an increasing function of x, bounded above by $\mathrm{Var}(F)$.
3) a) Moreover, if $x<y$, then $V_F(y)-V_F(x)=\sup\left(\{\sum_{k=1}^{n}|F(x_k)-F(x_{k-1})|;x\leq x_0<x_1<...<x_n=y\}\right)$.
   b) special case: $|F(y)-F(x)|\leq V_F(y)-V_F(x)\leq V_F(y)\leq\mathrm{Var}(F)$.
   c) consequence: $F\in BV\Rightarrow F$ is bounded.
4) An increasing $F:\mathbb{R}\to\mathbb{R}$ is in BV iff F is bounded.
5) $F:\mathbb{R}\to\mathbb{R}\in BV$ iff $F=F_1-F_2$, where $F_1,F_2:\mathbb{R}\to\mathbb{R}$ are bdd and increasing.
6) $F:\mathbb{R}\to\mathbb{C}\in BV$ iff $\mathrm{Re}\,F,\mathrm{Im}\,F\in BV$.
7) $F\in BV\Rightarrow F$ continuous except at countable many points, and for all $x\in\mathbb{R}$, $F(x+)=\lim_{t\to x^+}F(t)$ and $F(x-)=\lim_{t\to x^-}F(t)$, and $\lim_{x\to+\infty}F(x)$ and $\lim_{x\to-\infty}F(x)$ all exist and are in $\mathbb{R}$.
8) $F\in BV\Leftrightarrow F=F_1-F_2+iF_3-iF_4$, where $F_k:\mathbb{R}\to\mathbb{R}$, increasing, bounded, right continuous, and $\lim_{x\to-\infty}F_k(x)=0$ for all k.

Proof: ADD

The linear map $T=\nu\mapsto F_\nu$ from $\mathcal{M}(\mathbb{R})$ to NBV is an isomorphism. Thus it is bijective and $\mathrm{Var}(F_\nu)=||\nu||$ for all $\nu\in\mathcal{M}(\mathbb{R})$, which implies that NBV is a Banach space with norm $||F||=\mathrm{Var}(F)$, $||T(\nu)||=||\nu||$. This also applies to $\mathcal{M}([a,b])$ and $NBV([a,b]()$, by zero padding $F(x)$ and replacing ν by $\nu_{[a,b]}(E):=\nu(E\cap[a,b])$.

Proof: ADD

We say $F : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous*, or say $F \in AC$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that $a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n}|F(b_k) - F(a_k)| < \epsilon$. If $n = 1$ then this is uniformly continuous, so $F \in AC \Rightarrow F$ is uniformly continuous, also $AC \subset BV$. Define $NAC := NBV \cap AC$. Again this can apply to $F : [a, b] \to \mathbb{R}$.

Proof: ADD

$F \in NAC \Leftrightarrow \mu_F \ll \lambda$, where μ_F is the Lebesgue-Stieltjes measure from F, and λ is the Lebesgue measure.

Proof: ADD

(a Vitali covering lemma) Suppose $W \subset \mathbb{R}^k$, $W \subset \cup_{i=1}^{n} B(x_i, r_i)$, where $B(x, r)$ is the ball centered at $x \in \mathbb{R}^k$, with readius $r > 0$, then there exists $S \subset \{1, 2, ..., n\}$ such that:

a) $B(x_i, r_i) \cap B(x_j, r_j) = \phi$ if $i, j \in S$, $i \neq j$.

b) $W \subset \cup_{i \in S} B(x_i, 3r_i)$

c) $\lambda(w) \leq 3^k \sum_{i \in S} \lambda(B(x_i, r_i))$

Proof: ADD

For $\mu \in \mathcal{M}(\mathbb{R}^k)$, $x \in \mathbb{R}^k$, $r > 0$, define $(Q_r\mu)(x) = \frac{\mu(B(x,r)}{\lambda(B(x,r))}$. Call $M_\mu(x) := \sup\{(Q_r|\mu|)(x); 0 < r < \infty\}$, the *maximal function* of μ, $M_\mu : \mathbb{R}^k \to [0, \infty]$. A special case, for $F \in L^1(\mathbb{R}^k, \lambda)$, $\mu(E) := \int_E F \, d\lambda$, in this case write M_F for M_μ.

$F : \Omega \to [-\infty, \infty]$ is called *lower semi continuous* (lsc) if $F^{-1}((t, \infty])$ is open for all $t \in \mathbb{R}$, this makes sense if Ω is any topological space.

$\mu \in \mathcal{M}(\mathbb{R}^k) \Rightarrow M_\mu$ is lower semi continuous.
Proof: ADD

(Hardy Littlewood theorem) If $\mu \in \mathcal{M}(\mathbb{R}^k)$, $a < t < \infty$ then $\lambda(\{x \in \mathbb{R}^k; M_\mu(x) > t\}) \leq 3^k||\mu|| \div t$.
Proof: ADD

A function $f : \mathbb{R}^k \to \mathbb{F}$ is called *locally integralbe*, or $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$ if $F|_K \in L^1(K, \lambda)$ for all compact $K \subset \mathbb{R}^k$.

If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, $x \in \mathbb{R}^k$ is called a *lebesque point* for f if

$$\lim_{r \to 0} \frac{1}{\lambda(B(x, r))} \int_{B(x,r)} |f(y) - f(x)| \, d\lambda(y) = 0.$$

If x is a Lebesgue point for f then

$$f(x) = \lim_{r \to 0} \frac{1}{\lambda(B(x, r))} \int_{B(x,r)} f(y) \, d\lambda(y).$$

Proof: ADD

For $\mu \in \mathcal{M}(\mathbb{R}^k)$, define the *symmetric derivative* of μ as $D_\mu(x) = \lim_{r\to 0} \frac{\mu(B(x,r)}{\lambda(B(x,r))}$, wherever this limit exists, $x \in \mathbb{R}^k$.

Define

$$f^*(x) = \limsup_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\lambda(y).$$

then

1) $(f+g)^* \leq \int f^* + g^*$ for all $f, g \in L^1(\mathbb{R}^k, \lambda)$

2) If g is continuous at x then $g^*(x) = 0$.

3) If $f, g \in L^1(\mathbb{R}^k, \lambda)$, g continuous then $f^* = (f^* - g + g) \leq (f-g)^* + g^* = (f-g)^*$

4) If $f \in L^1(\mathbb{R}^k, \lambda)$ then $f^* \leq |f| + M_f$.

Proof: ADD

(Lebesue's theorem)

a) If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, then a.e. $x \in \mathbb{R}^k$ is a Lebesgue point.

b) $\mu \in \mathcal{M}(\mathbb{R}^k)$, $\mu \ll \lambda \Rightarrow D_\mu = \frac{d\mu}{d\lambda}$, λ-a.e.

Proof: ADD

Corollary: If $[f] \in L^1_{loc.}(\mathbb{R}^k, \lambda)$ then for any $g \in [f]$, $f(x) = g(x)$ for all Lebesgue points x for f. Thus, for all $[f] \in L^1(\mathbb{R}^k, \lambda)$, there is a cononical $\hat{f} \in [f]$ such that all points in $\mathbb{R}^k$ are lebesgue points for $\hat{f}$.

Proof: ADD

For $x \in \mathbb{R}^k$, a sequence $(E_k)_{k=1}^\infty$ of measurable sets in $\mathbb{R}^k$ is said to *shrink nicely* to x if there exists a $C > 0$, scalars $r_k \searrow 0$ such that $E_k \subset B(x, r_k)$ and $\lambda(B(x,r_k)) \leq C\lambda(B(x,r_k))$ for all $k \in \mathbb{N}$. In this case we write $E_k \xrightarrow{s.n.} x$.

If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, and x is a lebesgue point of f then

$$f(x) = \lim_{r\to 0} \frac{1}{\lambda(E_k)} \int_{E_k} f(y) \, d\lambda(y).$$

Proof: ADD

(first fundamental theorem of calculus)

If $[g] \in L^1([a,b],\lambda)$ resp. $[g] \in L^1(\mathbb{R},\lambda)$, let $G(x) = \int_a^x g(t)\,dt = \int_{[a,b]} f\,d\lambda$ resp. $G(X) = \int_{-\infty}^x g(t)\,dt = \int_{(-\infty,x)} g\,d\lambda$, then $G \in NAC([a,b])$ resp. $F \in NAC$, and G is differentialbe a.e. and $G' = g$ a.e.

(second fundamental theorem of calculus, version 1)

a) $F \in AC \Leftrightarrow ($ F is diff'able a.e. on $[a,b]$ and $F' \in L^1([a,b],\lambda)$ and $F(x) - F(a) = \int_z^x F'(t)\,dt$ for all $x \in [a,b]$ $)$.

b) $F \in AC \Leftrightarrow ($ F is diff'able a.e. on $\mathbb{R}$ and $F' \in L^1(\mathbb{R},\lambda)$ and $F(x) - F(a) = \int_z^x F'(t)\,dt$ for all $x \in \mathbb{R}$ $)$.

c) $F \in NAC$ or $F \in NAC([a,b])$ then $F' = \frac{d\mu_F}{d\lambda}$.

Proof: ADD

If $\mu \in \mathcal{M}(\mathbb{R}^k)$ then

a) $D_\mu(x)E$ exists for a.e. $x \in \mathbb{R}^k$ and $D_\mu = \frac{d\mu_a}{d\lambda}$ λ-a.e., where μ_a is the absolutely continuous part in the LDT of μ.

b) If $\mu \perp \lambda$ Then $D_\mu(x) = 0$ λ-a.e. and for λ-a.e. x, $\lim_{k\to 0} \mu(E_k)/\lambda(E_k) = 0$ if $E_k \overset{s.n.}{\to} x$.

c) For λ-a.e. x, $\lim_{k\to 0} \mu(E_k)/\lambda(E_k) = 0$ if $E_k \overset{s.n.}{\to} x$. ??? ADD.

Proof: ADD

Corollary: If $F \in NBV$ then $F' = \frac{d\mu_a}{d\lambda} = D_\mu$ λ-a.e. where $\mu \in \mathcal{M}(\mathbb{R})$ as $F(x) = \mu((-\infty, x])$. So a bigger class of functions is differentiable with this forumla.
Proof: ADD

If $\mu \in \mathcal{M}(\mathbb{R}^k)$ then

a) If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then F is differentiable λ-a.e.

b) if $F \in BV$, then F is differentiable λ-a.e.

c) if $F \in BV$ there exists a constant c, and $G \in NBV$ such that $F = C + G$ everywhere except at a countable number of points. May take $C = \lim_{x\to-\infty} F(x)$ and $G(x) = \lim_{y\to x^+} F(y) - C$ for all x. Then $F' = G' = D_\mu = \frac{d\mu_a}{d\lambda}$ a.e. where μ is the measure on $\mathbb{R}$ associated to G.

Proof: ADD

If $H \in BV$, $H \geq 0$ for all x, $H = 0$ except on a countable set, then H is differentiable a.e. and $H' = 0$ a.e.
Proof: ADD

Remark: A function $H : \mathbb{R} \to \mathbb{R}$ such that $H' = 0$ a.e. is called a *singular function*. Note: take any $\mu \in \mathcal{M}(\mathbb{R})$, $\mu \perp \lambda$, then defining $F_\mu(x) = \mu((-\infty, x])$, as usual, then $F_\mu \in NBV$, and $F' = \frac{d\mu_a}{d\lambda} = 0$ a.e. so F_μ is singular. Conversely, If $H \in NBV$ is singular, ADD.

$F \in BV[a,b]$

5,1) $F \in AC[a,b]$, take $F(x) = 0$ for $x \notin [a,b]$, then $\lim_{x\to-\infty} F(x) = 0$, and F is continuous because it's uniformly continuous, and $F \in BV[a,b]$, so $AC[a,b] \subset NBV$, and so $\mu_F \ll \lambda$ by theorem 5.1.6. $\mu_F \ll \lambda \Leftrightarrow |\mu_F| \ll \lambda$ by prop 4.1.5.e. F is not nescesarily in $AC[a,b]$ because $F(a)$ is not nescesarily 0.

Now that $|\mu_F| = \mu_{V_F}$. let $\alpha = \nu \mapsto |\nu|, \beta = F \mapsto V_F, \gamma = F \mapsto \mu_F$, and γ is invertible by 5.1.4. Want to show that $\alpha \circ \gamma = \gamma \circ \beta$, but then $\gamma^{-1} \circ (\alpha \circ \gamma) = \beta$, which says that $|\mu_F|((-\infty, x]) = V_F(x)$ for all x, which is true by last lines of proof of 5.1.4.

So $|\mu_F| = \mu_{V_F} \ll \lambda$. $F \in NBV \Rightarrow V_F \in NBV$ by lemma before 8 in 5.1.3, then again theorem 5.1.6 says that $\mu_{V_F} \ll \lambda \Rightarrow V_F \in NAC \Rightarrow V_F \in AC$, and so $V_F \in AC[a,b]$. $\square$

5,2) Suppose $F \in NBV[a,b]$, $F \geq 0$ and F is increasing, $F(a) = 0$. Let μ be the measure associated to F, then by corollary 5.2.5, $F' = \frac{d\mu_{F,a}}{d\lambda}$, where $\mu_F = \mu_{F,a} + \mu_{F,s}$, form the Lebesgue decomposition theorem, $\mu_{F,a} \ll \lambda$. By 5.1.4, $|\frac{d\mu_{F,a}}{d\lambda}| = \frac{d|\mu_{F,a}|}{d\lambda}$, so that $\int_{[a,b]} |F'|\, d\lambda = |\mu_{F,a}|((a,b])$. Now $F \geq 0 \Rightarrow \mu_F \geq= 0 \Rightarrow \mu_{F,a}, \mu_{F,s} \geq 0$, by the lebesgue decomposition theorem, so $\mu_{F,a} \leq \mu_F \Rightarrow |\mu_{F,a}|(a,b) \leq |\mu_F|(a,b) = V_F(a,b)$.

If $F \in BV[a,b]$, then by 5.2.6, $F(x) = F(a) + G(x)$, a.e. where $G \in NBV[a,b]$, and $F' = G'$ a.e. so that $\int_{[a,b]} |F'|\, d\lambda = \int_{[a,b]} |G'|\, d\lambda$.

5,3) In 5.2, have $\int_{[a,b]} |F'|\, d\lambda = |\mu_{F,a}|((a,b])$, and $\mu_F = \mu_{F,a} + \mu_{F,s}$, but $\mu_{F,s} = 0 \Leftrightarrow \mu_F \ll \lambda \Leftrightarrow F \in NAC[a,b]$.

5,4) If $F \in BV[a,b]$, then by 5.2.6, $F(x) = F(a) + G(x)$, a.e. where $G \in NBV[a,b]$, and $F' = G'$ a.e, so assume that $F \in NBV[a,b]$. In this case there is a $\mu \in \mathcal{M}(\mathbb{R})$ such that $\mu((-\infty, x]) = F(x)$ for all x. Apply lebesgue decomposition to obtain $\mu = \mu_a + \mu_s$, 5.2.6 says that $F' = \frac{d\mu_a}{d\lambda} = D_\mu$ a.e. So $|F'| = |D_\mu| = |\frac{d\mu_a}{d\lambda}| = \frac{d|\mu_a|}{d\lambda}$ by prop 1 in 4.1.7. Then again,

$$|F'| = \frac{d|\mu_a|}{d\lambda} = D_{|\mu|} = \lim_{r\to 0} \frac{|\mu_a|(B(x,r))}{\lambda(B(x,r))} = \lim_{r\to 0} \frac{V_F(x-r, x+r)}{2r} = V_F'(x) \quad \lambda - \text{a.e.}$$

$\square$

Folland 33) $F : \mathbb{R} \to \mathbb{R}$, By 5.2.6a F is differentiable a.e.. Given $a < b \in \mathbb{R}$, replace F with $F\mathbf{1}_{[a,b]}$. By problem 4, F' is measurable. If $F(x) < \infty$ then $F(y) < \infty$ for all $y \in [a,x]$ by F increasing; if $F(x) = \infty$ some $x \in [a,b]$ then $F(b) - F(a) = \infty >= \int_{[a,b]} F'\, d\lambda \in [-\infty, \infty]$ so the equality holds. assume $F(x) \in \mathbb{R}$ for all $x \in [a,b]$. Then F is bounded by $F(b)$, so by 5.1.3b $F \in BV[a,b]$. By 5.2.6c $F(x) = F(a) + G(x)$ a.e. where $G \in NBV[a,b]$, and $G' = F'$ a.e., also then by problem 5,2) $\int_{[a,b]} |F'| \leq V_F(a,b) < \infty$, so $G', F' \in L^1([a,b], \lambda)$. Let μ be the measure asssociated with F and G. This is the same measure for both F, G, because it is defined as the inf of sums of the numbers of the form $F(b_k) - F(a_k)$, but $F(b_k) - F(a_k) = G(b_k) - G(a_k)$. Now $G(a) = 0$ and G is increasing, so $G(x) \geq 0$ for $x \in [a,b]$, and $G \in NBV[a,b]$, so by the arguement in problem 5,2 , $\mu_a \leq \mu$ and $F' = \frac{d\mu_a}{d\lambda}$ a.e., so $\int_{[a,b]} F'\, d\lambda = \mu_a([a,b])$. On the other hand, $F(b) - F(a) = \mu((a,b])$, and $\mu \geq \mu_a$ from the lebesgue decomposition, i.e. $\mu = \mu_a + \mu_s$ and all these are positive finite measures, because $G \geq 0$. This gives $F(b) - F(a) = \mu((a,b]) \geq \mu_a([a,b]) = \int_{[a,b]} F'\, d\lambda$. a, b are arbitrary so this holds for all $a, b \in \mathbb{R}$. $\square$