Notes on Integration and differentiation on $\mathbb{R}^n$, etc.
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A measure defined on a Borel σ-algebra is called a Borel measure. Write $\mathcal{M}(\mathbb{R}^n)$ for the Borel measures on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$, then $\mathcal{M}(E)$ are the borel measures on E, but $\mathcal{M}(E) \subset \mathcal{M}(\mathbb{R}^n)$, by zero padding.

Definition: a positive finite measure μ on $\mathbb{R}^n$ is regular if

1. $\mu(E) = \inf\{\mu(U); U \text{ open}, U \supset E\}$
2. $\mu(E) = \sup\{\mu(K); K \text{ compact}, K \subset E\}$

Every positive finite measure on $\mathbb{R}^n$ is regular. For every measure $\nu \in \mathcal{M}(\mathbb{R}^n)$, for all $E \in \mathcal{B}\ell(\mathbb{R}^n)$, there exists a sequence of open sets $U_k \supset E$, and compact sets $K_n \subset E$ such that $\nu(U_n) \to \nu(E)$ and $\nu(K_n) \to \nu(E)$.

Proof: ADD

Definition: we say a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular if each positive ν_k, in the Jordan decomposition $\nu = \sum_{k=0}^{3} i^k \nu_k$, is regular. By the last result, every $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular and then the condition about sequences of sets holds.

If $\nu \in \mathcal{M}(\mathbb{R})$, we define its distribution function by $F_\nu(x) = \nu((\infty, x])$. $\nu \mapsto F_\nu$ is injective and linear on $\mathcal{M}(\mathbb{R})$.

Proof: ADD

Def: $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation, BV, or say $F \in BV$, if $\mathrm{Var}(F) < \infty$, where $\mathrm{Var}(F) := \sup\{V_F(x); x \in \mathbb{R}\}$, and

$$V_F(x) := \sup\left\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|;\ x_0 < x_1 < \dots < x_n = x, (x_k) \in \mathbb{R}, n \in \mathbb{N}\right\}.$$

Def: $F : \mathbb{R} \to \mathbb{R}$ is in NBV if $F \in BV$, F is right continuous at all $x \in \mathbb{R}$, and $\lim_{x\to-\infty} F(x) = 0$; normalized BV.

If $\nu \in \mathcal{M}(\mathbb{R}, \mathcal{B}\ell(\mathbb{R}))$, then $F_\nu \in NBV$.

Proof: ADD

(Folland 3.28) If $F \in BV$ then $\lim_{x\to-\infty} V_F(x) = 0$ and $F \in BV \Rightarrow V_F \in NBV$.
Proof: ADD

Properties of BV,

1) If $F, G : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, then $V_{F+G}(x) \le V_F(x) + V_F(x)$ and $V_{cF}(x) = |c|V_F(x)$. Hence BV is a vector space and if $F, G \in BV$, then $\mathrm{Var}(F + G) \le \mathrm{Var}(F) + \mathrm{Var}(G)$ and $\mathrm{Var}(cF) = |c|\mathrm{Var}(F)$. NBV is a subspace of BV.

2) If $F \in BV$, then $V_F(x)$ is an increasing function of x, bounded above by $\mathrm{Var}(F)$.

3) a) Moreover, if $x < y$, then $V_F(y) - V_F(x) = \sup\left(\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|; x \le x_0 < x_1 < ... < x_n = y\}\right)$.
   b) special capse: $F(y) - F(x) \le V_F(y) - V_F(x) \le V_F(y) \le \mathrm{Var}(F)$.
   c) consequence: $F \in BV \Rightarrow F$ is bounded.

4) An increasing $F : \mathbb{R} \to \mathbb{R}$ is in BV iff F is bounded.

5) $F : \mathbb{R} \to \mathbb{R} \in BV$ iff $F = F_1 - F_2$, where $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are bdd and increasing.

6) $F : \mathbb{R} \to 0\mathbb{C} \in BV$ iff $\mathrm{Re}\, F, \mathrm{Im}\, F \in BV$.

7) $F \in BV \Rightarrow F$ continuous except at countable many points, and for all $x \in \mathbb{R}$, $F(x+) = \lim_{t \to x^+} F(t)$ and $F(x-) = \lim_{t \to x^-} F(t)$, and $\lim_{x \to +\infty} F(x)$ and $\lim_{x \to -\infty} F(x)$ all exist and are in $\mathbb{R}$.

8) $F \in BV \Leftrightarrow F = F_1 - F_2 + iF_3 - iF_4$, where $F_k : \mathbb{R} \to \mathbb{R}$, increasing, bounded, right continuous, and $\lim_{x \to -\infty} F_k(x) = 0$ for all k.

Proof: ADD

The linear map $T = \nu \mapsto F_\nu$ from $\mathcal{M}(\mathbb{R})$ to NBV is an isomorphism. Thus it is bijective and $\mathrm{Var}(F_\nu) = ||\nu||$ for all $\nu \in \mathcal{M}(\mathbb{R})$, which implies that NBV is a Banach space with norm $||F|| = \mathrm{Var}(F)$, $||T(\nu)|| = ||\nu||$. This also applies to $\mathcal{M}([a,b])$ and $NBV([a,b]()$, by zero padding $F(x)$ and replacing ν by $\nu_{[a,b]}(E) := \nu(E \cap [a,b])$.

Proof: ADD

We say $F : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous*, or say $F \in AC$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that $a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$. If $n = 1$ then this is uniformly continuous, so $F \in AC \Rightarrow F$ is uniformly continuous, also $AC \subset BV$. Define $NAC := NBV \cap AC$. Again this can apply to $F : [a,b] \to \mathbb{R}$.

Proof: ADD

$F \in NAC \Leftrightarrow \mu_F \ll \lambda$, where μ_F is the Lebesgue-Stieltjes measure from F, and λ is the Lebesgue measure.

Proof: ADD

(a Vitali covering lemma) Suppose $W \subset \mathbb{R}^k$, $W \subset \cup_{i=1}^{n} B(x_i, r_i)$, where $B(x,r)$ is the ball centered at $x \in \mathbb{R}^k$, with readius $r > 0$, then there exists $S \subset \{1, 2, ..., n\}$ such that:

a) $B(x_i, r_i) \cap B(x_j, r_j) = \phi$ if $i, j \in S$, $i \neq j$.

b) $W \subset \cup_{i \in S} B(x_i, 3r_i)$

c) $\lambda(w) \le 3^k \sum_{i \in S} \lambda(B(x_i, r_i))$

Proof: ADD

For $\mu \in \mathcal{M}(\mathbb{R}^k)$, $x \in \mathbb{R}^k$, $r > 0$, define $(Q_r\mu)(x) = \frac{\mu(B(x,r)}{\lambda(B(x,r))}$. Call $M_\mu(x) := \sup\{(Q_r|\mu|)(x); 0 < r < \infty\}$, the *maximal function* of μ, $M_\mu : \mathbb{R}^k \to [0, \infty]$. A special case, for $F \in L^1(\mathbb{R}^k, \lambda)$, $\mu(E) := \int_E F\, d\lambda$, in this case write M_F for M_μ.

$F : \Omega \to [-\infty, \infty]$ is called *lower semi continuous* (lsc) if $F^{-1}((t, \infty])$ is open for all $t \in \mathbb{R}$, this makes sense if Ω is any topological space.

$\mu \in \mathcal{M}(\mathbb{R}^k) \Rightarrow M_\mu$ is lower semi continuous.
Proof: ADD

(Hardy Luttlewood theorem) If $\mu \in \mathcal{M}(\mathbb{R}^k)$, $a < t < \infty$ then $\lambda(\{x \in \mathbb{R}^k; M_\mu(x) > t\}) \leq 3^k ||\mu|| \div t$.
Proof: ADD

A function $f : \mathbb{R}^k \to \mathbb{F}$ is called *locally integralbe*, or $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$ if $F|_K \in L^1(K, \lambda)$ for all compact $K \subset \mathbb{R}^k$.

If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, $x \in \mathbb{R}^k$ is called a *lebesque point* for f if

$$\lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\lambda(y) = 0.$$

If x is a Lebesgue point for f then

$$f(x) = \lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) \, d\lambda(y).$$

Proof: ADD

For $\mu \in \mathcal{M}(\mathbb{R}^k)$, define the *symmetric derivative* of μ as $D_\mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r)}{\lambda(B(x,r))}$, wherever this limit exists, $x \in \mathbb{R}^k$.

Define

$$f^*(x) = \limsup_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\lambda(y).$$

then

1) $(f+g)^* \leq \int f^* + g^*$ for all $f, g \in L^1(\mathbb{R}^k, \lambda)$

2) If g is continuous at x then $g^*(x) = 0$.

3) If $f, g \in L^1(\mathbb{R}^k, \lambda)$, g continuous then $f^* = (f^* - g + g) \leq (f-g)^* + g^* = (f-g)^*$

4) If $f \in L^1(\mathbb{R}^k, \lambda)$ then $f^* \leq |f| + M_f$.

Proof: ADD

(Lebesue's theorem)

a) If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, then a.e. $x \in \mathbb{R}^k$ is a Lebesgue point.

b) $\mu \in \mathcal{M}(\mathbb{R}^k)$, $\mu \ll \lambda \Rightarrow D_\mu = \frac{d\mu}{d\lambda}$, λ-a.e.

Proof: ADD

Corollary: If $[f] \in L^1_{loc.}(\mathbb{R}^k, \lambda)$ then for any $g \in [f]$, $f(x) = g(x)$ for all Lebesgue points x for f. Thus, for all $[f] \in L^1(\mathbb{R}^k, \lambda)$, there is a cononical $\hat{f} \in [f]$ such that all points in $\mathbb{R}^k$ are lebesgue points for $\hat{f}$.

Proof: ADD

For $x \in \mathbb{R}^k$, a sequence $(E_k)_{k=1}^{\infty}$ of measurable sets in $\mathbb{R}^k$ is said to *shrink nicely* to x if there exists a $C > 0$, scalars $r_k \searrow 0$ such that $E_k \subset B(x, r_k)$ and $\lambda(B(x, r_k)) \leq C\lambda(B(x, r_k))$ for all $k \in \mathbb{N}$. In this case we write $E_k \overset{s.n.}{\to} x$.

If $f \in L^1_{loc.}(\mathbb{R}^k, \lambda)$, and x is a lebesgue point of f then

$$f(x) = \lim_{r \to 0} \frac{1}{\lambda(E_k)} \int_{E_k} f(y)\, d\lambda(y).$$

Proof: ADD

(first fundamental theorem of calculus)

If $[g] \in L^1([a,b],\lambda)$ resp. $[g] \in L^1(\mathbb{R},\lambda)$, let $G(x) = \int_a^x g(t)\,dt = \int_{[a,b]} f\,d\lambda$ resp. $G(X) = \int_{-\infty}^x g(t)\,dt = \int_{(-\infty,x)} g\,d\lambda$, then $G \in NAC([a,b])$ resp. $F \in NAC$, and G is differentialbe a.e. and $G' = g$ a.e.

(second fundamental theorem of calculus, version 1)

a) $F \in AC \Leftrightarrow ($ F is diff'able a.e. on $[a,b]$ and $F' \in L^1([a,b],\lambda)$ and $F(x) - F(a) = \int_z^x F'(t)\,dt$ for all $x \in [a,b]$ $)$.

b) $F \in AC \Leftrightarrow ($ F is diff'able a.e. on $\mathbb{R}$ and $F' \in L^1\mathbb{R},\lambda$ and $F(x) - F(a) = \int_z^x F'(t)\,dt$ for all $x \in \mathbb{R}$ $)$.

c) $F \in NAC$ or $F \in NAC([a,b])$ then $F' = \frac{d\mu_F}{d\lambda}$.

Proof: ADD

If $\mu \in \mathcal{M}(\mathbb{R}^k)$ then

a) $D_\mu(x)E$ exists for a.e. $x \in \mathbb{R}^k$ and $D_\mu = \frac{d\mu_a}{d\lambda}$ λ-a.e., where μ_a is the absolutely continuous part in the LDT of μ.

b) If $\mu \perp \lambda$ Then $D_\mu(x) = 0$ λ-a.e. and for λ-a.e. x, $\lim_{k\to 0} \mu(E_k)/\lambda(E_k) = 0$ if $E_k \overset{s.n.}{\to} x$.

c) For λ-a.e. x, $\lim_{k\to 0} \mu(E_k)/\lambda(E_k) = 0$ if $E_k \overset{s.n.}{\to} x$. ??? ADD.

Proof: ADD

Corollary: If $F \in NBV$ then $F' = \frac{d\mu_a}{d\lambda} = D_\mu$ λ-a.e. where $\mu \in \mathcal{M}(\mathbb{R})$ as $F(x) = \mu((-\infty, x])$. So a bigger class of functions is differentiable with this forumla.
Proof: ADD

If $\mu \in \mathcal{M}(\mathbb{R}^k)$ then

a) If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then F is differentiable λ-a.e.

b) if $F \in BV$, then F is differentiable λ-a.e.

c) if $F \in BV$ there exists a constant c, and $G \in NBV$ such that $F = C + G$ everywhere except at a countable number of points. May take $C = \lim_{x\to-\infty} F(x)$ and $G(x) = \lim_{y\to x^+} F(y) - C$ for all x. Then $F' = G' = D_\mu = \frac{d\mu_a}{d\lambda}$ a.e. where μ is the measure on $\mathbb{R}$ associated to G.

Proof: ADD

If $H \in BV$, $H \geq 0$ for all x, $H = 0$ except on a countable set, then H is differentiable a.e. and $H' = 0$ a.e.
Proof: ADD

Remark: A function $H : \mathbb{R} \to \mathbb{R}$ such that $H' = 0$ a.e. is called a *singular function*. Note: take any $\mu \in \mathcal{M}(\mathbb{R})$, $\mu \perp \lambda$, then defining $F_\mu(x) = \mu((-\infty, x])$, as usual, then $F_\mu \in NBV$, and $F' = \frac{d\mu_a}{d\lambda} = 0$ a.e. so F_μ is singular. Conversely, If $H \in NBV$ is singular, ADD.