Solutions, Exam 3

April 11, 2009

Q1. Show that every sequentially compact metric space (X, d) is complete. Is the converse true? (Prove or give a counterexample.)

Let (x_n) be a Cauchy sequence in X. Since X is compact, there exists a convergent subsequence (x_{n_k}) of (x_n). Set $\lim_{k\to\infty} x_{n_k} = x^\star$. We claim $\lim_{n\to\infty} x_n = x^\star$. We have

$$d(x_n, x^\star) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x^\star).$$

Since (x_n) is Cauchy, given $\varepsilon > 0$, there exists N such that if $n, n_k \geq N$ then $d(x_n, x_{n_k}) < \varepsilon$. Hence if $n, n_k \geq N$, $d(x_n, x^\star) \leq \varepsilon + d(x_{n_k}, x^\star)$. Letting $n_k \to \infty$, we see that $d(x_n, x^\star) \leq \varepsilon$ for all $n \geq N$. Hence $\lim_{n\to\infty} x_n = x^\star$.
(Note: acceptable to say that if a Cauchy sequence has a convergent subsequence, then the Cauchy sequence is convergent — coursework.)

False. $(\mathbb{R}, |\,|)$ is complete but not compact.

Q2. Let E, F, G be connected subsets of the metric space X. (1) Show that if $E \cap F \neq \emptyset$ and $F \cap G \neq \emptyset$, then $E \cup F \cup G$ is connected. (2) If $E \cap F \neq \emptyset$, must $E \cap F$ be connected? Would your answer change if $X = \mathbb{R}$? (For (1), work from the definition of connected given on the handout. For the second part of (2) you may cite any needed results about connected subsets of $\mathbb{R}$.)

(1) If E, F are connected and $E \cap F \neq \emptyset$ then $E \cup F$ is connected. Well, let U, V be open subset of X such that $E \cup F \subset U \cup V$ and $(U \cap (E \cup F)) \cap (V \cap (E \cup F)) = \emptyset$. It suffices to prove that either $E \cup F \subset U$ or $E \cup F \subset V$. Certainly $E \subset U \cup V$ and $(E \cap U) \cap (E \cap V) = \emptyset$. Hence, since E is connected, we must have either $E \subset U$ or $E \subset V$. Suppose $E \subset U$. Then, since $E \cap F \neq \emptyset$, we have $U \cap F \neq \emptyset$ and so, since F is connected, $F \subset U$ (by the same argument we used for E). This shows $E \cup F$ is connected. But now $G, E \cup F$ are connected, $(E \cup F) \cap G \neq \emptyset$ and so exactly the same argument shows that $E \cup F \cup G$ is connected.

(2) False. Let $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geq 0\}$, $F = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \leq 0\}$. Both E, F are connected but $E \cap F = \{(-1, 0)\} \cup \{(1, 0)\}$ is disconnected.

Q3. Let E be a connected subset of the metric space X (assume $E \neq X, \emptyset$ and that E contains at least 2 points). Which of the statements below are true, which are false? In each case either prove or provide a counterexample. (a) $\overset{\circ}{E}$ is connected. (b) $\overline{E}$ is connected. (c) If $z \in E'$, then $E \cup \{z\}$ is connected (E' is the set of limit points of E and $\overline{E} = E \cup E'$). (d) If $z \notin E'$, then $E \cup \{z\}$ is disconnected.

(a) False. Take $E = \overline{D}_1(-1,0) \cup \overline{D}_1(1,0)$. Then E is connected (since both $\overline{D}_1(-1,0), \overline{D}_1(1,0)$ are connected and $\overline{D}_1(1,0) \cap \overline{D}_1(-1,0) \neq \emptyset$. However, $\mathring{E} = D_1(-1,0) \cup D_1(1,0)$ is disconnected (take $U = D_1(-1,0)$, $V = D_1(1,0)$).

(b) True. Let U, V be open subsets of X such that $U \cup V \supset \overline{E}$ and $U \cap V \supset \overline{E} = \emptyset$. Observe that $U \cup V \supset E$ and $U \cap V \supset E = \emptyset$. Hence, since E is connected, either $E \subset U$ or $E \subset V$. WLOG suppose $E \subset U$. Then $V \cap E = \emptyset$ and so no point of V can lie in the closure $\overline{E}$ of E (for every $x \in V$, there exists $r > 0$ such that $D_r(x) \cap E = \emptyset$). If $V \cap \overline{E} = \emptyset$, then $\overline{E} \subset U$. Hence $\overline{E}$ is connected.

(c) True. Follow same argument as for (b) to get to $V \cap E = \emptyset$. Then $V \cap E' = \emptyset$ and so $V \cap (E \cup \{z\}) = \emptyset$ and so $U \supset E \cup \{z\}$.

(d) True. If z is an isolated point of E, then $E \cup \{z\} = E$ is disconnected since E contains at least two points, one of which is an isolated point. If $z \notin E' \cup E$, then $z \notin \overline{E}$, since $\overline{E} = E' \cup E$. Take $U = X \setminus \{z\}$ and $V = D_r(z)$, where $r > 0$ is chosen so that $D_r(z) \cap \overline{E} = \emptyset$. Then $U \cup V \subset E \cup \{z\}$, $U \cap V \cap (E \cup \{z\}) = \emptyset$ and $U \cap (E \cup \{z\}), V \cap (E \cup \{z\}) \neq \emptyset$. Therefore, $E \cup \{z\}$ is disconnected.

Q4. Let (X, d) be a complete metric space. Which of the statements below are true, which are false? In each case either prove or provide a counterexample. (a) If $x_0 \in X$, then $X \setminus \{x_0\}$ is never complete (you may assume X contains more than one point). (b) If $f : X \to Y$ is a continuous onto map, then Y is complete. (Hint: e^x.) (c) The space of continuous bounded functions $(B^0(X,Y), \rho)$ is complete for every metric space $(Y, \bar{d})$. (ρ denotes the uniform metric $\rho(f,g) = \sup_{x \in X} \bar{d}(f(x), g(x))$.)

(a) False. Suppose $x_0 \in X$ is an isolated point, then $X \setminus \{x_0\}$ is complete. (Note the result is true if X contains no isolated points.)

(b) False. For example, $Y = (0, \infty)$ and $f : \mathbb{R} \to Y$ defined by $f(x) = e^x$. Observe that Y is not complete (for example, the Cauchy sequence $(1/n) \subset Y$ does not converge).

(c) False. Take $Y = \mathbb{Q}$. Let $(q_n) \subset \mathbb{Q}$ be a Cauchy sequence converging to $\sqrt{2} \notin \mathbb{Q}$. Let $f_n : X \to \mathbb{Q}$ be the constant function taking value q_n. Then (f_n) is a Cauchy sequence in $(B^0(X,Y), \rho)$ which is not convergent (if it did converge, the limit function f would be constant with value $\sqrt{2}$ and so $f \notin B^0(X,Y)$. This is maybe a little easier if you take $X = \mathbb{R}$, then $B^0(\mathbb{R}, \mathbb{Q}) = \mathbb{Q}$ (all continuous functions must be constant by the intermediate value theorem).

Comments: A counter example means just that: an *explicit* example. I put a very high value on examples and a very, very low value on rote memorization of definitions and theorems with no examples... A question that says "working from the definition..." means that. Don't cite coursework. Use the definition on the handout. Most of the problems on connected sets require logical reasoning, working from the definition. Perhaps the easiest approach is to use the definition in the form given on the handout. In the final test there will be one or two questions on metric spaces — at least one on connected sets. I've included the handout with some extra comments.

Definition A
A metric space (X, d) is (sequentially) *compact* if every sequence (x_n) of points of X has a convergent subsequence.

Definition B
A metric space (X, d) is *complete* if every Cauchy sequence (x_n) of points of X converges.

Definition C
A subset E of the metric space X is *connected* if given open subsets U, V of X such that

(1) $U \cup V \supset E$.

(2) $(U \cap E) \cap (V \cap E) = \emptyset$,

then one of the sets $U \cap E, V \cap E$ must be the emptyset.
(If this condition fails, E is *disconnected*.)
Notice that (2) might just as well be written $U \cap V \cap E = \emptyset$. We do not require that $U \cap V$ is empty. If a set E is not connected, then there exist open sets U, V such that $E \subset U \cup V$, $U \cap V \cap E = \emptyset$ and $U \cap E, V \cap E \neq \emptyset$. Of course, the definition is easier if $E = X$. Then X is disconnected iff X can be written as the union of two disjoint non-empty open sets.

Definition D
A point $e \in X$ is a limit point of $E \subset X$ if every open neighborhood of N of e contains points of $E \setminus \{e\}$.
Notice: If $e \in E$ is not a limit point of E, then e is an isolated point of E.

Theorem 1
A subset A of $\mathbb{R}$ is connected iff A is an interval.
Notice: Since the intersection of two intervals is either an interval or empty, it is a consequence of Theorem 1 that if E, F are connected subsets of $\mathbb{R}$, then $E \cap F$ is connected if $E \cap F \neq \emptyset$.

Theorem 2
A subset A of $\mathbb{R}^n$ is compact iff A is closed and bounded.