MATH 4332
Homework 9, due Monday, April 20th

Q1. Let E be a subset of the metric space X. Show that $E' = (\overline{E})'$. Using the result that the closure of a connected set is connected, deduce that if E is connected, then E' is connected. (Hint: Isolated points?)

Q2. Find a contraction map $f : X \to X$, where $X = \mathbb{R} \setminus \{0\}$, which does not have a fixed point.

Q3. Consider the ODE $x' = x$. Fix $x \in \mathbb{R}$. Taking $\phi_0 : [-a, a] \to \mathbb{R}$ to be the constant map $\phi_0(t) = x$, compute the first three iterates ϕ_1, ϕ_2, ϕ_3 of $T\phi(t) = x + \int_0^t f(\phi(s))\, ds$, starting with $\phi = \phi_0$. Compare with the actual solution $x(\sum_{n=0}^{\infty} t^n/n!)$.

Q4. Let (X, d) be a metric space. A map $f : X \to X$ is an *expansion* if there exists $k > 1$ such that $d(f(x), f(y)) \geq kd(x, y)$ for all $x, y \in X$. Show

1. If $f : X \to X$ is an expansion and f has a fixed point, then the fixed point is unique.
2. If X is compact, then there are no expansions of X. (Hint: this is easy.)

Q5. Suppose that the metric space (X, d) is connected. Show that if $f : X \to \mathbb{R}$ is continuous and $a, b \in f(X)$, then f takes every value between a and b. Using this result, show that if X is *countable* then X is connected if and only if X consists of a single point.
(Hint for the second part. The case when X is finite is easy so suppose X is infinite. Let $X = \{x_n \mid n \in \mathbb{N}\}$. For $a > 1$, define $f : X \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} a^{-n} d(x, x_n)/(1 + d(x, x_n))$. Show that f is continuous and that for some sufficiently large $a > 1$, f not constant — unless X consists of a single point. This is a good illustration of a metric space result on connectedness which is definitely false for general topological spaces.)