Topology, Fall 2010

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1.2 Topology

1.2.1 Definition A topology on a set X is a collection $\tau \in \mathcal{P}(X)$ such that:

i τ is closed under arbitrary unions

ii τ is closed under finite intersections

iii $\phi, X \in \tau$.

We say (X, τ) is a topological space. A subset $U \subset X$ is called open iff $U \in \tau$. A subset $C \subset X$ is called closed iff $C^c \in \tau$.

Example

i Discrete toplogy: $\tau = \mathcal{P}(X)$, empty set is clopen (closed and open)

ii Indiscrete topology: $\tau = \{\phi, X\}$

iii $X := \{1\}$ has one topology, $\{\phi, \{1\}\}$ (discrete and indecrete are the same). $X := \{1, 2\}$ has four topologies, discrete, indiscrete, $\{\phi, \{1, 2\}, \{1\}\}$, and $\{\phi, \{1, 2\}, \{2\}\}$.

iv $(\mathbb{R}^n, \tau)$, with $\tau = \{U \subset X;\ \forall x \in U\ \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \subset U\}$. Here $B(x, \epsilon) = \{y \in \mathbb{R}^n; ||x - y|| < \epsilon\}$.

Definition For two topologies, σ, τ on a set X, we say σ is finer than τ if $\sigma \supset \tau$, in this case we also say τ is coarser than σ.

1.2.2 Definition Let (X, τ) be a topological space. A subset $\mathcal{B} \subset \tau$ is called a basis for τ, if for all $x \in X$, for all $\tau \ni U \ni x$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Example

i $\mathcal{B} = \tau$, (not powersert; $\mathcal{B} \subset \tau$)

ii $X = \mathbb{R}^n, \tau$ as in example 4, then $\mathcal{B} = \{B(x, r);\ x \in X, r > 0\}$.

Lemma 1 (X, τ) a topological space, with basis $\mathcal{B}$. $U \in \tau$, the following are equivalent

i $U \in \tau$ (U is open)

ii U is a union of elements in $\mathcal{B}$.

iii For all $x \in U$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Proof. $(2 \Rightarrow 1)$ Since $\mathcal{B} \subset \tau$, and 1 in definition of a topology. $(1 \Rightarrow 3)$ By definition of a basis. $(3 \Rightarrow 2)$ By 3, for all $x \in U$ there exists a set $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U \Rightarrow U = \cup_{x \in U}\{x\} \subset \cup_{x \in U} B_x \subset \cup_{x \in U} U = U$ $\Rightarrow U = \cup_{x \in U}$. $\square$

Notice, that if $\mathcal{B}$ is a basis for a topology τ on a set X, then we have:

(B1) for all $x \in X\ \exists B \in \mathcal{B}$ s.t. $x \in B$
(B2) for all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2\ \exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Proof.

(B1) follows from the definition of a basis, with $U = X$
(B2) follows from the definition of a basis, with $U = B_1 \cap B_2$

Lemma 2 Conversely, if X is any set, $\mathcal{B} \subset \mathcal{P}(X)$, with $\mathcal{B}$ satisfying (B1) and (B2) above, then there exists a unique topology τ such that $\mathcal{B}$ is a basis for τ, uniquely.

Proof. (uniqueness) this follows from lemma 1 (ii), the topology is forced to be the union of sets in $\mathcal{B}$. (existance) Define $\tau = \{U \subset X;\ \forall x \in U\ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$. $\phi \in \tau$ trivially, $X \in \tau$ by (B1). If $U_i \in \tau$, $\forall i \in I$, then $\forall x \in \cup_{i \in I} U_i \Rightarrow \exists j \in I$ s.t. $x \in U_j \overset{\text{def of}}{\underset{\tau}{\Rightarrow}} \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_i \subset \cup_{i \in I} U_i \Rightarrow$ $\cup_{i \in I} U_i \in \tau$, so τ is closed under arbitrary unions. Let $U, V \in \tau, x \in U \cap V$ by def of τ, $\exists B_1, B_2 \in \mathcal{B}$ s.t. $x \in B_1 \subset U, x \in B_2 \subset V$, so $x \in B_1 \cap B_2$, so by (B2) $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U \cap V \Rightarrow U \cap V \in \tau$, so τ is closed under finit intersections. Finally, $\mathcal{B}$ is a (unique ?) basis for τ, by the definition of τ and what a basis is. □

Example

1. Let $\mathcal{B} = \{(a, b); a, b \in \mathbb{R}, a < b\}$
2. Let $\mathcal{B} = \{B(x, \epsilon); x \in \mathbb{R}, \epsilon > 0\}$, check (B2) in a more general setting.
3. Metric spaces.
4. Normed (vector) spaces.

Proposition 1 Every normed space is a metric space with respect to $d = x, y \mapsto ||x - y||$.

Proof. (i) $d(x, y) = 0 \Leftrightarrow ||x - y|| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$ (ii) $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = ||y - x|| = d(y, x)$, (iii) $d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$.

Proposition 2 $\forall y \in B(x, r)\ \exists s > 0$ s.t. $B(y, s) \subset B(x, r)$

Proof. Take $s = r - d(x, y)$, then for all $z \in B(y, s)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = d(x, y) + r - d(x, y) = r$, so $d(x, z) < r \Rightarrow z \in B(x, r) \Rightarrow B(y, s) \subset B(x, r)$.

Proposition 3 $\{B(x, r); x \in X, r > 0\}$ forms a basis for a topology on the metric space (X, d).

Proof. We have to check (B1), (B2). (B1) is trivial, because $x \in B(x, r)$. (B2): if $z \in B(x, \epsilon) \cap B(y, \delta)$ by prop 2 $\exists r > 0$ s.t. $B(z, r) \subset B(x, \epsilon)$, similarly, $\exists s > 0$ s.t. $B(z, s) \subset B(y, \delta)$. Then $z \in B(z, \min(r, s)) \subset B(x, \epsilon) \cap B(y, \delta)$ so (B2) holds, so we are done by lemma 2.

Summary: Every metric space (X, d) has an associated topology, called the *metric topology*, write this as τ_d. It has as basis the set of open balls $\{B(x, r); x \in X, r > 0\}$. Putting this together with prop 1, every normed space, $(X, || \cdot ||)$, has an associated topology, called the *norm topology*, it is τ_d, where $d(x, y) = ||x - y||$.

Definition

A topological space (x, τ) is called *meterizable* if $\tau = \tau_d$, for some metric d on X. Because metric spaces are 'nice' or 'understood', in many cases it is of interest to show that a topology is meterizable; we will mention some tests later.

Proposition 4 $U \in \tau_d \Leftrightarrow \forall x \in U\ \exists r > 0$ s.t. $B(x,r) \subset U$.

Proof. Use the definition of τ in the proof of lemma 2, also use prop 3.

Applying all of this to the usual euclidian metrix in $\mathbb{R}^n$, the τ_d we get is precicely what we call the 'open sets' in undergraduate analysis.

1.2.4 Lemma 1 If $\mathcal{B}_i$ is a basis for a topology τ_i on X, $i \in \mathbb{N}$, $i \in \{1,2\}$, then
$\tau_1 \subset \tau_2 \Leftrightarrow \forall x \in X, \forall B \in \mathcal{B}_1, \exists C \in B_2$ s.t. $x \in C \subset B$.

Proof. ($\Leftarrow$) if $U \in \tau_1, x \in U$, then $\exists B \in \mathcal{B}$ such that $x \in B \in U$. By hypothesis
$\exists C \in \mathcal{B}_2$ s.t. $x \in C \subset B \subset U$. So $U \in \tau_2$, by lemma 1.

skipped some things here; left off at page 6, picking up at page 9.

1.2.6 Definition A *neighborhood* of a point x in a topological space (X,τ) is an open set $U \in \tau$ containing x (some authors sefine a neighborhood as any set containing x). The *neighborhood basis* of x, written $\mathcal{O}(x)$ is $\{U \in \tau; x \in U\}$.

1.2.7 Definition We say $A \subset X$ is *closed* if A^c is open.

Proposition

1. Arbitrary intersections of closed sets are closed.
2. Finite unoins of closed sets are closed.
3. ϕ, X are closed.

Proof. By De Morgan's laws

Example In a metric space, (X,d), $\overline{B}(x,\epsilon) = \{y \in X; d(x,y) \leq \epsilon\}$ is closed in the metric topology τ_d.

1.2.8 Definition The *closure*, $\overline{A}$, of A, is the intersection of all closed subsets of X, containing A. Note, $\overline{A}$ is the smallest closed subset of X containing A. A closed $\Leftrightarrow A = \overline{A}$.

Proposition 1

1. $x \in \overline{A} \Leftrightarrow U \cap A = \phi\ \forall x \in U \in \tau$.
2. If τ has a basis $\mathcal{B}$ then $x \in \overline{a} \Leftrightarrow U \cap A \neq \phi, \forall x \in U \in \mathcal{B}$.
3. In a metric space (X,d), $x \in \overline{A} \Leftrightarrow B(x,r) \cap A \neq \phi\ \forall r > 0$.

Proof. to do

Definition The *interior*, A^o or $\mathrm{int}(A)$ is the union of all upen sets contained inside A, which also equals the biggest open set inside A. Clearly A open $\Leftrightarrow A = A^o$.

Proposition 2

1. $x \in A^o \Leftrightarrow\ \exists U \in \tau$ s.t. $x \in U \subset A$
2. If τ has a basis $\mathcal{B}$ then $x \in A^o \Leftrightarrow B \in \mathcal{B}$ s.t. $x \in B \subset A$
3. In a metric space, $x \in A^o \Leftrightarrow\ \exists \epsilon > 0$ s.t. $B(x,\epsilon) \subset A$

Proof. Ex.

Proposition 3 For all $A, B \subset X$

1. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$, and $A^o \subset B^o$.
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}, (A \cup B)^o \supset A^o \cup B^o$
3. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and $(A \cap B)^o = A^o \cap B^o$.
4. $(\overline{A})^c = (A^c)^o$ and $(A^o)^c = \overline{A^c}$

Proof. Ex.

1.2.9 Definition Boundary and accumulation points: $\text{bdy}(A) = \partial A := \overline{A} \setminus A^o$, $\overline{\phi} = \phi, \overline{x} = X$. An accumulation (or limit or cluster) point of A is an element $x \in X$ such that every neighborhood of x contains at least one point in $A \setminus \{x\}$.

Proposition For a non-empty set A in a topological space (X, τ):

i $x \in \partial A \Leftrightarrow \forall$ open U comtaining x, $U \cap A \neq \phi$, and $U \cap A^c \neq \phi$

ii $\partial A = \partial(A^c) = \overline{A} \cap \overline{A^c}$

iii $\overline{A} = A \cup \partial A = A^o \cup \partial A$, $a^o = A \setminus \partial A = \overline{A} \setminus \partial A$

iv A open $\Leftrightarrow A \cap \partial A = \phi$, A closed $\Leftrightarrow \partial A \subset A$.

Let A' be the set of accumulation points of A, By (i), $x \in A' \Leftrightarrow x \in \overline{A \setminus \{x\}}$. As we saw in undergraduate analysis, $\partial A \not\subset A'$ nor $A' \not\subset \partial A$, in general.

1.2.10 Definition (X, τ) a topological space and Y a subset of X, the *subspace* or *relative topology* on Y is $\tau_Y = \{U \cap Y; U \in \tau\}$. This is called the *topology induced on Y from/by X*. Sets in τ_Y are called 'relatively open' sets. We also say that Y with its topology τ_Y is a *subspace* of X. Similarily, a subset A of Y is *relatively closed*, or *closed in subspace topology* if $Y \setminus A$ is relatively open (= open in subspace topology).

Proof. We need to show that τ_Y is a topology. $\phi = \phi \cap Y \in \tau_Y$, $Y = X \cap Y \in \tau_Y$, let $\{U_1, ..., U_n\} \in \tau_Y$, then $\cap_{k=1}^n (U_k \cap Y) = (\cap_{k=1}^n U_k) \cap Y \in \tau_Y$, since $\cap_{k=1}^n U_k \in \tau$. Finally, if $\{U_i; i \in I\} \subset \tau_Y$, then $\cup_{i \in I}(U_i \cap Y) = (\cup_{i \in I} U_i) \cap Y \in \tau_Y$ since $\cup_{i \in I} U_i \in \tau$.

Example $[0, 1)$ is an open set in $[0, 2]$ if $[0, 2]$ has a subspace topoogy. Because $[0, 1) = [0, 2] \cap (-1, 1)$. It is relatively open in $[0, 2]$.

Lemma

1. If $A \subset Y \subset (X, \tau)$, then A is relatively closed in $Y \Leftrightarrow A = Y \cap C, C$ closed in X.
2. A subspace of a subspace of (X, τ) is a subspace of (X, τ).
3. If $\mathcal{B}$ is a basis for τ, on X, then $\{U \cap Y; U \in \mathcal{B}\}$ is a basis for τ_Y.
4. If $A \subset Y \subset (X, \tau)$, then the closure of A with respect to τ_Y equals $\overline{A} \cap Y$, where $\overline{A}$ is the closure with respect to τ.
5. not finished.