MATH 4332 Homework 8 solutions

Q1. Show that the metric space (X, d) is complete if d is the discrete metric.

Suppose (x_n) is a Cauchy sequence in (X, d). Then there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. Since d is the discrete metric, it follows that $x_n = x_N$ for all $n \geq N$. Hence (x_n) converges (to x_N). Therefore (X, d) is complete.

Q2. Let $d, \bar{d}$ be equivalent metrics on X, show that (X, d) is complete iff $(x, \bar{d})$ is complete.

The metrics $d, \bar{d}$ are complete if there exist $m, M > 0$ such that

$$md(x, y) \leq \bar{d}(x, y) \leq Md(x, y), \text{ all } x, y \in X.$$

Let (x_n) be a Cauchy sequence in (X, d). Since $\bar{d}(x_n, x_m) \leq Md(x_n, x_m)$, (x_n) is also a Cauchy sequence in $(X, \bar{d})$. Conversely, every Cauchy sequence in $(X, \bar{d})$ is a Cauchy sequence in (X, d) (since $d(x_n, x_m) \leq m^{-1}\bar{d}(x_n, x_m)$). Suppose now that (x_n) is a Cauchy sequence (with respect to either d or $\bar{d}$). If (X, d) is complete, there exists $x^\star \in X$ such that $\lim_{n\to\infty} d(x_n, x^\star) = 0$. But $\bar{d}(x_n, x^\star) \leq Md(x_n, x^\star)$ and so $\lim_{n\to\infty} \bar{d}(x_n, x^\star) = 0$ and (x_n) converges in $(X, \bar{d})$ (to $x^\star$). Hence if (X, d) is complete so is $(X, \bar{d})$. The proof of the converse implication is similar.

Q3. Suppose that E, F are connected subsets of X. If $E \cap F \neq \emptyset$, need $E \cap F$ be connected?

No. For example, let $E = S^+ = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$ and $F = S^- = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \leq 0\}$. Then $S^\pm$ are connected but $S^+ \cap S^- = \{(-1, 0)\} \cup \{(1, 0)\}$ — two isolated points, which is disconnected.

Q4. Let $Y = \{0, 1\}$ with the discrete metric. Show that (X, d) is connected iff every continuous function $f : X \to Y$ is constant. Use this result to show that if (X_1, d_1), (X_2, d_2) are connected metric spaces then the product $(X_1 \times X_2, d)$ is connected (where we take the product metric $d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2))\}$ on $X_1 \times X_2$).

Suppose that $f : X \to Y$ is continuous and not constant. Set $U = f^{-1}(0)$, $V = f^{-1}(1)$. Then U, V are non-empty disjoint open subsets of X such that $U \cup V = X$. Therefore X is disconnected. Conversely, suppose X is disconnected. Then there exist non-empty disjoint open subsets U, V of X such that $U \cup V = X$. Define $f : X \to \{0, 1\}$ by $f(x) = 0$ if $x \in U$, $f(x) = 1$ if $x \notin U$. Then f is continuous (the inverse image of every open subset of $\{0, 1\}$ is an open subset of X).

We start by noting that if we take the product metric d on $X_1 \times X_2$ then for all $(a, b) \in X_1 \times X_2$, the metric induced on $X_1 \times \{b\}$ is d_1 and the metric induced on $\{a\} \times X_2$ is d_2. In particular, if U is an open subset of $X_1 \times X_2$, then $U \cap (X_1 \times \{b\})$ is an open subset of $X_1 \times \{b\} \approx X_1$. Similarly for intersection of U with $\{a\} \times X_2 \approx X_2$.
Suppose $f : X_1 \times X_2 \to Y$ is continuous. Then $U = f^{-1}(0)$, $V = f^{-1}(1)$ are open disjoint subsets of $X_1 \times X_2$ such that $U \cup V = X_1 \times X_2$. Without loss of generality, suppose

$U \neq \emptyset$. Let $(a,b) \in U$. Then $U \cap (X_1 \times \{b\}) = U_1$ is an open subset of $X_1 \times \{b\}$ as is $V \cap (X_1 \times \{b\}) = V_1$. Since $U_1 \cap V_1 = \emptyset$, $U_1 \cup V_1 = X_1 \times \{b\}$, we must have $V_1 = \emptyset$ since X_1, and therefore $X_1 \times \{b\}$, is connected. Hence $U_1 = X_1 \times \{b\} \subset U$. Similarly, $\{a\} \times X_2 \subset U$. We have shown that $U \supset (X_1 \times \{b\}) \cup (\{a\} \times X_2)$. Now let $(a', b) \in X_1 \times \{b\} \subset U$. Exactly the same argument shows that $U \supset \{a'\} \times X_2$. Since this is so for all $a' \in X_1$, we have shown that $U = X_1 \times X_2$ and $V = \emptyset$. Therefore $X_1 \times X_2$ is connected.

Q5. Find an example of a sequence of closed non-compact connected subsets F_n of $\mathbb{R}^2$ such that $F_1 \supset F_2 \supset \cdots$ and $\cap_{n\geq 1} F_n$ is disconnected.

For $i = 0, 1, \cdots$, define $R_i = \{(x, i) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$. Let $L = \{(0, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and $S = \{(1, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Define

$$F_n = L \cup S \cup \bigcup_{i \geq n} R_i.$$

Observe that F_0 is an infinite 'ladder' with rungs $R_0, R_1, \cdots$. F_n is obtained from F_0 by removing the first n-rungs. Each F_n is a closed connected subset of $\mathbb{R}^2$ and $F_0 \supset F_1 \supset \cdots$. However,

$$\bigcap_{n \geq 0} F_n = L \cup S,$$

which is disconnected.

Here is another example: Let $(-n, n) = \{(x, 0) \mid -n < x < n\} \subset \mathbb{R}^2$. Define $F_n = \mathbb{R}^2 \setminus (-n, n)$. In this case $\cap_{n\geq 0} F_n = \mathbb{R}^2 \setminus (-\infty, \infty)$