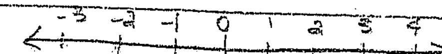
Linear Algebra 4377

www.math.uh.edu/~dblecher/4377.html

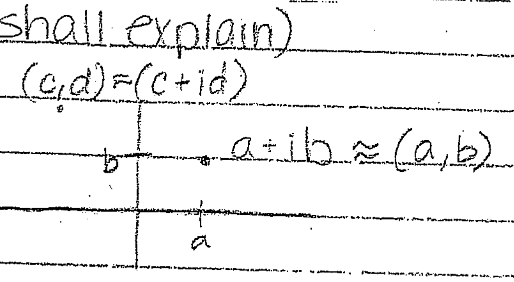
$\mathbb{R}$ = real numberline 

$\mathbb{C}$ = complex plane = $\mathbb{R}^2$ (shall explain)

addition in

$\mathbb{C}$ $\forall$ $(a+ib)+(c+id) =$

$(a+c)+i(b+d)$



Can multiply two points in $\mathbb{C}$:

multiplication $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

assume abcd = $\mathbb{R}$

notice $i^2 = (0+1i)(0+1i) = 0-1+i = -1 = i^2$

a number or scalar is a number in $\mathbb{R}$ or $\mathbb{C}$

both $\mathbb{R}$ and $\mathbb{C}$ are fields:

a field is a set $\mathbb{F}$ w/ a + and a multiplicati

st =

$a+b = b+a$ } communitivity

$a \cdot b = b \cdot a$

$a+(b+c) = (a+b)+c$ } associative

$a(bc) = (ab)c$

s.t. = such that

$\forall$ for all

$\exists$ there exist

$\forall$ a,b,c $\in \mathbb{F}$

$\exists\ 0 \in F$ s.t. $0+a = a$ $\forall \in \mathbb{F}$

$\exists\ 1 \in F$ s.t. $1a = a$ $\forall \in \mathbb{F}$

$\forall\ a \in \mathbb{F}\ \exists\ b \in \mathbb{F}$ st. $a+b=0$

b is additive inverse of a, & is written $-a$

$\forall\ a \in \mathbb{F}$, $a \neq 0$, $\exists\ c \in \mathbb{F}$ st. $ac = 1$, c is multiplicative inverse, written $\frac{1}{a}$

Read 1.2 → 1.6

1, 3, 4, 5, 9, 14, 15
pg 19
(HOMEWORK)

$a(b+c) = ab + ac$ distributive

When I write $\mathbb{F}$, I mean $\mathbb{R}$ or $\mathbb{C}$

subtraction $a-b = a+(-b)$
division $a \div b = \frac{a}{b}$ is $a \cdot \left(\frac{1}{b}\right)$

Def. A vector space is a set V with a $+$
$V \times V \to V$ and a 'scalar multiplication'
$\cdot : \mathbb{F} \times V \to V$ such that

$\forall v, w, u \in V$:

1) $u + v = v + u$ communitivity
$\boxed{(ab)v = a(bv)}$ $\forall a, b \in \mathbb{F}$ associativity
2) $u + (v + w) = (u + v) + w$
3) $\exists$ element $0 \in V$ st $0 + v = v$ $\forall v \in V$ zero element
4) $1v = v$ $\forall v \in V$
5) $\forall v \in V$ $\exists$ element $w \in V$ st $v + w = 0$; this is additive inverse $-v$

distributive laws
6) $a(v+w) = av + aw$, $(a+b)v = av + bv$ $\forall v, w \in V$, $\forall a, b \in \mathbb{F}$

if $\mathbb{F}$ is $\mathbb{R}$ we say V is a real vector space, or vector space over $\mathbb{R}$

if $\mathbb{F}$ is $\mathbb{C}$ we say V is a complex vector space, or vector space over $\mathbb{C}$

- VS = vector space
- V will always be a vector space
- element of V ~~are called 'vectors'~~ (but they dont need to have a length/direction)

Ex 1) $\mathbb{R}$ or $\mathbb{C}$ are vector spaces

Ex 2) $\mathbb{R}^n = \{(x_1, x_2, \dots x_n) : x_1, x_2, \dots x_n \in \mathbb{R}\}$ Calc 3

Case $n = 2, 3$

set

plus $+ : V \times V \to V$; $(x_1, x_2 \dots x_n) + (y_1, y_2, \dots y_n) = x_1 + y_1, x_2 + y_2, x_n + y_n$

scalar multi $c(y_1\ x_2 \dots x_n) = cx_1, cx_2\ cx_n$ $c \in \mathbb{R}$

1, 2, 3, 4, 5, 6

zero vector $\vec{0}$

$\mathbb{R}^n$ is a real vector space

We prefer to write elements of $\mathbb{R}^n$ as columns $\begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$, and vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$

$\mathbb{C}^n = \mathbb{R}^{2n}$

Ex 3) $\mathbb{C}^n = \{(x_1 \dots x_n)$ & $x_1\ x_2 \dots x_n \in \mathbb{C}\}$ - add & scalar multi as Ex 2

this is a $\mathbb{C}$ vector space

Ex 4) Any complex vector space is a real vector space

that is because every real number $\underline{a}$ is a complex number $\underline{a + i0}$

we think of $\mathbb{R} \subseteq \mathbb{C}$

Ex 5) $M_{m,n}$ = set of $m \times n$ matrices with entries in $\mathbb{F}$

$M_{m,n}(\mathbb{R})$ = " $\mathbb{R}$

$M_{m,n}(\mathbb{C})$ = " $\mathbb{C}$

$M_n = M_{n,n}$, $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$, etc

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \in M_{2,3}(\mathbb{R})$$

2×2 entry

Ex 6) $\mathbb{F}^\infty = \{(x_1, x_2, x_3, \ldots) : x_k \in \mathbb{F} \ \forall \ k = 1, 2, 3 \ldots\}$

Ex 7) $\mathcal{P}(\mathbb{F})$ = set of all polynomials

$a_0 + a_1 x + a_2 x^2 + a_n x^n$,

$n \in 1, 2, 3, \ldots$ $a_k \in \mathbb{F} \ \forall k = 1, 2, 3$

$\mathcal{P}(\mathbb{R})$ would contain things like $3 - 2x + x^2$

obvious add $(3 - 2x + x^2) + (x^3 - 1)$

$2 - 2x + x^2 + x^3$

scalar multiplication

$7(3 - 2x + x^2) = 21 - 14x + 7x^2$

Ex 8) $V = \{f : S \to \mathbb{F}\}$ functions from a set S to $\mathbb{F}$

add: $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$

s.m. $(cf)(x) = cf(x) \quad c \in \mathbb{F}$

"zero function" $0(x) = 0 \ \forall x \in S$

* memorize every definition

Ex 9) $\{f : [0,1] \to \mathbb{R}$ continuous$\}$ or
Ex 10) $\{f : (0,1) \to \mathbb{R}$ differentiable$\}$
w/ add & sc. multi. as in Ex 8

learn in Calc 1 for add
"sitting inside Ex 8"

WE NOW PROVE EASY THINGS ABOUT ANY VECTOR SPACE V

Prop 1: The zero element in a vector space is unique
proof: suppose 0_1 and 0_2 are 2 zero elements
then $0_1 = 0_1 + 0_2 = 0_2$ $\square$

Prop 2: The additive inverse is unique
proof: suppose $v + w_1 = 0$ and $v + w_2 = 0$
Then $w_1 \overset{\text{rule 3,1}}{=} w_1 + 0 = w_1 + (v + w_2) \overset{\text{rule 2}}{=} (w_1 + v) + w_2 =$
$0 + w_2 \underset{\text{rule 3}}{=} w_2$ $\square$

vector 0
scalar 0

Prop 3: $\overset{\text{scalar}}{0} \cdot v = \overset{\text{vector}}{0}$ $\forall v \in V$
proof: Let $0 \cdot v = z$ then $z = (0+0)v \overset{6}{=} 0v + 0v = z + z$
add $-z$ to both sides:
$0 = z + -z = z + z + (-z) \overset{3}{=} z + 0 \overset{3}{=} z$ $\square$

try to prove that $z =$ vector 0

Prop 4: $a \cdot 0 = 0$ $\forall a \in \mathbb{F}$
proof: Ex, almost identical to Prop 3

Prop 5: the additive inverse of $v \in V$ is $(-1)v$

proof: $v + (-1)v \overset{4}{=} 1v + (-1)v \overset{6}{=} (1 + -1)v = 0v \underset{\text{Prop 3}}{=} 0$

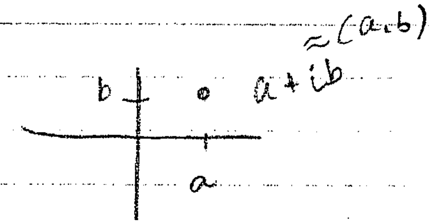
So $w = (-1)(v)$ satisfies rule 5

so is the additive inverse

unique by prop 2 $\square$

6/2

$\mathbb{R}$ = real number line



$\mathbb{C}$ = complex plane = $\mathbb{R}^2$

Addition in $\mathbb{C}$: $(a+ib) + (c+id) = (a+c) + i(b+d)$.

Multiplication in $\mathbb{C}$: $(a+ib)\cdot(c+id) =$

$$(ac-bd) + i(ad+bc),$$

assuming $a, b, c, d \in \mathbb{R}$.

Notice $i^2 = (0+1i)(0+1i) = 0-1+i(0+0) = \boxed{-1}$

A <u>number</u> or <u>scalar</u> for us is a member of $\mathbb{R}$ or $\mathbb{C}$.

Both $\mathbb{R}$ and $\mathbb{C}$ are <u>fields</u>:

A <u>field</u> is a set F with a + and a multiplication s.t.

$$\left.\begin{array}{l} - \; a+b = b+a \\ - \; a\cdot b = b\cdot a \end{array}\right\} \text{commutativity}$$

$$\left.\begin{array}{l} - \; a+(b+c) = (a+b)+c \\ - \; a(bc) = (ab)c \end{array}\right\} \text{associativity}$$

$\exists\, 0 \in \mathbb{F} \ni 0 + a = a \;\forall\, a \in \mathbb{F}$

$\exists\, 1 \in \mathbb{F} \ni 1\cdot a = a \;\forall\, a \in \mathbb{F}$

$\forall\, a \in \mathbb{F}, \exists\, b \in \mathbb{F} \ni a + b = 0$

b is the additive inverse of a, written a'

$\forall a \in \mathbb{F}, \exists\ c \in \mathbb{F} \ni ac = 1$
c is the multiplicative inverse, written $\frac{1}{a}$.

$a(b+c) = ab + ac$ ← distributive prop

★ $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Above rules are $\forall\ a, b, c \in \mathbb{F}$.

Subtraction $a - b = a + (-b)$

Division $a \div b$ or $\frac{a}{b} = a \cdot (\frac{1}{b})$

★★ Def: A vector space is a set V with a

$+ : V \times V \to V$ and a

scalar multiplication $\cdot : F \times V \to V$ such that

commutativity 1. $u + v = v + u$

Associativity 2. $u + (v + w) = (u + v) + w$; $(ab)v = a(bv)$ $\forall\ a, b \in F$.

Zero Element 3. $\exists$ an element $0 \in V \ni 0 + v = v\ \forall\ v \in V$

4. $1 \cdot v = v \quad \forall\ v \in V$

5. $\forall\ v \in V$, $\exists$ an element $w \in V \ni v + w = 0$; this is the additive inverse, written $-v$.

Distributive laws 6. $a(v + w) = av + aw$, $(a + b)v = av + bv$
$\forall\ v, w \in V, \ \forall\ a, b \in F$.

If $\mathbb{F} = \mathbb{R}$, we say V is a real vector space, or vector space over $\mathbb{R}$

If $\mathbb{F} = \mathbb{C}$, we say V is a complex vector space, a vector space over $\mathbb{C}$

- VS = vector space
- V will always be VS.
- elements of V are called vectors (but they don't need to have a length or direction)

Ex 1.

$\mathbb{R}$ or $\mathbb{C}$ are vector spaces.

Ex 2. $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$

$(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots y_n) = (x_1+y_1, x_2+y_2, \ldots x_n+y_n)$

$c(x_1, x_2, x_3, \ldots x_n) = (cx_1, cx_2, \ldots cx_n), \ c \in \mathbb{R}.$

Zero vector : $\vec{0} = (0, 0, \ldots, 0)$

This is a real vector space. We prefer to write elements of $\mathbb{R}^n$ as <u>columns</u>: $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and

write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Ex 3

$\mathbb{C}^n = \{(x_1, x_2, \ldots x_n), x_1, x_2, \ldots x_n \in \mathbb{C}\}$ $\quad (\mathbb{C}^n = \mathbb{R}^{2n})$

Addition + Scalar Mult as in Ex 2

This is a complex vector space.

Ex 4

Any complex vector space is a real vector space. This is because every real number $\underline{a}$ is a complex number $a + i(0)$. We think of $\mathbb{R} \subseteq \mathbb{C}$.

Ex 5

$M_{m\times n}$ = set of $m\times n$ matrices with entries in $\mathbb{F}$

$M_{m\times n}(\mathbb{R})$ = set of $m\times n$ matrices with entries in $\mathbb{R}$.
$M_{m\times n}(\mathbb{C})$ = set of $m\times n$ matrices with entries in $\mathbb{C}$.

$M_n = M_{n,n}$, $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$, etc.

$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \in M_{2,3}(\mathbb{R})$

$\nwarrow$ 2,2 entry of matrix

Ex 6

$\mathbb{F}^\infty = \{(x_1, x_2, x_3, \ldots) : x_k \in \mathbb{F} \ \forall k = 1, 2, 3, \ldots\}$.

Addition + Scalar Mult. as in ex 2.

Ex 7

$P(\mathbb{F})$ = set of all polynomials:
$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $n = 0, 1, 2, \ldots$,
$a_k \in \mathbb{F} \ \forall k = 0, 1, 2, \ldots$

$P(\mathbb{R})$ would contain things like $3 - 2x + x^2$
Obvious addition and scalar multiplication

Ex 8

* $V = \{f : S \to \mathbb{F}\}$, functions from a set S to $\mathbb{F}$.

Addition $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$
Scalar Mult: $(cf)(x) = cf(x), \quad \forall c \in \mathbb{F}$
zero function: $0(x) = 0 \quad \forall x \in S$.

Ex 9 $\{f : [0,1] \to \mathbb{R} \text{ continuous}\}$ or
Ex 10 $\{f : (0,1) \to \mathbb{R} \text{ differentiable}\}$, with

addition as in ex 8.
and scal. mult

We now prove some easy things about any vector space V

Prop 1

The zero element in a vector space is unique

Proof: Suppose 0_1 and 0_2 were two zero elements. Then

$$0_1 = 0_1 + 0_2 = 0_2 .$$

Prop 2

The additive inverse is unique

Proof

Suppose $v + w_1 = 0$ and $v + w_2 = 0$. Then

$$w_1 = w_1 + 0 = w_1 + (v + w_2)$$

$$= (w_1 + v) + w_2 = 0 + w_2 = w_2 .$$

Prop 3

$$0 \cdot v = 0 \quad \forall\, v \in V$$

Proof:

Let $0 \cdot v = z$. Then $z = (0+0)v \overset{6}{=} 0v + 0v$
$= z + z$.

(Trying to prove $z = 0$.)

Add $-z$ to both sides:

$0 = z + (-z) = z + z + (-z) \overset{5}{=} z + 0 \overset{3}{=} z$.

Prop 4

$$a0 = 0 \quad \forall\, a \in \mathbb{F}.$$

Proof: Ex, almost identical to prop 3.

Prop 5

The additive inverse of $v \in V$ is $(-1)v$.

Proof $v + (-1)v \overset{4}{=} 1v + (-1)v \overset{6}{=} (1+(-1))v$
$= 0v \overset{(\text{prop } 3)}{=} 0$ So $w = (-1)v$ satisfies rule 5 so is the additive inverse by prop 2.

Some notations from last class explained

s.t = such that

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$+ : V \times V \rightarrow V$

Means...

$\cdot : \mathbb{F} \times V \rightarrow V$

(I explained this verbally)

6/3/08

Checking in an example that V is a vector space requires checking 10 things.

Ex 11

$P_2(\mathbb{F})$ = the polynomials of degree 2 or less

$a_0 + a_1 x + a_2 x^2$, $a_0, a_1, a_2 \in \mathbb{F}$.

Let's show it is a vector space w/ the usual addition and scalar multiplication.

did not end up w/ x^3, sin, etc.

1. $(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \in P_2(\mathbb{F})$

2. $c(a_0 + a_1 x + a_2 x^2) = ca_0 + ca_1 x + ca_2 x^2 \in P_2(\mathbb{F})$

3. $(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) =$

$(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 =$

$(b_0 + b_1 x + b_2 x^2) + (a_0 + a_1 x + a_2 x^2)$

4. $((a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)) + (c_0 + c_1 x + c_2 x^2)$

$= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (c_0 + c_1 x + c_2 x^2) =$

$(a_0+b_0+c_0) + (a_1+b_1+c_1)x + (a_2+b_2+c_2)x^2 =$

$(a_0 + a_1 x + a_2 x^2) + (b_0+c_0) + (b_1+c_1)x + (b_2+c_2)x^2$

$= (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) + (c_0 + c_1 x + c_2 x^2)$

5. $ab(a_0+a_1x+a_2x^2) = aba_0 + aba_1x + aba_2x^2$

$= a(ba_0+ba_1x+ba_2x^2) = a(b(a_0+a_1x+a_2x^2))$

6. $0+(a_0+a_1x+a_2x^2) = (0+a_0)+a_1x+a_2x^2 =$

$a_0 + a_1x + a_2x^2$

7. $1(a_0+a_1x+a_2x^2) = a_0+a_1x+a_2x^2$

8. $a_0+a_1x+a_2x^2+(-a_0-a_1x-a_2x^2) =$

$0+0x+0x^2 = 0$

9. $(a+b)(a_0+a_1x+a_2x^2) = (a+b)a_0+(a+b)a_1x+(a+b)a_2x^2$
(here $a,b \in \mathbb{F}$).

$= aa_0+aa_1x+aa_2x^2+ba_0+ba_1x+ba_2x^2 =$

10. $a(a_0+a_1x+a_2x^2)+b(a_0+a_1x+a_2x^2)$

$= a(a_0+a_1x+a_2x^2+b_0+b_1x+b_2x^2) =$

$a((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2) =$

$a(a_0+b_0)+a(a_1+b_1)x+a(a_2+b_2)x^2$

$= a(a_0+a_1+a_2x^2)+a(b_0+b_1x+b_2x^2)$

Def:
a subspace (or linear subspace or vector subspace) of a vector space V is a subset $U \subseteq V$ satisfying:

1. $0 \in U$ where 0 is the zero element of V
2. $u+v \in U$ whenever u and v are in U.
3. $cu \in U$ whenever $u \in U$ and $c \in \mathbb{F}$.

Prop 6
Any subspace of a vector space is a vector space

Proof: let U be a subspace of a vector space V. (we have to check 10 things for U).

Condition (2) in last definition gives the first of the 10.

condition (3) in last definition gives the second of the 10.

the third of the ten holds since it's true in V!

Similarly for the fourth, fifth, seventh, ninth, and tenth.

By condition (1) in the last definition, the sixth of the ten also holds in U since it holds in V.

By Prop. 5, the additive inverse of v in V is $(-1)v$, which is in U if $v \in U$ by condition (3) in the last definition.

$v + (-1)v = 0$

Ex 1

V and 0 are subspaces of any vector space V.

Ex 2

$\{(x,y,z)\} \in \mathbb{R}^3 : 2x - y + 3z = 0$ is a subspace of $\mathbb{R}^3$ (can be $\mathbb{C}^3$; proof is same).

Check: (1) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the set ✓

(2) $2x - y + 3z = 0$ and $2a - b + 3c = 0$
$\Rightarrow 2(x+a) - (y+b) + 3(z+c) =$
$\ldots = 0$

So $(x,y,z) + (a,b,c)$ is in the set (if (x,y,z) and (a,b,c) are.

(3) (x,y,z) in this set $\Rightarrow 2x - y + 3z = 0$
$\Rightarrow 2cx - cy + 3cz = 0 = c(x,y,z)$ in set.

SIDEBAR: $2x - y + 3z^2 = 0$ $\quad (-1,1,1) \quad 2(-1,1,1) = (-2,2,2)$
$(-2,2,2)$ does not satisfy condition so condition (3) fails.

Ex3

The symmetric matrices in $M_{n,n}(\mathbb{R})$
ie $U = \{A \in M_n(\mathbb{R}) : A = A^T\}$ is a subspace of M_n

Check (1). Zero matrix $\in U$

(2) $A, B \in U \Rightarrow (A+B)^T = A^T + B^T = A+B$, so $A+B \in U$

(3) $A \in U$, $c \in \mathbb{R}$, $(cA)^T = cA^T = cA$, so $cA \in U$.

Ex4

The selfadjoint matrix in $M_n(\mathbb{C})$
ie $U = \{A = [a_{ij}] \in M_n(\mathbb{C}) : [a_{ij}] = [\overline{a_{ji}}]$

SIDEBAR: complex conjugate: $\overline{a+ib} = a - ib$, $a, b \in \mathbb{R}$

Note that (1) and (2) of last definition holds, for example for (2), if $[a_{ij}] = [\overline{a_{ji}}]$, $[b_{ij}] = [\overline{b_{ji}}]$ then $[a_{ij} + b_{ij}] = [\overline{a_{ji}} + \overline{b_{ji}}] = [\overline{a_{ji} + b_{ji}}]$.

look in $M_2 \mathbb{C}$: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is self adjoint, but

$i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ is not self adjoint.

So U is not a subspace, failing (3)

However, U is a subspace if we consider $M_2(\mathbb{C})$ as a vector space over $\mathbb{R}$.
Item (1) and (2) of last definition is still true.

for (3), If $[a_{ij}] = [\overline{a_{ji}}]$ and if $c \in \mathbb{R}$,
then $[ca_{ij}] = [\overline{ca_{ji}}] = [c\overline{a_{ji}}]$ so $c[a_{ij}] \in U$.

SIDEBAR: $\overline{zw} = \bar{z}\,\bar{w}$ so if c is real, $\overline{cz} = \bar{c}\bar{z} = c\bar{z}$

Ex5
$\{p \in P(\mathbb{R}) : p(3) = 0\}$ is a subspace of $P(\mathbb{R})$

For example for (3): $p(3) = 0 \Rightarrow (cp)(3) = c(p(3)) = c \cdot 0 = 0$.

Ex6
Let $S = [0,1]$, then last time/HW,
$V = \{f : [0,1] \to \mathbb{R}\}$ is a vector space.

Ex 9 from last time $\{f : (0,1) \to$ continuous$)$ is a subspace of V.

Check (1): Zero function is continuous

(2): Sum of two continuous functions is continuous (Cal. 1)

(3) cf is continuous if $c \in \mathbb{R}$, f cont (Cal 1)

Ex7
Let $S = (0,1)$, $V = \{f : (0,1) \to \mathbb{R}\}$,
$U = \{f : (0,1) \to \mathbb{R}$ differentiable$)$.

Just as in last example, U is a subspace of V.

Prop 7.

A nonempty subset U of a vector space V is a subspace iff $cv + w \in U$ whenever $c \in \mathbb{F}$ and $v, w \in U$

Proof: $(\Rightarrow)$

If U is a subspace of V and if v and $w \in U$ and $c \in \mathbb{F}$, then $cv \in U$ by condition (3) in last def, and so $cv + w \in U$ by condition (2

$(\Leftarrow)$

Assume the "$cv+w$" condition. Take $v \in U$, let $w = v$, $c = -1$. Then $cv + w = (-1)(v) + v = 0 \in U$. So (1) in last definition holds. Taking $c = 1$, $v, w \in U$ arbitrary says $v + w \in U$ which is (2) in last def. Taking $w = 0$, $c \in \mathbb{F}$ arbitrary, $v \in U$ arbitrary, gives $cv + w = cv \in U$ so (3) holds.

Prop 8 6/4/08

The intersection of any collection of subspaces of V is a subspace of V.

Proof: If $\{U_i : i \in I\}$ is a collection of subspaces of V, set $U = \bigcap_{i \in I} U_i$

Since $0 \in U_i$, $\forall\, i \in I$, $0 \in \bigcap U_i = U$.
If $v, w \in U$ and $c \in \mathbb{F}$ then $v, w \in U_i \forall i$
so $cv + w \in U_i \;\forall i$, and so
$cv + w \in U$. So U is a subspace by Prop. 7.

More notations used in course

$\mathbb{N} = \{1, 2, 3, \ldots\}$

If the symbol n or m is used, it is always a member of $\mathbb{N}$. Usually $j, k, i \in \mathbb{N}$.

List: $= (x_1, x_2, \ldots, x_n)$. This would be a list of length n. Say x_1 is the first coordinate/entry in list, x_2 is the second in list....
Sometimes called an n-tuple.

Order matters so $(3,5) \neq (5,3)$.
Repetition is allowed: $(4,4,4)$.

Def:

A **linear combination** of a list $(v_1, v_2, \ldots, v_n)$ of vectors in V is a vector of form $c_1v_1 + c_2v_2 + \ldots + c_nv_n$, where $c_k \in \mathbb{F}$ $\forall k = 1, 2, \ldots, n$

The **span** of this list is the set of all linear combinations, written span$(v_1, v_2, \ldots, v_n)$ or span(S) if S is the list $(v_1, v_2, \ldots, v_n)$.

We say that a list S is a spanning set for V, or spans V, or V is the span of S if $V = \text{span}(S)$.

Prop 9

For any list $(v_1, v_2, \ldots, v_n)$ in V, span$(v_1, v_2, \ldots$ is a subspace of V.

Proof: $0 = \sum_{k=1}^{n} 0 \cdot v_k \in \text{span}(v_1, \ldots, v_n)$

Also, $c_1v_1 + c_2v_2 + \ldots + c_nv_n + d_1v_1 + d_2v_2 + \ldots d_n$

$= (c_1+d_1)v_1 + (c_2+d_2)v_2 + \ldots + (c_n+d_n)v_n \in \text{span}(v_1, v_2, \ldots, v_n)$.

and $k(c_1v_1 + c_2v_2 + \ldots + c_nv_n) =$

$(kc_1)v_1 + (kc_2)v_2 + \ldots + (kc_n)v_n \in \text{span}(v_1, v_2, \ldots, v_n)$

for scalars $k, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n$, which verifies (2) and (3) of the definition of subspace.

Def: <u>Sums of subspaces:</u> If $U_1, U_2, \ldots, U_m$ are subspaces of V, we define $U_1 + U_2 + \ldots + U_m$ to be the set of sums $u_1 + u_2 + \ldots + u_m$ $\forall\, u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m$

Prop 10

With notation as above, $U_1 + U_2 + \ldots + U_m$ is also a subspace of V.

Proof: Essentially the same as Prop 9.

$$0 = \underbrace{0 + 0 + 0 + \ldots + 0}_{m} \in U_1 + U_2 + \ldots U_m.$$

$$(u_1 + u_2 + \ldots + u_m) + (u_1' + u_2' + \ldots + u_m') =$$
$$(u_1 + u_1') + (u_2 + u_2') + \ldots + (u_m + u_m') \in$$
$U_1 + U_2 + \ldots + U_m$ whenever $u_k, u_k' \in U_k$, $k = 1, 2, 3, \ldots, m$

Similarly $c(u_1 + u_2 + \ldots + u_m) =$
$(cu_1) + (cu_2) + \ldots + (cu_m) \in U_1 + U_2 + \ldots + U_m.$

Ex

let $V = \mathbb{R}^3$, let $U_1 = \{(x, x, 0) : x \in \mathbb{R}\}$,
$U_2 = \{(x, -x, 0) : x \in \mathbb{R}\}$.
Then $U_1 + U_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$, the xy plane in $\mathbb{R}$.

Indeed, given any $(x, y, 0) =$
$(a, a, 0) + (b, -b, 0) \iff$

$$\begin{cases} x = a + b \\ y = a - b \end{cases} \iff \begin{cases} \frac{x+y}{2} = a \\ \frac{x-y}{2} = b \end{cases}$$

so any $(x, y, 0) = \left(\frac{x+y}{2}, \frac{x+y}{2}, 0\right) +$

$\left(\frac{x+y}{2}, -\left(\frac{x-y}{2}\right), 0\right) \in U_1, U_2$.

Def: (Internal) direct sum: If $U_1, U_2, \ldots, U_n$ are subspaces of V, then we say V is the (internal) direct sum of $U_1, U_2, \ldots U_m$ and write $V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$, if every element $v \in V$ may be written as $v = u_1 + u_2 + \ldots + u_m$ in a unique (one and only one) way where $u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m$.

Remark: This implies, but is not usually, the same as $V = U_1 + U_2 + \ldots + U_m$.

Prop 11 If U, W are subspaces of V, then $V = U \oplus W$ iff $V = U + W$ and $U \cap W = (0)$.

Proof: ($\Rightarrow$) By the remark above, if $V = U \oplus W$, then $V = U + W$.

(more detail: $U + W \subseteq V$ since if $u \in U$, $w \in W$ then $u, w \in V$ so $u + w \in V$.
$V \subseteq U + W$: By definition, $v = u_1 + u_2$ with $u_1 \in U_1$, $u_2 \in W$, so $v \in U + W$).

Suppose $x \in U \cap W$. Then $x = \underset{\substack{\uparrow \\ u}}{x} + \underset{\substack{\uparrow \\ w}}{0} = \underset{\substack{\uparrow \\ u}}{0} + \underset{\substack{\uparrow \\ w}}{x}$

By the uniqueness in definition $x = 0$.
So $U \cap W \subseteq (0)$, and $(0) \subseteq U \cap W$ is obvious.

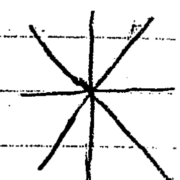
($\Leftarrow$) Assume $V = U + W$ and $U \cap W = 0$.
Then any v in V can be written as $v = u + w$, $u \in U$, $w \in W$. We need to show if $v = u' + w'$, $u' \in U$, $w' \in W$, then $u = u'$, $w = w'$

But $v = u + w = u' + w' \Rightarrow u = u' + w' - w \Rightarrow$
$\underbrace{u - u'}_{\in U} = \underbrace{w' - w}_{\in W} \in U \cap W = (0)$.

So $u - u' = 0$ and $w' - w = 0$ so
$u = u'$ and $w = w'$.

Ex1

$\mathbb{R}^2 = \{(x,x) : x \in \mathbb{R}\} \oplus \{(x,-x) : x \in \mathbb{R}\}$



By Prop 11, since the intersection of the two subspaces is origin (see picture) (0) and their sum $\{(x,x) : x \in \mathbb{R}\} + \{(x,-x) : x \in \mathbb{R}\}$ is $\mathbb{R}^2$ by slight change to previous exercise.

Ex 2
$\mathbb{R}^2 = \{(x,0) : x \in \mathbb{R}\} \oplus \{(0,x) : x \in \mathbb{R}\}$

Similarly $\mathbb{R}^3 = \{(x,0,0) : x \in \mathbb{R}\} \oplus \{(0,x,0) : x \in \mathbb{R}\} \oplus \{(0,0,x) : x \in \mathbb{R}\}$.

$\mathbb{R}^4 = \ldots$, etc.

Proof is same as ex 1 but easier since dealing with 0.

Ex 3
$P(\mathbb{R}) = U \oplus W$ where U consists of even polynomials $(a_0 + a_2x^2 + \ldots a_{2n}x^{2n})$ and W is odd polynomials $(a_1x + a_3x^3 + \ldots a_{2n+1}x^{2n+1})$

Proof: By Prop 11, this amounts to showing

(1) any polynomial = (even polynomial) + (odd polynomial)

(2) the only polynomial that is both even and odd is 0.

Check (2):

$$a_0 + a_2x^2 + \ldots + a_{2n}x^{2n} = b_1x + b_3x^3 + \ldots b_{2n+1}x^{2n+1}$$

Set $x = 0 \rightarrow a_0 = 0$

Differentiate, set $x = 0 \rightarrow b_1 = 0$:

$2a_2x + \ldots = b_1 + 3b_3x^2 + \ldots \rightarrow$

$x = 0 \longrightarrow b_1 = 0$.

Differentiate again and set $x = 0$:

$2a_2 + 12a_4x^2 + \ldots = 6b_3x + \ldots \rightarrow a_2 = 0$

Keep going, all the $a_k, b_k = 0$, so polynomial is 0.

CHAPTER 2
FINITE DIMENSIONAL VECTOR SPACES

Def: A vector space V is finite dimensional (f.d or f dim) if it has a spanning list $(v_1, v_2, \ldots, v_n)$. That is $V = (v_1, v_2, \ldots, v_n)$. If it is not f.d, say it is infinite dimensional.

ex

(Think of (i,j,k) for $\mathbb{R}^3$ from cal 3.)

Ex 1-3 on first class day $(= \mathbb{F}^n)$, a spanning list is $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Ex 5, $M_{m \times n}$ is f.d, with spanning list =

$\{E_{ij} : 1 \le i \le n, 1 \le j \le n\}$ where E_{ij} is the matrix whose entries are all 0 except for a 1 in the ij entry

$$E_{ij} = \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

eg $$\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{11}} + 3\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + (-1)\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + 2\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

Ex 6, 7 are infinite dimensional.

Ex 11 $P_2(\mathbb{R})$ is f.d. with spanning list = $(1, x, x^2)$

Ex 7

Prove that $P(\mathbb{R})$ is infinite dimensional (proof by contradiction).

Suppose it was f.d with spanning list $(p_1, p_2, \dots, p_n)$. Suppose that the highest power of x in any of these polynomials is x^k say. Since it is a spanning list,

$x^{k+1} = c_1 p_1 + c_2 p_2 + \dots + c_n p_n$. But RHS has highest power of x k or less. This is a contradiction so that proves that $P(\mathbb{R})$ is infinite dimensional

Def:

A list $(v_1, v_2, \dots, v_n)$ in V is <u>linearly independent</u> (l.i) if the only way $\sum_{k=1}^{n} c_k v_k = 0$ is if $c_1 = c_2 = \dots = c_n = 0$. (This is equivalent to (prove as exercise) no one of the v_k's can be written as a linear combination of the others.

Ex 1

In $\mathbb{R}^3$, $(\vec{i}, \vec{j}, \vec{k})$ is l.i.

Ex2
Is $\left(\begin{bmatrix}2\\3\\1\end{bmatrix}\right), \left(\begin{bmatrix}1\\-1\\2\end{bmatrix}\right), \left(\begin{bmatrix}7\\3\\8\end{bmatrix}\right)$ l.i in $\mathbb{R}^3$?

No: $2\begin{bmatrix}2\\3\\1\end{bmatrix} + 3\begin{bmatrix}1\\-1\\2\end{bmatrix} + (-1)\begin{bmatrix}7\\3\\8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

(solve $\begin{bmatrix}2 & 1 & 7\\3 & -1 & 3\\1 & 2 & 8\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\3\\-1\end{bmatrix}$)

Ex3
$\{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$. not l.i.

Ex4
$(1, x, x^2)$ in $P(\mathbb{R})$. is l.i

$a1 + bx + cx^2 = 0$. Set $x=0$

$x=0$ $a=0$

Diff, set $x=0$ $b=0$

$Dx'' = 1$ $\therefore c = 0$

Def: A list which is not linearly independent is called <u>linearly dependent</u>.

~~Say~~ Note (0) is <u>linearly dependent</u>

6/5/08

Ex

Prop 1 If $v \in V$ then $\{v\}$ is l.i $\Leftrightarrow v \neq 0$.

If $(v_1, v_2, \ldots, v_n)$ is l.i, then so is any nonempty subset of $\{v_1, \ldots, v_n\}$.

Lemma 1 (linear dependence lemma):

If $(v_1, v_2, \ldots v_m)$ is linearly dependent, and if $v_1 \neq 0$, then $\exists\, j \in \{2, 3, \ldots, m\} \ni$

a) $v_j \in \text{Span}(v_1, v_2, \ldots v_{j-1})$.
b) $\text{span}(v_1, v_2, \ldots, v_m) = \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_m)$

Proof: Since $(v_1, \ldots, v_m)$ is linearly dependent, $\exists$ scalars c_k with $\sum_{k=1}^{m} c_k v_k = 0$ and not all the scalars c_k are zero. This implies that not all of $c_1, c_2, \ldots c_m$ are zero (because if they were then $c_1 v_1 = 0 \overset{HW}{\Rightarrow} c_1 = 0$, so all c_k's are zero, a contradiction

Let $j = \max\{i : c_i \neq 0\}$. Then $\sum_{k=1}^{j} c_k v_k = 0$, so $c_j v_j = -\sum_{k=1}^{j-1} c_k v_k \Rightarrow$

$$v_j = \sum_{k=1}^{j-1} \left(\frac{-c_k}{c_j}\right) v_k \in \text{span}(v_1, v_2, \ldots, v_{j-1}),$$

proving a.

To prove (b) we need to show if $u \in \text{span}(v_1, v_2, \ldots, v_n)$ then $u \in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$.

But if $u = \sum_{k=1}^{n} c_k v_k = \sum_{k=1}^{j-1} c_k v_k + c_j v_j + \sum_{k=j+1}^{m} c_k v_k = \sum_{k=1}^{j-1} c_k v_k - \sum_{k=1}^{j-1} c_k v_k + \sum_{k=j+1}^{m} c_k v_k$.

Theorem)

Let S be a spanning list for V and let B be a l.i list. Then

length of $B \leq$ length of S.

Proof: Suppose $S = (w_1, w_2, \ldots, w_n)$, $B = (u_1, \ldots, u_m)$, we need to show $m \leq n$.

Multi-step process. At each step, remove one w and add one u.

Step 1: $(u_1, w_1, \ldots, w_n)$ is linearly dependent, since $u_1 \in \text{span}(S)$.
So by lemma 1(b), $\exists j \ni$ if we remove w_j from $(u_1, w_1, \ldots, w_n)$, the remaining list, call it B_1, still spans V.

Step k:

the list B at the end of step k-1 spans V, so if we adjoin u_k to it (after $u_1, u_2, \ldots, u_{k-1}$ in the list) it is linearly dependent.

By lemma 1(a), one of these vectors in this new list is a linear combination of the vectors appearing before it in the list. This vector cannot be a u since B is l.i. so it is a w. Remove this w, then by lemma 1(b), the remaining list, which we still call B, spans V (as in step 1).

After m steps, we have added all the process stops. Since at each step we removed one w and added one u, there must have been at least m w's. So $n \geq m$.

Prop 2

Every subspace of a f.d. v.s. V is f.d.

Proof: Let V be f.d., U a subspace. We use a multistep argument.

Step 1:

If $U = (0)$, then proposition is true.
If $U \neq (0)$, choose nonzero vector $v_1 \in U$.

Step j: If $U = \text{span}(v_1, v_2, \dots, v_{j-1})$, then we're done. If not, choose a vector $v_j \in U$ which is not in last span.

Keep doing these steps, and notice that $\{(v_1, v_2, \dots, v_j)\}$ is l.i. by lemma 1 (a). (because if $(v_1, \dots, v_j)$ linearly dependent, lemma 1 (a) says one of the v's is in span of previous v's, which is impossible by the way we constructed each v.)

★ If S is a spanning list for V of length r, then the process above must stop in r steps or fewer, since by Thm 1, $j = \text{length}(v_1, \dots, v_j) \leq r$. So some step j above is the last, which means $U = \text{span}(v_1, \dots v_j)$, so U is f.d.

Def: a basis for V is a spanning l.i. list.

Ex1

$\{\vec{i}, \vec{j}, \vec{k}\}$ in $\mathbb{R}^3$ or $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$ in $\mathbb{F}^n$.

Ex2

$\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m \times n}$.
(l.i. in M_2: $a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\begin{bmatrix}a&b\\c&d\end{bmatrix} \Rightarrow a=b=c=d=0.$)

Ex3

$(1, x, x^2)$ is a basis for $P_2(\mathbb{R})$.

last class we showed that this is l.i and spanning.

Ex4

$\{(1,1), (1,-1)$ is a basis for $\mathbb{R}^2$ (typical 2331 test question).

Prop3

A list $(v_1, \dots, v_n)$ is a basis of V iff every $v \in V$ can be written in one and only one way (uniquely) as

$$v = \sum_{k=1}^{n} c_k v_k \text{ with } c_k \in \mathbb{F}.$$

Proof: ($\Rightarrow$) Suppose $(v_1, \dots v_n)$ is a basis. Then it is spanning, so $v = \sum_{k=1}^{n} c_k v_k$, $c_k \in \mathbb{F}$.

Suppose also $v = \sum_{k=1}^{n} d_k v_k$, $d_k \in \mathbb{F}$.

Then subtracting $\sum_{k=1}^{n} c_k v_k - \sum_{k=1}^{n} d_k v_k = 0 =$

$\sum_{k=1}^{n} (c_k - d_k) v_k$. So $c_k - d_k = 0 \ \forall k$,

Since (v_k) l.i) So $c_1 = d_1, c_2 = d_2, \dots c_n = d_n$.

($\Leftarrow$) The condition on the right of the "iff" certainly implies $(v_1, \dots, v_n)$ is spanning.

If $\sum_{k=1}^{n} c_k v_k = 0$, then since $0 = \sum_{k=1}^{n} 0 v_k$, by the "only one" above, so $c_k = 0 \ \forall k$. So $(v_1, \dots, v_n)$ is linearly independent.

Theorem 2

Every spanning list for V contains a basis for V.

Proof: Let $(v_1, \dots, v_n)$ be a spanning list, call it B. Multistep process.

Step 1: If $v_1 = 0$, delete it from B, otherwise leave B unchanged.

Step j: If $v_j \in \text{span}(v_1, \dots, v_j)$, delete it from B, otherwise leave B unchanged.

Stop after n steps, after which, B still spans V (because if $v \in V$ is written as $\sum_{k=1}^{n} c_k v_k$, original v_k, and if v_j say was thrown away in step j, then $v_j = \sum_{i=1} d_i v_i$

$$v = \sum_{k=1}^{j-1} c_k v_k + c_j\left(\sum_{i=1} d_i v_i\right) + \sum_{k=j+1}^{n} c_k v_k$$

So we do not <u>need</u> v_j to span V.

Now B is also l.i, by lemma 1 (since if it were linearly dependent, then by lemma 1(a), one of the members of B, v_r say is a linear combination of previous members $v_1, \ldots, v_{r-1}$, which contradicts the step construction. So B is a basis.

Ex

Do the steps to $B = ((1,2),(3,6),(4,7),(5,9))$ in $\mathbb{R}^2$.

Soln:

Step 1: $\rightarrow B = ((1,2),(3,6),(4,7),(5,9))$.

Step 2: $\rightarrow B = ((1,2), -, (4,7),(5,9))$.

Step 3: $\rightarrow B = ((1,2), -, (4,7),(5,9))$. $\leftarrow (5,9) = (1,2)+(-1,7)$

Step 4: $\rightarrow B = ((1,2), -, (4,7), -)$, a basis.

Corollary 1

Every nontrivial f.d. v.s. has a basis.

Proof: V has a spanning list, which contains a basis, by Thm 2.

step 4: $\to B = ((1,2), \text{---}, (4,7), \text{---}))$ This is a basis.

Corollary: Every nontrivial f.d. v.s. has a basis.

proof: V has a spanning list, which contains a basis by Theorem 2.

<u>Theorem 3</u>: Every L.I. set in a f.d. v.s. V is a subset of a basis for B.

proof: Suppose $(v_1, \dots, v_m)$ is L.I. in f.d. v.s. V. Suppose $(w_1, \dots, w_n)$ is a spanning list

step 1: If $w_1 \in \text{span}(v_1, \dots, v_m)$ let $B = (v_1, \dots, v_m)$

If $w_j \notin$ " " let $B = (v_1, \dots, v_m, w_1)$

step 2: If $w_j \in \text{span}(B)$ leave B unchanged, otherwise add w_j to B to end of list.

After ea. step, B is still L.I. (since if it was L.D. then by lemma 1, one member in B is a linear combination of earlier terms in the list B, which is impossible by construction) after n steps we've added all the w's. That is $w_j \in \text{span}(B)$ $\forall j$. So $\text{Span}(B) \supseteq \text{span}(w_1, \dots, w_n) = V$, so B spans V. Thus B is a basis.

Friday June 6

Proposition 4: If U is a subspace of f.d. v.s. V, then $\exists$ subspace W of V with $U \oplus W = V$.

proof: Let $(u_1, u_2, \dots, u_n)$ be a basis for U. By Thm. 3, $\exists$ $u_{n+1}, u_{n+2}, \dots, u_m \in V$ s.t. $(u_1, u_2, u_3, \dots, u_n, u_{n+1}, \dots, u_m)$ is a basis for V. Let $W = \text{span}(u_{n+1}, \dots, u_m)$; any $v \in V$ can be written $v = \sum_{k=1}^{n} c_k u_k + \sum_{k=n+1}^{m} c_k u_k \in U + W$ so $V = U + W$.

By prop. 11, we only need to show $U \cap W = (0)$, but if $v \in U \cap W$ then $v = \sum_{k=1}^{n} c_k u_k$ since its in U and $v = \sum_{k=n+1}^{m} c_k u_k$ since $v \in W$, subtracting

$0 = \sum_{k=1}^{n} c_k u_k - \sum_{k=n+1}^{m} c_k u_k$, so $c_k = 0$ $\forall k$ so $v = 0$.

Theorem 4: Any two bases of a f.d. v.s. V have the same length.

proof:

Let B_1, B_2 be two bases, then B_1 is spanning and B_2 is spanning. B_2 is L.I. so length $(B_1) \geq$ length (B_2) by Theorem 1. Similarly, B_2 is spanning, B_1 is L.I. so the length $(B_2) \geq$ length (B_1).

Definition: Dimension $\dim(V)$ of a f.d. v.s. V is the length of any basis

examples: $\dim(\mathbb{F}^n) = n$

$\dim(M_{m,n}) = m \cdot n$

$\dim(\mathcal{P}_2(\mathbb{R})) = 3$

$\dim(\mathcal{P}_n(\mathbb{R})) = n+1$

~~Def.~~ Set $\dim(\{0\}) = 0$. (indeed $\dim(V) = 0 \Leftrightarrow V = \{0\}$)

Proposition 5: If U is a subspace of f.d. V, then $\dim(U) \leq \dim(V)$

proof: Redo the 1st line or so of Prop. 4, in the notation of that proof $\dim(U) = n \leq m = \dim(V)$.

Proposition 6: If $\dim(V) = n$, then any spanning list of length n is a basis.

proof: If S is a spanning list of length n, then by Thm. 2 $\exists$ subset $S' \subseteq S$ which is a basis. Since $\dim(V) = n$ length $(S') = n$. So $S = S'$ and this is a basis.

Proposition 7: If $\dim(V) = n$, then any L.I. list of length n is a basis.

proof: If S is a L.I. list of length n, then by Thm. 3, $\exists$ basis

$S' \supseteq S$. Since $\dim(V) = n$, length $(S') = n$, so $S' = S$ is a basis.

Theorem 5: If U_1 and U_2 are subspaces of f.d. V then
$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

proof: Let $(u_1, u_2, \ldots, u_m)$ be a basis for $U_1 \cap U_2$. By Thm. 3 $\exists$ $(v_1, v_2, \ldots, v_r)$ s.t. $(u_1, \ldots, u_m, v_1, \ldots, v_r)$ is a basis for U_1. Similarly by Thm 3, $\exists$ $(w_1, w_2, \ldots, w_s)$ s.t. $(u_1, \ldots, u_m, w_1, \ldots, w_s)$ is a basis for U_2.

Claim 1: $(u_1, \ldots, u_m, v_1, \ldots, v_r, w_1, \ldots, w_s)$ spans $U_1 + U_2$. To see this note a typical element in $U_1 + U_2$ is $x_1 + x_2$ where $x_1 \in U_1$, $x_2 \in U_2$, and $x_1 = \sum_{k=1}^{m} c_k u_k + \sum_{k=1}^{r} d_k v_k$ and $x_2 = \sum_{k=1}^{m} a_k v_k + \sum_{k=1}^{s} b_k w_k$, so $x_1 + x_2 = \sum_k c_k u_k + \sum d_k v_k +$

$\sum a_k u_k + \sum b_k w_k \in \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_r, w_1, \ldots, w_s)$

Claim 2: $(u_1, \ldots, u_m, v_1, \ldots, v_r, w_1, \ldots, w_s)$ is L.I. To see this, suppose $\sum_{k=1}^{m} c_k u_k + \sum_{k=1}^{r} d_k v_k + \sum_{k=1}^{s} b_k w_k = 0$. Then

$\sum b_k w_k = -\sum c_k u_k - \sum d_k v_k \in U_1$ and also is in U_2 since $w_k \in U_2$, So $\sum_{k=1}^{s} b_k w_k \in U_1 \cap U_2$, so we can write

$\sum_{k=1}^{s} b_k w_k = \sum_{k=1}^{m} a_k u_k$ so $\sum b_k w_k - \sum a_k u_k = 0$, so $b_k = 0$

$\forall k$ since the u's and w's form a basis for U_2.
So $\sum c_k u_k + \sum d_k v_k = 0$, so $c_k = 0 = d_k$ $\forall k$ since the u's and w's form a basis for U_1. This proves Claim 2.
By Claim 1 + Claim 2, $(u_1, \ldots, u_m, v_1, \ldots, v_r, w_1, \ldots, w_s)$ is a basis for $U_1 + U_2$. So $\dim(U_1 + U_2) = m + r + s =$
$(m+r) + (m+s) - m = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

Proposition 8: If $U_1, U_2, \ldots, U_m$ are subspaces of f.d. V with $V = U_1 + U_2 + \ldots + U_m$ then $V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$

if and only if $\dim V = \sum_{k=1}^{m} \dim(U_k)$

proof: $(\Longrightarrow)$ Homework Hint: Starts out the same way as below.

$(\Longleftarrow)$ ~~Assume m = 2, the general case is more complicated.~~

Let B_k be a basis for U_k, for $k = 1, \ldots, n$, and let $B = \bigcup_{k=1}^{m} B_k$. This B spans V (since any $v \in V$ is a sum of $u_k \in U_k$, and each u_k is a linear combination of elements in B_k and hence in B so B is a linear comb. of elements of B). Also B has length $= \sum_{k=1}^{m}$ length $(B_k) = \sum_{k=1}^{m} \dim(U_k) = \dim V$. By Proposition 6, B is a basis ~~so B linearly independent~~. We need to show that if $\sum_{k=1}^{m} u_k = \sum_{k=1}^{m} u_k'$ with $u_k, u_k' \in U_k \ \forall k = 1, \ldots, m$ then $u_k = u_k' \ \forall k = 1, \ldots, m$. To prove this let $m = 2$ for simplicity (the general case is identical) and suppose that $B_1 = (v_1, v_2, \ldots, v_r)$ and $B_2 = (w_1, w_2, \ldots, w_s)$. Write $u_1 = \sum_{i=1}^{r} c_i v_i$, $u_1' = \sum_{i=1}^{r} d_i v_i$, $u_2 = \sum_{j=1}^{s} a_j w_j$ and $u_2' = \sum_{j=1}^{s} b_j w_j$. Then $\sum_{i=1}^{r} c_i v_i + \sum_{j=1}^{s} a_j w_j = \sum_{i=1}^{r} d_i v_i + \sum_{j=1}^{s} b_j w_j$. Since $(v_1, \ldots, v_r, w_1, \ldots, w_s) = B$ is a basis for V, we deduce $c_i = d_i$, $a_j = b_j \ \forall i, j$. So $u_1 = u_1'$, $u_2 = u_2'$. $\square$

Ch. 3 Linear Transformations ~~(Linear maps, linear operators)~~

Def. A: In this chapter V, W are vector spaces. A function $T: V \to W$ is linear if

$$(1)\ T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall v_1, v_2 \in V$$

$$(2)\ T(cv) = cT(v) \quad \forall v \in V \quad c \in \mathbb{F}$$

Such T are also called linear transformations, linear operators, or linear maps. Note (1) $\Longrightarrow T(0) = 0$ $[T(0) = T(0+0) = T(0) + T(0) \Longrightarrow T(0) = 0]$

Write $\mathcal{L}(V, W)$ for set of all linear maps from V to W

" $\mathcal{L}(V) = \mathcal{L}(V, V)$

Example: The zero map $V \to W$ maps everything in V to W. We write this map as $0 \in \mathcal{L}(V, W)$