Definitions:

For a normed space X, write $\text{Ball}(X) := \{x \in X; ||x|| \leq 1\}$, and $\mathcal{U}_X := \{x \in X; ||X|| < 1\}$.

A function $f : X \to Y$, between topological spaces is continuous if $f^{-1}(U)$ is open for all open $U \subset Y$.

A function $f : X \to Y$, between topological spaces is an open map if $f(U)$ is open for all open $U \subset X$.

A homeomorphism or bicontinuous map is a bijective map which is also open, or f^{-1} is also continuous.

Prop: for a bijective map, $f : X \to Y$, f is an open map iff f^{-1} is continuous.

The Baire catagory theorem: If $(A_k)_{k\in\mathbb{N}}$ is a sequence of open dnse sets in a complete metric space X, then $\cap_{k\in\mathbb{N}} A_k$ is dense in X.

Proof: ADD

Corollary: A complete metric space X cannod be written as a countable union of closed sets, each of which have empty interior.

Proof: ADD

Lemma: If $T : X \to Y$ is a bounded operator between Banach spaces, X, Y, and if $r\mathcal{U}_Y \subset \overline{T(\mathcal{U}_X)}$, then $r\mathcal{U}_Y \subset T(U_X)$

Proof: ADD

Thoerem: (The open mapping theorem) If $T : X \to Y$ is a surjective bounded linear operator between Banach spaces X, Y, then T is open.

Proof: ADD

Corollary: If $T : X \to Y$ is an bijective, bounded linear operator between Banach spaces, then T is bicontinuous.

Proof: By the open mapping theorem, T is open, and hence bicontinuous.

The closed graph theorem: If $T : X \to Y$ is a linear operator between Banach spaces then TFAE;

i) T is bounded.

ii) The graph of T, namely $\mathcal{G}(T) := \{(x, T(x)); x \in X\}$, is closed in $X \times Y$, wrt to the product topology.

iii) Whenever $(X_k)_{k\in\mathbb{N}} \subset X$, with $X_k \to x \in X$, and $T(x_k) \to u \in Y$, then $y = T(x)$.

iv) Whenever $(X_k)_{k\in\mathbb{N}} \subset X$, with $X_k \to 0$, and $T(x_k) \to y \in Y$, then $y = 0$.

Theorem: (the principle of uniform boundedness(PUB)). Suppose that X, Y are Banach spaces, and $S \subset B(X, Y)$. Suppose that for every $x \in X$, $\{Tx;\ T \in S\}$ is bounded in Y. Then there is a constant M with $||T|| \leq M$ for all $T \in S$.

Proof: Let $E_n = \{x \in X;\ \sup(\{||Tx||; T \in S\}) \leq n\} = \cap_{T \in S}\{x \in X; ||Tx|| \leq n\}$. By the Baire catagory teorem, there must exist an n suc that E_n has an interior point x_0, say. Thus there exists $r > 0$ with $\overline{B(x_0, r)} \subset E_n$, beucase if $||x|| \leq r$, then $x + x_0 \in \overline{B(x_0, r)} \subset E_n$, so $||T|| \leq ||T(x + x_0)|| + ||Tx_0|| \leq 2n$ for any $T \in S$. If $||x|| \leq 1$ then $||rx|| \leq r$, so that $||T(rx)|| = r||Tx|| \leq 2n$, and $||Tx|| \leq 2n/r$. Thus $||T|| \leq 2n/r$. So $\sup(\{||T||;\ T \in S\}) \leq 2n/r$.

Definition: An orthonormal set is a subset $\{x_j;\ j \in J\}$ of a Hilbert space, if $\langle x_i, x_j \rangle = \delta_{i,j}$ for all $i, j \in J$.

If X is an orthonormal set and J is finite, then for all scalars $(c_k)_{k\in J} \in \mathbb{F}$,

$$||\sum_{k\in J} c_k b_k||^2 = \left\langle \sum_{j\in J} c_k x_k, \sum_{j\in J} c_k x_k \right\rangle = \sum_{i,j\in J} c_i c_j^* \langle x_i, x_j \rangle = \sum_{k\in J} |c_k|^2$$

Lemma: any orthonormal set is linearly independent.

Proof: Let $X = \{x_j;\ j \in J\}$ is an orthonormal set, $(c_k)_{k\in J} \in \mathbb{F}$. If $0 = \sum_{k\in J} c_k x_k$, then $\langle \sum_{k\in J} c_k x_k, x_j \rangle = c_j$.