Real Analysis

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CHAPTER 1

Sets, Functions and the real numbers

1.1. Introduction

In this chapter we start by reviewing some of the basic definitions and properties of sets and functions. Although much of this material should be familiar, notations vary and readers should at least skim through the sections on sets and functions so as to get familiar with the notational conventions we use throughout the notes. We conclude the chapter with leisurely but careful discussion of the real number system. Roughly speaking, we will think of a real number as defined by its decimal approximations.

We assume some familiarity with *proof by induction* and *recursive* or *inductive definitions*. We briefly recall the ideas; first, proof by induction. If for each natural number n, we are given a statement $P(n)$, then $P(n)$ will be true for all n if $P(1)$ is true and the truth of $P(n)$ implies the truth of $P(n+1)$ for all $n \geq 1$. For a recursive or inductive definition, we aim to define definitions or mathematical objects $S(n)$ for $n \geq 1$. We can do this if $S(1)$ is given, and $S(n+1)$ is uniquely determined by $S(n)$ for all $n \geq 1$. We most often use recursive definitions to define sequences. For example $x_1 = 1$, $x_{n+1} = \frac{1}{2}(1 + \frac{2}{x_n})$, $n \geq 1$. Typically the rule used to define x_{n+1} in terms of x_n may be more complicated, involve logical statements and not be given in terms of a simple mathematical formula.

1.2. Sets

Roughly speaking a set is a collection of 'objects'. Each object in the set is regarded as a *member* of the set. If we have a set X and x is an object, then we say x is a member of X if x is one of the objects comprising X. We write this symbolically as "$x \in X$".

Examples 1.2.1. (1) let $X = \{1, 2, 3\}$ be the set with the members $1, 2, 3$. We have $1 \in X$, $2 \in X$, $3 \in X$, $4 \notin X$, where the last notation means that '4 is not a member of X'.
(2) Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of strictly positive integers — the natural numbers. Note the use of the dots to signify that $\mathbb{N}$ consists

of all positive integers. We have $10 \in \mathbb{N}$ but $-1, 0, \frac{1}{2} \notin \mathbb{N}$. $\frac{1}{2} \notin \mathbb{N}$. We also let $\mathbb{N}^\circ = \{0, 1, 2, \ldots\}$ denote the set of positive integers (including zero).
(3) Let $\mathbb{Z}$ denote set of integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.
(4) Let $\mathbb{Q} = \{\frac{r}{s} \mid r, s \in \mathbb{Z}, s \neq 0\}$ denote the set of all rational numbers. We usually assume $s > 0$ and $(r, s) = 1$ (the notation $(r, s) = 1$ signifies that r, s have no common factors).
(5) We let $\mathbb{R}$ denote the set of all real numbers. For the present, we will be imprecise about the the exact nature of the members of $\mathbb{R}$. However, if $x \in \mathbb{N}$ or $x \in \mathbb{Z}$, then $x \in \mathbb{R}$.
(6) Let $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. We refer to $[0, 1]$ as the 'closed unit interval.' Observe the logical definition of $[0, 1]$: we impose a condition — $0 \leq x \leq 1$ — on the set of real numbers. The logical condition follows the $\mid$ symbol. In words: $[0, 1]$ is the set of real numbers satisfying the condition $0 \leq x \leq 1$.

Let $\emptyset$ denote the *empty set*. The empty set is the set with no members.

EXAMPLE 1.2.2. Consider the sets $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$. The second set $\{\emptyset\}$ is not empty: it has one member, the empty set $\emptyset$. Similarly, the third set has one member: $\{\emptyset\}$. Finally the fourth set has two members, the empty set and the set $\{\emptyset\}$.

1.2.1. Subsets. Let A, B be sets. We say that A is a *subset* of B, written symbolically as $A \subset B$, if every member of A is a member of B. In terms of the implication symbol $\Longrightarrow$, $A \subset B$ if

$$a \in A \Longrightarrow a \in B.$$

Note that in this definition we allow $A = B$. If $A \subset B$ but $A \neq B$ we write $A \subsetneq B$ and refer to A as a *proper* subset of B. If X is not a subset of B we write $A \not\subset B$. We also allow the notation $A \supset B$ which means that B is a subset of A (or A is a *superset* of B).

EXAMPLE 1.2.3.

$$[0, 1] \subset \mathbb{R}, \quad [0, 1] \not\subset \mathbb{N}, \quad \mathbb{N} \subset \mathbb{N}^\circ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

1.2.2. Operations on sets.
Unions and intersections. Let $A, B, \ldots$ be sets. The *union* $A \cup B$ of A and B is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

We observe that $A \subset A \cup B$, $B \subset A \cup B$ and $A \cup B = B \cup A$.
The *intersection* $A \cap B$ of A and B is defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

We have $A \cap B \subset A, B$ and $A \cap B = B \cap A$.

EXAMPLE 1.2.4. $A \cup A = A \cap A = A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Later we need to look at unions and intersections of families of sets. Suppose then that we are given a collection $\{A_i\}$ of sets indexed by by the (non empty) set I. That is $\{A_i \mid i \in I\}$ (or, in abbreviated form $\{A_i\}_{i\in I}$). If the index set $I = \mathbb{N}$, then we have a *sequence* of sets $A_1, A_2, \ldots$. We define unions and intersections for the family $\{A_i\}_{i\in I}$ by

$$\begin{aligned} \cup_{i\in I} A_i &= \{x \mid \exists i \in I \text{ with } x \in A_i\}, \\ \cap_{i\in I} A_i &= \{x \mid x \in A_i \ \forall i \in I\}. \end{aligned}$$

(Here we have made use of the shorthand symbols $\exists$ ('there exists') and $\forall$ ('for all')).

EXAMPLES 1.2.5. (1) $\cup_{n\in\mathbb{N}} \{n\} = \mathbb{N}$, $\cup_{n\in\mathbb{N}^\circ} \{\pm n\} = \mathbb{Z}$.
(2) For $x \in \mathbb{R}$, define $\delta_x = \frac{1}{1+|x|}$, $A_x = [x-\delta_x, x+\delta_x]$. Then $\cup_{x\in\mathbb{R}} A_x = \mathbb{R}$, $\cap_{x\in\mathbb{R}} A_x = \emptyset$. If instead we take $\delta_x = |x|$, $A_x = [x - \delta_x, x + \delta_x]$, then $\cup_{x\in\mathbb{R}} A_x = \mathbb{R}$, $\cap_{x\in\mathbb{R}} A_x = \{0\}$.

Complements. Fix a non empty set X. If A is a subset of X, we define the *complement* $X \setminus A$ of A (in X) by

$$X \setminus A = \{x \in X \mid x \notin A\}.$$

REMARK 1.2.6. There are several other notations commonly in use for the complement $X \setminus A$. For example, $X - A$, A' and A^c.

EXAMPLE 1.2.7. We have $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$.

LEMMA 1.2.8. *For all subsets A of X we have*

$$X \setminus (X \setminus A) = A.$$

PROOF. If $x \in A$ then $x \notin X \setminus A$. Hence x lies in the complement of $X \setminus A$. That is, $x \in X \setminus (X \setminus A)$. We have shown that $A \subset X \setminus (X \setminus A)$. Reversing the argument shows that $X \setminus (X \setminus A) \subset A$. Hence $X \setminus (X \setminus A) = A$. $\square$

The next result will prove useful when we begin our investigation of open and closed sets of metric spaces.

PROPOSITION 1.2.9. *Let $\{X_i\}_{i\in I}$ be a family of subsets of X. We have*

1. $X \setminus \cap_{i\in I} X_i = \cup_{i\in I}(X \setminus X_i)$.
2. $X \setminus \cup_{i\in I} X_i = \cap_{i\in I}(X \setminus X_i)$.

PROOF. The proof is left to the exercises. □

EXAMPLE 1.2.10. We define a subset A of $\mathbb{R}$ to be of type $\mathbf{C}$ if it is either finite, or empty or equal to $\mathbb{R}$. We say that a subset of $\mathbb{R}$ is of type $\mathbf{O}$ if it is the complement of a subset of type $\mathbf{C}$. Since $\mathbb{R}$ and $\emptyset$ are of type $\mathbf{C}$ it follows (by taking complements) that $\mathbb{R}$ and $\emptyset$ are are of type $\mathbf{O}$. These are the only subsets of $\mathbb{R}$ that are of type $\mathbf{C}$ *and* type $\mathbf{O}$. Indeed, all other subsets of type $\mathbf{O}$ are the complement of a finite set and so must be infinite. It follows from proposition 1.2.9 that the intersection (respectively, union) of any collection of sets of type $\mathbf{C}$ (respectively, type $\mathbf{O}$) is of type $\mathbf{C}$ (respectively, type $\mathbf{O}$). On the other hand, in general only *finite* unions (respectively, intersections) of type $\mathbf{C}$ (respectively, type $\mathbf{O}$) are of type $\mathbf{C}$ (respectively, type $\mathbf{O}$).

The power set. Let X be a set. We define the *power set of* X, $P(X)$, to be the set of all subsets of X.

EXAMPLES 1.2.11. (1) $P(\emptyset) = \{\emptyset\} \neq \emptyset$.
(2) If $X = \{1,2\}$, then $P(X) = \{\{1\},\{2\},\{1,2\},\emptyset\}$.

Note that $P(X)$ always contains $\emptyset$ and X. Hence, provided $X \neq \emptyset$, $P(X)$ must contain at least two members. It is easy to see that if X is finite and contains N members, then $P(X)$ contains exactly 2^N members.

Products of sets. Let X, Y be non empty sets. We define the (Cartesian) product $X \times Y$ to be the set of ordered pairs of elements of X and Y. That is,

$$X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}.$$

It is straightforward to extend this definition to finite products or products over an arbitrary indexing set. For example, if $X_1, \ldots, X_N$ are non empty sets we define

$$\Pi_{n=1}^N X_i = X_1 \times \ldots \times X_N = \{(x_1,\ldots,x_N) \mid x_i \in X_i, 1 \leq i \leq N\}.$$

REMARK 1.2.12. Provided that X and Y are non empty, $X \times Y \neq \emptyset$. This depends on the observation that if X and Y are non empty, then we can pick at least one element x_0 from X, and one element y_0 from Y. Hence, $(x_0,y_0) \in X \times Y$ and $X \times Y \neq \emptyset$. This argument becomes a little dangerous if we form a product over an *arbitrary* indexing set $\Pi_{i\in I} X_i$. Tacitly, we have to assume that given an arbitrary collection of sets, we can make a choice of one element from each set. This assumption is usually called the "Axiom of Choice". We will not go further into this matter here except to remark that to avoid contradictions and absurdities, it became necessary fairly early on in the development of set theory to give careful statements of exactly what operations could be performed on sets and give precise rules as to what objects can constitute sets. For example, it is prudent not to allow the collection of all sets to be a set. For more information, we refer the reader to one of the many books on the foundations of set theory.

EXERCISES 1.2.13.

(1) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(2) Complete the proof of proposition 1.2.9. (To show that $X \setminus \cap_{i \in I} X_i = \cup_{i \in I}(X \setminus X_i)$, prove that the left hand side is a subset of the right hand side and conversely.)
(3) Let A, B, C be subsets of X. Define the *symmetric difference* $A \triangle B$ by
$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$
Complete the sentence '$x \in A \triangle B$ iff $x \in A$ and ... or ... and ... $\notin$ ...'. Prove
 (a) $A \triangle B = \emptyset$ iff $A = B$.
 (b) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

1.3. Functions

Let X, Y be non empty sets. A function, or map, f from X to Y assigns to each $x \in X$, a unique point $f(x)$ in Y. We denote this assignment symbolically by $f : X \to Y$ ("f maps X to Y"). We call $f(x)$ the *value* of f at x. Every function $f : X \to Y$ has a *graph* $\Gamma_f \subset X \times Y$ defined by
$$\Gamma_f = \{(x, f(x)) \mid x \in X\}.$$
Conversely, if $G \subset X \times Y$ has the property that for every $x \in X$, there exists a unique point $y \in Y$ such that $(x, y) \in G$, then $G = \Gamma_f$, where the value of f at x is y — the unique point in Y such that $(x, y) \in G$.

If $f : X \to Y$, then the *range* or *image* of f is the subset $f(X)$ of Y defined by
$$f(X) = \{f(x) \mid x \in X\}.$$
More generally, if $A \subset X$, then $f(A)$ is the subset of Y defined by $f(A) = \{f(a) \mid a \in A\}$. inverse image (by f) of B.

EXAMPLE 1.3.1. If $B \subset Y \setminus f(X)$, then $f^{-1}(B) = \emptyset$. The converse also holds.

If $f : X \to Y$, $g : Y \to Z$, then the *composite* gf of f and g is the map $gf : X \to Z$ defined by
$$(gf)(x) = g(f(x)), \ (x \in X).$$

REMARK 1.3.2. The composite gf of f and g is not the multiplicative product of f and g. Of course, if (say) $f, g : X \to \mathbb{R}$, we can form the multiplicative product $f \times g$, defined by $(f \times g)(x) = f(x)g(x)$. As it is natural to abbreviate $f \times g$ as fg (especially if $f, g : \mathbb{R} \to \mathbb{R}$), it is sometimes useful to use a notation like $g \circ f$ for composites so as to make it clear that we are not dealing with $f \times g$.

EXAMPLE 1.3.3. If $C \subset Z$, then $(gf)^{-1}(C) = f^{-1}(g^{-1}(C))$ — note the reverse order. The proof is left to the exercises.

DEFINITION 1.3.4. Let $f : X \to Y$.

(1) f is *onto* (or *surjective*) if $f(X) = Y$.
(2) f is *1:1* (or *injective*) if $f(x) = f(x')$ iff $x = x'$.
(3) f is *1:1 onto* (or *bijective*) if f is 1:1 and onto.

If $f : X \to Y$ is a bijection, then we may define the *inverse map* $f^{-1} : Y \to X$ by defining the value of f^{-1} at $y \in Y$, to be the unique point $f^{-1}(y) \in X$ such that $f(f^{-1}(y)) = y$. Since f is onto, there always exists at least one point $x \in X$ such that $f(x) = y$. Since f is 1:1, the point x is unique. We define $f^{-1}(y) = x$.

REMARK 1.3.5. The reader should be aware of the ambiguity caused by using f^{-1} both for inverses of sets and inverse maps. In particular, if $f : X \to Y$, and $b \in Y$, then $f^{-1}(\{b\})$ is a *subset* of X. If f has an inverse, then $f^{-1}(b)$ is a *point* of X. In practice, one writes $f^{-1}(b)$ to cover *both* situations. That is, $f^{-1}(b) = \{x \in X \mid f(x) = b\}$. If f is a bijection, we *identify* the point $f^{-1}(b) \in X$ with the subset $\{f^{-1}(b)\}$ of X.

EXAMPLE 1.3.6. $f : X \to Y$, $g : Y \to Z$ are bijections, then $gf : X \to Y$ is a bijection and $(gf)^{-1} = f^{-1}g^{-1}$.

1.3.1. Equivalence of sets.

DEFINITION 1.3.7. The sets X, Y are *equivalent* if there exists a bijection $f : X \to Y$. We write this symbolically as "$X \sim Y$".

REMARK 1.3.8. We note the following properties of equivalence: $X \sim X$, $X \sim Y \implies Y \sim X$ and $X \sim Y$, $Y \sim Z \implies X \sim Z$ (transitivity).

We start with a general result on inequivalence that has an intriguing proof discovered by the creator of set theory: Cantor.

PROPOSITION 1.3.9. *Let X be a set. Then $X \not\sim P(X)$.*

The interest in this result lies in the case when X is not finite (see below). The result is trivially true if $X = \emptyset$ ($P(\emptyset) = \{\emptyset\} \neq \emptyset$) so we assume $X \neq \emptyset$.

PROOF. We shall prove that there does not exist a surjection from X to $P(X)$. Our proof goes by contradiction. Suppose that $f : X \to P(X)$ is surjective. Since $f(x)$ is a subset of X for all $x \in X$, we may define the following subset B of X:

$$B = \{x \in X \mid x \notin f(x)\}.$$

Observe that the definition makes sense even if $f(x) = \emptyset$ — a possibility since $\emptyset \in P(X)$. Since we assume f is onto, $\exists b \in X$ such that $f(b) = B$. There are exactly two possibilities: $b \in B$, or $b \notin B$. If $b \in B$ then, by definition of B, $b \notin f(b) = B$. Contradiction. Similarly, if $b \notin B = f(b)$, then $b \in B$, by definition of B. Contradiction. Either assumption, leads to an absurd conclusion and therefore our original assumption that f is onto must be false. $\square$

1.4. Finite and countable sets

If $n \in \mathbb{N}$, we let $\mathbf{n} = \{1, \ldots, n\}$.

DEFINITION 1.4.1. A non empty set X is *finite* if there exists $n \in \mathbb{N}$ such that

$$X \sim \mathbf{n}.$$

A non empty set is *infinite* if it is not finite.

We need to check that the n in our definition of finite is uniquely determined by X. For this we need a preliminary result.

LEMMA 1.4.2. *Let X, Y be equivalent sets containing at least two members. If $x_0 \in X$ and $y_0 \in Y$, then $X \setminus \{x_0\} \sim Y \setminus \{y_0\}$.*

PROOF. Let $f : X \to Y$ be a bijection. If $f(x_0) = y_0$, then f restricts to a bijection $f : X \setminus \{x_0\} \to Y \setminus \{y_0\}$ and so $X \setminus \{x_0\} \sim Y \setminus \{y_0\}$. If $f(x_0) \neq y_0$, we may choose $z_0 \in Y$, $z_0 \neq y_0$ (since Y contains at least two members). Now define $g : Y \to Y$ by $g(y) = y$ if $y \notin \{y_0, z_0\}$, $g(z_0) = y_0$, $g(y_0) = z_0$. The composite $gf : X \to Y$ is a bijection and $(gf)(x_0) = y_0$. We proceed as before. $\square$

LEMMA 1.4.3. *Let $n, m \in \mathbb{N}$. Then $\mathbf{n} \sim \mathbf{m}$ iff $n = m$.*

PROOF. Obviously, if $n = m$, we have $\mathbf{n} \sim \mathbf{m}$. So suppose $\mathbf{n} \sim \mathbf{m}$. The result is trivial if $n = 1$ or $m = 1$. So suppose $n, m > 1$. Without loss of generality we may suppose $n \leq m$. Apply the previous lemma $n-1$ times to get $\mathbf{1} \sim \mathbf{m} - \mathbf{n} + \mathbf{1}$. It follows that $1 = m - n + 1$. That is, $m = n$. $\square$

As an immediate corollary of lemma 1.4.3 we have

COROLLARY 1.4.4. *If X is finite and non empty there exists a unique $n \in \mathbb{N}$ such that $X \sim \mathbf{n}$. The integer n is called the* cardinality *of X.*

EXAMPLE 1.4.5. If the cardinality of X is n, then the cardinality of $P(X)$ is 2^n. Since $2^n > n$, for all $n \in \mathbb{N}$, we see that the cardinality of the power set of a finite set is always greater than the cardinality of the set (this holds true if $X = \emptyset$ as $P(\emptyset) = \{\emptyset\}$.)

DEFINITION 1.4.6. A set X is *countable* if either X is finite or $X \sim \mathbb{N}$. If $X \sim \mathbb{N}$ we sometimes say X is countably infinite.

EXAMPLES 1.4.7. (1) The set $\mathbb{Z}$ of all integers is countable. For this it is enough to note that the map $f : \mathbb{N} \to \mathbb{Z}$. defined by

$$\begin{aligned} f(n) &= n/2, \text{ if } n \text{ is even,} \\ &= -(n-1)/2, ; \text{ if } n \text{ is odd.} \end{aligned}$$

is a bijection.
(2) The set of prime numbers is countably infinite.
(3) Not all sets are countable. For example, $P(\mathbb{N}) \not\sim \mathbb{N}$ (proposition 1.3.9) and so, since $P(\mathbb{N})$ is infinite ($P(\mathbb{N}) \supset \{\{1\}, \{2\}, \ldots\}$), $P(\mathbb{N})$ cannot be countable.

PROPOSITION 1.4.8. *A subset of a countable set is countable.*

PROOF. Let Y be a subset of the countable set X. The result is immediate if X is finite so we shall assume that X is countably infinite. We may write $X = \{x_1, x_2, \ldots\}$. More precisely, there exists a bijection $f : \mathbb{N} \to X$, and so we may define $x_n = f(n)$, $n \in \mathbb{N}$. Let $n_1 \geq 1$ be the smallest integer such that $x_{n_1} \in Y$. Assume we have defined $n_1 < n_2 < \ldots < n_k$ such that $x_{n_j} \in Y$, $1 \leq j \leq k$ and $Y \cap \{x_1, \ldots, x_{n_k}\} = \{x_{n_1}, \ldots, x_{n_k}\}$. Then either $Y \cap \{x_1, \ldots, x_{n_k}\} = Y$ and so Y is finite or there exists a smallest $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in Y$. It follows that either Y is finite (the process terminates) or we may write $Y = \{x_{n_1}, x_{n_2}, \ldots\}$ and so Y is countable (a bijection $g : \mathbb{N} \to Y$ is defined by $g(k) = n_k$, $k \geq 1$). □

THEOREM 1.4.9. (1) *A countable union of countable sets is countable.*
(2) *A finite product of countable sets is countable.*

PROOF. (1) Let $\{A_i\}_{i \in I}$ be a countable family of countable sets. We assume that I is infinite (the case when I is finite is easier). Since $I \sim \mathbb{N}$, it is enough to consider the union $\cup_{n \in \mathbb{N}} A_n$, where the sets A_n are all countable. For each $n \in \mathbb{N}$, we may choose a sequence

$(a_{ni})_{i\geq 1}$ of elements of A_n such that $A_n = \cup_{i\in\mathbb{N}}\{a_{ni}\}$. Indeed, if $A_n = \{A_1, \ldots, A_N\}$ is finite, we define $a_{ni} = A_i$, $i < N$, $a_{ni} = A_N$, $i \geq N$. If A_n is infinite, we label the elements of A_n by the integers in the usual way: $A_n = \{a_{n1}, a_{n2}, \ldots\}$. We define the infinite array $\mathcal{A}$ by

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

Let us start by assuming that all the elements of the array $\mathcal{A}$ are different. That is, $a_{ij} = a_{i'j'}$ iff $i = i'$, $j = j'$ (this can occur only if $A_n \cap A_m = \emptyset$, $n \neq m$, and each A_n is countably infinite). We give an inductive definition for a bijection $F : \mathbb{N} \to \mathcal{A}$. Specifically, we define $F(1) = a_{11}$. Suppose we have defined $F(1), \ldots, F(n-1)$, $n > 1$. We define $F(n)$. Suppose $F(n-1) = a_{ij}$. If $i = 1$, we define $F(n) = a_{j+1\,1}$, otherwise take $F(n) = a_{i-1\,j+1}$. This complete the inductive definition of F. The reader should check the path traced out by $F(n)$ through $\mathcal{A}$ as we increase n $(a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, \ldots\})$. Obviously F is 1:1 onto and so defines an equivalence $\mathbb{N} \sim \mathcal{A}$. Finally, we allow for $A_n \cap A_m \neq \emptyset$, by modifying the definition of F so that we only take $a_{ij} = F(n)$ if $a_{ij} \notin \{F(1), \ldots, F(n-1)\}$.

(2) We prove by induction on the number of terms in the product. We start by considering the product $A \times B$ of two countably infinite sets A, B (the case where one of the sets is finite is easier). Write $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$ and define the array $\mathcal{A}$ whose ijth entry is (a_i, b_j). In this case the elements of $\mathcal{A}$ are all distinct ($(a, b) = (a', b')$ iff $a = a'$, $b = b'$) and obviously $\mathcal{A} \sim A \times B$. Hence the method of (1) shows that $A \times B$ is countable with no issues of duplication. Suppose we have proved the result for $2, \ldots, n-1$ factors. Consider the product $\Pi_{i=1}^n A_i$ of the countable sets A_i. We have

$$\begin{aligned} \Pi_{i=1}^n A_i &\sim (\Pi_{i=1}^{n-1} A_i) \times A_n, \\ &\sim \mathbb{N} \times \mathbb{N}, \\ &\sim \mathbb{N}, \end{aligned}$$

where the first line comes from the trivial bijection $(a_1, \ldots, a_n) \mapsto ((a_1, \ldots, a_{n-1}), a_n)$, the second line from the result for $n-1$ and the final line from the special case $n = 2$. $\square$

EXAMPLES 1.4.10. (1) For $m \geq 1$, $\mathbb{N}^m$, $\mathbb{Z}^m$ are countable.

(2) The set $\mathbb{Q}$ of rational numbers is countable. Every element of $\mathbb{Q}$ can be represented uniquely in the form r/s, where $s > 0$ and $(r, s) = 1$ (we write $0 = \frac{0}{1}$) . Hence we may represent $\mathbb{Q}$ as a subset of $\mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$.

Apply proposition 1.4.8.
(3) Let $A = \{a \in \mathbb{R} \mid \exists n, p_0 \neq 0, \ldots, p_n \in \mathbb{Z} \text{ with } p_0 a^n + \ldots p_{n-1}a + p_n = 0\}$. Then A — the set of algebraic numbers — is countable. This is a simple consequence of theorem 1.4.9 and we leave details to the reader as an exercise. In particular, $\{m^{1/n} \mid m, n \geq 1\} \subset A$ is countable.

EXERCISES 1.4.11.

(1) Prove that the following sets are countably infinite.
   (a) The set of positive odd integers.
   (b) The set of prime numbers.
   (c) The subset A of $\mathbb{R}$ defined by $a \in A$ iff there exist $p_1, \ldots, p_{n-1} \in \mathbb{Q}$ such that $a^n + p_1 a^{n-1} + \ldots + p_{n-1}a + \sqrt{2} = 0$.

(For (a) you should construct a bijection between the set and $\mathbb{N}$. For (b) you need to verify that the set can be represented as an *infinite* subset of a (known) countable set.)

1.5. The real numbers

1.5.1. Not all real numbers are rational. The original formulation of geometry by the Pythagorean school was based on ideas of proportion and tacitly assumed that all numbers were rational[1]. An advantage of this approach was that numbers could, in theory, all be constructed geometrically usually ruler and compass. It came as a shock to the Pythagorean school when it was discovered that some numbers that arose geometrically were not rational. The easiest example comes from Pythagoras's theorem: the hypotenuse of an isosceles right angle triangle with side length 1 is $\sqrt{2} \notin \mathbb{Q}$. In most cases, the square root of a positive integer is not rational. Indeed, the only time it is rational is when the integer is the square of another integer.

PROPOSITION 1.5.1. *Let $n \in \mathbb{N}$. Then $\sqrt{n} \in \mathbb{Q}$ iff $\sqrt{n} \in \mathbb{N}$. That is, the set of natural numbers with rational square root is precisely $\{1^2 = 1, 2^2 = 4, 3^2 = 9, \ldots\}$.*

PROOF. We prove a special case of this result and leave the general case (and extensions) to the exercises. We show that if $p > 1$ is prime then $\sqrt{p} \notin \mathbb{Q}$. Our proof goes by contradiction. Suppose that $\sqrt{p} \in \mathbb{Q}$, then we may write $\sqrt{p} = \frac{r}{s}$, where $r, s \in \mathbb{N}$ and $(r, s) = 1$ (recall that $(r, s) = 1$ means no common factors — the unique factorization of an integer into a product of primes that allows for this representation is used again in our proof). Since we assume $\sqrt{p} = \frac{r}{s}$ we have, on squaring

[1]Strictly speaking, strictly positive numbers; the concepts of negative and zero numbers were developed later in Indian and Arabian mathematics.

and multiplying by s^2.

$$ps^2 = r^2.$$

It follows that p is a factor of r^2 and so, since $p > 1$ is prime, p must be a factor of r (use the prime factorization of r). Hence we may write $r = pR$, where $R \in \mathbb{N}$. Substituting for r, we get $ps^2 = p^2R^2$ and so, after canceling p,

$$s^2 = pR^2.$$

Just as before, it follows that p is a factor of s. But we have shown that p is a factor of both r and s. This contradicts our assumption that $(r, s) = 1$. Hence $\sqrt{p}$ cannot be rational. $\square$

REMARK 1.5.2. As remarked above, the discovery that mathematics could not be done within the (countable) framework of rational numbers was of profound significance. It is no coincidence that numbers that are not rational are called *irrational* or that there is the word play between *surd* (root of number) and *absurd*. Irrational numbers cannot be expressed in finite terms — indeed, most irrational numbers correspond (in a sense that can be made very precise) to an infinite sequence of random numbers and so cannot be represented in any finite form. Allowing irrational numbers means the acceptance that randomness can and does play a pivotal role in mathematics — even in a precise and quantitative subject like real analysis.

1.5.2. The real numbers. In these notes we shall think of the real numbers as the set of all *signed decimal expansions*.

Formally, we start by defining

$$\mathbb{R} = \{\pm x_0.x_1x_2\ldots \mid x_0 \in \mathbb{N}^\circ;\ x_i \in \{0, 1, \ldots, 9\},\ i > 0\}.$$

Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x = +x_0.x_1x_2\ldots\}$ (the positive real numbers), and $\mathbb{R}^- = \{x \in \mathbb{R} \mid x = -x_0.x_1x_2\ldots\}$ (the negative real numbers). So as to simplify notation, we almost always drop the $+$ from the definition of $\mathbb{R}^+$ and just write $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x = x_0.x_1x_2\ldots\}$. We also identify the zero expansions $+0.000\ldots$ and $-0.000\ldots$. With this convention, $\mathbb{R}^+ \cap \mathbb{R}^- = \{0.000\ldots\}$. We usually set $\pm 0.000\ldots = 0$ ("zero").

In order to progress further, we need to relate this abstract definition of numbers in terms of sequences to the 'known' rational numbers. What we shall show is that the rational numbers $\mathbb{Q}$ can be naturally identified as a subset of $\mathbb{R}$. As part of our investigation we shall see that we need to be more restrictive in our definition of $\mathbb{R}$ as a set of decimal expansions. Basically we sometimes need to view different decimal expansions as defining the same real number. We have already seen this in case of the zero expansions.

A point $x \in \mathbb{R}$ has a *terminating* or finite decimal expansion if there exists a least $N \geq -1$ such that $x_n = 0$ all $n > N$. For simplicity start by assuming x is positive. Then we may write $x = x_0.x_1 \dots x_N 000\dots$. We usually abbreviate this to $x = x_0.x_1 \dots x_N\overline{0}$. Use of this convention always signifies that $x_N \neq 0$ (except in the case when $x = 0$, when we write $0.\overline{0}$). Even simpler, we may write $x = x_0.x_1 \dots x_N$ (though in this case we will not always insist that $X_N \neq 0$). Exactly the same conventions hold for negative real numbers except that now we write $x = -x_0.x_1 \dots x_N\overline{0} = -x_0.x_1 \dots x_N$.

The terminating decimal $x = x_0.x_1 \dots x_N$ corresponds to the rational number

$$\sum_{i=0}^{N} \frac{x_i}{10^i}.$$

Similarly, $-x_0.x_1 \dots x_N$ corresponds to $-\sum_{i=0}^{N} \frac{x_i}{10^i} \in \mathbb{Q}$. If $X_N \neq 0$, then $x = x_0.x_1 \dots x_N$ may also be written as

$$x = x_0.x_1 \dots x_{N-1}(x_N - 1)\overline{9}.$$

This follows since $\sum_{i=N+1}^{\infty} \frac{9}{10^i} = 10^{-N}$. Notice this is arithmetic within $\mathbb{Q}$ — if $a, r \in \mathbb{Q}$, $0 \leq |r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to the *rational* number $a/(1-r)$. Sometimes it will useful to restrict to those decimal expansions that do not end in recurring 9's (alternatively, expansions not ending in recurring 0's). That is, if x can be written as a terminating decimal or as an infinite decimal ending with $\overline{9}$, then we can elect to write x as a terminating decimal. If $x \in \mathbb{Q}$ then either x has a terminating decimal expansion or the decimal expansion of x is eventually periodic. For example,

$$\frac{3}{34} = 0.088235294117647058823529 41\dots = 0.0\overline{8823529411764705}$$

Conversely, it is not hard to show that if the decimal expansion of x is eventually periodic, then $x \in \mathbb{Q}$ (we leave this to the exercises). In brief, decimal expansions of rational numbers are either terminating or they are eventually periodic. In either case, only finitely many integers are needed to specify them exactly.

Summarizing, if we define $\mathbb{R}$ to consist of all signed decimal expansions which do not end in recurring 9's, then there is an natural injection $f : \mathbb{Q} \to \mathbb{R}$ with image the set of decimal expansions that are eventually periodic or end with recurring 0's. In particular, if $x \notin \mathbb{R} \setminus f(\mathbb{Q})$, then $\sum_{i=0}^{\infty} \frac{x_i}{10^i}$ will not sum to a rational number. That is, there will not exist $q \in \mathbb{Q}$ such that $\lim_{n\to\infty} |\sum_{i=0}^{n} \frac{x_i}{10^i} - q| = 0$. (This statement makes sense even if we cannot yet give meaning to the *infinite* sum $\sum_{i=0}^{\infty} \frac{x_i}{10^i}$.)

Now that we have at least a minimal description of the real numbers it is easy to prove Cantor's result that the set of real numbers is not countable.

THEOREM 1.5.3. *The set* $\mathbb{R}$ *is not countable.*

PROOF. We give the proof discovered by Cantor and based on his diagonal method. If $\mathbb{R}$ is countable then certainly the half-open interval $[0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ is countable (every subset of a countable set is countable). It is therefore enough to show that $[0,1)$ is not countable. Suppose the contrary. Then we may write $[0,1) = \{x_1, x_2, \ldots\}$. Each $x_n \in \{x_1, x_2, \ldots\}$ has a *unique* decimal expansion

$$x_n = 0.x_{n1}x_{n2}\ldots x_{nn}\ldots,$$

not ending in recurring 9's. We define $z = 0.z_1z_2\ldots$ by

$$\begin{aligned} z_n &= 4, \text{ if } x_{nn} = 5, \\ &= 5, \text{ if } x_{nn} \neq 5 \end{aligned}$$

Clearly $z \in [0,1)$ (the decimal expansion of z does not end in recurring 9's). On the other hand, $z \notin \{x_1, x_2, \ldots\}$ since $z_n \neq x_{nn}$, all $n \geq 1$ (decimal expansions are unique). Contradiction. □

1.5.3. Structure of the real numbers. We initiate a gentle exploration of the real numbers and show how we can proceed from our abstract definition of the real numbers as the set of decimal expansions to a more workable definition of the real numbers. Basically, we want to show how we can extend the usual order, absolute value, and the operations of addition and subtraction from $\mathbb{Q}$ to $\mathbb{R}$. All of this can be done in an abstract and very general way but we prefer here to carry out our constructions and definitions in the familiar context of decimal expansions. Although conceptually the process is quite simple, there are some slightly irritating special arguments we have to make mainly caused by the non-uniqueness of decimal expansions. The reader should approach this material in the following spirit: how do we go from our formal definition of real numbers to setting up the arithmetic ($\pm, \times$, etc) and defining the basic operations of analysis, such as limits?

Order on $\mathbb{R}$. Provided we deny either recurring 0's or 9's, it is easy to define an order $<$ on $\mathbb{R}$ that extends the usual order on $\mathbb{Q}$. Suppose then we restrict to decimal expansions that do not end with recurring 9's. If $x, y \in \mathbb{R}^+$, we write $x < y$ (equivalently, $y > x$) if there exists $n \geq 0$ such that $x_i = y_i$, $i < n$, and $x_n < y_n$. (Necessarily $x \neq y$ by uniqueness of decimal expansions!) If $x \in \mathbb{R}^-$, and $y \in \mathbb{R}^+$ and $x \neq y$ (so $x, y \neq 0$) we declare that $x < y$ and if $x, y \in \mathbb{R}^-$ then $x < y$ iff

$-y < -x$. Since we have unique decimal expansions, this restricts to the usual order on $\mathbb{Q}$ (if we did not have this restriction, then $x = 0.1$, $y = 0.0\overline{9}$ would cause a problem.) We extend notation in the usual way to $\leq, \geq$. With these conventions we have

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\},\ \mathbb{R}^- = \{x \in \mathbb{R} \mid x \leq 0\}.$$

REMARK 1.5.4. If instead we had restricted to decimal expansions of non-zero numbers that do not end with recurring 0's, we would have ended with the same order structure on $\mathbb{R}$. Thus $0.1 < 0.2$ (deny recurring 9's) and $0.0\overline{9} < 0.1\overline{9}$ deny recurring 0's).

EXAMPLE 1.5.5. If $x \in \mathbb{R}^+$, there exists $n \in \mathbb{N}$ such that $x < n$. The proof is immediate: if $x = x_0.x_1\ldots$ and we deny recurring 9's, define $n = (x_0 + 1).\overline{0}$. This property is known as the *Archimedean property* of real numbers.

Absolute Value.

DEFINITION 1.5.6. If $x \in \mathbb{R}$, the *absolute value* $|x|$ of x is defined to be x, if $x \geq 0$ and $-x$ if $x < 0$. That is $|x|$ is x 'unsigned'.

REMARK 1.5.7. Absolute value restricted to $\mathbb{Q}$ gives the usual absolute value on rationals (same definition).

LEMMA 1.5.8. *If $x \in \mathbb{R}$ and $x_0, \ldots, x_N = 0$, $x_{N+1} \neq 0$, then $10^{-N-1} \leq |x| \leq 10^{-N}$.*

PROOF. If we replace x_n by 9 for $n > N$, we have

$$|x| = 0.0\ldots 0x_{N+1} \leq 0.0\ldots 0\overline{9} = 10^{-N-1}\sum_{m=0}^{\infty}\frac{9}{10}^{-m} = 10^{-N}.$$

This shows $|x| \leq 10^{-N}$. On the other hand if $x_{N+1} \neq 0$, we can replace x_{N+1} by 1 and set $x_n = 0$, $n > N + 1$ to obtain

$$|x| = 0.0\ldots 0x_{N+1} \geq 0.0\ldots 01\overline{0} = 10^{-N-1}.$$

Hence $10^{-N-1} \leq |x|$. $\square$

EXAMPLE 1.5.9. We claim that if $x \in \mathbb{R}^+$ is non-zero, then there exists $z \in \mathbb{R}$ such that $0 < z < x$. Since $x \neq 0$, there exists a least N such that $x_{N+1} \neq 0$. By lemma 1.5.8, $x \geq 10^{-N-1}$. Since $10^{-N-1} > 10^{-N-2} > 0$, we may take $z = 10^{-N-2}$. In this case we constructed a rational z. We can find an irrational z by choosing any non-rational decimal expansion, for example define take $a = 10110011100011110000\ldots 1^n 0^n \ldots$ (where 1^n is shorthand for n repeated 1's). If we define $z = 0.0^{N+1}a$, then z is positive, irrational and $z < 10^{-N-1} \leq x$ by lemma 1.5.8. Hence $0 < z < x$. If we assume more structure on the reals (multiplication and division, it is easy to deduce this result from the Archimedean property of $\mathbb{R}$ (example 1.5.5). See also proposition 1.5.19.

REMARK 1.5.10. The reader should note that so far we have made no use of addition and subtraction of real numbers.

Real numbers as sequences of rational approximations. We shall think of an irrational number x as defined by its decimal (more generally, rational) approximations. That is, rather than attempting to *define* x by evaluating the infinite sum $\sum_{n=0}^{\infty} \frac{x_n}{10^n}$ (we cannot at this point), we think of x as defined by the set

$$x_0, x_0.x_1, \ldots, x_0.x_1 \ldots x_n, \ldots.$$

of decimal approximations to x. In practice, this is exactly the way we have to deal with irrational numbers: we cannot write down an irrational number in exact form — that requires an infinite string of integers — instead we write down a 'good enough' rational approximation to the number. In order to make this process work, we need some ideas based on limits. First a useful estimate.

LEMMA 1.5.11. *Let $x \in \mathbb{R}$ have decimal expansion $x = x_0.x_1 \ldots$. For $N \geq 0$, define the terminating decimal $x^N = x_0.x_1 \ldots x_N \in \mathbb{Q}$. Then for all $M > N \geq 0$ we have*

$$|x^N - x^M| < 1/10^N.$$

PROOF. We have

$$|x^N - x^M| = |\sum_{n=N+1}^{M} \frac{x_n}{10^n}| \leq \sum_{n=N+1}^{M} \frac{9}{10^n} < \frac{9}{10^{N+1}} \sum_{n=0}^{\infty} 10^{-n} = \frac{1}{10^N}.$$

(This argument is carried out within the set of rational numbers.) □

DEFINITION 1.5.12. Suppose that $x \in \mathbb{R}$ and that either $x = 0$ or x is expressed as a decimal expansion not ending in recurring 0's. Suppose that $(x^n)_{n \geq 1}$ is a sequence of real numbers. We write $\mathbf{lim}_{n \to \infty} x^n = x$ if

$$\text{For every } N \in \mathbb{N}\ ,\ \exists M \in \mathbb{N} \text{ such that } x_i^n = x_i, i \leq N, n \geq M.$$

In words, this definition says that (x^n) converges to x iff for any M, we can find N so that the first M terms of the decimal expansions of x^n and x agree for $n \geq N$. Note the symbolic character of this definition.

REMARKS 1.5.13. (1) We emphasize that we allow (x^n) to consist of terminating decimals — our restriction on the decimal expansion is only on x. (See the example (1) below.)

(2) Later we relate this definition of limit to the familiar definition in terms of $|x - x_n|$ — as yet, we have not defined either subtraction or absolute value on $\mathbb{R}$.

EXAMPLES 1.5.14. (1) Let $x = \pm x_0.x_1 \ldots \in \mathbb{R}$ and let $x^n = \pm x_0.x_1 \ldots x_n$ denote the nth. decimal approximation to x. Then $\mathbf{lim}_{n\to\infty} x^n = x$. In this case, given $N \in \mathbb{N}$, we may choose $M = N$ in our limit definition.
(2) The reason we need to take care with the decimal expansion of x in the limit definition is shown by the example $x = 1.\overline{0} = 0.\overline{9}$. If we write $x = 0.\overline{9}$, then $\mathbf{lim}_{n\to\infty} x^n = 0.\overline{9}$ but $\mathbf{lim}_{n\to\infty} x^n \neq 1.\overline{0}$. Notice that there is no problem if we restrict to irrational numbers.

In the next example we sketch how we can use definition 1.5.12 to define the operations of addition and subtraction of real numbers.

EXAMPLE 1.5.15. Consider the problem of defining *addition* of real numbers. To simplify arguments and notation, suppose first that $x, y \in \mathbb{R}^+$. If $x, y \in \mathbb{Q}$ we know how to define $x + y$. Indeed, if $x = \frac{r}{s}$, $y = \frac{p}{q}$, then $x + y = \frac{rq+ps}{sq}$. Alternatively, we can, and will, work with the subset of $\mathbb{Q}$ consisting of terminating decimal expansions.

Let $x, y \in \mathbb{R}^+$. For $n \in \mathbb{N}$, define $x^n = x_0.x_1 \ldots x_n$, $y^n = y_0.y_1 \ldots y_n$ (x_n, y_n may be zero). We want to define $x + y = \mathbf{lim}_{n\to\infty}(x^n + y^n)$, where the limit is in the sense of Definition 1.5.12. Set $z^n = x^n + y^n$. Then $z^n = z_0^n.z_1^n \ldots z_n^n$ where the integers $z_0^n, \ldots z_n^n$ may depend on n (for example, take $x = 0.44445\ldots$, $y = 0.55555\ldots$ and observe that $z_i^5 \neq z_i^n$, $n < 5$, $i = 0, 1, \ldots, 4$). Nevertheless, it follows from lemma 1.5.11 that if for some N there exists $m < N$ such that $z_m^N \neq 9$, then $z_j^n = z_j^N$ for all $n \geq N$, $j < m$. Consequently, if it is true that for some $m \in \mathbb{N}$, we can find $N > m$ such that $z_{N-1}^N \neq 9$, then $z_i^p = z_i^q$, for all $p, q \geq N$ and $i < m$. If this latter condition holds for all $m \in \mathbb{N}$, then it follows from our definition of limit that $\mathbf{lim}_{n\to\infty}(z^n = x^n + y^n)$ exists and we shall denote the limit to be $x + y$. If the condition fails, then there exists $N \in \mathbb{N}$ such that for $j, n \geq N$, $z_j^n = 9$. Hence the $\mathbf{lim}_{n\to\infty} z^n = x^n + y^n$ exists and the corresponding decimal expansion ends with recurring 9's. If both x and y are negative, we may define $x + y = -(-x + -y)$. If $x, y \in \mathbb{R}^+$, define $x - y$ in exactly the same was as we did above for $x + y$. Then we may define $x + y$ for general $x, y \in \mathbb{R}$ by writing $x + y = x - (-y)$ (in the case where $y < 0 < x$ and $-y < x$). It is now straightforward to define $x \pm y$ for all $x, y \in \mathbb{R}$.

We can define multiplication in a similar way but we will not do this here as there are easier ways to show multiplication (and division) on $\mathbb{Q}$ naturally extends to $\mathbb{R}$. We return to this point in the next chapter.

LEMMA 1.5.16. *Let (x^n) be a sequence of real numbers and $x \in \mathbb{R}$. Then $\mathbf{lim}_{n\to\infty} x^n = x$ (in the sense of definition 1.5.12) iff $\lim_{n\to\infty} |x - x^n| = 0$.*

PROOF. Suppose that $\mathbf{lim}_{n\to\infty} x_n = x$. This means that given $N \in \mathbb{N}$, we can find $M \in \mathbb{N}$ such that $x_i^n = x_i$, $n \geq M$, $i \leq N$. Hence, by lemma 1.5.8, $|x - x^n| \leq 10^{-N}$, for all $n \geq N$ and so $\lim_{n\to\infty} |x - x^n| = 0$. The converse is similar. $\square$

Using lemma 1.5.16, we have the following extension to lemma 1.5.11.

LEMMA 1.5.17. *Let $x \in \mathbb{R}$ have decimal expansion $x = x_0.x_1\ldots$. For $N \geq 0$, define the terminating decimal $x^N = x_0.x_1 \ldots x_N \in \mathbb{Q}$. Then for all $M > N \geq 0$ we have*

$$|x^N - x^M| < 1/10^N, \quad |x - x^N| \leq 1/10^N.$$

PROOF. The first inequality is lemma 1.5.11. For the second inequality, we can take the limit $\lim_{M\to\infty} |x^N - x^M|$ and use lemmas 1.5.16, 1.5.8. $\square$

EXAMPLE 1.5.18. Using lemma 1.5.16, the usual rules for absolute value, such as the triangle inequality, follow immediately from the corresponding rules for rational numbers: if $x, y \in \mathbb{R}$ then

$$|x + y| = \lim_{n\to\infty} |x^n + y^n| \leq \lim_{n\to\infty} |x^n| + \lim_{n\to\infty} |y^n| = |x| + |y|.$$

PROPOSITION 1.5.19. *Let $x, y \in \mathbb{R}$ and suppose $x < y$. Then there exists $z \in \mathbb{R}$ such that $x < z < y$. We may require z to be either rational or irrational.*

PROOF. Observe that $x < y$ iff $y - x > 0$. By example 1.5.9, there exists $a \in \mathbb{R}$ such that $0 < a < y - x$. Hence $x < x + a < y$. If we want z to be irrational, observe that either $x + a$ is irrational (and we done) or $x + a$ is rational. In the latter case we can choose an irrational b satisfying $0 < b < y - x - a$ (for example, $b = 10^{-M}\sqrt{2}$ for large enough M) and then define $z = b + x + a < y$. Arguments are similar if we require z to be rational. Finally, note that we cannot (yet!) define $z = (x + y)/2$ as we have not defined multiplication and division of real numbers. $\square$

Up until now we have regarded the infinite sum '$\sum_{n=0}^{\infty} \frac{x_n}{10^n}$' as more or less synonymous with the decimal expansion $x = x_0.x_1\ldots$. As a result of lemma 1.5.16, we have the *result* that

$$\sum_{n=0}^{\infty} \frac{x_n}{10^n} \stackrel{\text{def}}{=} \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x_n}{10^n} = x,$$

since $\lim_{N\to\infty} |x - \sum_{n=0}^{N} \frac{x_n}{10^n}| = 0$.

REMARKS 1.5.20. (1) Let $x \in \mathbb{R}$. We have shown that we can view x as the limit of the sequence (x^n) of decimal approximations to x: $x = \lim_{n\to\infty} x^n$. In this sense, we can think of a real number as defined by its decimal approximations. However, while the rational numbers $\mathbb{Q}$ are naturally defined (in terms of the integers), the restriction to *decimal* expansions is neither natural nor entirely satisfactory. For example, it is conceivable that working in binary or to base five would give a different set of real numbers. We will show in the next chapter that $\mathbb{R}$ does not depend on choice of base.
(2) It is worth emphasizing again the conceptual leap that is required in going from rational to irrational numbers. Rational numbers are given finitely; the specification of an irrational number depends on a limiting process and irrational numbers cannot be described finitely. About the best one can do is define an irrational number by a recursive process. For example, if we take $x_0 = 1$ and define $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$, $n \geq 0$, then $\lim_{n\to\infty} x_n = \sqrt{2}$. However, 'most' irrational numbers cannot be specified in this way.

EXERCISES 1.5.21.

(1) Complete the proof of proposition 1.5.1.
(2) Prove that an eventually periodic decimal expansion defines a rational number.
(3) Show that
   (a) $\mathbb{R} \sim (0,1)$. (Find an explicit bijection; the simpler the better.)
   (b) $(0,1) \cup C \sim (0,1)$ for any countable set C disjoint from $(0,1)$. (Hint: Choose a countable infinite subset of $(0,1)$.)
   (c) $P(\mathbb{N}) \sim (0,1)$. (Hint: regard $(0,1)$ as the set of binary expansions $b = 0.b_1\ldots$, where $b_i \in \{0,1\}$, $i > 0$ and we deny expansions ending in recurring 0's.)
   (d) $P(\mathbb{N}) \sim \mathbb{R}$.
(4) Using decimal expansions, find an *onto* map $F : [0,1] \to [0,1]^2$. Verify that your map is well-defined and is not 1:1. Show that it is possible to define a *bijection* $G : [0,1] \to [0,1]^2$ (the new map G is closely related to F).
   (Hint for the second part: The problem lies with non-uniqueness of decimal expansions. Let D denote the set of *all* decimal expansions $0.x_1x_2\ldots$. Show there is a bijection between D and D^2 — easy! Then verify $D \sim [0,1]$, $D^2 \sim [0,1]^2$, this will require handling countable sets of 'bad' points (use the result of Q1). The maps F, G will not be continuous but it is possible to construct continuous maps of $[0,1]$ onto $[0,1]^2$ (Peano curves) but not continuous bijections between $[0,1]$ and $[0,1]^2$.)
(5) Let $\mathcal{F}$ be the set of all functions $f : [0,1] \to \mathbb{R}$. Show that $\mathcal{F} \not\sim [0,1]$. (We have equivalence if we restrict to continuous functions.)

(6) Let X be a non-empty set and $\mathcal{F}$ denote the set of all functions $f : X \to \mathbb{R}$. Show that $\mathcal{F} \not\sim X$.

(7) Let X, Y be non-empty sets and suppose that Y contains contains at least two points. Let $\mathcal{F}$ denote the set of all functions $f : X \to Y$. Show that $\mathcal{F} \not\sim X$.

CHAPTER 2

Basic properties of real numbers, sequences and continuous functions

2.1. Introduction

In this chapter we prove a number of foundational results about real numbers, sequences and continuous functions. Central to our analysis is the Bolzano-Weierstrass theorem and its corollary that every bounded sequence has at least one convergent subsequence. After verifying the equivalence of continuity with sequential continuity, we establish some standard results about continuous functions on a closed bounded interval (boundedness, attainment of bounds, intermediate value theorem and uniform continuity). Next we define the concept of a Cauchy sequence and prove the fundamental result that a sequence is convergent iff it is Cauchy. As a corollary of this result we show we can define multiplication and division of real numbers as well as verify all the standard laws of arithmetic. We devote a section to the definitions and properties of the operations of $\limsup$ and $\liminf$ and show how we use may use these concepts to provide alternate proofs of some of our results. Finally, in an appendix, we provide a simple proof that every continuous function on a closed interval has a Riemann integral (the proof does not use uniform continuity).

2.2. Sequences

Let X be a non-empty set. A *sequence* of points of X is a function $x : \mathbb{N} \to X$. We invariably set $x(n) = x_n$, $n \geq 1$, and regard (x_n) as defining an ordered subset of X — the order being determined $\mathbb{N}$.

EXAMPLE 2.2.1. If $x \in X$ and we define $x_n = x$, for all $n \geq 1$, then (x_n) is a constant sequence. In particular, we do not require that the map $x : \mathbb{N} \to X$ be 1:1. ♠

2.2.1. Sequences of real numbers and convergence. In this chapter we will be mainly interested in sequences of real numbers: $(x_n) \subset \mathbb{R}$.

EXAMPLE 2.2.2. Since $\mathbb{Q}$ is countable, there exists a surjective map $x : \mathbb{N} \to \mathbb{Q}$. The sequence (x_n) has the property that $\cup_{n=1}^{\infty}\{x_n\} = \mathbb{Q}$. We can even require that every rational number occurs infinitely often in the sequence (x_n) (the set of all pairs $(r, s) \in \mathbb{Z}^2$, with $s \neq 0$ is countable and if $q = r/s$, then $(nr)/(ns) = q$ for all non-zero integers n.) ♠

DEFINITION 2.2.3. The sequence $(x_n) \subset \mathbb{R}$ *is convergent* if there exists $x \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x - x_n| < \varepsilon,\ n \geq N.$$

We write this as $\lim_{n\to\infty} x_n = x$ and say that x is the *limit* of the sequence (x_n) or that the sequence (x_n) *converges* to x.

REMARKS 2.2.4. (1) If the sequence (x_n) is convergent, then the limit is unique. We leave this as an easy exercise for the reader.
(2) The definition of convergence works perfectly well within the framework of the rational numbers. In this case, we require $(x_n) \subset \mathbb{Q}$ and $x \in \mathbb{Q}$. As we pointed out earlier, the geometric series $\sum ar^n$ always converges in $\mathbb{Q}$ if $a, r \in \mathbb{Q}$ and $|r| < 1$. As we shall soon see this is quite exceptional. In general, infinite sequences or series of rational numbers will not converge in $\mathbb{Q}$ even though they converge in $\mathbb{R}$.
(3) A serious defect of the definition of convergent sequence is that we need to know the limit x — x enters in the limit definition. Later we shall see that, providing we work with the real numbers $\mathbb{R}$ (as opposed to the rationals $\mathbb{Q}$), it is possible to give an *intrinsic* definition of a convergent sequence that does not depend on knowing the limit x.

We will often make use of the well-known *squeezing lemma.* We give the statement for reference and leave the simple proof to the exercises.

LEMMA 2.2.5. *If $(a_n), (b_n), (x_n)$ are sequences of real numbers which satisfy*

(1) *$a_n \leq x_n \leq b_n$ for all $n \geq 1$ (for large enough n suffices);*
(2) *$\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ exist and have the same limit, say $x^\star$,*

then (x_n) is convergent and has limit $x^\star$.

EXAMPLES 2.2.6. (1) If $\alpha > 0$, then $(n^{-\alpha})$ converges to zero. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > (1/\varepsilon)^{1/\alpha}$ (this uses the Archimedean property). We have $n^{-\alpha} < \varepsilon$, all $n \geq N$. Hence $\lim_{n\to\infty} n^{-\alpha} = 0$.
(2) If $x > 0$, then $(x^{1/n})$ converges to 1. If $x > 1$, then $x^{1/n} > 1$. Set $x_n = x^{1/n} - 1 > 0$. By the binomial theorem $x = (1 + x_n)^n \geq 1 + nx_n$,

$n \geq 1$, and so

$$0 < x_n \leq \frac{x-1}{n}.$$

The result follows by (1) and lemma 2.2.5. If $0 < x < 1$, apply the previous argument to $y = 1/x$.
(3) The sequence $(n^{1/n})$ converges to 1. Set $n^{1/n} = 1 + x_n$. Clearly $x_n \geq 0$. Applying the binomial theorem we have

$$n = (1 + x_n)^n \geq 1 + \frac{n(n-1)}{2} x_n^2,$$

and so

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}.$$

The result follows by (1) and lemma 2.2.5.
(4) If $r \in (-1, 1)$, the geometric sequence (r^n) converges and has limit zero. Suppose that $r \in (0, 1)$. Define $x > 0$ by $r = 1/(1 + x)$. Then $r^{-n} = (1 + x)^n \geq nx$ and so $0 \leq r^n \leq x^{-1}n^{-1}$. The result follows by (1) and lemma 2.2.5. If $r \in (-1, 0)$, then the same argument shows that $|r|^n \to 0$ and hence $\lim_{n\to\infty} r^n = 0$ (by definition of limit). ♠

There are many different ways of formulating the limit definition. The next omnibus lemma gives the equivalence of some of these definitions.

LEMMA 2.2.7. *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. The following statements are equivalent.*

1. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ *such that* $|x - x_n| < \varepsilon$ *for all* $n \geq N$.
2. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ *such that* $|x - x_n| \leq \varepsilon$ *for all* $n \geq N$.
3. $\forall m \in \mathbb{N}$, $\exists N \in \mathbb{N}$ *such that* $|x - x_n| < 10^{-m}$ *for all* $n \geq N$.
4. *There exists a sequence (κ_n) of strictly positive numbers converging to zero such that* $\forall m \in \mathbb{N}$, $\exists N \in \mathbb{N}$ *such that* $|x - x_n| < \kappa_m$ *for all* $n \geq N$.
5. *For every sequence (κ_n) of strictly positive numbers converging to zero,* $\forall m \in \mathbb{N}$, $\exists N \in \mathbb{N}$ *such that* $|x - x_n| < \kappa_m$ *for all* $n \geq N$.

(In statements (3,4,5), we can replace $<$ by $\leq$ as in (2).)

PROOF. We need to show that if $n, m \in \mathbf{5}$, $n \neq m$, then (n) $\Longrightarrow$ (m). That is, if the sequence converges according to (n), then it converges according to (m).

We start by proving the equivalence of (1) and (2). (1) $\Longrightarrow$ (2) is obvious since $\varepsilon \leq \varepsilon$. For the converse, suppose convergence according to (2). Given $\varepsilon/2 > 0$, we can choose $N \in \mathbb{N}$ such that $|x - x_n| \leq \varepsilon/2$

for all $n \geq N$. Since $\varepsilon/2 < \varepsilon$, we have $|x - x_n| < \varepsilon$ for all $n \geq N$ and so we have convergence according to (1).

Turning to the remaining statements, we have (5) $\Longrightarrow$ (3,4) and (3) $\Longrightarrow$ (4) ((5) is the strongest statement, (4) the weakest), Hence, it suffices to show that (1) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1). For (1) $\Longrightarrow$ (4), take $\kappa_n = \frac{1}{n}$. and apply (1) with $\varepsilon = \kappa_n$. Next suppose (4) holds with the sequence (κ_n). Let (ρ_n) be any sequence of strictly positive numbers converging to zero. Given $m \in \mathbb{N}$, $\rho_m > 0$ and so, since (κ_n) converges to zero, there exists $\ell \in \mathbb{N}$ such that $0 < \kappa_\ell \leq \rho_m$. Hence, by (4), there exists $N \in \mathbb{N}$ such that $|x - x_n| < \kappa_\ell$ for all $n \geq N$. Since $\kappa_\ell \leq \rho_m$, $|x - x_n| < \rho_m$ for all $n \geq N$, proving that (5) holds. Finally, we show that (5) $\Longrightarrow$ (1). For this, it is enough to define $\kappa_n = \varepsilon/n$ and apply (5) with $m = 1$. ☐

Remark 2.2.8. Statements (3,4) of the lemma are the easiest to work with as they only require verification of a countable number of conditions. On the other hand, (1,2,5) require verification of a non-countable number of conditions.

Example 2.2.9. Let $x \in \mathbb{R}$ have decimal expansion $x = x_0.x_1x_2\ldots$. For $n \geq 1$, define $z_n = x_0.x_1\ldots x_n$. Then (z_n) is convergent and $\lim_{n\to\infty} z_n = x$. This follows since

$$|x - z_n| \leq 10^{-n},\ n \geq 1$$

and so we may use the convergence statement (3) of lemma 2.2.7 (with $<$ replaced by $\leq$). ♠

2.2.2. Subsequences.

Definition 2.2.10. Let (x_n) be a sequence of real numbers. A subsequence (x_{n_j}) of (x_n) is a sequence of the form $x_{n_1}, x_{n_2}, \ldots$ where $1 \leq n_1 < n_2 < \ldots$. That is, it is a sequence (z_j) where $z_j = x_{n_j}$, $j \geq 1$.

Remark 2.2.11. If $x : \mathbb{N} \to \mathbb{R}$ is a sequence then every countably infinite subset K of $\mathbb{N}$ determines a subsequence. Indeed, if K is a countably infinite subset of $\mathbb{N}$, we may write K uniquely as $K = \{n_j \mid j \in \mathbb{N}\}$, where $n_1 < n_2 < \ldots$ and define $z : \mathbb{N} \to \mathbb{R}$ by $z(j) = x(n_j) = x_{n_j}$, $j \in \mathbb{N}$.

We leave the next lemma as an exercise.

Lemma 2.2.12. *Let (x_n) be a convergent sequence with limit x. Every subsequence (x_{n_j}) of (x_n) is convergent with limit x.*

Examples 2.2.13. (1) Define the sequence (x_n) by $x_n = n$, $n \geq 1$. As a subsequence we could take (x_{n_p}) where x_{n_p} denotes the pth prime number (so (x_{n_p}) is the sequence $2, 3, 5, 7, 11, \ldots$). It is clear that (x_n) has no convergent subsequences.

(2) Let (x_n) be a sequence such that $\{x_n \mid n \geq 1\} = \mathbb{Q}$. Not surprisingly, (x_n) is not convergent. We claim far more: for every $x \in \mathbb{R}$, we can construct a subsequence (x_{n_j}) of (x_n) which converges to x. (By lemma 2.2.12, if (x_n) is convergent all subsequences have the same limit.) Suppose then that $x \in \mathbb{R}$. We give an inductive construction for a subsequence converging to x which will (repeatedly) use the fact that $\mathbb{Q} \cap (a, b)$ is countably infinite for all open intervals (a, b), $a < b$. We define x_{n_1} by taking $n_1 \geq 1$ to be the smallest integer such that $x_{n_1} \in (x - 1, x + 1)$. Suppose we have constructed $x_{n_1}, \ldots, x_{n_m}$ so that $1 \leq n_1 < \ldots < n_m$ and

$$x_{n_j} \in (x - 10^{-j+1}, x + 10^{-j+1}), \quad j = 1, \ldots, m.$$

Choose n_{m+1} to be the smallest integer greater than n_m such that $x_{n_{m+1}} \in (x - 10^{-m}, x + 10^{-m})$, That we can choose n_{m+1} follows since $(x - 10^{-m}, x + 10^{-m}) \cap \{x_{n_m+j} \mid j \geq 1\}$ contains $(x - 10^{-m}, x + 10^{-m}) \cap (\mathbb{Q} \setminus \{x_1, \ldots, x_{n_m}\})$ which is countably infinite. This completes the inductive construction of (x_{n_j}). Since $|x - x_{n_j}| < 10^{-j+1}$, $j \geq 1$, it follows from lemma 2.2.7 (statement (4) this time) that (x_{n_j}) converges to x. ♠

Exercises 2.2.14.

(1) Prove lemma 2.2.12.
(2) Find a countable infinite subset X of $\mathbb{R}$ such that if $(x_n) \subset X$ is convergent, then (x_n) is eventually constant and the limit of (x_n) lies in X ((x_n) is *eventually constant* if $\exists x$, $\exists N \in \mathbb{N}$ such that $x_n = x$, $n \geq N$).
(3) Let X be a non-empty subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is a *closure point* of X if we can find a sequence $(x_n) \subset X$ which converges to x. Denote the set of closure points of X by $\overline{X}$. Why is it true that $\overline{X} \supset X$?
   (a) Find an example of a countably infinite unbounded set X of $\mathbb{R}$ such that $\overline{X} = X$.
   (b) Find an example of a countably infinite bounded subset X of $\mathbb{R}$ such that $\overline{X} = X$.
   (c) Find an example of a countably infinite bounded subset of $[0, 1]$ such that $\overline{X} \setminus X = \{0, \frac{1}{2}, 1\}$,
   (d) Find an example of a countably infinite subset X of $\mathbb{R}$ such that $\overline{X} = \mathbb{R}$.
(4) Suppose that (x_n) s convergent. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Prove that $(x_{\sigma(n)})$ is convergent and has the same limit as (x_n).
(5) Write the set $\mathbb{Q}$ of all rational numbers as a sequence $(q_n)_{n \geq 1}$ where we assume $q_n \neq q_m$ if $n \neq m$. Given $\varepsilon > 0$, define $I_n = (q_n - \varepsilon 2^{-(n+1)}, q_n + \varepsilon 2^{-(n+1)})$, $n \geq 1$.
   (a) If $|I_n|$ denotes the length of I_n, show that $\sum_{n=1}^{\infty} |I_n| = \varepsilon$.
   (b) Show that the set $X = \mathbb{R} \setminus I$ contains no proper subintervals even though the 'length' of the complement I is at most ε.
   (c) The set X consists of irrational numbers. Does X contain all the irrational numbers? Why/Why not?

2.3. Bounded subsets of $\mathbb{R}$ and the supremum and infimum

DEFINITION 2.3.1. Let A be a subset of $\mathbb{R}$.

(1) A is *bounded above* if $\exists M \in \mathbb{R}$ such that $M \geq x$ for all $x \in A$. We call M an *upper bound* for A.
(2) A is *bounded below* if $\exists m \in \mathbb{R}$ such that $m \leq x$ for all $x \in A$. We call m a *lower bound* for A.
(3) A is *bounded* is A is bounded above and below.

EXAMPLES 2.3.2. (1) $\mathbb{N}$ is bounded below ($m \leq 1$ works) but not bounded above.
(2) $\mathbb{Z}$ is unbounded.
(3) If $a < b$ are real numbers, then (a, b), $[a, b]$ are bounded. In both cases we can take as upper bound any $M \geq b$ and as lower bound any $m \leq a$.
(4) A is bounded iff $\exists R \geq 0$ such that $A \subset [-R, R]$. ♠

DEFINITION 2.3.3. Let A be a subset of $\mathbb{R}$.

(1) Suppose A is bounded above. A *least upper bound* for A, or *supremum* for A is a real number α such that
   (a) α is an upper bound for A.
   (b) If M is any upper bound for A, then $\alpha \leq M$.
(2) Suppose A is bounded below. A *greatest lower bound* for A, or *infimum* for A is a real number β such that
   (a) β is a lower bound for A.
   (b) If m is any lower bound for A, then $\beta \geq m$.

LEMMA 2.3.4. *Suppose A is bounded above.*

(1) *If the supremum of A exists, it is unique. Similarly, for the infimum if A is bounded below.*
(2) *If both* $\inf(A)$ *and* $\sup(A)$ *exist, the same is true for* $-A = \{-a \mid a \in A\}$ *and* $\inf(A) = -\sup(-A)$.

PROOF. (1) Suppose that α, α' are supremums of A. Then by property (a), both α and α' are upper bounds of A. Since α is a supremum of A, it follows by (b) that $\alpha \leq \alpha'$. Applying same argument with the roles of α, α' interchanged, we get $\alpha' \leq \alpha$. Hence $\alpha = \alpha'$. We may apply a similar argument to prove the uniqueness of the infimum. (2) We leave this as an exercise. □

Henceforth, if the subset A has a supremum, we denote it by $\sup(A)$. Similarly, we denote the infimum by $\inf(A)$.

We have the following necessary and sufficient condition for a supremum.

LEMMA 2.3.5. *Let A be a subset of $\mathbb{R}$ which is bounded above and $\alpha \in \mathbb{R}$. Suppose that*

(1) *α is an upper bound for A.*
(2) *For every $\varepsilon > 0$, there exists $x \in A$ such that $x > \alpha - \varepsilon$.*

Then $\alpha = \sup(A)$. Conversely, if $\alpha = \sup(A)$, then (1,2) are satisfied. A similar criterion holds for the infimum of A.

PROOF. Observe that if (2) fails then $\alpha - \varepsilon \geq x$ for all $x \in A$. Hence $\alpha - \varepsilon$ is an upper bound for A and α cannot be the supremum of A. The converse is equally simple. □

THEOREM 2.3.6. *Let $A \subset \mathbb{R}$ be bounded above. Then $\sup(A)$ exists. Similarly, if A is bounded below, $\inf(A)$ exists.*

PROOF. Let A be bounded above. We use an inductive technique to construct the decimal expansion $x_0.x_1x_2\dots$ of $\sup(A)$. Specifically, we construct an increasing sequence $z_n = x_0.x_1x_2\dots x_n$, $n \geq 0$, such that

(a) z_n is not an upper bound of A, $n \geq 0$.
(b) $z_n + 10^{-n}$ is an upper bound of A, $n \geq 0$.
(c) If $m > n \geq 0$, then z_n and z_m agree to the first n decimal places.

(We start with $n = 0$ rather than $n = 1$ for notational convenience.)
Let N be the smallest integer which is an upper bound for A. Define $z_0 = x_0 = N - 1$. In particular, $z_0 + 1$ is an upper bound for A and there exists $x \in A$, $x > z_0$ so z_0 is not an upper bound for A.
Proceeding inductively, suppose that we have constructed numbers $z_j = x_0.x_1x_2\dots x_j$ satisfying (a,b,c) for $j < n$. Consider $Z_p = z_{n-1} + p10^{-n}$, $0 \leq p \leq 10$. We have that X_{10} is an upper bound of X (by (b) for z_{n-1}), but $X_0 = z_{n-1}$ is not (by (a). Choose $p \in \{0, \dots, 9\}$ so that X_{p+1} is an upper bound but X_p is not. Define $x_n = p$, $z_n = x_0.x_1x_2\dots x_n = z_{n-1} + z_n 10^{-n}$. This completes the inductive step.

It is immediate from the construction that $\lim_{n\to\infty} z_n$ converges to a real number $x^\star = x_0.x_1x_2\dots$. We claim that $x^\star = \sup(A)$.

Since $x_0.x_1x_2\dots x_n + 10^{-n} \geq x$, for all $x \in A$, $x_0.x_1x_2\dots x_n + 10^{-n}$ is an upper bound for A for all $n \geq 0$. Since $(x_0.x_1x_2\dots x_n)$ is an increasing sequence and $x_0.x_1x_2\dots x_n \leq x^\star$ for all n, we have that $x^\star + 10^{-n}$ is an upper bound for A for all $n \geq 0$. Hence $x^\star$ must be an upper bound for A. We need to show $x^\star$ is the least upper bound. Suppose M is an upper bound of A. Then $M > x_0.x_1x_2\dots x_n$, $n \geq 0$ (property (a)). Hence, $M \geq \lim_{n\to\infty} x_0.x_1x_2\dots x_n = x^\star$. The result for infimums can be proved along the same lines or, more simply, by using lemma 2.3.4(2). □

REMARKS 2.3.7. (1) The proof of theorem 2.3.6 is carefully constructed so as to make transparent the convergence of the sequence (z_n) to the supremum of A. We do this in two ways. First, the sequence (z_n) is an increasing sequence which obviously converges to the decimal expansion of $x^\star$. Secondly, z_n is not an upper bound but $z_n + 10^{-n}$ is an upper bound.
(2) The proof would not work over the rational numbers. There is no reason why the sequence (z_n) should converge to a rational number.
(3) The existence of the supremum is sometimes taken as an Axiom (basic assumption) about the real numbers. The point of the proof is that if one thinks of real numbers as being decimal expansions, then it is straightforward to construct the supremum directly. In particular, we construct the supremum as a sequence of rational approximations.

EXAMPLES 2.3.8. (1) Theorem 2.3.6 fails if we work with rational numbers. The easiest example is got by defining $A = \{x_0.x_1 \dots x_n \mid n \geq 1\}$, where $\sqrt{2} = x_0.x_1 \dots$. In this case, if $\sup(A)$ existed it would have to be $\sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$. Other examples can be found by constructing (according to some simple rule) a non-periodic non-terminating decimal expansion and then taking the set of finite decimal approximations. For example, $x^\star = 1.0^{1!}10^{2!}10^{3!}1 \dots 0^{n!}1 \dots$ (here 0^p means a string of p zeros).
(2) Let $\{I_j = (a_j, b_j) \mid j \in J\}$ be a set of open intervals of $\mathbb{R}$. Suppose that $\cap_{j \in J} I_i \neq \emptyset$ — that is, the intervals I_i share at least one common point, say x_0. Then $I = \cup_{j \in J} I_i$ is an open interval (we regard $(-\infty, b)$, (a, ∞) and $\mathbb{R}$ as open intervals). Suppose the sets $\{a_j \mid j \in J\}$, $\{b_j \mid j \in J\}$ are bounded subsets of $\mathbb{R}$ (we leave the case where one or both of these sets is unbounded to the exercises). Set $a^\star = \inf\{a_j \mid j \in J\}$, $b^\star = \sup\{b_j \mid j \in J\}$. We claim $I = (a^\star, b^\star)$. By definition of the supremum and infimum, given $\varepsilon > 0$, there exist $\ell, m \in J$ such that $a^\star \geq a_\ell - \varepsilon$, $b^\star \leq b_m + \varepsilon$. Since I_ℓ, I_m share the common point x_0, and $a_\ell < x_0 < b_m$, $I_\ell \cup I_m \subset I$ and so $(a^\star + \varepsilon, b^\star - \varepsilon) \subset I$. Since this is true for all $\varepsilon > 0$, we see that $(a^\star, b^\star) \subset I$. On the other hand, $a^\star, b^\star \notin I$. Indeed, if $a^\star \in I$, this would imply that there exists $m \in J$ such that $a^\star \in I_m$. But then $a_m < a^\star$ and so $a^\star$ could not be a lower bound for $\{a_j \mid j \in J\}$. A similar argument applies to $b^\star$. We have shown that $I = (a^\star, b^\star)$. The reader should note that this result is *false* if we work over the rational numbers. We leave the construction of an explicit counter-example to the exercises. ♠

2.3.1. Applications to sequences and series. Let (a_n) be a sequence of real numbers. Recall that the sequence (a_n) *diverges to* $+\infty$ if for every $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $a_n \geq M$ for all $n \geq N$. We often write this $\lim_{n\to\infty} a_n = +\infty$. We may similarly define divergence to $-\infty$.

We have already made use of increasing sequences in the proof of theorem 2.3.6. For completeness, we give some formal definitions.

DEFINITION 2.3.9. A sequence (a_n) of real numbers is *increasing* if $a_1 \leq a_2 \leq a_3 \leq \ldots$. That is, $a_n \leq a_m$ whenever $n < m$. The sequence is *strictly increasing* if $a_n < a_m$ whenever $n < m$ and is *eventually increasing* if there exists $N \in \mathbb{N}$ such that $a_n < a_m$ whenever $N \leq n < m$. We similarly may define *decreasing*, *strictly decreasing* and *eventually decreasing* sequences.

THEOREM 2.3.10. *Let (a_n) be an increasing sequence of real numbers.*

(1) *If $\{a_n \mid n \in \mathbb{N}\}$ is not bounded above, then $\lim_{n\to\infty} a_n = +\infty$.*
(2) *If $\{a_n \mid n \in \mathbb{N}\}$ is bounded above then (a_n) is convergent and $\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}$.*

A similar result holds for decreasing sequences. The results also hold for eventually increasing (or decreasing) sequences provided that we take the supremum (or infimum) over the increasing (or decreasing) part of the sequence.

PROOF. Suppose first that $\{a_n \mid n \geq 1\}$ is not bounded. Then for every $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $a_N \geq M$. Since (a_n) is increasing, $a_n \geq M$, for all $n \geq N$. Hence $\lim_{n\to\infty} a_n = +\infty$. If $\{a_n \mid n \geq 1\}$ is bounded, theorem 2.3.6 applies and we can define $a^\star = \sup(A)$. We claim (a_n) is convergent with limit $a^\star$. Certainly $a_n \leq a^\star$ for all $n \geq 1$ ($a^\star$ is an upper bound). Further, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_N > a^\star - \varepsilon$ (otherwise $a^\star$ would not be the least upper bound). Since (a_n) is increasing, and bounded above by $a^\star$, we have $a^\star \geq a_n > a^\star - \varepsilon$ for all $n \geq N$. That is, $|a^\star - a_n| < \varepsilon$, $n \geq N$. Hence $\lim_{n\to\infty} a_n$ exists and equals $a^\star$. We leave the proofs of the remaining parts of the theorem to the reader. □

This result has the following important and useful corollary.

THEOREM 2.3.11. *Let $\sum_{i=1}^{\infty} a_i$ be a series of (eventually) positive terms. Then either $\sum_{i=1}^{\infty} a_i$ diverges to $+\infty$ or $\sum_{i=1}^{\infty} a_i$ converges. In particular, $\sum_{i=1}^{\infty} a_i$ converges iff the sequence $(\sum_{i=1}^{n} a_n)$ of partial sums is bounded.*

PROOF. For $n \geq 1$, define the nth partial sum $S_n = \sum_{i=1}^{\infty} a_i$. We recall that, by definition, $\sum_{i=1}^{\infty} a_i$ converges if and only if the sequence

(S_n) of partial sums converges. Since it is assumed that the terms in the series are (eventually) positive, it follows that the sequence (S_n) is (eventually) increasing. The result follows by theorem 2.3.10. $\square$

REMARK 2.3.12. The significance of theorems 2.3.10, 2.3.11 is that they give a criterion for convergence that does not require us to know the limit. As we shall shortly see, we can do much better.

For the remainder of this section, we show how we can use theorem 2.3.10 to establish the convergence properties of some basic geometric sequences. Notice that the use of theorem 2.3.10 is *not* necessary for these proofs — we have already given elementary proofs (that work over $\mathbb{Q}$) for most of the next lemma in examples 2.2.6.

LEMMA 2.3.13. *Let $x \in \mathbb{R}$ and consider the sequence (x^n).*

(1) *If $x \in (0,1)$, (x^n) converges to* 0.
(2) *If $x = 1$, (x^n) converges to* 1.
(3) *if $x > 1$, (x^n) diverges to $+\infty$.*
(4) *If $x \leq -1$, (x^n) is divergent.*

PROOF. Statements (2,4) are obvious; we prove (1,3) using theorem 2.3.10 together with standard facts about limits .

If $x = 0$, (1) is immediate. Suppose $x \in (0,1)$. Then (x^n) is a (strictly) decreasing sequence bounded below by 0. Hence, by theorem 2.3.10 (x^n) converges with limit $x^\star \geq 0$. We have

$$xx^\star = x \lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{n+1} = x^\star$$

and so $xx^\star = x^\star$. Since $0 < x < 1$, $x^\star = 0$. If $x \in (-1,0)$, then $|x|^n \to 0$ since $|x| \in (0,1)$. Hence, by definition of the limit, $\lim_{n\to\infty} x^n = 0$. We prove (3) by observing that if $x > 1$, then $y = x^{-1} \in (0,1)$, hence by (1), $\lim_{n\to\infty} y^n = 0$. Hence for any $M > 0$, there exists $N \in \mathbb{N}$ such that $y^n \leq 1/M$, for all $n \geq N$. That is, $x^n \geq M$, $n \geq N$. Hence (x^n) diverges to $+\infty$. $\square$

As a useful and immediate corollary of lemmas 2.3.13, 2.2.5, we have

LEMMA 2.3.14. *Let (a_n) be a sequence. Suppose that there exist $C \geq 0$ and $r \in (0,1)$ such that*

$$0 \leq |a_n| \leq Cr^n,$$

for all sufficiently large n. Then (a_n) converges and $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 2.3.15. Let $\alpha \in \mathbb{R}$, $r \in (-1,1)$, then the sequence $(n^\alpha r^n)$ is convergent with limit zero. We may assume $r \neq 0$. Set $x_n = n^\alpha r^n$.

We have

$$|\frac{x_{n+1}}{x_n}| = \left(1+\frac{1}{n}\right)^\alpha |r|,\ n \geq 1.$$

Choose $N \in \mathbb{N}$ so that $(1+\frac{1}{N})^\alpha |r| \leq (|r|+1)/2 < 1$. Since $((1+\frac{1}{n})^\alpha)$ is a decreasing sequence, we have

$$\left(1+\frac{1}{n}\right)^\alpha |r| \leq (|r|+1)/2,\ n \geq N.$$

Therefore

$$|x_n| \leq |x_N|(\frac{(|r|+1)}{2})^{n-N} = |x_N|2^N(|r|+1)^{-N}\left(\frac{(|r|+1)}{2}\right)^n,\ n \geq N.$$

The result follows from lemma 2.3.14. ♠

All the examples we have given so far have a rational limit (zero or one) and can be analyzed without using theorem 2.3.10. We end this section with examples where the limit is not rational and elementary arguments do not suffice to prove convergence.

EXAMPLES 2.3.16. (1) Define $x_1 = 2$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$, $n \geq 1$. Clearly $x_n > 0$, and $x_n \in \mathbb{Q}$ for all $n \geq 1$. If $a, b > 0$ then $(a+b)^2 \geq 4ab$ with equality iff $a = b$. Applying the inequality with $a = x_n$, $b = \frac{2}{x_n}$, we get

$$x_{n+1}^2 \geq 2,\ n \geq 1,$$

with equality iff $x_n^2 = 2$. Since $\sqrt{2} \notin \mathbb{Q}$, we must have $x_{n+1}^2 > 2$ for all $n \geq 1$. Since we assumed $x_1 = 2$, we have $x_n^2 > 2$ for all $n \in \mathbb{N}$. A simple computation shows that

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{2}(1 - \frac{2}{x_n x_{n-1}}),\ n \geq 2.$$

Since $x_n^2, x_{n-1}^2 > 2$, $n \geq 2$, we have $1 - \frac{2}{x_n x_{n-1}} > 0$ and so the sign of $x_{n+1} - x_n$ is the same as that of $x_n - x_{n-1}$ for all $n \geq 2$. Computing we see that $x_2 = 3/2 < x_1 = 2$ and so, (x_n) must be a decreasing sequence of strictly positive numbers (necessarily bounded below by 0). It follows from theorem 2.3.10 that (x_n) converges and that the limit z of (x_n) is positive. If we let $n \to \infty$ in $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$, we see that z satisfies

$$z = \frac{1}{2}(z + \frac{2}{z}).$$

That is, $z^2 = 2$. Since $z \geq 0$, $z = \sqrt{2}$.
(2) Suppose $x_n = (1 + \frac{1}{n})^n$, $n \geq 1$. By the Binomial theorem

$$\begin{aligned}(1+\frac{1}{n})^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2}\frac{1}{n^2} + \ldots + \frac{1}{n^n},\\ &= 1 + 1 + \sum_{j=2}^{n} K_n(j),\end{aligned}$$

where

$$K_n(j) = \frac{1}{j!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{j-1}{n}).$$

Observe that for fixed j, $K_n(j)$ increases with n as do the number of terms in the expansion of $(1+\frac{1}{n})^n$. Hence $(1+\frac{1}{n})^n$ is an increasing sequence and so either converges or diverges to $+\infty$. But since $K_n(j) < \frac{1}{j!}$, we have

$$\begin{aligned}(1+\frac{1}{n})^n &< 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}\\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} < 3\end{aligned}$$

proving that $(1+\frac{1}{n})^n$ is bounded and therefore converges. (Later we show that the limit is e.) ♠

Exercises 2.3.17.

(1) Find an example of a *countably infinite* subset A of $\mathbb{R}$ such that
  (a) $\sup(A) \in A$.
  (b) For every $\varepsilon > 0$, $\exists x \in A$ such that $x > \sup(A) - \varepsilon$.
  (c) $\inf(A) \notin A$.
  (d) For every $\varepsilon > 0$, $\exists x \in A$ such that $x < \inf(A) + \varepsilon$.
  Which of (a,b,c,d) could hold if A were finite?
(2) Find an explicit example of a *countably infinite* subset $A = \{a_n \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ such that the following four properties hold:
  (a) $\sup(A) = +\infty$.
  (b) $\inf(A) = 0 \notin A$.
  (c) For every $\varepsilon > 0$, $\exists x \in A$ such that $x < \inf(A) + \varepsilon$.
  (d) If (a_{n_k}) is a subsequence of (a_n), then *either* $\lim_{k\to\infty} a_{n_k} = 0$ *or* $\lim_{k\to\infty} a_{n_k} = +\infty$ *or* (a_{n_k}) is not convergent.
  (You should construct A so that it consists of distinct points: $a_n \neq a_m$, $n \neq m$. In particular, you need an explicit expression for a_n. This is essential for (d).)
(3) Construct an explicit example of a *countably infinite* subset $A = \{a_n \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ such that the following four properties hold:
  (a) $\inf(A) = -\infty$.
  (b) $\sup(A) = 1 \notin A$.
  (c) For every $\varepsilon > 0$, $\exists x \in A$ such that $x > \sup(A) - \varepsilon$.

(d) If (a_{n_k}) is a subsequence of (a_n), then *either* $\lim_{k\to\infty} a_{n_k} = -\infty$ *or* $\lim_{k\to\infty} a_{n_k} = 1$ *or* (a_{n_k}) is not convergent. Construct explicit subsequences to show that each of these possibilities can occur.

(You should construct A so that it consists of distinct points: $a_n \neq a_m$, $n \neq m$. In particular, you need an explicit expression for a_n. This is essential for (d).)

2.4. The Bolzano-Weierstrass Theorem

THEOREM 2.4.1 (Bolzano-Weierstrass theorem). *Let X be an infinite bounded subset of $\mathbb{R}$. Then there exists a convergent sequence (x_n) consisting of distinct points of X.*

PROOF. Since X is bounded, there exists a closed interval $I_0 = [a_0, b_0]$ which contains X: $I_0 \supset X$. We construct a sequence of closed intervals $I_n = [a_n, b_n]$, $n \geq 0$, with the following properties

(1) $I_{n+1} \supset I_n$, $n \geq 0$.
(2) (a_n) is an increasing sequence, (b_n) is a decreasing sequence.
(3) $a_n < b_n$, all $n \geq 0$.
(4) $|I_n| = |b_n - a_n| = 2^{-n}|b_0 - a_0|$, $n \geq 0$.
(5) $X \cap I_n$ is infinite, $n \geq 0$.

Our construction of the sequence (I_n) is inductive. When $n = 0$, conditions (3,4,5) are automatically satisfies (conditions (1,2) are empty). So suppose we have constructed intervals $I_0, \ldots, I_n$ satisfying (1—5). Let $J = [a_n, \frac{a_n+b_n}{2}]$, $K = [\frac{a_n+b_n}{2}, b_n]$. Note that $|J| = |K| = 2^{-1}|I_n| = 2^{-(n+1)}|b_0 - a_0|$ by (4). Since $J \cup K = I_n$ and $I_n \cap X$ is infinite, one (at least) of $J \cap X$, $K \cap X$ must be infinite. Choose one of J, K so that the intersection is infinite. Denote the corresponding interval by $I_{n+1} = [a_{n+1}, b_{n+1}]$. Since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, we have $a_n \leq a_{n+1} < b_{n+1} \leq b_n$. This completes the inductive step and the construction of the intervals I_n.

Since (a_n) is bounded above by b_m, $m \geq 0$, and (b_n) is bounded below by a_m, $m \geq 0$, both sequences (a_n), (b_n) converge by theorem 2.3.10 and $a_m \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq b_m$, for all $m \geq 0$. Applying (4), we see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

It remains to construct the required convergent sequence (x_n) of distinct points of X. Since $I_0 \cap X$ is infinite (non-empty suffices here), we can choose $x_0 \in I_0 \cap X$. Proceeding inductively, suppose we have constructed distinct points $x_j \in I_j \cap X$, $0 \leq j \leq n$. Since $I_{n+1} \cap X$ is infinite, we can choose $x_{n+1} \in (X \cap I_{n+1}) \setminus \{x_0, \ldots, x_n\}$. This completes the inductive construction of (x_n). Since $x_n \in I_n$, $n \geq 0$, we have

$$a_n \leq x_n \leq b_n,\ n \geq 0,$$

and so by lemma 2.2.5, (x_n) is convergent. $\square$

REMARK 2.4.2. Theorem 2.4.1 fails if we work over the rational numbers. The condition that X is bounded is also necessary. A simple counterexample is given by $X = \mathbb{N}$.

We have a very useful application of theorem 2.4.1 to sequences.

PROPOSITION 2.4.3. *Let (x_n) be a bounded sequence. Then there exists a convergent subsequence (x_{n_k}) of (x_n).*

PROOF. We leave the details of the proof to the reader: the result can either be deduced using theorem 2.4.1 or proved directly along the lines of the proof of theorem 2.4.1. The (easy) case when $\{x_n \mid n \geq 1\}$ is finite needs to be handled separately. □

2.4.1. Applications to continuous functions. We start by recalling the standard definition of a continuous function.

DEFINITION 2.4.4. If X is a non-empty subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$, then f is *continuous* at the point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ whenever } x \in X, \text{ and } |x - x_0| < \delta.$$

We say f is continuous on X if f is continuous at every point of X.

REMARKS 2.4.5. (1) For most of our initial applications, X will either be an open or closed interval of $\mathbb{R}$. Later we will need to work with rather more general subsets of $\mathbb{R}$.
(2) The definition of continuity has some unpleasant and subtle features. For example, it appears to require the verification of non-countably many conditions (that is, for each $\varepsilon > 0 \ldots$). As in lemma 2.2.7, we can easily show that it suffices to verify the conditions just for $\varepsilon = 10^{-n}$, $n \geq 1$ (indeed, any sequence converging to zero will do). We give below an alternate, but equivalent, formulation of continuity that is, in many cases, much easier to work with. In spite of the simplifications obtained either by working with a countable set of conditions or with sequential continuity, the fact remains that the concept of continuity is highly non-intuitive. Contrary to the often made suggestion that the graph of a continuous function is what one gets by 'drawing a line without breaks', the reality is that the graph of a 'typical' continuous function is very jagged at all scales and the function is *nowhere* differentiable. In practice, the functions usually encountered in analysis and its applications have more structure than just continuity. Finally, there is a far more elegant and natural definition of continuity that applies in many contexts (including algebra) and which avoids the arid and uninformative ε, δ notation. This definition does, however, require

another significant layer of abstraction. We revisit this issue later.
(3) Just as in lemma 2.2.7, we can replace $< \varepsilon$ in the definition by $\leq \varepsilon$ (of course we *cannot* replace the condition that δ is *strictly* positive).

DEFINITION 2.4.6. If X is a non-empty subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$, then f is *sequentially continuous* at the point $x_0 \in X$ if for every sequence $(x_n) \subset X$ converging to x_0 we have

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

We say f is sequentially continuous on X if f is sequentially continuous at every point of X.

REMARK 2.4.7. At first sight the definition of sequential continuity requires even more to be checked than does the definition of continuity. However, the power of the definition lies in the application to convergent sequences. If f is continuous and $x_n \to x_0$ then $f(x_n) \to f(x_0)$. As we shall see this property is very useful, especially in context where we can apply the Bolzano-Weierstrass theorem (or its corollary proposition 2.4.3).

EXAMPLE 2.4.8. Let $X = \mathbb{Z} \subset \mathbb{R}$. Every function $f : \mathbb{Z} \to \mathbb{R}$ is sequentially continuous. This follows since if $(x_n) \subset \mathbb{Z}$ is convergent to $x_0 \in \mathbb{Z}$ then (x_n) is eventually constant (that is, there exists $N \in \mathbb{N}$ such that $x_n = x_N$, for all $n \geq N$). Of course, it is easy to give a direct proof that $f : \mathbb{Z} \to \mathbb{R}$ is continuous — take $\delta < 1$ in the definition. ♠

THEOREM 2.4.9. *If X is a non-empty subset of $\mathbb{R}$, $f : X \to \mathbb{R}$ and $x_0 \in X$, then f is continuous at x_0 iff f is sequentially continuous at x_0.*

PROOF. We start by proving that if f is continuous at x_0 then f is sequentially continuous at x_0. If f is continuous at x_0 then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ (here, and below, we always assume without further comment that $x \in X$). Let $(x_n) \subset X$ be a sequence converging to x_0. Given $\delta > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_0| < \delta$. But then $|f(x_n) - f(x_0)| < \varepsilon$, for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} f(x_n) = f(x_0)$.

It remains to prove the trickier converse that the sequential continuity of f at x_0 implies the continuity of f at x_0. We prove this by contradiction. Suppose that f is not continuous at x_0. This means that there exists some $\varepsilon > 0$ for which we cannot find a $\delta > 0$ satisfying the conditions of the definition. In particular, if we take $\delta = 1/n$, we can find an $x_n \in X$ such that $|x_n - x_0| < 1/n$ and $|f(x_0) - f(x_n)| \geq \varepsilon$. By construction $\lim_{n\to\infty} x_n = x_0$ and so, by sequential continuity $\lim_{n\to\infty} f(x_n) = f(x_0)$. But this means that given our $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f(x_0) - f(x_n)| < \varepsilon$, for all $n \geq N$. This contradicts our assumption that $|f(x_0) - f(x_n)| \geq \varepsilon$, for all $n \in \mathbb{N}$. Hence f must be continuous at x_0. $\square$

With these preliminaries out of the way, we can now state and prove a basic theorem which gives some of the key properties of a continuous function defined on a closed interval.

THEOREM 2.4.10. *Let $f : [a, b] \to \mathbb{R}$ be continuous ($-\infty < a \leq b < \infty$). Then*

(1) *$f([(a, b)])$ is a bounded subset of $\mathbb{R}$ ("continuous functions are bounded on closed bounded intervals").*
(2) *If $\alpha = \inf(f([(a, b)]))$, $\beta = \sup(f([(a, b)]))$, then there exist $x, y \in [a, b]$ such that $f(x) = \alpha$, $f(y) = \beta$ ("a continuous function on a closed and bounded interval attains its bounds").*
(3) *$f([a, b]) \supset [f(a), f(b)]$ (the intermediate value theorem). In particular, $f([a, b]) = [\alpha, \beta]$ (notation of (2)).*

PROOF. (1) Suppose that f is not bounded above on $[a, b]$. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) \geq n$. Applying proposition 2.4.3, we can choose a convergent subsequence sequence (x_{n_k}) of (x_n). Let $\lim_{k\to\infty} x_{n_k} = x^\star \in [a, b]$. By sequential continuity, $\lim_{k\to\infty} f(x_{n_k}) = f(x^\star)$. But the sequence $(f(x_{n_k}))$ is unbounded by construction and so cannot converge. Contradiction. Hence f must be bounded above on $[a, b]$. Applying this result to $-f$ shows that f is bounded below on $[(a, b]$.
(2) Let $\beta = \sup(f([a, b])) < \infty$. For each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) > \beta - 1/n$ (definition of the supremum). Using proposition 2.4.3 again, we can pick a convergent subsequence (x_{n_k}) of (x_n). If $\lim_{k\to\infty} x_{n_k} = x^\star$, then by sequential continuity we have $f(x^\star) = \beta$. The result for the infimum is obtained by applying the result to $-f$.
(3) We have to prove that $f([a, b]) \supset [f(a), f(b)]$ (without loss of generality we assume $f(a) < f(b)$ — the result is trivial if $f(a) = f(b)$ and if $f(a) > f(b)$, replace f by $-f$).

Suppose the result is false. Then we can find $y \in [f(a), f(b)] \setminus f([a, b])$. Note that $y \neq f(a), f(b)$ and so $f(a) < y < f(b)$. We claim that we can find an open interval $I \subset [f(a), f(b)]$ containing y such that $I \subset [f(a), f(b)] \setminus f([a, b])$ (in other words, if we miss one point, we miss at least an open intervals worth of points — suggesting a 'jump' discontinuity in f). If not, then we can choose a sequence $(x_n) \subset [a, b]$ such that $f(x_n) \to y$. Apply proposition 2.4.3 to extract a subsequence (x_{n_k}) of (x_n) which converges to a point $x \in [a, b]$. Then, by sequential continuity, $f(x) = \lim_{k\to\infty} f(x_{n_k}) = y$, contradicting our assumption that y is not in the image of f. Let I be the union of all open intervals in $[f(a), f(b)] \setminus f([a, b])$ which contain y. It follows from examples 2.3.8 (2) that $I = (u, v)$, where $f(a) \leq u < v \leq f(b)$. Notice that $u, v \in f([a, b])$ (else we could make I bigger by the previous argument). Set $X = \{x \in [a, b] \mid f(x) \leq u\}$. Since $a \in X$, $X \neq \emptyset$ and we may define $\xi = \sup(X)$. Obviously $f(\xi) \leq u$ (indeed $f(\xi) = u$ but we do not need that). Noting that f cannot take values in (u, v), we see that for $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|x_n - \xi| < 1/n$ and $f(x_n) \geq v$ (else ξ would not be the least upper bound of X). Since $\lim_{n\to\infty} x_n = \xi$, it follows by sequential continuity that $\lim_{n\to\infty} f(x_n) = f(\xi) \leq u$ but by construction $f(x_n) \geq v$ and so $\lim_{n\to\infty} f(x_n)$ must be a least $v > u$. Contradiction. Hence f must take all values in the range $[f(a), f(b)]$.

Finally, we need to show that $f([a, b]) = [\alpha, \beta]$. By (2), we can find $a', b' \in [a, b]$ such that $f(a') = \alpha$, $f(b') = \beta$. By what we have just proved we have $f([a', b']) \supset [\alpha, \beta]$. But $f([a, b]) \subset [\alpha, \beta]$ (by the definition of α, β) and so we have equality: $f([a, b]) = f([a', b']) = [\alpha, \beta]$. □

Remark 2.4.11. The proofs of (1,2) and, in particular, the intermediate value theorem are a little different (and easier) than the proofs given in many texts. The proof of the intermediate value theorem that we give has the merit of showing clearly the implications of missing a value in the range. We indicate alternative proofs of these results in the exercises.

Example 2.4.12. All three parts of theorem 2.4.10 *fail* if we work over the rational numbers or consider real valued functions defined on intervals of rational numbers. For example, if we set $[0, 1]_{\mathbb{Q}} = [0, 1] \cap \mathbb{Q}$ and define $f : [0, 1]_{\mathbb{Q}} \to \mathbb{Q}$ (or $\mathbb{R}$) by $f(x) = 2x^2 - 1$, then $0 \notin f([0, 1]_{\mathbb{Q}})$ even though $f(0) = -1 < 0 < 1 = f(1)$. ♠

We conclude this review of basic properties of continuous functions with a definition and result that shows the utility of working with sequential continuity.

Definition 2.4.13. If X is a non-empty subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$, then f is *uniformly continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_0) - f(x)| < \varepsilon, \text{ whenever } x_0, x \in X, \text{ and } |x_0 - x| < \delta.$$

Remarks 2.4.14. (1) A uniformly continuous function on X is continuous.
(2) The point of the definition of uniform continuity is that δ can be chosen to be independent of x_0.

THEOREM 2.4.15. *Every continuous real valued function defined on a closed and bounded interval $[a,b]$ is uniformly continuous.*

PROOF. Suppose $f : [a,b] \to \mathbb{R}$ is continuous but not uniformly continuous. If f is not uniformly continuous, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there is a pair $x, y \in [a,b]$, with $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Choose $\delta = 1/n$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, we can find points $x_n, y_n \in [a,b]$ such that

$$|f(x_n) - f(y_n)| \geq \varepsilon, \text{ and } |x_n - y_n| < \frac{1}{n}.$$

By proposition 2.4.3, $(x_n) \subset [a,b]$ has a convergent subsequence, say (x_{n_k}). Let $\lim_{k\to\infty} x_{n_k} = x^\star \in [a,b]$. Since $|x_{n_k} - y_{n_k}| < 1/n_k$, we have

$$\begin{aligned} |x^\star - y_{n_k}| &= |(x^\star - x_{n_k}) + (x_{n_k} - y_{n_k})|, \\ &\leq |x^\star - x_{n_k}| + |x_{n_k} - y_{n_k}|, \\ &< |x^\star - x_{n_k}| + \frac{1}{n_k}. \end{aligned}$$

Letting $k \to \infty$, we see that (y_{n_k}) is convergent with limit $x^\star$. By the sequential continuity of f, we have $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(x^\star)$ and hence $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$, contradicting our assumption that $|f(x_n) - f(y_n)| \geq \varepsilon$, all $n \in \mathbb{N}$. Hence f must be uniformly continuous. □

EXERCISES 2.4.16.

(1) Let (x_n) be a bounded sequence of real numbers. Give a proof based on the subdivision method used in the proof of the Bolzano-Weierstrass theorem to show that (x_n) has a convergent subsequence. (For your proof you should not need to distinguish the cases where $\{x_n \mid n \in \mathbb{N}\}$ is finite or infinite — as a subset of $\mathbb{R}$.)
(2) Find a countable infinite subset X of $\mathbb{R}$ such that if $(x_n) \subset X$ is convergent, then (x_n) is eventually constant and the limit of (x_n) lies in X ((x_n) is *eventually constant* if $\exists x$, $\exists N \in \mathbb{N}$ such that $x_n = x$, $n \geq N$. Eventually constant sequences always converge).
(3) Let X be a non-empty subset of $\mathbb{R}$. We say that $x \in \mathbb{R}$ is a *closure point* of X if we can find a sequence $(x_n) \subset X$ which converges to x. Denote the set of closure points of X by $\overline{X}$. Why is it true that $\overline{X} \supset X$?
    (a) Find an example of a countably infinite unbounded set X of $\mathbb{R}$ such that $\overline{X} = X$.
    (b) Find an example of a countably infinite bounded subset X of $\mathbb{R}$ such that $\overline{X} = X$.
    (c) Find an example of a countably infinite bounded subset of $[0,1]$ such that $\overline{X} \setminus X = \{0, \frac{1}{2}, 1\}$,
    (d) Find an example of a countably infinite subset X of $\mathbb{R}$ such that $\overline{X} = \mathbb{R}$.
(4) By theorem 2.4.10, if f is a **continuous** $\mathbb{R}$-valued map on a **closed** and **bounded** interval, then f is *bounded* and *attains its bounds*. Show by means of examples that each of the conditions **continuous**, **closed**, and **bounded** is necessary for either of

the conclusions *bounded*, *attains its bounds* to hold. (Note: You should construct a total of six examples to cover all the possibilities. For example, a discontinuous map on a closed and bounded interval may be unbounded or bounded but not attain its bounds.)

(5) Show that a continuous map $f : [0,1] \cap \mathbb{Q} \to \mathbb{R}$ need not be uniformly continuous.

(6) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= 10^{-s}, \text{ if } x = \frac{r}{s} \in \mathbb{Q},\ (r,s) = 1, s > 0, \text{ and } s = 1 \text{ if } r = 0 \\ &= 0, \text{ if } x \notin \mathbb{Q}. \end{aligned}$$

Prove that f is continuous at x iff x is irrational. (Hint: You may assume that every rational r can be approximated by irrationals — $r + 10^{-n}\sqrt{2}$; that will help you prove that f is not continuous at rational points.)

(7) For $A \subset \mathbb{R}$, define $A_{\mathbb{Q}} = A \cap \mathbb{Q}$. True of false? In each case either *prove* the result or provide a *simple explicit counter-example*.

(a) If $f : [0,1]_{\mathbb{Q}} \to \mathbb{Q}$ is continuous, then f is bounded.
(b) If $f : [0,1]_{\mathbb{Q}} \to \mathbb{Q}$ is continuous and bounded, then f attains its bounds.
(c) The Intermediate Value Theorem holds for continuous functions $f : [a,b]_{\mathbb{Q}} \to \mathbb{Q}$.

How would your answers change if instead we looked at continuous maps $f : [0,1] \to \mathbb{Q}$? (Be advised: the answers change! As a hint: Since $\mathbb{Q} \subset \mathbb{R}$, every continuous $f : [0,1] \to \mathbb{Q}$ determines a continuous $\mathbb{R}$-valued map $F : [0,1] \to \mathbb{R}$ with image $f([0,1])$ consisting of rational numbers.)

(8) Show that $f(x) = 1/x$ is not uniformly continuous on $(0,1)$ but is uniformly continuous on $(1,2)$.

(9) Find examples of functions $f : \mathbb{R} \to \mathbb{R}$ which are (a) uniformly continuous, (b) not uniformly continuous.

(10) A common proof of theorem 2.4.10(1) proceeds along the following lines. Let

$$X = \{x \in [a,b] \mid f \text{ bounded on } [a,x]\}.$$

Show that (a) $X \neq \emptyset$, (b) $\sup(X) \in X$, (c) $\sup(X) = b$. Fill in details and use similar methods to prove parts (2,3) of theorem 2.4.10. (Comment: The deficit of this approach is that it does not extend well to functions defined on more general sets, for example, subsets of $\mathbb{R}^n$, since it makes use of the order structure on $\mathbb{R}$.)

2.4.2. Application to Cauchy sequences. Equipped with the Bolzano-Weierstrass theorem we can now give a satisfactory intrinsic definition of convergent sequence which does not depend on knowing the limit.

DEFINITION 2.4.17. A sequence (x_n) of real numbers is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon, \text{ for all } n, m \geq N.$$

REMARKS 2.4.18. (1) Roughly speaking, a Cauchy sequence has the property that terms in the sequence eventually get arbitrarily close to one another.
(2) Just as in lemma 2.2.7, we can replace $< \varepsilon$ by $\leq \varepsilon$ in the definition.

Indeed it is enough the test the truth of the definition for any sequence (κ_n) of strictly positive numbers converging to zero.

EXAMPLE 2.4.19. Let $x \in \mathbb{R}$. The sequence of decimal approximations $(x^n = x_0.x_1 \ldots x_n)$ to x defines a Cauchy sequence — $|x^n - x^m| \leq 10^{-n}$, $m \geq n$. ♠

We need the following elementary lemma about Cauchy sequences (this result is also true if we work over $\mathbb{Q}$).

LEMMA 2.4.20. *Let (x_n) be a sequence of real numbers.*

(1) *If (x_n) is Cauchy, then $\{x_n \mid n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$.*
(2) *If (x_n) is convergent, then (x_n) is Cauchy.*
(3) *If (x_n) is Cauchy and (x_n) has a convergent subsequence, then (x_n) is convergent.*

PROOF. (1) Take $\varepsilon = 1$ in definition 2.4.17. Then there exists $N \in \mathbb{N}$ so that $|x_n - x_m| \leq 1$, for all $n, m \geq N$. Taking $m = N$, we see that $|x_n - x_N| \leq 1$, for all $n \geq N$ and so $|x_n| \leq |x_N| + 1$, $n \geq N$. Hence $|x_n| \leq \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\}$ for all $n \geq 1$ proving that $\{x_n \mid n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$.
(2) Suppose $\lim_{n\to\infty} x_n = x^\star$. Let $\varepsilon > 0$. Since (x_n) converges to $x^\star$ we can choose $N \in \mathbb{N}$ such that $|x^\star - x_n| < \varepsilon/2$, $n \geq N$. We have

$$\begin{aligned} |x_n - x_m| &= |x^\star - x_m + x_n - x^\star|, \\ &\leq |x^\star - x_m| + |x^\star - x_n|, \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ if } n, m \geq N. \end{aligned}$$

(3) Finally suppose that (x_{n_k}) is a convergent subsequence of (x_n) with limit $x^\star$. Given $\varepsilon > 0$, we can choose $N_1 \in \mathbb{N}$ such that $|x^\star - x_{n_k}| < \varepsilon/2$, provided $n_k \geq N_1$ (it is easier to work with n_k here as opposed to the index k). Since (x_n) is Cauchy, we can choose $N_2 \in \mathbb{N}$ so that $|x_n - x_m| < \varepsilon/2$, $n, m \geq N_2$. Set $N = \max\{N_1, N_2\}$. For all $n, n_k \in \mathbb{N}$ we have

$$|x^\star - x_n| = |x^\star - x_{n_k} + x_{n_k} - x_n| \leq |x^\star - x_{n_k}| + |x_{n_k} - x_n|.$$

Fix $n_k \geq N$. Then for all $n \geq N$, we have $|x^\star - x_{n_k}| + |x_{n_k} - x_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, proving that (x_n) converges to $x^\star$. □

We can now state and prove our main result on Cauchy sequences.

THEOREM 2.4.21. *A sequence (x_n) of real numbers is convergent iff (x_n) is Cauchy.*

PROOF. By lemma 2.4.20(2), if (x_n) is convergent, then (x_n) is Cauchy. Conversely, if (x_n) is Cauchy then by lemma 2.4.20(1), (x_n)

is bounded and so, by proposition 2.4.3, (x_n) has a convergent subsequence. Apply lemma 2.4.20(3). $\square$

REMARKS 2.4.22. (1) Theorem 2.4.21 fails over $\mathbb{Q}$. Indeed, the sequence of finite decimal approximations to an irrational number provides an example of a non-convergent Cauchy sequence in $\mathbb{Q}$.
(2) We can use theorem 2.4.21 as the basis for a more intrinsic (though perhaps less transparent) definition of the real numbers that does not depend on working to a particular base. Specifically, consider the set $\mathcal{C}$ of all Cauchy sequences of rational numbers. We define an equivalence relation $\sim$ on $\mathcal{C}$ by $(x_n) \sim (y_n)$ iff $\lim_{n\to\infty} |x_n - y_n| = 0$. In particular, if one or other sequence converges, then both do with the same limit. We define the set of real numbers as the set of $\sim$ equivalence classes and then prove theorem 2.4.21 directly without recourse to the Bolzano-Weierstrass theorem. Modulo the abstraction of using equivalence classes, what we are doing with this general construction is defining real numbers by (all of) their rational approximations.

An application to multiplication and division. Once we know Cauchy sequences of real numbers converge it is easy to define the operations of multiplication and division on $\mathbb{R}$.

Suppose $x, y \in \mathbb{R}$. Let (x_n), (y_n) be sequences of rational numbers converging to x, y respectively. We want to define $xy = \lim_{n\to\infty} x_n y_n$. For this to work we need to check that (a) $(x_n y_n)$ is convergent and (b) the limit of $(x_n y_n)$ is independent of the choice of sequences (x_n), (y_n) converging to x, y. We verify (a) and leave (b) to the exercises. What we shall show is that $(x_n y_n)$ is a Cauchy sequence. For this, observe that

$$\begin{aligned} |x_n y_n - x_m y_m| &\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m|, \\ &= |x_n||y_n - y_m| + |y_m||x_n - x_m|. \end{aligned}$$

By lemma 2.4.20(1), there exists $M > 0$ such that $|x_n|, |y_m| \leq M$, for all $n, m \in \mathbb{N}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $|y_n - y_m|, |x_n - x_m| < \frac{\varepsilon}{2M}$, $n, m \geq N$. We have

$$\begin{aligned} |x_n y_n - x_m y_m| &\leq |x_n||y_n - y_m| + |y_m||x_n - x_m|, \\ &< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon, \text{ for } n, m \geq N. \end{aligned}$$

Hence $(x_n y_n)$ is Cauchy.

We use exactly the same process to define division of real numbers. Finally, we may deduce all the standard laws of arithmetic for real numbers from the corresponding laws for rational numbers. For example, the distributive law $x(y+z) = xy + xz$ follows from

$$x(y+z) = \lim_{n\to\infty} x_n(y_n + z_n) = \lim_{n\to\infty} x_n y_n + \lim_{n\to\infty} x_n z_n = xy + yz.$$

EXERCISES 2.4.23.

(1) Verify that (x^n) is a Cauchy sequence if $|x| < 1$.
(2) Find examples of sequences (x_n) such that
(a) for every $p \in \mathbb{N}$, $\lim_{n\to\infty} |x_{n+p} - x_n| = 0$, (b) (x_n) is not Cauchy. (Hint: try $x_n = \log(n+1)$.)
(3) Suppose that (x_n) is a sequence of real numbers and there exists $k \in (0,1)$ such that $|x_{n+1} - x_n| < k|x_n - x_{n-1}|$ for all $n \geq 2$. Show that (x_n) is a Cauchy sequence. Show by means of an example that if we allow $k \in (0,1)$ to depend on n, then this result may fail and (x_n) may not converge. (You may assume the infinite series $\sum_{n=1}^{\infty} n^{-1}$ diverges to $+\infty$.)
(4) Complete the proof of the definition of multiplication on $\mathbb{R}$ by showing that the limit of $(x_n y_n)$ is the same for all rational sequences (x_n) converging to x and (y_n) converging to y.
(5) Show how we define division by non-zero real numbers.
(6) Suppose that $f : \mathbb{Q} \to \mathbb{R}$ is uniformly continuous. Show that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in \mathbb{Q}$ (we say "F is a continuous *extension* of f to $\mathbb{R}$"). Find an example of a continuous (but not uniformly continuous) function $g : \mathbb{Q} \to \mathbb{R}$ which does *not* extend to $\mathbb{R}$. (Hint for first part: Show that every Cauchy sequence $(q_n) \subset \mathbb{Q}$ is mapped by f to a Cauchy sequence $(f(q_n)) \subset \mathbb{R}$. Remember to verify F is well-defined and does not depend on the particular choice of Cauchy sequence.)

2.5. $\limsup$ and $\liminf$

Suppose that $(x_n) \subset \mathbb{R}$ is a bounded sequence. For $n \geq 1$, let $X_n = \{x_m \mid m \geq n\}$. We define

$$\alpha_n = \inf(X_n), \quad \beta_n = \sup(X_n).$$

Since (x_n) is a bounded sequence, we have

$$-\infty < \alpha_n \leq \beta_n < +\infty,$$

for all $n \geq 1$. Moreover, since $X_1 \supset X_2 \supset \ldots$, we have

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1.$$

It follows by theorem 2.3.10 that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ exist and that

$$\lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \beta_n. \tag{2.1}$$

We define $\liminf x_n = \lim_{n\to\infty} \alpha_n$ and $\limsup x_n = \lim_{n\to\infty} \beta_n$. Alternative (and commonly used) notations are $\overline{\lim}\ x_n$ for $\limsup x_n$ and $\underline{\lim}\ x_n$ for $\liminf x_n$.

LEMMA 2.5.1. *If (x_n) is a bounded sequence of real numbers, then*

(1) $\liminf x_n \leq \limsup x_n$.
(2) $\liminf x_n = \limsup x_n$ *iff* (x_n) *is convergent. If* (x_n) *is convergent then the limit of* (x_n) *must be the common value of* $\liminf x_n$ *and* $\limsup x_n$.
(3) *There exists a subsequence of* (x_n) *converging to* $\liminf x_n$. *Similarly for* $\limsup x_n$.
(4) *If* (x_{n_k}) *is a convergent subsequence of* (x_n) *then*

$$\lim_{k\to\infty} x_{n_k} \in [\liminf x_n, \limsup x_n].$$

PROOF. (1) is immediate from (2.1). We leave the remainder of the proof to the exercises. □

REMARKS 2.5.2. (1) Observe that lemma 2.5.1(3) gives an alternative proof of proposition 2.4.3. It does not, however, give a proof of the Bolzano-Weierstrass theorem.
(2) We can extend the definition of $\limsup$ and $\liminf$ to unbounded sequences and give expressions like $\limsup x_n = +\infty$ the obvious meaning.

EXAMPLE 2.5.3. Suppose that (x_n) is a Cauchy sequence. By lemma 2.4.20, (x_n) is bounded. It follows from the definition of Cauchy sequence that for every $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so that $|\inf\{x_n \mid n \geq N\} - \sup\{x_n \mid n \geq N\}| < \varepsilon$. Hence $\liminf x_n = \limsup x_n$ and (x_n) is convergent by lemma 2.5.1(2). ♠

EXERCISES 2.5.4.

(1) Complete the proof of lemma 2.5.1.

2.6. Complex numbers

In section we briefly recall some elementary definitions and properties of complex numbers and then indicate how our results on sequences (and series) of real numbers extend straightforwardly to complex numbers).

2.6.1. Review of complex numbers. A *complex number* $z = x + \imath y$ may be identified with the point $(x, y) \in \mathbb{R}^2$. We define addition and subtraction of complex numbers $z_1 = x_1 + \imath y_1$ and $z_2 = x_2 + \imath y_2$ by

$$z_1 \pm z_2 = (x_1 \pm x_2) + \imath(y_1 \pm y_2).$$

This corresponds to vector addition in $\mathbb{R}^2$. We define multiplication of complex numbers by

$$z_1 z_2 = (x_1 + \imath y_1)(x_2 + \imath y_2) = (x_1 x_2 - y_1 y_2) + \imath(x_1 y_2 + x_2 y_1).$$

On $\mathbb{R}^2$, multiplication is given by $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. It is easy to check that multiplication is commutative ($z_1z_2 = z_2z_1$), associative ($(z_1(z_2z_3) = (z_1z_2)z_3$) and that the distributive law holds

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$

If we let $\mathbb{C} \approx \mathbb{R}^2$ denote the set of complex numbers, then the real numbers $\mathbb{R}$ are naturally defined as the subset $\{(x, 0) \mid x \in \mathbb{R}\}$ of $\mathbb{C}$. Thus $z \in \mathbb{C}$ is real iff we can write $z = x + \imath 0$. With this convention, $1 = (1, 0) = 1+\imath 0, 0 = (0, 0)$ and we have $1z = z1 = z$, $z+0 = 0+z = z$ for all complex numbers z.

Since $\imath^2 = (0, 1) \times (0, 1) = -(1, 0) = -1$, we have $\imath^2 = -1$. We say a complex number z is *(pure) imaginary* if $z = \imath y$ for some $y \in \mathbb{R}$. The square of every imaginary number is negative.

We define the *modulus* $|z|$ of $z = x + \imath y \in \mathbb{C}$ by

$$|z| = \sqrt{x^2 + y^2}.$$

Of course, $|z|$ is the Euclidean length of the vector $(x, y) \in \mathbb{R}^2$. It is straightforward to verify that $|z_1z_2| = |z_1||z_2|$, for all $z_1, z_2 \in \mathbb{C}$. If $z \in \mathbb{C}$ is real, then $|z|$ is the absolute value of z. We have the important triangle-inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|, \text{ for all } z_1, z_2 \in \mathbb{C}.$$

Let $c : \mathbb{C} \to \mathbb{C}$ be the (real) linear map defined by

$$c(x + \imath y) = x - \imath y.$$

We refer to c as *complex conjugation* and write $c(z) = \bar{z}$. Observe that $\bar{z} = z$ iff z is real and $\bar{z} = -z$ iff z is imaginary. Since $z\bar{z} = (x + \imath y)(x - \imath y) = x^2 - \imath^2 y^2 = x^2 + y^2$, we have

$$|z|^2 = z\bar{z}.$$

If $|z| = 1$, then z is a point on the unit circle $x^2 + y^2 = 1$.

Referring to figure 1, z defines a unique $\theta \in [0, 2\pi)$ such that $z = \cos\theta + \imath\sin\theta$ (that is, the Cartesian coordinates of z are $(\cos\theta, \sin\theta)$). If we *define* $e^{\imath\theta} = \cos\theta + \imath\sin\theta$, then $e^{-\imath\theta} = \cos\theta - \imath\sin\theta$ and so

$$\cos\theta = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2}, \ \sin\theta = \frac{e^{\imath\theta} - e^{-\imath\theta}}{2\imath}.$$

Remark 2.6.1. If we substitute $x = \imath\theta$ in the exponential series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then we recover the standard infinite series for $\cos\theta$ and $\sin\theta$. We return to this point in chapter 5.

We may use standard trigonometric identities to verify

$$e^{\imath 0} = 1, \ \overline{e^{\imath\theta}} = e^{-\imath\theta}, \ e^{\imath\theta}e^{\imath\phi} = e^{\imath(\theta+\phi)}$$

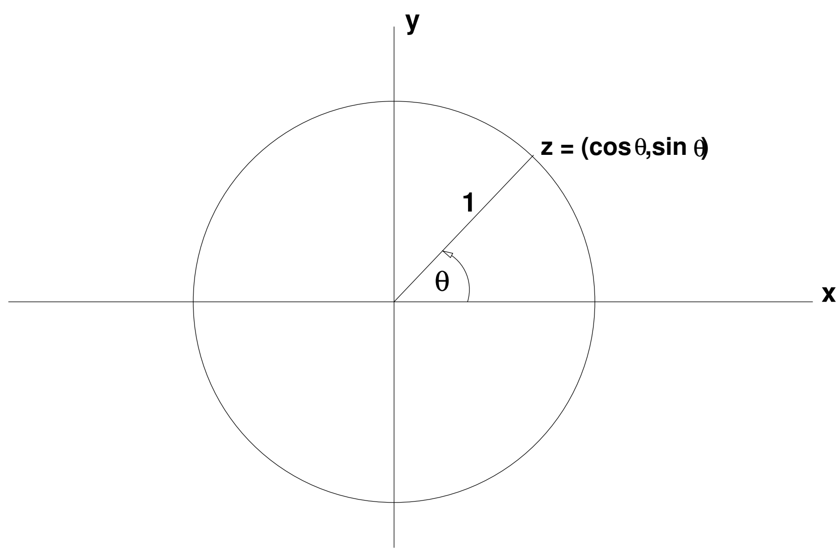


FIGURE 1. Complex number of unit modulus

In particular, for $n \in \mathbb{Z}$ we have De Moivre's formula

$$(e^{\imath\theta})^n = e^{\imath n\theta}.$$

We leave to the exercises the proof that if $a \in \mathbb{C}$ and $ae^{\imath\theta} \neq 1$, then

$$\sum_{p=0}^{n} a^p e^{\imath p\theta} = \frac{(1 - a^{n+1}e^{\imath(n+1)\theta})}{1 - ae^{\imath\theta}}.$$

If $z \neq 0$, there exists a unique $\theta \in [0, 2\pi)$ such that

$$z = |z|e^{\imath\theta}.$$

For this we observe that $u = z/|z|$ lies on the unit circle and so defines a unique $\theta \in [0, 2\pi)$ as described above. We call $z = |z|e^{\imath\theta}$ the *modulus and argument* form of z. If $z = x + \imath y$, then $r = |z|, \theta$ are the polar coordinates of (x, y).

Multiplication takes a particularly simple form if we use the modulus and argument representation of complex numbers. If $z_1 = |z_1|e^{\imath\theta_1}$ and $z_2 = |z_1|e^{\imath\theta_1}$, then

$$z_1 z_2 = |z_1||z_2|e^{\imath\theta_1}e^{\imath\theta_1} = |z_1 z_2|e^{\imath(\theta_1+\theta_2)}.$$

2.6.2. Sequences of complex numbers.

DEFINITION 2.6.2. A sequence (z_n) of complex numbers is *convergent* if there exists $z \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |z - z_n| = 0.$$

We call z the *limit* of the sequence (z_n) and write $\lim_{n\to\infty} z_n = z$.

REMARK 2.6.3. The definition is formally identical to that of convergence of a real sequence with the proviso that we replace absolute value by the modulus.

The next lemma allows us to switch easily between real and complex sequences.

LEMMA 2.6.4. *let (z_n) be a sequence of complex numbers. If we write $z_n = x_n + \imath y_n$, then (z_n) is convergent iff both the real sequences (x_n) and (y_n) are convergent.*

PROOF. We start by observing that if $z = x + \imath y$ then

$$|x|, |y| \le |z| \le |x| + |y|. \tag{2.2}$$

Suppose that (z_n) is convergent with limit $z = x + \imath y$. By definition, $\lim_{n\to\infty} |z - z_n| = 0$. By the left hand inequality of (2.2), we have $|x - x_n|, |y - y_n| \le |z - z_n|$, for all $n \in \mathbb{N}$. Hence, by the squeezing lemma, $\lim_{n\to\infty} |x - x_n|, |y - y_n| = 0$ and the sequences (x_n), (y_n) converge with respective limits x and y. The converse is equally simple using the right hand inequality of (2.2). $\square$

DEFINITION 2.6.5. A sequence (z_n) complex numbers is a *Cauchy sequence* if $\lim_{n,m\to\infty} |z_n - z_m| = 0$.

THEOREM 2.6.6. *A sequence (z_n) complex numbers is Cauchy iff it is convergent.*

PROOF. We leave the proof, which is an easy consequence of theorem 2.4.21 and lemma 2.6.4, to the exercises. $\square$

REMARK 2.6.7. Lemma 2.6.4 and theorem 2.4.21 enable us to extend most of our results on real sequences (and series) to complex sequences and series. In subsequent chapters, we generally indicate these extensions by remarks rather than developing the complex theory separately.

EXERCISES 2.6.8.

(1) Verify that $|z_1 z_2| = |z_1||z_2|$ and $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ for all $z_1, z_2 \in \mathbb{C}$.
(2) Complete the proof of theorem 2.6.6.
(3) Verify the formula for the sum of a geometric series. (Hint: multiply both sides by $1 - ae^{\imath\theta}$.)

(4) A subset A of $\mathbb{C}$ is *bounded* if there exists $C \geq 0$ such that $|z| \leq C$ for all $z \in A$. Show that if A is an infinite bounded subset of $\mathbb{C}$ then there exists a convergent subsequence (z_n) consisting of distinct points of A. (Hint: Use the Bolzano Weierstrass theorem twice and lemma 2.6.4.)
(5) Show that every bounded sequence of complex numbers has a convergent subsequence.
(6) Show that a continuous function $f : [a, b] \to \mathbb{C}$ is bounded and attains its bounds.

2.7. Appendix: the Riemann integral

Suppose that f is a *continuous* function defined on a closed and bounded interval. In this appendix we show how to define, construct and compute the Riemann integral of f. We make use of two results: theorem 2.4.10(1,2) (f is bounded on a closed bounded interval and attains its bounds) and the Mean Value theorem (that uses theorem 2.4.10(2), see the exercises).

Rather than defining the integral of f by approximating upper and lower sums, we instead state two very simple properties that the integral should possess. We show these properties are reasonable by verifying that they give the correct areas under the graph of a constant function (area of a rectangle) and under the graph of $y = x$ (area of a triangle). We then prove that these properties *uniquely determine* the integral if it exists. It is then almost a *triviality* to observe that if f has an anti-derivative F ($F' = f$) then the integral from a to x of f exists and is equal to $F(x) - F(a)$ (the fundamental theorem of calculus). We conclude with an elementary proof of the main theoretical result that every continuous function defined on a closed interval has an anti-derivative, This result amounts to an existence theorem for solutions of the ordinary differential equation $\frac{dy}{dx} = f(x)$. We briefly indicate how to extend our definition of the integral to include bounded functions with at most countably many discontinuities.

If f is everywhere positive then we think of the integral of f as the area under the graph of f. If f is everywhere negative, then the corresponding integral will be negative and equal to minus the area under the graph of $-f$.

2.7.1. Two basic properties required of the integral. Let f be a real valued continuous function with domain $\mathcal{D} \subset \mathbb{R}$, where $\mathcal{D}$ is an *interval* which may be open, closed, half-open or unbounded. We assume f is bounded on all closed intervals $[a, b] \subset \mathcal{D}$. In particular, any continuous function f will satisfy this condition by theorem 2.4.10.

DEFINITION 2.7.1. A function $\mathcal{I}(x, y)$, with domain $\mathcal{D} \times \mathcal{D}$, is a (definite) integral for f if

(1) Given $a \leq b \leq c$, $a, b, c \in \mathcal{D}$, we have

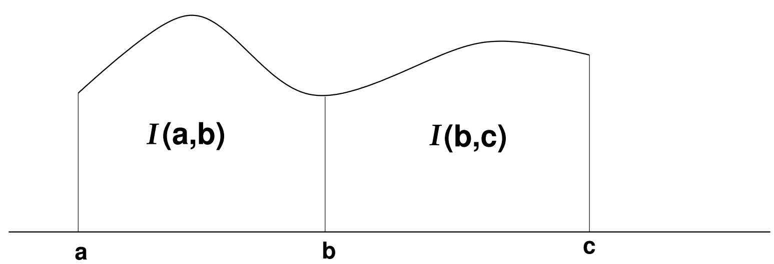
$$\mathcal{I}(a,c) = \mathcal{I}(a,b) + \mathcal{I}(b,c). \tag{2.3}$$

(2) Given $a < b$, $a, b \in \mathcal{D}$,

$$m(b-a) \leq \mathcal{I}(a,b) \leq M(b-a), \tag{2.4}$$

where m is any lower bound for f on $[a,b]$ and M is any upper bound for f on $[a,b]$.

Remark 2.7.2. We should emphasize that $\mathcal{I}(x,y)$ depends on f — we could have written $\mathcal{I}_f$ rather than $\mathcal{I}$ but we prefer the simpler notation with the understanding that the function f remains fixed.



I(a,c) = I(a,b) + I(b,c)

Figure 2. Condition (2.3)

In figure 2 we show the meaning of (2.3). The condition clearly implies that if we choose any finite sequence $a = x_0 < x_1 \ldots < x_N = b$, then

$$\mathcal{I}(a,b) = \sum_{n=0}^{N-1} \mathcal{I}(x_n, x_{n+1}). \tag{2.5}$$

Turning to the second condition, assume for the moment that f is positive (as in figure 3). The first inequality $m(b-a) \leq \mathcal{I}(a,b)$ of (2.4) says that whatever $\mathcal{I}(a,b)$ is, it cannot be smaller than the area of the *largest* rectangle with base $[a,b]$ that we can fit *under* the graph of f. Similarly, the second inequality $\mathcal{I}(a,b) \leq M(b-a)$ implies that $\mathcal{I}(a,b)$ can be no larger than the area of the smallest rectangle with base $[a,b]$ that contains the graph of f.

Examples 2.7.3. (1) Let $f(x) = C$ be a constant function. Then $\mathcal{I}(a,b) = C(b-a)$: may take $m = M = C$ and the result is immediate from (2.4). Notice that this gives the (signed) area of the rectangle

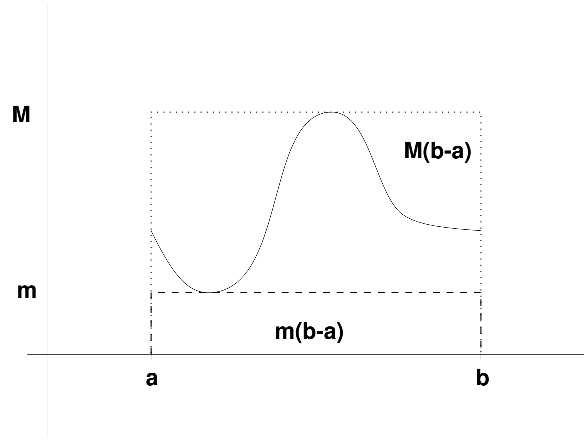


FIGURE 3. Condition (2.4)

with base $[a, b]$ and height C. Of course, if $C > 0$ we get the usual unsigned area. If f is negative, then $\mathcal{I}(a, b)$ is 'signed' and negative.
(2) If f takes positive and negative values there can be cancellation between the positive and negative parts of $\mathcal{I}(a, b)$ and so the inequality (2.4) is weaker — see figure 4 where $m = -1, M = +1$ and $\mathcal{I}(a, b) = 0$ (using (2.3) and the previous example). ♠

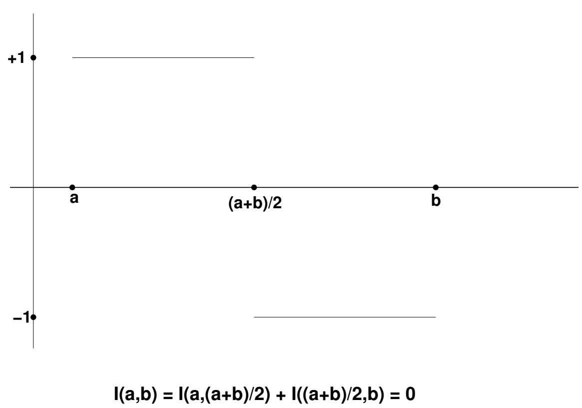


FIGURE 4. A case where $\mathcal{I}(a, b) = 0$

As we shall soon see (2.3, 2.4) uniquely characterize the function $\mathcal{I}(x, y)$ when f is continuous.

In practice, it is useful to extend (2.3) to allow for arbitrary triples $a, b, c \in \mathcal{D}$. First note that if $a = b = c$ we get

$$\mathcal{I}(a, a) = \mathcal{I}(a, a) + \mathcal{I}(a, a)$$

and so $\mathcal{I}(a,a) = 0$. It follows that if we want to define $\mathcal{I}(a,b)$ when $b < a$ we must take $\mathcal{I}(a,b) = -\mathcal{I}(a,b)$ since for (2.3) to hold (for a, b, a) we need

$$0 = \mathcal{I}(a,a) = \mathcal{I}(a,b) + \mathcal{I}(b,a)$$

With these conventions, it is easy to check that if (2.3) holds then we have

$$\mathcal{I}(a,c) = \mathcal{I}(a,b) + \mathcal{I}(b,c),$$

for *all* $a, b, c \in \mathcal{D}$.

Summarizing, we henceforth suppose that $\mathcal{I}(x,y)$ satisfies

(**I**) For all $a, b, c \in \mathcal{D}$ we have

$$\mathcal{I}(a,c) = \mathcal{I}(a,b) + \mathcal{I}(b,c).$$

(**II**) Given $a < b$, $a, b \in \mathcal{D}$,

$$m(b-a) \leq \mathcal{I}(a,b) \leq M(b-a),$$

where m is any lower bound for f on $[a,b]$ and M is any upper bound for f on $[a,b]$.

EXAMPLE 2.7.4. Conditions (**I,II**) allow us to do some simple computations. For example, suppose $f(x) = x$ and we take $a = 0$, $b = 1$. We compute $\mathcal{I}(0,1)$. Take the subdivision $0, 1/N, 2/N, \ldots, (N-1)/N, 1$ of $[0,1]$. Applying (2.5), we have

$$\mathcal{I}(0,1) = \sum_{n=0}^{N-1} \mathcal{I}(\frac{n}{N}, \frac{n+1}{N}).$$

On the interval $[\frac{n}{N}, \frac{n+1}{N}]$ we have the bounds $\frac{n}{N} \leq x \leq \frac{n+1}{N}$. Hence $n/N^2 \leq \mathcal{I}(\frac{n}{N}, \frac{n+1}{N}) \leq (n+1)/N^2$. Summing from $n = 0$ to $N = 1$, we obtain the estimate

$$\sum_{n=0}^{N-1} \frac{n}{N^2} \leq \mathcal{I}(0,1) \leq \sum_{n=0}^{N-1} \frac{n+1}{N^2}.$$

The arithmetic progressions $0, 1, \ldots, N-1$ and $1, 2, \ldots, N$ have respective sums $N(N-1)/2$ and $N(N+1)/2$ and so we have (cancelling an N)

$$\frac{N-1}{2N} \leq \mathcal{I}(0,1) \leq \frac{N+1}{2N}$$

This estimate holds for all $N \geq 1$. Letting $N \to \infty$, the squeezing lemma implies that $\mathcal{I}(0,1) = \frac{1}{2}$ (the area of the triangle of base 1 and height 1). ♠

2.7.2. Existence of $\mathcal{I}(x, y)$, part 1. Recall that $f : \mathcal{D} \to \mathbb{R}$ has an *anti-derivative* if there exists a differentiable function $F : \mathcal{D} \to \mathbb{R}$ such that $F' = f$ on $\mathcal{D}$. We show that if f is continuous and has an anti-derivative, then we can define $\mathcal{I}(\ ,\ )$ satisfying properties **I, II**.

This is very easy. Suppose f has anti-derivative F. Define

$$\mathcal{I}(a, b) = F(b) - F(a),\ a, b \in \mathcal{D}$$

Since $(F(b)-F(a))+(F(c)-F(b)) = (F(c)-F(a))$ (for all $a, b, c \in \mathcal{D}$), it is obvious that $\mathcal{I}(a, b)$ satisfies **I**. It remains to show that $\mathcal{I}$ satisfies **II**. Suppose $a, b \in \mathcal{D}$, $a < b$. Let M, m be upper and lower bounds for f on $[a, b]$. By the Mean Value theorem, we can find $x \in (a, b)$ so that

$$\mathcal{I}(a, b) = F(b) - F(a) = F'(x)(b - a) = f(x)(b - a).$$

Since $m \leq f(x) \leq M$ on $[a, b]$, it follows immediately that

$$m(b - a) \leq f(x)(b - a) = \mathcal{I}(a, b) \leq M(b - a),$$

proving **II**.

2.7.3. Uniqueness of $\mathcal{I}(x, y)$, f continuous. Suppose that given a continuous function f with domain $\mathcal{D}$, we can find $\mathcal{I}(a, b)$ satisfying **I, II**. We show that $\mathcal{I}(x, y)$ is unique.

Fix $a \in \mathcal{D}$ and define $F(x) = \mathcal{I}(a, x)$, $x \in \mathcal{D}$. We claim that F is an anti-derivative of f: $F'(x) = f(x)$, $x \in \mathcal{D}$.

For this, we need to apply the definition of derivative. Fix $x \in \mathcal{D}$ and choose $h \in \mathbb{R}$ so that $x + h \in \mathcal{D}$. (If x is not an end-point of $\mathcal{D}$, we have $x + h \in \mathcal{D}$ for sufficiently small h. Otherwise we restrict to positive or negative values of h as appropriate.) By **I**, we have

$$\mathcal{I}(a, x) + \mathcal{I}(x, x + h) = \mathcal{I}(a, x + h),$$

Hence

$$\mathcal{I}(a, x + h) - \mathcal{I}(a, x) = \mathcal{I}(x, x + h).$$

Therefore, if $h \neq 0$,

$$\frac{\mathcal{I}(a, x + h) - \mathcal{I}(a, x)}{h} = \frac{\mathcal{I}(x, x + h)}{h}.$$

Let m_h and M_h respectively denote the infimum and supremum of f on $[x, x + h]$. Since f is continuous, $-\infty < m_h \leq M_h < \infty$ and $\lim_{h \to 0} M_h, m_h = 0$. Suppose first that $h > 0$. From **II**, we have

$$m_h h \leq \mathcal{I}(x, x + h) \leq M_h h.$$

and so

$$m_h \leq \mathcal{I}(x, x + h)/h \leq M_h.$$

Letting $h \to 0+$, we see

$$\lim_{h \to 0+} \mathcal{I}(x, x+h)/h = f(x).$$

If $h < 0$, then $\frac{\mathcal{I}(x,x+h)}{h} = \frac{\mathcal{I}(x+h,x)}{-h}$ (since $\mathcal{I}(x, x+h) = -\mathcal{I}(x+h, x)$). The argument now proceeds as above using **II** applied to the interval $[x+h, x]$ (note $x + h < x$) to give

$$\lim_{h \to 0-} \mathcal{I}(x, x+h)/h = f(x).$$

Hence we have shown that

$$\lim_{h \to 0} \frac{\mathcal{I}(a, x+h) - \mathcal{I}(a, x)}{h} = f(x).$$

and $F(x) = \mathcal{I}(a, x)$ is an anti-derivative of f.

We have shown that if f is continuous and we can find a function $\mathcal{I}(x, y)$ satisfying properties **I, II**, then $\mathcal{I}(a, x)$ is an anti-derivative of f for all a in the $\mathcal{D}$. On the other hand, if f has an anti-derivative F then $\mathcal{I}(a, b) = F(b) - F(a)$ satisfies properties **I, II**.

Since any two anti-derivatives of f differ by a constant[1], we see that if the function $\mathcal{I}(x, y)$ exists then it must be unique. In particular, if f has an anti-derivative $F(x)$ then f has a unique integral given by $\mathcal{I}(a, b) = F(b) - F(a)$.

Summarizing our arguments, we have proved

THEOREM 2.7.5. *Let $f : \mathcal{D} \to \mathbb{R}$ be a continuous function with anti-derivative F. Then there exists a unique function $\mathcal{I}(x, y)$ satisfying properties* **I, II**. *Moreover,*

(a) $\mathcal{I}(a, b) = F(b) - F(a)$, $a, b \in \mathcal{D}$.
(b) *Given $a \in \mathcal{D}$, the function $G(x) = \mathcal{I}(a, x)$ is an anti-derivative for f (fundamental theorem of the calculus).*

REMARK 2.7.6. Note that theorem 2.7.5 suffices for all the standard applications and examples in a first calculus course: all the functions considered invariably have an anti-derivative and so the definite (or indefinite) integral is given by the anti-derivative. No arguments needing approximating sums are needed.

In future we adopt the standard notation and set $\mathcal{I}(a, b) = \int_a^b f(x)\,dx$ and refer to $\int_a^b f(x)\,dx$ as the Riemann integral of f from a to b.

[1]By the Mean Value theorem, $F' - G' = 0 \Longrightarrow G = F + c$

2.7.4. Existence of the integral, part 2. In this section we prove

THEOREM 2.7.7. *Every continuous function $f : \mathcal{D} \to \mathbb{R}$ has an anti-derivative.*

Our proof of theorem 2.7.7 proceeds by constructing a function $L(x, y)$ that satisfies conditions **I, II**. This construction is quite straightforward and uses only parts (1,2) of theorem 2.4.10 (in particular, no use is made of results on uniform continuity).

Proof of theorem 2.7.7 Fix an interval $[a, b]$ and suppose that f is continuous on $[a, b]$. A *partition* $\mathcal{P}$ of $[a, b]$ consists of a finite number of points $t_0, \ldots, t_N$ satisfying

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b.$$

Given a partition $\mathcal{P}$, let m_j denote the minimum value of f on $[t_j, t_{j+1}]$. We define the (lower) sum $L(\mathcal{P}, f)$ by

$$L(\mathcal{P}, f) = \sum_{j=0}^{N-1} m_j(t_{j+1} - t_j).$$

If we let m, M denote the minimum and maximum values of f on $[a, b]$ then $m \leq m_j \leq M$ and so

$$m(b - a) \leq L(\mathcal{P}, f) \leq M(b - a). \tag{2.6}$$

If we add new points to $\mathcal{P}$, say to form $\mathcal{P}'$, then the reader may easily check that $L(\mathcal{P}', f) \geq L(\mathcal{P}, f)$. By (2.6), $M(b - a)$ is an upper bound for $\{L(\mathcal{P}, f) \mid \text{all partitions } \mathcal{P} \text{ of } [a, b]\}$ and so we may define

$$L(a, b) = \sup\{L(\mathcal{P}, f) \mid \text{all partitions } \mathcal{P} \text{ of } [a, b]\}.$$

By (2.6) $\{L(\mathcal{P}, f) \mid \text{all partitions } \mathcal{P} \text{ of } [a, b]\}$ is bounded above by $M(b-a)$ and below by $m(b - a)$. Hence

$$m(b - a) \leq L(a, b) \leq M(b - a).$$

This shows $L(a, b)$ satisfies **II**. Since we can always add a point b to a partition of $[a, c]$ $(a \leq b \leq c)$, it is easy to see that

$$L(a, b) + L(b, c) = L(a, c).$$

Thus $L(a, b)$ satisfies **I**. Hence $L(a, x)$ must be an anti-derivative of f. ☐

REMARKS 2.7.8. (1) We could have done the construction using 'upper' sums $U(\mathcal{P}, f)$. Since we have proved already the integral is unique, we get that $\sup_{\mathcal{P}} L(\mathcal{P}, f) = \inf_{\mathcal{P}} U(\mathcal{P}, f) = \int_a^b f(x)\, dx$.
(2) If we use uniform partitions $\mathcal{P}$ $(t_{j+1} - t_j = \delta$ is independent

of j), then we need the uniform continuity of f in order to prove that $\lim_{\delta\to 0} U(\mathcal{P}, f), L(\mathcal{P}, f) = \int_a^b f(x)\, dx$. But this assumption is *not* needed to prove the existence of the integral. Indeed, a great virtue of the Riemann integral is that once you know the integrand f has an anti-derivative F, then you can write down the integral in terms of F. Nothing is needed about approximating sums. The only technical difficulty with the Riemann integral is proving its existence for general continuous functions. This may be regarded as an existence theorem for ordinary differential equations: given a continuous function f, then the ordinary differential equation

$$\frac{dy}{dx} = f(x)$$

has a solution $y = F(x)$. If we specify the value of F at a point, then the solution is unique. (Later we address this existence theorem when f is a function of y, rather than x.)

2.7.5. Extensions. We can prove the existence of an integral satisfying **I, II** for any *bounded* function on a domain $\mathcal{D}$ which has at most countably many discontinuities. This is easy to do when there are finitely many discontinuities of f (see the exercises). We can also weaken the boundedness condition to allow for functions that grow slowly enough near singular points (for example $1/\sqrt{|x|}$ near zero) as well as allow for the definition of the integral on unbounded domains. We address these issues in more detail as and when they arise in the text.

Exercises 2.7.9.

(1) Suppose that the function $\mathcal{I}(x, y)$, $x, y \in \mathcal{D}$, satisfies condition **I** whenever $a \le b \le c$. Show that if we define $\mathcal{I}(a, b) = -\mathcal{I}(b, a)$ for $b < a$, then **I** holds for all $a, b, c \in \mathcal{D}$.
(2) Suppose that G is differentiable on (a, b), continuous on $[a, b]$ and $G(a) = G(b) = 0$. Show that if G takes strictly positive values and $G(x) = \sup G([a, b])$, then $G'(x) = 0$. Deduce that there always exists an $x \in (a, b)$ such that $G'(x) = 0$. Apply this result to
$$G(x) = f(x) - f(a) - \left(\frac{x-a}{b-a}\right)(f(b) - f(a))$$
to deduce the Mean Value theorem.
(3) Show that if $f : \mathcal{D} \to \mathbb{R}$ is continuous and $\mathcal{I}(x, y)$ satisfies **I, II**, then $\mathcal{I}(a, x)$ is a continuous function of $x \in \mathcal{D}$, a a fixed point of $\mathcal{D}$.
(4) Let $\mathcal{P}, \mathcal{P}'$ be two (finite) partitions of $[a, b]$. Show that if we define $\mathcal{Q} = \mathcal{P} \cup \mathcal{P}'$, then $L(\mathcal{Q}, f) \ge \max\{L(\mathcal{P}, f), L(\mathcal{P}', f)\}$.
(5) Let $n \in \mathbb{N}$ and let $\mathcal{P}_n$ be the partition of $[a, b]$ defined by $t_j = a + \frac{j(b-a)}{n}$, $0 \le j \le n$. Using the uniform continuity of f (theorem 2.4.15), verify that $\lim_{n\to\infty} L(\mathcal{P}_n, f) = \int_a^b f(t)\, dt$. (Hint: Show $\lim_{n\to\infty} U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f) = 0$.)

(6) Suppose $f : [a, b] \to \mathbb{R}$ has finitely many discontinuities. Show how we may define the definite integral $\int_a^b f(t)\,dt$ and verify that the derivative of $\int_a^x f(t)\,dt$ exists and equals $f(x)$ at all points x where $f(x)$ is continuous.

(7) Show that we can define $\int_a^b f(t)\,dt$ when f has countably many discontinuities so that (a) $\int_a^x f(t)\,dt$ is continuous on $[a, b]$, and (b) the derivative of $\int_a^x f(t)\,dt$ exists and equals $f(x)$ at all points x where $f(x)$ is continuous. (Hint: For $\varepsilon > 0$, construct a continuous function f_ε which equals f outside of a set of open intervals of length at most ε/M, where $|f| \leq M$ on $[a, b]$.)

CHAPTER 3

Infinite Series

3.1. Introduction

In this chapter we investigate convergence of infinite series.

We start by recalling some results from chapter 2. Let (a_n) be a sequence of real numbers. For $n \in \mathbb{N}$, we define the *partial sum* $S_n = \sum_{i=1}^{n} a_i$. We say the infinite series $\sum_{n=1}^{\infty} a_n$ is *convergent* if the sequence (S_n) of partial sums is convergent. If (S_n) is convergent then we define $\sum_{n=1}^{\infty} a_n$ to be equal to $\lim_{n\to\infty} S_n$.

REMARKS 3.1.1. (1) In general, the infinite series $\sum_{n=1}^{\infty} a_n$ should be thought of symbolically — as shorthand for the sequence of partial sums. When (and only when) the sequence is known to be convergent, we identify $\sum_{n=1}^{\infty} a_n$ with the limit of the corresponding sequence of partial sums. Of course, this is what we did previously in our description of real numbers. If x has decimal expansion $x_0.x_1\ldots$, then we identify x with the infinite sum $\sum_{n=0}^{\infty} x_n 10^{-n}$.
(2) We sometimes write "$\sum_{n=1}^{\infty} a_n < \infty$" to signify that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. A statement like "$\sum_{n=1}^{\infty} a_n = 5$" should be interpreted as saying that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and that the limit (of the sequence of partial sums) is equal to 5. We often say "the sum of the infinite series is 5".
(3) Although we shall not spell out the details, all the usual limit laws for sequences carry over to infinite series. For example, *if*[1] the infinite series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are both convergent, then so is the infinite series $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

There is one general necessary condition for convergence of an infinite series.

LEMMA 3.1.2. *Suppose the the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. Then $\lim_{n\to\infty} a_n = 0$. (This result holds without any restriction on the signs of the a_n.)*

PROOF. If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$. Hence $\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} a_n = 0$. $\square$

[1] An essential 'if'.

REMARK 3.1.3. Everything above extends immediately to infinite series of complex terms. In the next section we study series of postive terms and the results have no analog for complex series (basically because there is no natural order relation on the complex numbers).

3.2. Series of eventually positive terms

If the elements of (a_n) are all positive (respectively, eventually positive), then (S_n) is increasing (respectively, eventually increasing). As a consequence of theorem 2.3.10, we see that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent iff the sequence (S_n) of partial sums is bounded.

EXAMPLES 3.2.1. (1) The infinite series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ is convergent and $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1$. For $n \geq 2$ we have $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} > 0$. Hence

$$\sum_{n=2}^{N} \frac{1}{n(n-1)} = \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 - \frac{1}{N}.$$

Letting $N \to \infty$, we see that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges to 1.
(2) The infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$. For $n \geq 1$, we set $N = 2^{n+1} = 1 + \sum_{i=0}^{n} 2^i$. We have

$$\begin{aligned} \sum_{i=1}^{N} \frac{1}{i} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \ldots + \frac{1}{8} \right) + \ldots \\ &\quad + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n} \right), \\ &\geq 1 + \frac{1}{2} + 2\frac{1}{2^2} + \ldots + 2^j \frac{1}{2^{j+1}} + \ldots + 2^{n-1} \frac{1}{2^n}, \\ &= 1 + n\frac{1}{2} = \frac{n+2}{2}. \end{aligned}$$

This estimate shows that the increasing sequence (S_N) is not bounded above and so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$. ♠

Our aim in the remainder of this section is to develop some convergence tests for infinite series of (eventually) positive terms. These tests range from the highly practical (comparison, ratio and Cauchy integral test) to the rather theoretical (D'Alembert and Cauchy test). Readers are cautioned to avoid using the theoretical tests on actual series. They rarely work any better than the simpler tests.

3.2.1. The comparison test.

PROPOSITION 3.2.2 (The comparison test). *Let $(u_n), (v_n)$ be sequences of real numbers satisfying $0 \leq u_n \leq v_n$, for all $n \in \mathbb{N}$.*.

(1) *If* $\sum_{n=1}^{\infty} v_n$ *is convergent, then (a)* $\sum_{n=1}^{\infty} u_n$ *is convergent, and (b)* $0 \leq \sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} v_n$.

(2) *If* $\sum_{n=1}^{\infty} u_n$ *is divergent, then* $\sum_{n=1}^{\infty} v_n$ *is divergent (in either case to* $+\infty$*).*

(The result applies with minor changes in statements if $(u_n), (v_n)$ *are eventually positive and* $u_n \leq v_n$ *for all sufficiently large* n*.)*

Proof. For $n \in \mathbb{N}$, define $S_n = \sum_{i=1}^{n} u_i$, $T_n = \sum_{i=1}^{n} v_i$. Since $0 \leq u_n \leq v_n$, we have

$$0 \leq S_n \leq T_n, \text{ for all } n \in \mathbb{N}$$

Suppose that $\sum_{n=1}^{\infty} v_n$ is convergent, with limit T. Then $0 \leq S_n \leq T_n \leq T$, for all $n \in \mathbb{N}$. Hence the increasing sequence (S_n) is bounded above by T and so $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} v_n$, proving (1).

If $\sum_{n=1}^{\infty} u_n$ is divergent, then the series must diverge to $+\infty$ (theorem 2.3.11). Hence for all $K \geq 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{m} u_n \geq K$, $m \geq N$. Hence $\sum_{n=1}^{m} v_n \geq \sum_{n=1}^{m} u_n \geq K$, for all $m \geq N$ and so $\sum_{n=1}^{\infty} v_n$ diverges to $+\infty$. $\square$

Examples 3.2.3. (1) Using the comparison test, we show that $\sum_{n=1}^{\infty} n^{-p}$ is convergent for $p \geq 2$. If $p > 2$, then $n^{-p} \leq n^{-2}$ for all $n \in \mathbb{N}$ and so, by (1) of the comparison test, it suffices to show that $\sum_{n=1}^{\infty} n^{-2}$ is convergent. Observe that for $n \geq 2$ we have

$$\frac{1}{n^2} < \frac{1}{n(n-1)}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ is convergent by examples 3.2.1(1) and so, by the comparison test, $\sum_{n=1}^{\infty} n^{-2} = 1 + \sum_{n=2}^{\infty} n^{-2}$ is convergent with sum at most 2.

(2) Using the comparison test, we show that $\sum_{n=1}^{\infty} n^{-p}$ is divergent for $p \leq 1$. If $p \leq 1$, we have $n^{-p} \geq n^{-1}$, for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} n^{-1}$ is divergent by examples 3.2.1(2), the divergence of $n^{-p} \geq n^{-1}$ is immediate from (2) of the comparison test. ♠

3.2.2. The ratio test.

Proposition 3.2.4 (The ratio test). *Let* (a_n) *be a sequence of positive real numbers and suppose that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ *exists.*

(1) *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$*, the series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

(2) *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$*, the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. We prove (1) and leave (2) to the exercises. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = s < 1$, then there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq r = (s+1)/2 < 1$

for all $n \geq N$. Consequently, $a_{N+p} \leq ra_{N+p-1} \leq \ldots \leq r^p a_N$ for all $p \in \mathbb{N}$. The series $\sum_{p=0}^{\infty} a_{N+p}$ therefore converges by comparison with the geometric series $\sum_{p=0}^{\infty} a_N r^p$. If $\sum_{p=0}^{\infty} a_{N+p}$ converges, then obviously $\sum_{n=1}^{\infty} a_n$ converges. □

REMARK 3.2.5. We emphasize that for the ratio test to apply, it is necessary that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists.

EXAMPLES 3.2.6. (1) Convergence *does not* follow if $a_{n+1}/a_n < 1$ for all $n \in \mathbb{N}$. It is essential to compute the limit (if it exists). As a simple example, take $a_n = 1/n$. Then $a_{n+1}/a_n = n/(n+1) < 1$ for all $n \in \mathbb{N}$, yet $\sum_{n=1}^{\infty} 1/n$ diverges (examples 3.2.1(2)).
(2) The classic area of application of the ratio test is to *power series*. As an example, consider the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $x \in \mathbb{R}^+$. If $x = 0$, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1$ and there is nothing to prove. If we fix $x > 0$, and define $a_n = x^n/n!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1} n!}{x^n (n+1)!} = \frac{x}{n+1}.$$

Since $\lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$, the ratio test applies and so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent. ♠

3.2.3. D'Alembert's test.

PROPOSITION 3.2.7 (D'Alembert's test). *Let (a_n) be a sequence of positive real numbers.*

(1) *If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent.*
(2) *If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.*

PROOF. The proof is almost identical to that of the ratio test. For example, the condition $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ implies that there exists $0 < r < 1$, $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq r < 1$ for all $n \geq N$ the proof then proceeds exactly as in the ratio test. □

EXAMPLE 3.2.8. It is quite difficult to find interesting examples of series where the ratio test fails to apply but D'Alembert's test is applicable. As a somewhat contrived example, define (a_n) by

$$\begin{aligned} a_{2n} &= \left(\frac{1}{2}\right)^n \left(\frac{1}{3}\right)^{n-1}, \ n \geq 1, \\ a_{2n-1} &= \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{3}\right)^{n-1}, \ n \geq 1. \end{aligned}$$

We have $a_{2n}/a_{2n-1} = 1/2 \neq 1/3 = a_{2n+1}/a_{2n}$ and so the limit as $n \to \infty$ of a_{n+1}/a_n does not exist. However, D'Alembert's test applies, since

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1/2$, and the infinite series $\sum_{n=1}^{\infty} a_n$ converges. Of course, the convergence is easily seen by comparison with the geometric series $\sum_{n=1}^{\infty} 2^{-n}$. ♠

3.2.4. Cauchy's test.

PROPOSITION 3.2.9 (Cauchy's test). *Let (a_n) be a sequence of positive real numbers.*

(1) *If $\limsup a_n^{\frac{1}{n}} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent.*
(2) *If $\limsup a_n^{\frac{1}{n}} > 1$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.*

PROOF. We prove (1) and leave (2) to the exercises. If $\limsup a_n^{\frac{1}{n}} = s < 1$, then there exists $N \in \mathbb{N}$ such that $\sup\{a_n^{\frac{1}{n}} \mid n \geq N\} \leq r = (s+1)/2 < 1$. We therefore have $a_n \leq r^n$, all $n \geq N$. Hence $\sum_{n=1}^{\infty} a_n$ converges by comparison with the geometric series $\sum_{n=1}^{\infty} r^n$. □

REMARKS 3.2.10. (1) The Cauchy test is of great theoretical importance as we see when we look at power series. However, in most practical applications it is usually best to start by trying the ratio test and only invoke Cauchy's test as a last resort.
(2) The divergence condition for Cauchy's test uses the lim sup not (the weaker) lim inf.

EXAMPLE 3.2.11. We examine the convergence of the series $\sum_{n=1}^{\infty} nx^n$, $x \in \mathbb{R}^+$. We have $\lim_{n\to\infty}(nx^n)^{1/n} = \lim_{n\to\infty} n^{1/n}x = x$, by examples 2.6 (3). Hence, by Cauchy's test, $\sum_{n=1}^{\infty} nx^n$ converges if $x \in [0,1)$. We see by inspection that the series diverges if $x \geq 1$ (lemma 3.1.2). Of course, the same results follow easily using the ratio test. ♠

3.2.5. Cauchy's integral test. If $f : [a,\infty) \to \mathbb{R}$ is continuous, then the infinite integral $\int_a^{\infty} f(t)\,dt$ is defined to equal $\lim_{x\to+\infty} \int_a^x f(t)\,dt$ if the limit exists (and is finite). A necessary (but not sufficient) condition for the existence of $\int_a^{\infty} f(t)\,dt$ is that $\lim_{x\to+\infty} f(x) = 0$. We refer to $\int_a^{\infty} f(t)\,dt$ as an *improper integral* (see the exercises for more definitions and examples). When $\int_a^{\infty} f(t)\,dt$ exists, we often say the integral converges and write $\int_a^{\infty} f(t)\,dt < \infty$.

PROPOSITION 3.2.12 (Cauchy's integral test). *Let $f : [1,\infty) \to \mathbb{R}$ be a positive, continuous and monotone decreasing function. A necessary and sufficient condition for the convergence of $\sum_{n=1}^{\infty} f(n)$ is the convergence of the improper integral $\int_1^{\infty} f(t)\,dt$. If either series or integral converge, then we have the estimates*

(1) $f(n) + \int_1^n f(t)\,dt \leq \sum_{j=1}^n f(j) \leq f(1) + \int_1^n f(t)\,dt$, *$n > 1$.*

(2) $\int_1^\infty f(t)\,dt \le \sum_{j=1}^\infty f(j) \le f(1) + \int_1^\infty f(t)\,dt$.

PROOF. For $n > 1$, we have (property **I** of the Riemann integral)

$$\int_1^n f(t)\,dt = \sum_{j=1}^{n-1} \int_j^{j+1} f(t)\,dt.$$

Since f is monotone decreasing, we have (property **II** of the Riemann integral)

$$f(j) \ge \int_j^{j+1} f(t)\,dt \ge f(j+1).$$

Using these estimates we easily verify (1) of the proposition. Since $f \ge 0$, the sequence $(\sum_{j=1}^n f(j))$ of partial sums is increasing and so, by theorem 2.3.10, $\sum_{j=1}^\infty f(j)$ converges iff $\int_1^\infty f(t)\,dt$ converges. Finally, (2) follows by letting $n \to \infty$ in (1) and noting that $\lim_{n\to\infty} f(n) = 0$ (lemma 3.1.2). $\square$

EXAMPLE 3.2.13. We consider the convergence of $\sum_{n=1}^\infty \frac{1}{n^p}$, where $p \in \mathbb{R}^+$. Define the continuous function $f(x) = 1/x^p$, $x \in [1, \infty)$. Since $p \ge 0$, f is monotone decreasing and so Cauchy's integral test applies. If $p \ne 1$, we have

$$\int_1^n \frac{1}{t^p}\,dt = \frac{1}{p-1}(1 - \frac{1}{n^{p-1}}).$$

If $p > 1$, $\lim_{n\to\infty} \int_1^n \frac{1}{t^p}\,d = (p-1)^{-1}$ and so the improper integral converges. Hence $\sum_{n=1}^\infty \frac{1}{n^p}$ converges by Cauchy's integral test and

$$(p-1)^{-1} \le \sum_{n=1}^\infty \frac{1}{n^p} \le 1 + (p-1)^{-1}.$$

On the other hand if $p < 1$, the improper integral diverges and so $\sum_{n=1}^\infty \frac{1}{n^p}$ diverges by Cauchy's integral test. There remains the case $p = 1$. We have

$$\int_1^n \frac{1}{t}\,dt = \log n - 1.$$

Since $\lim_{n\to+\infty} \log n = +\infty$ as $n \to \infty$, the improper integral diverges and so $\sum_{n=1}^\infty \frac{1}{n}$ diverges by Cauchy's integral test. We remark for future reference the useful estimate

$$\frac{1}{n} + \log n \le \sum_{j=1}^n \frac{1}{j} \le 1 + \log n. \tag{3.1}$$

We provide a number of other examples of applications of Cauchy's integral test in the exercises. ♠

EXERCISES 3.2.14.

(1) Complete the proofs of D'Alembert's and Cauchy's test — take particular care with the divergence statement (2) in Cauchy's test.
(2) Let $T_n = \sum_{j=1}^{n} 1/(2j-1)$ and $S_n = \sum_{j=1}^{n} 1/j$, $n \geq 1$. Show that $T_n > S_n/2$ and deduce that the series $\sum_{j=1}^{\infty} 1/(2j-1)$ is divergent. Show also how this result can derived from Cauchy's integral test.
(3) Cauchy's test is stronger than D'Alembert's test which is stronger than the ratio test. For each of the following series, determine the weakest test that proves convergence (implicit in the question is showing why the weaker tests fail; you do not have to prove that tests stronger than the weakest test that works also work.)
   (a) $1 + \frac{(a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \ldots + \frac{(a+1)\ldots(na+1)}{(b+1)\ldots(nb+1)} + \ldots$, where $b > a > 0$.
   (b) $1 + \alpha + \beta^2 + \alpha^3 + \beta^4 + \ldots$, where $0 < \alpha < \beta < 1$.
(4) Show that $\sum_{n=1}^{\infty} q^{n^2} x^n$ converges for all positive values of x if $0 < q < 1$. What happens if $q > 1$?
(5) Determine whether or not the followings eries converge
   (a) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{-n^2}$.
   (b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} 5^{-n}$.
   (c) $\sum_{n=1}^{\infty} n^{-1/2}(\sqrt{n+1} - \sqrt{n})$.
   (d) $\sum_{n=1}^{\infty} n^{-2/3}(\sqrt{n+1} - \sqrt{n})$.
(6) Show that
   (a) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is divergent. (Start at $n = 2$ as $\log 1 = 0$.)
   (b) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p > 1$. (Start at $n = 2$ as $\log 1 = 0$.)
   (c) $\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$ is divergent.

   ($\log n$ is the logarithm to base e, also denoted by $\ln x$.) Show also that
$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \in [\frac{1}{\log 2}, \frac{1}{\log 2} + \frac{1}{2(\log 2)^2}].$$
(7) Show that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges (it is always assumed that (a_n) is a sequence of positive numbers).
   (a) Show that if (a_n) is a decreasing sequence, then the convergence of $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.
   (b) Find an example where $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges but $\sum_{n=1}^{\infty} a_n$ diverges (by (a), (a_n) cannot be decreasing).
(8) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive terms.
   (a) Show that $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges.
   (b) Show that if $\sum_{n=1}^{\infty} a_n^2$ converges then so does $\sum_{n=1}^{\infty} a_n/n$.
   (c) Show, by means of an example, that the converse of (b) is false, even if (a_n) is decreasing.
(9) Show that
   (a) $S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \in [0, 1]$, $n \geq 1$.
   (b) (S_n) is an increasing sequence.

   Deduce that $\lim_{n\to\infty}(\sum_{j=1}^{n} 1/j - \log n)$ exists and lies in $[0, 1]$. (The limit is usually denoted by γ and referred to as *Euler's constant*. The value of γ is approximately $0.5772\ldots$. It is not yet known whether γ is rational or irrational.

3.3. General Principle of Convergence

In the remaining sections of this chapter we will study series where the terms are not necessarily all of the same sign. Consequently, the sequence of partial sums will no longer be monotone and we will not be able to apply theorems 3.10, 3.11. Instead, we will need to use the result that if the sequence of partial sums is Cauchy, then it converges. More formally, we have

THEOREM 3.3.1 (General principle of convergence). *Let (a_n) be a sequence of real numbers. Then $\sum_{n=1}^{\infty} a_n$ is convergent iff for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|x_m + x_{m-1} + \ldots + x_n| < \varepsilon, \text{ for all } m \geq n \geq N.$$

PROOF. The sequence (S_n) of partial sums is convergent iff it is a Cauchy sequence. That is, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|S_m - S_n| = |x_m + x_{m-1} + \ldots + x_{n+1}| < \varepsilon$, for all $m \geq n \geq N$. Replacing N by $N + 1$ gives the condition of the theorem. □

REMARK 3.3.2. Theorem 3.3.1 extends to infinite series of complex numbers. The proof is formally the same as that of theorem 3.3.1 but using theorem 2.6.6.

3.4. Absolute Convergence

DEFINITION 3.4.1. Let (a_n) be a sequence of real numbers. The infinite series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

REMARK 3.4.2. The usual rules for sums and differences of convergent series apply immediately to absolute convergent series using the corresponding results for series of positive terms. For example, if $\sum a_n$ and $\sum b_n$ are absolutely convergent then so are the series $\sum(a_n \pm b_n)$.

THEOREM 3.4.3. *Every absolutely convergent series is convergent.*

PROOF. Our proof makes essential use of theorem 3.3.1 (general principle of convergence). Suppose that $\sum_{n=1}^{\infty} |a_n|$. Then, by theorem 3.3.1, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_m| + |a_{m-1}| + \ldots + |a_n| < \varepsilon, \text{ for all } m \geq n \geq N.$$

But $|a_m + a_{m-1} + \ldots + a_n| \leq |a_m| + |a_{m-1}| + \ldots + |a_n|$ and so

$$|a_m + a_{m-1} + \ldots + a_n| < \varepsilon, \text{ for all } m \geq n \geq N.$$

Hence, $\sum_{n=1}^{\infty} a_n$ is convergent by theorem 3.3.1. □

REMARKS 3.4.4. (1) The reader is cautioned that the converse to theorem 3.4.3 is *false*: a convergent series need not be absolutely convergent. We give examples in the next section.
(2) If $\sum_{n=1}^{\infty} a_n$ is an infinite series of complex numbers, then the series is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n| < \infty$ (where $|\ |$ now denotes modulus of a complex number). Without exception, all of the results on absolutely convergent real series extend to absolutely convergent complex series.

Theorem 3.4.3 allows us to translate easily results on convergent series of positive terms to absolutely convergent series.

EXAMPLE 3.4.5. Let $p > 1$. The series $\sum_{n=1}^{\infty}(-1)^{n+1}n^{-p}$ is absolutely convergent by example 3.2.13. Hence $\sum_{n=1}^{\infty}(-1)^{n+1}n^{-p}$ is convergent, for $p > 1$. (Later in this chapter we shall prove that $\sum_{n=1}^{\infty}(-1)^{n+1}n^{-p}$ is convergent for $p > 0$ — of course, absolute convergence fails if $p \leq 1$). ♠

3.4.1. The exponential series. The exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all $x \in \mathbb{R}$ by examples 3.2.6(2). Hence, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. We define the *exponential function* e^x or $\exp(x)$ by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in \mathbb{R}.$$

As an illustration of some of the methods used to study convergence we prove the following result which we shall use later in chapter 4 (see also examples 2.3.16(2)).

PROPOSITION 3.4.6. *For all* $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

PROOF. Fix $x \in \mathbb{R}$. In order to prove the result it suffices to show that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|(1 + \frac{x}{n})^n - e^x| < \varepsilon$, for all $n \geq N$.

By the binomial theorem, we have

$$\begin{aligned}(1 + \frac{x}{n})^n &= 1 + n\frac{x}{n} + \frac{n(n-1)}{2}\frac{x^2}{n^2} + \ldots + \frac{x^n}{n^n}, \\ &= 1 + \sum_{j=1}^{n} \frac{x^j}{j!} K_n(j),\end{aligned}$$

where $K_n(1) = 1$ and

$$K_n(j) = \frac{1}{j!}(1 - \frac{1}{n})(1 - \frac{2}{n})\ldots(1 - \frac{j-1}{n}),\ j > 1.$$

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent, we can choose $N_1 \in \mathbb{N}$ so that

$$\sum_{j=p}^{q} \frac{|x|^j}{j!} < \varepsilon/3, \text{ for all } \infty \geq q \geq p > N_1. \tag{3.2}$$

Since $1 \geq K_n(j) \geq 0$ for all $n \geq j \geq 1$, (3.2) implies that

$$\sum_{j=N_1+1}^{n} K_n(j) \frac{|x|^j}{j!} < \varepsilon/3, \text{ for all } n > N_1. \tag{3.3}$$

For fixed j, $\lim_{n\to\infty} K_n(j) = 1$. Hence, we may choose $N > N_1$ so that

$$|1 + \sum_{j=1}^{N_1} \frac{x^j}{j!} K_n(j) - \sum_{j=0}^{N_1} \frac{x^j}{j!}| < \varepsilon/3, \ n \geq N. \tag{3.4}$$

For $n > N_1$, we may write

$$(1 + \frac{x}{n})^n - e^x = I_1 + I_2 + I_3,$$

where

$$I_1 = 1 + \sum_{j=1}^{N_1} \frac{x^j}{j!} K_n(j) - \sum_{j=0}^{N_1} \frac{x^j}{j!}, \ I_2 = \sum_{j=N_1+1}^{n} \frac{x^j}{j!} K_n(j), \ I_3 = \sum_{j=N_1+1}^{\infty} \frac{x^j}{j!}.$$

We have

$$\begin{aligned} |I_1| &< \varepsilon/3, \text{ for all } n \geq N \text{ (by (3.4))}, \\ |I_2| &< \varepsilon/3, \text{ for all } n > N_1 \text{ (by (3.3))}, \\ |I_3| &< \varepsilon/3, \text{ for all } n > N_1 \text{ (by (3.2))}. \end{aligned}$$

Hence $|(1 + \frac{x}{n})^n - e^x| < \varepsilon$, for all $n \geq N$. □

REMARK 3.4.7. This result continues to hold if we allow for complex variables: $\lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$, for all $z \in \mathbb{C}$. The proof is exactly the same with 'absolute value' replaced everywhere by 'modulus'.

3.4.2. Tests for absolutely convergent series. Next we present versions of the comparison, ratio and Cauchy test appropriate for absolutely convergent series. Proofs are all immediate from theorem 3.4.3 together with the corresponding result for series of positive terms. The results also apply to complex series with 'absolute value' replaced by the 'modulus'.

PROPOSITION 3.4.8 (The comparison test). *Let $(u_n), (v_n)$ be sequences of real numbers satisfying $|u_n| \leq |v_n|$, for all $n \in \mathbb{N}$.*

(1) *If $\sum_{n=1}^{\infty} v_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} u_n$ is absolutely convergent (and so convergent).*

(2) *If* $\sum_{n=1}^{\infty} u_n$ *is not absolutely convergent then* $\sum_{n=1}^{\infty} v_n$ *is not absolutely convergent.*

REMARK 3.4.9. For the second statement, $\sum_{n=1}^{\infty} v_n$ may converge even if $\sum_{n=1}^{\infty} u_n$ is divergent.

PROPOSITION 3.4.10 (The ratio test). *Let* (a_n) *be a sequence of real numbers and suppose that* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \ell$.

(1) *If* $\ell < 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and so convergent).*
(2) *If* $\ell > 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

PROPOSITION 3.4.11 (Cauchy's test). *Let* (a_n) *be a sequence of real numbers.*

(1) *If* $\limsup |a_n|^{\frac{1}{n}} < 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and so convergent).*
(2) *If* $\limsup |a_n|^{\frac{1}{n}} > 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

DEFINITION 3.4.12. Let (a_n) be a sequence of real numbers. The series $\sum_{n=1}^{\infty} b_n$ is a *rearrangement* of $\sum_{n=1}^{\infty} a_n$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$, all $n \in \mathbb{N}$. That is, a rearrangement of $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ where σ is a "permutation" of $\mathbb{N}$.

EXAMPLE 3.4.13. The series $x + x^3 - x^2 + x^5 + x^7 - x^4 + \dots$ is a rearrangement of $\sum_{n=1}^{\infty}(-1)^{n+1}x^n$.

We end this section with an important result that fails dramatically when the series is convergent but not absolutely convergent.

THEOREM 3.4.14. *Every rearrangement* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *of an absolutely convergent series* $\sum_{n=1}^{\infty} a_n$ *is convergent and*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n.$$

PROOF. It follows from the general principle of convergence that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_m| + \dots + |a_n| < \varepsilon$, for all $m > n \geq N$. Let $M = \max\{\sigma^{-1}(1), \dots \sigma^{-1}(N-1)\} \in \mathbb{N}$. Observe that if $n \geq M$, then $\sigma(n) \geq N$. Let $m > n \geq M$ and set $m' = \max\{\sigma(m), \dots, \sigma(n)\}$ and $n' = \min\{\sigma(m), \dots, \sigma(n)\}$. We have $m' > n' \geq N$ and so

$$|a_{\sigma(m)}| + \dots + |a_{\sigma(n)}| \leq |a_{\sigma(m')}| + |a_{\sigma(m')-1}| + \dots + |a_{\sigma(n')}| < \varepsilon.$$

Hence by the general principle of convergence $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is absolutely convergent. It remains to prove that the series have the same sum. Let

$\sum_{n=1}^{\infty} a_n = \ell$. With the same notation used above, suppose $p > M$ and set $q = \max\{\sigma^{-1}(N), \dots \sigma^{-1}(p)\}$. We have $|\sum_{n=1}^{N-1} a_n - \ell| \leq \varepsilon$ and so

$$|\sum_{n=1}^{p} a_{\sigma(n)} - \ell| \leq |\sum_{n=1}^{N-1} a_n - \ell| + |a_N + \dots + a_q| < 2\varepsilon.$$

This estimate holds for all $p > M$ and so $\sum_{n=1}^{\infty} a_{\sigma(n)} = \ell$. □

Exercises 3.4.15.

(1) Suppose that (a_n) is a sequence of real numbers such that (a) $\sum_{n=1}^{\infty} a_n$ is convergent, and (b) the a_n are eventually of the same sign. Show that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(2) Show that the rearrangment theorem also holds for absolutely convergent series of complex terms.

3.5. Conditionally Convergent Series

In this section we look at convergent real series that are not absolutely convergent. The results we give do not have (simple) extensions to complex series.

Definition 3.5.1. Let (a_n) be a sequence of real numbers. The series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if it is convergent but not absolutely convergent.

Remark 3.5.2. If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then the series must have have infinitely many terms of positive and negative signs. Else the signs would be eventually positive or negative and the series would be absolutely convergent.

Example 3.5.3. The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is not absolutely convergent. Since

$$S_{2n} = \left(1 - \frac{1}{2}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \geq 0,\ n \geq 1,$$

the sequence (S_{2n}) of partial sums is an increasing sequence of positive numbers. On the other hand,

$$S_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2n} - \frac{1}{2n+1}\right) = S_{2n} - \frac{1}{2n+1},$$

is a decreasing sequence bounded above by 1. Now $S_{2n} = S_{2n+1} + \frac{1}{2n+1} < 1$, $n \geq 1$, and so (S_{2n}) is an increasing sequence of positive numbers bounded above by $1 + 1/(2m+1)$, all $m \geq 1$. Hence (S_{2n}) converges to $\ell \in (0, 1]$. Since $S_{2n+1} = S_{2n} - \frac{1}{2n+1}$, (S_{2n+1}) also converges to ℓ. Therefore, $\lim_{n\to\infty} S_n = \ell \in (0, 1]$. (In the next chapter we show that $\ell = \log 2$.) ♠

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Define sequences (u_n), (v_n) of positive real numbers by

$$u_n = \max\{0, a_n\}, \quad v_n = -\min\{0, a_n\}.$$

Observe that

$$a_n = u_n - v_n, \quad |a_n| = u_n + v_n.$$

The next result will be useful when we prove Riemann's theorem on rearrangements of a conditionally convergent series.

PROPOSITION 3.5.4. *If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series and define the sequences (u_n), (v_n) as above, then $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ both diverge to $+\infty$.*

PROOF. If $\sum_{n=1}^{\infty} v_n$ converges then $\sum_{n=1}^{\infty}(a_n + v_n) = \sum_{n=1}^{\infty} u_n$ converges. Therefore, $\sum_{n=1}^{\infty}(u_n+v_n) = \sum_{n=1}^{\infty} |a_n|$ converges, contradicting the conditional convergence of $\sum_{n=1}^{\infty} a_n$. Hence $\sum_{n=1}^{\infty} v_n = +\infty$. A similar argument shows that $\sum_{n=1}^{\infty} u_n = +\infty$. □ □

EXAMPLE 3.5.5. The series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} + \ldots$ is divergent. For this series, $\sum_{n=1}^{\infty} u_n$ is divergent (exercises 3.2.14(2)) but $\sum_{n=1}^{\infty} v_n$ is convergent (by comparison with $\sum 1/n^2$. Hence the series cannot be absolutely or conditionally convergent (proposition 3.5.4) and therefore must diverge (in this case to $+\infty$). ♠

3.5.1. Alternating series.

DEFINITION 3.5.6. The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is called an *alternating series* if (a_n) is a sequence of positive real numbers.

PROPOSITION 3.5.7 (Leibniz alternating series test). *Let (a_n) be a sequence of positive numbers. The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if*

(a) *(a_n) is a decreasing sequence.*
(b) $\lim_{n\to\infty} a_n = 0$.

If (a,b) hold then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n \in [0, a_1]$.

PROOF. The proof is formally identical to the argument used in example 3.5.3 and we leave details to the reader. □

EXAMPLE 3.5.8. The alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$, $\sum_{n=3}^{\infty} \frac{(-1)^n}{\log \log n}$ are convergent. Note that the convergence of these series is very slow. For example, we need to take n at least 1.8×10^{65} to ensure that the nth term of the second series is less than $1/5$. ♠

3.5.2. Riemann's theorem. We this section we look at rearrangements of a conditionally convergent series. We start with a simple example.

EXAMPLE 3.5.9. Consider the conditionally convergent series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \ldots + \frac{1}{n} - \frac{1}{n} + \ldots$. The series trivially converges to zero. Take the rearrangement

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \ldots$$

It is easy to see that $S_{3n} = \sum_{j=1}^{n} \frac{1}{2j(2j-1)}$. Using the identity $\frac{1}{2j(2j-1)} = \frac{1}{2j-1} - \frac{1}{2j}$, we find that $S_{3n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n}$. We deduce easily that the rearranged series converges with sum equal to $\log 2 > 0$. This example shows the failure of the rearrangement theorem when the series is not absolutely convergent. ♠

THEOREM 3.5.10 (Riemann's rearrangement theorem). *Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.*

(a) *For every $x \in \mathbb{R} \cup \{-\infty, +\infty\}$ there exists a rearrangement σ such that*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = x.$$

(b) *There exist rearrangements σ such that $\sum_{n=1}^{\infty} a_{\sigma(n)}$ does not converge (even to $\pm\infty$).*

PROOF. We prove (a) and leave (b) to the exercises. Our first step is to define sequences (p_k) and (q_k) by requiring that (p_k) is the subsequence of (a_n) defined by the positive terms and (q_k) is the subsequence of (a_n) defined by the strictly negative terms. Note that the sequence (p_k) may contain zeros and that for each $k \in \mathbb{N}$, there exist unique $n_k, m_k \in \mathbb{N}$ such that $p_k = a_{n_k}$, $q_k = a_{m_k}$.

Let $x \in \mathbb{R}$. For simplicity, suppose $x \geq 0$. Since $\sum_{k=1}^{\infty} p_k$ diverges to $+\infty$, there exists a unique $k_1 \geq 1$ such that $p_1 + \ldots + p_{k_1 - 1} \leq x < p_1 + \ldots + p_{k_1} = P_1$. Since $\sum_{k=1}^{\infty} q_k$ diverges to $-\infty$, there exists a unique $\ell_1 \geq 1$ such that $Q_1 = P_1 + q_1 + \ldots + q_{\ell_1} < x \leq P_1 + q_1 + \ldots + q_{\ell_1 - 1} = Q_1 - q_{\ell_1}$. In the obvious way, we may inductively define increasing sequences (k_n), (ℓ_n) and sequences (P_n), (Q_n) so that $P_n = \sum_{j=k_{n-1}+1}^{k_n} p_j + Q_{n-1}$, $Q_n = P_n + \sum_{j=\ell_{n-1}+1}^{\ell_n} q_j$ and $P_n - p_{k_n - 1} \leq x < P_n$, $Q_n < x \leq Q_n - q_{\ell_n - 1}$, for all $n \geq 1$. Since $k_j, \ell_j \geq 1$, there are at least $2n - 1$ terms from the series $\sum a_n$ in P_n, and at least $2n$ terms in Q_n. This construction defines the rearrangement

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = a_{n_1} + \ldots + a_{n_{k_1}} + a_{m_1} + \ldots + a_{m_{\ell_1}} + a_{n_{k_1+1}} + \ldots$$

We claim the rearranged series converges to x. Since $\sum a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$ and so $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 0$. Hence there exists $N \geq 1$ so that $|p_n|, |q_n| < \varepsilon$ for all $n \geq N$. Choose $M \in \mathbb{N}$ so that if $n \geq M$, then $a_{\sigma(n)}$ is either p_k, with $k \geq N$ or q_k with $k \geq N$. It follows from the construction of the rearrangement that $|\sum_{n=1}^{m} a_{\sigma(n)} - x| < \varepsilon$ for all $m \geq M$. Hence $\sum_{n=1}^{\infty} a_{\sigma(n)} = x$.

If $x = +\infty$, we modify the construction by requiring that $P_n - p_{k_n-1} \leq n < P_n$ and $Q_n < n - 1 \leq Q_n - q_{\ell_n - 1}$. The argument when $x = -\infty$ is similar. □

Remark 3.5.11. The statement of Riemann's rearrangement theorem can be strengthened along the following lines: let $-\infty \leq x_1 < x_2 < \ldots < x_N \leq +\infty$. Then there exists a rearrangement σ of $\sum_{n=1}^{\infty} a_n$ such that for each x_j there exists a subsequence (S_{n_k}) of (S_n) which converges to x_j. (See also the exercises.)

Exercises 3.5.12.

(1) Show that the rearrangement $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} + \ldots$ of $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges to $\frac{1}{2}\log 2$. (Hint: Look at the partial sum to $3n$ terms of the series.)
(2) Show that the rearrangement $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \ldots$ of $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges to $\frac{3}{2}\log 2$. (Hint: Work with the partial sum to $3n$ terms. At some point you will need to show that $\frac{1}{n+1} + \ldots + \frac{1}{2n}$ is equal to the partial sum to $2n$ terms of the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$.)
(3) Prove part (b) of Riemann's rearrangement theorem.
(4) Prove the result indicated in remark 3.5.11.

3.6. Abel's and Dirichlet's tests

In this section we state and prove two powerful tests that can be used to determine the convergence of non-absolutely convergent series. We start with a useful lemma.

Lemma 3.6.1 (Abel's lemma). *If the sequence (S_n) of partial sums of the infinite series $\sum_{n=1}^{\infty} a_n$ satisfies the bounds*

$$m \leq S_n \leq M,\ n \in \mathbb{N},$$

then for any decreasing sequence (u_n) of positive real numbers we have the bounds

$$mu_1 \leq \sum_{j=1}^{n} a_j v_j \leq Mu_1,\ n \in \mathbb{N} \tag{3.5}$$

PROOF. Since $a_n = S_{n+1} - S_n$, $n \geq 1$, we have

$$\begin{aligned}\sum_{j=1}^{n} a_j u_j &= S_1 u_1 + (S_2 - S_1)u_2 + \ldots + (S_n - S_{n-1})u_n, \\ &= S_1(u_1 - u_2) + \ldots + S_{n-1}(u_{n-1} - u_n) + S_n u_n.\end{aligned}$$

Since (u_n) is a decreasing sequence of positive numbers, we have $u_1 - u_2, \ldots, u_{n-1} - u_n, u_n \geq 0$. Hence, upper and lower bounds for $\sum_{j=1}^{n} a_j u_j$ are given respectively by

$$\begin{aligned}M((u_1 - u_2) + (u_2 - u_3) + \ldots + (u_{n-1} - u_n) + u_n) &= Mu_1, \\ m((u_1 - u_2) + (u_2 - u_3) + \ldots + (u_{n-1} - u_n) + u_n) &= mu_1,\end{aligned}$$

and so $mu_1 \leq \sum_{j=1}^{n} a_j u_j \leq Mu_1$. □

PROPOSITION 3.6.2 (Abel's test). *If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and (v_n) is a bounded monotone sequence of positive numbers, then $\sum_{n=1}^{\infty} a_n v_n$ is convergent.*

PROOF. Since (v_n) is monotone and bounded, (v_n) is convergent, say to v. If (v_n) is increasing, set $u_n = v - v_n$ and if (v_n) is decreasing set $u_n = v_n - v$. In both cases (u_n) is a monotone decreasing sequence of positive numbers. Since $a_n v_n = a_n v - a_n u_n$ or $a_n v_n = a_n u_n - a_n v$ and $\sum a_n v = v \sum a_n$ is convergent, it is enough to prove that $\sum_{n=1}^{\infty} a_n u_n$ converges.

Fix $n \geq 2$ and define $K_n = \sup\{|S_m - S_{n-1}| \mid m \geq n\}$. By the general principle of convergence, $\lim_{n\to\infty} K_n = 0$. Applying Abel's lemma to $\sum_{j=n}^{m} a_j u_j$ gives

$$|\sum_{j=n}^{m} a_j u_j| \leq K_n v_n \leq K_n v_1,\ m \geq n.$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} K_n = 0$, there exists $N \in \mathbb{N}$ such that $|K_n| < \varepsilon/v_1)$, Therefore,

$$|\sum_{j=n}^{m} a_j u_j| \leq K_n v_1 < \varepsilon,\ \text{if } m \geq n \geq N.$$

The result is follows by the general principle of convergence. □

EXAMPLES 3.6.3. (1) Consider the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \ldots$ which trivially converges to zero (see example 3.5.9). Take the bounded increasing sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \ldots$. Multiplying the terms of the series by the corresponding term of the decreasing sequence yields the infinite series

$$1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{3}+\frac{2}{3^2}-\frac{1}{4}+\frac{3}{4^2}-\dots$$

Abel's test implies that this series converges. Note that the alternating series test does not apply to this series as the terms in the series do not define a decreasing sequence. (It is not difficult to show that the series converges to $2-\sum_{n=1}^{\infty} n^{-2}$.)
(2) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges if $x \geq 0$. ♠

PROPOSITION 3.6.4 (Dirichlet's test). *Suppose that the sequence of partial sums of the infinite series $\sum_{n=1}^{\infty} a_n$ is bounded and (u_n) is a decreasing sequence of positive numbers. If $\lim_{n\to\infty} u_n = 0$, then $\sum_{n=1}^{\infty} a_n u_n$ is convergent.*

PROOF. Suppose that $m \leq S_n \leq M$ for all $n \in \mathbb{N}$. If we set $K=\max\{|M|,|m|\}$, then $|S_n| \leq K$ for all $n \geq 1$. We have $|S_m - S_n| \leq |S_m|+|S_n|$ and so $|S_m - S_n| \leq 2K$ for all $m \geq n \geq 1$. Applying Abel's lemma gives the estimate

$$|\sum_{j=n}^{m} a_j u_j| \leq 2K u_n, \text{ for all } m \geq n \geq 1.$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $u_n < \varepsilon/2K$ for all $n \geq N$. We have

$$|\sum_{j=n}^{m} a_j u_j| \leq 2K u_n < \varepsilon,\ n \geq N.$$

It follows from the general principle of convergence that $\sum_{n=1}^{\infty} a_n u_n$ is convergent. □

EXAMPLES 3.6.5. (1) The series $\sum_{n=1}^{\infty}(-1)^{n+1}$ is not convergent but the partial sums are bounded. If (u_n) is a decreasing sequence of positive numbers converging to zero then Dirichlet's test implies that $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ converges (Leibniz test).
(2) If the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded then $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges if $x > 0$.
(3) Dirichlet's test is often useful for the study of trigonometric series. For example, consider the series $\sum_{n=1}^{\infty} \frac{\cos n\theta}{n}$. Using Dirichlet's test we show that the series converges provided that θ is not an integer multiple of 2π. We start by noting that if θ is an integer multiple of 2π then the series is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent. Note that if θ is an odd multiple of π, then the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test. For other values of θ, in particular irrational multiples of π, the issue of convergence is quite subtle.

The main ingredient in the proof of convergence of $\sum_{n=1}^{\infty} \frac{\cos n\theta}{n}$ is the trigonometric identity

$$\sum_{j=1}^{n} \cos(n\theta) = \frac{\cos\left(\frac{n+1}{2}\theta\right)\sin\left(\frac{n}{2}\theta\right)}{\sin(\frac{\theta}{2})},\ \theta \neq 2n\pi. \tag{3.6}$$

(We give the proof of this identity in an appendix at the end of the chapter.) Provided θ is not an integral multiple of 2π, (3.6) gives the estimate

$$|\sum_{j=1}^{n} \cos(n\theta)| \leq \frac{|\sin\left(\frac{n+1}{2}\theta\right)\sin\left(\frac{n}{2}\theta\right)|}{|\sin(\frac{\theta}{2})|} \leq \frac{1}{|\sin(\frac{\theta}{2})|},\ n \geq 1.$$

Take $a_n = \cos n$, $u_n = 1/n$ in Dirichlet's test. ♠

EXERCISES 3.6.6.

(1) Suppose that $\sum_{n=1}^{\infty} na_n$ converges. Show that $\sum_{n=1}^{\infty} a_n$ converges. What about $\sum_{n=1}^{\infty} \sqrt{n}a_n$? $\sum(-1)^{n+1}a_n$? (You may not assume all the terms are of the same sign. Either prove true of find a counter example.)
(2) Prove that $\sum_{n=0}^{\infty} \frac{\sin((2n+1)\theta)}{(2n+1)}$ converges for all $\theta \in \mathbb{R}$.
(3) For what values of $\theta \in \mathbb{R}$ is the series with nth term

$$(1 + \frac{1}{2} + \ldots + \frac{1}{n})\frac{\sin(\theta)}{n}?$$

convergent. (You may assume $(1 + \frac{1}{2} + \ldots + \frac{1}{n})/n \to 0$.)

3.7. Infinite Products

Suupose that (a_n) is a sequence of real numbers. The infinite product $\Pi_{n=1}^{\infty} a_n$ is defined to be the sequence (P_n) of partial products where $P_n = a_1 \times a_2 \times \ldots \times a_n$, $n \in \mathbb{N}$. Roughly speaking, the infinite product $\Pi_{n=1}^{\infty} a_n$ *converges* if the sequence (P_n) converges. In practice, we want to avoid situations where, for example, $a_n = 0$ for some n or more generally $\lim_{n\to\infty} P_n = 0$. If $\lim_{n\to\infty} P_n$ exists and is either 0 or $\pm\infty$, the infinite product is said to diverge. So as to make a stronger connection with the theory of infinite series, it turns out that it is more useful to consider infinite products of the form $\Pi_{n=1}^{\infty}(1 + a_n)$.

DEFINITION 3.7.1. Let (a_n) be a sequence of real numbers. The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ is *convergent* if the sequence (P_n) of partial products is convergent and does not converge to either 0 or $+\infty$. If $\Pi_{n=1}^{\infty}(1 + a_n)$ is not convergent, it is *divergent*.

REMARKS 3.7.2. (1) The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ converges iff the infinite sum $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges.
(2) We can define infinite products $\Pi_{n=1}^{\infty}(1 + a_n)$ with $a_n \in \mathbb{C}$. The

definition of convergence is the same though we no can longer relate the convergence of the infinite product with the convergence of $\sum_{n=1}^{\infty} \log(1+a_n)$. Most of the results and tests we give below do not apply to the complex case.

EXAMPLES 3.7.3. (1) The infinite products $\Pi_{n=1}^{\infty}(1+\frac{1}{n})$ and $\Pi_{n=2}^{\infty}(1-\frac{1}{n})$ are both divergent. We have $(1+1)(1+1/2)\dots(1+1/n) \geq 1+1/2+\dots+1/n$. Since the series $\sum_{n=1}^{\infty} 1/n$ is divergent to $+\infty$, $\Pi_{n=1}^{\infty}(1+\frac{1}{n})$ diverges to $+\infty$. Since $\Pi_{j=1}^{n}(1-a_j) < (\Pi_{j=1}^{n}(1+a_j))^{-1}$ if $a_j \in (0,1)$, we obviously have $\Pi_{n=2}^{\infty}(1-\frac{1}{n}) < (\Pi_{j=2}^{n}(1+1/n))^{-1}$ and so $\Pi_{n=2}^{\infty}(1-1/n) = 0$.
(2) The infinite product $\Pi_{n=2}^{\infty}(1-\frac{1}{n^2})$ is convergent. Observe that $1-\frac{1}{n^2} = \frac{(n-1)(n+1)}{n^2}$. Consquently, $P_n = \Pi_{j=2}^{n}\frac{(n-1)(n+1)}{n^2} = \frac{1}{2}\frac{n+1}{n}$. Hence $\Pi_{n=2}^{\infty}(1-\frac{1}{n^2})$ is convergent and $\Pi_{n=2}^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$. ♠

3.7.1. Tests for convergence of an infinite product.

LEMMA 3.7.4. *Suppose that (a_n) is a subsequence of $(0,1)$. Then $\sum_{n=1}^{\infty} a_n$ converges iff either the infinite product $\Pi_{n=1}^{\infty}(1+a_n)$ converges or the infinite product $\Pi_{n=1}^{\infty}(1-a_n)$ converges.*

PROOF. For $n > N \geq 1$ we have

$$\Pi_{j=N}^{n}(1+a_j) \geq 1+\sum_{j=N}^{n} a_j, \quad \Pi_{j=N}^{n}(1-a_j) \geq 1-\sum_{j=N}^{n} a_j.$$

The first inequality is obvious, the second equality is a simple inductive argument. Noting that $(1+a) < (1-a)^{-1}$ if $a \in (0,1)$, we deduce the additional estimates

$$\Pi_{j=N}^{n}(1+a_j) \leq (1-\sum_{j=N}^{n} a_j)^{-1}, \quad \Pi_{j=N}^{n}(1-a_j) \leq (1+\sum_{j=N}^{n} a_j)^{-1},$$

which hold if $\sum_{j=N}^{n} a_j < 1$. Hence provided $\sum_{j=N}^{n} a_j < 1$ we have for all $n > N$ the estimates

$$\begin{aligned}
(1-\sum_{j=N}^{n} a_j)^{-1} > \quad & \Pi_{j=N}^{n}(1+a_j) \quad > 1+\sum_{j=N}^{n} a_j, \\
(1+\sum_{j=N}^{n} a_j)^{-1} > \quad & \Pi_{j=N}^{n}(1-a_j) \quad > 1-\sum_{j=N}^{n} a_j,
\end{aligned}$$

Suppose that $\sum_{j=1}^{\infty} a_j$ converges. Certainly we can choose $N \in \mathbb{N}$ so that $\sum_{j=N}^{\infty} a_j < 1$. Since $(\Pi_{j=N}^{n}(1+a_j))_{n>N}$ is an increasing sequence, the estimates imply that $\Pi_{j=N}^{\infty}(1+a_j)$, and therefore $\Pi_{n=1}^{\infty}(1+a_n)$, converges. Exactly the same argument handles all the remaining cases. We leave details to the reader. ☐

EXAMPLES 3.7.5. (1) The infinite products $\Pi_{n=2}^{\infty}(1 + n^{-p})$ and $\Pi_{n=2}^{\infty}(1 - n^{-p})$ converge if $p > 1$.
(2) The infinite products $\Pi_{n=2}^{\infty}(1 + n^{-p})$ and $\Pi_{n=2}^{\infty}(1 - n^{-p})$ diverge if $p < 1$. ♠

THEOREM 3.7.6 (General principle of convergence for products). *Let (a_n) be a sequence of real numbers none of which equals -1. The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ converges iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|\Pi_{j=m}^{n}(1 + a_n) - 1| < \varepsilon, \textit{ for all } m, n \geq N.$$

PROOF. We leave the proof to the exercises. ☐

LEMMA 3.7.7. *Let (a_n) be a sequence of real numbers none of which equals -1. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then*

(a) $\Pi_{n=1}^{\infty}(1 + a_n)$ *converges if* $\sum_{n=1}^{\infty} a_n$ *converges.*
(b) $\Pi_{n=1}^{\infty}(1 + a_n)$ *diverges to* $+\infty$ *if* $\sum_{n=1}^{\infty} a_n$ *diverges to* $+\infty$.
(c) $\Pi_{n=1}^{\infty}(1 + a_n)$ *diverges to* 0 *if* $\sum_{n=1}^{\infty} a_n$ *diverges to* $-\infty$.

PROOF. A simple application of the calculus shows that

$$\begin{aligned} 0 < u - \log(1 + u) &< u^2/2, \text{ if } u > 0, \\ &< \frac{1}{2}u^2/(1 + u), \text{ if } 0 > u > -1 \end{aligned}$$

We may assume that there exists $N \in \mathbb{N}$, $c \in (0, 1)$, such that $|a_n| \leq c$, $n \geq N$ (otherwise $\Pi_{n=1}^{\infty}(1 + a_n)$ will diverge). Define

$$\gamma = \min\{1, \inf\{(1 + a_n) \mid n \geq N\}\}.$$

Using the inequalities above, we have for $n > m \geq N$

$$\begin{aligned} 0 &< (a_{m+1} + \ldots + a_n) - \log\left[\Pi_{i=m+1}^{n}(1 + a_i)\right], \\ &< (\frac{1}{2}\sum_{i=m+1}^{n} a_i^2)/\gamma. \end{aligned}$$

Hence if $\sum a_n^2$ is convergent then $(a_{m+1}+\ldots+a_n)-\log\left[\Pi_{i=m+1}^{n}(1 + a_i)\right]$ converges to zero as $n \geq m \to \infty$. Statement (a,b,c) now follow by the general principle of convergence. ☐

EXAMPLE 3.7.8. Lemma 3.7.7 implies that $\Pi_{n=1}^{\infty}(1+\frac{1}{n})$ is divergent while $\Pi_{n=1}^{\infty}(1 + \frac{(-1)^n}{n})$ is convergent. ♠

DEFINITION 3.7.9. The infinite product $\Pi_{n=1}^{\infty}(1 + a_n)$ is *absolutely convergent* if $\Pi_{n=1}^{\infty}(1 + |a_n|)$ is convergent.

LEMMA 3.7.10. *Let (a_n) be a sequence of real numbers none of which equals -1.*

(a) $\Pi_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent iff* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
(b) *If* $\Pi_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent then* $\Pi_{n=1}^{\infty}(1+a_n)$ *is convergent.*

PROOF. (a) Lemma 3.7.4 implies that $\Pi_{n=1}^{\infty}(1+|a_n|)$ is convergent iff $\sum_{n=1}^{\infty}|a_n|$ is convergent. It remains to prove (b). For this, it suffices by theorem 3.7.6 to show that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\Pi_{j=m}^{n}(1+a_j)-1| < \varepsilon$ for all $m, n \geq N$. Since the product is absolutely convergent, we may find $N \in \mathbb{N}$ so that $\Pi_{j=m}^{n}(1+|a_j|)-1 < \varepsilon$ for all $m, n \geq N$. Observe that $|\Pi_{j=m}^{n}(1+a_j)-1| < \Pi_{j=m}^{n}(1+|a_j|)-1$. $\square$

REMARK 3.7.11. Theorem 3.7.6 and lemma 3.7.10 continue to hold if (a_n) is a sequence of complex numbers, none of which equals -1. The proofs are the formally the same as in the real case.

3.7.2. Infinite product for $\sin x$.

PROPOSITION 3.7.12. *For all $z \in \mathbb{C}$*

$$\sin z = z\,\Pi_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

PROOF. We give a proof of proposition 3.7.12 that uses a minimal amount of complex variable theory. We give an alternative and simpler real variable proof based on Fourier series in chapter 5. Most readers will probably prefer to omit the details following the partial result (3.7).

For $z \in \mathbb{C}$ we have (by definition)

$$\sin z = \frac{e^{\imath z}-e^{-\imath z}}{2\imath}.$$

Applying proposition 3.4.6 to $e^{\pm \imath z}$ gives

$$\begin{aligned} \sin z &= \lim_{n\to\infty}\left[\frac{(1+\frac{\imath z}{n})^n-(1-\frac{\imath z}{n})^n}{2\imath}\right], \\ &= \lim_{n\to\infty} P_n(z), \end{aligned}$$

where $P_n(z)$ is a polynomial of degree (at most) n. Our approach will be to factorize $P_n(z)$ and for this we need to find the solutions of

$P_n(z) = 0$. Observe that

$$\begin{aligned} P_n(z) = 0 \iff & (1 + \frac{\imath z}{n})^n = (1 - \frac{\imath z}{n})^n, \\ \iff & 1 + \frac{\imath z}{n} = u(1 - \frac{\imath z}{n}), \text{ where } u^n = 1, \\ \iff & z = \frac{n}{\imath}\left(\frac{u-1}{u+1}\right). \end{aligned}$$

From now we assume that n is odd and so $u \neq -1$. The solutions of $u^n = 1$ are given by

$$u = e^{\frac{2k\pi\imath}{n}},\ k = -\frac{n-2}{2}, \ldots, -1, 0, 1, \ldots, \frac{n-1}{2}.$$

For $k \in \{-\frac{n-2}{2}, \ldots, -1, 0, 1, \ldots, \frac{n-1}{2}\}$ we have $P(z_k) = 0$, where

$$\begin{aligned} z_k &= \frac{n}{\imath}\left(\frac{e^{\frac{2k\pi\imath}{n}} - 1}{e^{\frac{2k\pi\imath}{n}} + 1}\right), \\ &= n\frac{(e^{\frac{k\pi\imath}{n}} - e^{\frac{-k\pi\imath}{n}})/2\imath}{(e^{\frac{k\pi\imath}{n}} + e^{\frac{k\pi\imath}{n}})/2}, \\ &= n\tan(\frac{k\pi}{n}). \end{aligned}$$

Since $\tan x$ is an odd function, the roots of $P_n(z) = 0$ are

$$0, \pm n\tan(\frac{\pi}{n}), \pm n\tan(\frac{2\pi}{n}), \ldots, \pm n\tan(\frac{\frac{n-1}{2}\pi}{n}),$$

and so (for n odd) we have

$$P_n(z) = Cz\Pi_{j=1}^{\frac{n-1}{2}}\left(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}\right).$$

The coefficient of z in $P_n(z)$ is easily verified to be 1 and so $C = 1$. For fixed j, we have $\lim_{n\to\infty} n^2\tan^2(\frac{j\pi}{n}) = j^2\pi^2$. Hence for $Q \geq 1$, we have

$$\lim_{n\to\infty} z\Pi_{j=1}^{Q}\left(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}\right) = z\Pi_{j=1}^{Q}\left(1 - \frac{z^2}{j^2\pi^2}\right). \tag{3.7}$$

At this point we need to be careful: we would like to take the limit as $Q \to \infty$ in (3.7) and then claim that the limit on the left hand side is equal to $\lim_{n\to\infty} P_n(z) = \sin z$. The problem is that Q depends on n — $Q \leq (n-1)/2$ in the product formula on the left-hand side. The basic idea we use is that if z is fixed then for Q large enough, the product $\Pi_{j=Q+1}^{\frac{n-1}{2}}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})$ can be made arbitrarily close to 1.

Now for the details. Fix $z \in \mathbb{C}$. We may assume $z \neq 0$ since the result is immediate if $z = 0$. Since $\sum_{j=1}^{\infty} j^{-2} < \infty$, the infinite product $z\Pi_{j=1}^{\infty}\left(1 - \frac{z^2}{j^2\pi^2}\right)$ is absolutely convergent and therefore convergent by lemma 3.7.10. If we define $M = \Pi_{j=1}^{\infty}(1 + \frac{|z|^2}{j^2\pi^2})$ then M is a bound for the partial products:

$$|\Pi_{j=p}^{q}(1 - \frac{z^2}{j^2\pi^2})| \leq M, \text{ for all } \infty \geq q \geq p \geq 1$$

Since $\tan x \geq x$ for $x \in [0, \pi/2)$, $|\frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}| \leq \frac{|z|^2}{j^2\pi^2}$ and so M is also a bound for the partial products of $\Pi_{j=1}^{\frac{n-1}{2}}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})$. Let $\varepsilon > 0$. Since the infinite product converges, we can choose $N \in \mathbb{N}$ such that

$$|\Pi_{j=p}^{q}\left(1 - \frac{z^2}{j^2\pi^2}\right) - 1| < \varepsilon/(2M|z|), \ \infty \geq q > p \geq N. \tag{3.8}$$

Noting that $|\frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}| \leq \frac{|z|^2}{j^2\pi^2}$. we may also require

$$|\Pi_{j=p}^{q}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}) - 1| < \varepsilon/(2M|z|), \ (n-1)/2 \geq q \geq p \geq N. \tag{3.9}$$

With $Q \geq 1$, it follows from the convergence of $z\Pi_{j=1}^{\infty}\left(1 - \frac{z^2}{j^2\pi^2}\right)$ that

(3.10)

$$\lim_{n\to\infty} z\Pi_{j=1}^{Q}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})\Pi_{j=Q+1}^{\infty}(1 - \frac{z^2}{j^2\pi^2}) = z\Pi_{j=1}^{\infty}(1 - \frac{z^2}{j^2\pi^2}).$$

Now if $n \geq Q \geq N$, (3.8), (3.9) imply that

$$|\Pi_{j=Q+1}^{\infty}(1 - \frac{z^2}{j^2\pi^2}) - \Pi_{j=Q+1}^{\frac{n-1}{2}}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})| < \varepsilon/(M|z|) \tag{3.11}$$

Hence for $n \geq Q \geq N$,

$$|z\Pi_{j=1}^{Q}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})\Pi_{j=Q+1}^{\infty}(1 - \frac{z^2}{j^2\pi^2}) - z\Pi_{j=1}^{\frac{n-1}{2}}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})})|$$

is bounded by $|z|M\varepsilon/(|z|M) = \varepsilon$. Hence by (3.10), there exists $N_1 \geq N$ such that for $n \geq N_1$

$$|z\Pi_{j=1}^{\frac{n-1}{2}}(1 - \frac{z^2}{n^2\tan^2(\frac{j\pi}{n})}) - z\Pi_{j=1}^{\infty}(1 - \frac{z^2}{j^2\pi^2})| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

EXERCISES 3.7.13.

(1) Prove theorem 3.7.6.

(2) Prove that if $\Pi(1+a_n)$ is absolutely convergent, then the value of the product is independent of the order of the factors.
(3) State and prove an analog of Riemann's rearrangement theorem for infinite products that are not absolutely convergent.

3.8. Appendix: trigonometric identities

In this appendix we prove some very useful trigonometric identities.

THEOREM 3.8.1. *Let $\alpha, \beta \in \mathbb{R}$ and suppose that β is not an integer multiple of 2π. For $n \geq 0$ we have*

$$\begin{aligned}
\sum_{k=0}^{n} \cos(\alpha + k\beta) &= \frac{\sin\left(\frac{(n+1)\beta}{2}\right) \cos\left(\alpha + \frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}, \\
\sum_{k=0}^{n} \sin(\alpha + k\beta) &= \frac{\sin\left(\frac{(n+1)\beta}{2}\right) \sin\left(\alpha + \frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}, \\
\sum_{k=1}^{n} \cos(k\beta) &= \frac{\cos\left(\frac{(n+1)\beta}{2}\right) \sin\left(\frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}, \\
\sum_{k=1}^{n} \sin(k\beta) &= \frac{\sin\left(\frac{(n+1)\beta}{2}\right) \sin\left(\frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}.
\end{aligned}$$

PROOF. By DeMoivre's theorem we have

$$\cos(\alpha + k\beta) + \imath \sin(\alpha + k\beta) = e^{\imath\alpha + \imath k\beta} = e^{\imath\alpha} e^{\imath k\beta}.$$

Therefore

$$\sum_{k=0}^{n} \cos(\alpha + k\beta) + \imath \sin(\alpha + k\beta) = e^{\imath\alpha} \sum_{k=0}^{n} e^{\imath k\beta}.$$

Provided that β is not an integer multiple of 2π, we have

$$\sum_{k=0}^{n} e^{\imath k\beta} = \frac{1 - e^{\imath(n+1)\beta}}{1 - e^{\imath\beta}}.$$

(This is most easily verified by multiplying both sides by $1 - e^{\imath\beta}$. Alternatively, divide.) Taking real and imaginary parts gives us

$$\begin{aligned}
\sum_{k=0}^{n} \cos(\alpha + k\beta) &= \text{Real}\left(e^{\imath\alpha} \frac{1 - e^{\imath(n+1)\beta}}{1 - e^{\imath\beta}}\right), \\
\sum_{k=0}^{n} \sin(\alpha + k\beta) &= \text{Im}\left(e^{\imath\alpha} \frac{1 - e^{\imath(n+1)\beta}}{1 - e^{\imath\beta}}\right).
\end{aligned}$$

We have

$$\begin{aligned} e^{\imath\alpha}\frac{1-e^{\imath(n+1)\beta}}{1-e^{\imath\beta}} &= e^{\imath\alpha}\frac{(1-e^{\imath(n+1)\beta})(1-e^{-\imath\beta})}{2-e^{\imath\beta}-e^{-\imath\beta}} \\ &= e^{\imath\alpha}\frac{(1-e^{\imath(n+1)\beta})(1-e^{-\imath\beta})}{2-2\cos\beta}, \\ &= \frac{A+\imath B}{4\sin^2(\frac{\beta}{2})}, \end{aligned}$$

where

$$\begin{aligned} A &= \cos\alpha+\cos(n\beta+\alpha)-\cos((n+1)\beta+\alpha)-\cos(\alpha-\beta), \\ B &= \sin\alpha+\sin(n\beta+\alpha)-\sin((n+1)\beta+\alpha)-\sin(\alpha-\beta). \end{aligned}$$

Using the trigonometric identities $\cos a+\cos b = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$ and $\cos a-\cos b = 2\sin(\frac{a+b}{2})\sin(\frac{b-a}{2})$, it is straightforward to show that

$$A = 4\cos(\alpha+\frac{n\beta}{2})\sin(\frac{(n+1)\beta}{2})\sin(\beta/2).$$

Hence $\sum_{k=0}^{n}\cos(\alpha+k\beta) = A/4\sin^2(\frac{\beta}{2}) = \frac{\sin\left(\frac{(n+1)\beta}{2}\right)\cos\left(\alpha+\frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}$. A similar analysis using the identities $\sin a\pm\sin b = 2\sin(\frac{a\pm b}{2})\cos(\frac{a\mp b}{2})$ gives the result for the sum of sines. Alternatively, replace α by $\alpha-\pi/2$ in the cosine sum formula.

Finally we need to show $\sum_{k=1}^{n}\cos(k\beta) = \frac{\cos\left(\frac{(n+1)\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)}{\sin(\frac{\beta}{2})}$. This follows from the expression for the sum from $k=0$ to n (with $\alpha = 0$) if we subtract the initial term 1 ($\cos 0$) and then use the formula $\sin(\frac{(n+1)\beta}{2})\cos(\frac{n\beta}{2})-\cos(\frac{(n+1)\beta}{2})\sin(\frac{n\beta}{2}) = \sin(\frac{\beta}{2})$. $\square$

CHAPTER 4

Uniform Convergence

4.1. Introduction

In this section we begin our study of continuous and differentiable functions. Our strategy will be to build functions as infinite sums (or products) of simpler functions like x^n or $\sin nx$ and $\cos nx$. For example, we develop techniques that enable us to give conditions for a *power series* $\sum_{n=0}^{\infty} a_n x^n$ to converge to a continuous or differentiable function. We also investigate continuity properties of *trigonometric* or *Fourier series* such as the sine series $\sum_{n=1}^{\infty} b_n \sin(nx)$. The aim in this chapter is to develop the tools — which are mainly based on the idea of *uniform convergence*. In the next chapter, we study a number of important classes of function using these tools. Although in this and the following chapter we work almost exclusively with real valued functions defined on subsets, usually subintervals, of the real line, the ideas and methods we develop have very general applicability and most of the results we give apply to complex valued functions.

We start by looking at convergence of *sequences* of functions. We then apply our results to the partial sums of infinite series of functions. This is very much the approach of the previous chapter and much of our work will be making the translation from sequences/series of real numbers to sequences/series of functions. A new, and very important, feature will be the relation between infinite series and term by term integration and differentiation. For example, when can we find the integral of the function defined by an infinite series by integrating term-by-term? Many foundational theorems in analysis are about problems of this type: when can we interchanging two limiting operations?

4.2. Pointwise Convergence

Let $I \subset \mathbb{R}$ be an interval, possibly unbounded, which may be open, closed, or half-open (indeed, all of what we say works perfectly well if I is any subset of $\mathbb{R}$). Suppose that we are given a sequence (u_n) of real valued functions on I. That is, for each $n \in \mathbb{N}$, $u_n : I \to \mathbb{R}$. At this point we do not assume any additional properties of the functions u_n (such as continuity). Observe that for each $x \in I$, $(u_n(x))$

is a sequence of real numbers. The next definition gives a natural definition of convergence of the sequence of functions (u_n) in terms of the sequences $(u_n(x))$, $x \in I$.

Definition 4.2.1. (Notation and assumptions as above) The sequence (u_n) of functions on I is *pointwise convergent* (on I), if there exists a function $u : I \to \mathbb{R}$ such that for every $x \in I$ we have

$$\lim_{n\to\infty} u_n(x) = u(x).$$

We refer to u as the *pointwise limit* of the sequence (u_n).

Examples 4.2.2. (1) Take $I = [0,1]$, let $f : I \to \mathbb{R}$ be any function and define $u_n = f/n$, $n \in \mathbb{N}$. That is, for each $x \in I$, $n \in \mathbb{N}$, $u_n(x) = f(x)/n$. Although f may not be bounded on I (we are not assuming f is continuous), it is true that for every (fixed) $x \in I$, $f(x) \in \mathbb{R}$, and so $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} f(x)/n = 0$. Hence (u_n) is pointwise convergent on I with pointwise limit the zero function. In this case, the pointwise limit is continuous even though the terms in the sequence may be discontinuous at every point of I.
(2) Take $I = [0,1]$ and let $u_n(x) = x^n$, $x \in I$, $n \in \mathbb{N}$. If $0 \leq x < 1$, we have $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} x^n = 0$. On the other hand $\lim_{n\to\infty} u_n(1) = 1$. The pointwise limit u is continuous on $[0,1)$ and has discontinuity at $x = 1$. In this case, we see that a pointwise limit of continuous functions need not be continuous. A feature of this example is that as x gets close to 1, convergence to $u(x)$ gets very slow. More specifically, given $x \in [0,1)$, $1 > \varepsilon > 0$, let $N(x) \in \mathbb{N}$ be the smallest integer such that $u_n(x) = x^n < \varepsilon$. Clearly $N(0) = 1$ and if $0 < x < 1$, $N(x)$ is the smallest integer bigger than $\frac{\log \varepsilon}{\log x}$. Consequently, the convergence of $u_n(x)$ to 0 is very slow when x is close to 1.
(3) Even if the pointwise limit of a sequence of continuous functions is continuous, the convergence can have unpleasant features. For example, take $I = [0,1]$, $p \in \mathbb{R}$ and define

$$u_n(x) = n^p x^n (1-x),\ x \in [0,1],\, n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} n^p x^n(1-x) = 0$, if $x \in [0,1]$, we see that (u_n) is pointwise convergent on I with pointwise limit the zero function (note that $u_n(1) = 0$, all n). A straightforward application of the differential calculus shows that the maximum value of u_n on I is $n^{p-1}(n/(n+1))^n$ and is attained when $x = n/(n+1)$. We see that if $p < 1$, then $\lim_{n\to\infty} \sup_{x\in I} u_n(x) = \lim_{n\to\infty} n^{p-1}(n/(n+1))^n = 0$. If $p = 1$, then $\lim_{n\to\infty} \sup_{x\in I} u_n(x) = e^{-1}$ (where we have used $\lim_{n\to\infty}(n/(n+1))^n = 1/(1+1/n)^n = e^{-1}$). If $p > 1$, then $\lim_{n\to\infty} \sup_{x\in I} u_n(x) = +\infty$. In the cases $p \geq 1$, even though (u_n) converges pointwise to the zero function, the graph of u_n does not approach that of the zero function. It is also natural to consider the area under the graph of u_n. We have

$$\int_0^1 u_n(x) = \int_0^1 n^p x^n (1-x)\,dx = \frac{n^p}{(n+1)(n+2)},\ n \geq 1.$$

Clearly $\lim_{n\to\infty} \int_0^1 u_n(x) = 0 = \int_0^1 \lim_{n\to\infty} u_n(x)\,dx$ if and only if $p < 2$. Indeed, if $p > 2$, $\lim_{n\to\infty} \int_0^1 u_n(x) = +\infty$. This shows that without further conditions on the convergence of functions we cannot interchange the order of limit and integration. ♠

Remark 4.2.3. All of the results and definitions in this section extend immediately to sequences of complex valued functions.

4.3. Uniform convergence of sequences

What we seek is a good definition of convergence of functions defined on a subset I of $\mathbb{R}$. In the previous section we indicated through examples that pointwise convergence of functions allows for some nasty pathology. We need a better definition. Suppose that $f, g : I \to \mathbb{R}$. What does it mean for f and g to be 'close'? A natural approach is to require that $|f(x) - g(x)|$ is small for *all* $x \in I$. In terms of the graphs of f and g, we are asking that their graphs are close as subsets of $\mathbb{R}^2$. In figure 1, we display graphically what it means for the graphs of f

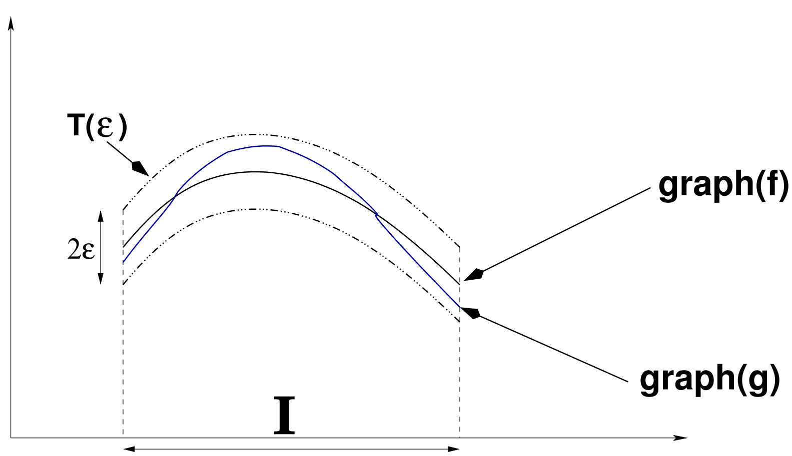


Figure 1. Graphs of f and g within ε

and g to be within ε of each other. We define a tube $T(\varepsilon)$ of width 2ε

containing the graph of f. Formally,

$$T(\varepsilon) = \{(x, y) \mid x \in I, |f(x) - y| < \varepsilon\}.$$

In order for the graph of g to be within ε of the graph of f we require

$$\mathrm{graph}(g) \subset T(\varepsilon).$$

That is, $|f(x) - g(x)| < \varepsilon$ for all $x \in I$.

In the remainder of this section we formalize this idea of closeness or *uniform approximation*. We do this by first restricting to the class of bounded functions (defined on any subset of $\mathbb{R}$) and then giving a precise definition of what we mean by the distance between two functions f, g such that $f - g$ is bounded. This will enable us to give a good definition of convergence for sequences of continuous functions. We develop these ideas further in the next chapter where we show how we can approximate a continuous function, which may be nowhere differentiable, by more regular functions, such as polynomials. Almost everything we do extends immediately to complex valued functions.

4.3.1. Spaces of bounded functions. If I is a non-empty subset of $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is *bounded* if there exists $M \geq 0$ such that

$$|f(x)| \leq M, \text{ for all } x \in I.$$

(We do not assume at this point that I is an interval or f is continuous.) If $f : I \to \mathbb{R}$ is bounded, we define

$$\|f\| = \sup\{|f(x)| \mid x \in I\} < \infty.$$

REMARK 4.3.1. The number $\|f\|$ is often called the *uniform* or C^0 or ∞-norm of f. It is also commonly denoted by $\|f\|_\infty$.

DEFINITION 4.3.2. Suppose that I is a non-empty subset of $\mathbb{R}$. Let $B(I)$ denote the set of all bounded functions $f : I \to \mathbb{R}$.

EXAMPLE 4.3.3. Constant functions are bounded and so $B(I)$ contains all the constant functions, including the zero function. ♠

LEMMA 4.3.4. *(Notation as above.) Let $f, g \in B(I)$.*

1. *For all $c \in \mathbb{R}$, we have $f + cg \in B(I)$.*
2. *$\|f + g\| \leq \|f\| + \|g\|$, and $\|cf\| = |c|\|f\|$, all $c \in \mathbb{R}$.*
3. *$\|f\| = 0$ iff $f \equiv 0$.*

PROOF. We give a careful proof of this result, missing none of the details. We start by showing that if $f \in B(I)$, then $cf \in B(I)$, for all $c \in \mathbb{R}$, and $\|cf\| = |c|\|f\|$. We have $|f(x)| \leq \|f\|$ for all $x \in M$. Hence $|c|\|f\|$ is an upper bound for cf and so $cf \in B(I)$. We claim that $\|cf\| = |c|\|f\|$. If not, there exists $M < |c|\|f\|$ such that M is an upper bound for cf. But then $M/|c| < \|f\|$ would be an upper bound for f, contradicting the definition of $\|f\|$.

Next we prove that if $f, g \in B(I)$ then $f+g \in B(I)$ and $\|f+g\| \leq \|f\|+\|g\|$ (this will complete the proof of (1,2)). For all $x \in I$, we have

$$|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|+\|g\|.$$

Therefore $\|f\|+\|g\|$ is an upper bound for $\{|f(x)+g(x)| \mid x \in I\}$ and so $f+g \in B(I)$ and $\|f+g\| \leq \|f\|+\|g\|$.

Finally, suppose $\|f\| = 0$. Then $\sup\{|f(x)| \mid x \in I\} = 0$. Hence $f(x) = 0$, for all $x \in I$ and so $f \equiv 0$. The converse is trivial. ☐

DEFINITION 4.3.5. Suppose that $I \subset \mathbb{R}$ is non-empty, $f, g : I \to \mathbb{R}$ and $f - g \in B(I)$. We define the distance between f and g, $d(f, g)$, by

$$d(f, g) = \|f - g\|.$$

LEMMA 4.3.6. *(Notation as above.) Suppose that $f, g, h \in B(I)$. We have*

(1) *$d(f, g) \geq 0$ and $d(f, g) = 0$ iff $f = g$.*
(2) *$d(f, g) = d(g, f)$.*
(3) *$d(f, h) \leq d(f, g) + d(g, h)$ (triangle inequality).*

PROOF. The result is immediate from lemma 4.3.4. ☐

REMARK 4.3.7. It is enough that $f - g, g - h, f - h \in B(I)$.

4.3.2. Spaces of continuous functions. Suppose that $I \subset \mathbb{R}$ is non-empty. We let $C^0(I)$ denote the space of continuous real-valued functions on I. In general $C^0(I) \not\subset B(I)$ (for example, take $I = (0, 1)$). However, there is a large class of subsets I of $\mathbb{R}$ for which $C^0(I) \subset B(I)$. We concentrate on the best known case.

THEOREM 4.3.8. *If I is a closed and bounded interval, then $C^0(I) \subset B(I)$.*

PROOF. This is a restatement of the theorem that continuous functions on a closed and bounded interval are bounded. ☐

4.3.3. Convergence of functions.

DEFINITION 4.3.9. Let I be a non-empty subset of $\mathbb{R}$. If (u_n) is a sequence of functions on I, then (u_n) *converges uniformly* to $u : I \to \mathbb{R}$ if

$$\lim_{n\to\infty} d(u, u_n) = 0.$$

Remarks 4.3.10. (1) If (u_n) converges uniformly to $u : I \to \mathbb{R}$, then we must have $u - u_n \in B(I)$, at least for large enough n. In particular, if $(u_n) \subset B(I)$ then $u \in B(I)$ since $d(u, u_n) < \infty$ implies that $u - u_n \in B(I)$ and so, $u = (u - u_n) + u_n \in B(I)$ (lemma 4.3.4).
(2)The use of the term 'uniform' in the definition should be clear. The sequence (u_n) converges uniformly to u, if for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that if $n \geq N$ then $|u_n(x) - u(x)| < \varepsilon$ for all $x \in I$. This is a much stronger condition than pointwise convergence, where N may depend strongly on x — see examples 4.2.2.

The next result gives a formal proof that uniform convergence is at least as strong as pointwise convergence.

Proposition 4.3.11. *(Notation as above.) If (u_n) converges uniformly to u, then (u_n) converges pointwise to u.*

Proof. We are required to prove that for each $x \in I$, $\lim_{n\to\infty} u_n(x) = u(x)$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(u_n, u) = 0$, there exists $N \in \mathbb{N}$ such that $d(u_n, u) < \varepsilon$, for all $n \geq N$. That is,

$$d(u_n, u) = \sup\{|u_n(y) - u(y)| \mid y \in I\} < \varepsilon,\ n \geq N.$$

Since $|u_n(x) - u(x)| \leq d(u_n, u)$, we have $|u_n(x) - u(x)| < \varepsilon$ for all $n \geq N$ and so $\lim_{n\to\infty} u_n(x) = u(x)$. $\square$

Theorem 4.3.12. *Let I be a non-empty subset of $\mathbb{R}$ (for example an interval) and let (u_n) be a sequence of continuous functions on I which converges uniformly to u. Then $u \in C^0(I)$. If, in addition, $(u_n) \subset B(I)$, then $u \in B(I) \cap C^0(I)$.*

Proof. Suppose that $(u_n) \subset C^0(I)$ converges uniformly to u. We are required to prove that if $x_0 \in I$ and $\varepsilon > 0$, then there exists $\delta > 0$ such that $|u(x_0) - u(x)| < \varepsilon$, for all $x \in I$ such that $|x_0 - x| < \delta$. The idea of the proof is to approximate u sufficiently closely by a continuous function u_N (how large we need to take N depends on ε) and then use the continuity of u_N to deduce the estimate we require on u. In more detail, choose $N \in \mathbb{N}$ such that $d(u_N, u) < \varepsilon/3$. By definition of $d(u_N, u)$, we have

$$|u_N(y) - u(y)| < \varepsilon/3, \text{ for all } y \in I.$$

Since u_N is continuous on I, there exists $\delta > 0$ such that

$$|u_N(x_0) - u_N(x)| < \varepsilon/3, \text{ for all } x \in I \text{ such that } |x_0 - x| < \delta.$$

Now we use the triangle inequality. Suppose $x \in I$, then

$$\begin{aligned} |u(x_0) - u(x)| &= |u(x_0) - u_N(x_0) + u_N(x_0) - u_N(x) + u_N(x) - u(x)|, \\ &\leq |u(x_0) - u_N(x_0)| + |u_N(x_0) - u_N(x)| + |u_N(x) - u(x)|, \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where the last inequality holds provided $|x_0 - x| < \delta$. □

COROLLARY 4.3.13. *Let (u_n) be a sequence of continuous functions on the closed and bounded interval $I = [a, b]$. Suppose that (u_n) converges uniformly to u, then u is continuous and bounded.*

PROOF. An immediate corollary of theorem 4.3.12 since every continuous function on $[a, b]$ is bounded. □

Next we give a number of examples of uniform and non-uniform convergence after which we give a general criterion for uniform convergence.

EXAMPLES 4.3.14. (1) Take $I = [0, 1]$ and let $u_n(x) = x^n$, $x \in I$ (as in examples 4.2.2(2)). Recall that the pointwise limit u denote the pointwise limit of (u_n) is the function which is equal to zero on $[0, 1)$ and 1 at $x = 1$. We claim that $d(u, u_n) = 1$ for all $n \in \mathbb{N}$ and so the convergence is not uniform. It suffices to show that for every $\varepsilon > 0$, there exists $x \in [0, 1)$ such that $|u_n(x) - u(x)| = |u_n(x)| = x^n > 1 - \varepsilon$. This is immediate from the continuity of u_n at $x = 1$. (Of course, since u is not continuous, we can deduce that (u_n) does not converge uniformly to u using corollary 4.3.13.)
(2) Take $I = [0, 1]$, $p \in \mathbb{R}$ and let $u_n(x) = n^p x^n (1 - x)$, $x \in I$ (as in examples 4.2.2(3)). Recall that the pointwise limit u of (u_n) is identically zero. We have (see examples 4.2.2(3)), $d(u, u_n) = n^{p-1}(n/(n + 1))^n$, $n \in \mathbb{N}$. Hence $d(u, u_n) \to 0$ iff $p < 1$. If $p = 1$, $d(u, u_n) \to e^{-1}$, and if $p > 1$, $d(u, u_n) \to +\infty$. Hence we only have uniform convergence when $p < 1$. ♠

4.3.4. General principle of convergence. Just as we did for sequences of real numbers we may define Cauchy sequences of functions. This definition has the advantage of not requiring information about the limit.

DEFINITION 4.3.15. If (u_n) is a sequence of functions on the non-empty subset I of $\mathbb{R}$, then (u_n) is a *Cauchy sequence* if $d(u_m, u_n) \to 0$ as $m, n \to \infty$. That is, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(u_m, u_n) < \varepsilon, \text{ for all } m, n \geq N.$$

THEOREM 4.3.16 (General principle of (uniform) convergence). *Let I be a non-empty subset of $\mathbb{R}$ and let (u_n) be a sequence of functions* *on I. Then (u_n) is uniformly convergent on I if and only if (u_n) is a Cauchy sequence. If either condition holds and the limit function is u, then u will be bounded (respectively, continuous) if $(u_n) \subset B(I)$ (respectively $(u_n) \subset C^0(I)$).*

Proof. We start by showing that for each $x \in I$, $(u_n(x))$ is a Cauchy sequence of real numbers. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $d(u_m, u_n) < \varepsilon$ for all $m, n \geq N$. If $x \in I$, we have $|u_m(x) - u_n(x)| \leq d(u_m, u_n) < \varepsilon$, for all $m, n \geq N$. Hence $(u_n(x))$ is a Cauchy sequence and by the general principle of convergence for sequences of real numbers, there exists $u(x) \in \mathbb{R}$ such that $\lim_{n\to\infty} u_n(x) = u(x)$. This construction defines a function $u : I \to \mathbb{R}$. Observe that u is the pointwise limit of the sequence (u_n). The estimate

$$|u_m(x) - u_n(x)| < \varepsilon,\ m, n \geq N, \tag{4.1}$$

holds for all $x \in I$. That is, the integer N does not depend on the choice of $x \in I$. Letting $m \to \infty$ in (4.1) gives

$$|u(x) - u_n(x)| \leq \varepsilon,\ n \geq N, \text{ for all } x \in I,$$

and so $d(u, u_n) \leq \varepsilon$, for all $n \geq N$. Hence (u_n) converges uniformly to u. We leave the proof that a uniformly convergent sequence is Cauchy to the exercises. The final statements follow from remarks 4.3.10(1) and theorem 4.3.12. □

Our main applications and examples of the general principle of uniform convergence will be to infinite series and are described in the next section.

Exercises 4.3.17.

(1) Find the pointwise limit of the following sequences of functions on the specified domain. In each case describe the continuity properties of the limit function.
  (a) $f_n(x) = \tan^{-1}(nx)$, $x \geq 0$.
  (b) $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$.
  Is the convergence for either of these sequences uniform? Why/Why not?
(2) Suppose that (f_n) is a sequence of continuous functions which is pointwise convergent to f on the open interval (a, b). Suppose that the convergence of (f_n) to f is *uniform* on every closed subinterval of (a, b). Does it follow that f is continuous on (a, b)? Proof or counterexample.
(3) Complete the proof of theorem 4.3.16 by showing that a uniformly convergent sequence of functions is a Cauchy sequence.
(4) Determine whether or not the following sequences converge uniformly on the specified domains. It is a good idea to start by finding pointwise limits.
  (a) $f_n(x) = \frac{1}{n+x}$, $x \geq 0$, $n \geq 1$.
  (b) $f_n(x) = \frac{x^n}{1+x^n}$, $x \in [0, 1]$, $n \geq 1$.

4.4. Uniform convergence of infinite series

In this section I will always denote a subinterval of $\mathbb{R}$ (open, closed, half-open, bounded or unbounded). However, most of the results we give in this section apply to functions defined on any non-empty subset of $\mathbb{R}$.

Let (u_n) be a sequence of functions defined on I. For $n \geq 1$, we define the sequence (S_n) of partial sums by

$$S_n(x) = \sum_{j=1}^{n} u_j(x),\ x \in I.$$

Note that (S_n) is a sequence of functions defined on I.

DEFINITION 4.4.1. (Notation as above.)

(a) The infinite series $\sum_{n=1}^{\infty} u_n$ is *pointwise convergent* (on I) to the function $S : I \to \mathbb{R}$ if the sequence of partial sums (S_n) is pointwise convergent to S. (That is, for all $x \in I$, $\lim_{n\to\infty} S_n(x) = S(x)$.)

(b) The infinite series $\sum_{n=1}^{\infty} u_n$ is *uniformly convergent* (on I) to the function $S : I \to \mathbb{R}$ if the sequence of partial sums (S_n) is uniformly convergent to S. (That is, $\lim_{n\to\infty} d(S, S_n) = 0$.)

EXAMPLES 4.4.2.

As an immediate consequence of our results on the uniform convergence of sequences, we have the first of our main results on uniform convergence of infinite series.

THEOREM 4.4.3. *Let (u_n) be a sequence of continuous functions on I. If the infinite series $\sum_{n=1}^{\infty} u_n$ is uniformly convergent to the function $S : I \to \mathbb{R}$, then S is continuous. If the partial sums define a sequence of bounded functions on I, then $S \in B(I) \cap C^0(I)$.*

For applications, it is useful to have a slightly stronger version of 4.4.3.

THEOREM 4.4.4. *Let (u_n) be a sequence of continuous functions on I. If the infinite series $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on every closed and bounded subinterval of I, then $\sum_{n=1}^{\infty} u_n$ converges to a continuous function on I. More generally, if $\{I_i \mid i \in I\}$ is a family of closed and bounded subintervals of I such that (a) $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on I_i for all $i \in I$, and (b) $\cup_{i\in I} I_i = I$, then $\sum_{n=1}^{\infty} u_n$ converges to a continuous function on I.*

PROOF. The result is immediate from the previous theorem if I is a closed and bounded interval. So we assume that I is not a closed and bounded interval. The hypotheses of the theorem imply that $\sum_{n=1}^{\infty} u_n$ converges pointwise on I to a function $S : I \to \mathbb{R}$. Indeed, given $x \in I$, apply the uniform convergence hypothesis of the theorem to the closed interval $[x, x]$. In order to show that S is continuous it is enough to prove that S is continuous on every closed and bounded subinterval of I. But this follows from the hypotheses of the theorem and theorem 4.4.3. (If $I = [a, +\infty)$ then we prove S continuous on any bounded interval $[a, a + c]$ and that suffices for continuity at a. For all other points $x \in I$, we can choose $a < x < b$ so that $[a, b] \subset I$ and then S is continuous on $[a, b]$ and therefore certainly continuous at x.) We leave the final statement to the exercises. □

REMARKS 4.4.5. (1) We do not claim in theorem 4.4.4 that S is bounded.
(2) If I is an arbitrary non-empty subset of $\mathbb{R}$, we ask that $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[a, b] \cap I$ for all $-\infty < a \le b < +\infty$.

One last, but key, result before we give some examples.

THEOREM 4.4.6 (General principle of uniform convergence for series). *Let (u_n) be a sequence of functions on I. The infinite series $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on I iff he sequence of partial sums (S_n) is Cauchy. More formally, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\|u_m + \ldots + u_n\| < \varepsilon, \text{ for all } m \ge n \ge N,$$

then there exists a function $S : I \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} u_n$ converges uniformly to S. If the sequence (u_n) consist of continuous functions, then S is continuous.

PROOF. Apply theorem 4.3.16 to the sequence of partial sums. □

EXAMPLES 4.4.7. (1) Let $u_n(x) = \sin nx/n^2$, $n \ge 1$. We claim that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent on $\mathbb{R}$ and $S(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is continuous on $\mathbb{R}$. To see this, observe that for all $x \in \mathbb{R}$, we have $|u_n(x)| \le 1/n^2$. We know that $\sum_{n=1}^{\infty} 1/n^2 < \infty$ and so, by the general principle of convergence for series (of real numbers) given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\frac{1}{m^2} + \ldots + \frac{1}{n^2}| < \varepsilon$, for all $m \ge n \ge N$. Hence

$$|\sum_{j=n}^{m} \frac{\sin nx}{n^2}| \le \sum_{j=n}^{m} \frac{1}{j^2} < \varepsilon, \text{ for all } m \ge n \ge N.$$

Therefore $d(S_m, S_{n-1}) = \|\sum_{j=n}^{m} u_j\| < \varepsilon$, for all $m \ge n \ge N$ and the sequence of partial sums is Cauchy. The result follows from theorem 4.4.6. The reader should note how this proof is a mix of the theory of series of real numbers (using in this case the convergence of

$\sum 1/n^2$) and results on uniform convergence. The method we used to deduce uniform convergence by comparing with a 'known' series of real numbers is very powerful and due to Weierstrass (it is a special case of the Weierstrass M-test).

(2) Let $u_n(x) = x^n$, $n \geq 0$, Let $a \in (0,1)$ and regard u_n as defined on $[-a,a]$. For all $x \in [-a,+a]$, $m \geq n$ we have

$$|x^m + \ldots + x_n| = |x|^n|1 + \ldots + x^{m-n}| \leq a^n \sum_{j=0}^{\infty} a^j \leq a^n/(1-a).$$

Hence $\|u_m + \ldots + u_n\| \leq a^n/(1-a) \to 0$ as $m \geq n \to \infty$. By the general principle of uniform convergence for series (theorem 4.4.3), $\sum_{n=0}^{\infty} x^n$ is uniformly convergent to a continuous function on $[-a,a]$. This argument is valid for all $a \in (0,1)$ and so $\sum_{n=0}^{\infty} x^n$ converges to a continuous function S on $(-1,1)$ (theorem 4.4.4). In this case we know that $S(x) = 1/(1-x)$. The reader should note that $\sum_{n=0}^{\infty} x^n$ is *not* uniformly convergent to $1/(1-x)$ on $(-1,1)$. This is easily seen since given any $m \geq n \geq 1$, we can make $a^m + \ldots + a^n > 1/2$ by taking a sufficiently close to 1. In particular, the sequence of partial sums cannot be Cauchy on $(-1,1)$. ♠

THEOREM 4.4.8 (Weierstrass M-test). *Suppose that (u_n) is a sequence of functions defined on I and that there exists a sequence (M_n) of positive real numbers such that*

(a) $|u_n(x)| \leq M_n$ *for all* $x \in I$, $n \in \mathbb{N}$.
(b) $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on I. If the (u_n) are all continuous, so is $S = \sum_{n=1}^{\infty} u_n$

PROOF. The proof is similar to the method used in examples 4.4.7(1). We prove that the sequence (S_n) of partial sums is Cauchy. Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} M_n < \infty$, there exists $N \in \mathbb{N}$ such that $M_m + \ldots + M_n < \varepsilon$ for all $m \geq n \geq N$. It follows from assumption (a) that for $m \geq n \geq N$ and $x \in I$ we have

$$|\sum_{j=n}^{m} u_n(x)| \leq \sum_{j=n}^{m} M_n < \varepsilon.$$

Hence $d(S_m, S_{n-1}) = \| \sum_{j=n}^{m} u_j\| \leq \sum_{j=n}^{m} M_n < \varepsilon$, for all $m \geq n \geq N$, and so (S_n) is a Cauchy sequence. Now apply theorem 4.4.6. □

We give some characteristic applications of the M-test in the next set of examples (more examples appear in the following section).

EXAMPLES 4.4.9. (1) Consider the series $\sum_{n=1}^{\infty} \frac{n}{1+x^2+n^3}$. We have $0 < \frac{n}{1+x^2+n^3} \leq \frac{n}{1+n^3} < \frac{1}{n^2}$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$. Taking $M_n = \frac{1}{n^2}$ in the M-test, we see that $\sum_{n=1}^{\infty} \frac{n}{1+x^2+n^3}$ converges uniformly to a continuous function on $\mathbb{R}$.
(2) Let (a_n) be any sequence of real numbers and $p > 1$. The infinite series $\sum_{n=1}^{\infty} \frac{\sin(a_n x)}{n^p}$ converges uniformly to a continuous function on $\mathbb{R}$. For this, we note that $|\frac{\sin(a_n x)}{n^p}| \leq n^{-p}$ and take $M_n = n^{-p}$ in the M-test (since $p > 1$, $\sum n^{-p} < \infty$).
(3) Consider the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. This series does not converge uniformly on $\mathbb{R}$. We show that the series converges uniformly on every closed and bounded interval $[-R, R]$, $R \geq 0$. Certainly $|\frac{x^n}{n!}| \leq \frac{R^n}{n!}$ for all $x \in [-R, R]$. We take $M_n = \frac{R^n}{n!}$ in the M-test. Since $\sum_{n=0}^{\infty} \frac{R^n}{n!} < \infty$, it follows by the M-test that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-R, R]$ for all $R \geq 0$. As a consequence $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ defines a continuous function on $\mathbb{R}$. ♠

EXERCISES 4.4.10.

(1) Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$.
   (a) For what values of x is the series convergent?
   (b) On what closed intervals does it converge uniformly.
   (c) Is f continuous on the set of points where the series converges?
(2) Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $\mathbb{R}$.
(3) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ is uniformly convergent on $\mathbb{R}$.
(4) Complete the proof of theorem 4.4.4.

4.5. Power series

In this section we consider the convergence properties of series of the form $\sum_{n=0}^{\infty} a_n x^n$ (more generally, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$). This type of series is called a *power series.*

LEMMA 4.5.1. *Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = z \neq 0$. Then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all $x \in (-|z|, |z|)$. If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = z$, then $\sum_{n=0}^{\infty} a_n x^n$ will be divergent for all x satisfying $|x| > |z|$.*

PROOF. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is convergent. Since $\lim_{n\to\infty} a_n z^n = 0$, the sequence $(a_n z^n)$ must be bounded. Hence we can find $C \geq 0$ such that $|a_n||z|^n \leq C$ for all $n \geq 0$ and so

$$|a_n| \leq C|z|^{-n},\ n \geq 0.$$

Therefore

$$|a_n x^n| \leq C \left(\frac{|x|}{|z|}\right)^n,\ n \geq 0,$$

and $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < |z|$ by comparison with the geometric series $C \sum_{n=0}^{\infty} \left(\frac{|x|}{|z|}\right)^n$.

Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is divergent and that there exists x, $|x| > |z|$, such that $\sum_{n=0}^{\infty} a_n x^n$ is convergent. Then by the first part of the lemma, $\sum_{n=0}^{\infty} a_n y^n$ will be convergent if $|y| < |x|$, contradicting the divergence of the series at z. Therefore, the series is divergent for all x satisfying $|x| > |z|$. □

DEFINITION 4.5.2. The *radius of convergence* R of $\sum_{n=0}^{\infty} a_n x^n$ is defined by

$$R = \sup\{|x| \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}.$$

EXAMPLES 4.5.3. (1) The exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $R = +\infty$.
(2) Using the ratio test, it is easy to verify that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ is convergent if $|x| < 1$ and divergent if $|x| > 1$. Hence the radius of convergence $R = 1$. (Note that for this example, the series converges if $x = 1$ and diverges if $x = -1$.) ♠

PROPOSITION 4.5.4. *Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then $S(x) = \sum_{n=0}^{\infty} a_n x^n$ defines a continuous function on $(-R, R)$.*

PROOF. Let $a \in (0, R)$ and choose b, $a < b < R$. Since $\sum_{n=0}^{\infty} a_n b^n$ is convergent, $(a_n b^n)$ is bounded and there exists $C \geq 0$ such that $|a_n| \leq C b^{-n}$. Therefore $|a_n a^n| \leq C(\frac{a}{b})^n$, for all $n \geq 0$. Since $0 \leq b/a < 1$, $\sum_{n=0}^{\infty} C(\frac{a}{b})^n < \infty$. Take $M_n = C(\frac{a}{b})^n$. Then $|a_n x^n| \leq M_n$ for all $x \in [-a, a]$ and so, by the M-test, $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-a, a]$. Since the functions $a_n x^n$ are continuous, it follows that $\sum_{n=0}^{\infty} a_n x^n$ converges to a continuous function on $[-a, a]$. This holds for all $a \in (-R, R)$, and so $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$ (see theorem 4.4.4). □

EXAMPLE 4.5.5. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ defines a continuous function on $(-1, 1)$ since the radius of convergence is 1 by examples 4.5.3(2). This series converges at $x = 1$ and we shall show in the next section that the series converges uniformly on $[0, 1]$ (this uses Abel's test for uniformly convergent series). ♠

DEFINITION 4.5.6. The *product* $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$ of two power series is defined to be the power series $\sum_{n=0}^{\infty} c_n x^n$, where for

$n \geq 0$,

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

PROPOSITION 4.5.7. *If the power series* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *have radii of convergence* R *and* S *respectively, then the product of the series has radius of convergence at least* $\min\{R, S\}$.

PROOF. We assume $\min\{R, S\} > 0$ otherwise the result is trivial. Fix $0 < r < \min\{R, S\}$ and choose $s \in (r, \min\{R, S\})$. As in the proof of lemma 4.5.1, there exists $C > 0$ such that $|a_n|, |b_n| \leq C s^{-n}$ for all $n \geq 0$. It follows that $|c_n| \leq \sum_{j=0}^{n} |a_j||b_{n-j}| \leq C^2 s^{-n}(n+1)$, $n \geq 0$. Hence $|c_n x^n| \leq C^2(n+1)(\frac{r}{s})^n = M_n$, if $|x| \leq r$. Since $\sum M_n < \infty$, it follows by the M-test that $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-r, r]$. This holds for all $0 < r < \min\{R, S\}$ and so the radius of convergence of the product is at least $\min\{R, S\}$. $\square$

The final proposition of this section gives an explicit formula for the radius of convergence in terms of the coefficients a_n of a power series.

PROPOSITION 4.5.8. *The radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$ *is equal to* $1/(\limsup |a_n|^{1/n})$.

PROOF. Set $\ell = \limsup |a_n|^{1/n}$, Then $\limsup |a_n x^n|^{1/n} = \ell |x|$, and so $\sum_{n=0}^{\infty} a_n x^n$ converges if $|x|\ell < 1$ and diverges if $|x|\ell > 1$ by Cauchy's test. Hence $R = 1/(\limsup |a_n|^{1/n})$. $\square$

4.5.1. Products and quotients of power series. Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series with radii of convergence R and S respectively, where $R, S > 0$, We define the product $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$ to be the power series $\sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, $n \geq 0$.

PROPOSITION 4.5.9. *(Notation as above.) The product series* $\sum_{n=0}^{\infty} c_n x^n$ *has radius of convergence* $T \geq \min\{R, S\}$ *and for all* $x \in (-T, T)$ *we have*

$$\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n).$$

4.6. Abel and Dirichlet's test for uniform convergence

PROPOSITION 4.6.1 (Abel's test for uniform convergence). *Given sequences* (a_n), (u_n) *of functions defined on* $I \subset \mathbb{R}$, *the series* $\sum_{n=1}^{\infty} a_n u_n$ *is uniformly convergent on* I *if*

(1) *The series* $\sum_{n=1}^{\infty} a_n$ *is uniformly convergent on* I.
(2) $\exists K \geq 0$ *such that* $\|u_n\| \leq K$, $n \geq 1$.

(3) *$(u_n(x))$ is either decreasing for all $x \in I$ or increasing for all $x \in I$.*

In particular, if a_n, u_n are continuous, $n \geq 1$, then $U = \sum_{n=1}^{\infty} a_n u_n$ is continuous on I.

PROOF. The proof is obtained by using the argument of the proof of Abel's test for infinite series. We leave the proof to the exercises (see also the proof of the Dirichlet test for uniform convergence which we give below). □

PROPOSITION 4.6.2 (Dirichlet's test for uniform convergence). *Given sequences (a_n), (u_n) of functions defined on $I \subset \mathbb{R}$, the series $\sum_{n=1}^{\infty} a_n u_n$ is uniformly convergent on I if*

(1) *$\exists K \geq 0$ such that $\|a_1 + \ldots + a_n\| \leq K$ for all $n \geq 1$.*
(2) *$(u_n(x))$ is decreasing for all $x \in I$.*
(3) *(u_n) is uniformly convergent to the zero function on I.*

In particular, if a_n, u_n are continuous, $n \geq 1$, then $S = \sum_{n=1}^{\infty} a_n u_n$ is continuous on I.

PROOF. We apply the argument of the proof of Dirichlet's test for infinite series pointwise to the infinite series $\sum_{n=1}^{\infty} a_n u_n$. Thus, using Abel's lemma, we have the estimate

$$|\sum_{j=n}^{m} a_j(x)u_j(x)| \leq 2Ku_n(x), \text{ for all } m \geq n \geq 1, x \in I.$$

Hence $|\sum_{j=n}^{m} a_j(x)u_j(x)| \leq 2K\|u_n\|$ for all $m \geq n \geq 1$, $x \in I$ and so

$$\|\sum_{j=n}^{m} a_j u_j\| \leq 2K\|u_n\|, \text{ for all } m \geq n \geq 1.$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\|u_n\| < \varepsilon/2K$, all $n \geq N$. Hence

$$\|\sum_{j=n}^{m} a_j u_j\| \leq \varepsilon, \text{ for all } m \geq n \geq N.$$

It follows from the general principle of uniform convergence (theorem 4.4.6) that $\sum_{n=1}^{\infty} a_n u_n$ is uniformly convergent on I. □

EXAMPLES 4.6.3. (1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ is uniformly convergent on $[0,1]$. In particular, $\lim_{x\to 1-} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. This is an application of Abel's test with $a_n = (-1)^{n+1}/n$ and $u_n(x) = x^n$.
(2) We claim that for all $\varepsilon \in (0,\pi)$ and $m \in \mathbb{Z}$, the series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$ is uniformly convergent on $[2m\pi + \varepsilon, 2(m+1)\pi - \varepsilon]$. In particular,

$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$ defines a continuous function on $(2m\pi, 2(m+1)\pi)$ for all $m \in \mathbb{Z}$. Similar results hold for $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$. To prove the claim, suppose that x is not an integer multiple of 2π. We have

$$\sum_{k=1}^{n}\cos(kx) = \frac{\cos\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{nx}{2}\right)}{\sin(\frac{x}{2})}.$$

If we suppose $x \in [2m\pi+\varepsilon, 2(m+1)\pi-\varepsilon]$, where $\varepsilon \in (0,\pi)$ and $m \in \mathbb{Z}$, then the minimum value of $|\sin(\frac{x}{2})|$ is taken at $x = 2m\pi+\varepsilon$ (or $2(m+1)\pi-\varepsilon$). Hence we have the estimate

$$|\sum_{k=1}^{n}\cos(kx)| \leq 1/|\sin(\frac{\varepsilon}{2})|, \text{ for all } x \in [2m\pi+\varepsilon, 2(m+1)\pi-\varepsilon].$$

We now apply Dirichlet's test with $K = 1/|\sin(\frac{\varepsilon}{2})|$ and $u_n = 1/n$. ♠

Examples 4.6.3(1) is a special case of Abel's theorem which we now state and prove.

THEOREM 4.6.4 (Abel's theorem). *If the infinite series $\sum_{n=0}^{\infty} a_n$ is convergent, then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0,1]$ and*

$$\lim_{x\to 1-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

PROOF. The method is the same as that used for examples 4.6.3(1) and depends on Abel's test. □

EXERCISES 4.6.5.

(1) Show that if $\sum_{n=1}^{\infty} a_n$ converges then
(a) $\sum_{n=1}^{\infty} a_n \frac{x^n}{1+x^n}$ converges uniformly on $[0,1]$,
(b) $\sum_{n=1}^{\infty} a_n \frac{nx^n(1-x)}{1-x^n}$ converges uniformly on $[0,1]$
(for (b), the x-dependent terms are defined to be equal to 1 at $x=1$). Deduce that these infinite series define continuous functions on $[0,1]$.
(2) Show that if the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded then $\sum_{n=1}^{\infty}\frac{a_n}{n^x}$ defines a continuous function on $(0,\infty)$.
(3) Show that if $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty}\frac{a_n}{n^x}$ defines a continuous function on $[0,\infty)$.
(4) Show that $\sum_{n=1}^{\infty}\frac{\sin(nx)}{\sqrt{n}}$ defines a continuous function on $(2m\pi, 2(m+1)\pi)$ for all $m \in \mathbb{Z}$.
(5) Show that $\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{2n+1}$ defines a continuous function on $(m\pi,(m+1)\pi)$ for all $m \in \mathbb{Z}$.

4.7. Integrating and differentiating term-by-term

In this section we address the question of when we can interchange the order of integration or differentiation with summation.

THEOREM 4.7.1. *Given a sequence (u_n) of continuous functions defined on the interval $I \subset \mathbb{R}$, suppose that $\sum_{n=1}^{\infty} u_n$ is uniformly convergent to $U : I \to \mathbb{R}$. Given $a \in I$, we have for all $x \in I$,*

$$\int_a^x U(t)\,dt = \int_a^x \left(\sum_{n=1}^{\infty} u_n(t)\right)\,dt = \sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt,$$

and the series $\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$ is uniformly convergent on every closed and bounded subinterval of I.

REMARKS 4.7.2. (1) Since $\sum_{n=1}^{\infty} u_n$ is a uniformly convergent series of continuous functions, $U = \sum_{n=1}^{\infty} u_n$ is continuous and therefore integrable.
(2) For theorem 4.7.1, and the result we shortly give for differentiation, it is important that I is an interval.
(3) The series $\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$ may not converge uniformly on I if I is not bounded (for a simple example, take $u_1 \equiv 1$, $u_n \equiv 0$, $n > 1$, and $I = \mathbb{R}$).

Proof of theorem 4.7.1. Since I is an interval, $[a, x] \subset I$. Set $S_n = \sum_{j=1}^{n} u_j$, $n \geq 1$. It suffices to show that given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$|\int_a^x U(t)\,dt - \int_a^x S_n(t)\,dt| = |\int_a^x U(t)\,dt - \sum_{j=1}^{n} \int_a^x u_j(t)\,dt| < \varepsilon,$$

for all $n \geq N$. Since the result is trivial if $a = x$, we may assume $x \neq a$. Since $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on I, we can choose $N \geq 1$ such that $\|S_n - U\| < \varepsilon/|x - a|$ for all $n \geq N$. Integrating from a to x we have

$$\begin{aligned}
|\int_a^x U(t)\,dt - \int_a^x S_n(t)\,dt| &\leq \int_a^x |U(t) - S_n(t)|\,dt \\
&\leq \int_a^x \|S_n - U\|\,dt \\
&\leq |a - x|\|S_n - U\| \\
&= \varepsilon,\ n \geq N.
\end{aligned}$$

Hence $\int_a^x (\sum_{n=1}^{\infty} u_n(t))\,dt = \sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$. The uniform convergence of $\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$ on closed and bounded subintervals of I follows easily from the estimate $|\int_a^x U(t)\,dt - \int_a^x S_n(t)\,dt| \leq |a - x|\|S_n - U\|$. $\square$

EXAMPLES 4.7.3. (1) We have $(1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$, $x \in (-1,1)$. Convergence is uniform on every closed subinterval $[-x,x] \subset (-1,1)$. Take $a = 0$ in the statement of theorem 4.7.1. We have $U(x) = 1/(1+x)$, $x \in (-1,1)$, and

$$\begin{aligned}\log(1+x) = \int_0^x \frac{1}{1+t}\,dt &= \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^n\right) dt \\ &= \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n\,dt, \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n}.\end{aligned}$$

It follows from Abel's theorem (see also examples 4.6.3(1)) that convergence of $\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n}$ is uniform on $[0,1]$ and so $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n}$ for $x \in (-1,1]$. Taking $x = 1$, we get the series formula for $\log 2$.
(2) We have $(1+x^2)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ on $(-1,1)$ and convergence is uniform on $[-x,x]$, for all $x \in [0,1)$. Applying theorem 4.7.1, we have for $x \in (-1,1)$,

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

It follows from Abel's theorem that convergence of $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ is uniform on $[0,1]$ and so, taking $x = 1$ (strictly, talking the limit of both sides as $x \to 1-$) we get

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

♠

DEFINITION 4.7.4. Let I be an interval. A function $u : I \to \mathbb{R}$ is C^1, or *once continuously differentiable*, (on I) if u is continuous, differentiable and $u' : I \to \mathbb{R}$ is continuous. More generally, if $r \in \mathbb{N}$, u is C^r, or *r-times continuously differentiable*, if the first r derivatives of u all exist and are continuous on I. If u is C^r for all $r \in \mathbb{N}$, we say u is C^∞ or *infinitely differentiable*.

REMARKS 4.7.5. (1) We allow the interval I to be open, closed, half-open and unbounded. When I is closed, we interpret continuity and differentiability in the usual way. For example, $u : [a,b] \to \mathbb{R}$ is differentiable at $x = a$ if $\lim_{h\to 0+}(u(a+h) - u(a))/h$ exists and the

value of the limit is defined to be $u'(a)$.
(2) If $r \in \mathbb{N}$, we denote the rth derivative (map) of u by $u^{(r)}$. If $r = 1, 2$ we usually write u', u''. Note that if u is C^r, $r > 1$, we require $u^{(r-1)}$ to exist and be continuous and we define $u^{(r)}$ to be the derivative of $u^{(r-1)}$.

THEOREM 4.7.6. *Given a sequence (u_n) of C^1 functions defined on the interval $I \subset \mathbb{R}$, suppose that*

(a) *$\sum_{n=1}^{\infty} u_n'$ is uniformly convergent on I.*
(b) *There exists $a \in I$ such that $\sum_{n=1}^{\infty} u_n(a)$ is convergent.*

Then $\sum_{n=1}^{\infty} u_n$ converges pointwise on I to a C^1 function $U : I \to \mathbb{R}$ and

$$U' = \sum_{n=1}^{\infty} u_n'.$$

The convergence of $\sum_{n=1}^{\infty} u_n$ is uniform on all closed and bounded subintervals of I.

REMARK 4.7.7. Condition (b) is clearly necessary. For example, take $u_n \equiv 1$ for all $n \in \mathbb{N}$.

Proof of theorem 4.7.6. Since u_n is C^1, u_n' is continuous for all $n \in \mathbb{N}$. Therefore $V = \sum_{n=1}^{\infty} u_n'$ defines a continuous function on I. Apply theorem 4.7.1 to get for all $x \in I$

$$\begin{aligned}
\int_a^x V(t)\,dt &= \int_a^x \sum_{n=1}^{\infty} u_n'(t)\,dt \\
&= \sum_{n=1}^{\infty} \int_a^x u_n'(t)\,dt, \\
&= \sum_{n=1}^{\infty} (u_n(x) - u_n(a)), \\
&= \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(a),
\end{aligned}$$

where the final step follows by condition (b) and the convergence of $\sum_{n=1}^{\infty}(u_n(x) - u_n(a))$. Hence $\sum_{n=1}^{\infty} u_n$ converges pointwise on I and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on all closed and bounded subintervals of I by theorem 4.7.1. Our argument shows that for all $x \in I$ we have

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n(a) + \int_a^x V(t)\,dt$$

Since V is continuous, it follows by the fundamental theorem of calculus that the right hand side of this equation is differentiable at x with derivative $V(x)$. Hence

$$\left(\sum_{n=1}^{\infty} u_n\right)'(x) = V(x) = \sum_{n=1}^{\infty} u_n'(x),$$

and so we obtain the derivative by term-by-term differentiation. $\square$

We end this section with an important application of our results on term-by-integration and differentiation to power series.

THEOREM 4.7.8. *Suppose that the power series $U(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$ (as usual we allow $R = +\infty$). Then U is a C^∞ function on $(-R, R)$ and the derivatives and definite integrals of U may be computed by term-by-term differentiation and integration. Furthermore, the power series giving the derivatives and integrals of U all have radius of convergence R.*

PROOF. It suffices to show that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R iff $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R (note that the first series is obtained, up to a constant, by term-by-term integration of the second series). We may prove this either by estimating $|a_n|$ along the lines of the proof of lemma 4.5.1 or, more simply, by using the root test: $R^{-1} = \limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n}$ since $\lim_{n\to\infty} n^{1/n} = 1$. $\square$

EXERCISES 4.7.9.

(1) We gave results on term-by-term differentiation and integration for infinite series. State and prove the corresponding results for sequences of functions.

(2) Let $(I_n)_{n=1}^{\infty}$ be a sequence of non-empty, mutually disjoint open intervals and (α_n) be a sequence of strictly positive real numbers.
Suppose that (f_n) is a sequence of positive continuous functions on $\mathbb{R}$ such that for $n \geq 1$, (a) f_n is non-zero precisely on I_n and (b) the maximum value of f_n is α_n.
   (a) Is $\sum_{n=1}^{\infty} f_n$ always *pointwise* convergent on $\mathbb{R}$?
   (b) Find a necessary and sufficient condition on the sequence (α_n) that allows the M-test to be applied to $\sum_{n=1}^{\infty} f_n$.
   (c) Is it true that if the M-test does not apply, then $\sum_{n=1}^{\infty} f_n$ cannot be uniformly convergent on $\mathbb{R}$? If false, provide an example.
   (d) Denote the length of the interval I_n by ℓ_n. Show that if there exists $A > 0$ such that $\ell_n \geq A$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} f_n$ is always continuous on $\mathbb{R}$.

(3) Let $(I_n)_{n=1}^{\infty}$ be a sequence of non-empty, mutually disjoint open intervals (for example: $I_n = (n, n+1)$ or $I_n = (\frac{1}{n+1}, \frac{1}{n})$, $n \geq 1$). Suppose that (f_n) is a sequence of positive continuous functions on $\mathbb{R}$ such that for $n \geq 1$, (a) f_n is non-zero precisely on I_n and (b) the maximum value of f_n is 1.
   (a) Show that $\sum_{n=1}^{\infty} f_n$ is *pointwise* convergent on $\mathbb{R}$.
   (b) Show that the M-test can never be applied to $\sum_{n=1}^{\infty} f_n$.

(c) Show that $\sum_{n=1}^{\infty} f_n$ is never uniformly convergent but there exist choices of (I_n) for which $\sum_{n=1}^{\infty} f_n$ is *always* continuous on $\mathbb{R}$ (you choose the (I_n); the (f_n) satisfy conditions (a,b) listed above).

(d) Find a choice of (I_n) for which $\sum_{n=1}^{\infty} f_n$ *never* converges to a continuous function on $\mathbb{R}$ (you choose the (I_n); the (f_n) satisfy conditions (a,b) listed above).

4.8. A continuous nowhere differentiable function

Consider the infinite series $\sum_{n=1}^{\infty} a^n \sin(b^n x)$, where $0 < a < 1 < b$ and $ab > 1$. Since $|a^n \sin(b^n x)| \leq a^n$, for all $x \in \mathbb{R}$, it follows from the M-test (with $M_n = a^n$) that $\sum_{n=1}^{\infty} a^n \sin(b^n x)$ converges uniformly to a continuous function U on $\mathbb{R}$. If we differentiate term-by-term, we obtain the infinite series $\sum_{n=1}^{\infty} (ab)^n \cos(b^n x)$. Since $ab > 1$, it looks unlikely that this series converges uniformly or even pointwise on $\mathbb{R}$. This *suggests* that U may not be differentiable anywhere on $\mathbb{R}$. Weierstrass showed in 1872 that if $ab > 1 + \frac{3}{2}\pi$, then U was indeed nowhere differentiable on $\mathbb{R}$. Weierstrass' proof is a little tricky and we will instead give a much simpler example, due to Landau, of a nowhere differentiable continuous function. Like Weiestrass' example, Landau's function is defined using a uniformly convergent series of continuous functions.

Define

$$\begin{aligned} u_1(x) &= x,\ 0 \leq x \leq \frac{1}{2}, \\ &= 1 - x,\ \frac{1}{2} \leq x \leq 1. \end{aligned}$$

Extend u_1 to $\mathbb{R}$ as a 1-periodic function. That is, if $x \in [n, n+1]$, $n \in \mathbb{Z}$, then $u_1(x) = u_1(x - n)$ (note $x - n \in [0, 1]$). For all $m \in \mathbb{Z}$, $x \in \mathbb{R}$ we have

$$u_1(x + m) = u_1(x).$$

We show the graph of u_1 in figure 2.

For $n > 1$, define

$$u_n(x) = \frac{1}{10^{n-1}} u_1(10^{n-1} x),\ x \in \mathbb{R}.$$

The function u_n is $\frac{1}{10^{n-1}}$-periodic. Indeed, for all $m \in \mathbb{Z}$, $x \in \mathbb{R}$,

$$u_n(x + \frac{m}{10^{n-1}}) = u_n(x).$$

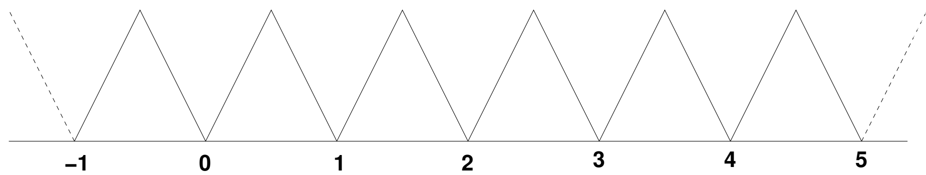


FIGURE 2. The graph of u_1

To see this, observe that

$$\begin{aligned} u_n(x+\frac{m}{10^{n-1}}) &= \frac{1}{10^{n-1}}u_1(10^{n-1}(x+\frac{m}{10^{n-1}})) \\ &= \frac{1}{10^{n-1}}u_1(10^{n-1}x+m) \\ &= \frac{1}{10^{n-1}}u_1(10^{n-1}x) \\ &= u_n(x). \end{aligned}$$

Let $\mathbb{Z}^{\frac{1}{2}} = \{\frac{m}{2} \mid m \in \mathbb{Z}\}$. So $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots \in \mathbb{Z}^{\frac{1}{2}}$. Observe that the set of points where u_1 is not differentiable is precisely $\mathbb{Z}^{\frac{1}{2}}$. Elsewhere the derivative of u_1 is ± 1. The set of points where u_n is not differentiable is $\frac{1}{10^{n-1}}\mathbb{Z}^{\frac{1}{2}} = \{\frac{m}{2\times 10^{n-1}} \mid m \in \mathbb{Z}\}$. Elsewhere the derivative of u_n is ± 1.

Define $U : \mathbb{R} \to \mathbb{R}$ by

$$U(x) = \sum_{n=1}^{\infty} u_n(x).$$

Since $|u_n(x)| \leq \frac{1}{10^{n-1}}$, it follows by the M-test that $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on $\mathbb{R}$ and U is continuous. We claim that U is nowhere differentiable. We prove the nowhere differentiability of U by showing that for each $x_0 \in \mathbb{R}$, there exists a sequence (x_N) converging to x_0 such that the limit as $N \to \infty$ of $\frac{U(x_N)-U(x_0)}{x_N - x_0}$ does not exist (if U is differentiable at x_0 then $\lim_{N\to\infty} \frac{U(x_N)-U(x_0)}{x_N-x_0} = U'(x_0)$ if $x_N \to x_0$.)

Let $x_0 \in \mathbb{R}$ and $N \geq 1$. Then there exists a unique $m \in \mathbb{Z}^{\frac{1}{2}}$ such that $x_0 \in [\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}})$. The length of the interval $[\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}})$ is $\frac{1}{2}\frac{1}{10^{N-1}}$. Certainly either $x_0 - \frac{m}{10^{N-1}} > 10^{-N}$ or $\frac{m+\frac{1}{2}}{10^{N-1}} - x_0 > 10^{-N}$ (as

$\frac{1}{2}\frac{1}{10^{N-1}} > \frac{2}{10^N}$). Define $x_N = x_0 \pm \frac{1}{10^N}$ so that $x_N \in [\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}}]$. This completes the construction of the sequence (x_N).

For $n \le N$, we have

$$\frac{u_n(x_N) - u_n(x_0)}{x_N - x_0} = \pm 1.$$

(The set of points where u_n is not differentiable is a proper subset of the set of points where u_N is not differentiable if $n < N$.) On the other hand if $n > N$ then $u_n(x_N) = u_n(x_0 \pm \frac{1}{10^N}) = u_n(x_0)$ by the $\frac{1}{10^{n-1}}$ periodicity of u_n ($\frac{1}{10^N}$ is an integer multiple of $\frac{1}{10^{n-1}}$ if $n > N$). Hence if $n > N$,

$$\frac{u_n(x_N) - u_n(x_0)}{x_N - x_0} = 0.$$

We have

$$\begin{aligned} \frac{U(x_N) - U(x_0)}{x_N - x_0} &= \sum_{n=1}^{\infty} \frac{u_n(x_N) - u_n(x_0)}{x_N - x_0}, \\ &= \sum_{n=1}^{N} \frac{u_n(x_N) - u_n(x_0)}{x_N - x_0}, \\ &= \sum_{n=1}^{N} \pm 1, \\ &= Q_N, \end{aligned}$$

where Q_N must be an even integer if N is even and an odd integer if N is odd. Hence the limit of $\frac{U(x_N)-U(x_0)}{x_N - x_0}$ as $N \to \infty$ does not exist and so U cannot be differentiable at x_0.

Remark 4.8.1. Are these examples of nowhere differentiable continuous functions exceptional and pathological? Pathological perhaps yes but certainly not exceptional. ‘Most’ — in a sense that can be made precise — continuous functions are not differentiable at any point of the real line.

Exercises 4.8.2.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous 2-periodic function defined by

$$\begin{aligned} f(x) &= 0, \text{ if } x \in [0, 1/2], \\ &= 6x - 3, \text{ if } x \in [1/2, 2/3], \\ &= 1, \text{ if } x \in [2/3, 1], \\ &= 2 - x, \text{ if } x \in [1, 2]. \end{aligned}$$

Define $E = (X, Y) : \mathbb{R} \to \mathbb{R}^2$ by

$$E(t) = (\sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t)).$$

(a) Show that E is continuous and E maps the unit interval $[0,1]$ onto $[0,1]\times[0,1]$. (Hint: Given $(x,y) \in [0,1]\times[0,1]$, write x, y in binary form as $x = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}$, $y = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$, where $a_i \in \{0,1\}$, $i \geq 1$. Show that if $t = 2\sum_{i=1}^{\infty} 3^{-1-t} a_i$, then $f(3^k t) = a_k$ and so $E(t) = (x,y)$.)

(b) Show that the result of (a) does not depend on the values of f on $(1,2)$ — subject to $f(1) = 1$, $f(2) = 0$.

(c) Modifying f on $(1,2)$ as needed, show that we can require that E to be nowhere differentiable.

(This elementary example of a space filling curve was given by I J Schoenberg in 1938. No such examples exist if E is differentiable.)

CHAPTER 5

Functions

5.1. Introduction

In this chapter we consider and compare various classes of functions defined on the real line. Before getting into the details, we start by giving a general overview. In subsequent sections, we study particular classes of functions in greater detail.

The most regular, and familiar, class of functions on the real line is the space $P(\mathbb{R})$ of *polynomials* on $\mathbb{R}$. Recall that if $p \in P(\mathbb{R})$, then either $p \equiv 0$ or we may write $p(x) = \sum_{j=0}^{n} a_j x^{n-j}$ where $a_0 \neq 0$, n is the *degree* of p, and the expression for p is unique. If $p \in P(\mathbb{R})$ then p is smooth (that is, infinitely differentiable or C^∞) and the derivatives and integrals of p are obtained by term-by-term differentiation and integration of p. At the other extreme we have the space $C^0(\mathbb{R})$ of continuous functions on $\mathbb{R}$. As we indicated in the previous chapter, functions in $C^0(\mathbb{R})$ typically have unpleasant properties such as nowhere differentiability. If for $1 \leq r \leq \infty$ we let $C^r(\mathbb{R})$ denote the space of C^r functions on $\mathbb{R}$ then we have the inclusions

$$C^0(\mathbb{R}) \supset C^1(\mathbb{R}) \supset \ldots \supset C^r(\mathbb{R}) \supset \ldots \supset C^\infty(\mathbb{R}) \supset P(\mathbb{R}).$$

There is another class of functions, intermediate between polynomials and C^∞ functions, that play an important historic role in analysis (especially complex analysis). Recall that if $f \in C^\infty(\mathbb{R})$ then the *Taylor series* Tf_{x_0} of f at $x_0 \in \mathbb{R}$ is defined by

$$Tf_{x_0}(x) = \sum_{j=0}^{\infty} \frac{f^{(n)}}{n!}(x - x_0)^n.$$

In general, the Taylor series of f at x_0 may have zero radius of convergence and even if it converges it may not converge to f — we give examples shortly. However, for many standard functions of analysis (such as e^x, $\sin x$, $\cos x$), we find that the Taylor series at x_0 does converge to f for all x close enough to x_0. More formally, we make the following definition.

Definition 5.1.1. Let $I \subset \mathbb{R}$ be an open and non-empty interval. A function $f : I \to \mathbb{R}$ is (real) *analytic* if for every $x_0 \in I$, there exists

$\delta = \delta(x_0) > 0$ such that

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(n)}}{n!}(x - x_0)^n \text{ for all } x \text{ satisfying } |x - x_0| < \delta.$$

We let $C^\omega(\mathbb{R})$ denote the space of real analytic functions on $\mathbb{R}$. Evidently we have

$$C^0(\mathbb{R}) \supset C^\infty(\mathbb{R}) \supset C^\omega(\mathbb{R}) \supset P(\mathbb{R}).$$

We start by developing the theory of C^∞ functions and, in particular, show that a C^∞ function need not be analytic. Next we show that even though a continuous function f may be nowhere differentiable, we can uniformly approximate f as close as we wish by polynomials (the "Weierstrass approximation theorem"). We then develop a little of the classical theory of analytic functions and show, for example, that e^x, $\sin x$ and $\cos x$ all define analytic functions on $\mathbb{R}$. Finally, we conclude the chapter with a section on Fourier series — this will use many results from the previous chapter on uniform convergence and trigonometric identities.

5.2. Smooth functions

We start by constructing an example of a smooth non-analytic function. Specially, we construct a smooth bounded function $\Phi : \mathbb{R} \to \mathbb{R}$ that is strictly positive on $x > 0$ and is zero on $x \leq 0$. We then use this function to construct a range of smooth non-analytic functions satisfying various properties — an illustration of how to build a smooth function with specified properties. The next example shows the construction of Φ has to be done with some care.

Example 5.2.1. Define $F(x) = x^4$, $x \geq 0$ and $F(x)0$, $x < 0$. The graph of F looks 'smooth' near $x = 0$. See figure 1. However, although it is easily checked that F is C^3 (we have $F'(0) = F''(0) = F'''(0) = 0$), F is not 4-times differentiable at $x = 0$. Indeed, for $x > 0$, $F'''(x) = 24x$, and if $x < 0$, $F'''(x) = 0$. Therefore

$$\lim_{h \to 0+} \frac{F'''(h) - F'''(0)}{h} = 24 \neq 0 = \lim_{h \to 0-} \frac{F'''(h) - F'''(0)}{h}$$

and so F''' is not differentiable at $x = 0$. The moral of this example is that piecing together bits of standard functions like polynomials and trigonometric functions will not work. ♠

Before we construct our smooth non-analytic function we need a technical lemma.

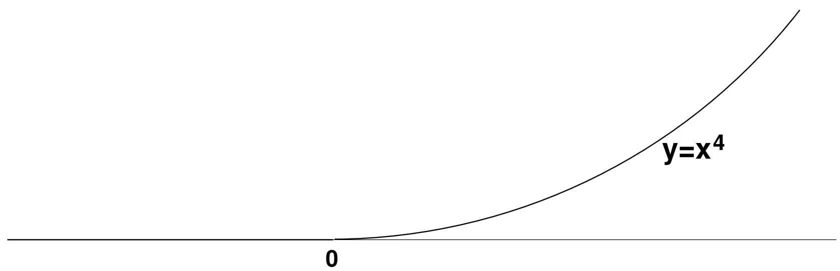


FIGURE 1. Graph of C^3 but not four times differentiable function

LEMMA 5.2.2. *If $q \in P(\mathbb{R})$ is a polynomial of degree m and $p \in \mathbb{Z}$, then*

$$\lim_{x \to 0+} \frac{q(x)}{x^p} e^{-\frac{1}{x}} = 0.$$

PROOF. If we write $q(x) = \sum_{j=0}^m b_j x^{m-j}$, then $\frac{q(x)}{x^p} = \sum_{j=0}^m b_j x^{m-j-p}$. Since the limit of a finite sum is the sum of the limits of the terms in the sum, it is enough to show that $\lim_{x \to 0+} x^{-k} e^{-\frac{1}{x}} = 0$, for all $k \in$ *intg*. Since the case $k \le 0$ is easy, we always assume $k \in \mathbb{N}$. Setting $y = 1/x$, it suffices to show

$$\lim_{y \to +\infty} y^k e^{-y} = 0,$$

for all $k \in \mathbb{N}$. The function $y^k e^{-y}$ is positive on $[0, \infty)$ and decreasing for $y \ge k$ (for example, use calculus). Hence $\lim_{y \to +\infty} y^k e^{-y}$ exists and is positive. But $\lim_{y \to +\infty} y^k e^{-y} = \lim_{n \to \infty} n^k e^{-n}$ and so $\lim_{y \to +\infty} y^k e^{-y} = 0$ by standard results on sequences (see example 2.3.15). $\square$

PROPOSITION 5.2.3. *Define $\Phi : \mathbb{R} \to \mathbb{R}$ by*

$$\begin{aligned} \Phi(x) &= 0, \text{ if } x \le 0, \\ &= e^{-\frac{1}{x}}, \text{ if } x > 0. \end{aligned}$$

Then

(1) $\Phi \in C^\infty(\mathbb{R})$.
(2) $\Phi^{(j)}(0) = 0$, *for all* $j \ge 0$.

PROOF. It is clear that Φ restricted to either $(-\infty, 0)$ or $(0, \infty)$ is C^∞. We have to show that Φ is infinitely differentiable with all derivatives continuous at $x = 0$. We start by finding expressions for the derivatives of Φ at non-zero points of $\mathbb{R}$. Let $j \geq 1$. We claim that

$$\begin{aligned} \Phi^{(j)}(x) &= 0, \text{ if } x < 0, \\ &= \frac{q_j(x)}{x^{2j}} e^{-\frac{1}{x}}, \text{ if } x > 0, \end{aligned}$$

where q_j is a polynomial in x of degree less than $2j$ with constant term $+1$. To see this, note $\Phi^{(j)}(x) = 0$ when $x < 0$ since Φ vanishes identically on $(-\infty, 0)$. The expression for $x > 0$ is an easy inductive argument that we leave to the reader. We prove that for $j \geq 0$, $\Phi^{(j)}(0)$ exists and is equal to zero and $\Phi^{(j)}$ is continuous at $x = 0$. If $j = 0$, $\Phi^{(0)}(0) = \Phi(0) = 0$ (by definition of Φ) and Φ will be continuous at $x = 0$ since $\lim_{x \to 0+} e^{-\frac{1}{x}} = 0$ by lemma 5.2.2. Proceeding inductively, suppose that we have shown for $j < n$ that $\Phi^{(j)}(0)$ exists and is equal to zero and that $\Phi^{(j)}$ is continuous at $x = 0$. First we show that $\Phi^{(n-1)}$ is differentiable at $x = 0$ with zero derivative. We have

$$\lim_{x \to 0-} \frac{\Phi^{(n-1)}(x) - \Phi^{(n-1)}(0)}{x} = \frac{0 - 0}{x} = 0.$$

It remains to consider the limit as $x \to 0+$. We have

$$\begin{aligned} \lim_{x \to 0+} \frac{\Phi^{(n-1)}(x) - \Phi^{(n-1)}(0)}{x} &= \lim_{x \to 0+} \frac{\frac{q_{n-1}(x)}{x^{2n-2}} e^{-\frac{1}{x}} - 0}{x} \\ &= \lim_{x \to 0+} \frac{q_{n-1}(x)}{x^{2n-1}} e^{-\frac{1}{x}} \\ &= 0, \quad \text{by lemma 5.2.2, with } k = 2n-1. \end{aligned}$$

Hence $\Phi^{(n-1)}$ is differentiable at $x = 0$ with zero derivative and so Φ is n times differentiable at $x = 0$ with $\Phi^{(n)}(0) = 0$. To complete the inductive step, we must show that $\Phi^{(n)}$ is continuous at $x = 0$; that is, $\lim_{x \to 0} \Phi^{(n)}(x) = 0$. Obviously, $\lim_{x \to 0-} \Phi^{(n)}(x) = 0$. Since $\Phi^{(n)}(x) = \frac{q_n(x)}{x^{2n}} e^{-\frac{1}{x}}$ if $x > 0$, we have $\lim_{x \to 0+} \Phi^{(n)}(x) = 0$ by lemma 5.2.2. $\square$

EXAMPLE 5.2.4. The C^∞ function $\Phi : \mathbb{R} \to \mathbb{R}$ defined in proposition 5.2.3 is not analytic. Indeed, Φ is strictly positive on $x > 0$ and, in particular is non-zero on $x > 0$. On the other hand the Taylor series $T\Phi_0$ of Φ at the origin is $\sum_{n=0}^{\infty} \frac{\Phi^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 x^n = 0$. Hence the Taylor series of Φ at the origin does not converge to Φ on any interval $(a, -a)$, $a > 0$, and therefore Φ cannot be analytic. ♠

REMARK 5.2.5. In general, the Taylor series of a smooth function bears little relation to the function. There is a classical result of E Borel (1895) that shows that given *any* sequence $(a_n)_{n \geq 0}$ of real numbers, there exists a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with Maclaurin series (that is, the Taylor series at zero) given by $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ (so $f^{(n)}(0) = a_n$). If we choose a rapidly increasing sequence such as $a_n = n^{n^n}$, the radius of convergence of the Maclaurin series will be zero even though f is defined on all of $\mathbb{R}$.

5.2.1. Constructing functions. We shall use the smooth function constructed in proposition 5.2.3 as a building block to construct many other smooth but non-analytic functions. Technically speaking it is often quite hard or impossible to construct analytic, or even polynomial functions, with simple specified properties. The most studied case is the construction of an analytic function f on $\mathbb{R}$ with a specified zero set X[1]. This is easy if X is finite (we can then use a polynomial). If X is not finite, then X must be countable and consist of isolated points (see the section on analytic functions for some more details).

EXAMPLES 5.2.6 (Bump functions). (1) Given $a, b \in \mathbb{R}$ with $a < b$, we construct a smooth function $\Psi_{a,b}$ such that

$$\begin{aligned} \Psi_{a,b}(x) &= 0, \text{ if } x \notin (a,b) \\ &> 0 \text{ if } x \in (a,b) \end{aligned}$$

To this end we define

$$\Psi_{a,b}(x) = \Phi(b-x)\Phi(x-a), \ x \in \mathbb{R}.$$

Observe that $\Phi(x-a) = 0$ iff $x < a$ and $\Phi(b-x) = 0$ iff $x > b$. Hence $\Psi_{a,b}(x) = 0$ iff $x \notin (a,b)$. Since $\Phi(x) > 0$ if $x > 0$, we have $\Psi_{a,b}(x) > 0$ if $x \in (a,b)$. Since $\Psi_{a,b}$ is the product of C^∞ functions, $\Psi_{a,b}$ is C^∞. Note that $\Psi_{a,b}^{(j)}(z) = 0$ for all $j \in \mathbb{N}$ if $x = a, b$.
We show the graph of $\Psi_{a,b}$ in figure 2 (the graph is symmetrical about the mid-point $(a+b)/2$).

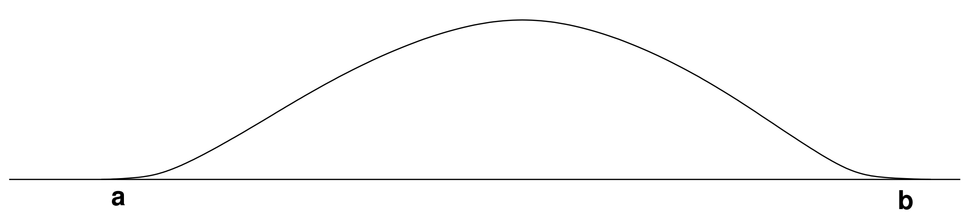


FIGURE 2. Smooth positive bump function, non-zero on (a,b)

[1]That is, X is to equal $f^{-1}(0)$

(2) Given $a, b \in \mathbb{R}$ with $0 < a < b < \infty$, we construct a smooth function $\Theta_{a,b}$ such that

$$\begin{aligned}\Theta_{a,b}(x) &= 0, \text{ if } |x| \geq b\\ \Theta_{a,b}(x) &= 1, \text{ if } |x| \leq a\\ \Theta_{a,b}(x) &\in (0,1), \text{ if } |x| \in (a,b).\end{aligned}$$

. For this we define

$$\Theta_{a,b}(x) = \frac{\Phi(b^2 - x^2)}{\Phi(b^2 - x^2) + \Phi(x^2 - a^2)},\ x \in \mathbb{R}.$$

Since $0 < a < b$, the denominator is never zero and so $\Theta_{a,b}$ is well-defined and C^∞. If $|x| > b$, then the numerator is zero; if $|x| < a$, the denominator is equal to the numerator and so $\Theta_{a,b}(x) = 1$. If $|x| \in (a,b)$, then the numerator is strictly less than the denominator and so $\Theta_{a,b}(x) \in (0,1)$. We remark that all the derivatives of $\Theta_{a,b}$ at x are zero if $x = \pm a, \pm b$. In particular, $\Theta_{a,b}$ is not analytic. We show the graph of $\Theta_{a,b}$ in figure 3.

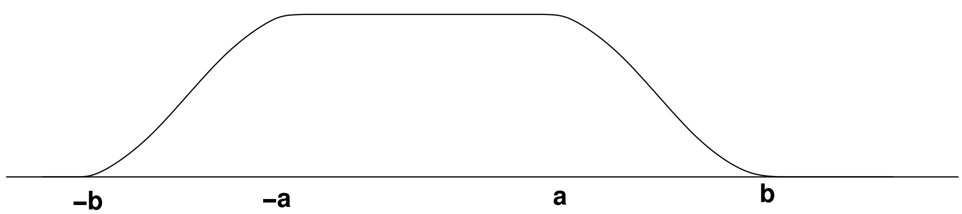


FIGURE 3. Smooth positive bump function non-zero on $(-b, b)$

REMARK 5.2.7. The two functions constructed in the previous examples are usually called "bump" functions. Basically their construction depends more on simple logic than difficult analysis.

EXAMPLES 5.2.8. (1) We construct a smooth function with zero set equal to $\{0\} \cup \{\pm 1/n \mid n \geq 1\}$. As a first try, we might consider $f(x) = x \sin(\pi/x)$, $x \neq 0$, $f(0) = 0$. This function is continuous and has the specified zero set but it is not differentiable at $x = 0$ as $\lim_{x \to 0}(f(x) - f(0))/x = \lim_{x \to 0} \sin(\pi/x)$ which does not exist. If we instead try $f(x) = x^2 \sin(\pi/x)$, $x \neq 0$, $f(0) = 0$, we find that f is differentiable at $x = 0$ but not C^1. More generally, if we define $f(x) = x^{2n+1} \sin(\pi/x)$, $x \neq 0$, and $f(0) = 0$, then f can be shown to be C^n, $n \geq 1$ (we leave this to the exercises). In order to find a C^∞ function with the correct properties, we try

$$\begin{aligned}f(x) &= \Phi(x^2)\sin(\pi/x),\ x \neq 0\\ &= 0,\ x = 0.\end{aligned}$$

Just as in the proof of proposition 5.2.3, we may use lemma 5.2.2 to show that f is C^∞ and all the derivatives of f vanish at zero. In particular, f is not analytic. Notice the way we use Φ to 'smooth' out the irregularities near $x = 0$ of $\sin(\pi/x)$.
(2) We show how to construct a C^∞ function $F : \mathbb{R} \to \mathbb{R}$ which satisfies

(a) $F(x) = 2$, $x \le -2$.
(b) $F(x) \in (0, 1)$ for $x \in (-2, -1)$.
(c) $F(x) = 0$, for $x \in [-1, 0]$.
(d) $F(x) \ge 0$ on $[0, 1]$ and $F(x) = 0$ iff $x = 1/n$ or $1 - 1/n$ for some $n \in \mathbb{N}$.
(e) $F(x) \in (-1, 0)$, for $x \in (1, 5)$.
(f) $F(x) = -1$, for $x \ge 5$.

We express F as a sum of functions $F_1 + F_2 + F_3$, where

$$
\begin{aligned}
F_1(x) &= 2\frac{\Phi(-x-1)}{\Phi(-x-1)+\Phi(x+2)}, \\
F_2(x) &= \Phi(x)\Phi(1-x)\sin^2(\frac{\pi}{x})\sin^2(\frac{\pi}{1-x}),\ 0 < x < 1, \\
&= 0,\ x \notin (0,1), \\
F_3(x) &= -\frac{\Phi(x-5)}{\Phi(x-5)+\Phi(x-1)}.
\end{aligned}
$$

The denominator of F_1 is never zero and so F_1 defines a smooth function on $\mathbb{R}$ which satisfies (a,b). Further, $F_1(x) = 0$, for all $x \ge -1$. The function F_2 is zero outside $[0, 1]$ and is positive on $[0, 1]$ with zeros at $1/2, 1/3, 2/3, 1/4, 3/4, \ldots$. The factors $\Phi(x)$, $\Phi(1 - x)$ ensure that F_2 is smooth at $x = 0, 1$. Finally, F_3 vanishes for $x \le 1$ and satisfies (e,f). Since the denominator of F_3 is never zero, F_3 defines a smooth function on $\mathbb{R}$. The function $F = F_1 + F_2 + F_3$ is a sum of smooth functions and therefore defines a smooth function on $\mathbb{R}$ which satisfies (a–f). ♠

EXERCISES 5.2.9.

(1) Define $f(x) = x^3 \sin(\frac{\pi}{x})$, $x \ne 0$, and $f(0) = 0$. Show that
(a) f is continuous on $\mathbb{R}$ (you may assume that f is C^1 on $x \ne 0$).
(b) f is differentiable at $x = 0$ and $f'(0) = 0$ (you will need to work from the definition of the derivative as a limit).
(c) f' is continuous on $\mathbb{R}$. (You will need to find $\lim_{x\to 0} f'(x)$.)
(d) Is f' differentiable at $x = 0$?
More generally, show that if $f(x) = x^{2n} \sin(\frac{\pi}{x})$, $x \ne 0$, and $f(0) = 0$, then f is C^{n-1} and n-times differentiable but not C^n ($f^{(n)}$ is not continuous at $x = 0$). What about if $f(x) = x^{2n+1} \sin(\frac{\pi}{x})$, $x \ne 0$, and $f(0) = 0$?

(2) Define

$$
\begin{aligned}
f(x) &= x^2 \sin\left(\frac{\pi}{\sqrt{x}}\right), x > 0, \\
&= 0,\ x \le 0.
\end{aligned}
$$

You may assume f is smooth on $x \neq 0$. Show that
 (a) f is continuous on $\mathbb{R}$.
 (b) f is differentiable at $x = 0$ and $f'(0) = 0$.
 (c) f' is continuous on $\mathbb{R}$.
 (d) f is not twice differentiable at $x = 0$.

What is the zero set $(f^{-1}(0))$ of f?

(3) Find (explicit) smooth (C^∞) functions $f, g : \mathbb{R} \to \mathbb{R}$ such that
 (a) $f(0) = 0$ and $f(\frac{1}{n}) = 0$, $n \ge 1$. Elsewhere $f > 0$.
 (b) $g(x) \in (0, 1)$, for all $x \in (0, 1) \cup (2, 3) \cup (3, 4) \cup (5, 6)$, $g = 1$ on $[1, 2] \cup [4, 5]$, elsewhere $g = 0$.

(4) Let $a, b \in \mathbb{R}$, $a < b$. Find a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that

$$
\begin{aligned}
f(x) &= 0,\ x \le a \\
&\in (0, 1),\ x \in (a, b) \\
&= 1,\ x \ge b
\end{aligned}
$$

(5) Let $-\infty < a < b < c < d < +\infty$. Using the function Φ find a C^∞ $\rho \in C^\infty(\mathbb{R})$ such that
 (a) $\rho(x) = 0$ if $x \le a$ or $x \ge b$.
 (b) $\rho(x) = 1$ if $x \in [b, c]$.
 (c) For all other $x \in \mathbb{R}$, $\rho(x) \in (0, 1)$.

Extend the definition of ρ as far as you can so as to remove the strict inequalities in $-\infty < a < b < c < d < +\infty$ (for example, $-\infty \le a < b \le c < d < +\infty$).

(6) Using the function Φ
 (a) Find a C^∞ function e such that $e > 0$ on $(-\infty, 0) \cup (1, \infty)$ and $e \equiv 0$ on $[0, 1]$.
 (b) Find a C^∞ function f such that $f(0) = 0$, elsewhere $f < 0$ and $f^{(j)}(0) = 0$, $j \ge 0$.
 (c) Find a C^∞ function g such that the zero set of g is $\{\pm n^3 | n \in \mathbb{Z}\}$, elsewhere $g < 0$.
 (d) Find a C^∞ function h such that $h(x) = 0$, $x \le 0$, and
  (a) $h(x) = n + 1$, if $x \in [2n + 1, 2n + 2]$, $n \ge 0$.
  (b) $h(x) \in (n, n + 1)$, if $x \in (2n, 2n + 1)$, $n \ge 0$.

 (You are advised to draw the graph first. One step at a time.)

(7) Using the function Φ
 (a) Find a C^∞ function e such that (a) $e > 0$ on $(-\infty, 0)$, (b) $e \equiv 0$ on $[0, \infty)$.
 (b) Find a C^∞ function f such that (a) $f > 0$ on $(-\infty, 1)$, (b) $f \equiv 0$ on $[1, \infty)$.
 (c) Find a C^∞ function g such that (b) $g > 0$ on $(0, 1)$, (b) $g(x) = 0$ if $x \notin (0, 1)$, (c) g has a unique maximum value at $x = \frac{1}{2}$. (In particular, g is not a tabletop function — it is simpler). What are $g^{(n)}(1)$, $g^{(n)}(0)$, $n \ge 0$?
 (d) Find a C^∞ function $F(x)$ such that $F(\frac{1}{n}) = F(1 - \frac{1}{n}) = 0$, $n \ge 1$; elsewhere F is strictly positive. What are $F^{(n)}(0), F^{(n)}(1)$, $n \ge 0$? (For this problem it suffices to give a brief indication of why your function F is infinitely differentiable at the points $0, 1$.)

(8) Using the C^∞ function Φ, find a C^∞ function $G : \mathbb{R} \to \mathbb{R}$ which satisfies all of the following conditions:
   (a) $G(x) = 2$ if either $x \le -1$ or $x \ge 2$.
   (b) $G(x) \in (0, 2)$ if either $x \in (-1, 0)$ or $x \in (1, 2)$.
   (c) $G(x) \ge 0$ on $[0, 1]$ and equals zero iff $x = \frac{1}{n}$ or $1 - \frac{1}{n}$ for some $n \in \mathbb{N}$.

   Indicate briefly why your function G is smooth at $x = 0, 1$.

5.3. The Weierstrass approximation theorem

If $I \subset \mathbb{R}$ is a (non-empty) interval, we have the sequence of strict inclusions

$$P(\mathbb{R}) = P(I) \subset C^\omega(I) \subset C^\infty(I) \subset \ldots \subset C^r(I) \subset \ldots \subset C^1(I) \subset C^0(I)$$

It is reasonable to ask whether or not we can "approximate" functions in one of these classes by more regular functions. In this section we prove an approximation theorem that allows us to approximate continuous functions on a closed interval by polynomials. The type of approximation we use is *uniform approximation* which is defined in terms of the greatest difference in values of two functions. The result is known as the *Weierstrass Approximation Theorem*. We give an elementary constructive proof using Bernstein polynomials.

THEOREM 5.3.1 (The Weierstrass Approximation Theorem). *Every continuous function on $I = [0, 1]$ can be uniformly approximated by polynomials. That is, if $f \in C^0(I)$ and $\varepsilon > 0$, then there exists a polynomial p such that*

$$d(f, p) = \sup_{x \in I} |f(x) - p(x)| < \varepsilon.$$

REMARK 5.3.2. We have stated the theorem for the closed interval $I = [0, 1]$ but the result holds for any closed and bounded interval $[a, b] \subset \mathbb{R}$. See the exercises.

5.3.1. Bernstein polynomials. Let $f \in C^0(I)$ and $n \ge 1$. The *nth Bernstein polynomial* $B_n(f)$ of f is the polynomial defined by

$$B_n(f)(x) = \sum_{p=0}^{n} \binom{n}{p} f(\frac{p}{n}) x^p (1 - x)^{n-p}.$$

LEMMA 5.3.3. *We have for $n \ge 1$*
1. $B_n(cf) = cB_n(f)$, $f \in C^0(I)$, $c \in \mathbb{R}$.
2. $B_n(f + g) = B_n(f) + B_n(g)$, $f, g \in C^0(I)$.
3. $B_n(f) > 0$ *on I if* $f > 0$ *on I*.
4. $B_n(1) = 1$.
5. $B_n(t)(x) = x$ *(here $f(t) = t$)*.
6. $B_n(t^2)(x) = x^2 + \frac{x - x^2}{n}$ *(here $f(t) = t^2$)*.

REMARKS 5.3.4. (1) Statements (1,2) amount to the linearity of $B_n : C^0(I) \to C^0(I)$.
(2) Statement (3) implies that if $f > g$ then $B_n(f) > B_n(g)$ (always on I).
(3) When we replace f by an actual function the variable for f will always be t — as in $f(t)$. The Bernstein polynomial will always be a function of $x \in I$.

Proof of Lemma 5.3.3 (1,2,3) are obvious (note for (3) that $x^p(1-x)^{n-p} > 0$ on $(0,1)$).
(4) $B_n(1)(x) = \sum_{p=0}^{n} \binom{n}{p} 1 x^p (1-x)^{n-p} = (x + (1-x))^n = 1$.
(5) We assume $n \geq 2$ — the result is easy if $n = 1$. We have

$$
\begin{aligned}
B_n(t)(x) &= \sum_{p=0}^{n} \binom{n}{p} \frac{p}{n} x^p (1-x)^{n-p}, \\
&= \sum_{p=1}^{n} \frac{n!}{p!(n-p)!} \frac{p}{n} x^p (1-x)^{n-p}, \\
&= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!(n-p)!} x^p (1-x)^{n-p}, \\
&= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!((n-1)-(p-1))!} x x^{p-1} (1-x)^{(n-1)-(p-1)}, \\
&= x \sum_{q=0}^{n-1} \binom{n-1}{q} x^q (1-x)^{(n-1)-q},\ (q = p-1) \\
&= x B_{n-1}(1)(x) = x.
\end{aligned}
$$

(6) Again we assume $n \geq 3$, the result is easy if $n = 1, 2$. We have

$$\begin{aligned}
B_n(t^2)(x) &= \sum_{p=0}^{n} \binom{n}{p} (\frac{p}{n})^2 x^p (1-x)^{n-p}, \\
&= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!(n-p)!} \frac{p}{n} x^p (1-x)^{n-p}, \text{ as in (5)}, \\
&= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!(n-p)!} (\frac{p-1}{n} + \frac{1}{n}) x^p (1-x)^{n-p}, \\
&= A + B, \text{ where} \\
A &= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!(n-p)!} \frac{p-1}{n} x^p (1-x)^{n-p}, \\
B &= \sum_{p=1}^{n} \frac{(n-1)!}{(p-1)!(n-p)!} \frac{1}{n} x^p (1-x)^{n-p}.
\end{aligned}$$

Checking the proof of (5), we see that $B = \frac{1}{n} B_{n-1}(t)(x) = \frac{x}{n}$. It remains to evaluate A. Cancelling the factor $(p-1)$ and taking out factors $(n-1)/n$ and x^2 we have (just as in the proof of (5))

$$\begin{aligned}
A &= x^2 \frac{(n-1)}{n} \sum_{p=2}^{n} \frac{(n-2)!}{(p-2)!((n-2)-(p-2))!} x^{p-2} (1-x)^{(n-2)-(p-2)}, \\
&= x^2 \frac{(n-1)}{n} B_{n-2}(1)(x) \\
&= x^2 \frac{(n-1)}{n}.
\end{aligned}$$

(Note that if $n = 2$, $\sum_{p=2}^{2} \frac{(2-2)!}{(p-2)!((2-2)-(p-2))!} x^{p-2} (1-x)^{(2-2)-(p-2)} = 1$.)
Finally,

$$A + B = x^2 \frac{(n-1)}{n} + \frac{x}{n} = x^2 + \frac{x - x^2}{n}.$$

and so $B_n(t^2)(x) = x^2 + \frac{x-x^2}{n}$. $\square$

Proof of Theorem 5.3.1 Let $f \in C^0(I)$ and $\varepsilon > 0$. Since I is closed and bounded, $f : I \to \mathbb{R}$ is uniformly continuous (theorem 2.4.15) and so $\exists \delta > 0$ such that for all $t, x \in I$ satisfying $|x - t| < \delta$ we have

$$-\varepsilon/2 < f(t) - f(x) < \varepsilon/2 \text{ (ie } |f(t) - f(x)| < \varepsilon/2). \tag{5.1}$$

Since f is continuous on I, $M = \sup_{s \in I} |f(s)| < \infty$. The next inequality follows from the triangle inequality.

$$-2M < f(t) - f(x) < 2M, \text{ for all } t, x \in I. \tag{5.2}$$

Observe that the function $\frac{2M}{\delta^2}(t-x)^2$ is greater than or equal to $2M$ provided that $|t-x| \geq \delta$. It follows from (5.1,5.2) that for all $t, x \in I$ we have

$$-\varepsilon/2 - \frac{2M}{\delta^2}(t-x)^2 < f(t) - f(x) < \frac{2M}{\delta^2}(t-x)^2 + \varepsilon/2. \tag{5.3}$$

Regard each term in this inequality as a function of t (so x is fixed). Noting property (3) of Bernstein polynomials we have for all $n \geq 1$ the inequality between *functions* (of x)

$$B_n(-\varepsilon/2 - \frac{2M}{\delta^2}(t-x)^2) < B_n(f) - B_n(f(x)) < B_n(\frac{2M}{\delta^2}(t-x)^2 + \varepsilon/2).$$

(What are we doing? We fix x, set $t = \frac{p}{n}$ in (5.3), multiply by $\binom{n}{p}x^p(1-x)^{n-p}$ and sum from $p = 0$ to $p = n$. In particular, $B_n(f(x)) = f(x)B_n(1) = f(x)$, using property (1)).

Using properties (1,2,4), we have

$$\begin{aligned} B_n(\frac{2M}{\delta^2}(t-x)^2 + \varepsilon/2) &= \frac{2M}{\delta^2}B_n((t-x)^2) + \varepsilon/2, \\ B_n(-\frac{2M}{\delta^2}(t-x)^2 - \varepsilon/2) &= -\frac{2M}{\delta^2}B_n((t-x)^2) - \varepsilon/2. \end{aligned}$$

Hence for all $x \in I$ we have

(5.4)

$$-\varepsilon/2 - \frac{2M}{\delta^2}B_n((t-x)^2)(x) < B_n(f)(x) - f(x) < \frac{2M}{\delta^2}B_n((t-x)^2)(x) + \varepsilon/2.$$

We claim that $\exists N$ such that for $n \geq N$, $|\frac{2M}{\delta^2}B_n((t-x)^2)(x)| < \varepsilon/2$ for all $x \in I$. It then follows from (5.4) that for $n \geq N$, $x \in I$, $|B_n(f)(x) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and we are done.

In order to prove the claim we evaluate $B_n((t-x)^2)$. Since $(t-x)^2 = t^2 - 2tx + x^2$, we have

$$B_n((t-x)^2) = B_n(t^2) - 2xB_n(t) + x^2B_n(1).$$

Evaluating at x, this gives us (using (4,5,6))

$$\begin{aligned} B_n((t-x)^2)(x) &= B_n(t^2)(x) - 2xB_n(t)(x) + x^2B_n(1)(x), \\ &= (x^2 + \frac{x-x^2}{n}) - 2xx + x^2 1, \\ &= \frac{x-x^2}{n}. \end{aligned}$$

The maximum value of $x - x^2$ on $[0,1]$ is $1/4$ and so $0 \leq B_n((t - x)^2)(x) < 1/4n$. Hence for $x \in I$,

$$0 < \frac{2M}{\delta^2} B_n((t-x)^2) + \varepsilon/2 < \frac{M}{2n\delta^2} + \varepsilon/2.$$

Now choose N so that $\frac{M}{2n\delta^2} < \varepsilon/2$, $n \geq N$. $\square$

5.3.2. An application of the Weierstrass approximation theorem.

PROPOSITION 5.3.5. *Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose that $\int_a^b f(x)x^n \, dx = 0$, for all $n \geq 0$. Then $f \equiv 0$.*

PROOF. Since f is continuous, it suffices to show that for all $\varepsilon > 0$, $\int_a^b f(x)^2 \, dx < \varepsilon$ (that is, $\int_a^b f(x)^2 \, dx = 0$).

We start by observing that if $p(x) = a_n x^n + \ldots + a_1 x + a_0$, then

$$\int_a^b f(x)p(x)\, dx = \sum_{j=0}^n a_j \int_a^b f(x)x^j \, dx = 0,$$

by our assumption.

Let $M = 1 + \sup_{x \in [a,b]} |f(x)| \geq 1$. By the Weierstrass approximation theorem, there exists a polynomial p such that $\sup_{x\in[a,b]} |f(x) - p(x)| < \varepsilon/(M(b-a))$. We have

$$\begin{aligned} \int_a^b f(x)^2 \, dx &= \int_a^b f(x)(f(x) - p(x))\, dx + \int_a^b f(x)p(x)\, dx, \\ &= \int_a^b f(x)(f(x)-p(x))\, dx. \end{aligned}$$

Now

$$\begin{aligned} \left|\int_a^b f(x)(f(x)-p(x))\, dx\right| &\leq \int_a^b |f(x)||f(x)-p(x)|\, dx \\ &< (b-a)M\frac{\varepsilon}{M(b-a)} = \varepsilon, \end{aligned}$$

proving that $f \equiv 0$. $\square$

EXERCISES 5.3.6.

(1) Let $f(x) = |x - \frac{1}{2}|$, $x \in [0,1]$. Compute $B_n(f)$, $n = 1,2,3$.
   (a) Sketch the graph of f, together with the graphs of the approximations $B_n(f)$, $n = 1,2,3$.
   (b) Where is the approximation poor?
   (c) Suppose we take $\varepsilon = 1/10$. Find a value of N for which $d(f, B_n(f)) < 1/10$, for all $n \geq N$. (Note: Do not strive for the best estimate of N. Just get a value — even if it is quite large. You may want to look back over the proof of the Weierstrass approximation theorem.)

(2) Let $g \in C^0([a,b])$, $-\infty < a < b < +\infty$. Find an explicit surjective linear function $L : [a,b] \to [0,1]$ and hence, by considering $f(t) = g(h(t))$, $t \in [0,1]$, verify that the Weierstrass approximation theorem holds for $C^0([a,b])$.

(3) Let $C^r(I)$ denote the space of r-times continuously differentiable functions on $I = [0,1]$, $0 \leq r < \infty$. Show that given $\varepsilon > 0$, there exists a polynomial p such that

$$d(f^{(s)}, p^{(s)}) < \varepsilon,\ 0 \leq s \leq r.$$

(Uniform approximation of a function and its first r-derivatives.) Hint: Start by approximating $f^{(r)}$ and then work back to f.

(4) For $\eta > 0$, define $\phi_\eta = \frac{1}{\sqrt{2\pi\eta}} \exp(-\frac{x^2}{2\eta})$. Show that if $f \in C^0(\mathbb{R})$ is bounded and we define $f_\eta(x) = \int_{-\infty}^{\infty} f(t)\phi_\eta(x-t)\,dt$, then
   (a) f_η is C^∞, $\eta > 0$.
   (b) f_η converges uniformly to f on all closed bounded subintervals of $\mathbb{R}$ (that is, given $[a,b]$ and $\varepsilon > 0$, there exists $\eta_0 > 0$ such that $\sup_{x\in[a,b]} |f(x) - f_\eta(x)| < \varepsilon$, for all $\eta \in (0, \eta_0)$).

   (For part (b) you will need (A) $\int_{-\infty}^{\infty} \phi_\eta(t)\,dt = 1$ for all $\eta > 0$, and (B) if $\delta, \varepsilon > 0$, there exists $\eta_0 > 0$ such that $\int_{-\delta}^{\delta} \phi_\eta(t)\,dt > 1 - varepsilon$ for all $\eta \in (0, \eta_0]$.) Show how this result can be used to prove that we can uniformly approximate continuous functions on $[a,b]$ by smooth functions.

(5) Suppose that $f : [0,1] \times [a,b] \to \mathbb{R}$ is continuous. For $\lambda \in [a,b]$, define $f_\lambda : [0,1] \to \mathbb{R}$ by $f_\lambda(x) = f(x,\lambda)$ (we think of f_λ as a parameterized family of continuous functions). Prove
   (a) Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_\lambda(x) - f_\lambda(y)| < \varepsilon$ if $|x-y| < \delta$ and $\lambda \in [a,b]$.
   (b) There exists $M \geq 0$ such that $|f(x,\lambda)| \leq M$ for all $(x,\lambda) \in [0,1] \times [a,b]$.
   (c) If we define $p_\lambda^n(x) = B_n(f_\lambda)(x)$, $n \geq 1$, $\lambda \in [a,b]$, then the polynomials p_λ^n uniformly approximate f_λ in the sense that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(p_\lambda^n, f_\lambda) < \varepsilon$, $n \geq N$ and all $\lambda \in [a,b]$.

   (Properties (a,b) are analogs of the corresponding properties for continuous functions of one variables defined on a closed and bounded interval. Granted (a,b) the proof of (c) follows that of the Weierstrass approximation theorem.)

5.4. Analytic functions

Definition 5.4.1. A function $f : (a,b) \to \mathbb{R}$ is *analytic* if for each $x_0 \in (a,b)$ there exists $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,\ |x-x_0| < r.$$

That is, a function f is analytic if for every point x_0 in the domain of f, f is equal to the Taylor series of f at x_0 on some open interval containing x_0.

Example 5.4.2. If $f : (a,b) \to \mathbb{R}$, $g : (c,d) \to \mathbb{R}$ are analytic then $f \pm g$ is an analytic function on $(a,b) \cap (c,d)$. ♠

PROPOSITION 5.4.3. *Suppose that the power series $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$. Then $f(x) = \sum_{n=0}^\infty a_n x^n$ defines an analytic function on $(-R, R)$. More generally, if $c \in \mathbb{R}$, then $\sum_{n=0}^\infty a_n (x-c)^n$ defines an analytic function on $(c - R, c + R)$.*

PROOF. We are required to show that if $x_0 \in (-R, R)$, then there exists $r > 0$ such that the Taylor series $Tf_{x_0}(x) = \sum_{n=0}^\infty \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ of f at x_0 converges to $f(x)$, for all $x \in (x_0 - r, x_0 + r)$.

Since the derivatives of f on $(-R, R)$ are obtained by term-by-term differentiation of the power series $\sum_{n=0}^\infty a_n x^n$, we have

$$\frac{f^{(n)}(x_0)}{n!} = \sum_{m=n}^\infty \binom{m}{n} a_m x_0^{m-n},\ n \geq 0.$$

We start by noting a special case of the result: if f is a polynomial of degree p, then we have

$$\sum_{n=0}^p a_n x^n = \sum_{n=0}^p \left(\sum_{m=n}^p \binom{m}{n} a_m x_0^{m-n} \right) (x - x_0)^n,$$

since it is easy to check that both sides of the equation are polynomials of degree at most p and have the same derivatives of order $\leq p$ at $x = x_0$.

Our proof of the general case has two parts. We need to estimate $\frac{|f^{(n)}(x_0)|}{n!}$ so as to show that Tf_{x_0} has a non-zero radius of convergence. Then we need to prove that the partial sums of $\sum_{n=0}^\infty a_n x^n$ converge to $Tf_{x_0}(x)$ — this will use the special case together with estimates on remainders.

Fix $b \in (|x_0|, R)$. Exactly as in lemma 4.5.1, there exists $C \geq 0$ such that

$$|a_n| \leq C b^{-n},\ n \geq 0.$$

Using this estimate it is easy to show that $\sum_{m=n}^\infty \binom{m}{n} a_m x_0^{m-n}$ is absolutely convergent and that we have the estimate

$$\begin{aligned} \frac{|f^{(n)}(x_0)|}{n!} &\leq C b^n \left(\sum_{m=n}^\infty \binom{m}{n} \frac{|x_0|}{b}^{m-n} \right), \\ &= C b^n (1 - \frac{|x_0|}{b})^{-(n+1)}, \end{aligned}$$

where the last equality follows from the binomial theorem. Choose $r > 0$ so that

$$\frac{br}{1 - \frac{|x_0|}{b}} < 1, \text{ and } [x_0 - r, x_0 + r] \subset (-R, R).$$

We claim that the Taylor series Tf_{x_0} converges on $[x_0-r, x_0+r]$. We have

$$\begin{aligned}
|\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n| &\leq Cb^n(1-\frac{|x_0|}{b})^{-(n+1)}|x-x_0|^n, \\
&< Cb^n(1-\frac{|x_0|}{b})^{-(n+1)}r^n, \text{ if } x \in (x_0-r, x_0+r), \\
&= D\left(\frac{rb}{1-\frac{|x_0|}{b}}\right)^n,
\end{aligned}$$

where $D = C/(1-\frac{|x_0|}{b})$. Since $\sum_{n=0}^{\infty}(\frac{rb}{1-\frac{|x_0|}{b}})^n$ is convergent (by our choice of r), the Taylor series converges for all $x \in [x_0-r, x_0+r]$.

Finally, we need to show that $Tf_{x_0}(x)$ converges to $f(x)$ for all $x \in [x_0-r, x_0+r]$. For this it suffices to show that if $\varepsilon > 0$ then there exists $N\mathbb{N}$ such that $|T_{x_0}(x) - \sum_{n=0}^{p} a_n x^n| < \varepsilon$, for all $p \geq N$ and $x \in [x_0-r, x_0+r]$.

Let $\varepsilon > 0$. Fix $x \in [x_0-r, x_0+r]$ and choose $N \in \mathbb{N}$ so that for all $p \geq N$ we have

(5.5)

$$|\sum_{n=p+1}^{\infty}(\sum_{m=n}^{\infty}\binom{m}{n}|a_m|r^{m-n})r^n|, |\sum_{n=0}^{p}(\sum_{m=p+1}^{\infty}\binom{m}{n}|a_m|r^{m-n})r^n| < \varepsilon/2.$$

For $p \geq N$ define

$$\begin{aligned}
I_1 &= \sum_{n=0}^{p}(\sum_{m=n}^{p}\binom{m}{n}a_m x_0^{m-n})(x-x_0)^n, \\
I_2 &= \sum_{n=p+1}^{\infty}(\sum_{m=n}^{\infty}\binom{m}{n}a_m x_0^{m-n})(x-x_0)^n, \\
I_3 &= \sum_{n=0}^{p}(\sum_{m=p+1}^{\infty}\binom{m}{n}a_m x_0^{m-n})(x-x_0)^n.
\end{aligned}$$

For $x \in [x_0-r, x_0+r]$, we have (by absolute convergence)

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = I_1 + I_2 + I_3.$$

Now $I_1 = \sum_{n=0}^{p} a_n x^n$ (special case: $\sum_{n=0}^{p} a_n x^n$ is a polynomial of degree p). Since $x \in [x_0-r, x_0+r]$, we have by (5.5), $|I_2|, |I_3| < \varepsilon/2$, if $p \geq N$. Hence

$$|Tf_{x_0}(x) - \sum_{n=0}^{p} a_n x^n| = |I_1 + I_2| < \varepsilon, \text{ for all } p \geq N.$$

Hence the sequence $(\sum_{n=0}^{p} a_n x^n)$ of partial sums converges pointwise to $Tf_{x_0}(x)$ on $[x_0 - r, x_0 + r]$. □

REMARK 5.4.4. As we show in the next examples, the radius of convergence of Tf_{x_0} may well be strictly bigger than R. It is straightforward to show that it is always at least $\min\{R - x_0, x_0 + R\}$.

EXAMPLES 5.4.5. (1) The exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ defines an analytic function $\exp(x)$ on $\mathbb{R}$. We claim that (a) $\exp(0) = 1$, (b) $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$, (c) $\exp(x)\exp(-x) = 1$, for all $x \in \mathbb{R}$, and (d) $\exp(x + y) = \exp(x)\exp(y)$, for all $x, y \in \mathbb{R}$. (a) is immediate from the series definition and (b) follows by term-by-term differentiation of the power series defining $\exp(x)$. By the chain rule $\frac{d}{dx}(\exp(-x)) = -\exp(-x)$ and so $\frac{d}{dx}(\exp(x)\exp(-x)) = 0$ for all $x \in \mathbb{R}$. Hence $\exp(x)\exp(-x)$ is constant and, taking $x = 0$, we have $\exp(x)\exp(-x) = 1$ for all $x \in \mathbb{R}$. Finally, using (b) again, we have $\frac{d}{dx}(\exp(x+y)\exp(-x)\exp(-y)) = 0$ and so $\exp(x+y)\exp(-x)\exp(-y)$ is constant as a function of x. Take $x = -y$ and use (a,c) to deduce that $\exp(x + y)\exp(-x)\exp(-y) = 1$ for all $x \in \mathbb{R}$. Hence, applying (c) again, we deduce that $\exp(x + y) = \exp(x)\exp(y)$. If we set $\exp(1) = e \approx 2.718\ldots$, then (c,d) imply that we may write $\exp(x) = e^x$ where e^x satisfies the exponent laws for a power.

(2) The power series $\sum_{n=0}^{\infty} x^n$ has radius of convergence 1 and converges to $f(x) = (1 - x)^{-1}$ on $(-1, 1)$. Given $a \in (-1, 1)$, we have $f^{(n)}(a) = n!(1-a)^{-n}$. We have $Tf_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty}(\frac{x-a}{1-a})^n$ and the radius of convergence of this series is $1-a$. Observe that $1-a > 1$ if $a < 0$ and so the the radius of convergence of the Taylor series of a power series can be strictly bigger than the radius of convergence of the power series. In this example, the analytic function defined by $\sum_{n=0}^{\infty} x^n$ on $(-1, 1)$ is equal to $(1 - x)^{-1}$ and the latter function is naturally defined on $(-\infty, 1)$ as an analytic function. ♠

PROPOSITION 5.4.6. *Suppose that* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$ *are analytic.*

(1) *The product* $f \times g : I \cap J \to \mathbb{R}$ *is analytic.*
(2) *If* g *is non-zero on* $I \cap J$ *then the quotient* $f/g : I \cap J \to \mathbb{R}$ *is analytic.*
(3) *If* $f(I) \subset J$ *then the composite* $g \circ f : I \to \mathbb{R}$ *is analytic.*

PROOF. Statement (1) can be proved using proposition 5.4.3 and the result on products of power series given (proposition 4.5.7). We omit the proofs of (2,3) — see the exercises and the references cited there. □

PROPOSITION 5.4.7. *Suppose that* $f : (a, b) \to \mathbb{R}$ *is analytic and not identically zero. Then*

(1) *The zeros of* f *are* isolated*: if* $f(x_0) = 0$*, then there exists* $s > 0$ *such that the only zero of* f *on* $(x_0 - s, x_0 + s)$ *is* x_0*.*
(2) *If* $f(x_0) = 0$ *then there exists a unique* $p \in \mathbb{N}$ *and analytic function* g *on* (a, b) *such that* $g(x_0) \neq 0$ *and*

$$f(x) = (x - x_0)^p g(x)$$

PROOF. Suppose that $f(x_0) = 0$. Without loss of generality, take $x_0 = 0$. For some $r > 0$ we may write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \ x \in (-r, r).$$

Since $f(0) = 0$, we must have $a_0 = 0$. Let p be the smallest integer for which $a_p \neq 0$. Then

$$f(x) = \sum_{n=p}^{\infty} a_n x^n = x^p \sum_{n=0}^{\infty} a_{n+p} x^n = x^p g(x),$$

where $g(x) = \sum_{n=0}^{\infty} a_{n+p} x^n$, $x \in (-r, r)$. Since $a_p \neq 0$, $g(0) \neq 0$. Moreover, the radius of convergence of the power series defining g is at least r and so g is analytic on $(-r, r)$. In particular, g is continuous on $(-r, r)$ and non-zero at $x = 0$. Hence there exists $s > 0$ such that $g \neq 0$ on $(-s, s)$. Therefore the only zero of f on $(-s, s)$ is at $x = 0$, proving (1). We define $g : (a, b) \to \mathbb{R}$ by $g(0) = a_p$ and $g(x) = x^{-p} g(x)$, $x \neq 0$. We leave it to the exercises for the reader to verify that g is analytic. ☐

5.4.1. Analytic continuation. The next result is very special to analytic functions — it fails completely for C^∞ functions.

PROPOSITION 5.4.8. *Let* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$ *be analytic functions defined on the open intervals* I, J*. If there exists* $x_0 \in I \cap J$ *such that*

$$f^{(n)}(x_0) = g^{(n)}(x_0), \ \textit{for all } n \geq 0,$$

then $f = g$ *on* $I \cap J$*. Otherwise said, if the analytic functions* f *and* g *have the same Taylor series at some point then* $f = g$ *on their common domain.*

PROOF. Let $X = \{x \in I \cap J \mid f^{(n)}(x) = g^{(n)}(x) \text{ for all } n \geq 0\}$. It suffices to prove $X = I \cap J$. Since $x_0 \in X$, $X \neq \emptyset$. Moreover, if $x \in X$, then f and g have the same power series representation on an open interval $K \subset I \cap J$ containing x and therefore $K \subset I \cap J$.

Suppose $X \neq I \cap J$. Without loss of generality suppose there exists $z \in I \cap J$, $z < x_0$. Let $z_0 = \sup\{z < x_0 \mid z \notin X\}$. Clearly, $(z_0, x_0) \subset X$. Choose a sequence $(y_j) \subset (z_0, x_0)$ such that $\lim_{j\to\infty} y_j = z_0$. By sequential continuity of $f^{(n)}, g^{(n)}$, we have $\lim_{j\to\infty} f^{(n)}(y_j) = f^{(n)}(z_0)$, $\lim_{j\to\infty} g^{(n)}(y_j) = g^{(n)}(z_0)$ for all $n \geq 0$. But since $(y_j) \subset X$, we have $f^{(n)}(y_n) = g^{(n)}(y_n)$ for all $n \geq 0$ and so $f^{(n)}(z_0) = g^{(n)}(z_0)$, $n \geq 0$. Hence $z_0 \in X$. But if $z_0 \in X$, then there is an open interval $(z_0 - r, z_0 + r) \subset X \cap (I \cap J)$, contradicting the definition of z_0 as the supremum of points $z < x_0$ not in X. Hence $X = I \cap J$. ☐

The next result is an immediate corollary of proposition 5.4.8.

Corollary 5.4.9. *If $f, g : I \to \mathbb{R}$ are analytic functions which are equal on a non-empty open interval contained in I then $f = g$ on I.*

Definition 5.4.10. Let $f : I \to \mathbb{R}$ be an analytic function. An analytic function $g : J \to \mathbb{R}$ is called an *analytic continuation* of f if (a) $J \supset I$, and (b) $g = f$ on $I \cap J$.

Proposition 5.4.11. *Every analytic function $f : I \to \mathbb{R}$ has a maximal analytic continuation $F : J \to \mathbb{R}$.*

Proof. let $\mathcal{A} = \{g_i : J_i \to \mathbb{R} \mid i \in I\}$ denote the set of all analytic continuations of f. Define $J = \cup_{i\in I} J_i$. Given $x \in J$, there exists $i \in I$ such that $x \in J_i$ and we define $F(x) = g_i(x)$. An an immediate consequence of corollary 5.4.9, the value $F(x)$ is independent of the choice of $i \in I$ such that $x \in J_i$ (note that if $x \in J_i, J_k$ then $x \in J_i \cap J_k \supset I$). The map F is analytic (since $F = g_i$ on each J_i) and obviously F is the maximal analytic continuation of f. ☐

Example 5.4.12. The analytic function $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$, has maximal analytic continuation $F(x) = 1/(1 + x)$ defined on $(-1, \infty)$. ♠

5.4.2. Analytic functions and ordinary differential equations. A natural way of constructing analytic functions is as solutions to linear ordinary differential equations.

Example 5.4.13. Consider the linear differential equation $y' = ay$, where $a \in \mathbb{R}$ and $y' = \frac{dy}{dx}$. We search for a solution $y(x)$ which satisfies the initial condition $y(0) = y_0$ (the analysis is the same if we specify $y(x_0)$, $x_0 \neq 0$).

We start by observing that if $y : \mathbb{R} \to \mathbb{R}$ is a C^∞ solution to $y' = ay$, then all the derivatives $y^{(n)}(0)$ are all uniquely determined by the initial condition. Indeed, since $y' = ay$ we have $y'(0) = ay(0) = ay_0$. Differentiating once, y must satisfy $y'' = ay'$ and so $y''(0) = ay'(0) =$

$a^2 y_0$. Proceeding inductively, it is clear that for $n \geq 0$ we have

$$y^{(n)}(0) = a^n y_0.$$

Assume that y is analytic. Then $y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$ for $x \in (-r, r)$, where $r > 0$. Using our computed values of $y^{(n)}(0)$ we see that

$$y(x) = y_0 \sum_{n=0}^{\infty} \frac{(ax)^n}{n!}.$$

This power series has radius of convergence $R = \infty$. Using our results on term-by-term differentiation of a power series, we see easily that $y(x) = y_0 \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = y_0 e^{ax}$ is a solution of $y' = ay$ which is defined for all $x \in \mathbb{R}$ and satisfies the initial condition $y(0) = y_0$. Moreover, the solution is unique. To see this, suppose that $u(x)$ is a differentiable function defined on an open interval I containing $x = 0$ which satisfies $u' = au$ on I and $u(0) = y_0$. Define $v(x) = e^{-ax} u(x)$. For $x \in I$ we have

$$v'(x) = -e^{-ax} u(x) + e^{-ax} u'(x) = -e^{-ax} u(x) + a e^{-ax} u(x) = 0.$$

Therefore, v is constant on I. We have $v(0) = u(0) = y_0$ and so $v(x) = y_0$ for all $x \in I$. That is, $u(x) = y_0 e^{ax}$, $x \in I$. ♠

REMARK 5.4.14. It is worth summarizing the method used in the previous example. Given the initial condition, all the higher derivatives of a solution are uniquely determined. As a result the Taylor series of the solution at the origin is uniquely determined. We show that the Taylor series has non-zero radius of convergence and observe, using term-by-term differentiation, that the Taylor series defines a solution to the differential equation with the correct initial condition. Finally, we compare a solution with the right initial condition to the constructed solution and so verify uniqueness. In practice, for higher order linear constant coefficient differential equations it is usually best to work over the complex numbers though in some cases it is possible to work using just real numbers (see the exercises at the end of the section).

The next proposition gives a general result on the existence of analytic solutions to a linear ordinary equation. We omit the proof — which is most easily done using complex variable methods.

PROPOSITION 5.4.15. *Consider the second order linear constant coefficient differential equation*

$$y'' + a(x)y' + b(x)y = 0, \tag{5.6}$$

where $a, b \in C^\omega(\mathbb{R})$. Given $y_0, y_0' \in \mathbb{R}$, there exist solutions $y_1, y_2 \in C^\omega(\mathbb{R})$ to (5.6) satisfying

(a) $y_1(0) = y_0$, $y_1'(0) = 0$; $y_2(0) = 0$, $y_2'(0) = y_0'$
(b) *If* $y : I \to \mathbb{R}$ *is a solution to (5.6) such that* $y(0) = y_0$, $y'(0) = y_0'$ *(so* $0 \in I$*), then* $y = y_0 y_1 + y_0' y_2$ *on* I *(in particular, solutions are uniquely specified by their initial conditions).*

EXERCISES 5.4.16.

(1) Consider the ordinary differential equation $y'' = -y$. Suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution of the equation (assume a non-zero radius of convergence — you will verify this assumption later). Show that $y(x)$ is uniquely determined by $y(0)$ and $y'(0)$
 (a) If $y(0) = 1$, $y'(0) = 0$, denote the solution by $c(x)$. Verify that the power series you get has radius of convergence $R = \infty$.
 (b) If $y(0) = 0$, $y'(0) = 1$, denote the solution by $s(x)$. Verify that the power series you get has radius of convergence $R = \infty$.
 (c) Verify that $s' = c$, $c' = -s$ and hence that $s^2 + c^2 \equiv 1$ on $\mathbb{R}$.

5.5. Trigonometric and Fourier series

In this section we consider the problem of approximating functions by trigonometric polynomials and representing functions by a trigonometric or *Fourier* series.

DEFINITION 5.5.1. A function $T : \mathbb{R} \to \mathbb{R}$ is a *trigonometric polynomial* if T can be written in the form

$$T(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx).$$

We call N the *degree* of T.

Trigonometric functions lie in the important class of periodic functions.

DEFINITION 5.5.2. A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period $\tau > 0$ if

$$f(x + \tau) = f(x), \text{ for all } x \in \mathbb{R}.$$

We say f is τ-periodic.

EXAMPLES 5.5.3. (1) A trigonometric polynomial is 2π-periodic.
(2) If $f : \mathbb{R} \to \mathbb{R}$ is τ-periodic then $f(x + n\tau) = f(x)$ for all $n \in \mathbb{Z}$. ♠

REMARKS 5.5.4. (1) We always suppose that the period τ is the smallest strictly positive real number such that $f(x + \tau) = f(x)$ for all $x \in \mathbb{R}$. Of course, if $f(x + \tau) = f(x)$ for all $x \in \mathbb{R}$ and $\tau > 0$ then f is constant.
(2) We generally assume that the period $\tau = 2\pi$ — if not, we can change coordinates to as to make the period equal to 2π.

We used the Weierstrass approximation theorem to uniformly approximate continuous functions on a closed bounded interval by polynomials. We may also use the Weierstrass approximation theorem to show that continuous periodic functions can be uniformly approximated by trigonometric polynomials. We give the proof of the next result in the appendix to this chapter.

THEOREM 5.5.5 (Second Weierstrass approximation theorem). *Every continuous 2π-periodic function on $\mathbb{R}$ can be uniformly approximated by trigonometric polynomials.*

In practice it is far more useful to represent periodic functions by trigonometric series.

DEFINITION 5.5.6. A *trigonometric series* is a series of the form

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

We will mainly be interested in the class of piecewise continuous (or differentiable) functions. These are functions which have only jump discontinuities. For reference, we give the precise definition we use.

DEFINITION 5.5.7. A function $f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there exist a finite subset $\{d_j \mid j = 1, \ldots N\}$ of $[a, b]$ such that

(a) $a < d_1 < \ldots < d_N < b$.
(b) f is continuous, except at $x = d_1, \ldots, d_N$.
(c) For each j, $\lim_{x\to d_j-} f(x) = f(d_j-)$ and $\lim_{x\to d_j+} f(x) = f(d_j+)$ exist and are finite.

If f is defined on $\mathbb{R}$, then f is piecewise continuous if it is piecewise continuous restricted to every bounded closed interval $[a, b]$. A function f is piecewise C^1 if both f and f' are piecewise continuous.

REMARK 5.5.8. We often refer to the type of discontinuity described in definition 5.5.7 as a *jump discontinuity*. The jump at a jump discontinuity d of f is defined to be $f(d+) - f(d-)$.

EXAMPLES 5.5.9. (1) If $f(x) = \sin(1/x)$, $x \neq 0$, and $f(0) = 0$, then f is *not* piecewise continuous.
(2) If we define $F(x) = 1$, $x \in [2n\pi, (2n + 1)\pi)$ and $F(x) = -1$, $x \in [(2n + 1)\pi, (2n + 2)\pi)$, $n \in \mathbb{Z}$, then F is piecewise continuous (indeed, piecewise smooth) and 2π-periodic. The function F defines a *square wave.* ♠

If the trigonometric series $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges for all $x \in \mathbb{R}$ then $U(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is

2π-periodic: $U(x+2\pi) = U(x)$ for all $x \in \mathbb{R}$. In this section, we will be interested in representing 2π-periodic functions as trigonometric series. Most of the time, we obtain results on pointwise convergence. Indeed, uniform convergence turns out not to be so useful in the study of trigonometric series. Much better is the concept of *mean square convergence* which we address later in the section.

DEFINITION 5.5.10. Let f be a 2π-periodic function on $\mathbb{R}$ and assume that f is piecewise continuous (so f has finitely many jump discontinuities on $[0, 2\pi]$). The *Fourier series* $\mathcal{F}(f)$ of f is defined to be the infinite series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n \ge 1 \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n \ge 1 \end{aligned}$$

We refer to a_n, b_n as the *Fourier coefficients* of f.

EXAMPLE 5.5.11. Let F be the 2π-periodic function defined in examples 5.5.9(2). Since F is odd ($F(-x) = -F(x)$), we have $a_n = 0$, $n \ge 0$ (using the integral with limits $\pm\pi$). On the other hand

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx.$$

It follows easily that for $n \ge 0$ we have

$$\begin{aligned} b_{2n} &= 0, \\ b_{2n+1} &= \frac{4}{(2n+1)\pi}. \end{aligned}$$

Hence the Fourier series is $\mathcal{F}(F) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$. ♠

5.5.1. The orthogonality relations. We compute the Fourier coefficients of $\cos px, \sin px$, $p \ge 0$.

Lemma 5.5.12.

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} \cos px \cos nx\, dx &= 1,\ \textit{if } p = n,\ p, n \geq 1,\\
&= 0,\ \textit{if } p \neq n,\\
&= 2,\ \textit{if } p = n = 0,\\
\frac{1}{\pi}\int_0^{2\pi} \sin px \cos nx\, dx &= 0,\ \textit{if } p \geq 1,\ n \geq 0,\\
\frac{1}{\pi}\int_0^{2\pi} \sin px \sin nx\, dx &= 0,\ \textit{if } p = n,\ p, n \geq 1,\\
&= 1,\ \textit{if } p = n \geq 1.
\end{aligned}$$

Proof. All of the statements follow using standard trigonometric identities such as $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ and we leave details to the reader. □

As an important corollary of the second Weierstrass approximation theorem (theorem 5.5.5) we have

Theorem 5.5.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic. If all the Fourier coefficients of f are zero, then $f \equiv 0$. In particular, if continuous and 2π-periodic functions $f, g : \mathbb{R} \to \mathbb{R}$ have the same Fourier series, then $f = g$.*

Proof. We leave to the exercises at the end of the section. □

Theorem 5.5.13 shows that the Fourier coefficients are important invariants of the function f — notwithstanding any issues about convergence of the Fourier series. We can give additional justification for our definition of the Fourier coefficients a_n, b_n when the Fourier series converges uniformly.

Proposition 5.5.14. *Suppose that $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly on $[0, 2\pi]$ to the function f. Then the a_n and b_n must be the Fourier coefficients of f.*

Proof. If $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly on $[0, 2\pi]$ to f then f is continuous on $[0, 2\pi]$ since the partial sums $S_N = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ are continuous. If (S_N) converges uniformly to f then $S_N(x) \cos nx$ converges uniformly to $f(x) \cos nx$ on $[0, 2\pi]$ for all $n \geq 0$. Hence

$$\frac{1}{\pi}\int_0^{2\pi} S_N(x) \cos nx\, dx \to \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx.$$

Using the orthogonality relations (lemma 5.5.12), we have

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} S_N(x)\cos nx\,dx &= a_n, \text{ if } N \geq n > 0,\\ &= 2a_0 \text{ if } n = 0.\end{aligned}$$

Hence $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$ and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$, $n \geq 1$. A similar analysis applies to the coefficients b_n, $n \geq 0$. □

REMARK 5.5.15. Theorem 5.5.13 shows that a continuous 2π-periodic function is uniquely determined by its Fourier coefficients. Proposition 5.5.14 gives conditions under which we can reconstruct f given the Fourier coefficients. A general resolution of this 'inverse' problem motivates much of the more advanced foundational work on Fourier series.

5.5.2. The Riemann-Lebesgue lemma.

LEMMA 5.5.16 (The Riemann-Lebesgue lemma). *Let $f : [a, b] \to \mathbb{R}$ be piecewise continuous. Then*

$$\begin{aligned}\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\,dx &= 0,\\ \lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda x\,dx &= 0.\end{aligned}$$

PROOF. We prove that $\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\,dx = 0$, the analysis for the second integral is similar. We start by assuming f is C^1 on $[a, b]$. Integrating by parts, we have

$$\int_a^b f(x)\cos\lambda x\,dx = \frac{1}{\lambda}(f(b)\sin\lambda b - f(a)\sin\lambda a) - \frac{1}{\lambda}\int_a^b f'(x)\sin\lambda x\,dx.$$

Let C be an upper bound for $|f|$ and $|f'|$ on $[a, b]$ (this uses the continuity of f, f'). We have the estimate

$$|\int_a^b f(x)\cos\lambda x\,dx| \leq 2C/\lambda + C(b-a)/\lambda.$$

Letting $\lambda \to \infty$ we have $\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\,dx = 0$. Next, assume only that f is continuous. Given $\varepsilon > 0$, it suffices to show that there exists $\lambda_0 > 0$ such that $|\int_a^b f(x)\cos\lambda x\,dx| < \varepsilon$, for all $\lambda \geq \lambda_0$. By the Weierstrass approximation theorem, we can find a polynomial $p : [a, b] \to \mathbb{R}$ such that

$$\|f - p\| = \sup_{x\in[a,b]}|f(x) - p(x)| < \frac{\varepsilon}{2(b-a)}.$$

Since p is C^1, we can find $\lambda_0 > 0$ such that

$$|\int_a^b p(x)\cos\lambda x\,dx| \le \frac{\varepsilon}{2}, \text{ for all } \lambda \ge \lambda_0.$$

We have

$$\begin{aligned}
|\int_a^b f(x)\cos\lambda x\,dx| &= |\int_a^b (f-p)\cos\lambda x\,dx + \int_a^b p\cos\lambda x\,dx|,\\
&\le |\int_a^b (f-p)(x)\cos\lambda x\,dx| + |\int_a^b p(x)\cos\lambda x\,dx|,\\
&\le \int_a^b |f(x)-p(x)|\,dx + |\int_a^b p(x)\cos\lambda x\,dx|,\\
&< (b-a)\frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } \lambda \ge \lambda_0.
\end{aligned}$$

It remains to prove the case when f is only piecewise continuous. Suppose that f has discontinuities at $a < d_1 \le \ldots \le d_N < b$. Given $\varepsilon > 0$, we can approximate f by a continuous function F so that $\int_a^b |f(x) - F(x)|\,dx < \varepsilon/(b-a)$ — see figure 4. (For this to work we need the left- and right-hand limits at d_j to exist and be finite.) We

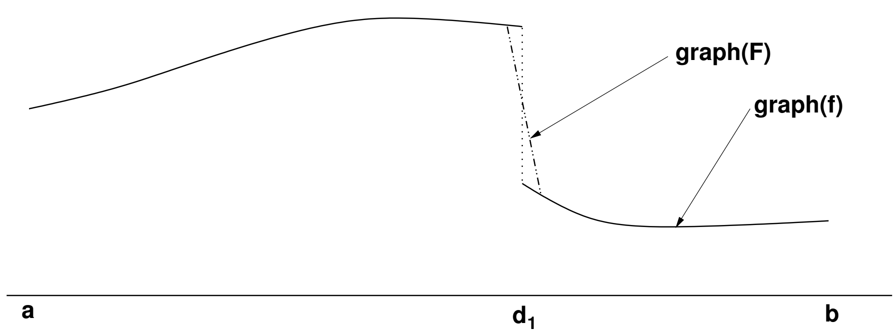


FIGURE 4. Approximating a piecewise continuous function.

have

$$\int_a^b f(x)\cos\lambda x\,dx = \int_a^b F(x)\cos\lambda x\,dx + \int_a^b (F(x)-f(x))\cos\lambda x\,dx$$

Therefore $|\int_a^b f(x)\cos\lambda x\,dx| \leq |\int_a^b F(x)\cos\lambda x\,dx| + \varepsilon$ and

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\,dx \leq \varepsilon$$

for all $\varepsilon > 0$. $\square$

REMARK 5.5.17. It is useful to have a slightly stronger version of the Riemann-Lebesgue lemma that holds for families of continuous functions. Suppose $f : [a,b]\times[c,d] \to \mathbb{R}$ is continuous. Set $f_\lambda(x) = f(x,\lambda)$, $x \in [a,b]$, $\lambda \in [c,d]$. If we define $C(\lambda) = \int_a^b f_\lambda(x)\cos\lambda x\,dx$, $S(\lambda) = \int_a^b f_\lambda(x)\sin\lambda x\,dx$, then given $\varepsilon > 0$, there exists λ_0 such that

$$|C(\lambda)|, |S(\lambda)| < \varepsilon$$

for all $\lambda \geq \lambda_0$. The proof is basically the same as that given above except that we use the Weierstrass approximation theorem for families (exercises 5.3.6(5)).

5.5.3. Integral formula for the partial sum of a Fourier series.

We start with a useful trigonometric identity.

LEMMA 5.5.18. *If for $n \geq 0$ we define $D_n(x) = 1 + 2\sum_{j=1}^n \cos jx$, then*

(1) $\int_0^{2\pi} D_n(x)\,dx = 2\pi$.
(2) $D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}$, *if x is not an integer multiple of 2π.*

PROOF. The first statement is an immediate consequence of the orthogonality relations. Next, from the trigonometric identities in the appendix to chapter 3, we have

$$1 + 2\sum_{j=1}^n \cos jx = 1 + \frac{2\cos(\frac{n+1}{2}x)\sin(\frac{nx}{2})}{\sin(\frac{x}{2})}.$$

Since $2\cos A\sin B = \sin(A+B) - \sin(A-B)$, it is easy to verify that the right hand side is equal to $\frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}$. $\square$

REMARK 5.5.19. As we shall soon see the function $D_n(x)$ plays an important role in the convergence theory of Fourier series. The collection $\{D_n \mid n \geq 1\}$ is usually called the *Dirichlet kernel*. The integral $\int_{-\pi}^{\pi}|D_n(x)|\,dx$ grows like $\log n$ and this lack of convergence is reflected in the fact that the Fourier series of a continuous function may not converge pointwise at every point. It can be shown that the Fourier series of a continuous function does converge at 'most' points. However, this result, due to Carleson (1966), is very difficult to prove.

As we shall see adding a little regularity to the function, improves the convergence properties of the function.

Suppose f is a piecewise continuous function on $[0, 2\pi]$. For $n \geq 0$ define the partial sums

$$S_n(f)(x) = a_0 + \sum_{j=1}^{n}(a_j \cos jx + b_j \sin jx),$$

where a_j, b_j are the Fourier coefficients of f.

LEMMA 5.5.20. *(Notation as above.) For $n \geq 0$ we have*

$$S_n(f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)D_n(t)\,dt.$$

PROOF. We have

$$\begin{aligned}
S_n(f)(x) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx + \sum_{j=1}^{n}\frac{1}{\pi}\left[(\int_0^{2\pi} f(t)\cos jt\,dt)\cos jx\right] \\
&\quad + \sum_{j=1}^{n}\frac{1}{\pi}\left[(\int_0^{2\pi} f(t)\sin jt\,dt)\sin jx\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)[1 + 2\sum_{j=1}^{n}\cos(j(x-t))]\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)D_n(x-t)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x-t)D_n(t)\,dt,
\end{aligned}$$

where we have made the change of variable $t \to x - t$. $\square$

THEOREM 5.5.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic piecewise continuous function and let $x_0 \in \mathbb{R}$. Set $f(x_0+) = \lim_{x\to x_0+} f(x)$, $f(x_0-) = \lim_{x\to x_0-} f(x)$. Assume that*

$$\begin{aligned}
D_R &= \lim_{t\to 0+}\frac{f(x_0+t) - f(x_0+)}{t}, \\
D_L &= \lim_{t\to 0-}\frac{f(x_0+t) - f(x_0-)}{t},
\end{aligned}$$

exist. Then the Fourier series of f is convergent at x_0 and

$$\mathcal{F}(f)(x_0) = \frac{1}{2}\left[f(x_0-) + f(x_0+)\right].$$

In particular, if f is continuous and piecewise differentiable on $\mathbb{R}$ then $\mathcal{F}(f)(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$.

Before we start the proof of the theorem we need a technical lemma that allows us to use the differentiability properties of f at x_0.

LEMMA 5.5.22. *Suppose that f satisfies the conditions of theorem 5.5.21. If we define $g : [-\pi, \pi] \to \mathbb{R}$ by*

$$\begin{aligned} g(t) &= \frac{f(x_0 - t) - f(x_0)}{\sin \frac{t}{2}},\ t \neq 0, \\ &= -(D_L + D_R),\ t = 0, \end{aligned}$$

then $\lim_{t\to 0+} g(t) = -2D_L$, $\lim_{t\to 0-} g(t) = -2D_R$. In particular,

(a) *g is piecewise continuous on $[-\pi, \pi]$ with at most one discontinuity at $t = 0$.*
(b) *If f is differentiable at x_0, then g is continuous at zero and $g(0) = -2f'(x_0)$.*

PROOF. For $t > 0$, we have

$$\frac{f(x_0 - t) - f(x_0)}{\sin \frac{t}{2}} = -2\frac{f(x_0 - t) - f(x_0)}{-t}\frac{\frac{t}{2}}{\sin \frac{t}{2}} \to -2D_L, \text{ as } t \to 0+.$$

The same argument shows that $\lim_{t\to 0-} g(t) = -2D_R$. $\square$

Proof of Theorem 5.5.21. By the partial sum formula (lemma 5.5.20), we have

$$S_n(f)(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x_0 - t) - f(x_0)}{\sin(\frac{t}{2})} \sin((n + \frac{1}{2})t)\ dt.$$

Applying lemma 5.5.22,

$$\int_{-\pi}^{\pi} \frac{f(x_0 - t) - f(x_0)}{\sin \frac{t}{2}} \sin((n + \frac{1}{2})t)\ dt = \int_{-\pi}^{\pi} g(t) \sin((n + \frac{1}{2})t)\ dt,$$

where g is piecewise continuous on $[-\pi, \pi]$.

We have $\lim_{n\to\infty} \int_{-\pi}^{\pi} g(t) \sin((n + \frac{1}{2})t)\ dt = 0$ (by the Riemann-Lebesgue lemma) and so $\lim_{n\to\infty} S_n(f)(x_0) = f(x_0)$. $\square$

EXAMPLES 5.5.23. (1) Let F be the 2π-periodic square wave function with Fourier series $\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{2n+1}$ (example 5.5.11). As a result of theorem 5.5.21, we see that

$$\begin{aligned} \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{2n+1} &= 1, \text{ if } x \in \bigcup_{n\in\mathbb{Z}} (2n\pi, (2n+1)\pi), \\ &= -1, \text{ if } x \in \bigcup_{n\in\mathbb{Z}} ((2n+1)\pi, (2n+2)\pi), \\ &= 0, \text{ if } x \text{ is an integer multiple of } \pi. \end{aligned}$$

THEOREM 5.5.24. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic and piecewise C^1, then the Fourier series of f is uniformly convergent to f.*

PROOF. (Sketch.) Suppose first that f is C^1. Then, as in the proof of theorem 5.5.21, we may write $S_n(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,t)\sin((n+\frac{1}{2})t)\,dt$, where $g(x,t)$ is continuous. We regard $g(x,t) = g_x(t)$ as a parameterized family of continuous functions and apply the Riemann-Lebesgue lemma for families (remark 5.5.17) to get the required estimate for uniform convergence. If we assume that f is continuous and piecewise C^1, then the same argument gives the uniform estimates needed for the Riemann-Lebesgue lemma on any closed interval not containing a discontinuity of f' as an interior point. $\square$

5.5.4. The infinite product formula for $\sin \mathbf{x}$. We show how to derive the infinite product formula $\sin x = x\Pi_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2})$, $x \in \mathbb{R}$, using methods based on Fourier series.

We start by finding the Fourier series of the 2π-periodic continuous piecewise C^1 function f on $\mathbb{R}$ defined by

$$f(x) = \cos\frac{\lambda x}{\pi},\ x \in [-\pi, \pi].$$

Since $\cos x$ is even, f is an even function of x and so all the Fourier sine coefficients $b_n = 0$. We have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \cos\frac{\lambda x}{\pi}\,dx = \frac{\sin\lambda}{\lambda}.$$

and

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}\cos\frac{\lambda x}{\pi}\cos nx\,dx, \\
&= \frac{1}{\pi}\int_0^{\pi}\cos(\frac{\lambda}{\pi}+n)x + \cos(\frac{\lambda}{\pi}-n)x\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin(\frac{\lambda}{\pi}+n)x}{\frac{\lambda}{\pi}+n} + \frac{\sin(\frac{\lambda}{\pi}-n)x}{\frac{\lambda}{\pi}-n}\right]_{x=0}^{x=\pi} \\
&= \frac{1}{\pi}\left[\frac{\sin(\frac{\lambda}{\pi}+n)\pi}{\frac{\lambda}{\pi}+n} + \frac{\sin(\frac{\lambda}{\pi}-n)\pi}{\frac{\lambda}{\pi}-n}\right] \\
&= \left[\frac{(-1)^n\sin\lambda}{\frac{\lambda}{\pi}+n} + \frac{(-1)^n\sin\lambda}{\frac{\lambda}{\pi}-n}\right] \\
&= (-1)^n\frac{2\lambda\sin\lambda}{\lambda^2-\pi^2n^2}.
\end{aligned}$$

Since f is continuous and piecewise differentiable, the Fourier series of f converges pointwise to f and so for all $x \in \mathbb{R}$ we have

$$\cos \frac{\lambda x}{\pi} = \frac{\sin \lambda}{\lambda} + \sum_{n=1}^{\infty} (-1)^n \frac{2\lambda \sin \lambda}{\lambda^2 - \pi^2 n^2} \cos nx.$$

If we take $x = \pi$ and divide both sides by $\sin \lambda$ we get the partial fraction expansion for $\cot \lambda$:

$$\cot \lambda = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - n^2\pi^2}, \text{ for all } \lambda \in \mathbb{R} \setminus 2\pi\mathbb{Z}.$$

Let $\varepsilon \in (0, 2\pi)$. For $x \in [\varepsilon, 2\pi]$ we have

$$\begin{aligned} \int_{\varepsilon}^{x} (\cot \lambda - \frac{1}{\lambda})\, d\lambda &= \int_{\varepsilon}^{x} \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - n^2\pi^2}\, d\lambda, \\ &= -\sum_{n=1}^{\infty} \frac{2\lambda}{n^2\pi^2 - \lambda^2}\, d\lambda, \end{aligned}$$

since it follows easily from the M-test that the series $\sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - n^2\pi^2}$ is uniformly convergent on $[\varepsilon, x]$. Integrating, we see that

$$[\log(\sin \lambda) - \log \lambda]_{\lambda=\varepsilon}^{\lambda=x} = \sum_{n=1}^{\infty} \left[\log(n^2\pi^2 - \lambda^2)\right]_{\lambda=\varepsilon}^{\lambda=x}.$$

Evaluating, we obtain the identity

$$\log\left(\frac{\sin x}{x}\right) - \log\left(\frac{\sin \varepsilon}{\varepsilon}\right) = \sum_{n=1}^{\infty} \log\left(\frac{n^2\pi^2 - x^2}{n^2\pi^2 - \varepsilon^2}\right).$$

Letting $\varepsilon \to 0+$, we get

$$\log\left(\frac{\sin x}{x}\right) = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Exponentiating this expression and multiplying both sides by x gives

$$\sin x = x \Pi_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Remark 5.5.25. This proof only applies when $x \in \mathbb{R}$. The proof we gave in chapter 3 holds for $x \in \mathbb{C}$.

5.5.5. Failure of uniform convergence: Gibbs phenomenon. The *Gibbs phenomenon* is the appearance of quite large oscillations in the partial sums $S_n(x)$ to the left and right of a jump discontinuity. The resulting 'overshoot' in the partial sums does not die out as $n \to \infty$. We illustrate the phenomenon with an investigation of the convergence properties of the Fourier series of the square wave function.

The 2π-periodic square wave $F(x)$ defined in examples 5.5.9(2) does not satisfy the conditions of theorem 5.5.24 as F has discontinuities at integer multiples of 2π. We showed in example 5.5.11 that F had Fourier series

$$\mathcal{F}(F) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$$

and in examples 5.5.23(1) that the series converges pointwise to $F(x)$ except if x is an integer multiple of π. Since the pointwise limit of $\mathcal{F}$ is not continuous, convergence of $\mathcal{F}(F)$ cannot be uniform. On the other hand, a straightforward application of Dirichlet's test shows that convergence of $\mathcal{F}(F)$ is uniform on every closed interval $[a, b]$ which does not contain an integer multiple of 2π (see the section on Dirichlet and Abel's tests in chapter 4, especially examples 4.6.3(2)).

We have $S_n(F)(x) = \frac{4}{\pi}\sum_{j=1}^{n}\frac{1}{2j-1}\sin((2j-1)x)$. Taking $x = \frac{\pi}{2n}$, we compute that

$$\begin{aligned} S_n(F)(\frac{\pi}{2n}) &= \frac{4}{\pi}\sum_{j=1}^{n}\frac{1}{2j-1}\sin(\frac{(2j-1)\pi}{2n}), \\ &= 2\sum_{j=1}^{n}\frac{1}{n}G(\frac{2j-1}{2n}), \end{aligned}$$

where $G(0) = 1$ and $G(x) = \frac{\sin \pi x}{\pi x}$, $x \neq 0$. Now $\sum_{j=1}^{n}\frac{1}{n}G(\frac{2j-1}{2n})$ is an approximating Riemann sum to

$$\int_0^1 G(x)\,dx = \int_0^1 \frac{\sin \pi x}{\pi x}\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{\sin u}{u}\,du.$$

($\sum_{j=1}^{n}\frac{1}{n}G(\frac{2j-1}{2n})$ is the sum from $j = 0, \dots, n-1$ of the value of G at the mid-point of $[j/n, (j+1)/n]$ times the length of the interval — $1/n$.) Since G is continuous, we therefore have

$$\lim_{n\to\infty} S_n(F)(\frac{\pi}{2n}) = \frac{2}{\pi}\int_0^{\pi}\frac{\sin u}{u}\,du.$$

The integral $\int_0^{\pi}\frac{\sin u}{u}\,du$ may be computed numerically and has approximate value 1.8519. Hence $\lim_{n\to\infty} S_n(F)(\frac{\pi}{2n}) \approx 3.7038/\pi \approx 1.179$. The jump in F at $x = 0$ is equal to 2 and so we see that

$$\lim_{n\to\infty} S_n(F)(\frac{\pi}{2n}) \approx 1.179 \approx 1 + 0.0895 * 2.$$

In other words, the overshoot is (at least) 8.95% of the jump at $x = 0$. It turns this is a universal phenomena: wherever there is a jump discontinuity in a piecewise C^1-function, there will be an overshoot in the partial sums near the discontinuity and this overshoot in the limit is approximately 9% of the jump at the discontinuity. In the exercises we give another example where one can compute fairly easily a lower bound on the overshoot.

Remarks 5.5.26. (1) The ratio of the limiting overshoot to the size of the jump discontinuity is a universal constant approximately equal to 0.089490. The phenomenon was originally described (partly incorrectly) by Gibbs in 1848 and corrected by him later in 1898.
(2) It is work remarking that the rate at which the Fourier coefficients of a function f converge to zero depends on the smoothness of f. If f is C^∞ (or analytic) the coefficients decay very rapidly. If the function is only piecewise smooth then the series of coefficients is never absolutely convergent (else the M-test would imply convergence to a continuous function).

Exercises 5.5.27.

(1) Let f be continuous and 2π-periodic. Show that if all the Fourier coefficients of f are zero then $f \equiv 0$ (Hint: use the second Weierstrass approximation theorem).
(2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic with Fourier series $\mathcal{F}(f) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Suppose that (A) $\mathcal{F}(f)$ converges at one point of $\mathbb{R}$, (B) the series $\sum_{n=1}^{\infty}(-na_n \sin nx + nb_n \cos nx)$ is uniformly convergent on $\mathbb{R}$. Show that f is C^1 and that $\mathcal{F}(f)$ converges uniformly to f on $\mathbb{R}$. (Hints: $\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(f)$ and the result of (1) above).
(3) Define the continuous 2π-periodic function $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \pi - |x|,\ x \in [-\pi, \pi].$$

   (a) Sketch the graph of T.
   (b) Find the Fourier series of T.
   (c) Does the Fourier series converge pointwise to T? uniformly on $\mathbb{R}$ to T? Why/Why not?
(4) Define the piecewise continuous 2π-periodic function $S : \mathbb{R} \to \mathbb{R}$ on $[-\pi, \pi]$ by

$$\begin{aligned} S(x) &= x,\ x \in (-\pi, \pi), \\ S(\pm\pi) &= 0. \end{aligned}$$

   (a) Sketch the graph of S.
   (b) Find the Fourier series of S.
   (c) Does the Fourier series converge pointwise to S? uniformly on $\mathbb{R}$ to S?

If you have access to a program like *Maple* or *Matlab*, plot the graphs of the partial sums $S_{20}(S)$ and $S_{50}(S)$ over the range $[-3\pi, 3\pi]$ and estimate the overshoot as a percentage of the jump 2π.

(5) Show that the Fourier series of the 2π-periodic sawtooth function defined by

$$\begin{aligned} S(x) &= \frac{\pi - x}{2},\ x \in (0, 2\pi), \\ &= 0,\ x = 0, 2\pi \end{aligned}$$

is given by

$$\mathcal{F}(S) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

(a) Show the partial sums S_n of the Fourier series of S satisfy

$$S_n(x) = \frac{1}{2}\int_0^x D_n(t)\,dt - \frac{x}{2}.$$

(b) Using the approximation $\sin t \approx t$, for t small, deduce that there exists $C \geq 0$ (independent of n) such that

$$S_n(x) = \int_0^x \frac{\sin(n+\frac{1}{2})t}{t}\,dt + e_n(x)$$

where $|e_n(x)| \leq Cx$, $x \in (0, 2\pi)$.

(c) Take $x = \pi/(n+\frac{1}{2})$ in (b) and deduce that

$$S_n(\frac{\pi}{n+\frac{1}{2}}) = \int_0^{\pi} \frac{\sin u}{u}\,du + \alpha_n,$$

where $|\alpha_n| \leq c/n$. Using the approximate value 1.852 for $\int_0^{\pi} \frac{\sin u}{u}\,du$, deduce that $S_n(\pi/(n+\frac{1}{2})) \approx \pi/2 + 0.09 \times \pi$ for n large. That is, for large n the overshoot is (at least) 9% of the size of the jump at the discontinuity.

(6) Let $F : \mathbb{R} \to \mathbb{R}$ be the piecewise continuous 2π-periodic function defined on $[-\pi, \pi]$ by

$$\begin{aligned} F(x) &= 0,\ \text{if } x \in [-\pi, -\pi/2] \cup [\pi/2, \pi], \\ &= \pi,\ \text{if } x \in (-\pi/2, \pi/2). \end{aligned}$$

(a) Find the Fourier series of F.

(b) At what points of $[-\pi, \pi]$ does $\mathcal{F}(F)$ converge pointwise to F? If $\mathcal{F}(F)$ does not converge pointwise to F at x_0, what is $\mathcal{F}(F)(x_0)$? Justify your answer with reference to course work.

(c) Using (a,b), find $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$.

(7) Suppose that the 2π-periodic continuous function $f : \mathbb{R} \to \mathbb{R}$ has Fourier series $\mathcal{F}(f) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Assuming that (a) $\mathcal{F}(f)$ converges at at least one point, and (b) $\sum_{n=1}(-na_n \sin nx + nb_n \cos nx)$ is uniformly convergent, explain why the series $\mathcal{F}(f)$ converges (uniformly) to f on $\mathbb{R}$.

(8) Suppose $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and C^∞. Show that the Fourier coefficients of f decay faster than any power of $1/n$. Specifically, show that for each $m \geq 1$, there exists $C_m \geq 0$ such that $|a_n|, |b_n| \leq C_m n^{-m}$, for all $n \geq 1$. Conversely, show that if f is continuous and this condition holds then f is C^∞. (The decay is exponentially fast if f is analytic.)

5.6. Mean square convergence

So far we have focused on the question of whether or not the Fourier series of a continuous function f converges pointwise to f. It turn out that a more natural notion of convergence for Fourier series is *mean-square* or L^2-convergence: $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - S_n(f)(x)|^2\, dx = 0$. Although the full development of this theory depends on using a more sophisticated version of integration such as the Lebesgue integral, we can at least indicate why mean-square convergence is a natural concept for Fourier series.

Definition 5.6.1. Given continuous 2π-periodic functions $f, g : \mathbb{R} \to \mathbb{R}$, the *scalar* or *inner product* of f and g is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

We leave the proof of the next lemma to the exercises.

Lemma 5.6.2. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic.*

(1) $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$ *for all* $a, b \in \mathbb{R}$.
(2) $\langle f, g\rangle = \langle g, f\rangle$
(3) $\langle f, f\rangle \geq 0$ *and* $\langle f, f\rangle = 0$ *iff* $f = 0$.

Definition 5.6.3. Given a continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, define the *norm* of f by

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

Lemma 5.6.4. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic. We have*

(1) $\|f\| \geq 0$ *and* $\|f\| = 0$ *iff* $f = 0$.
(2) $\|af\| = |a|\|f\|$ *for all* $a \in \mathbb{R}$.
(3) $|\langle f, g\rangle| \leq \|f\|\|g\|$ *(Cauchy-Schwarz inequality).*
(4) $\|f + g\| \leq \|f\| + \|g\|$ *(triangle inequality).*

Proof. (1,2) are immediate from lemma 5.6.2. In order to prove (3) we shall use the necessary and sufficient condition $A, C \geq 0$ and $B^2 < AC$ for a quadratic form $Ax^2 + 2Bxy + Cy^2$ to be positive semi-definite. Let $x, y \in \mathbb{R}$. By lemma 5.6.2,

$$\begin{aligned}\langle xf + yg, xf + yg\rangle &= x^2\langle f, f\rangle + 2xy\langle f, g\rangle + y^2\langle g, g\rangle,\\ &= x^2\|f\|^2 + 2xy\langle f, g\rangle + y^2\|g\|^2.\end{aligned}$$

Since $\langle xf + yg, xf + yg\rangle \geq 0$, the quadratic form $x^2\|f\|^2 + 2xy\langle f, g\rangle + y^2\|g\|^2$ is positive for all $x, y \in \mathbb{R}$, and so we must have $\langle f, g\rangle^2 \leq \|f\|^2\|g\|^2$, proving (3). Finally we have $\|f + g\|^2 = \|f\|^2 + 2\langle f, g\rangle +$

$\|g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$ by (3). That is, $\|f+g\|^2 \leq (\|f\|+\|g\|)^2$, proving (4). □

DEFINITION 5.6.5. The continuous non-zero 2π-periodic functions $f, g : \mathbb{R} \to \mathbb{R}$ are *orthogonal* if $\langle f, g\rangle = 0$.

LEMMA 5.6.6. *The set*

$$1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots$$

of 2π-periodic functions are pairwise orthogonal.

PROOF. Use the orthogonality relations (lemma 5.5.12). □

LEMMA 5.6.7 (Pythagoras' theorem). *Suppose that $f_1, \ldots, f_n$ are pairwise orthogonal (that is $\langle f_i, f_j\rangle = 0$, $i \neq j$). Then*

$$\|f_1 + \ldots + f_n\|^2 = \|f_1\|^2 + \ldots + \|f_n\|^2.$$

PROOF. We have

$$\begin{aligned} \langle \sum_{i=1}^n f_i, \sum_{j=1}^n f_j\rangle &= \sum_{i,j=1}^n \langle f_i, f_j\rangle, \\ &= \sum_{i=1}^n \langle f_i, f_i\rangle, \\ &= \sum_{i=1}^n \|f_i\|^2. \end{aligned}$$

Since $\langle \sum_{i=1}^n f_i, \sum_{j=1}^n f_j\rangle = \|\sum_{i=1}^n f_i\|^2$, the result follows. □

We define a new distance function $D(f,g)$ on piecewise continuous 2π-periodic functions by

$$D(f,g) = \|f-g\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2\,dx}.$$

It follows from lemma 5.6.4 that $D(f,g)$ satisfies the usual properties of a distance function; in particular, the triangle inequality: $D(f,h) \leq D(f,g) + D(g,h)$.

Since $d(f,g) = \sup_{x\in[-\pi,\pi]} |f(x) - g(x)|$, we have

$$\begin{aligned} D(f,g) &= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2\,dx} \\ &\leq \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} d(f,g)^2, dx} \leq d(f,g), \end{aligned}$$

for all piecewise continuous 2π-periodic functions f, g.

EXAMPLE 5.6.8. In general, $D(f,g)$ may be much smaller than $d(f,g)$. For example, if we define

$$\begin{aligned} f_N(x) &= 1,\ x \in [-1/N, 1/N], \\ &= 0,\ x \in (-\pi, \pi] \setminus [-1/N, 1/N], \end{aligned}$$

then $d(f_N, 0) = 1$ for all $N \geq 0$ but $D(f_N, 0) = \sqrt{1/N\pi}$. ♠

PROPOSITION 5.6.9. *Let f be a piecewise continuous 2π-periodic function. For $n \geq 0$, let S_n denote the partial sum $a_0 + \sum_{j=1}^{n}(a_j \cos jx + b_j \sin jx)$ of the Fourier series of f. The infimum of $D(f,g)$ over all trigonometric polynomials of degree less than or equal to n is given by $D(f, S_n)$ and is attained only when $g = S_n$. Moreover*

$$\|f - S_n\|^2 = \|f\|^2 - (a_0^2 + \frac{1}{2}\sum_{j=1}^{n}(a_j^2 + b_j^2)).$$

PROOF. Let $T(x) = A_0 + \sum_{j=1}^{n}(A_j \cos jx + B_j \sin jx)$ be any trigonometric polynomial of degree at most n. We have

$$D(f,T)^2 = \|f - T\|^2 = \|(f - S_n) + (S_n - T)\|^2.$$

Now $f - S_n$ is orthogonal to $\cos jx, \sin jx$, $0 \leq j \leq n$ since, for example, if $j \leq n$,

$$\begin{aligned} \langle f - S_n, \cos jx \rangle &= \langle f, \cos jx \rangle - \langle S_n, \cos jx \rangle, \\ &= a_j - a_j = 0, \end{aligned}$$

by the orthogonality relations and the definition of a_j. It follows by Pythagoras' theorem (lemma 5.6.7) that

$$\|f - T\|^2 = \|f - S_n\|^2 + \|S_n - T\|^2,$$

and so $D(f,T)^2 = D(f,S_n)^2 + D(S_n,T)^2 \geq D(f,S_n)^2$ with equality iff $T = S_n$. The final statement follows taking $T = 0$. □

LEMMA 5.6.10. *Let f be a piecewise continuous 2π-periodic function. Given $\varepsilon > 0$, there exists a trigonometric polynomial T such that*

$$D(f,T) < \varepsilon$$

PROOF. If f is continuous, then by the second Weierstrass approximation theorem we can choose a trigonometric polynomial T such that $d(f,T) < \varepsilon$. But $D(f,T) \leq d(f,T)$ and so the result is proved if f is continuous. If f is piecewise continuous, we may choose a continuous 2π-periodic function g such that $D(f,g) < \varepsilon/2$ (we may require $f = g$ outside of small intervals containing the discontinuity points). As we did above, we may choose a trigonometric polynomial T such that $D(g,T) < \varepsilon/2$. Now $D(f,T) \leq D(f,g) + D(g,T) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

THEOREM 5.6.11. *Let f a piecewise continuous 2π-periodic function with Fourier coefficients a_n, b_n. Then*

(a) $D(f, S_n) \to 0$ *as* $n \to \infty$.
(b) $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

PROOF. Immediate from proposition 5.6.9 and lemma 5.6.10. □

REMARKS 5.6.12. (1) Statement (b) of theorem 5.6.11 is known as *Parseval's identity*.
(2) Theorem 5.6.11 suggests a natural inverse problem: Given sequences $(a_n), (b_n)$ such that $a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) < \infty$, does there exist a function f with Fourier coefficients a_n, b_n and which satisfies Parseval's identity? In order to give a satisfactory answer to the problem we have to expand the class of functions to allow for functions which may not be continuous anywhere on $\mathbb{R}$ but which are nevertheless square integrable. For this to make sense we need to work with a more powerful version of the integral that allows for functions which may have no points of continuity. All of this can be, and has been, done but lies beyond the scope of these notes.

5.7. Appendix: Second Weierstrass approximation theorem

In this appendix we prove theorem 5.5.5: every continuous 2π-periodic function on $f : \mathbb{R} \to \mathbb{R}$ can be uniformly approximated by trigonometric polynomials (the second Weierstrass approximation theorem).

Since f is 2π-periodic, it is enough to show that we can uniformly approximate f by trigonometric polynomials on $[-\pi, \pi]$. We break the proof into a number of lemmas.

LEMMA 5.7.1. *If $f : \mathbb{R} \to \mathbb{R}$ is even ($f(-x) = f(x)$, for all $x \in \mathbb{R}$), then we can uniformly approximate f by trigonometric polynomials.*

PROOF. Since f is even the values of f on $[-\pi, \pi]$ are uniquely determined by the values of f on $[0, \pi]$. Therefore it suffices to uniformly approximate f on $[0, \pi]$ by *even* trigonometric polynomials.

Define $g(t) = f(\cos^{-1} t)$, $t \in [-1, 1]$. Since $\cos^{-1} : [-1, 1] \to [0, \pi]$ is continuous, g is continuous on $[-1, 1]$. By the Weierstrass approximation theorem, we may uniformly approximate g on $[-1, 1]$ by polynomials. That is, given $\varepsilon > 0$, there exists $p \in P(\mathbb{R})$ such that

$$\sup_{t\in[-1,1]} |g(t) - p(t)| < \varepsilon. \tag{5.7}$$

Set $t = \cos x$, $x \in [0, \pi]$. We can rewrite (5.7) as $\sup_{x\in[0,\pi]} |g(\cos x) - p(\cos x)| < \varepsilon$. Since $g(\cos x) = f(\cos^{-1}(\cos x)) = f(x)$, we have

$$\sup_{x\in[0,\pi]} |f(x) - p(\cos x)| < \varepsilon.$$

Using standard trigonometric identities it is well-known (and easy) to show that every power of $\cos x$ can be written as linear combinations of $\cos jx$ (j positive integer). Hence $p(\cos x)$ can be written as a trigonometric polynomial with no sine terms:

$$p(\cos x) = a_0 + \sum_{j=1}^{n} a_j \cos jx.$$

This function is even and so we have uniformly approximated f on $[0, \pi]$ by an even trigonometric polynomial. $\square$

LEMMA 5.7.2. *If f is even, then $\sin^2 x f(x)$ can be uniformly approximated by trigonometric polynomials.*

PROOF. Using lemma 5.7.1, we first uniformly approximate f by trigonometric polynomials then we use standard trigonometric identities to obtain the required uniform approximations of $\sin^2 x f(x)$ by trigonometric polynomials. $\square$

LEMMA 5.7.3. *If f is odd ($f(-x) = -f(x)$) then $\sin x f(x)$ can be uniformly approximated by trigonometric polynomials.*

PROOF. Since f is odd, $g(x) = \sin x f(x)$ is even and so we may apply lemma 5.7.1. $\square$

LEMMA 5.7.4. *Every continuous function $f : \mathbb{R} \to \mathbb{R}$ may be written uniquely as a sum $f_{\mathrm{e}} + f_{\mathrm{o}}$ of even and odd continuous functions. If f is 2π-periodic, so are $f_{\mathrm{e}}, f_{\mathrm{o}}$.*

PROOF. Define $f_{\mathrm{e}}(x) = \frac{f(x)+f(-x)}{2}$, $f_{\mathrm{o}}(x) = \frac{f(x)-f(-x)}{2}$. $\square$

LEMMA 5.7.5. *If f is 2π-periodic then we can uniformly approximate $\sin^2 x f(x)$ by trigonometric polynomials.*

PROOF. Using lemmas 5.7.4 and 5.7.2, we reduce to the case when f is odd. Now apply lemma 5.7.3 to $\sin x f(x)$ and finally multiply by approximating trigonometric polynomials by $\sin x$ and apply the trigonometric identities $\sin x \cos jx = \frac{1}{2}(\sin(j+1)x - \sin(j-1)x)$ to obtain the required uniform approximations to $\sin^2 x f(x)$. $\square$

LEMMA 5.7.6. *If f is 2π-periodic then we can uniformly approximate $\sin^2 x f(\frac{\pi}{2} - x)$ by trigonometric polynomials.*

Proof. Apply lemma 5.7.4 to $\tilde{f}(x) = f(\frac{\pi}{2} - x)$. $\square$

Proof of Theorem 5.5.5. Taking $y = \frac{\pi}{2} - x$ in lemma 5.7.6, we see that $\cos^2 x f(x)$ can be uniformly approximated by trigonometric polynomials. Hence, by lemma 5.7.2, $\sin^2 x f(x) + \cos^2 x f(x) = f(x)$ can be uniformly approximated by trigonometric polynomials. $\square$