MATH 4332
Homework 7, due Monday, March 30th

Q1. Suppose that $E \subset X$, $\overline{E}$ is compact and $f : X \to Y$ is continuous. Show that $f(\overline{E}) = \overline{f(E)}$. (Hint: In Q4 of exam 2, you were asked to prove that $f(\overline{E}) \subset \overline{f(E)}$. Now you have to show that if $\overline{E}$ is compact then the reverse inclusion $f(\overline{E}) \supset \overline{f(E)}$ holds.)

Q2. Show that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected subsets of $\mathbb{R}$.

Q3. Find examples of subsets of $\mathbb{R}$ which are (a) compact, perfect, not totally disconnected, (b) compact, not perfect, totally disconnected, (c) not compact, perfect, totally disconnected.

Q4. Find all the points $x \in \mathbf{C}$ such that $T(x) = x$ (x is a *fixed point* of T). Find a point $x \in \mathbf{C}$ such that $T^2(x) = x$, but $T(x) \neq x$ (we call x a point of prime period two for T). Can you find a point $x \in \mathbf{C}$ which is of prime period three for T? (Hint: It is probably not a very good idea to try and directly solve $T^3(x) = x$. Use the ternary.)

Q5. If you construct a middle fifths Cantor set $\subset [0, 1]$ — the middle fifth interval of each closed subinterval is removed — what is the total length of all middle fifth intervals that are removed? Prove that the resulting set is compact, perfect and totally disconnected. More generally, define a Cantor set by removing middle $x$ths, starting with $[0, 1]$, $x \in (0, 1)$. Show that the total length of all the intervals removed is one. (Hint: this is much easier than you might expect; focus on finding the stuff removed at each step rather than on counting intervals etc.)