MATH 4332 Homework 10 solutions

Q1. Let $\{f_1, f_2\}$ be the IFS given by

$$
\begin{aligned}
f_1(x,y) &= \begin{pmatrix} 0.4000 & -0.3733 \\ 0.0600 & 0.6000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.3533 \\ 0.0000 \end{pmatrix}, \\
f_2(x,y) &= \begin{pmatrix} -0.8000 & -0.1867 \\ 0.1371 & 0.8000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1.1000 \\ 0.1000 \end{pmatrix}
\end{aligned}
$$

Verify that f_1 and f_2 are contractions. If you have access to a computer with Matlab or Mathematica, plot the resulting image you get with this IFS.

The fractal constructed using this IFS is shown in figure 1 (see also figure 2).

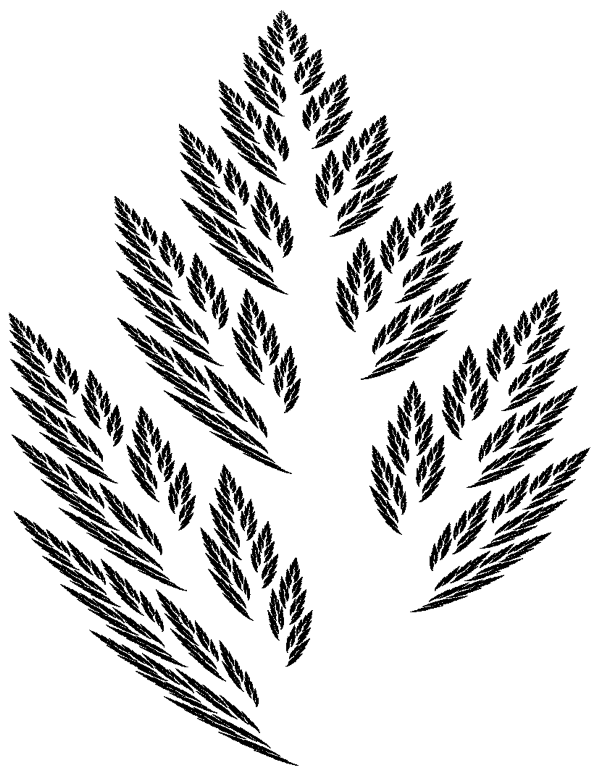

Figure 1: Fractal leaf

The proof that the maps f_1 and f_2 are contractions follows using lemma 8.3.1 of the notes. For example for f_1, we have $a^2 + c^2 = 0.1636 < 1$, $b^2 + d^2 = 0.49935289 < 1$ and $a^2 + b^2 + c^2 + d^2 = 0.66295289 < 1.06885271 = 1 + (ac - bd)^2$.

Q2. Find an explicit example of a smooth map $f : \mathbb{R} \to \mathbb{R}$ such that (a) $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathbb{R}$, $x \neq y$, and (b) f has no fixed point. Why does your example not contradict the contraction mapping lemma? (Hint: Look for an example of the form $f(x) = x + \phi(x)$ where $\phi(x) > 0$ for all $x \in \mathbb{R}$ and $-1 < \phi'(x) < 0$ for all $x \in \mathbb{R}$.)

We start by finding a map $\psi : \mathbb{R} \to \mathbb{R}$ such that $-1 < \psi(x) < 0$. For this we try $\psi(x) = -\frac{1}{2(1+x^2)}$ $(-\frac{1}{2} \leq \psi < 0)$. To get ϕ, we integrate ψ and choose the constant so that $\phi(x) > 0$

for all $x \in \mathbb{R}$. We find that

$$\phi(x) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$$

satisfies all of the required conditions. (Note: $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ is 1:1 onto and so ϕ is never zero.)
Another solution is $\phi(x) = 1/(1 + e^x)$.

We estimate $|f(x) - f(x')|$. By the Mean Value Theorem, we have for some $z \in [x, y]$,

$$|f(x) - f(x')| = |x - x'||f'(z)|,$$

But $f'(z) = 1 + \phi'(z)$ and so, since $\phi'(z) \in (-1, 0)$, we have $|f'(z)| < 1$. Hence f satisfies the estimate

$$|f(x) - f(x')| < |x - x'|,\ x, x' \in \mathbb{R}.$$

We claim f does not have a fixed point: $f(x) = x$ iff $x + \phi(x) = x$ iff $\phi(x) = 0$ — but $\phi(x) \neq 0$ for all $x \in \mathbb{R}$.

This example does not contradict the contraction mapping lemma as that requires a $k < 1$ such that $|f(x) - f(x')| \leq k|x - x'|$. (For the example, note that we can make $\phi'(z)$ close to zero by taking z large.)

Q3. Suppose that $\eta : \mathbb{R} \to \mathbb{R}$ is a contraction: $|\eta(x) - \eta(y)| \leq k|x - y|$, where $0 \leq k < 1$. Show that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \eta(x)$ is 1:1 onto and that the inverse map $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition $|f^{-1}(x) - f^{-1}(y)| \leq A|x - y|$ where $A = 1/(1 - k)$.

We follow the hints. For $y \in \mathbb{R}$, define $\Phi_y : \mathbb{R} \to \mathbb{R}$ by $\Phi_y(x) = x - f(x) + y$. Observe that $\Phi_y(x) = x$ iff $f(x) = y$. In other words, a fixed point of Φ_y gives a solution $x = x(y)$ to $f(x) = y$. This solution will determine the inverse map: $f^{-1}(y) = x$ — the fixed point of Φ_y.

We start by proving that $\Phi_y : \mathbb{R} \to \mathbb{R}$ is a contraction mapping. For $x, x' \in \mathbb{R}$ we have

$$|\Phi_y(x) - \Phi_y(x')| = |x - f(x) + y - (x' - f(x') + y))| = |\eta(x) - \eta(x')| \leq k|x - x'|,$$

where we used $f(x) = x + \eta(x)$ and the Lipschitz property of η. Since $k < 1$, we have shown that for all $y \in \mathbb{R}$, Φ_y is a contraction mapping. Let $\phi(y)$ denote the unique fixed point of Φ_y. Since Φ_y obviously satisfies the conditions of the contraction mapping lemma with parameters ($\Phi_y(x) = y - \eta(x)$ is continuous in y for fixed x and k does not depend on y), $\phi : \mathbb{R} \to \mathbb{R}$ is continuous.
We have shown that for every $y \in \mathbb{R}$, there exists x $(= \phi(y))$ such that $f(x) = y$. Hence f is onto. Moreover, f is 1:1 since if $f(x) = f(x') = y$ then $x = x' = \phi(y)$ by the uniqueness part of the contraction mapping lemma. It follows that $\phi = f^{-1}$ and f^{-1} is continuous. It remains to prove that $f^{-1} = \phi$ is Lipschitz.

For $y, y' \in \mathbb{R}$, we have

$$\begin{aligned} |\phi(y) - \phi(y')| &= |\Phi_y(\phi(y)) - \Phi_{y'}(\phi(y'))| \\ &= |\eta(\phi(y')) - \eta(\phi(y)) + y - y'| \\ &\leq |\eta(\phi(y')) - \eta(\phi(y))| + |y - y'| \\ &\leq k|\phi(y) - \phi(y')| + |y - y'|. \end{aligned}$$

Therefore, $(1-k)||\phi(y) - \phi(y')| \leq |y - y'|$ or $|\phi(y) - \phi(y')| \leq (1/(1-k))|y - y'|$.

Q4. Show that a metric space (X, d) is connected iff for every proper non-empty subset E of X, $\partial E \neq \emptyset$. (By 'proper' we mean $E \neq \emptyset, X$.)

METHOD 1: Suppose there exists a proper subset E of X such that $\partial E = \emptyset$. By definition of ∂E, this implies that for every $x \in E$, there exists an open neighborhood N_x of x such that $N \cap (X \setminus E) = \emptyset$. Let $U = \cup_{x \in E} N_x$. Then $U = E$ (since $N_x \subset E$ for all $x \in E$) is a non-empty open subset of X such that $U \supset E$ and $U \cap (X \setminus E) = \emptyset$. The same argument applied to $(X \setminus E)$, gives us a non-empty open subset $V = X \setminus E$ of X such that $V \supset (X \setminus E)$ and $V \cap E = \emptyset$. We have written X as a union of two disjoint non-empty open sets. Hence X is not connected. For the converse, suppose X is not connected. We can write $X = U \cup V$ where U, V are disjoint non-empty open sets. Now take $E = U$ and observe that $\partial E = \emptyset$.

METHOD 2: Use the course result $\partial E = \overline{E} \setminus \mathring{E}$. Noting that $\overline{E} \supset E \supset \mathring{E}$, we have $\partial E = \emptyset$ iff $\overline{E} = \mathring{E}$ iff E is open and closed. Hence if there is a proper subset E with $\partial E = \emptyset$, X is not connected (take $U = E$, $V = X \setminus E$).

Figure 2: Fractal leaf magnified — note image is inverted compared with figure 1