MATH 4332 Homework 9 solutions

Q1. Let E be a subset of the metric space X. Show that $E' = (\overline{E})'$. Using the result that the closure of a connected set is connected, deduce that if E is connected, then E' is connected.

Since $E \subset \overline{E}$, we have $E' \subset (\overline{E})'$. Suppose $x \in (\overline{E})'$. Since $(\overline{E})' \subset \overline{E}$ (by the general result that $A' \subset \overline{A}$), $x \in \overline{E}$ and x is not an isolated point of $\overline{E}$. It suffices to show that for every open neighborhood N of x, $N \cap E \supsetneq \{x\}$. Suppose the contrary. Then there exists an open neighborhood N of x such that $N \cap E = \{x\}$. This implies that x is an isolated point of E and therefore of $\overline{E}$. Contradiction.

Here is an alternative (easier) proof that uses the result $(A \cup B)' = A' \cup B'$. We have $\overline{E} = E \cup E'$ (coursework) and so $\overline{E}' = E' \cup (E')' = E'$, since $(E')' \subset E'$.

Suppose that E is connected. Then $\overline{E}$ is connected (course result). Suppose that E contains more than one point (else $E' = \emptyset$). By the first part of the question, it suffices to prove that $(\overline{E})' = \overline{E}$. Since $(\overline{E})' \subset \overline{E}$, equality fails only if $\overline{E}$ contains an isolated point. Since we assume that E connected and contains more than one point, E and $\overline{E}$ contain no isolated points and so $(\overline{E})' = \overline{E}$.

Q2. Find a contraction map $f : X \to X$, where $X = \mathbb{R} \setminus \{0\}$, which does not have a fixed point.

Take $f(x) = \frac{x}{2}$. Since $|f(x) - f(y)| = |x - y|/2$, f is a contraction mapping with contraction constant $k = 1/2$. Obviously f has no fixed point in $\mathbb{R} \setminus \{0\}$.

Q3. Consider the ODE $x' = x$. Fix $x \in \mathbb{R}$. Taking $\phi_0 : [-a, a] \to \mathbb{R}$ to be the constant map $\phi_0(t) = x$, compute the first three iterates ϕ_1, ϕ_2, ϕ_3 of $T\phi(t) = x + \int_0^t f(\phi(s))\, ds$, starting with $\phi = \phi_0$. Compare with the actual solution $x(\sum_{n=0}^{\infty} t^n/n!)$.

Define the iteration by $T\phi_n(t) = \phi_{n+1}(t) = x + \int_0^t f(\phi_n(s))\, ds$. If we take $\phi_0 \equiv x$, then

$$\phi_1(t) = x + \int_0^t f(\phi_0(s))\, ds = x + \int_0^t f(x)\, ds = x + \int_0^t x\, ds = x + |sx|_0^t = x + tx.$$

(Note that $f(x) = x$ — the ODE is $x' = f(x) = x$). Since $\phi_1(t) = x + tx$ we have

$$\phi_2(t) = x + \int_0^t x + sx\, ds = x + \left|sx + s^2x/2\right|_0^t = x + tx + t^2x/2.$$

Finally,

$$\phi_3(t) = x + \int_0^t x + sx + s^2x/2\, ds = x + tx + t^2x/2! + t^3x/3!.$$

The actual solution of $x' = x$ with initial condition x is $xe^t = x(1 + t + t^2/2! + t^3/3! + \cdots)$.

Q4. Let (X, d) be a metric space. A map $f : X \to X$ is an *expansion* if there exists $k > 1$ such that $d(f(x), f(y)) \geq kd(x, y)$ for all $x, y \in X$. Show (1) If $f : X \to X$ is an expansion and f has a fixed point, then the fixed point is unique. (2) If X is compact, then there are no expansions of X.

Note: we assume (or should have) that X has at least two points.
(1) Suppose $x^\star, y^\star$ are fixed points of f. Then

$$d(x^\star, y^\star) = d(f(x^\star), f(y^\star)) \geq kd(x^\star, y^\star).$$

Since $k > 1$, the only way this inequality can hold is if $x^\star = y^\star$. Therefore, if a fixed point exists it is unique.
(2) METHOD 1: Let $x, y \in X$, $x \neq y$. Then $d(f^n(x), f^n(y)) \geq k^n d(x, y)$, for all $n \geq 0$. Since $k > 1$, $(d(f^n(x), f^n(y)))$ is unbounded. But X is assumed compact and so X is bounded. In particular, there exists $C \geq 0$, such that $d(u, v) \leq C$ for all $u, v \in X$. Now choose n so that $k^n d(x, y) > C$.
METHOD 2: Let $D(X) = \sup_{x,y \in X} d(x, y)$. Since X is compact $D < \infty$. Choose $x^\star, y^\star \in X$ such that $d(x^\star, y^\star) = D(X)$ (use the compactness of X). Now $d(f(x^\star), f(y^\star)) \geq kd(x^\star, y^\star) > D(X)$. Contradiction.

Q5. Suppose that the metric space (X, d) is connected. Show that if $f : X \to \mathbb{R}$ is continuous and $a, b \in f(X)$, then f takes every value between a and b. Using this result, show that if X is *countable* then X is connected if and only if X consists of a single point.

Since f is continuous and X is assumed connected, $f(X) \subset \mathbb{R}$ is connected. Therefore, $f(X)$ is an interval. In particular, if $u, v \in f(X)$ then $[u, v] \subset f(X)$ (since $f(X)$ is an interval).

METHOD 1: Not using the suggestion (and much easier). Fix $x_0 \in X$ and define $f(x) = d(x, x_0)$. The map $f : X \to \mathbb{R}$ is continuous and, since X contains at least two points, f is not constant. Therefore by the first part, if X is connected, f must take non-countably many different values. Hence if X is connected and not a single point, X cannot be countable.

METHOD II: Following the suggestion. Suppose X is countable. Let $X = \{x_n \mid n \in \mathbb{N}\}$. For $a > 1$, define $f : X \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} a^{-n} d(x, x_n)/(1 + d(x, x_n))$. Observe that $d(x, x_n)/(1 + d(x, x_n)) < 1$ for all $n \in \mathbb{N}$ and so, since $a > 1$, the series is uniformly convergent (M-test) and so $f : X \to \mathbb{R}$ is continuous. For all $x \in X$, $f(x) = a^{-1} d(x, x_1)/(1 + d(x, x_1)) + \sum_{n=2}^{\infty} a^{-n} d(x, x_n)/(1 + d(x, x_n))$. It follows that $f(x) \geq a^{-1} d(x, x_1)/(1 + d(x, x_1))$. We also have the estimate $\sum_{n=2}^{\infty} a^{-n} d(x, x_n)/(1 + d(x, x_n)) \leq \sum_{n=2}^{\infty} a^{-n} = 1/a(a-1)$. Fix $z \in X$, $z \neq x_1$. Then $f(z) \geq a^{-1} d(z, x_1)/(1 + d(z, x_1)) = a^{-1} C > 0$. On the other hand, $f(x_1) \leq 1/a(a-1)$. Hence if we choose a sufficiently large $f(z) \geq a^{-1} C > 1/a(a-1) \geq f(x_1)$ and so f is not constant. Now if X is connected then $f(X)$ is connected. If f is not constant, then $f(X)$ contains an interval $[u, v]$, $u < v$. Hence f takes non-countably many values. Contradiction — X is countable and so any function on X takes at most countably many distinct values.

This result is not true for general topological spaces. For example, every set X has a topology consisting of X and $\emptyset$. With this topology, X is connected.