MATH 4332
Homework 10, due Monday, April 27th

Q1. Let $\{f_1, f_2\}$ be the IFS given by

$$\begin{aligned} f_1(x,y) &= \begin{pmatrix} 0.4000 & -0.3733 \\ 0.0600 & 0.6000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.3533 \\ 0.0000 \end{pmatrix}, \\ f_2(x,y) &= \begin{pmatrix} -0.8000 & -0.1867 \\ 0.1371 & 0.8000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1.1000 \\ 0.1000 \end{pmatrix} \end{aligned}$$

Verify that f_1 and f_2 are contractions. If you have access to a computer with Matlab or Mathematica, plot the resulting image you get with this IFS.

Q2. Find an explicit example of a smooth map $f : \mathbb{R} \to \mathbb{R}$ such that (a) $|f(x)-f(y)| < |x-y|$ for all $x, y \in \mathbb{R}$, $x \neq y$, and (b) f has no fixed point. Why does your example not contradict the contraction mapping lemma? (Hint: Look for an example of the form $f(x) = x + \phi(x)$ where $\phi(x) > 0$ for all $x \in \mathbb{R}$ and $-1 < \phi'(x) < 0$ for all $x \in \mathbb{R}$.)

Q3. Suppose that $\eta : \mathbb{R} \to \mathbb{R}$ is a contraction: $|\eta(x) - \eta(y)| \leq k|x - y|$, where $0 \leq k < 1$. Show that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \eta(x)$ is 1:1 onto and that the inverse map $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschtiz condition $|f^{-1}(x) - f^{-1}(y)| \leq A|x - y|$ where $A = 1/(1 - k)$. (Hint: For $y \in \mathbb{R}$, define $\Phi_y : \mathbb{R} \to \mathbb{R}$ by $\Phi_y(x) = x - f(x) + y$. Start by showing that for all $y \in \mathbb{R}$, Φ_y is a contraction mapping. You might also read through the second part of 7.9.2 in the notes.)

Q4. Show that a metric space (X, d) is connected iff for every proper non-empty subset E of X, $\partial E \neq \emptyset$. (By 'proper' we mean $E \neq \emptyset, X$.)