6/17 Classnotes www.math.uh.edu/~dblecher/4377h1.pdf
/4377h2.pdf.

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Last time: defined an <u>inner product</u>, <u>inner product space</u> (i.p., i.p.s.) This is a v.s. V with $\langle \cdot, \cdot \rangle$ satisfying 5 or 6 properties

Remarks:

(1) $\langle 0, w \rangle = 0$ (since $\langle \cdot, w \rangle$ linear in 1st variable)
$\implies \langle w, 0 \rangle = \overline{\langle 0, w \rangle} = 0$

(2) I.p. satisfies $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$, by taking $\overline{\langle v, w \rangle} = \langle w, v \rangle$ and $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$
$\implies \overline{\langle v_1 + v_2, w \rangle} = \overline{\langle v_1, w \rangle + \langle v_2, w \rangle}$ so by $\overline{\langle v, w \rangle} = \langle w, v \rangle$
we have $\langle w, v_1 + v_2 \rangle = \langle w, v_1 \rangle + \langle w, v_2 \rangle$

(3) $\langle v, cw \rangle = c \langle v, w \rangle$ <u>for</u> $c \in \mathbb{R}$
proof: $\langle v, cw \rangle = \overline{\langle cw, v \rangle} = c \overline{\langle w, v \rangle} = c \langle v, w \rangle$ but
$\langle v, cw \rangle = \bar{c} \langle v, w \rangle$ if $c \in \mathbb{C}$, by tiny modification of last proof.

Consequently, the i.p. is linear in 2nd variable if $\mathbb{F} = \mathbb{R}$, but if $\mathbb{F} = \mathbb{C}$, it is "<u>conjugate</u> <u>linear</u>".

<u>Norm</u>: $\|v\| \overset{\text{def.}}{=} \sqrt{\langle v, v \rangle} \quad \forall v \in V$

<u>note</u>: $\|v\| \geq 0$, and $\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0$

<u>fact</u>: $\|cv\| = |c| \, \|v\|$

proof: $\|cv\|^2 = \langle cv, cv \rangle = c\bar{c} \langle v, v \rangle = |c|^2 \|v\|^2 \overset{\sqrt{}}{=} \|cv\| = |c| \|v\|$

<u>consequence</u>: $\forall v \in V$, $v \neq 0$, we have $\frac{v}{\|v\|}$ has norm <u>1</u> (call it a unit vector) Process is called normalizing or scaling.

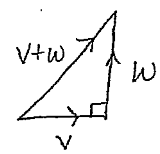
proof: $\left\| \frac{v}{\|v\|} \right\| = \left\| v \cdot \frac{1}{\|v\|} \right\| \overset{\text{fact}}{=} \frac{1}{\|v\|} \cdot \|v\| = \underline{1}$

Orthogonality - Say v is orthogonal (perpendicular) to w if $\langle v, w \rangle = 0$. We write $v \perp w$.

note: $v \perp w \Rightarrow w \perp v$ $\quad (\langle w, v \rangle = \overline{\langle v, w \rangle} = \overline{0} = 0)$

by remark (1), $0 \perp w \quad \forall w \in V$

Pythagorean Identity -

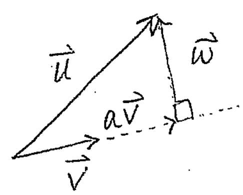


$\|v+w\|^2 = \|v\|^2 + \|w\|^2$ if $v \perp w$

proof:

$$\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v+w \rangle + \langle w, v+w \rangle =$$
$$\langle v, v \rangle + \cancel{\langle v, w \rangle} + \cancel{\langle w, v \rangle} + \langle w, w \rangle = \|v\|^2 + \|w\|^2 \quad \square$$

Orthogonal Decomposition -



Write $u = av + w$, where $w \perp v$, and a is a scalar.

Formula: $a = \dfrac{\langle u, v \rangle}{\|v\|^2}$, $\quad \vec{w} = u - av$

check to see if it works: we only need to check $(u - av) \perp v$, where $a = \dfrac{\langle u, v \rangle}{\|v\|^2}$ since if this holds, set $w = u - av \Longrightarrow$ $u = av + w$ as desired.

Check: $\langle u - av, v \rangle = \langle u, v \rangle - \langle av, v \rangle = \langle u, v \rangle - \dfrac{\langle u, v \rangle}{\cancel{\|v\|^2}} \cdot \cancel{\langle v, v \rangle} = 0$

So $v \perp (u - av)$

Lemma 1: (Cauchy-Schwarz Inequality) - $|\langle u, v \rangle| \leq \|u\| \, \|v\|$

Also $|\langle u, v \rangle| = \|u\| \, \|v\|$ if and only if (u, v) is linearly dependent.

proof:

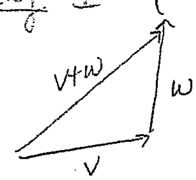
If $v = 0$ there is nothing to prove. So assume $v \neq 0$.

Orthogonal decomposition above: $u = av + w$, where $a = \dfrac{\langle u, v \rangle}{\|v\|^2}$

$w \perp v$. So by Pythagorean identity $\|u\|^2 = \|av\|^2 + \|w\|^2$
$\geq \|av\|^2 \overset{\text{fact}}{=} |a|^2 \|v\|^2$. So $\|u\| \geq |a|\,\|v\| = \dfrac{|\langle u,v\rangle|}{\|v\|^2}\,\|v\|$

multiply by $\|v\| \Longrightarrow \|u\|\,\|v\| \geq |\langle u,v\rangle|$
Changing $\geq$ to $=$, and reversing the argument, we see that $|\langle u,v\rangle| = \|u\|\,\|v\| \Longleftrightarrow \|u\|^2 = \|av\|^2 + \|w\|^2 = \|av\|^2 = |a|^2\|v\|^2$, which happens iff $\|w\|^2 = 0$ iff $w = 0 \Longleftrightarrow u = av + 0 \Longleftrightarrow (u,v)$ linearly dependent. $\square$

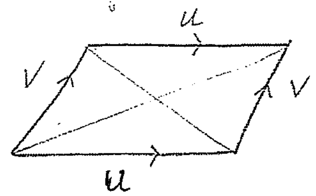
<u>Corollary 1</u>: (Triangle (Δ) Inequality) $\|v+w\| = \|v\| + \|w\|$



*also $\|v+w\| = \|v\| + \|w\|$ iff one of v and w is a positive scalar multiple of the other.

proof: $\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle$
$= \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} = \|v\|^2 + \|w\|^2 + 2\,\mathrm{Re}\,\langle v,w\rangle$
$\leq \|v\|^2 + \|w\|^2 + 2\,|\langle v,w\rangle| \overset{\text{Lemma 1}}{\leq} \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\| =$
$(\|v\| + \|w\|)^2$. Taking $\sqrt{\ }$: $\|v+w\| \leq \|v\| + \|w\|$.
Finally $\|v+w\| = \|v\| + \|w\|$ iff $|\langle v,w\rangle| = \|v\|\,\|w\| \Longrightarrow$ one (say v) is a scalar multiple of other:
$v = cw$, $c \in \mathbb{F}$. If $v = cw$ then $\|v+w\| = \|v\| + \|w\|$
$\Longleftrightarrow \|cw + w\| = \|cw\| + \|w\| \Longleftrightarrow \|(c+1)w\| = |c|\,\|w\| + \|w\|$
$\Longleftrightarrow |c+1|\,\cancel{\|w\|} = (|c|+1)\cancel{\|w\|}$ *assume $w \neq 0$
$\Longleftrightarrow |c+1| = |c| + 1 \overset{\text{square}}{\Longleftrightarrow} |c+1|^2 = |c|^2 + 2|c| + 1 \Longleftrightarrow$
$(c+1)(\bar{c}+1) = |c|^2 + 2\,\mathrm{Re}\,c + 1 = |c|^2 + 2|c| + 1 \Longleftrightarrow \mathrm{Re}\,c = |c|$
$\Longleftrightarrow c \geq 0$. Reversing the argument gives other half of 'iff'. $\square$

<u>Lemma 2</u>: (parallelogram identity) - $\|u+v\|^2 + \|u-v\|^2 = (\|u\|^2 + \|v\|^2)\,2$



proof: $\langle u+v, u+v\rangle + \langle u-v, u-v\rangle \iff \langle u, u+v\rangle + \langle v, u+v\rangle + \langle u, u-v\rangle - \langle v, u-v\rangle \iff \langle u,u\rangle + \cancel{\langle u,v\rangle} + \cancel{\langle v,u\rangle} + \langle v,v\rangle + \langle u,u\rangle - \cancel{\langle u,v\rangle} - \cancel{\langle v,u\rangle} + \langle v,v\rangle \iff 2\langle u,u\rangle + 2\langle v,v\rangle \iff 2\|u\|^2 + 2\|v\|^2$ $\square$

Definition: Say $(v_1, \dots, v_n)$ is orthonormal (o.n.) if $v_j \perp v_j$ $\forall i \neq j$, and $\|v_k\| = 1$ $\forall k = 1, \dots, n$.

example: (i, j, k) in $\mathbb{R}^3$, more generally, the standard basis of $\mathbb{R}^n$.

Slight generalization of Pythagorean identity: <u>Proposition 1</u>: if $(v_1, \dots, v_n)$ is an orthonormal then $\|\sum_{k=1}^{n} c_k v_k\| = \sqrt{\sum_{k=1}^{n} |c_k|^2}$ $\forall c_1, \dots, c_n \in \mathbb{F}$

proof: By induction

$n=1$: $\|c_1 v_1\| \overset{\text{fact}}{=} |c_1|\,\|v_1\| = |c_1| = \sqrt{|c_1|^2}$. Suppose true ~~for $n=k$. Then for $n=k+1$,~~

$$\cancel{\left\|\sum_{k=1}^{n+1} c_k v_k\right\| = \left\|\sum_{k=1}^{n} c_k v_k + c_{n+1} + v_{n+1}\right\|}$$

$$\cancel{= \|a w + c_{n+1} v_{n+1}\|}$$

~~where~~ $\cancel{w = \dfrac{\sum_{k=1}^{n} c_k v_k}{\left\|\sum_{k=1}^{n} c_k v_k\right\|}}$, ~~and~~ $\cancel{a = \left\|\sum_{k=1}^{n} c_k v_k\right\|}$

~~claim: (w, v_{n+1}) is o.n.~~

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6/18 Last time: some changes:

Suppose true for $k=n$. Then for $k=n+1$,

$\left\| \sum_{i=1}^{n+1} c_i v_i \right\|^2 = \left\| \sum_{i=1}^{n} c_i v_i + c_{n+1} v_{n+1} \right\|^2 = \left\| \sum_{i=1}^{n} c_i v_i \right\|^2 +$

$\| c_{n+1} v_{n+1} \|^2$ by Pythagorean Identity $=$

$\sum_{i=1}^{n} |c_i|^2 + |c_{n+1}|^2$ by induction hypothesis.

Throughout rest of this chapter, V is f.d. i.p.s., unless we say otherwise.

Definition: a list $(v_1, \ldots, v_n)$ is called an orthonormal (o.n.b) basis for V if it is orthonormal and is a basis for V.

example: Standard basis for $\mathbb{R}^n$ (eg (i,j,k) in $\mathbb{R}^3$)

Note: by the last result any orthonormal list of length $\dim(V)$ is an o.n.b.

Theorem 1: If $(v_1, \ldots, v_n)$ is an orthonormal basis for V then $\boxed{v = \sum_{k=1}^{n} \langle v, v_k \rangle v_k}$ $\forall v \in V$

also $\|v\|^2 = \sum_{k=1}^{n} |\langle v, v_k \rangle|^2$.

proof:

Let $v \in V$, then $\exists\, c_k \in \mathbb{F}$ s.t. $v = \sum_k c_k v_k$. So $\langle v, v_j \rangle =$

$\langle \sum_k c_k v_k, v_j \rangle = \sum_k c_k \langle v_k, v_j \rangle = c_j \langle v_j, v_j \rangle = c_j \|v_j\|^2 = c_j$

Hence $v = \sum_j c_j v_j = \sum_j \langle v, v_j \rangle v_j$. Second assertion follows from the first and Prop. 1.

Lemma 3: (Gram-Schmidt procedure)

If $(u_1, \ldots, u_n)$ is L.I in V, then $\exists$ orthonormal set $(v_1, \ldots, v_n)$ s.t. $\text{Span}(u_1, \ldots, u_j) = \text{Span}(v_1, \ldots, v_j)$ $\forall j = 1, \ldots, n$.

proof:

Let $v_1 = \frac{u_1}{\|u_1\|}$. Suppose at the end of step $j-1$ that $(v_1, \ldots, v_{j-1})$ has been found is orthonormal and satisfies spanning condition involving $u_1, \ldots, u_{j-1}$. Step j: first set $w_j = u_j - \sum_{k=1}^{j-1} \langle u_j, v_k \rangle v_k$ and let $v_j = \frac{w_j}{\|w_j\|}$. Note: if $i < j$ then $\langle w_j, v_i \rangle = \langle u_j - \sum_{k=1}^{j-1} \langle u_j, v_k \rangle v_k, v_i \rangle =$ $\langle u_j, v_j \rangle - \sum_{k=1}^{j-1} \langle \langle u_j, v_k \rangle v_k, v_i \rangle = \langle u_j, v_i \rangle - \langle u_j, v_i \rangle \langle v_i, v_i \rangle = 0$

Hence $\langle v_j, v_i \rangle = \langle \frac{w_j}{\|w_j\|}, v_i \rangle = \frac{1}{\|w_j\|} \langle w_j, v_i \rangle = 0$. So $(v_1, \ldots, v_j)$ are orthonormal. Note: $w_j \neq 0$, since if $w_j = 0$ then $u_j \in \text{span}(v_1, \ldots, v_{j-1}) \underset{\text{hypothesis}}{\overset{\text{Induction}}{=}} \text{span}(u_1, \ldots, u_{j-1})$ which is impossible since $(u_1, \ldots, u_j)$ are L.I.

Note: $w_j \in \text{span}(u_1, \ldots, u_j)$ by the definition of w_j, and because $v_k \in \text{span}(u_1, \ldots, u_{j-1})$ by the induction hypothesis. Thus $v_j \in \text{span}(u_1, \ldots, u_j)$, so $\text{span}(v_1, \ldots, v_j) \subseteq \text{span}(u_1, \ldots, u_j)$. Since all the u's, v's are all L.I. these spans are two V.S. of dimension j, so they are equal by HW Ch.2 #11. This completes the step j. So keep iterating till step n. $\square$

Corollary 3: Every f.d. i.p.s. has an o.n.b.

proof:

Take a basis and apply Gram-Schmidt (Lemma 3) to get an o.n. set which has the same length $= \dim V$. This set is L.I. by Cor. 2, so its a basis by Ch.2 Prop. 7. $\square$

Corollary 4: Every o.n. list in V is a subset of an o.n.b. for V.

proof:

If $(u_1,\dots,u_m)$ is orthonormal, it is L.I by Cor. 2 so $\exists\ v_1,\dots,v_n \in V$ s.t. $(u_1,\dots,u_m,v_1,\dots v_n)$ is a basis for V, by Ch. 3 Thm. 3. Apply Gram-Schmidt to this list. It is easy to see that the algorithm does not change the u vectors at all. (ie $w_2 = u_2 - \langle u_2,u_1\rangle u_1 = u_2 = v_2$) So we produce an orthonormal set $u_1,\dots,u_m,d_1,\dots,d_n$. This is the same number of elements as the earlier basis for V (ie $\dim V$ elements) and since they are L.I. (by Cor. 2), they are a basis by Ch. 2 prop. 7. $\square$

Corollary 5: If $T\in\mathcal{L}(V)$ and $\mathcal{M}(T,B,B)$ is upper Δ with respect to some basis B, then $\mathcal{M}(T,C,C)$ is upper Δ with respect to some o.n.b. C.

proof:

If $B=(u_1,\dots,u_n)$ apply Gram-Schmidt to get $C=(v_1,\dots,v_n)$ an o.n.b. By Prop. 1 Ch. 5, $T(u_j)\in \operatorname{span}(u_1,\dots,u_j)$ $\forall j=1,\dots,n$. By Gram-Schmidt lemma, $\operatorname{span}(u_1,\dots,u_j)=\operatorname{span}(v_1,\dots,v_j)$ and T and so $v_j = \sum_{k=1}^{j} c_k u_k$ then $T(v_j) = \sum_{k=1}^{j} c_k T(v_k) \in \operatorname{span}(u_1,\dots,u_j) = \operatorname{span}(v_1,\dots,v_j)$. By Prop. 1 Ch. 5, $\mathcal{M}(T,C,C)$ is upper Δ. $\square$

Corollary 6: If $T\in\mathcal{L}(V)$ then $\exists$ o.n.b. s.t. matrix of T with respect to this basis is upper Δ.

proof:

Corollary 5 + Ch. 5 Thm. 3. $\square$

Orthogonal Projections: Let $U\subseteq V$ and define $U^{\perp}$ "perp" $=\{v\in V : v\perp u\ \ \forall u\in U\}$

facts:

- $V^{\perp}=(0)$ $[v\in V^{\perp} \Rightarrow \langle v,v\rangle = 0 \Rightarrow v=0]$

- $(0)^\perp = V$
- $U_1 \subseteq U_2 \subseteq V \implies U_2^\perp \subseteq U_1^\perp$
- $U \subseteq (U^\perp)^\perp$
- $U^\perp$ is a subspace of V

  proof: $0 \in U^\perp$, $c \in F$ $v, w \in U^\perp$ then $\langle cv + w, u \rangle =$
  $c\langle v, u\rangle + \langle w, u \rangle = c0 + 0 = 0$ if $u \in U$.
- $\langle u, v\rangle = \langle w, v \rangle \ \forall v \in V \Rightarrow u = w$ [Proof: $\langle u-w, v\rangle = 0 \ \forall v \in V$, so $u - w \in V^\perp = (0)$]

<u>Theorem 2</u>: If U is a subspace of V then $V = U \oplus U^\perp$

proof:

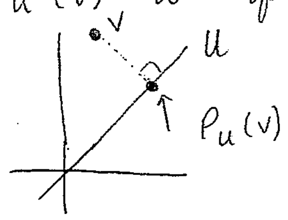
If $v \in U \cap U^\perp$ then $\langle v, v \rangle = 0 \implies v = 0$. Thus $U \cap U^\perp = (0)$. Let $(u_1, \dots, u_n)$ be an o.n.b. for U, and extend it by Cor. 4 to an o.n.b. $(u_1, \dots, u_n, v_1, \dots, v_m)$ for V. Then write any $v = \left(\sum_{k=1}^n \langle v, u_k\rangle\right) + \left(\sum_{k=1}^m \langle v, v_k\rangle v_k\right)$ by Theorem 1. The first parenthesis is in U. The second parenthesis is in $U^\perp$, since $v_k \in U^\perp$. (since $\langle \sum^n c_j u_j, v_k\rangle = \sum c_j \langle u_j, v_k \rangle = 0$) So $V = U + U^\perp$, so $V = U \oplus U^\perp$. □

Corollary 7: If U is a subspace of V then $U = (U^\perp)^\perp$

proof:

Already said $U \subseteq (U^\perp)^\perp$. Let $v \in (U^\perp)^\perp$, by Thm. 2. $\exists\, u \in U, w \in U^\perp$ s.t. $v = u + w$. Hence $w = v - u \in (U^\perp)^\perp$ (since both v and u live in $(U^\perp)^\perp$) On the other hand $w \in U^\perp$, then $w = 0$ ($\langle w, w\rangle = 0$). So $v = u \in U$ so $(U^\perp)^\perp \subseteq U$ so these sets are equal. □

Definition: If U is a subspace of V, the <u>orthogonal projection</u> P_U of V onto U is defined by $P_U(v) = u$ if $v = u + w$ with $u \in U$, $w \in U^\perp$



Lemma 4: (1) $P_U(v) = \sum_{k=1}^{n} \langle v, u_k \rangle u_k \quad \forall v \in V$ if $(u_1, \ldots, u_n)$ is an o.n.b. for U.

(2) $P_U \in \mathcal{L}(V)$

(3) $\operatorname{Ran} P_U = U$

(4) $\operatorname{Ker} P_U = U^\perp$

(5) $P_U^{\,2} = P_U$

(6) $\|P_U(v)\| \leq \|v\| \quad \forall v \in V$

(1) proof: follows from proof of Thm. 2 (wrote $v = (\checkmark) + (\ \ )$)

(2) proof: follows from (1) eg. $P_U(cv) = \sum_{k=1}^{n} \langle cv, u_k \rangle u_k = c \sum_{k} \langle v, u_k \rangle u_k = c P_U(v)$

(3) proof: Clearly $\operatorname{Ran} P_U \subseteq U$, but if $u \in U$, then $P_U(u) = u$ so $u \in \operatorname{Ran}(P_U)$

(5) proof: by proof of (3) if $P_U(v) = u$ then $P_U^{\,2}(v) = P_U(u) = u = P_U(v)$.

(4) proof: $v \in \ker(P_U) \iff P_U(v) = 0 \iff v = w \in W$ (see def. of P_U)

(6) proof: by Pythagorean identity $\|v\|^2 = \|u+w\|^2 = \|u\|^2 + \|w\|^2 \geq \|u\|^2 = \|P_U(v)\|^2$ where u, w are as in def. of P_U above. $\square$

<u>Proposition 2</u>: If P_U is as above and $v \in V$ then $\|v - P_U(v)\| \leq \|v - u\| \quad \forall u \in U$ (This is saying that $P_U(v)$ is the closest point in U to v.

proof: $\|v - P_U(v)\|^2 \leq \|\underbrace{v - P_U(v)}_{\in U^\perp}\|^2 + \|\underbrace{P_U(v) - u}_{\in U}\|^2$ and

$v - P_U(v) = v - u = w \in U^\perp$ in the def. of P_U. By the Pythagorean $= \|v - \cancel{P_U(v)} + \cancel{P_U(v)} - u\|^2$

$= \|v - u\|^2$ $\square$

6/19 <u>Linear functionals</u> are elements of $\mathcal{L}(V, \mathbb{F})$, V any v.s. over $\mathbb{F}$. (eg. $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $f(x,y,z) = 3x - 2y + z$)

Example: $f(p) = \int_0^1 p\,dx$ for $p \in \mathcal{P}(\mathbb{R})$

Example: $f(g) = \int_0^1 g(x) \sin x\,dx$ for $g \in V = \{f : [0,1] \to \mathbb{R}$ continuous$\}$

(4) There are n 'special' linear functionals on $\mathbb{F}^n$, defined by $\pi_k\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = x_k$. In $\mathbb{R}^3$ for example $\pi_2\left(\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}\right) = -3$.

* Note that these have the property that if $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ is Standard Basis then $\pi_i(\vec{e}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

*An important fact for general v.s's, V is that if $(v_1, \ldots, v_m)$ is L.I. in V, $\exists$ $(f_1, \ldots, f_m) \subseteq \mathcal{L}(V, \mathbb{F})$ s.t. $f_i(v_j) = \delta_{ij}$ as in (4) above.

proof: assume $B = (v_1, \ldots, v_m)$ is a basis
Let $S(v) = [v]_B$ in notation above Ch.3 Prop. 6, for $v \in V$ and let $f_i \stackrel{\text{def.}}{=} \pi_i \circ S$, where π_i is as above, Check $f_i(v_j) = \delta_{ij}$

(5) Example: Let V be an i.p.s. and $w \in V$ and define $f(v) = \langle v, w \rangle$, for $\forall v \in V$.

Theorem 3: If V is a f.d. i.p.s. then every linear functional on V is of the form $f(v) = \langle v, w \rangle$ $\forall v \in V$. (This w is unique)

proof:
If f is the functional and $(v_1, \ldots, v_n)$ is an o.n.b. then $\forall v \in V$, $f(v) = f\left(\sum_{k=1}^{n} \langle v, v_k \rangle v_k\right) = \sum_{k=1}^{n} \langle v, v_k \rangle f(v_k) = \langle v, \sum_{k=1}^{n} \overline{f(v_k)} v_k \rangle$ and let $w = \sum_{k=1}^{n} \overline{f(v_k)} v_k$. If $\langle v, w \rangle = \langle v, u \rangle$

$\forall v \in V$ then $\langle v, w-u\rangle = \langle v, w\rangle - \langle v, u\rangle = 0$ $\forall v \in V \implies w-v=0$ (by setting $v = w-u$) So, $w = u$.

Henceforth, let V, W be f.d. nonzero i.p.s.'s over $\mathbb{F}$.

Definition: The <u>adjoint</u> T^* of $T \in \mathcal{L}(V,W)$ is defined as follows; Fix $w \in W$:

- Noticing the function on V defined by $f(v) = \langle T(v), w\rangle$ is linear (= composition of T with Example 5 above).

  By Thm. 3, $\exists$ a unique $u \in W$ s.t. $f(v) = \langle T(v), w\rangle = \langle v, u\rangle$ $\forall v \in V$. Define $T^*(w) = u$.

Key Formula: $\boxed{\langle Tv, w\rangle = \langle v, T^*w\rangle \quad \forall v \in V, w \in W}$

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (y+3z, 2x)$

Find $T^*: \mathbb{R}^2 \to \mathbb{R}^3$.

soln: $\langle T(x,y,z), \underbrace{(v,w)}_{\in \mathbb{R}^2}\rangle = \langle (y+3z, 2x), (v,w)\rangle = (y+3z)v + 2xw$

$= 2xw + yv + 3zv = \langle (x,y,z), (2w, v, 3v)\rangle \implies$

$T^*(v,w) = (2w, v, 3v)$

Proposition 3: $\forall\, S, T \in \mathcal{L}(V,W)$, we have

1) $T^* \in \mathcal{L}(W,V)$
2) $(S+T)^* = S^* + T^*$
3) $(cT)^* = \bar{c}T^*$ $\forall c \in \mathbb{F}$
4) $(T^*)^* = T$
5) $I_V^* = I_V$
6) $(RT)^* = T^*R^*$ if $R \in \mathcal{L}(W,Z)$ where Z is another i.p.s.

proof:

HOMEWORK

For example, to see 3) $\langle cT(v), w\rangle = c\langle T(v), w\rangle =$

$c\langle v, T^*w\rangle = \langle v, \bar{c}T^*w\rangle \implies (cT)^* = \bar{c}T^*$. $\square$

Proposition 4: If $T \in \mathcal{L}(V,W)$ then
(a) $\ker(T^*) = (\operatorname{Ran} T)^{\perp}$
(b) $\operatorname{Ran}(T^*) = \ker(T)^{\perp}$
(c) $\ker(T) = \operatorname{Ran}(T^*)^{\perp}$
(d) $\operatorname{Ran}(T) = \ker(T^*)^{\perp}$

proof:
(a) $w \in \ker(T^*) \iff T^*(w) = 0 \iff \langle v, T^* w\rangle = 0 \quad \forall v \in V$
$\iff \langle Tv, w\rangle = 0 \quad \forall v \in V \iff w \in \operatorname{Ran}(T)^{\perp}$
Take $\perp$ of (a): $\ker(T^*)^{\perp} = (\operatorname{Ran}(T)^{\perp})^{\perp} = \operatorname{Ran}(T) \Longrightarrow$ (d). Switching T and T^* in (a) and (d) gives (b) and (c). $\square$

Definition: The conjugate transpose A^* of a matrix $A = [a_{ij}]$ is $[\overline{a_{ji}}]$.

Example: $\begin{bmatrix} 1 & i \\ 2 & 3i \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ -i & -3i \end{bmatrix}$

Proposition 5: If $T \in \mathcal{L}(V,W)$ and $\mathcal{B}, \mathcal{C}$ are o.n.b. for V and W respectively, then $\mathcal{M}(T,\mathcal{B},\mathcal{C})^* = \mathcal{M}(T^*, \mathcal{C}, \mathcal{B})$

proof:
If $\mathcal{B} = (u_1, \dots, u_n)$, $\mathcal{C} = (f_1, \dots, f_m)$ then by Thm. 1 we have $Tu_j = \sum_{i=1}^{m} \langle Tu_j, f_i\rangle f_i \Longrightarrow \mathcal{M}(T,\mathcal{B},\mathcal{C}) = [\langle Tu_j, f_i\rangle]$

Also $T^* f_j \overset{\text{Thm. 1}}{=} \sum_{i=1}^{n} \langle T^* f_j, u_i\rangle u_i = \sum_{i=1}^{n} \overline{\langle u_i, T^* f_j\rangle} u_i =$

$\sum_{i=1}^{n} \overline{\langle Tu_i, f_j\rangle} u_i$ so $\mathcal{M}(T^*, \mathcal{C}, \mathcal{B}) = [\overline{\langle Tu_i, f_j\rangle}] = \mathcal{M}(T,\mathcal{B},\mathcal{C})^*$ $\square$

CHAPTER 7: Operators on inner product spaces

Throughout this chapter, V is a nonzero f.d. i.p.s. over $\mathbb{F}$.

Definition: $T \in \mathcal{L}(V)$ is <u>selfadjoint</u> if $T = T^*$

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x,y) = (2x+3y,\ 3x+7y)$
Show $T = T^*$

soln: compute $\mathcal{M}(T)$ with respect to standard basis
$\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$ So $\mathcal{M}(T)^* = \mathcal{M}(T)$. By Prop. 5 Ch. 6,
$\mathcal{M}(T^*) = \mathcal{M}(T)$ so $\theta(T^*) = \theta(T)$; θ as in Ch. 3 Thm. 2
$\implies T^* = T$ since θ is one-to-one.

Proposition 1: Eigenvalues of a selfadjoint $T \in \mathcal{L}(V)$ are real.

proof:
Let $v \neq 0$, $Tv = \lambda v$, $\implies \lambda \|v\|^2 = \lambda \langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, T^* v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle = \bar{\lambda}\|v\|^2$. Dividing by $\|v\|^2$, $\lambda = \bar{\lambda}$ so $\lambda \in \mathbb{R}$. □

Proposition 2: If V is an i.p.s. over $\mathbb{C}$, $T \in \mathcal{L}(V)$ with $\langle Tv, v\rangle = 0 \ \forall v \in V$, then $T = 0$.

proof:
$$\langle Tu, w\rangle = \frac{\langle T(u+w)\rangle - \langle T(u-w), u-w\rangle}{4} + i\,\frac{\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle}{4}$$
$= 0 \quad \forall v, w$
$\implies Tu \in V^\perp = (0) \implies Tu = 0 \quad \forall u \implies T = 0$ □

Corollary 1: If V is an i.p.s. over $\mathbb{C}$, $T \in \mathcal{L}(V)$, then $T = T^* \iff \langle Tv, v\rangle \in \mathbb{R} \ \forall v \in V$

proof:
$\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = \langle Tv, v\rangle - \overline{\langle v, T^* v\rangle} = \langle Tv, v\rangle - \langle T^* v, v\rangle = \langle (T - T^*)v, v\rangle \quad \forall v \in V$. So $\langle Tv, v\rangle \in \mathbb{R} \ \forall v \in V \iff 0 = \langle (T-T^*)v, v\rangle \ \forall v \in V \overset{\text{Prop. 2}}{\iff} T - T^* = 0 \iff T = T^*$ □

Definition: $T \in \mathcal{L}(V)$ is <u>normal</u> if $TT^* = T^*T$

- Every selfadjoint T is normal. (but converse is false)
  * $T(x,y) = (y,0)$ is not normal.

Proposition 3: $T \in \mathcal{L}(V)$ is normal if and only if
$\|Tv\| = \|T^*v\| \quad \forall v \in V.$

proof:
$\|Tv\|^2 = \|T^*v\| \ \forall v \iff \langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle \iff$
$\langle T^*Tv, v \rangle = \langle TT^*, v \rangle \ \forall v \iff \langle (T^*T - TT^*)v, v \rangle = 0$
$\forall v \in V \overset{\text{Prop 2}}{\iff} T^*T - TT^* = 0 \iff T$ normal. $\square$

Corollary 2: If $T \in \mathcal{L}(V)$ is normal with eigenvector v corresponding to e-value λ, then v is an e-vector corresponding to e-value $\bar{\lambda}$.

proof:
We can assume V is an i.p.s. over $\mathbb{C}$. If T is normal, so is $T - \lambda I$.
$(T-\lambda I)^*(T-\lambda I) = (T^* - \bar{\lambda} I^*)(T - \lambda I) = TT^* - \bar{\lambda} T - \lambda T^* + \lambda\bar{\lambda} I$
$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda} T - \lambda T^* + \lambda\bar{\lambda} I$
which are the same. Since $(T-\lambda I)v = 0$, $\|(T-\lambda I)^* v\| \overset{\text{Prop. 3}}{=} \|(T-\lambda I)v\| = 0$. So $(T^* - \bar{\lambda} I)v = 0 \implies T^*v = \bar{\lambda} v$
$\implies v$ e-vector corresponding to $\bar{\lambda}$. $\square$

Corollary 3: $T \in \mathcal{L}(V)$ is normal and if v, w are e-vectors corresponding to two distinct e-values, then $v \perp w$.

proof:
Suppose $Tv = \alpha v$, $Tw = \beta w$, $\alpha \neq \beta$, then $(\alpha - \beta)\langle u, v \rangle =$
$\alpha \langle v, w \rangle - \beta \langle v, w \rangle = \langle \alpha v, w \rangle - \langle v, \bar{\beta} w \rangle \overset{\text{Cor. 2}}{=} \langle Tv, w \rangle - \langle v, T^*w \rangle$
$= \langle Tv, w \rangle - \langle Tv, w \rangle = 0$

Divide by $\alpha - \beta$: $\langle v, w \rangle = 0 \quad \square$

<u>Theorem 1</u>: The Spectral Theorem – $\mathbb{C}$ version)

Let V be an i.p.s. over $\mathbb{C}$, $T \in \mathcal{L}(V)$, the following are equivalent

(1) T is normal

(2) $\exists$ o.n.b. for V with respect to which $\mathcal{M}(T)$ is diagonal.

(3) $\exists$ o.n.b. for V consisting entirely of e-vectors of T

(4) $T = \sum_{k=1}^{n} \lambda_k P_k$, where P_k are orthogonal projections with $P_i P_j = 0$ if $i \neq j$.

In (4) $\lambda_1, \ldots, \lambda_n$ are the distinct e-values and P_k is the orthogonal projection onto e-space E_{λ_k}.

proof:

(3) implies (2) $\rightarrow$ obvious by definition of $\mathcal{M}(T)$

(2) implies (1) if $\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & \lambda_n \end{bmatrix}$, then by Prop. 5 Ch. 6. $\mathcal{M}(T^*) = \mathcal{M}(T)^* = \begin{bmatrix} \bar{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \bar{\lambda}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\lambda}_n \end{bmatrix}$ and so clearly $\mathcal{M}(T)\mathcal{M}(T^*) = \mathcal{M}(T^*)\mathcal{M}(T)$. By another result in Ch. 3 we get $\mathcal{M}(TT^*) = \mathcal{M}(T^*T)$ so in language of Ch. 3 Thm 2 $\theta(TT^*) = \theta(T^*T) \Longrightarrow TT^* = T^*T$ $\Longrightarrow T$ is normal.

(1) implies (3). Suppose T is normal, by Ch. 6 Cor. 6 $\exists$ o.n.b. $(u_1, \ldots, u_n)$ with respect to which $\mathcal{M}(T)$ is upper triangular: $\mathcal{M}(T) = \begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ 0 & a_{22} & & \\ \vdots & 0 & a_{33} & \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

$\|Tu_1\|^2 = \|a_{11}u_1\|^2 = |a_{11}|^2 \cdot 1 \overset{\text{Prop. 3}}{=} \|T^*(u_1)\|^2 = \sum_{k=1}^{n} |a_{1k}|^2$ since $\mathcal{M}(T^*) = \mathcal{M}(T)^* = \begin{bmatrix} \bar{a}_{11} & 0 & \cdots & 0 \\ \bar{a}_{12} & \bar{a}_{22} & & \vdots \\ \vdots & & \ddots & \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{bmatrix} \Longrightarrow |a_{1k}|^2 = 0 \quad \forall k = 2, \ldots, n$

$\Longrightarrow a_{1k} = 0 \quad \forall k = 2, \ldots, n.$

$\|Tu_2\|^2 = |a_{22}|^2 \overset{\text{Prop. 3}}{=} \|T^* u_2\| = \sum_{k=2}^{n} |a_{2k}|^2 \Longrightarrow a_{2k} = 0 \quad \forall k = 3, 4, \ldots, n.$

Similarly, $a_{kj} = 0 \quad \forall j > k$. So $\mathcal{M}(T)$ is diagonal.

(4) implies (1) : $T^*T = \left(\sum_{k=1}^{n} \lambda_k P_k\right)^* \left(\sum_{j=1}^{n} \lambda_j P_j\right)$

$$= \left(\sum_{k=1}^{n} \bar{\lambda}_k P_k\right)\left(\sum_{j=1}^{n} \lambda_j P_j\right)$$

$$= \sum_{k=1}^{n} \bar{\lambda}_k \lambda_k P_k^2$$

$$= \sum_{k=1}^{n} |\lambda_k|^2 P_k$$

and a similar argument shows this also equals TT^*.
(3) implies (4) : Let $(u_1, \ldots, u_m)$ be an o.n.b. of e-vectors and suppose $u_1, \ldots, u_{k_1}$ correspond to e-value λ_1, $u_{k+1}, \ldots, u_{k_2}$ correspond to e-value λ_2, etc. Let P_j be projection onto e-space for λ_j. $T(v) = T\left(\sum_{k=1}^{m} c_k u_k\right) = \sum_{k=1}^{n} c_k T(u_k) = \sum_{k=1}^{m} c_k \lambda_{jk} u_k$ where λ_{jk} is e-value for e-vector u_k. Similarly,

$\left(\sum_{j=1}^{n} \lambda_j P_j\right)\left(\sum_{k=1}^{m} c_k u_k\right) = \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j c_k P_j(u_k) = \sum_{k=1}^{m} c_k \lambda_{jk} u_k$. So

$T = \sum_{j=1}^{n} \lambda_j P_j$. $\square$

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HW. Ch.6 to be graded

Axler Q4 if $\|u\| = 3$, $\|u+v\| = 4$, $\|u-v\| = 6$, what's $\|v\|$?

(10) $P_2(\mathbb{R})$ with $\langle f, g \rangle = \int^1 f\, g\, dx$ apply Gram-Schmidt to $\{1, x, x^2\}$ to find an o.n.b.

(14) Find an o.n.b. for $P_2(\mathbb{R})$ s.t. 'differentiation' has matrix upper Δ.

(27) $T(z_1, \dots, z_n) = (0, z_1, z_2, \dots, z_{n-1})$ Find $T^*: \mathbb{R}^n \to \mathbb{R}^n$

(32) $A \in M_{m \cdot n}(\mathbb{R})$, show dim span(columns of A)
= " " (rows of A)

Fact I should have emphasized earlier (use often silently)

- In an i.p.s. V, if $\langle u, v \rangle = \langle w, v \rangle \ \forall v \in V \implies u = w$.

proof:
$\langle u-w, v \rangle = 0 \ \forall v \in V \implies u-w \in V^\perp = (0) \implies u-w=0 \implies u=w$. $\square$

Add a line of explanation to end of last proof
($\mathbb{C}$ spectral theorem) $P_j(u_k) \neq (0)$ iff u_k is the j^{th} eigenspace $E_{\lambda_j} \iff \lambda_j = \lambda_{j_k}$ in the notation of proof above so $\lambda_j P_j(u_k) = \lambda_{j_k} u_k$. $\square$

for selfadjoint operators)

Theorem 2: (The ~~Real~~ Spectral theorem (works even if $\mathbb{F} = \mathbb{R}$)) If V is an i.p.s. over $\mathbb{R}$ and $T \in \mathcal{L}(V)$ then TFAE:

(i) T is selfadjoint ($T = T^*$)

(ii) $\exists$ o.n.b. of V with respect to which $\mathcal{M}(T)$ is diagonal with real entries.

(iii) All e-values of T are real and $\exists$ o.n.b. for V consisting of e-vectors of T.

(iv) $T = \sum_{k=1}^{n} \lambda_k P_k$ where P_k are orthogonal projections with $P_i P_j = 0$ if $i \neq j$ and $\lambda_k \in \mathbb{R}$ $\forall k = 1, \ldots, n$.

In (iv) we can take $\lambda_1, \ldots, \lambda_n$ to be the distinct e-values of T, and P_k projection onto eigenspace E_{λ_k} $\forall k = 1, \ldots, n$.

proof:

(iii) $\Longrightarrow$ (ii) obvious as in Thm. 1

(ii) $\Longrightarrow$ (i) essentially just as in proof of Thm. 1, with slight adjustments. Assuming (ii) we have $\mathcal{M}(T) = \mathcal{M}(T)^* = \mathcal{M}(T^*) \Longrightarrow \theta(T) = \theta(T^*) \Longrightarrow T = T^*$

(iii) $\Longrightarrow$ (iv) as in Thm. 1.

(iv) $\Longrightarrow$ (iii) $T^* = (\sum \lambda_k P_k)^* = \sum \bar{\lambda}_k P_k^* \overset{\text{HW}}{=} \sum \lambda_k P_k = T$

(i) $\Longrightarrow$ (iii) omitted. See text. $\square$