Notes on probability, etc.
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Main idea:

$(\Omega, \mathcal{F}, P),\ P(\Omega) = 1$ a probability space.

If picking n points, $\{\omega_{n,k}\}_{k=1}^n$ "at random" from Ω, so all $\omega_{n,k} \in \Omega$, then the following will be true

$$\lim_{n\to\infty} \frac{\#\{k \in \{1, 2, ..., n\}; \omega_{n,k} \in E\}}{n} = P(E),\ \text{for all } E \in \mathcal{F},$$

where # is the counting measure.

Nonsense:

$\alpha : \mathbb{N} \to \Omega$, $n \in \mathbb{N}$, onto, but not one to one. Define $\#_n = \frac{1}{n}\#$, $N = \{1, 2, ..., n\}$. Then $(N, \mathcal{P}(N), \#_n)$ is a porobability space. Let $\alpha_n = \alpha|_N$, then this is an Ω valued random varaible. Now we can define

$$P(E) = \lim_{n\to\infty} \#_n \alpha_n^{-1}(E),\ \text{for all } E \in \mathcal{F}$$

Random varaibles:

$(S, \mathcal{S})$ a measurable space, $X : \Omega \to S$ is called a random variable when it is $(\mathcal{F}, \mathcal{S})$-measurable.

Define $P_X : \mathcal{S} \to [0, +\infty]$ by $P_X(E) = P(\{\omega \in \Omega; X(\omega) \in E\})$, this is the image measure by X.

$$P_X(S) = P(\{\omega \in \Omega; X(\omega) \in S\}) = P(\Omega) = 1$$

So the image measure induced by a random variable is a probability measure on its state space.

P_X is called the dirstribution of X.

Define $F_X : \mathbb{R} \to [0, 1] = x \mapsto P_X((-\infty, x])$, this is called the cumulative distribution function.

From wikipedia:
"The probability density function of a random variable is the RadonNikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables)."
ADD many details here

Expectation:

Define the expectation value of X as $E(X) = \int_\Omega X\, dP$, the integral of X.

Suppose X is a simple function, then $X(\omega) = \sum_{k=1}^n c_k \chi_{E_k}(\omega)$, $c_k \in S$, unique, and $E_k \in \mathcal{F}$ disjoint.

$$E(X) = \sum_{k=1}^n c_k P(E_k)$$

Discrete rv:

A discrete random variable X is one whose state space is countable. In this case there is a bijective map, $\gamma : S \to \mathbb{N}$, and clearly the function $\gamma \circ X$ is $(\mathcal{F}, \mathcal{P}(\mathbb{N}))$-measurable.
We may write $S = \{x_k := \gamma^{-1}(k)\}_{k=1}^{\infty}$, and may define $E_k := X^{-1}(x_k)$, $X(\omega) = \sum_{k=1}^{\infty} x_k \chi_{E_k}$.
If we temporarily adopt the notation "$p(x_k) = P(X = x_k)$"$:= P(E_k)$, then

$$E(X) = \sum_{k \in \mathbb{N}} x_k p(x_k)$$

In this simple case Ω may not really be nescesary, as $(\{x_k\}, \mathcal{P}(\{x_k\}), p)$ is a probability space in it's own right, and note, with $\beta_n := X \circ \alpha_n$, α_n as in the above nonsense,

$$p(x_k) = \lim_{n \to \infty} \#_n \beta_n^{-1}(x_k), \text{ for all } x_k$$

Conditional Expectation

$(X, \mathcal{A}, \mu)$ a σ-finite measure space, and $(X, \mathcal{F})$ a measurable space, $\mathcal{F} \subset \mathcal{A}$. For any $f : X \to \mathbb{R}$, $\mathcal{A}$-measurable, $h : X \to \mathbb{R}$ is the conditional expectation of f with respect to $\mathcal{F}$ if it is $\mathcal{F}$-measurable and for all $A \in \mathcal{F}$,

$$\int_A f \, d\mu = \int_A h \, d\mu,$$

we write $E[f|\mathcal{F}]$ for the conditional expectation of f w.r.t. $\mathcal{F}$.

If in addition, $f \in L^1(X, \mathcal{A}, \mu)$, then $\nu \in \mathcal{M}(X, \mathcal{A})$, where $\nu(E) := \int_E f \, d\mu$. $(X, \mathcal{F}, \mu|_{\mathcal{F}})$ is a measure space, $\nu|_{\mathcal{F}} \in \mathcal{M}(X, \mathcal{F})$, and $\nu|_{\mathcal{F}} \ll \mu|_{\mathcal{F}}$, then by the RNT, there exists a unique $h \in L^1(X, \mathcal{F}, \mu|_{\mathcal{F}})$ such that $\nu(E) = \int_E h \, d\mu|_{\mathcal{F}}$.

$$E[f|\mathcal{F}] = \frac{d\nu}{d\mu|_{\mathcal{F}}}, \text{ where } \nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{A},\ f \in L^1(X, \mathcal{A}, \mu),\ \mathcal{F} \subset \mathcal{A}.$$

Remarks:

1. For $(X, \mathcal{A}, \mu)$ a measure space, $f, g : X \to \mathbb{R}$, $\mathcal{A}$-measurable. Then $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \mathcal{A} \Rightarrow f = g$, $\mu-$a.e. Proof: let $h = f - g$, $A^+ = h^{-1}([0, \infty])$, $A^- = h^{-1}([-\infty, 0))$. Then $h \geq 0$ on A^+ so $\int_{A^+} h \, d\mu = 0 \Rightarrow h = 0$ μ-a.e. on A^+ by the vanishing principle. Similarly, $h^- = 0$ μ-a.e. on A^-.

2. In (1), what is really needed for the result is that the integrals of f and g agree on $(f - g)^{-1}(\mathbb{R}^+) \in \mathcal{A}$ and $(f - g)^{-1}(\mathbb{R}^-) \in \mathcal{A}$.

3. Let h be the conditional expectation of f w.r.t. $\mathcal{F}$, with $F \subset \mathcal{A}$, $\Delta := f - h$. h is $\mathcal{F}$-measurable, so is $\mathcal{A}$-measurable, so Δ is too, and so $\Delta^{-1}(\mathbb{R}^{+,0}) \in \mathcal{A}$, and $\Delta^{-1}(\mathbb{R}^-) \in \mathcal{A}$. By (2), if in addition, $\Delta^{-1}(\mathbb{R}^{+,0}) \in \mathcal{F}$, and $\Delta^{-1}(\mathbb{R}^-) \in \mathcal{F}$, then $f = h$ $\mu-$a.e., so in some sense, the larger $\mathcal{F}$ is, the closer an approximation h is of f.