MATH 4332 Homework 7 solutions

Q1. Suppose that $E \subset X$, $\overline{E}$ is compact and $f : X \to Y$ is continuous. Show that $f(\overline{E}) = \overline{f(E)}$.

Since f is continuous and $\overline{E}$ is compact, $f(\overline{E})$ is compact (a continuous image of a compact set is compact). Therefore $f(\overline{E})$ is closed (compact sets are closed). Since $f(\overline{E}) \supset f(E)$ (as $\overline{E} \supset E$), $f(\overline{E}) \supset \overline{f(E)}$ (the closure of a set is the smallest closed set containing the set). By Q4 of exam 2, we know that $f(\overline{E}) \subset \overline{f(E)}$ always. Hence if $\overline{E}$ is compact, $f(\overline{E}) = \overline{f(E)}$.

Q2. Show that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected subsets of $\mathbb{R}$.

Since every open interval contains rational and irrational points, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ have empty interiors and hence are totally disconnected.

Q3. Find examples of subsets of $\mathbb{R}$ which are (a) compact, perfect, not totally disconnected, (b) compact, not perfect, totally disconnected, (c) not compact, perfect, totally disconnected.

(a) $[a, b]$.
(b) $\{0, 1\}$.
(c) $\cup_{n \in \mathbb{Z}}\, n + \mathbf{C}$. (ie $\{x + c \mid n \in \mathbb{Z}, x \in \mathbf{C}\}$)

Q4. Find all the points $x \in \mathbf{C}$ such that $T(x) = x$ (x is a *fixed point* of T). Find a point $x \in \mathbf{C}$ such that $T^2(x) = x$, but $T(x) \neq x$ (we call x a point of prime period two for T). Can you find a point $x \in \mathbf{C}$ which is of prime period three for T?

The only fixed points of $T : I_0 \to I_0$ are $x = 0$ and $x = 3/4$ (solve $Tx = x$).
A point of prime 2 is given by $x = 3/10$).
The point with ternary expansion $0.\overline{220}$ (12/13)has period 3 for T:

$$0.220220220\cdots \mapsto 0.02002002\cdots \mapsto 0.2002002\cdots \mapsto .2202202\cdots$$

(Other solutions are $9/13 = Tx$, and $3/13 = T^2x$).
There is also another point with prime period 3: $y = \frac{3}{28}$. This has ternary expansion $y = 0.00\overline{222000}$. Also of period 3 are $Ty = \frac{9}{28}$ and $T^2y = \frac{27}{28}$.

Q5. If you construct a middle fifths Cantor set $\subset [0, 1]$ — the middle fifth interval of each closed subinterval is removed — what is the total length of all middle fifth intervals that are removed? Prove that the resulting set is compact, perfect and totally disconnected. More generally, define a Cantor set by removing middle $x$ths, starting with $[0, 1]$, $x \in (0, 1)$. Show that the total length of all the intervals removed is one.

At step 1, we remove x from I_0, leaving $(1 - x)$. At step 2 we remove $x(1 - x)$, leaving $(1 - x) - x(1 - x) = (1 - x)^2$. Proceeding inductively, at step n we remove $x(1 - x)^{n-1}$ leaving $(1 - x)^n$. So total amount removed is

$$\sum_{n=0}^{\infty} x(1-x)^n = x\sum_{n=0}^{\infty}(1-x)^n = x\frac{1}{1-(1-x)} = x\frac{1}{x} = 1.$$