CHAPTER 2

First Order Differential Equations

Introduction

It follows from the discussion in Section 1.3 that any first order differential equation can be written as

$$F(x, y, y') = 0$$

by moving all nonzero terms to the left hand side of the equation. To be a first order differential equation, y' must appear explicitly in the expression F. Our study of first order differential equations requires an additional assumption, namely that the equation can be solved for y'. This means that we can write the equation in the form

$$y' = f(x, y). \tag{$*$}$$

The study of first order differential equations is somewhat like the treatment of techniques of integration in Calculus II. There we had a variety of ”integration methods” (integration by parts, trigonometric substitutions, partial fraction decomposition, etc.) which were applied according to the form of the integrand. In this chapter we will learn some solution methods for $(*)$ which will correspond to the form of the function f.

2.1 Linear Differential Equations

A first order differential equation $y' = f(x, y)$ is a linear equation if the function f is a "linear expression" in y. That is, the equation is linear if the function f has the form

$$f(x, y) = P(x)y + q(x).$$

(c.f. the linear function $y = mx + b$.)

The solution method for linear equations is based on writing the equation as

$$y' - P(x)y = q(x) \qquad \text{which is the same as} \qquad y' + p(x)y = q(x)$$

where $p(x) = -P(x)$. We will focus on the latter form. The precise definition of a linear equation that we will use is:

FIRST ORDER LINEAR DIFFERENTIAL EQUATION: The first order differential equation $y' = f(x, y)$ is a *linear equation* if it can be written in the form

$$y' + p(x)y = q(x) \tag{1}$$

where p and q are continuous functions on some interval I. Equations that are not linear are called *nonlinear differential equations.*

Example 1. Some examples of linear equations are:

(a) $y' = 2xy$. Written in the form (1):

$$y' - 2xy = 0,$$

where $p(x) = -2x,\ q(x) = 0$ are continuous functions on $(-\infty, \infty)$.

(b) $y' + 2xy = x$ is already in the form (1): $p(x) = 2x,\ q(x) = x$ continuous on $(-\infty, \infty)$.

(c) $xy' + 2y = \dfrac{e^{3x}}{x}$. Written in the form (1):

$$y' + \frac{2}{x}\,y = \frac{e^{3x}}{x^2},$$

where $p(x) = 2/x,\ q(x) = e^{3x}/x^2$ are continuous functions on any interval that does not contain 0. For example, $(0, \infty)$ or $(-\infty, 0)$ each satisfy this requirement.

(d) $(1 - x^2)y' + 2xy = x^2 - 1$. Written in the form (1):

$$y' + \frac{2x}{1 - x^2}\,y = -1,$$

where $p(x) = \dfrac{2x}{1 - x^2},\ q(x) = -1$ are continuous on any interval that does not contain 1 and -1. For example, $(-1, 1)$ or $(1, \infty)$ each satisfy this requirement. ■

Solution Method for First Order Linear Equations:

Step 1. Identify: Can you write the equation in the form (1): $y' + p(x)\,y = q(x)$? If yes, do so.

Step 2. Calculate

$$h(x) = \int p(x)\,dx$$

(omitting the constant of integration) and form $e^{h(x)}$.

Step 3. Multiply the equation by $e^{h(x)}$ to obtain

$$e^{h(x)}y' + e^{h(x)}\,p(x)\,y = e^{h(x)}\,q(x).$$

(Note: Since $e^{h(x)} \neq 0$, this equation is equivalent to the original equation; no solutions have been added.)

Verify that the left side of this equation is $\left[e^{h(x)}\,y\right]'$.

Thus we have

$$\left[e^{h(x)}\,y\right]' = e^{h(x)}\,q(x).$$

Step 4. The equation in Step 3 implies that

$$e^{h(x)}\,y = \int e^{h(x)}\,q(x)\,dx + C$$

and

$$y = e^{-h(x)}\left[\int e^{h(x)}\,q(x)\,dx + C\right] = e^{-h(x)}\int e^{h(x)}\,q(x)\,dx + Ce^{-h(x)}.$$

Therefore, the general solution of (1) is:

$$y = e^{-h(x)}\int e^{h(x)}\,q(x)\,dx + Ce^{-h(x)} \tag{2}$$

where $h(x) = \int p(x)\,dx$.

INTEGRATING FACTOR: The key step in solving $y' + p(x)y = q(x)$ is multiplication by $e^{h(x)}$. It is multiplication by this factor, called an *integrating factor*, that enables us to write the left side of the equation as a derivative (the derivative of the product $e^{h(x)}y$) from which we get the general solution in Step 4. ■

EXISTENCE AND UNIQUENESS: Obviously solutions of first order linear equations exist; you've just seen how to construct the general solution. It follows from Steps (3) and (4) that the general solution (2) represents all solutions of the equation (1). As you will see, if an initial condition is specified, then the constant C will be uniquely determined. Thus, a first order linear initial-value problem will have a unique solution. This will also be confirmed by a Theorem in Section 2.4.

Example 2. Find the general solution of

$$y' + 2xy = x.$$

SOLUTION

(1) The equation is linear; it is already in the form (1); $p(x) = 2x$, $q(x) = x$ are continuous functions on $(-\infty, \infty)$.

(2) Calculate the integrating factor:

$$h(x) = \int 2x\,dx = x^2 \qquad \text{and} \qquad e^{h(x)} = e^{x^2}.$$

(3) Multiply the equation by the integrating factor:

$$\begin{aligned} e^{x^2} y' + 2x\, e^{x^2} y &= x\, e^{x^2} \\ \left[e^{x^2} y\right]' &= x\, e^{x^2} \qquad \text{(verify this)} \end{aligned}$$

(4) Integrate:

$$e^{x^2} y = \int x\, e^{x^2}\, dx = \tfrac{1}{2}\, e^{x^2} + C$$

and

$$y = e^{-x^2}\left[\tfrac{1}{2}\, e^{x^2} + C\right] = \tfrac{1}{2} + C\, e^{-x^2}.$$

Thus, $y = \frac{1}{2} + C\, e^{-x^2}$ is the general solution of the equation. ■

Example 3. Find the general solution of

$$xy' + 2y = \frac{e^{3x}}{x}.$$

SOLUTION

(1) After dividing the equation by x, we get

$$y' + \frac{2}{x}\, y = \frac{e^{3x}}{x^2}.$$

The equation is linear; $p(x) = 2/x$, $q(x) = e^{3x}/x^2$, continuous functions on $(0, \infty)$.

(2) Calculate the integrating factor:

$$h(x) = \int 2/x\, dx = 2\, \ln\, x = \ln\, x^2 \qquad \text{and} \qquad e^{h(x)} = e^{\ln\, x^2} = x^2.$$

(3) Multiply by the integrating factor:

$$\begin{aligned} x^2\, y' + 2x\, y &= e^{3x} \\ \left[x^2\, y\right]' &= e^{3x} \qquad \text{(verify this)} \end{aligned}$$

(4) Integrate:

$$x^2\, y = \int e^{3x}\, dx = \tfrac{1}{3}\, e^{3x} + C$$

and

$$y = \frac{e^{3x}}{3x^2} + \frac{C}{x^2}.$$

Thus, $y = \dfrac{e^{3x}}{3x^2} + \dfrac{C}{x^2}$ is the general solution of the equation. ■

Example 4. Find the solution of the initial-value problem

$$x^2\, y' - x\, y = x^4 \cos 2x, \quad y(\pi) = 2\pi.$$

SOLUTION The first step is to find the general solution of the differential equation. After dividing the equation by x^2, we obtain

$$y' - \frac{1}{x}\, y = x^2 \cos 2x, \tag{$*$}$$

a linear equation with $p(x) = -1/x$ and $q(x) = x^2 \cos 2x$, continuous functions on $(0, \infty)$.

Set $h(x) = \int (-1/x)\, dx = -\ln x = \ln x^{-1}$. Then $e^{h(x)} = e^{\ln x^{-1}} = x^{-1}$.

Multiplying $(*)$ by x^{-1} we get

$$x^{-1}y' - x^{-2}y = x \cos 2x \quad \text{which is} \quad \left[x^{-1}y\right]' = x \cos 2x. \quad \text{(verify this)}$$

It now follows that

$$x^{-1}y = \int x \cos 2x\, dx + C = \tfrac{1}{2}\, x \sin 2x + \tfrac{1}{4} \cos 2x + C. \quad \text{(integration by parts)}$$

Thus, the general solution of the differential equation is

$$y = \tfrac{1}{2}\, x^2 \sin 2x + \tfrac{1}{4}\, x \cos 2x + Cx.$$

We now apply the initial condition:

$$\begin{aligned} y(\pi) = 2\pi \quad \text{implies} \quad \tfrac{1}{2}\, \pi^2 \sin 2\pi + \tfrac{1}{4}\, \pi \cos 2\pi + C\pi &= 2\pi \\ \tfrac{1}{4}\pi + C\pi &= 2\pi \\ C &= \tfrac{7}{4} \end{aligned}$$

The solution of the initial-value problem is $y = \frac{1}{2}\, x^2 \sin 2x + \frac{1}{4}\, x \cos 2x + \frac{7}{4}\, x.$ ■

A Special Case:

There is a special case of equation (1) which will be useful later. If $q(x) = 0$ for all $x \in I$, then (1) becomes

$$y' + p(x)y = 0. \tag{3}$$

Following our solution procedure, we multiply this equation by $e^{\int p(x)\, dx}$, to obtain

$$e^{\int p(x)\, dx}\, y' + p(x)\, e^{\int p(x)\, dx}\, y = 0$$

which is the same as

$$\left[e^{\int p(x)\,dx}\,y\right]' = 0.$$

It follows from this that

$$e^{\int p(x)\,dx}\,y = C \qquad \text{and} \qquad y = C\,e^{-\int p(x)\,dx}.$$

Let $y = y(x)$ be a solution of (3). Since $e^{-\int p(x)\,dx} \neq 0$ for all x, we can conclude that:

(1) If $y(a) = 0$ for some $a \in I$, then $C = 0$ and $y(x) = 0$ for all $x \in I$; i.e., $y \equiv 0$.

(2) If $y(a) \neq 0$ for some $a \in I$, then $C \neq 0$ and $y(x) \neq 0$ for all $x \in I$. In fact, since y is continuous, $y(x) > 0$ for all x if $C > 0$; $y(x) < 0$ for all x if $C < 0$.

Therefore, if y is a solution of (3), then either $y \equiv 0$ on I, or $y(x) \neq 0$ for all $x \in I$.

Final Remarks:

1. The general solution (2) of a first order linear differential equation involves two integrals

$$h(x) = \int p(x)\,dx \quad \text{and} \quad \int f(x)e^{h(x)}\,dx.$$

It will not always be possible to carry out the integration steps as we did in the preceding examples. Even simple equations can lead to integrals that cannot be calculated in terms of elementary functions. In such cases you will either have to leave your answer in the integral form (2) or apply some type of numerical approximation method. ■

Example 5. Consider the linear equation

$$y' + 2xy = \cos 2x.$$

We set $h(x) = \int 2x\,dx = x^2$ and multiply the equation by $e^{h(x)} = e^{x^2}$. This gives

$$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}\cos 2x \quad \text{which is the same as} \quad \left[e^{x^2}y\right]' = e^{x^2}\cos 2x.$$

It now follows that

$$e^{x^2}y = \int e^{x^2}\cos 2x\,dx + C$$

and

$$y = e^{-x^2}\int e^{x^2}\cos 2x\,dx + Ce^{-x^2}. \qquad (*)$$

The integral cannot be calculated in terms of elementary functions; the integrand $e^{x^2}\cos 2x$ does not have an "elementary" antiderivative. Therefore, we'll accept $(*)$ as the general solution. ■

2. What does "linear" really mean? Recall from Calculus I that the "operation" of calculating a derivative has the properties:

(*i*) $[f(x) + g(x)]' = f'(x) + g'(x)$ (*"the derivative of a sum is the sum of the derivatives."*)

(*ii*) $[c\,f(x)]' = c\,f'(x)$, c a constant (*"the derivative of a constant times a function is that constant times the derivative of the function."*)

Any "operation" that has these two properties is said to be a *linear operation.* Another example of a linear operation is integration:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \quad \text{and} \quad \int c\,f(x)\,dx = c\int f(x)\,dx, \quad c \text{ constant.}$$

Now consider the first order linear equation

$$y' + p(x)y = q(x).$$

We can regard the left-hand side of the equation, $L[y] = y' + p(x)y$, as an "operation" that is performed on the function y. That is, the left-hand side says "take a function y, calculate its derivative, and then add that to $p(x)$ times y." The equation asks us to find a function y such that the operation $L[y] = y' + p(x)y$ produces the function q. ■

Example 6. Consider the differential equation $y' + \dfrac{1}{x}\,y = 3x$. Here

$$L[y] = y' + \frac{1}{x}y.$$

Let's calculate $L[y]$ for some functions $y = y(x)$:

If $y(x) = x^3$, then

$$L[x^3] = [x^3]' + \frac{1}{x}[x^3] = 3x^2 + x^2 = 4x^2.$$

Thus, $L[x^3] = 4x^2$.

If $y(x) = xe^x$, then

$$L[xe^x] = [xe^x]' + \frac{1}{x}[xe^x] = xe^x + e^x + e^x = xe^x + 2e^x.$$

Thus, $L[xe^x] = xe^x + 2e^x$.

If $y(x) = x^2$, then

$$L[x^2] = [x^2]' + \frac{1}{x}[x^2] = 2x + x = 3x;$$

that is, $y = x^2$ is a solution of the given differential equation; $y = x^3$ and $y = xe^x$ are not solutions of the equation. ■

The "operation" L defined by $L[y] = y' + p(x)y$ where p is a given function, is a linear operation:

$$\begin{aligned} L[f(x)+g(x)] = [f(x)+g(x)]' + p(x)[f(x)+g(x)] &= f'(x)+g'(x)+p(x)f(x)+p(x)g(x) \\ &= f'(x)+p(x)f(x)+g'(x)+p(x)g(x) \\ &= L[f(x)]+L[g(x)] \end{aligned}$$

and

$$L[cf(x)] = [cf(x)]' + p(x)[cf(x)] = cf'(x) + cp(x)f(x) = c[f'(x)+p(x)f(x)] = cL[f(x)].$$

The fact that the operation $L[y] = y' + p(x)y$ is a linear operation is the reason for calling $y' + p(x)y = q(x)$ a linear differential equation. Also, in this context, L is called a *linear differential operator.*

Exercises 2.1

Find the general solution.

1. $y' - 2y = 1$.
2. $y' - \dfrac{1}{x}\, y = x\, e^{x}$.
3. $y' + 2xy = 2x$.
4. $xy' - 2y = -x$.
5. $\dfrac{dy}{dx} - y = -2e^{-x}$.
6. $y' - 2y = 1 - 2x$.
7. $xy' + 2y = \dfrac{\cos x}{x}$.
8. $y' - 2y = e^{-x}$.
9. $(x+1)\dfrac{dy}{dx} + 2y = (x+1)^{5/2}$.
10. $xy' - y = 2x \ln x$.
11. $\dfrac{dy}{dx} + y \tan x = \cos^2 x$.
12. $\dfrac{dy}{dx} + y \cot x = \csc^2 x$.
13. $xy' + y = (1+x)e^{x}$.
14. $y' + 2xy = xe^{-x^2}$.

15. $xy' - y = 2x \ln x$.

16. $y' - e^x y = 0$.

17. $\dfrac{dy}{dx} + e^x\, y = e^x$.

18. $\cos x\, y' + y = \sec x$.

Find the solution of the initial-value problem.

19. $y' + y = x, \quad y(0) = 1$.

20. $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{4}{x}, \quad y(1) = 6$.

21. $y' + y = \dfrac{1}{1 + e^x}, \quad y(0) = e$.

22. $xy' + y = x\, e^x, \quad y(-1) = e^{-1}$.

23. $\dfrac{dy}{dx} + y \cot x = 2 \cos x, \quad y(\pi/2) = 3$.

24. $xy' - 2y = x^3 e^x, \quad y(1) = 0$.

Exercises 25 – 27 are concerned with the linear equation (a) $y' + p(x)y = 0$ where p is a continuous function on some interval I.

25. Show that if y_1 and y_2 are solutions of (a), then $u = y_1 + y_2$ is also a solution of (a).

26. Show that if y is a solution of (a) and α is a real number, then $u = \alpha\, y$ is also a solution of (a).

27. Show that if y_1 and y_2 are solutions of (a) such that $y_1(a) = y_2(a)$ for some $a \in I$, then $y_1 \equiv y_2$ on I.

Exercises 28 – 29 are concerned with the linear equation (b) $y' + p(x)y = q(x)$ where p and q are continuous functions on some interval I.

28. Let $c \in I$ and let $h(x) = \int_c^x p(t)\, dt$. Show that

$$y(x) = e^{-h(x)} \int_c^x q(t) e^{h(t)}\, dt$$

is the solution of (b) that satisfies the initial condition $y(c) = 0$.

29. Show that if y_1 and y_2 are solutions of (b), then $u = y_1 - y_2$ is a solution of (a).

2.2 Separable Equations

A first order differential equation $y' = f(x, y)$ is a *separable equation* if the function f can be expressed as the product of a function of x and a function of y. That is, the equation is separable if the function f has the form

$$f(x, y) = p(x)\, h(y).$$

where p and h are continuous functions.

The solution method for separable equations is based on writing the equation as

$$\frac{1}{h(y)}\, y' = p(x)$$

or

$$q(y)\, y' = p(x) \tag{1}$$

where $q(y) = 1/h(y)$.

Of course, in dividing the equation by $h(y)$ we have to assume that $h(y) \neq 0$. Any numbers r such that $h(r) = 0$ may result in *singular solutions* of the form $y = r$.

If we write y' as dy/dx and interpret this symbol as "differential y" divided by "differential x," then a separable equation can be written in differential form as

$$q(y)\, dy = p(x)\, dx.$$

This is the motivation for the term "separable," the variables are separated.

Solution Method for Separable Equations:

Step 1. Identify: Can you write the equation in the form (1). If yes, do so.

In expanded form, equation (1) is

$$q(y(x))\, y'(x) = p(x).$$

Step 2. Integrate this equation with respect to x:

$$\int q(y(x))\, y'(x)\, dx = \int p(x)\, dx + C \qquad C \text{ an arbitrary constant}$$

which can be written

$$\int q(y)\, dy = \int p(x)\, dx + C$$

by setting $y = y(x)$ and $dy = y'(x)\, dx$. Now, if P is an antiderivative for p, and if Q is an antiderivative for q, then this equation is equivalent to

$$Q(y) = P(x) + C. \tag{2}$$

INTEGRAL CURVES: Equation (2) is a one-parameter family of curves called the *integral curves* of equation (1). In general, the integral curves define y implicitly as a function of x. These curves are solutions of (1) since, by implicit differentiation,

$$\begin{aligned} \frac{d}{dx}\left[Q(y)\right] &= \frac{d}{dx}[P(x)] + \frac{d}{dx}[C] \\ q(y)\, y' &= p(x). \end{aligned}$$

Example 1. The differential equation

$$y' = -\frac{x}{y} \qquad (y \neq 0)$$

is separable since $f(x, y) = -(x/y) = (-x)(1/y)$. Writing the equation in the form (1)

$$y\, y' = -x \qquad \text{or} \qquad y\, dy = -x\, dx$$

and integrating

$$\int y\, dy = -\int x\, dx + C,$$

we get

$$\tfrac{1}{2}\, y^2 = -\tfrac{1}{2}\, x^2 + C \qquad \text{or} \qquad \tfrac{1}{2}\, x^2 + \tfrac{1}{2}\, y^2 = C$$

which, after multiplying by 2, gives

$$x^2 + y^2 = 2C \quad \text{or} \quad x^2 + y^2 = C.$$

(Since C is an arbitrary constant, $2C$ is arbitrary and so we'll just call it C again. This "manipulation" of arbitrary constants is standard in differential equations courses; you'll have to become accustomed to it.)

The set of integral curves is the family of circles centered at the origin. Note that for each positive value of C, the resulting equation defines y implicitly as a function of x. ■

Remark. we may or may not be able to solve the implicit relation (2) for y. This is in contrast to linear differential equations where the solutions $y = y(x)$ are given explicitly as a function of x. (See equation (2) in Section 2.1.) When we can solve (2) for y, we will.

As we shall illustrate below, the set of integral curves of a separable equation *may not* represent the set of all solutions of the equation and so it is not technically correct to use the term "general solution" as we did with linear equations. However for our purposes here this is a minor point and so we shall also call (2) the general solution of (1). As noted above, if $h(r) = 0$, then $y = r$ may be a singular solution of the equation; we will have to check for singular solutions.

Example 2. Show that the differential equation

$$y' = \frac{xy - y}{y + 1} \qquad (y \neq -1)$$

is separable. Then

1. Find the general solution and any singular solutions.
2. Find a solution which satisfies the initial condition $y(2) = 1$.

SOLUTION Here

$$f(x, y) = \frac{xy - y}{y + 1} = \frac{y(x - 1)}{y + 1} = (x - 1)\,\frac{y}{y + 1}.$$

Thus, f can be expressed as the product of a function of x and a function of y so the equation is separable.

Writing the equation in the form (1), we have

$$\frac{y + 1}{y}\,y' = x - 1 \qquad (y \neq 0)$$

or

$$\left(1 + \frac{1}{y}\right)\,y' = x - 1;$$

the variables are separated. Integrating with respect to x, we get

$$\int \left(1 + \frac{1}{y}\right)\,dy = \int (x - 1)\,dx + C$$

and

$$y + \ln|y| = \tfrac{1}{2}\,x^2 - x + C$$

is the general solution. Again we have y defined implicitly as a function of x. Note that $y = 0$ is a solution of the differential equation (verify this), but this function is not included in the general solution ($\ln 0$ does not exist). Thus, $y = 0$ is a singular solution of the equation.

To find a solution that satisfies the initial condition, set $x = 2$, $y = 1$ in the general solution:

$$1 + \ln 1 = \tfrac{1}{2}\,(2)^2 - 2 + C \quad \text{which implies} \quad C = 1.$$

A particular solution that satisfies the initial condition is: $y + \ln|y| = \frac{1}{2}\,x^2 - x + 1.$ ■

Example 3. Show that the differential equation

$$y' = xe^{y - x}$$

is separable and find the general solution.

SOLUTION Since $e^{y-x} = e^{-x}e^{y}$ we have $f(x,y) = xe^{y-x} = xe^{-x}e^{y}$. Thus, f can be expressed as the product of a function of x and a function of y; the equation is separable.

We have

$$y' = xe^{y-x} = xe^{-x}e^{y}.$$

To write the equation in the form (1), divide by e^{y} (i.e., multiply by e^{-y}) to get

$$e^{-y}y' = xe^{-x}.$$

(Note: Since $e^{-y} \neq 0$ for all y, there will be no singular solutions.)

Integrating this equation with respect to x we get

$$\begin{aligned}
\int e^{-y}\,dy &= \int xe^{-x}\,dx + C \\
-e^{-y} &= -xe^{-x} - e^{-x} + C \quad \text{(integration by parts)} \\
e^{-y} &= xe^{-x} + e^{-x} + C \quad \text{(the general solution).}
\end{aligned}$$

Again, the general solution gives y implicitly as a function of x. However, in this case we can solve for y by applying the natural log function:

$$\begin{aligned}
-y &= \ln\left(xe^{-x} + e^{-x} + C\right) \\
y &= -\ln\left(xe^{-x} + e^{-x} + C\right). \qquad \blacksquare
\end{aligned}$$

Example 4. As you can check, the differential equation

$$y' = xy + 2x$$

is both linear and separable so it can be solved using either method. We'll solve it as a separable equation. You should also solve it as a linear equation and compare the two approaches.

Rewriting the equation in the form (1), we have

$$\frac{1}{y+2}\,y' = x \qquad (y \neq -2).$$

Integrating this equation, we get

$$\begin{aligned}
\int \frac{1}{y+2}\,dy &= \int x\,dx + C \\
\ln|\,y+2| &= \tfrac{1}{2}\,x^2 + C \quad \text{(general solution)}
\end{aligned}$$

We can solve the latter equation for y as follows:

$$\begin{aligned} \ln|y+2| &= \tfrac{1}{2}x^2 + C \\ |y+2| &= e^{x^2/2+C} = e^C e^{x^2/2} = K e^{x^2/2} \quad (K = e^C, \text{ (an arbitrary constant)} \end{aligned}$$

By allowing K to take on both positive and negative values, we can write the general solution as

$$y + 2 = K e^{x^2/2} \quad \text{or} \quad y = K e^{x^2/2} - 2.$$

If we set $K = 0$ we get $y = -2$ so this solution is included in the general solution; $y = -2$ is not a singular solution.

Exercises 2.2

Find the general solution and any singular solutions. If possible, express your general solution in the form $y = f(x)$.

1. $y' = xy^{1/2}$.
2. $y' = y\sin(2x+3)$.
3. $y' = 3x^2(1+y^2)$.
4. $y' - xy^2 = x$.
5. $\dfrac{dy}{dx} = \dfrac{\sin^2 y}{1-x^2}$.
6. $y' = \dfrac{y^2+1}{xy+y}$.
7. $y' = xe^{x+y}$.
8. $y' = xy^2 - x - y^2 + 1$.
9. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.
10. $\ln x\,\dfrac{dy}{dx} = \dfrac{y}{x}$.
11. $(y\ln x)y' = \dfrac{y^2+1}{x}$.
12. $\dfrac{dy}{dx} = -\dfrac{\sin 1/x}{x^2 y\cos y}$.

Find a solution of the initial-value problem.

13. $x^2y' = y - xy, \quad y(-1) = -1.$

14. $y' = 6e^{2x-y}, \quad y(0) = 0.$

15. $xy' - y = 2x^2y, \quad y(1) = 1.$

16. $\dfrac{dy}{dx} = \dfrac{e^{x-y}}{1+e^x}, \quad y(1) = 0.$

17. $y' = \dfrac{x^2y - y}{y+1}, \quad y(3) = 1.$

18. $y' = x\sqrt{\dfrac{1-y^2}{1-x^2}}, \quad y(0) = 0.$

2.3 Extensions to Other First Order Equations

Sections 2.1 and 2.2 cover the two basic classes of first order differential equations: linear equations and separable equations. There are types of equations which are neither linear nor separable but which can be transformed into one or the other by means of a change of variable. We will look at two such types here.

Bernoulli Equations

The first order equation $y' = f(x, y)$ is a *Bernoulli equation* if it can be written in the form

$$y' + p(x)y = q(x)y^r \tag{B}$$

where p and q are continuous functions on some interval I and r is a real number, $r \neq 0, 1$. The reason for the restrictions $r \neq 0$ and $r \neq 1$ is that in each of these cases (B) reduces to a linear equation. As we show below, the change of dependent variable given by $v = y^{-r}$ transforms (B) into a linear equation.

Solution Method for Bernoulli equations:

Step 1. Multiply (B) by y^{-r} to obtain

$$y^{-r}y' + p(x)y^{1-r} = q(x). \tag{1}$$

Step 2. Introduce a new dependent variable v by setting $v = y^{1-r}$. Then, by the chain-rule, $v' = (1-r)y^{-r}y'$ and $y^{-r}y' = \frac{1}{1-r}v'$. With this change of variable (1) becomes

$$\frac{1}{1-r}v' + p(x)v = q(x) \quad \text{or} \quad v' + (1-r)p(x)v = (1-r)q(x),$$

a linear equation in x and v.

Step 3. Find the general solution of the linear equation found in Step 2.

Step 4. Find the general solution of (B) by setting $v = y^{1-r}$.

Example 1. Find the general solution of

$$y' - \frac{1}{x}y = 2xy^3.$$

SOLUTION This is a Bernoulli equation with $r = 3$. We multiply the equation by y^{-3} to obtain

$$y^{-3}y' - \frac{1}{x}y^{-2} = 2x.$$

We transform this equation into a linear equation by setting $v = y^{-2}$. Then $v' = -2y^{-3}y'$, and $y^{-3}y' = -\frac{1}{2}\,v'$. With this change of variable, we get the equation

$$-\frac{1}{2}\,v' - \frac{1}{x}v = 2x$$

or

$$v' + \frac{2}{x}v = -4x \qquad (*)$$

a linear equation in x and v which we solve by the method of Section 2.1:

Set $h(x) = \int (2/x)\,dx = 2\ln x = \ln x^2$ and multiply by $e^{\ln x^2} = x^2$. We get

$$x^2v' + 2xv = -4x^3 \quad \text{which implies} \quad (x^2v)' = -4x^3 \quad \text{and} \quad x^2v = -x^4 + C.$$

Therefore, the general solution of $(*)$ is

$$v = -x^2 + \frac{C}{x^2} = \frac{C - x^4}{x^2}.$$

We return to the original variables by setting $v = y^{-2}$:

$$y^{-2} = \frac{C - x^4}{x^2} \quad \text{or} \quad y^2 = \frac{x^2}{C - x^4}.$$

Thus, $y^2 = \dfrac{x^2}{C - x^4}$ is the general solution of the given equation. ■

Homogeneous Equations

The first order equation $y' = f(x,y)$ is a *homogeneous equation* if the function f has the property that

$$f(\lambda x, \lambda y) = f(x,y) \quad \text{for every } \lambda > 0.$$

An equivalent definition is: $y' = f(x,y)$ is a homogeneous equation if the function $f(x,y)$ can be expressed as a function of y/x; that is, if

$$f(x,y) = g(y/x).$$

The latter definition can sometimes be a little difficult to verify while verification of the former is straight forward.

Remark: In general, a function $h = h(x,y)$ of two variables is said to be homogeneous of degree k if

$$h(\lambda x, \lambda y) = \lambda^k h(x,y) \quad \text{for every } \lambda > 0.$$

For example: $h(x,y) = x^2 - y^2 + 2x^3/y$ is homogeneous of degree 2 since

$$h(\lambda x, \lambda y) = (\lambda x)^2 - (\lambda y)^2 + \frac{2(\lambda x)^3}{(\lambda y)} = \lambda^2\left[x^2 - y^2 + \frac{2x^3}{y}\right] = \lambda^2 h(x,y).$$

Roughly speaking, a function $h = h(x,y)$ is homogeneous of degree k if all its terms have degree k.

This concept can get a little tricky. For example, $H(x,y) = \dfrac{e^{x/y}}{\sqrt{x+y}}$ is homogeneous of degree $-1/2$ since

$$H(\lambda x, \lambda y) = \frac{e^{\lambda x/\lambda y}}{\sqrt{\lambda x + \lambda y}} = \frac{e^{x/y}}{\lambda^{1/2}\sqrt{x+y}} = \lambda^{-1/2} H(x,y).$$

Phrased in terms of homogeneous functions, the first order differential equation $y' = f(x,y)$ is homogeneous if f is a homogeneous function of degree 0. ■

If $y' = f(x,y)$ is a homogeneous equation, then the change of dependent variable given by $v = y/x$ (i.e., $y = vx$) transforms the equation into a separable equation.

Solution Method for Homogeneous Equations

Suppose that $y' = f(x,y)$ is a homogeneous equation.

Step 1. Introduce a new dependent variable v by means of the equation $y = vx$. Then $y' = v + xv'$ (the product rule).

Step 2. Substitute into the differential equation:

$$\begin{aligned} y' &= f(x,y) \\ v + xv' &= f(x, vx) = f(1, v) \\ v' &= \frac{f(1,v) - v}{x}, \end{aligned}$$

a separable equation in x and v.

Step 3. Find the general solution of the separable equation found in Step 2.

Step 4. Find the general solution of the original equation by setting $v = y/x$.

Example 2. Show that the differential equation

$$y' = \frac{x^2 + 3y^2}{2xy}$$

is homogeneous and find the general solution.

SOLUTION

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + 3(\lambda y)^2}{2(\lambda x)(\lambda y)} = \frac{\lambda^2(x^2 + 3y^2)}{\lambda^2(2xy)} = \frac{x^2 + 3y^2}{2xy} = f(x,y).$$

The equation is homogeneous.

Set $y = vx$. Then $y' = v + xv'$ and

$$\begin{aligned}
y' &= \frac{x^2+3y^2}{2xy} \\
v + xv' &= \frac{x^2+3v^2x^2}{2x(vx)} = \frac{x^2(1+3v^2)}{x^2(2v)} = \frac{1+3v^2}{2v} \\
xv' &= \frac{1+3v^2}{2v} - v = \frac{1+v^2}{2v} \\
\frac{dv}{dx} &= \frac{1+v^2}{2vx},
\end{aligned}$$

a separable equation

Separating the variables and integrating, we get

$$\begin{aligned}
\frac{2v}{1+v^2}\,dv &= \frac{1}{x}\,dx \\
\ln(1+v^2) &= \ln x + C = \ln x + \ln K \quad (C = \ln K,\ K \text{ arbitrary}) \\
\ln(1+v^2) &= \ln Kx \\
1+v^2 &= Kx \quad \text{and} \quad v^2 = Kx - 1.
\end{aligned}$$

This is the general solution of the separable equation in x and v. Setting $v = y/x$, we get

$$\frac{y^2}{x^2} = Kx - 1 \quad \text{or} \quad y^2 = Kx^3 - x^2.$$

Thus, $y^2 = Kx^3 - x^2$, K an arbitrary constant, is the general solution of the given equation. ■

Exercises 2.3

Find the general solution of the Bernoulli equation.

1. $y' + \frac{1}{x}\,y = 3x^2y^2$.
2. $y' + xy = xy^3$.
3. $y' - 4y = 2e^x\sqrt{y}$.
4. $2xy\,y' = 1 + y^2$
5. $3y' + 3x^{-1}y = 2x^2y^4$.

6. $(x-2)y' + y = 5(x-2)^2 y^{1/2}$.

Show that each of the following differential equations is homogeneous and find the general solution of the equation.

7. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$.

8. $\dfrac{dy}{dx} = \dfrac{y}{x+\sqrt{xy}}$.

9. $\dfrac{dy}{dx} = \dfrac{x^2 e^{y/x} + y^2}{xy}$.

10. $y' = \dfrac{x^4+2y^4}{xy^3}$.

11. $y' = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$.

12. $y' = \dfrac{y+\sqrt{x^2-y^2}}{x}$.

Find the general solution. (These equations are a mixture of linear, separable, Bernoulli and homogeneous equations.)

13. $x(1+y^2) + y(1+x^2)y' = 0$.

14. $xy' = y + x^2 e^x$.

15. $xy' + y - \sec x = 0$.

16. $y' = \dfrac{2xy}{x^2-y^2}$.

17. $(xy+y)y' = x - xy$.

18. $\dfrac{dy}{dx} = 2x - 2xy$.

19. $\dfrac{dy}{dx} = \dfrac{xe^{y/x}+y}{x}$.

20. $y' = (y/x) + \sin(y/x)$.

21. $(3x^2+1)y' - 2xy = 6x$.

22. $x(1-y) + y(1+x^2)\dfrac{dy}{dx} = 0$.

23. $xy' + y = y^2 \ln x$.

24. $y' = -\dfrac{3y}{x} + x^4 y^{1/3}$.

25. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{3xy^2}$.

26. Show that the change of variable $u = \ln y$ transforms

$$y' + y\,p(x)\,\ln y = q(x)\,y$$

into a linear equation.

27. Find the general solution of

$$y' - \frac{y}{x}\ln y = xy$$

using the change of variable indicated in Exercise 26.

28. (a) Determine a change of variable that will transform

$$y' \cos y + p(x) \sin y = q(x)$$

into a linear equation.

(b) Find the general solution of

$$y' \cos y + 2x \sin y = 4e^{-x^2}.$$

2.4 Some Applications

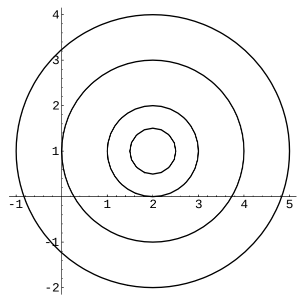
In this section we give some examples of applications of first order differential equations.

1. Orthogonal Trajectories

The one-parameter family of curves

$$(x-2)^2 + (y-1)^2 = C \qquad (C \geq 0) \tag{a}$$

is a family of circles with center at the point $(2, 1)$ and radius $\sqrt{C}$.



If we differentiate this equation with respect to x, we get

$$2(x-2) + 2(y-1)\,y' = 0$$

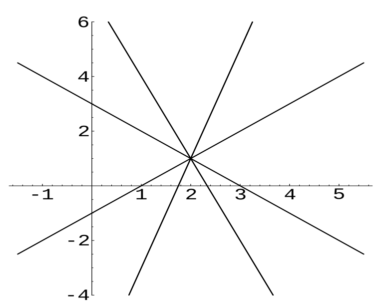
and

$$y' = -\frac{x-2}{y-1} \tag{b}$$

This is the differential equation for the family of circles. Note that if we choose a specific point (x_0, y_0), $y_0 \neq 1$ on one of the circles, then (b) gives the slope of the tangent line at (x_0, y_0).

Now consider the family of straight lines passing through the point $(2, 1)$:

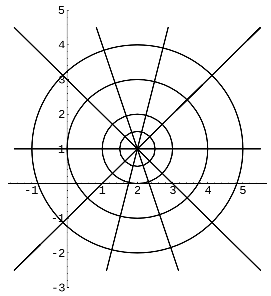
$$y - 1 = K(x-2). \tag{c}$$



The differential equation for this family is

$$y' = \frac{y-1}{x-2} \qquad \text{(verify this)} \tag{d}$$

Comparing equations (b) and (d) we see that right side of (b) is the negative reciprocal of the right side of (d). Therefore, we can conclude that if $P\,(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, then the line and the circle are perpendicular (orthogonal) to each other at P. The following figure shows the two families drawn in the same coordinate system.



A curve that intersects each member of a given family of curves at right angles (orthogonally) is called an *orthogonal trajectory* of the family. Each line in (c) is an orthogonal trajectory of the family of circles (a) [and conversely, each circle in (a) is an orthogonal trajectory of the family of lines (c)]. In general, if

$$F(x, y, c) = 0 \qquad \text{and} \qquad G(x, y, K) = 0$$

are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be *orthogonal trajectories.*

A procedure for finding a family of orthogonal trajectories $G(x, y, K) = 0$ for a given family of curves $F(x, y, C) = 0$ is as follows:

Step 1. Determine the differential equation for the given family $F(x, y, C) = 0$.

Step 2. Replace y' in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

Step 3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Example Find the orthogonal trajectories of the family of parabolas $y = Cx^2$.

SOLUTION You can verify that the differential equation for the family $y = Cx^2$ can be written as

$$y' = \frac{2y}{x}.$$

Replacing y' by $-1/y'$, we get the equation

$$-\frac{1}{y'} = \frac{2y}{x} \qquad \text{which simplifies to} \qquad y' = -\frac{x}{2y}$$

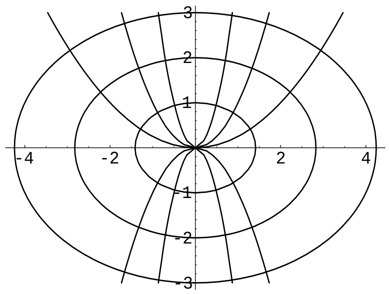
a separable equation. Separating the variables, we get

$$2y\,y' = -x \qquad \text{or} \qquad 2y\,dy = -x\,dx.$$

integrating with respect to x, we have

$$y^2 = -\frac{1}{2}\,x^2 + C \qquad \text{or} \qquad \frac{x^2}{2} + y^2 = C.$$

This is a family of ellipses with center at the origin and major axis on the x-axis. ■



Exercises 2.4.1

Find the orthogonal trajectories for the family of curves.

1. $y = Cx^3$.
2. $x = Cy^4$.
3. $y = Cx^2 + 2$.
4. $y^2 = 2(C - x)$.

   Find the orthogonal trajectories for the family of curves.

5. The family of parabolas symmetric with respect to the y-axis and vertex at the origin.
6. The family of parabolas with vertical axis and vertex at the point $(1, 2)$.
7. The family of circles that pass through the origin and have their center on the x-axis.
8. The family of circles tangent to the x-axis at $(3, 0)$.

   Show that the given family is *self-orthogonal.*

9. $y^2 = 4C(x + C)$.
10. $\dfrac{x^2}{C^2} + \dfrac{y^2}{C^2 - 4} = 1$.

2. Exponential Growth and Decay

Radioactive Decay: It has been observed and verified experimentally that the rate of decay of a radioactive material at time t is proportional to the amount of material present at time t. Mathematically this says that if $A = A(t)$ is the amount of radioactive material present at time t, then

$$A' = rA$$

where r, the constant of proportionality, is negative. To emphasize the fact that A is decreasing, this equation is often written

$$A' = -kA \qquad \text{or} \qquad \frac{dA}{dt} = -kA, \quad k > 0 \text{ constant.}$$

This is the form we shall use. The constant of proportionality k is called the *decay constant* .

Note that this equation is both linear and separable and so we can use either method to solve it. It is easy to show that the general solution is

$$A(t) = Ce^{-kt}$$

If $A_0 = A(0)$ is amount of material present at time $t = 0$, then $C = A_0$ and

$$A(t) = A_0\, e^{-kt}.$$

Note that $\lim_{t\to\infty} A(t) = 0$.

Half-Life: An important property of a radioactive material is the length of time T it takes to decay to one-half the initial amount. This is the so-called *half-life* of the material. Physicists and chemists characterize radioactive materials by their half-lives. To find T we solve the equation

$$\tfrac{1}{2}\, A_0 = A_0\, e^{-kT}$$

for T:

$$\begin{aligned}
\tfrac{1}{2}\, A_0 &= A_0\, e^{-kT} \\
e^{-kT} &= \tfrac{1}{2} \\
-kT &= \ln(1/2) = -\ln 2 \\
T &= \frac{\ln 2}{k}
\end{aligned}$$

Conversely, if we know the half-life T of a radioactive material, then the decay constant k is given by

$$k = \frac{\ln 2}{T}.$$

Example Cobalt-60 is a radioactive element that is used in medical radiology. It has a half-life of 5.3 years. Suppose that an initial sample of cobalt-60 has a mass of 100 grams.

(a) Find the decay constant and determine an expression for the amount of the sample that will remain t years from now.

(b) How long will it take for 90% of the sample to decay?

SOLUTION (a) Since the half-life $T = (\ln 2)/k$, we have

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5.3} \cong 0.131.$$

With $A(0) = 100$, the amount of material that will remain after t years is

$$A(t) = 100\, e^{-0.131t}.$$

(b) If 90% of the material decays, then 10%, which is 10 grams, remains. Therefore, we solve the equation

$$100\, e^{-0.131t} = 10$$

for t:

$$e^{-0.131t} = 0.1, \qquad -0.131t = \ln(0.1), \qquad t = \frac{\ln(0.1)}{-0.131} \cong 17.6.$$

It will take approximately 17.6 years for 90% of the sample to decay. ■

Population Growth; Growth of an Investment: It has been observed and verified experimentally that, under ideal conditions, a population (e.g., bacteria, fruit flies, humans, etc.) tends to increase at a rate proportional to the size of the population. Therefore, if $P = P(t)$ is the size of a population at time t, then

$$\frac{dP}{dt} = rP, \quad r > 0 \text{ (constant)} \tag{1}$$

In this case, the constant of proportionality r is called the *growth constant* .

Similarly, in a bank that compounds interest continuously, the rate of increase of funds at time t is proportional to the amount of funds in the account at time t. Thus equation (1) also represents the growth of a principal amount under continuous compounding. Since the two cases are identical, we'll focus on the population growth case.

The general solution of equation (1) is

$$P(t) = Ce^{rt}.$$

If $P(0) = P_0$ is the size of the population at time $t = 0$, then

$$P(t) = P_0\, e^{rt}$$

is the size of the population at time t. Note that $\lim_{t\to\infty} P(t) = \infty$. In reality, the rate of increase of a population does not continue to be proportional to the size of the population. After some time has passed, factors such as limitations on space or food supply, introduction of diseases, and so forth, affect the growth rate; the mathematical model is not valid indefinitely. In contrast, the model does hold indefinitely in the case of the growth of an investment under continuous compounding.

Doubling time: The analog of the half-life of a radioactive material is the so-called *doubling time,* the length of time T that it takes for a population to double in size. Using the same analysis as above, we have

$$\begin{aligned} 2\,A_0 &= A_0\,e^{rT} \\ e^{rT} &= 2 \\ rT &= \ln 2 \\ T &= \frac{\ln 2}{r} \end{aligned}$$

In the banking, investment, and real estate communities there is a standard measure, called the *rule of 72*, which states that the length of time (approximately) for a principal invested at $r\%$, compounded continuously, to double in value is $72/r\%$. We know that the doubling time is

$$T = \frac{\ln 2}{r} \approx \frac{0.69}{r} = \frac{69}{r\%} \approx \frac{72}{r\%}.$$

This is the origin of the "rule of 72;" 72 is used rather than 69 because it has more divisors. ■

Example Scientists have observed that a small colony of penguins on a remote Antarctic island obeys the population growth law. There were 2000 penguins initially and 3000 penguins 4 years later.

(a) How many penguins will there be after 10 years?

(b) How long will it take for the number of penguins to double?

SOLUTION Let $P(t)$ denote the number of penguins at time t. Since $P(0) = 2000$ we have

$$P(t) = 2000\,e^{rt}.$$

We use the fact that $P(4) = 3000$ to determine the growth constant r:

$$3000 = 2000\,e^{4r}, \qquad e^{4r} = 1.5, \qquad 4r = \ln 1.5,$$

and so
$$r = \frac{\ln 1.5}{4} \cong 0.101.$$

Therefore, the number of penguins in the colony at any time t is
$$P(t) = 2000\, e^{0.101t}.$$

(a) The number of penguins in the colony after 10 years is (approximately)
$$P(10) = 2000\, e^{(0.101)10} = 2000\, e^{1.01} \cong 5491.$$

(b) To find out how long it will take the number of penguins in the colony to double, we need to solve
$$2000\, e^{0.101t} = 4000$$
for t:
$$e^{0.101t} = 2, \qquad 0.101t = \ln 2, \qquad t = \frac{\ln 2}{0.101} \cong 6.86 \text{ years}.$$

Note: There is another way of expressing P that uses the exact value of r. From the equation $3000 = 2000\, e^{4r}$ we get $r = \frac{1}{4} \ln \frac{3}{2}$. Thus
$$P(t) = 2000\, e^{\frac{t}{4} \ln [3/2]} = 2000\, e^{\ln [3/2]^{t/4}} = 2000 \left(\frac{3}{2}\right)^{t/4}. \qquad \blacksquare$$

Exercises 2.4.2

1. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

    (a) An expression for the mass of the material remaining at any time t.

    (b) The mass of the material after 4 hours.

    (c) The half-life of the material.

2. What is the half-life of a radioactive substance it takes 5 years for one-third of the material to decay?

3. The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.

    (a) Estimate the size of the colony t hours after the initial measurement.

(b) What is the expected size of the colony after 12 hours?

(c) Find the doubling time of the colony.

4. A biologist observes that a certain bacterial colony triples every 4 hours and after 12 hours occupies 1 square centimeter.

   (a) How much area did the colony occupy when first observed?

   (b) What is the doubling time for the colony?

5. In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the world population at each time t increases at a rate proportional to the population at time t. Measure t in years after 1980.

   (a) Find the growth constant and give the world population at any time t.

   (b) How long will it take for the world population to reach 9 billion (double the 1980 population)?

   (c) The world population for 2002 was reported to be about 6.2 billion. What population does the formula in (a) predict for the year 2002?

6. It is estimated that the arable land on earth can support a maximum of 30 billion people. Extrapolate from the data given in Exercise 5 to estimate the year when the food supply becomes insufficient to support the world population.

3. Newton's Law of Cooling/Heating

Newton's Law of Cooling states that the rate of change of the temperature u of an object is proportional to the difference between u and the (constant) temperature σ of the surrounding medium (e.g., air or water), called the *ambient temperature.* The mathematical formulation of this statement is:

$$\frac{du}{dt} = m(u - \sigma), \quad m \text{ constant.}$$

The constant of proportionality, m, in this model must be negative; for if the object is warmer than the ambient temperature $(u - \sigma > 0)$, then its temperature will decrease $(du/dt < 0)$, which implies $m < 0$; if the object is cooler than the ambient temperature $(u - \sigma < 0)$, then its temperature will increase $(du/dt > 0)$, which again implies $m < 0$.

To emphasize that the constant of proportionality is negative, we write Newton's Law of Cooling as

$$\frac{du}{dt} = -k(u - \sigma), \quad k > 0 \text{ constant.} \tag{1}$$

This differential equation is both linear and separable so either method can be used to solve it. As you can check, the general solution is

$$u(t) = \sigma + Ce^{-kt}.$$

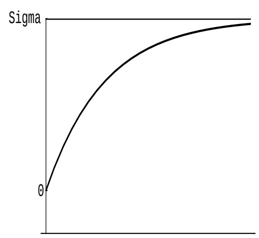
If the initial temperature of the object is $u(0) = u_0$, then

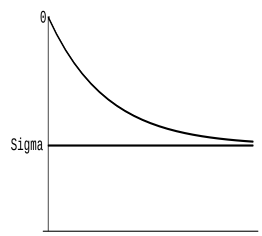
$$u_0 = \sigma + Ce^0 = \sigma + C \quad \text{and} \quad C = u_0 - \sigma.$$

Thus, the temperature of the object at any time t is given by

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}. \tag{2}$$

The graphs of $u(t)$ in the cases $u_0 < \sigma$ and $u_0 > \sigma$ are given below. Note that $\lim_{t\to\infty} u(t) = \sigma$ in each case. In the first case, u is increasing and its graph is concave down; in the second case, u is decreasing and its graph is concave up.





Example A metal bar with initial temperature $25^o\,C$ is dropped into a container of boiling water $(100^o\,C)$. After 5 seconds, the temperature of the bar is $35^o\,C$.

(a) What will the temperature of the bar be after 1 minute?

(b) How long will it take for the temperature of the bar to be within $0.5^o\,C$ of the boiling water?

SOLUTION Applying equation (2), the temperature of the bar at any time t is

$$T(t) = 100 + (25 - 100)\,e^{-kt} = 100 - 75\,e^{-kt}.$$

The first step is to determine the constant k. Since $T(5) = 35$, we have

$$35 = 100 - 75\,e^{-5k}, \quad 75\,e^{-5k} = 65, \quad -5k = \ln(65/75), \quad k \cong 0.0286.$$

Therefore,

$$T(t) = 100 - 75\,e^{-0.0286\,t}.$$

(a) The temperature of the bar after 1 minute is, approximately:

$$T(60) = 100 - 75\,e^{-0.0286(60)} \cong 100 - 75\,e^{-1.7172} \cong 86.53^o.$$

(b) We want to calculate how long it will take for the temperature of the bar to reach 99.5^o. Thus, we solve the equation

$$99.5 = 100 - 75\,e^{-0.0286\,t}$$

for t:

$99.5 = 100 - 75\,e^{-0.0286\,t} \quad -75\,e^{-0.0286\,t} = -0.5, \quad -0.0826\,t = \ln(0.5/75), \quad t \cong 60.66$ seconds. ■

Exercises 2.4.3

1. A thermometer is taken from a room where the temperature is $72^o\,F$ to the outside where the temperature is $32^o\,F$. After $1/2$ minute, the thermometer reads $50^o\,F$.
   (a) What will the thermometer read after it has been outside for 1 minute?
   (b) How many minutes does the thermometer have to be outside for it to read $35^o\,F$?
2. A metal ball at room temperature $20^o\,C$ is dropped into a container of boiling water $(100^o\,C)$. given that the temperature of the ball increases 2^o in 2 seconds, find:
   (a) The temperature of the ball after 6 seconds in the boiling water.
   (b) How long it will take for the temperature of the ball to reach $90^o\,C$.
3. Suppose that a corpse is discovered at 10 p.m. and its temperature is determined to be $85^o\,F$. Two hours later, its temperature is $74^o\,F$. If the ambient temperature is $68^o\,F$, estimate the time of death.

4. The Logistic Equation

In the mid-nineteenth century the Belgian mathematician P.F. Verhulst used the differential equation

$$\frac{dy}{dt} = ky(M - y) \tag{1}$$

where k and M are positive constants, to study the population growth of various countries. This equation is now known as the *logistic equation* and its solutions are called *logistic functions.* Life scientists have used this equation to model the spread of an infectious disease through a population, and social scientists have used it to study the flow of information. In the case of an infectious disease, if M denotes the number of people in the population and $y(t)$ is the number of infected people at time t, then the differential equation states that the rate of change of infected people is proportional to the product of the number of people who have the disease and the number of people who do not.

The constant M is called the *carrying capacity* of the environment. Note that $dy/dt > 0$ when $0 < y < M$, $dy/dt = 0$ when $y = M$, and $dy/dt < 0$ when $y > M$. The constant k is the *intrinsic growth rate.*

The differential equation (1) is separable. (It is also a Bernoulli equation.) We write the equation as

$$\frac{1}{y(M-y)}\,y' - k = 0$$

and integrate

$$\begin{aligned}
\int \frac{1}{y(M-y)}\,dy - \int k\,dt &= C_1 \\
\int \left(\frac{1/M}{y} + \frac{1/M}{M-y}\right) dy - kt &= C_1 \qquad \text{(partial fraction decomposition)} \\
\frac{1}{M}\ln|y| - \frac{1}{M}\ln|M-y| &= kt + C_1
\end{aligned}$$

We can solve this equation for y as follows:

$$\begin{aligned}
\frac{1}{M}\ln\left|\frac{y}{M-y}\right| &= kt + C_1 \\
\ln\left|\frac{y}{M-y}\right| &= Mkt + MC_1 = Mkt + C_2, \qquad (C_2 = MC_1) \\
\left|\frac{y}{M-y}\right| &= e^{Mkt+C_2} = e^{C_2}e^{Mkt} = Ce^{Mkt} \qquad (C = e^{C_2})
\end{aligned}$$

Now, in the context of this discussion, $y = y(t)$ satisfies $0 < y(t) < M$. Therefore, $y/(M-y) > 0$ and we have

$$\frac{y}{M-y} = C\,e^{Mkt}.$$

Solving this equation for y, we get

$$y(t) = \frac{CM}{C + e^{-Mkt}}.$$

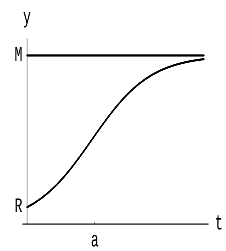
Finally, if $y(0) = R$, $R < M$, then

$$R = \frac{CM}{C+1} \qquad \text{which implies} \qquad C = \frac{R}{M-R}$$

and

$$y(t) = \frac{MR}{R + (M-R)e^{-Mkt}}. \tag{2}$$

The graph of this particular solution is shown below.



Note that y is an increasing function. In the Exercises you are asked to show that the graph is concave up on $[0, a)$ and concave down on $(a, \infty]$. This means that the disease is spreading at an increasing rate up to time a; after a, the disease is still spreading, but at a decreasing rate. Note, also, that $\lim_{t\to\infty} y(t) = M$.

Example An influenza virus is spreading through a small city with a population of 50,000 people. Assume that the virus spreads at a rate proportional to the product of the number of people who have been infected and the number of people who have not been infected. If 100 people were infected initially and 1000 were infected after 10 days, find:

(a) The number of people infected at any time t.

(b) How long it will take for half the population to be infected.

SOLUTION (a) Substituting the given data into equation (b), we have

$$y(t) = \frac{100(50,000)}{100 + 49,900\, e^{-50,000kt}} = \frac{50,000}{1 + 499\, e^{-50,000kt}}.$$

We can determine the constant k by applying the condition $y(10) = 1000$. We have

$$\begin{aligned} 1000 &= \frac{50,000}{1 + 499\, e^{-500,000k}} \\ 499\, e^{-500,000k} &= 49 \\ -500,000\, k &= \ln(49/499) \\ k &\cong 0.0000046. \end{aligned}$$

Thus, the number of people infected at time t is (approximately)

$$y(t) = \frac{50,000}{1 + 499\, e^{-0.23t}}.$$

(b) To find how long it will take for half the population to be infected, we solve $y(t) =$

25,000 for t:

$$\begin{aligned} 25,000 &= \frac{50,000}{1 + 499\, e^{-0.23\,t}} \\ 499\, e^{-0.23\,t} &= \frac{1}{499} \\ t &= \frac{\ln(1/499)}{-0.23} \cong 27 \text{ days.} \qquad \blacksquare \end{aligned}$$

Exercises 2.4.4

1. A rumor spreads through a small town with a population of 5,000 at a rate proportional to the product of the number of people who have heard the rumor and the number who have not heard it. Suppose that 100 people initiated the rumor and that 500 people heard it after 3 days.

   (a) How many people will have heard the rumor after 8 days?

   (b) How long will it take for half the population to hear the rumor.

2. Let y be the logistic function (4). Show that dy/dt increases for $y < M/2$ and decreases for $y > M/2$. What can you conclude about dy/dt when $y = M/2$?

3. Solve the logistic equation by means of the change of variables

$$y(t) = v(t)^{-1}, \quad y'(t) = -v(t)^{-2}\, v'(t).$$

   Express the constant of integration in terms of the initial value $y(0) = y_0$.

4. Suppose that a population governed by a logistic model exists in an environment with carrying capacity of 800. If an initial population of 100 grows to 300 in 3 years, find the intrinsic growth rate k.

2.5 Direction Fields; Existence and Uniqueness

First-order differential equations of the general form

$$y' = f(x, y) \tag{1}$$

can be solved only in very special cases. We have looked at two such cases, linear equations and separable equations. Two other cases, Bernoulli equations and homogeneous equations, were introduced in the exercise sets. It is important to understand that there are *no* methods for solving equation (1) in general.

If a given first-order equation does not fall into one of the special cases for which there is a solution method, then some other approach must be used. In such situations numerical methods or approximation methods are typically employed. These methods are studied in more advanced courses. In this section we give a geometric interpretation of equation (1) and then consider the basic questions of existence and uniqueness of solutions.

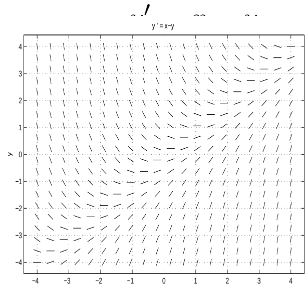
Direction Fields

Here we introduce a geometric approach to the first order differential equation (1) that enables us to produce sketches of solution curves without actually solving the equation. The approach does not produce equations in x and y; it produces pictures, pictures from which we can gather information on the qualitative behavior of solutions, behavior such as boundedness, concavity, possible maxima and minima, and so forth.

If a solution curve for equation (1) $y = y(x)$ passes through the point (x_0, y_0) then it does so with slope $f(x_0, y_0)$ since $y'(x_0) = f(x_0, y_0)$. We can indicate this by drawing a short line segment through (x_0, y_0) with slope $f(x_0, y_0)$. By repeating this process over and over, we construct a *direction field* for the differential equation (1). That is, we select a grid of points (x_i, y_i), $i = 1, 2, \ldots, n$, and draw at each of these points a short line segment with slope $f(x_i, y_i)$. While this is tedious to do by hand, it is a simple task for a computer algebra/graphing utility system (e.g., Mathematica, Maple, MatLab). Such systems typically include a feature for sketching direction fields.

Example 1. The direction field for the differential equation

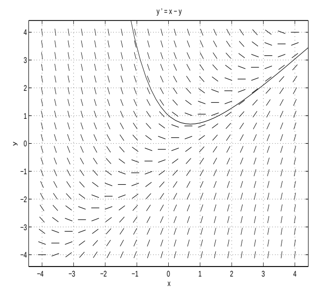
is



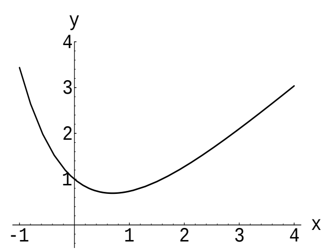
We can use a direction field to sketch the solution of an initial-value problem:

$$y' = f(x, y), \quad y(a) = b.$$

We start at the point (a, b) and follow the line segments in both directions. A sketch of the solution of $y' = x - y$ that satisfies the initial condition $y(0) = 1$ is shown in the next figure.



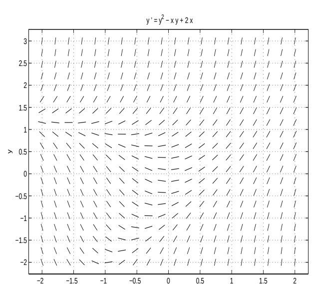
The differential equation in this case is linear and so you can find the general solution. As you can check, the general solution is $y = x - 1 + Ce^{-x}$, and the solution satisfying the initial condition is $y = x - 1 + 2e^{-x}$. The graph of is shown below. ■



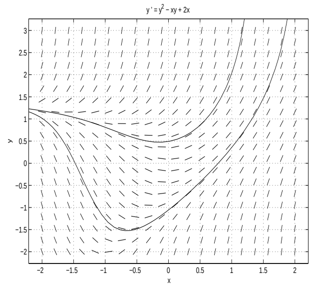
Example 2. There is no method for finding the general solution of the differential equation

$$y' = y^2 - xy + 2x$$

However, we can draw the direction field for the equation and get some idea about the solutions and their behavior. The direction field is



In the next figure, we sketch solution curves generated by the initial conditions $y(-2) = 1$ and $y(1) = 2$. ■



Existence and Uniqueness of Solutions

The questions of existence and uniqueness of solutions of initial-value problems are of fundamental importance in the study of differential equations. We'll illustrate these concepts with some simple examples, and then we'll state an existence and uniqueness theorem for first-order initial-value problems. A proof of the theorem is beyond the scope of this course.

Consider the differential equation

$$y' = -\frac{y^2}{x^2}$$

together with the three initial conditions:

(a) $y(0) = 1$,

(b) $y(0) = 0$,

(c) $y(1) = 1$.

Since the differential equation is separable, we can calculate the general solution.

$$\begin{aligned} -\frac{1}{y^2}\, y' &= \frac{1}{x^2} \\ -\int \frac{1}{y^2}\, dy &= \int \frac{1}{x^2}\, dx + C \\ \frac{1}{y} &= -\frac{1}{x} + C \quad \text{or} \quad \frac{1}{y} + \frac{1}{x} = C. \end{aligned}$$

Solving for y we get

$$y = \frac{x}{Cx - 1}.$$

To apply the initial condition (a), we set $x = 0,\ y = 1$ in the general solution. This gives

$$1 = \frac{0}{C \cdot 0 - 1} = 0.$$

We conclude that there is no value of C such that $y(0) = 1$; there is no solution of the initial-value problem $y' = -\dfrac{y^2}{x^2},\ y(0) = 1.$

Next we apply the initial condition (b) by setting $x = 0,\ y = 0$ in the general solution. In this case we obtain the equation

$$0 = \frac{0}{C \cdot 0 - 1} = 0$$

which is satisfied by all values of C. The initial-value problem $y' = -\dfrac{y^2}{x^2},\ y(0) = 0$ has infinitely many solutions.

Finally, we apply the initial condition (c) by setting $x = 1,\ y = 1$ in the general solution:

$$1 = \frac{1}{C \cdot 1 - 1} \quad \text{which implies} \quad C = 2.$$

This initial-value problem $y' = -\dfrac{y^2}{x^2},\ y(1) = 1$ has a unique solution, namely

$$y = x/(2x - 1).$$

Existence and Uniqueness Theorem Given the initial-value problem

$$y' = f(x, y) \quad y(a) = b. \tag{4}$$

If f and $\partial f/\partial y$ are continuous on a rectangle $R:\ a - \alpha \le x \le a + \alpha,\ b - \beta \le y \le b + \beta,\ \alpha,\ \beta > 0,$ then there is an interval $a - h \le x \le a + h,\ h \le \alpha$ on which the initial-value problem (2) has a unique solution $y = y(x)$.

Going back to our example, note that $f(x, y) = -y^2/x^2$ is not continuous on any rectangle that contains $(0, b)$ in its interior. Thus, the existence and uniqueness theorem does not apply in the cases $y(0) = 1$ and $y(0) = 0$.

In the case of the linear differential equation

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on some interval $I = [\alpha, \beta]$, we have

$$f(x, y) = q(x) - p(x)y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = p(x)$$

and these functions are continuous on every rectangle R of the form $\alpha \leq x \leq \beta,\ -\gamma \leq y \leq \gamma$ where γ is any positive number; that is f and $\partial f/\partial y$ are continuous on the "infinite" rectangle $\alpha \leq x \leq \beta,\ -\infty < y < \infty$. Thus, every linear initial-value problem has a unique solution .

Exercises 2.5

1. Given the initial-value problem $y' = y;\ y(0) = 1$.
   (a) Draw a direction field in the rectangle $R:\ -3 \leq x \leq 1.5,\ -1 \leq y \leq 3$.
   (b) Use this direction field to sketch the solution curve that satisfies the initial condition. Experiment with other rectangles to obtain additional views of the solution curve.
   (c) Find the exact solution of the initial-value problem using the methods of this chapter and then compare the graph of your solution with the curve you obtained in part (b).
2. Given the initial-value problem $y' = x + 2y;\ y(0) = 1$.
   (a) Draw a direction field in the rectangle $R:\ -1 \leq x \leq 2,\ -1 \leq y \leq 9$.
   (b) Use this direction field to sketch the solution curve that satisfies the initial condition. Experiment with other rectangles to obtain additional views of the solution curve.
   (c) Find the exact solution of the initial-value problem using the methods of this chapter and then compare the graph of your solution with the curve you obtained in part (b).
3. Given the initial-value problem $y' = 2xy;\ y(0) = 1$.
   (a) Draw a direction field in the rectangle $R:\ -1.5 \leq x \leq 3,\ -1 \leq y \leq 8$.
   (b) Use this direction field to sketch the solution curve that satisfies the initial condition. Experiment with other rectangles to obtain additional views of the solution curve.
   (c) Find the exact solution of the initial-value problem using the methods of this chapter and then compare the graph of your solution with the curve you obtained in part (b).
4. Given the initial-value problem $y' = -4x/y;\ y(1) = 1$.
   (a) Draw a direction field in the rectangle $R:\ -2 \leq x \leq 2,\ -3 \leq y \leq 3$.
   (b) Use this direction field to sketch the solution curve that satisfies the initial condition. Experiment with other rectangles to obtain additional views of the solution curve.

(c) Find the exact solution of the initial-value problem using the methods of this chapter and then compare the graph of your solution with the curve you obtained in part (b).

5. Draw a direction field for the differential equation $y' = 1 - y^2$ in the rectangle $R:\ 0 \leq x \leq 2,\ 0 \leq y \leq 2$. Plot three solution curves on the field by choosing three sets of initial conditions. Give the initial conditions that you chose.

6. Draw a direction field for the differential equation $y' = \frac{1}{10}y(5-y)$ in the rectangle $R:\ 0 \leq x \leq 10,\ 0 \leq y \leq 8$. Plot three solution curves on the field by choosing three sets of initial conditions. Give the initial conditions that you chose.

2.6 Some Numerical Methods

As indicated previously, there are only a few types of first order differential equations for which there are methods for finding exact solutions. Consequently, we have to rely on numerical methods to find approximate solutions in situations where the differential equation can not be solved. In this section we illustrate two elementary numerical methods.

Our focus here is on the initial-value problem

$$y' = f(x, y); \quad y(x_0) = y_0 \tag{1}$$

where f and $\partial f/\partial y$ are continuous functions on a rectangle R and $(x_0, y_0) \in R$. That is, the initial-value problem satisfies the conditions of the existence and uniqueness theorem.

EULER'S METHOD Although this method is rarely used in practice, we present it because it has the essential features of more advanced methods. We begin by setting a *step size* $h > 0$. Then we define x-values

$$x_k = x_0 + kh, \qquad \text{where } k \text{ is a natural number.}$$

The values x_k are the values of x where we try to approximate the solution to (1).

Next, we give some notation for our approximations to $y(x_k)$. We will use the notation

$$y_k \approx y(x_k).$$

By the definition of the derivative, we know that

$$y'(x_k) \approx \frac{y(x_{k+1}) - y(x_k)}{h},$$

which, using our notation, leads to the approximation

$$y'(x_k) \approx \frac{y_{k+1} - y_k}{h} \tag{2}$$

Substituting this approximation into (1) gives the *approximate equations*

$$\frac{y_{k+1} - y_k}{h} = f(x_k, y_k), \quad \text{with } y_0 \text{ given.}$$

Rearranging terms gives

$$y_{k+1} = y_k + hf(x_k, y_k), \quad \text{for } k = 0, 1, \ldots \text{ where } y_0 \text{ is given.} \tag{3}$$

This method is known as *Euler's Method with step size* h .

Example 1. Use Euler's method with a step size of 0.05 to approximate the solution to the initial value problem

$$y' = -y + \sin x; \quad y(0) = 1$$

Before we begin, note that we can give the exact solution to this initial value problem. You might naturally ask why we are going to bother approximating the solution if we know how to solve it. The answer is simple. We want to illustrate how well (or poorly) Euler's Method works. The exact solution is

$$y(x) = -\tfrac{1}{2}\cos x + \tfrac{1}{2}\sin x + \tfrac{3}{2}e^{-x}.$$

Now we'll use Euler's Method with a step size of $h = 0.05$ to approximate this solution. From (3) we have

$$y_{k+1} = y_k + 0.05\left[-y_k + \sin(0.05k)\right], \quad \text{for} \;\; k = 0, 1, \ldots \;\; \text{where} \;\; y_0 = 1.$$

Some simple calculations give

$$\begin{aligned} u_1 &= 0.95 \\ u_2 &= 0.904998958 \\ u_3 &= 0.864740681 \\ &\;\vdots \\ u_{20} &= 0.686056580 \end{aligned}$$

Noting that y_{20} is supposed to approximate

$$y(x_{20}) = y\,(1) = 0.702403501 \quad \text{(to 9 decimal places)},$$

we can see that our approximation of $y(1)$ has an error of

$$y(1) - y_{20} = 0.702403501 - 0.686056580 = 0.016346921$$

Actually, that's not too bad! ■

IMPROVED EULER'S METHOD Here we give an improvement to Euler's method. As above, we define $h > 0$ to be the *step size* of the method and take

$$x_k = x_0 + kh, \qquad \text{where} \;\; k \;\; \text{is a natural number.}$$

Also, we continue to use the notation $y_k \approx y(x_k)$ for the approximations to $y(x)$ when $x = x_k$. We can use two different approximations for the derivative. Namely,

$$y'(x_k) \approx \frac{y_{k+1} - y_k}{h}$$

and

$$y'(x_{k+1}) \approx \frac{y_{k+1} - y_k}{h}.$$

Since $y'(x_k) = f(x_k, y(x_k))$ and $y'(x_{k+1}) = f(x_{k+1}, y(x_{k+1}))$, substitution gives

$$\frac{y_{k+1} - y_k}{h} \approx f(x_k, y(x_k))$$

and

$$\frac{y_{k+1} - y_k}{h} \approx f(x_{k+1}, y(x_{k+1})).$$

Adding these equations and solving for y_{k+1} gives

$$y_{k+1} = y_k + \frac{h}{2}\left(f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right)$$

Unfortunately, this scheme is not easy to implement because y_{k+1} occurs on both the left and right side of the equation (for this reason it is called an implicit scheme). We avoid the implicit nature of this equation by replacing y_{k+1} on the right side by its Euler approximate (using y_k as a guess). That is

$$y_{k+1} = y_k + \frac{h}{2}\left(f(x_k, y_k) + f(x_{k+1}, M)\right) \quad \text{where} \quad M = y_k + hf(x_k, y_k) \quad \text{and} \quad y_0 \text{ is given.} \tag{4}$$

This method (4) is commonly known as the *Improved Euler's Method with step size h.*

Example 2. Apply the Improved Euler's Method with step size 0.05 to the initial value problem

$$y' = -y + \sin x; \quad y(0) = 1$$

As noted in Example 1, this initial value problem has the exact solution

$$y(x) = -\tfrac{1}{2}\cos x + \tfrac{1}{2}\sin x + \tfrac{3}{2}e^{-x}$$

Using the Improved Euler's method with a step size of $h = 0.05$ gives the values

$$\begin{aligned} y_1 &= 0.952499479 \\ y_2 &= 0.909747970 \\ y_3 &= 0.871504753 \\ &\vdots \\ y_{20} &= 0.702609956 \end{aligned}$$

Again, y_{20} is an approximation to $y(t_{20}) = y(1) = 0.7024035012$. In this case, the error at $x = 1$ is given by

$$|y(1) - y_{20}| \approx 2.06454773 \times 10^{-4}$$

This is much better than our earlier result from Euler's method. ■

The accuracy of these methods can be predicted. In general,

$$\text{Euler's Method:} \quad |y(x_k) - y_k| \leq L_1 k h^2$$

and

$$\text{Improved Euler's Method:} \quad |y(x_k) - y_k| \leq L_2 k h^3$$

where the constants L_1 and L_2 are dependent upon the actual solution, but independent of the step size h and the number k. Notice that the error estimates above imply Euler's

Method has an error which is a factor of h over every x-interval (there are essentially $k = 1/h$ steps of size h across each x-interval) and the Improved Euler's Method has an error which is a factor of h^2 over every x-interval. It is not hard to see that small values of h should give a much smaller error for Improved Euler's Method than for Euler's Method.

Exercises 2.6

1. Use both the Euler and Improved Euler Methods with a step size of $h = 0.01$ to estimate $y(2)$ where y is the solution of the initial-value problem

$$y' = \frac{1}{2y}; \quad y(1) = 2.$$

   Compare your values with those of the exact solution.

2. Use both the Euler and Improved Euler Methods with a step size of $h = 0.02$ to estimate $y(1)$ where y is the solution of the initial-value problem

$$y' = x + y; \quad y(0) = 2.$$

   Compare your values with those of the exact solution.

3. Use both the Euler and Improved Euler Methods with a step size of $h = 0.05$ to approximate the solution of

$$y' = y(4 - y); \quad y(0) = 2.$$

   Compare your values with those of the exact solution

$$y(x) = \frac{4}{1 + e^{-4x}}$$

   for $x = 1/20,\ 1/10,\ 3/20,\ \ldots,\ 19/20,\ 1.$

4. Use the Improved Euler's Method with a step size of $h = 0.1$ to approximate $y(0.2)$ where $y(t)$ is the unique solution of

$$y' = \sin\, x - y^3; \quad y0) = 1.$$