

A measure defined on a Borel  $\sigma$ -algebra is called a Borel measure. Write  $\mathcal{M}(\mathbb{R}^n)$  for the Borel measures on  $\mathbb{R}^n$ . If  $E \subset \mathbb{R}^n$ , then  $\mathcal{M}(E)$  are the borel measures on  $E$ , but  $\mathcal{M}(E) \subset \mathcal{M}(\mathbb{R}^n)$ , by zero padding.

Definition: a positive finite measure  $\mu$  on  $\mathbb{R}^n$  is regular if

1.  $\mu(E) = \inf\{\mu(U); U \text{ open}, U \supset E\}$
2.  $\mu(E) = \sup\{\mu(K); K \text{ compact}, K \subset E\}$

Every positive finite measure on  $\mathbb{R}^n$  is regular. For every measure  $\nu \in \mathcal{M}(\mathbb{R}^n)$ , for all  $E \in \mathcal{B}(\mathbb{R}^n)$ , there exists a sequence of open sets  $U_k \supset E$ , and compact sets  $K_n \subset E$  such that  $\nu(U_n) \rightarrow \nu(E)$  and  $\nu(K_n) \rightarrow \nu(E)$ .

Proof: ADD

Definition: we say a measure  $\nu \in \mathcal{M}(\mathbb{R}^n)$  is regular if each positive  $\nu_k$  in the Jordan decomposition  $\nu = \sum_{k=0}^3 i^k \nu_k$  is regular. By the last result, every  $\nu \in \mathcal{M}(\mathbb{R}^n)$  is regular and then the condition about sequences of sets holds.

If  $\nu \in \mathcal{M}(\mathbb{R})$ , we define its distribution function by  $F_\nu(x) = \nu((-\infty, x])$ .  $\nu \mapsto F_\nu$  is injective and linear on  $\mathcal{M}(\mathbb{R})$ .

Def:  $F : \mathbb{R} \rightarrow \mathbb{R}$  is of bounded variation, BV, or say  $F \in BV$ , if  $\text{Var}(F) < \infty$ , where  $\text{Var}(F) := \sup\{V_F(x); x \in \mathbb{R}\}$ , and

$$V_F(x) := \sup \left\{ \sum_{k=1}^n |F(x_k) - F(x_{k-1})|; x_0 < x_1 < \dots < x_n = x, \{x_k\} \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

Def:  $F : \mathbb{R} \rightarrow \mathbb{R}$  is in  $NBV$  if  $F \in BV$ ,  $F$  is right continuous at all  $x \in \mathbb{R}$ , and  $\lim_{x \rightarrow -\infty} F(x) = 0$ ; normalized  $BV$ .

If  $\nu \in \mathcal{M}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , then  $F_\nu \in NBV$ .

Proof: ADD

Properties of  $BV$ ,

1. If  $F : \mathbb{R} \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$ , then  $V_{cF}(x) \leq V_F(x) + V_F(x)$  and  $V_{cF}(x) = |c|V_F(x)$ . Hence  $BV$  is a vector space and if  $F, G \in BV$ , then  $\text{Var}(F + G) \leq \text{Var}(F) + \text{Var}(G)$  and  $\text{Var}(cF) = |c|\text{Var}(F)$