Notes on Integration and differentiation on $\mathbb{R}^n$, etc.
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A measure defined on a Borel σ-algebra is called a Borel measure. Write $\mathcal{M}(\mathbb{R}^n)$ for the Borel measures on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$, then $\mathcal{M}(E)$ are the borel measures on E, but $\mathcal{M}(E) \subset \mathcal{M}(\mathbb{R}^n)$, by zero padding.

Definition: a positive finite measure μ on $\mathbb{R}^n$ is regular if

1. $\mu(E) = \inf\{\mu(U); U \text{ open}, U \supset E\}$
2. $\mu(E) = \sup\{\mu(K); K \text{ compact}, K \subset E\}$

Every positive finite measure on $\mathbb{R}^n$ is regular. For every measure $\nu \in \mathcal{M}(\mathbb{R}^n)$, for all $E \in \mathcal{B}\ell(\mathbb{R}^n)$, there exists a sequence of open sets $U_k \supset E$, and compact sets $K_n \subset E$ such that $\nu(U_n) \to \nu(E)$ and $\nu(K_n) \to \nu(E)$.

Proof: ADD

Definition: we say a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular if each positive ν_k, in the Jordan decomposition $\nu = \sum_{k=0}^{3} i^k \nu_k$, is regular. By the last result, every $\nu \in \mathcal{M}(\mathbb{R}^n)$ is regular and then the condition about sequences of sets holds.

If $\nu \in \mathcal{M}(\mathbb{R})$, we define its distribution function by $F_\nu(x) = \nu((\infty, x])$. $\nu \mapsto F_\nu$ is injective and linear on $\mathcal{M}(\mathbb{R})$.

Proof: ADD

Def: $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation, BV, or say $F \in BV$, if $\mathrm{Var}(F) < \infty$, where $\mathrm{Var}(F) := \sup\{V_F(x); x \in \mathbb{R}\}$, and

$$V_F(x) := \sup\left\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|;\ x_0 < x_1 < \ldots < x_n = x, (x_k) \in \mathbb{R}, n \in \mathbb{N}\right\}.$$

Def: $F : \mathbb{R} \to \mathbb{R}$ is in NBV if $F \in BV$, F is right continuous at all $x \in \mathbb{R}$, and $\lim_{x\to-\infty} F(x) = 0$; normalized BV.

If $\nu \in \mathcal{M}(\mathbb{R}, \mathcal{B}\ell(\mathbb{R}))$, then $F_\nu \in NBV$.

Proof: ADD

(Folland 3.28) If $F \in BV$ then $\lim_{x\to-\infty} V_F(x) = 0$ and $F \in BV \Rightarrow V_F \in NBV$.
Proof: ADD

Properties of BV,

1) If $F, G : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, then $V_{F+G}(x) \le V_F(x) + V_F(x)$ and $V_{cF}(x) = |c|V_F(x)$. Hence BV is a vector space and if $F, G \in BV$, then $\mathrm{Var}(F+G) \le \mathrm{Var}(F) + \mathrm{Var}(G)$ and $\mathrm{Var}(cF) = |c|\mathrm{Var}(F)$. NBV is a subspace of BV.

2) If $F \in BV$, then $V_F(x)$ is an increasing function of x, bounded above by $\mathrm{Var}(F)$.

3) a) Moreover, if $x < y$, then $V_F(y) - V_F(x) = \sup\left(\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|; x \leq x_0 < x_1 < ... < x_n = y\}\right)$.

   b) special capse: $F(y) - F(x) \leq V_F(y) - V_F(x) \leq V_F(y) \leq \mathrm{Var}(F)$.

   c) consequence: $F \in BV \Rightarrow F$ is bounded.

4) An increasing $F : \mathbb{R} \to \mathbb{R}$ is in BV iff F is bounded.

5) $F : \mathbb{R} \to \mathbb{R} \in BV$ iff $F = F_1 - F_2$, where $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are bdd and increasing.

6) $F : \mathbb{R} \to 0\mathbb{C} \in BV$ iff $\mathrm{Re}\, F, \mathrm{Im}\, F \in BV$.

7) $F \in BV \Rightarrow F$ continuous except at countable many points, and for all $x \in \mathbb{R}$, $F(x+) = \lim_{t \to x^+} F(t)$ and $F(x-) = \lim_{t \to x^-} F(t)$, and $\lim_{x \to +\infty} F(x)$ and $\lim_{x \to -\infty} F(x)$ all exist and are in $\mathbb{R}$.

8) $F \in BV \Leftrightarrow F = F_1 - F_2 + iF_3 - iF_4$, where $F_k : \mathbb{R} \to \mathbb{R}$, increasing, bounded, right continuous, and $\lim_{x \to -\infty} F_k(x) = 0$ for all k.

Proof: ADD

The linear map $T = \nu \mapsto F_\nu$ from $\mathcal{M}(\mathbb{R})$ to NBV is an isomorphism. Thus it is bijective and $\mathrm{Var}(F_\nu) = ||\nu||$ for all $\nu \in \mathcal{M}(\mathbb{R})$, which implies that NBV is a Banach space with norm $||F|| = \mathrm{Var}(F)$, $||T(\nu)|| = ||\nu||$.

Proof: ADD