$(\Omega, \mathcal{U}, P)$ a probability space.

Problem 1

$Z : \Omega \to \{0,1\}$, measurable, $P(Z^{-1}(\{1\})) = P(Z^{-1}(\{0\})) = 0.5$, so let $A = Z^{-1}(\{1\}), B = A^c$, then $\mathcal{A} := \{\phi, A, B, A\cup B\}$ is a σ-algebra, $\Omega = A \cup B, A \cap B = \phi$, and Z is $\mathcal{A}$-measurable, so we have $Z = \mathbf{1}_A$.

Let $X_n : \Omega^n \to \{1, 2, ..., n\}$, for $\omega \in \Omega^n$, write $\omega = (\omega_k)$, then set $X_n((\omega_k)) = \sum_{k=1}^n Z(\omega_k)$.

Then $X_n^{-1}(\{n\}) = A \times A \times ... \times A$, n-times. $X_n^{-1}(\{n-1\}) = B \times A \times ... \times A \cup A \times B \times A \times ... \times A \cup ... \cup A \times A \times ... \times A \times B$, ..., $X_n^{-1}(\{0\}) = B \times B \times ... \times B$. So for any $0 \le k \le n$, $X_n^{-1}(\{k\})$ is a union of all permutations of cartesian products of k many sets A, and $n-k$ many sets B, and so is a union of measurable rectangles of sets from $\mathcal{A}$, and so $X_n^{-1}(\{k\}) \in \mathcal{A} \times \mathcal{A} \times ... \times \mathcal{A} =: \mathcal{A}^n$, the product σ-algebra.

Let P^n be the product measure on $\mathcal{A}^n$, i.e., $P^n = P \times P \times ... \times P$, n-times. Then $E_{P^n}[f] = \int_{\Omega^n} f \, dP^n = \int$. By Fubini's theorem, $E_{P^n}[\mathbf{1}_A] = \int_\Omega \left(... \left(\int_\Omega \mathbf{1}_A \, dP\right) ...\right) dP = \int_\Omega \left(... \int_\Omega (P(A)) \, dP...\right) dP = P(A) \int_\Omega \left(... \int_\Omega (1) \, dP...\right) dP$ $= P(A) \cdot 1 = P(A)$. Thus $E_{P^n}[X_n] = \sum_{k=1}^n E_{P^n}[Z] = \sum_{k=1}^n E_{P^n}[\mathbf{1}_A] = \sum_{k=1}^n P(A) = \frac{1}{2}n$

So, with $X(\omega', (\omega_k)) = \sum_{k=1}^{N(\omega')} Z(\omega_k)$, where $N(\omega') = \sum_{k=1}^4 k \mathbf{1}_{A_k}(\omega')$, $P(A_1) = 0.5, P(A_2) = 0.1, P(A_3) = 0.2, P(A_4) = 0.2$, and $\{A_k\}$ are independent events, $\Omega = \cup_k A_k$. So $E[X|N] = \sum_{k=1}^4 \frac{1}{P(A_k)} E[\mathbf{1}_{A_k} X] \mathbf{1}_{A_k}$. Now $E[\mathbf{1}_{A_k} X] = \int_{\Omega \times \Omega^4} \mathbf{1}_{A_k} X \, d(P \times P^4) = \int_{\Omega \times \Omega^4} \mathbf{1}_{A_k}(\omega') \sum_{k=1}^{N(\omega')} \mathbf{1}_A(\omega_k) \, d(P \times P^4)$

Problem 2

Consider the discrete stochastic process, $X : \mathbb{N} \times \Omega \to \mathbb{Z}$, where $X_0 > 0$, $X_{n+1} = 0$ if $X_n = 0$, and if $X_n > 0$, then $X_{n+1} = X_n \pm 1$ with each half probability. X_0 is a parameter of the process; it is not a random variable.

1) X is a non-negative martingale. First, we already have that X_0 is positive, suppose that $X_n > 0$, then by definition, $X_{n+1} = X_n \pm 1 > 0$, then and if $X_n = 0$ then $X_{n+1} = 0 \ge 0$, so by induction, $X_n(\omega) \ge 0$ for all $x \in \mathbb{N}$, $X_n(\omega) = |X_n(\omega)|$.

Problem 3

$X : \mathbb{N} \times \Omega \to \mathbb{N}$, let $\mathbb{P}_k = (P_{k,i,j}), P_{k,i,j} = P(\{X_{k+1} = i\}|\{X_k = j\})$, $x_k = (x_{k,i}), x_{k,i} = P(\{X_k = i\})$. Claim: $x_{k+1} = \mathbb{P}_k x_k$, so $x_{k+1,i} = \sum_j P_{k,i,j} x_{k,j}$. Proof: $P_{k,i,j} = P(\{X_{k+1} = i\}|\{X_k = j\}) = \frac{P(\{X_{k+1}=i\} \cap \{X_k=j\})}{P(\{X_k=i\})}$, so $\sum_j P_{k,i,j} x_{k,j} = \sum_j \frac{P(\{X_{k+1}=i\} \cap \{X_k=j\})}{P(\{X_k=j\})} \cdot P(\{X_k = j\}) = \sum_j P(\{X_{k+1} = i\} \cap \{X_k = j\})$, now $\{X_k = j_1\} \cap \{X_k = j_2\} = \phi$ when $i_1 \ne j_2$, becasue inverse images commute with intersections, then $\sum_j P_{k,i,j} x_{k,i} = P\left(\cup_j \left(\{X_{k+1} = i\} \cap \{X_k = j\}\right)\right) = P\left(\{X_{k+1} = i\} \cap \cup_j \{X_k = j\}\right)$, by countable additivity. Now, $\cup_{j \in \mathbb{N}} \{X_k = j\} = \Omega$, so $\sum_j P_{k,i,j} x_{k,i} = P\left(\{X_{k+1} = i\}\right) = x_{k+1}$, and the claim is proved.

If $\Omega = \cup_k E_k, \{E_k\}$ is disjoint, $A, E_k \in \mathcal{U}$, then $\sum_k P(E_k|A) = \sum_k \frac{(P(E_k \cap A)}{P(A)} = \frac{1}{P(A)} P\left(\cup_k (E_k \cap A)\right) = \frac{1}{P(A)} P\left(A \cap \cup_k (E_k)\right) = \frac{1}{P(A)} P\left(A \cap \Omega\right) = 1$. Then, $X_k^{-1}(i), i \in \mathbb{N}$ generates a measurable disjoint partition of Ω, thus $\sum_{i \in \mathbb{N}} P_{k,i,j} = \sum_{i \in \mathbb{N}} P(\{X_{k+1} = i\}|\{X_k = j\}) = 1$. So the columns of $\mathbb{P}$ are are normalized, i.e. they sum to one. Then, if x_k is normalized in the same sense, so $\sum_i x_i = 1$, and all $x_i \ge 0$, then $x_{k+1} = \mathbb{P} x_k$, $\sum_i x_{k+1,i} = \sum_i \sum_j P_{k,i,j} x_{k,j} = \sum_j \sum_i P_{k,i,j} x_{k,j} = \sum_j x_{k,j} \sum_i P_{k,i,j} = \sum_j x_{k,j} \cdot 1 = 1$, because all pobabilities are non-negative, as are the entries in x_k, and clearly $x_{k+1,i} \geq 0$ for all i. This shows that $\mathbb{P}$ preserves normalization, we we'd expect.

Problem 4

Problem 5