High-Dimensional Measures and Geometry

Lecture Notes from Mar 3, 2010

taken by Nick Maxwell

0.0.1 Theorem. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $E[f] = 0$, then $E[f] \le E\left[\exp(\frac{\pi^2}{8}||\nabla f||^2)\right]$, assuming this RHS is finite.*

Proof.

$$F(x, y) := f(x) - f(y)$$

$$E\left[e^f\right] \le E\left[e^F\right] = E\left[\exp\left(-\int_0^{\pi/2} \frac{\partial}{\partial\theta} G(x, y, \theta)\, d\theta\right)\right] = E\left[\exp\left(-\int_0^{\pi/2} \nabla f(x(\theta)) \cdot x'(\theta)\, d\theta\right)\right]$$

with $G(x, y, \theta) = f(x(\theta)) = f(x\cos\theta, y\cos\theta)$, and $x'(\theta) = -x\sin\theta + y\cos\theta = y(\theta)$, which looks like rotation.

$$E\left[e^F\right] = E\left[\exp\left(-\frac{2}{\pi}\int_0^{\pi/2} \frac{\pi}{2}\frac{\partial}{\partial\theta} G(x, y, \theta)\, d\theta\right)\right] \underset{\text{Fubini}}{\overset{\text{Jensen}}{\le}} \frac{2}{\pi}\int_0^{\pi/2} E\left[\exp\left(-\frac{\pi}{2}\nabla f(x(\theta)) \cdot x'(\theta)\right)\right] d\theta.$$

Fix θ, and change variavles,

□