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Homework

chapter 1

Folland problem 1.1:

A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a ring if it is clused under finite unions and differences. A ring that is closed under countable unions is called a σ-ring. By definition, a ring is also closed under symmetric differences.
a.) $A, B, E_n \in \mathcal{R}, n \in \mathbb{N}$.
$B \cap A = [B \cup (B \cap A)] \cap [A \cup (A \cap B)] = [B \cup (B \cup (B \cap A))] \cap [A \cup (A \cup (A \cap B))] =$
$[B \cup (B \cup (B \cap A))] \cap [A \cup (A \cup (A \cap B))] = [B \cup ((B \cap A) \cup (B \cap B))] \cap [A \cup ((A \cap B) \cup (A \cap A))] =$
$[(B \cap (A \cup B)) \cup ((B \cap A) \cup (B \cap B))] \cap [(A \cap (A \cup B)) \cup ((A \cap B) \cup (A \cap A))] =$
$[(B \cap (A \cup B)) \cup ((B \cap A) \setminus A)] \cap [(A \cap (A \cup B)) \cup ((A \cap B) \setminus B))] =$
$[(A \cup B) \setminus (A \setminus B)] \cap [(a \cup B) \setminus (B \setminus A)] = (A \cup B) \setminus ((A \setminus B) \cup (B \setminus A)) = (A \cup B) \setminus (A \Delta B) \in \mathcal{R}$
Let $P_n = \cap_{k=1}^{n} E_k$. $P_1 = E_1 \in \mathcal{R}$. Suppose $P_n \in \mathcal{R}, P_n \cap E_{n+1} = P_{n+1} \in \mathcal{R}$, as we have shown that $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$, and $E_{k+1} \in \mathcal{R}$, thus $\cap_{k \in \mathbb{N}} E_k \in \mathcal{R}$.
b.) $A, B, E_n \in \mathcal{R}, n \in \mathbb{N}$. Then $A \setminus A = \phi \in \mathcal{R}$. This satisfies (1) in the definition of an (σ)-algebra. The (σ)-ring is already closed under (countable) unions, satisfying (3). If $A, X \in \mathcal{R}$, where $\mathcal{R} \subset \mathcal{P}(X)$, then $A \subset X$, and then by definition, $A^c = X \setminus A \in \mathcal{R}$, satisfying condition (2), and thus $\mathcal{R}$ is a (σ)-algebra if it contains X. If it does not, the complement of A may reach outside of $\cup_{E \in \mathcal{R}} E$. If $\mathcal{R}$ is a (σ)-algebra, $\phi \in \mathcal{R}$, and $\phi^c = X \in \mathcal{R}$, the closure under (countable) union requirement is again automatic. Let $A, B \in \mathcal{R}$, the same (σ)-algebra, then $A \setminus B = A \cap B^c \in \mathcal{R} \Leftarrow \mathcal{R}$ closed under intersections, via De Morgan's laws. Thus a (σ)-algebra is a (σ)-ring, containing its parent set, X.
So, $\mathcal{R}$ a (σ)-ring, $X \in \mathcal{R} \Leftrightarrow \mathcal{R}$ a (σ)-algebra.
c.) $\mathcal{R}$ a σ-ring over X, $\mathcal{A} = \{E \subset X : E \in \mathcal{R}$ or $E^c \in \mathcal{R}\}$. We've already shown that any σ-ring contains ϕ. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, as this still satisfies the "or" condition in the definition of $\mathcal{A}$. If that condition was an exclusive or, then this would be false. Let $A_n \in \mathcal{A}$, then $\cup_{n \in \mathbb{N}} A_n \in$

Folland problem 1.4:

$\mathcal{A}$ an algebra.
$\mathcal{A}$ a σ-algebra $\Rightarrow$ if $E_k \in \mathcal{A}, k \in \mathbb{N}$. Let $B_k = \cup_{j=1}^{k} E_j$. Then by construction $B_k \subset B_{k+1}$.

Folland problem 1.7:

$\mu_1, \mu_2, ..., \mu_n$ measures on $(X, \mathcal{A})$, and $a_1, a_2, ..., a_n \in [0, \infty)$. Let $\mu(A) = \sum_{k=1}^{n} a_k\, \mu_k(A)$, for $A \in \mathcal{A}$.
a) $\mu(\phi) = \sum_{k=1}^{n} a_k\, \mu_k(\phi) = \sum_{k=1}^{n} 0 = 0$.
b) $A_j \in \mathcal{A}, j \in \mathbb{N}, A_j$ disjoint. $\mu(\cup_{j \in \mathbb{N}} A_j) = \sum_{k=1}^{n} a_k\, \mu_k(\cup_{j \in \mathbb{N}} A_j) = \sum_{k=1}^{n} a_k \sum_{j \in \mathbb{N}} \mu_k(A_j) =$
$\sum_{k=1}^{n} \sum_{j \in \mathbb{N}} a_k\, \mu_k(A_j) = \sum_{j \in \mathbb{N}} \sum_{k=1}^{n} a_k\, \mu_k(A_j) = \sum_{j \in \mathbb{N}} \mu(A_j)$.

Folland problem 1.12:

a) $(X, \mathcal{A}, \mu)$ a finite measure space, $A, B \in \mathcal{A}$. $\mu(A \Delta B) = 0 \Rightarrow \mu((A \setminus B) \cup (B \setminus A)) = 0 \Rightarrow$
$\mu(A \setminus B) + \mu(B \setminus A) = 0$, by $(A \setminus B) \cap (B \setminus A) = A \cap B^c \cap B \cap A^c = \phi$. Let $x = \mu(A \setminus B), y = \mu(B \setminus A)$, then $x + y = 0$. Now μ is finite on $\mathcal{A}$, so $x, y \in [0, \infty)$. Then $x = -y$. Now because x, y are positive or zero, the only solutions to $x = -y$ are $x = y = 0$, so $\mu(A \setminus B) = 0, \mu(B \setminus A) = 0$. And again becuase μ is finite on $\mathcal{A}$ we can rearange Caratheodory to get $\mu(A \setminus B) = \mu(A) - \mu(A \cap B) = 0 \Rightarrow \mu(A) = \mu(A \cap B)$, and also $\mu(B) = \mu(B \cap A)$, then by subtracting these equations, we have $\mu(A) = \mu(B)$.

Folland problem 1.8:

$(X, \mathcal{A}, \mu)$ a measure space, E_n a sequence of sets, $E_n \in \mathcal{A}$. Let $A_k = \cap_{n=k}^{\infty} E_n$, then $A_k = E_k \cap A_{k+1}$, then $x \in A_k \Rightarrow x \in E_k \cap A_{k+1} \Rightarrow x \in A_{k+1} \Rightarrow A_k \subset A_{k+1}$, and $A_k \in \mathcal{A}$. Then by continuity from below, $\mu\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$, $\mu\left(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n\right) = \lim_{k\to\infty} \mu(\cap_{n=k}^{\infty} E_n)$, $\mu\left(\liminf E_n\right) = \lim_{k\to\infty} \mu(\cap_{n=k}^{\infty} E_n)$, (not complete)

Folland problem 1.9:

$(X, \mathcal{A}, \mu)$ a measure space, $A, B \in \mathcal{A}$. If either $\mu(A) = \infty$ or $\mu(B) = \infty$ or both, then $\mu(A) + \mu(B) = \infty$, and $\mu(A \cup B) = \infty$, and thus, in these cases, $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$. Otherwise, $\mu(A) < \infty$ and $\mu(B) < \infty$, and hence $\mu(A \cup B) < \infty$ by subaditivity, and then $A \cap B \subset A \cup B$, so by monotonicity, $\mu(A \cap B) < \infty$. Then we need that $A \cup B = A \Delta B \cup A \cap B$, $A = A \cap B \cup A \cap B^c$, $B = B \cap A \cup B \cap A^c$. Then $\mu(A \cup B) = \mu(A \Delta B) + \mu(A \cap B)$, $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c)$, $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$, and $\mu(A \Delta B) = \mu(A \cap B^c) + \mu(B \cap A^c)$. Then we add, $\mu(A) + \mu(B) = 2\,\mu(A \cap B) + \mu(A \Delta B)$, and rearange $\mu(A \cup B) - \mu(A \cap B) = \mu(A \Delta B)$, which is valid becuase we verified that $\mu(A \cap B) < \infty$. Then adding these two, $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$.
Thus, for any $A, B \in \mathcal{A}$, $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$.

Folland problem 1.13:

$(X, \mathcal{A}, \mu)$ σ-finite. Suppose $E \in \mathcal{A}$, $\mu(E) = \infty$, then $E \neq \phi$. σ-finite implies $X = \cup_{k\in\mathbb{N}} E_k$, $E_k \in \mathcal{A}$, $\mu(E_k) < \infty$. Let $B_k = E_k \cap E$. Let $K = \{k \in \mathbb{N} : \mu(B_k) > 0\}$. Clearly $K \neq \phi$; if it was then all $\mu(B_k) = 0$, in which case $\mu(X \cap E) = \mu(\cup_{k\in\mathbb{N}} B_k) \leq \sum_{k\in\mathbb{N}} \mu(B_k) = 0$, which is false, because $E = E \cap X$, $\mu(E) > 0$. Also, for all $k \in \mathbb{N}$, $\mu(B_k) < \infty$; $B_k = E_k \cap E \subset E_k \Rightarrow \mu(B_k) \leq \mu(E_k) < \infty$, by monotonicity. So, for any $E \in \mathcal{A}, \mu(E) = \infty$, $\exists B_k \in \mathcal{A}$, $0 < \mu(B_k) < \infty$, $B_k \in \mathcal{A}$, $B_k \subset E$, for any $k \in K \neq \phi$. Thus σ-finite $\Rightarrow$ semifinite.

supplementary problem 3:

c) $A \subset \mathbb{R}^n, A$ open. Let $j, k_1, k_2, ..., k_n \in \mathbb{Z}$, write $(k_1, k_2, ..., k_n) = (k_i)$, using Π for cartesian pruducts, define $R_{j,(k_i)} = \Pi_{i=1}^{n} [k_i\, 2^{-j}, (k_i + 1)\, 2^{-j})$, Then $\mathbb{R}^n = \cup_{(k_i)\in\mathbb{Z}^n} R_{j,(k_i)}$ for any j. Let $\Gamma_{j,(k_i)} = R_{j,(k_i)}$ if $R_{j,(k_i)} \subset A$, $\Gamma_{j,(k_i)} = \phi$, otherwise. Let $\Omega_j = \cup_{(k_i)\in\mathbb{Z}^n} \Gamma_{j,(k_i)}$ Let $\Upsilon_0 = \Omega_0, \Upsilon_j = \Omega_j \setminus \cap_{j'=0}^{j-1} \Upsilon_{j'}$. Then define $\tilde{A} = \cup_{j\in\mathbb{N}} \Upsilon_j$. Then, $\tilde{A}$ is a countable union of finite cartesian products of half open intervals, which are disjoint. Also, $\Upsilon_j \subset \Upsilon_{j+1}$ by construction; the $R_{j,(k_i)}$ are dyadic intervals. If $x \in \tilde{A}$ then $x \in \Gamma_{j,(k_i)} \subset A$, some $j, (k_i)$, so $\tilde{A} \subset A$.
Proposition: If we have a ball, $B(\epsilon, x) \subset \mathbb{R}^n$, the n-cubes (the $R_{j,(k_i)}$ above) we could fit in it would have as the length of one of their sides ℓ, such that $\epsilon \geq \sqrt{\sum_{i=1}^{n} (\ell/2)^2} \to \ell \leq \frac{2\epsilon}{\sqrt{n}}$. Now, due to alignment problems, the center of such a cube and the ball may different; the distance between the two centers in any direction may be up to $\frac{1}{2}\ell$ by periodicity, so we half the size of the cubes. Then we'll always be able to fit in the ball, some cube from the mesh $\cup_{(k_i)\in\mathbb{Z}^n} R_{j,(k_i)} = \mathbb{R}^n$, with $2^{-j} \leq \frac{\epsilon}{\sqrt{n}} \to j \geq \text{ceil}\left(\frac{1}{2}\log_2 n - \log_2 \epsilon\right)$.
If $x \in A$, then by A open, there exists a ball, $B(\epsilon, x) \subset A \subset \mathbb{R}^n, \epsilon > 0$. By the proposition above, we can always find a $j, (k_i)$ such that $x \in R_{j,(k_i)} \subset B(\epsilon, x)$, which means we'll be able to find a $\Gamma_{j,(k_i)}$ with $x \in \Gamma_{j,(k_i)} \subset B(\epsilon, x)$, because that $R_{j,(k_i)}$ is contained in the ϵ ball, which is contained in A, which means by contruction that $\Gamma_{j,(k_i)} = R_{j,(k_i)}$. This being the case we can claim by the construction of $\tilde{A}$, that $\Gamma_{j,(k_i)} \subset \tilde{A}$. So we have that $x \in A \Rightarrow x \in \tilde{A}$, so $A \subset \tilde{A}$. We've already shown that $\tilde{A} \subset A$, so we can say that $A = \tilde{A}$.

supplementary problem 6:

Def: E_n a sequence of sets, $n \in \mathbb{N}$. Take the statement " $x \in E_n$ for all but finitely many n" to precicely mean "$\exists k \in \mathbb{N}$ s.t. $x \in \cap_{n=k}^{\infty} E_n$". Then, " $x \in E_n$ for infinitely many n" means "$x \in \cup_{n=k}^{\infty} E_n$, $\forall k \in \mathbb{N}$".

Prop 1: (Folland, p2)
$\limsup E_n = \{x : x \in E_n \text{ for infinitely many } n\}$

$$x \in \limsup E_n = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n \Leftrightarrow (x \in \cup_{n=k}^{\infty} E_n,\ \forall k \in \mathbb{N}) \quad (\text{ by the definition of intersection})$$

Prop 2: (Folland, p2)
$\liminf E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$

$$x \in \liminf E_n = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n \Leftrightarrow (x \in \cap_{n=k}^{\infty} E_n,\ \text{some } k \in \mathbb{N}) \quad (\text{ by the definition of union})$$

$(X, \mathcal{A}, \mu)$ a measure space, E_n a sequence of sets, $E_n \in \mathcal{A}$, μ a finite measure, and $\mu(E_n) > \alpha > 0$. Then we take $\limsup(E_n)$, by Folland problem 1.8,

$$\mu(\limsup(E_n)) \geq \limsup \mu(E_n) \geq \liminf \mu(E_n) \geq \inf \mu(E_n) \geq \alpha > 0,$$

$$\text{so } \mu(\limsup(E_n)) > 0 \Rightarrow \limsup(E_n) \neq \phi \Rightarrow \exists x \in \limsup(E_n) \Rightarrow \exists x \in E_n \subset X \text{ for infinitely many } n$$

supplementary problem 8:

$(X, \mathcal{A}, \mu)$ a measure space. $f : Y \to X$. $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$, $f^{-1}(A) = \{y \in Y : f(y) \in A\}$

Let $\mathcal{B} = f^{-1}(\mathcal{A})$. We have three functions here; $f : Y \to X$, maps elements in Y to elements in X, $f^{-1}(\mathcal{A})$ maps sigma algebras to sigma algebras, and $f^{-1}(A) : \mathcal{A} \to \mathcal{B}$ maps elements in one sigma algebra to elements in another. Write $F^{-1} : \mathcal{A} \to \mathcal{B}$, $F^{-1}(A) = \{y \in Y : f(y) \in A\}$.

8a) $\phi \in \mathcal{A}$, $\nexists f(y) \in \phi \Rightarrow f^{-1}(\phi) = \phi \Rightarrow \phi \in \mathcal{B}$.

8b) $B \in \mathcal{B} \Rightarrow f(B) = A$, some $A \in \mathcal{A}$. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \Rightarrow (f(B))^c \in \mathcal{A}$. Then by the definition of $\mathcal{B}, F^{-1}(f(B))^c) \in \mathcal{B}$, and the commutativity of complements and inverse images,
$F^{-1}(f(B))^c) = F^{-1}(f(B)))^c \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$. We've used that
$F^{-1}(f(B)) = B \Leftrightarrow F^{-1}(\{f(x) : x \in B\}) = B \Leftrightarrow \{y \in Y : f(y) \in \{f(x) : x \in B\}\} = B$.

8c) B_k a sequence in $\mathcal{B}$. $x \in f(\cup_{k \in \mathbb{N}} B_k) \Leftrightarrow (x \in f(B_k),\ \text{some } k \in \mathbb{N}) \Leftrightarrow x \in \cup_{k \in \mathbb{N}} f(B_k)$. Thus $f(\cup_{k \in \mathbb{N}} B_k) = \cup_{k \in \mathbb{N}} f(B_k)$. Now, $f(B_k) \in \mathcal{A}$, which is a sigma algebra, so $f(\cup_{k \in \mathbb{N}} B_k) = \cup_{k \in \mathbb{N}} f(B_k) \in \mathcal{A}$, so $f(\cup_{k \in \mathbb{N}} B_k) \in \mathcal{B}$ by the definition of $\mathcal{B}$.

Thus, $\mathcal{B}$ is a sigma algebra on Y. Let f be bijective and $\nu(B) = \mu(f(B)), B \in \mathcal{B}$. Then
$\nu(\phi) = \mu(f(\phi)) = \mu(\{f(x) : x \in \phi\}) = \mu(\phi) = 0$.
$B_k \in \mathcal{B}, k \in \mathbb{N}, B_k$ disjoint, then $f(B_k) = A_k \in \mathcal{A}$,
$\nu(\cup_{k \in \mathbb{N}} B_k) = \mu(f(\cup_{k \in \mathbb{N}} B_k)) = \mu(\cup_{k \in \mathbb{N}} f(B_k)) = \mu(\cup_{k \in \mathbb{N}} A_k) = \sum_{k \in \mathbb{N}} \mu(A_k)$
$= \sum_{k \in \mathbb{N}} \mu(f(B_k)) = \sum_{k \in \mathbb{N}} \nu(B_k)$. We've used that the direct images of f commutes with unions (Folland, p.8). We also needed that the A_k are disjoint; let $g : X \to Y$, $A, B \subset Y, A \cap B = \phi$, then
$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$, using commutativity of inverse images and intersections, thus the A_k inherit their disjointedness from the B_k. Thus, $(Y, \mathcal{B}, \nu)$ is a measure space.

supplementary problem 9:

$(X, \mathcal{A}, \mu)$ a measure space. $f : X \to Y$. $\mathcal{B} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$.

9a) $\phi \in \mathcal{A}, f(\phi) = \{f(x) : x \in \phi\} = \phi \Rightarrow \phi \in \mathcal{B}$.

9b) $B \in \mathcal{B} \Leftrightarrow A = f^{-1}(B) \in \mathcal{A}$. $A^c \in \mathcal{A} \Leftrightarrow (f^{-1}(B))^c \in \mathcal{A} \Rightarrow f^{-1}(B^c) \in \mathcal{A} \Leftrightarrow B^c \in \mathcal{B}$.

9c) $B_k \in \mathcal{B}, k \in \mathbb{N} \Leftrightarrow A_k = f^{-1}(B_k) \in \mathcal{A}$.
$\cup_{k\in\mathbb{N}} A_k \in \mathcal{A} \Leftrightarrow \cup_{k\in\mathbb{N}} f^{-1}(B_k) \in \mathcal{A} \Leftrightarrow f^{-1}(\cup_{k\in\mathbb{N}} B_k) \in \mathcal{A} \Leftrightarrow \cup_{k\in\mathbb{N}} B_k \in \mathcal{B}$, using the commutativity of unions and inverse images.

Thus, $\mathcal{B}$ is a σ-algebra. Let $\nu(B) = \mu(f^{-1}(B))$. We showed in problem 8 that $f^{-1}(\phi) = \phi$, so $\nu(\phi) = \mu(\phi) = 0$.
$B_k \in \mathcal{B}, k \in \mathbb{N}, B_k$ disjoint, then $f^{-1}(B_k) = A_k \in \mathcal{A}$,
$\nu(\cup_{k\in\mathbb{N}} B_k) = \mu(f^{-1}(\cup_{k\in\mathbb{N}} B_k)) = \mu(\cup_{k\in\mathbb{N}} f^{-1}(B_k)) = \mu(\cup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mu(A_k)$
$= \sum_{k\in\mathbb{N}} \mu(f^{-1}(B_k)) = \sum_{k\in\mathbb{N}} \nu(B_k)$. We've used that the inverse images of f commutes with unions, and we needed that the A_k are disjoint; let $g : X \to Y$, $A, B \subset Y, A \cap B = \phi$, then $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$, using commutativity of inverse images and intersections, thus the A_k inherit their disjointedness from the B_k. Thus, $(Y, \mathcal{B}, \nu)$ is a measure space.

<u>supplementary problem 10:</u>

Let $\mathcal{B}(X)$ denote the Borel sets on X, $X = \mathbb{R}^n, X = \mathbb{R}^m$, $f : X \to Y$.

i) f continuous, $B \in \mathcal{B}(Y)$ write $B = \cup$

<u>supplementary problem 11:</u>

$\mathcal{A}$ a sigma algebra on X, $E \subset X$, $\mathcal{C}$ a sigma algebra on E. $\mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}$, $\mathcal{F} = \{A \in \mathcal{A} : A \cap E \in \mathcal{C}\}$.

a.1) Let $A_k \in \mathcal{A}_E$, $k \in \mathbb{N}$, $A_{k+1} \subset A_k$. Then $A_k = B_k \cap E$, some $B_k \in \mathcal{A}$.
$\cap_{k\in\mathbb{N}} A_k = \cap_{k\in\mathbb{N}}(B_k \cap E) = E \cap (\cap_{k\in\mathbb{N}} B_k)$ by associativity of intersection.
$A_{k+1} \subset A_k \Rightarrow B_{k+1} \cap E \subset B_k \cap E \Rightarrow B_{k+1} \subset B_k$ as intersection ditributes over set inclusion. Now, by supplementary problem 5.i, $\cap_{k\in\mathbb{N}} B_k \in \mathcal{A} \Rightarrow E \cap (\cap_{k\in\mathbb{N}} B_k) \in \mathcal{A}_E \Rightarrow \cap_{k\in\mathbb{N}}(E \cap B_k) \in \mathcal{A}_E \Rightarrow \cap_{k\in\mathbb{N}} A_k \in \mathcal{A}_E$, then again by supplementary problem 5.i, $\mathcal{A}_E$ is a sigma algebra.
a.2) Let $A_k \in \mathcal{F}$, $k \in \mathbb{N}$, $A_{k+1} \subset A_k$. Then $\exists B_k = A_k \cap E \in \mathcal{C}$, $x \in A_{k+1} \Rightarrow x \in A_k$,
$x \in A_{k+1} \cap E = B_{k+1} \Rightarrow (x \in A_{k+1}$ and $x \in E) \Rightarrow (x \in A_k$ and $x \in E) \Rightarrow (x \in A_k \cap E = B_k)$, so $B_{k+1} \subset B_k$. Because $\mathcal{C}$ is a σ-algebra, by 5.i, $\cap_{k\in\mathbb{N}} B_k \in \mathcal{C}$, $\cap_{k\in\mathbb{N}}(A_k \cap E) \in \mathcal{C}$, $E \cap (\cap_{k\in\mathbb{N}} A_k) \in \mathcal{C}$. Then also, $A_k \in \mathcal{F} \Rightarrow A_k \in \mathcal{A}$, then because $A_{k+1} \subset A_k$, and by 5.i, $\cap_{k\in\mathbb{N}} A_k \in \mathcal{A}$, this with $E \cap (\cap_{k\in\mathbb{N}} A_k) \in \mathcal{C}$ implies $\cap_{k\in\mathbb{N}} A_k \in \mathcal{F}$, and then by 5.i $\mathcal{F}$ is a sigma algebra.
b) $(X, \mathcal{A}, \mu)$ a measure space, $\mu_E(A) = \mu(A \cap E)$. Already shown that $\mathcal{A}_E$ is a sigma algebra.
$\mu_E(\phi) = \mu(\phi \cap E) = \mu(\phi) = 0$. $A_k \in \mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}, k \in \mathbb{N}, A_k$ disjoint.
$\mu_E(\cup_{k\in\mathbb{N}} A_k) = \mu(E \cap (\cup_{k\in\mathbb{N}} A_k)) = \mu(\cup_{k\in\mathbb{N}}(E \cap A_k))$. $(E \cap A_{k1}) \cap (E \cap A_{k2}) = E \cap A_{k1} \cap A_{k2} = \phi$ when $k1 \neq k2$, so $E \cap A_k$ are also disjoint. Then, $\mu(\cup_{k\in\mathbb{N}}(E \cap A_k)) = \sum_{k\in\mathbb{N}} \mu(E \cap A_k) = \sum_{k\in\mathbb{N}} \mu_E(A_k)$. Thus $(E, \mathcal{A}_E, \mu_E)$ is a measure space.

<u>supplementary problem 15:</u>

$E \subset \mathbb{R}^n$. Let $\Upsilon = \{\Pi_{i=1}^n [a_i, b_i] : a_i, b_i \in \mathbb{R}, b_i \geq a_i\}$. Let $\Gamma = \{\mathcal{E} \subset \Upsilon : \mathcal{E}$ countable $\}$. For $\mathcal{C} \in \Gamma$, let $\lambda(\mathcal{C}) = \sum\{\Pi_{i=1}^n (b_i - a_i) : \Pi_{i=1}^n [a_i, b_i] \in \mathcal{C}\}$, where $\sum A$ denotes the sum of all elements in A. Define $C(E)$ such that $C(E) = \{\lambda(X) : X \in \Gamma \text{ and } E \subset \cup X\}$, for any $E \subset \mathbb{R}^n$, where $\cup X$ denotes the union of all elements in X. Then $\lambda^*(E) = \inf C(E)$.
If X is a set of objects which can be multiplied by a real number, let $k\,X = \{k\,x : x \in X\}$, for any $k \in \mathbb{R}$, and $k\,[a, b] = [k\,a, k\,b]$, and $k\,\Pi_{i=1}^n [a_i, b_i] = \Pi_{i=1}^n [k\,a_i, k\,b_i]$. Then for any $\mathcal{C} \in \Gamma$,
$\lambda(k\,\mathcal{C}) = \sum\{\Pi_{i=1}^n (k\,b_i - k\,a_i) : \Pi_{i=1}^n [a_i, b_i] \in \mathcal{C}\} = k^n \sum\{\Pi_{i=1}^n (b_i - a_i) : \Pi_{i=1}^n [a_i, b_i] \in \mathcal{C}\}$, so $\lambda(k\,\mathcal{C}) = k^n\,\lambda(\mathcal{C})$. Then, $C(k\,E) = \{\lambda(k\,X) : X \in \Gamma \text{ and } E \subset \cup X\} = \{k^n\,\lambda(X) : X \in \Gamma \text{ and } E \subset \cup X\}$.
Then finally, $\lambda^*(k\,E) = \inf C(k\,E) = k^n \inf C(E) = k^n\,\lambda^*(E)$.

chapter 2

supplementary problem 2:

$(X, \mathcal{A}, \mu)$ a measure space.
1.) Let $A \in \mathcal{A}$, and $B \subset X, B \notin \mathcal{A}$, and $A \cap B = \phi$. Then let $f^+ = \chi_{_A}$, clearly f^+ is meaurable, let $f^- = \chi_{_B}$, clearly f^- is not $\mathcal{A}$-meaurable. Then $f = f^+ - f^-$ is well defined on X, and not $\mathcal{A}$-measurable, but $|f| = |f^+ - f^-|$ is measurable.
2.) Let N be a non measurable set and N^c measurable, let $f(x) = x\,\chi_{_N}$, then f^{-1}
3.) $f : X \to [-\infty, \infty], \mathcal{A}-$measurable, $E \in \mathcal{A}$, let f_E be f restricted to E. By Supp. problem 11 on HW1 $\mathcal{A}_E = \{A \cap E, A \in \mathcal{A}\}$ is a sigma algebra, with measure $\mu_E(A) = \mu(A \cap E)$. Then for any $t \in \mathbb{R}$, $B_t = (t, \infty]$, we have $f^{-1}(B_t) \in \mathcal{A}$, and we take $f_E^{-1}(B_t) = E \cap f^{-1}(B_t)$. Then by $f^{-1}(B_t) \in \mathcal{A}$ we see that $E \cap f^{-1}(B_t) \in \mathcal{A}_E$, so that f_E inherits it's $\mathcal{A}_E$-measurability from the $\mathcal{A}$-measurability of f.
4.) $f : E \to [-\infty, \infty], \mathcal{A}_E-$measurable, $\tilde{f} : X \to [-\infty, \infty]$, $\tilde{f}(x) = 0, x \notin E, \tilde{f}(x) = f(x)$, else. Then f is the restriction of $\tilde{f}$ to E, so $\tilde{f}$ $\mathcal{A}-$measurable implies f $\mathcal{A}_E-$measurable, by (3). Let $B_t = (t, \infty]$, then for $t > 0$, $\tilde{f}^{-1}(B_t) \in \mathcal{A}_E \subset \mathcal{A}$ if f is $\mathcal{A}_E-$measurable. If $t \leq 0$, $\tilde{f}^{-1}(B_t) \in \mathcal{A}_E \cup \{E^c\} \subset \mathcal{A}$, by closure under complements. So, f $\mathcal{A}_E-$measurable iff $\tilde{f}$ $\mathcal{A}-$measurable.
5.) $g : X \to [-\infty, \infty]$, $\mathcal{A}-$measurable, $E \in \mathcal{A}$. Let $\tilde{g}(x) = f(x)$ if $x \in E, \tilde{g}(x) = 0$, else. Then we may take $\tilde{g} = \tilde{f}$ in (4), and f be g restricted to E. By (3), f is $\mathcal{A}_E-$measurable, which by (4) implies $\tilde{g} = \tilde{f}$ is $\mathcal{A}-$measurable

supplementary problem 4:

Lemma1: Given a sequence of sets in a sigma algebra, $\mathcal{A}$ over a set X, $(A_k)_{k\in\mathbb{N}}$, we can find a sequence also in $\mathcal{A}$, $(E_k)_{k\in\mathbb{N}}$, such that $\cup_{k\in\mathbb{N}} A_k = \cup_{k\in\mathbb{N}} E_k$ and the E_k are disjoint.
Lemma2: Given a simple function, $f = \sum_{k\in\mathbb{N}} a_k\,\chi_{_{A_k}}$, with (A_k) not nescesarily disjoint, and $A_k \in \mathcal{A}$, a sigma algebra on X, we can find a sumple funciton $g = \sum_{k\in\mathbb{N}} e_k\,\chi_{_{E_k}}$, such that E_k are disjoing and $f = g$, and the e_k are unique.
construction: First, using lemma 1, generate $(B_k)_{k\in\mathbb{N}}$, so that $\cup_{k\in\mathbb{N}} A_k = \cup_{k\in\mathbb{N}} B_k$, B_k disjoint, and $B_k \in \mathcal{A}$. Let $b_k = f(x)$, choosing any $x \in B_k$. Now, generate the sequences e_k, E_k as follows, let $e_1 = b_1$, for subsequent $e_k, k \in \mathbb{N}$, let $e_k = b_j$, with $j = \min\{i \in \mathbb{N} : b_i \neq e_l, l \in \{1, 2, ..., k-1\}\}$, then let $E_k = \cup\{B_j : b_j = e_k, j \in \mathbb{N}\}$.
proof: to do

Sum of simple functions is simple. First, assume we have two simple functions, f, g on $(X, \mathcal{A})$, $f = \sum_{k\in\mathbb{N}} f_k$, $g = \sum_{k\in\mathbb{N}} g_k$, with $f_k = a_k\,\chi_{_{A_k}}$, $g_k = b_k\,\chi_{_{B_k}}$. Define $h = \sum_{k\in\mathbb{N}} h_k$, where $h_k = f_{k/2}$ for even k, and $h_k = g_{(k-1)/2}$ for odd k, clearly h is a simple function on $\mathcal{A}$. Now apply lemma 2 to h. Then, given a sequence $(s_k)_{k\in\mathbb{N}}$ of simple functions on $(X, \mathcal{A})$, let $s = s_1 + s_2$, we've just shown that this is a simple function on $(X, \mathcal{A})$. Then starting with $k = 3$, redefine $s = s + s_k$, then again, s a simple function on $(X, \mathcal{A})$. Do this for all $k \in \mathbb{N}$, and so, inductively, s is a simple function on $(X, \mathcal{A})$, $s = \sum_{k\in\mathbb{N}} s_k$, and s is defined on disjoint intervals, and it's coefficients are unique.

Product of simple functions is simple. First, assume we have two simple functions, f, g on $(X, \mathcal{A})$, $f = \sum_{k\in\mathbb{N}} a_k\,\chi_{_{A_k}}$, $g = \sum_{j\in\mathbb{N}} b_j\,\chi_{_{B_j}}$. Now, clearly $\chi_{_A}\,\chi_{_B} = \chi_{_{A\cap B}}$,
$f\,g = \sum_{k\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} b_j\,\chi_{_{B_j}}\right) a_k\,\chi_{_{A_k}} = \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} b_j\,\chi_{_{B_j}} a_k\,\chi_{_{A_k}} = \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} b_j\,a_k\,\chi_{_{A_k\cap B_j}}$. Then, using a bijective map $M : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, let $e_i = a_k\,b_j, (k, j) = M(i)$, $E_i = A_k \cap B_j, (k, j) = M(i)$, and $E_i \in \mathcal{A}$ by closure under intersections of sets in sigma algebras. We need to be carefull with the product $e_i = a_k\,b_j, (k, j) = M(i)$; if we are workign with the extended real numbers, this product may be ill-defined, such as $a_k = 0, b_j = +\infty$. In such a case, the function $f\,g$ is ill-defined on $A_k \cap B_j$, and we omit e_i, E_i from the sequence, so that now $f\,g$ is undefined on $A_k \cap B_j$. Now $f\,g = \sum_{k\in\mathbb{N}} e_k\,\chi_{_{E_k}}$, a simple function on $\mathcal{A}$, and we may again pass it through Lemma 2. Using the same inductive arguement as for countable sums of simple functions, countable products of simple functions are simple.

supplementary problem 5:

$(X, \mathcal{A}, \mu)$, $\{f_k\} : X \to [-\infty, \infty]$, each f_k is finite a.e., so letting $F_k = \{x \in X : f_k(x) = \pm\infty\}$, $\mu(F_k) = 0$. Then by subadditivity, $\mu(\cup_{k\in\mathbb{N}} F_k) = 0$, but $\cup_{k\in\mathbb{N}} F_k = \{x \in X : f_k(x) = \pm\infty, \text{some } k \in \mathbb{N}\}$, so $(\cup_{k\in\mathbb{N}} F_k)^c = \{x \in X : |f_k(x)| < \infty, \text{for all } k \in \mathbb{N}\}$, and $\mu(\cup_{k\in\mathbb{N}} F_k)^c = 1$, so for a.e. $x \in X$, $f_k(x)$ is finite for all $k \in \mathbb{N}$.

supplementary problem 6:

$(X, \mathcal{A}, \mu)$ a complete measure space, $\mathcal{A}$ contains the Borel sets in X, $f : X \to \mathbb{R}$, continuous a.e. Then let $D = \{x \in X : f \text{ continuous at } x\}$. Let $A = f^{-1}(B)$, B any open set in $\mathbb{R}$, then $A = (A \cap D) \cup (A \cap D^c)$, $\mu(A \cap D^c) \le \mu(D^c) = 0 \Rightarrow A \cap D^c \in \mathcal{A}$ by completeness. Then for any $x \in A \cap D$, $f(x) \in U$, an open subset of a neighborhood of $f(x)$ in B, then $x \in f^{-1}(U)$ which is open in A. Using this, for each $x \in A \cap D$, find an open subset of A, V_x, with $x \in V_x$. Then we may take the arbitrary union $W = \cup_{x\in A\cap D} V_x$, by X a topology, and W is an open Borel set, with $A \cap D \subset W$. Now let $A' = A \cap W$, clearly $A \cap D = A' \cap D$, $A' \in \mathcal{A}$, $D \in \mathcal{A} \Rightarrow A \cap D \in \mathcal{A}$, thus $A \in \mathcal{A}$, and so f is measurable.

supplementary problem 8:

$(X, \mathcal{A}, \mu)$ a measure space, $\{f_n\}_{n\in\mathbb{N}} : X \to [0, \infty]$, converging pointwise to f, not nescesarily integrable, $f_n \le f$. Generate the increasing sequence $\{g_n\}_{n\in\mathbb{N}}$ by $g_n = \inf\{f_k\}_{k\ge n}$, then $f_n \ge g_n$, for all $n \in \mathbb{N}$. $\lim_n g_n = \lim_n f_n = f$, so we can use LMCT to get $\lim_n \int_X g_n = \int_X f$. By 2.2.2.c, $\int_X g_n \le \int_X f_k \le \int_X f$, $k \ge n$, and $\int_X g_n \le \int_X g_{n+1}$, so by the squeezing lemma, $\lim_n \int_X f_n = \int_X f$.

supplementary problem 13:

$(X, \mathcal{A}, \mu)$ a measure space, $f : X \to [0, \infty]$ integrable, with respect to μ. Given $\epsilon > 0$, show that there exists a $\delta > 0$, so that if $E \in \mathcal{A}, \mu(E) < \delta$, then $\int_E f < \epsilon$
1.) $f = \sum_{k\in\mathbb{N}} a_k \chi_{A_k}$, with A_k disjoint and $a_k \in (0, \infty]$, unique, then by definition $\int_X f = \sum_{k\in\mathbb{N}} a_k \mu(A_k)$. Now χ_E is a simple function, and by problem 4, $\chi_E f$ is also simple, with $\chi_E f = \sum_{k\in\mathbb{N}} a_k \chi_{E\cap A_k}$, and thus $\int_E f = \int_X f\chi_E = \sum_{k\in\mathbb{N}} a_k \mu(E \cap A_k)$. Then, $A_k \cap E \subset E \Rightarrow \mu(A_k \cap E) \le \mu(E)$, and $A_k \cap E \subset A_k \Rightarrow \mu(A_k \cap E) \le \mu(A_k)$. Then we can see that $\sum_{k\in\mathbb{N}} a_k\, \mu(E \cap A_k)$ converges when $\sum_{k\in\mathbb{N}} a_k\, \mu(A_k)$ does, which does because f is integrable. This is by due to the M-test, $a_k\, \mu(A_k \cap E) \le a_k\, \mu(F_k)$.

supplementary problem 15:

$(X, \mathcal{A}, \mu)$ a measure space. Let $X = \cup_{k\in\mathbb{N}} E_k$, with $E_k \in \mathcal{A}$ disjoint, f integrable on X. We can write $\chi_X = \chi_{\cup E_k} = \sum_n \chi_{E_k}$, then $\int_X f = \int \chi_X f = \int \sum_n \chi_{E_n} f = \sum_n \int \chi_{E_n} f = \sum_n \int_{E_n} f$. We need to justify $\int \sum_n \chi_{E_n} f = \sum_n \int \chi_{E_n} f$, let $F_n = \sum_{k\le n} \chi_{E_k} f$, then $\lim_n F_n \to F = \sum_{k\in\mathbb{N}} \chi_{E_k}$. Let $G = |f|$, $G \ge 0$, clearly $G \ge F_n$, all $n \in \mathbb{N}$, and G is integrable by 2.2.11. Then by LDCT, $\lim_n \int_X F_n = \int_X F$ $\Rightarrow \lim_n \int_X \sum_{k\le n} \chi_{E_k} f = \int_X \sum_{k\in\mathbb{N}} \chi_{E_k} f = \sum_{k\in\mathbb{N}} \int_X \chi_{E_k} f$.

Lemma: convergent sequences in a normed space are cauchy sequences.
pf: Suppose $||x_n - x|| \to 0$ as $n \to \infty$, then given $\epsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $||x_n - x|| < \epsilon/2$, and $\exists m \in \mathbb{N}$ s.t. $||x_n - x|| < \epsilon/3$. then by the triangle inequality,
$||x_n - x_m|| = ||(x_n - x) + (x - x_m)|| \le ||x_n - x|| + ||x - x_m|| < \epsilon/2 + \epsilon/3 < \epsilon$

supplementary problem 21:

Lemma: fix any $\vec{y} \in \mathbb{R}^n$, and $E \subset \mathbb{R}^n$, then $\chi_E(\vec{x} + \vec{y}) = \chi_{E-\vec{y}}(\vec{x})$. This follows from $\vec{x} + \vec{y} \in E \Leftrightarrow \vec{x} \in E - \vec{y}$

$f : \mathbb{R}^n \to [-\infty, \infty]$, fix any $\vec{y} \in \mathbb{R}^n$, and let $g(\vec{x}) = f(\vec{x} + \vec{y})$

$\int_{\mathbb{R}^n} f(\vec{x} + \vec{y}) = \int_{\mathbb{R}^n} f(\vec{x})$

f Lebesgue measurable, then $A_t := f^{-1}([t, \infty])$, $A_t \in \mathcal{L}(\mathbb{R}^n)$; $B_t := g^{-1}([t, \infty])$, clearly $B_t = A_t + \vec{y}$, as $g(B_t) = f(A_t + \vec{y})$, and by theorem 1.5.5, $B_t \in \mathcal{L}(\mathbb{R}^n)$. If $f(\vec{x}) \geq 0 \; \forall \vec{x}$ then $f(\vec{x} + \vec{y}) \geq 0 \; \forall \vec{x} + \vec{y}$.

By the remark after theorem 2.2.6, we may use the construction at the end of section 2.1 of simple functions, $s_n \leq s_{n+1} \in S_+; \lim_n s_n = f$, and by LMCT $\int f = \lim_n \int s_n$ as the definition of the integral for non-negative functions. Thus, if we can show translation invariance for these $\{s_n\}$, then we can use the usual arguement of taking $f = f^+ - f^-$ to show translation invariance for general functions.

If $s \in S_+$, and $s(\vec{x}) = \sum_{k=1}^{n} c_k \chi_{E_k}(\vec{x})$ in the standard representation, then $s(\vec{x} + \vec{y}) = \sum_{k=1}^{n} c_k \chi_{E_k}(\vec{x} + \vec{y})$ $= \sum_{k=1}^{n} c_k \chi_{E_k - \vec{y}}(\vec{x})$ by the lemma, and $E_k - \vec{y} \in \mathcal{L}(\mathbb{R}^n)$ as noted before. Then $\lambda(E_k - \vec{y}) = \lambda(E_k)$ by 1.5.5. and thus $\int s(\vec{x}) d\vec{x} = \int s(\vec{x} + \vec{y}) d\vec{x}$

supplementary problem 22:

Let μ denote the couting measure on $(X, \mathcal{A})$. For $E \in \mathcal{A}$, $\mu(E) := \sum_{x \in E} 1$.

a) Let $f, g : X \to \mathbb{R}$. $f = g \;\; \mu$−a.e. $\Leftrightarrow \mu(E) = 0, E = \{x \in X : f(x) \neq g(x)\}$.
$\mu(E) = \sum_{x \in E} 1 = 0 \Leftrightarrow E = \phi$, then $f(x) = g(x) \;\; \forall x \in X \Leftrightarrow f \equiv g$.
Thus $[f] := \{g : X \to \mathbb{R}, \text{s.t. } f = g \;\; \mu - \text{a.e.}\} = \{f\}$, and so $L^p(X, \mu) = \mathcal{L}^p(X, \mu)$.

b) $f \in \ell^1(X) \Leftrightarrow \int_X |f| \, d\mu < \infty \Leftrightarrow \int_X f \, d\mu < \infty$ by 2.2.11.
$\int_X f \, d\mu = \sup\{\int_X \sum_{k=1}^{n} c_k \chi_{E_k} \, d\mu : c_k > 0, \{c_k\}_{k=1}^{n} \text{distinct}, \{E_k\}_{k=1}^{n} \subset X \text{ disjoint}, \sum_{k=1}^{n} c_k \chi_{E_k} \leq f\} < \infty$
$\int_X f \, d\mu = \sup\{\sum_{k=1}^{n} c_k \sum_{x \in E_k} 1. : c_k > 0, \{c_k\}_{k=1}^{n} \text{distinct}, \{E_k\}_{k=1}^{n} \subset X \text{ disjoint}, \sum_{k=1}^{n} c_k \chi_{E_k} \leq f\} < \infty$
Now, if any E_k in this set contains infinitely many elements, then the corresponding sum, $\sum_{x \in E_k} 1 = \infty$, and then $\int_X f \, d\mu = \infty$ a contradiction; thus all E_k are finite. Given this, we relax the requirment that $\{c_k\}_{k=1}^{n}$ are distinct, which allows us to write
$\int_X f \, d\mu = \sup\{\int_X \sum_{k=1}^{n} c_k \chi_{x_k} \, d\mu : c_k > 0, \{x_k\}_{k=1}^{n} \in X \text{unique}, \sum_{k=1}^{n} c_k \chi_{x_k} \leq f\} < \infty$
$\int_X f \, d\mu = \sup\{\sum_{k=1}^{n} c_k : c_k > 0, \{x_k\}_{k=1}^{n} \in X \text{unique}, \sum_{k=1}^{n} c_k \chi_{x_k} \leq f\} < \infty$
Next, for the sake of notation, intruduce the set S, and the indexing set A, so that
$\int_X f \, d\mu = \sup_{\alpha \in A}\{\int_X s_\alpha \, d\mu : s_\alpha \in S\} < \infty$, $S = \{s_\alpha : s_\alpha = \sum_{k=1}^{n_\alpha} c_{\alpha,k} \chi_{x_{\alpha,k}},$
$c_{\alpha,k} > 0, \{x_{\alpha,k}\}_{k=1}^{n_\alpha} \in X \text{unique}\}$ So S is the set of approximating simple functions of f, and the integral of f is the sup of their integrals, and A indexes S.
• Now, pick an $\alpha \in A$; if $0 < c_{\alpha,k} < f(x_{\alpha,k})$, then $\exists \beta \in A$ s.t. $n_\alpha = n_\beta$, for $k_1, k_2 \leq n_\alpha$ s.t. $x_{\alpha,k1} = x_{\alpha,k2}$, $c_{\alpha,k1} < c_{\beta,k2} \leq f(x_{\beta,k_2})$, becasue it is always possible to find such an $s_\beta \in S$, given that $c_{\alpha,k} \neq f(x_{\alpha,k})$. Then it is clear that $\int_X s_\alpha \, d\mu < \int_x a_\beta \, d\mu$, so $\int_X s_\alpha \neq \int_X f \, d\mu$, and hence we may delete this particular α from A. Carrying this on, we can see that we can define $B \subset A$, such that B indexes the set
$\{s_\beta : s_\beta = \sum_{k=1}^{n_\beta} c_{\beta,k} \chi_{x_{\beta,k}}, c_{\beta,k} = f(x_{\beta,k}), \{x_{\beta,k}\}_{k=1}^{n_\beta} \in X \text{unique}\}$.

b) $f \in \ell^1(X) \Leftrightarrow \int_X |f| \, d\mu < \infty \Leftrightarrow \sup\{\int_X s \, d\mu : s \text{ simple}, 0 < s \leq |f|\} < \infty$
$\Leftrightarrow \sup\{\sum_{k=1}^{n} c_k \sum_{x \in E_k} 1 : s = \sum_{k=1}^{n} c_k \chi_{E_k}, s \text{ simple}, 0 < s \leq |f|\} < \infty$.
Now, if E_k is infinite, then $\sum_{x \in E_k} 1 = \infty$ and then $\int_X |f| \, d\mu = \infty$, a contradiction.

c) For $X = \mathbb{N}$, $a : \mathbb{N} \to \mathbb{R}$, then a can be written as $\sum_{k \in \mathbb{N}} a(k) \chi_{\{k\}}$, so that a is simple by default. Thus $\int_X a \, d\mu = \sum_{k \in \mathbb{N}} a(k) \mu(\{k\}) = \sum_{k \in \mathbb{N}} a(k)$, and ordinary sum.

d) X uncountable. Say $(\alpha_x)_{x \in X} \in \ell^p(X)$. Then suppose $\alpha_x \neq 0$ for uncountably many

?) If card$(X) \geq$ card$(\mathbb{N})$, $a : X \to \mathbb{R}$ measurable, then a can be written as $\sum_{x\in X} a(x)\chi_{\{x\}}$. Suppose $a \geq 0$, then $\int_X a\, d\mu = \sup\{\int_X s\, d\mu : s \text{ simple}, 0 < s \leq a\}$. Now $s = \sum_{k=1}^n c_k\chi_{E_k}$, $c_k > 0$ unique and E_k disjoint; $\int_X s\, d\mu = \sum_{k=1}^n c_k \sum_{x\in E_k} 1$. If any E_k is infinite then $\mu(E_k) = \infty$, and then $\int_X a\, d\mu = \infty$.

supplementary problem 24:

$f_n \in \mathcal{L}^\infty(X, \mu)$. $f_n \to f$ in $||\cdot||_\infty \Leftrightarrow f_n \to f$ uniformly a.e.
pf: $f_n \to f$ in $||\cdot||_\infty \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N, ||f - f_n||_\infty < \epsilon$
$\Leftrightarrow ||f - f_n||_\infty = \inf\{K \geq 0 : |f_n(x) - f(x)| \leq K, \forall x \in E\} < \epsilon$, with $E \subset X, \mu(E^c) = 0$
$\Leftrightarrow |f_n(x) - f(x)| < K_0 < \epsilon, \forall x \in E,\ K_0 := \frac{1}{2}(\epsilon - ||f - f_n||_\infty)$
$\Leftrightarrow \sup\{|f_n(x) - f(x)| : x \in E\} < \epsilon$
$\Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N,\ \sup\{|f_n(x) - f(x)| : x \in E\} < K_0 < \epsilon,\ \mu(E^c) = 0$
$\Leftrightarrow f_n \to f$ uniformly a.e.

supplementary problem 26:

Lemma: $(a_k) \in \mathbb{R}$, if $\sum_{k=1}^\infty |a_k| < \infty$ then $\sum_{k=1}^\infty |a_k|^n < \infty$ for $\infty > n \geq 1$, $n \in \mathbb{R}$.
pf: by undergrad math, $\sum_{k=1}^\infty |a_k| < \infty \Rightarrow \lim_{k\to\infty} |a_k| = 0 \Rightarrow \exists\, K \in \mathbb{N}$ s.t. $|a_k| < 1\ \ \forall k \geq K$. Then, by properties of $\mathbb{R}$, $|a_k| < 1 \Rightarrow |a_k|^n \leq |a_k|$ for $n \geq 1$. ($x^1 \leq x$, and for $x \in (0, 1), \epsilon \in \mathbb{R},$ and $\epsilon > 0$, $-\infty < \log(x) < 0$, $x^\epsilon = e^{\epsilon \log x} < 1 \Rightarrow x^{1+\epsilon} < x$). So by the Weierstrass M-test, $\sum_{k=K}^\infty |a_k| < \infty \Rightarrow \sum_{k=K}^\infty |a_k|^n < \infty$ for $n \geq 1$, and that $\sum_{k=1}^{K-1} |a_k| < \infty \Rightarrow \sum_{k=1}^{K-1} |a_k|^n < \infty$ is obvious if $n < \infty$.

$(X, \mathcal{A}, \mu)$, $\ell^p \subset \ell^q$ for $p \leq q$.
pf: By problem 22 and comments in class, if X is uncounable and $f \in \ell^p(X)$, then $f(x) = 0$ for all but countably many x, and so write $f(x) = \{f(x_k) \text{ if } x = x_k \in (x_1, x_2, ...); 0 \text{ else }\}$.
Clearly then for any X, $p \leq q < \infty$ and $f \in \ell^p(X)$, $(||f||_p)^p = \int_X |f|^p\, d\mu = \sum_{k\in\mathbb{N}} |f(x_k)|^p < \infty$ and then by the lemma (using $n = q - p \geq 1$), $(||f||_p)^p < \infty \Rightarrow \sum_{k\in\mathbb{N}} |f(x_k)|^q = (||f||_q)^q < \infty$
For any X, $p < q = \infty$ and $f \in \ell^p(X)$,
$(||f||_p)^p = \int_X |f|^p\, d\mu = \sum_{k\in\mathbb{N}} |f(x_k)|^p < \infty \Rightarrow |f(x_k)|^p < \infty\ \forall k \in \mathbb{N}$ and $\lim_{k\to\infty} |f(x_k)|^p = 0$
$\Rightarrow \sup_{k\in\mathbb{N}}\{|f(x_k)|\} = ||f||_\infty < \infty \Rightarrow f \in \ell^\infty(X)$

Folland problem 5:

$(X, \mathcal{A}, \mu)$, $X = A \cup B, A, B \in \mathcal{A}$, $f : X \to [-\infty, \infty]$, let $B_t = [\infty, t)$. If f is measurable on X, then $E_t = f^{-1}(B_t) \in \mathcal{A}$, taking intersections, $A \cap E_t \in \mathcal{A}, B \cap E_t \in \mathcal{A}$, by closure under intersection. Then $A \cap E_t$ is $f^{-1}(B_t)$ with f restricted to A, and $B \cap E_t$ is $f^{-1}(B_t)$ with f restricted to B. Conversely, if f is measurable if restriced to A, $f^{-1}(B_t) \in A$, and to B, $f^{-1}(B_t) \in B$, combining the two and taking unions, $f^{-1}(B_t) \in A \cup B = X$, with f unrestricted.

Folland problem 14:

$(X, \mathcal{A}, \mu)$, $f \in L^+$, for any $E \in \mathcal{A}$, define $\lambda(E) = \int_E f$. Then, $\lambda(\phi) = \int \chi_\phi f = 0$ as $\chi_\phi \equiv 0$. Let $\cup_{k\in\mathbb{N}} E_k = E$, $E)k$ disjoint. Then, by supplementary problem 15,
$\int_E f = \sum_{k\in\mathbb{N}} \int_{E_k} f \Rightarrow \mu(E) = \sum_{k\in\mathbb{N}} \mu(E_k)$, this was for arbitrary $E \in \mathcal{A}$, so μ is a measure on $\mathcal{A}$.
Suppose $s = \sum_{k\leq n} a_k\chi_{A_k}$, in the standard representation, then
$\int s\, d\lambda = \sum_{k\leq n} a_k\lambda(A_k) = \sum_{k\leq n} a_k \int \chi_{A_k} f d\mu = \int f \sum_{k\leq n} a_k\chi_{A_k} d\mu = \int f s d\mu$.
Suppose $g : X \to [0, \infty]$, by remark 2.2.6 and construction 2.1.6, we may find a sequence, s_k of simple functions, $0 \leq s_k \leq s_{k+1} \leq g$, and $s_k \to f$ uniformly, and by LMCT
$\int_X g\, d\lambda = \lim_k \int_X s_k\, d\lambda = \lim_k \int_X s_k f\, d\mu$. Then, clearly $\lim_k s_k\, f = g\, f$, and with both $f, g \in L^+$, and with $f\, s_k \leq f\, s_{k+1}$, we can again use LMCT to have $\int_X g\, d\lambda = \lim_k \int_X s_k f\, d\mu = \int_X f g\, d\mu$.

Folland problem 20:

$(X, \mathcal{A}, \mu)$. $f_n, g_n, f, g : X \to [-\infty, \infty]$, if these are complex valued on the other hand, following step 4 in the definition of integration, take the real and imaginary parts seperatly. then further $g_n \to g$, $f_n \to g$ a.e. $\int g_n \to \int g$, and $|f| \leq g$, show that $\int f_n \to \int f$. Now, if $|f_k| = 0$ for $k \geq$ some n, then the problem is trivial, i.e. $\int 0 \to \int 0$, so we may safely delete the n for which $|f_n| = 0$, and then we have $0 < |f_n| \leq g_n$ for all $n \in \mathbb{N}$. Following the proof of 2.2.13, we may assume the convergences are everywhere, not just a.e. and that f, g are finite everywhere, by redefining all the functions to be 0 on the set where these are not true, which, having measure zero, does not affect the integrals.
Then, as $g_n \geq |f_n|$, $f_n + g_n \geq 0$, as the only way this could fail is if $-f_n > g_n$, which is false, similarly, $g_n - f_n \geq 0$. Then $g_n \pm f_n \geq 0$, measurable, and $g_n \pm f_n \to g + f$, as limits may be added, and using Fatou and 2.2.10, $\int (g \pm f) \leq \liminf_n \int (g_n \pm f_n) = \liminf_n \int g_n + \liminf_n \pm \int f_n = \int g + \liminf_n \pm \int f_n$, because if a limit exists, it equals the correspoinding lim inf and lim sup. Subtracting $\int g$ then $\pm \int f \leq \liminf_n \pm \int f_n \Rightarrow \limsup_n \int f_n \leq \int f \leq \liminf_n \int f_n \Rightarrow \lim_n \int f_n = \int f$.

Folland problem 40:

Show that "$\mu(X) < \infty$ " can be replaced with "$|f_n| \leq h \; \forall n \in \mathbb{N}, g \in L^1(\mu)$ " in Egoroff's theorem.
Following the construction in Folland:

Folland constructs the sets E_n, and uses that $E_n \supset E_{n+1}$, $\cap_{n \in \mathbb{N}} E_n = \phi$, and that for all n, $\mu(E_n) \leq \mu(X) < \infty$, then by continuity from below, $\mu(\cap_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \mu(A_n) = 0$.

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$g_n(x) := \sup_{j > n}\{|f_j(x) - f(x)|\}$, then $g_n \to 0$ a.e. and monotonically, $g_n \geq 0$.
$E_n(k) := \cup_{m=n}^{\infty}\{x \in X : g_n(x) \geq k^{-1}\}$

Then, Folland uses the fact that $E_n \supset E_{n+1}$, because $g_n \to 0$ monotonically.

Let $\epsilon > 0$ given. $g_n(x) := \sup_{j > n}\{|f_j(x) - f(x)|\}$, then $g_n \to 0$ a.e. and monotonically, which implies by 2.6.2 that given $k \in \mathbb{N}$ (this is equivalent to picking an $\epsilon' \geq \epsilon\, k^{-1}$), we can find an integer n_k so that $\mu(B_k) < \epsilon\, k^{-1}$ with $B_k := \{x \in X : g_{n_k}(x) \geq k^{-1}\} \Rightarrow B_k^c = \{x \in X : g_{n_k}(x) < k^{-1}\}$. Then clearly $B_k \supset B_{k+1}$, and $\cap_{k \in \mathbb{N}} B_k = \phi$, because $y < \frac{1}{k+1} \Rightarrow y < \frac{1}{k}$, and

Folland problem 6.9:

$(X, \mathcal{A}, \mu)$. $1 \leq p < \infty$, $||f_n - f||_p \to 0$ then $f_n \to f$ in measure, and some subsequence converges to f a.e.
Conversely, $f_n \to f$ in measure, $|f_n| \leq f \in L^p \;\; \forall n$ then $||f_n - f||_p \to 0$.

pf: Choose $\epsilon > 0$, $E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon > 0\}$, then $\epsilon \chi_{E_{n,\epsilon}} \leq |f_n - f| \Rightarrow$ $\epsilon\, \mu(E_{n,\epsilon}) \leq \int_X |f_n - f|^p \, d\mu \Rightarrow \mu(E_{n,\epsilon}) \leq \frac{1}{\epsilon}||f_n - f||_p^p$. Then, $||f_n - f||_p \to 0 \Rightarrow$ $\frac{1}{\epsilon}||f_n - f||_p^p \to 0 \Rightarrow \lim_{n \to \infty} \mu(E_{n,\epsilon}) = 0$. That there exists a convergent a.e. subsequence then follows by applying 2.6.4.

Folland problem 6.10:

$(X, \mathcal{A}, \mu)$. $1 \leq p < \infty$, $f_n, f \in L^p$, $f_n \to f$ a.e, then $||f_n - f||_p \to 0 \Leftrightarrow ||f_n||_p \to ||f||_p$

The triangle inequality is $||x|| + ||y|| \geq ||x + y||$, by letting $a = x + y$, $y = a - b$, $x = b$, we get $||a - b|| \geq ||a|| - ||b||$, when $||a|| > ||b||$, more generally, $||x - y|| \geq |(||x|| - ||y||)| \Rightarrow$ $-||x - y|| \leq |(||x|| - ||y||)| \leq ||x - y||$, for a normed space.

pf: By the above, $||f_n - f||_p \to 0 \;\&\; - ||f_n - f||_p \leq |(||f_n||_p - ||f||_p)| \leq ||f_n - f||_p$
$\Rightarrow |(||f_n||_p - ||f||_p)| \to 0 \Rightarrow ||f_n||_p \to ||f||_p$.

Chapter 3

supplementary problem 3:

Lemma: $A_k \times B_k \neq \phi \; \forall k \in \mathbb{N}$ & $A \times B = \cup_{k\in\mathbb{N}} A_k \times B_k \Rightarrow A = \cup_{k\in\mathbb{N}} A_k$ & $B = \cup_{k\in\mathbb{N}} B_k$.
pf: for any $x \in A$, $\exists \, y \in B$ s.t. $(x, y) \in A \times B \Rightarrow (x, y) \in \cup_{k\in\mathbb{N}} A_k \times B_k \Rightarrow (x, y) \in A_k \times B_k$, some $k \in \mathbb{N} \Rightarrow x \in A_k \Rightarrow x \in \cup_{k\in\mathbb{N}} A_k$, so $A \subset \cup_{k\in\mathbb{N}} A_k$.
If $x \in \cup_{k\in\mathbb{N}} A_k$, then $x \in A_k$, some $k \in \mathbb{N}$. Then again by $A_k \times B_k \neq \phi$, $\exists \, y \in B_k$ s.t. $(x, y) \in A_k \times B_k \subset \cup_{k\in\mathbb{N}} A_k \times B_k = A \times B$, so $(x, y) \in A \times B \Rightarrow x \in A$. This completes $A = \cup_{k\in\mathbb{N}} A_k$.
That $B = \cup_{k\in\mathbb{N}} B_k$ follows from symmetry.
Corrollary: $A_k \times B_k \neq \phi \; \forall k \in \mathbb{N}$ & $A \times B = \cup_{k\in\mathbb{N}} A_k \times B_k \Rightarrow \cup_{k\in\mathbb{N}} A_k \times B_k = \cup_{k\in\mathbb{N}} A_k \times \cup_{k\in\mathbb{N}} B_k$

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ sigma finite measure spaces. If $\phi \neq E \subset \mathcal{A} \times \mathcal{B}$, then $E = \cup_{k\in\mathbb{N}} A_k \times B_k$, with $A_k \in \mathcal{A}$ $B_k \in \mathcal{B}$, $A_k \times B_k$ disjoint, so far by definition, then we may impose that $A_k \times B_k \neq \phi$, otherwise we could delete this entry from the union. If $E = X \times Y$, with $A \in X, B \in Y$, then $A \times B = \cup_{k\in\mathbb{N}} A_k \times B_k$, and then by the lemma $A = \cup_{k\in\mathbb{N}} A_k$ & $B = \cup_{k\in\mathbb{N}} B_k$, then by closure under countable union, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $A = \phi$, then $A \in \mathcal{A}$ automatically, similarly for B.

supplementary problem 3:

Lemma: $(X, \mathcal{A}, \mu)$, $g : X \to \mathbb{R}$ measureable, then $G(x, \cdot) := g(x)$, G is $\mathcal{A} \times \mathcal{B}$ measurable, where $\mathcal{B}$ is any signa algebra over any measure space $(Y, \mathcal{B}, \nu)$.
pf: Writing $B_t = [t, \infty]$, then $g^{-1}(B_t) = \{x \in X : g(x) < t\} \in \mathcal{A}$. Then $G^{-1}(B_t) = \{(x, y) \in X \times Y : G(x, y) < t\} = \{(x, y) \in X \times Y : g(x) < t\} = \{x \in X : g(x) < t\} \times Y = g^{-1}(B_t) \times Y \in \mathcal{A} \times \mathcal{B}$, because $g^{-1}(B_t) \in \mathcal{A}$, and $Y \in \mathcal{B}$.

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$, $h \in L^1(X, \mu)$, $g \in L^1(Y, \nu)$, then $f(x, y) := h(x)g(y)$, f is $\mathcal{A} \times \mathcal{B}$ measurable.
pf: By the lemma, h and g are both $\mathcal{A} \times \mathcal{B}$ measurable (by taking $H(x, \cdot) := h(x)$, and $G(\cdot, y) := g(y)$), then f is the product of two $\mathcal{A} \times \mathcal{B}$ measurable functions, and is thus $\mathcal{A} \times \mathcal{B}$ measurable by 2.1.3 (k).