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Math 6320 - Dr. Blecher
Homework 1

Folland problem 1.8:

$(X, \mathcal{A}, \mu)$ a measure space, E_n a sequence of sets, $E_n \in \mathcal{A}$. Let $A_k = \cap_{n=k}^{\infty} E_n$, then $A_k = E_k \cap A_{k+1}$, then $x \in A_k \Rightarrow x \in E_k \cap A_{k+1} \Rightarrow x \in A_{k+1} \Rightarrow A_k \subset A_{k+1}$, and $A_k \in \mathcal{A}$. Then by continuity from below, $\mu\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$, $\mu\left(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n\right) = \lim_{k\to\infty} \mu(\cap_{n=k}^{\infty} E_n)$, $\mu\left(\liminf E_n\right) = \lim_{k\to\infty} \mu(\cap_{n=k}^{\infty} E_n)$, (not complete)

supplementary problem 6:

Def: E_n a sequence of sets, $n \in \mathbb{N}$. Take the statement " $x \in E_n$ for all but finitely many n" to precicely mean "$\exists k \in \mathbb{N}$ s.t. $x \in \cap_{n=k}^{\infty} E_n$". Then, " $x \in E_n$ for infinitely many n" means "$x \in \cup_{n=k}^{\infty} E_n$, $\forall k \in \mathbb{N}$".

Prop 1: (Folland, p2)
$\limsup E_n = \{x : x \in E_n \text{ for infinitely many } n\}$

$$x \in \limsup E_n = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n \Leftrightarrow (x \in \cup_{n=k}^{\infty} E_n,\ \forall k \in \mathbb{N}) \quad (\text{ by the definition of intersection})$$

Prop 2: (Folland, p2)
$\liminf E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$

$$x \in \liminf E_n = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n \Leftrightarrow (x \in \cap_{n=k}^{\infty} E_n,\ \text{some } k \in \mathbb{N}) \quad (\text{ by the definition of union})$$

$(X, \mathcal{A}, \mu)$ a measure space, E_n a sequence of sets, $E_n \in \mathcal{A}$, μ a finite measure, and $\mu(E_n) > \alpha > 0$. Then we take $\limsup(E_n)$, by Folland problem 1.8,

$$\mu(\limsup(E_n)) \geq \limsup \mu(E_n) \geq \liminf \mu(E_n) \geq \inf \mu(E_n) \geq \alpha > 0,$$

$$\text{so } \mu(\limsup(E_n)) > 0 \Rightarrow \limsup(E_n) \neq \phi \Rightarrow \exists x \in \limsup(E_n) \Rightarrow \exists x \in E_n \subset X \text{ for infinitely many } n$$

supplementary problem 8:

$(X, \mathcal{A}, \mu)$ a measure space. $f : Y \to X$. $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$, $f^{-1}(A) = \{y \in Y : f(y) \in A\}$

Let $\mathcal{B} = f^{-1}(\mathcal{A})$. We have three functions here; $f : Y \to X$, maps elements in Y to elements in X, $f^{-1}(\mathcal{A})$ maps sigma algebras to sigma algebras, and $f^{-1}(A) : \mathcal{A} \to \mathcal{B}$ maps elements in one sigma algebra to elements in another. Write $F^{-1} : \mathcal{A} \to \mathcal{B}$, $F^{-1}(A) = \{y \in Y : f(y) \in A\}$.

claim 1: F^{-1} is injective
Let $A, A' \in \mathcal{A}$ such that $F^{-1}(A) = F^{-1}(A')$. Let $B = F^{-1}(A), B' = F^{-1}(A')$, so $B = B'$. This means that $b \in B \Leftrightarrow b \in B'$, which is to say $b \in \{y \in Y : f(y) \in A\} \Leftrightarrow b \in \{y \in Y : f(y) \in A'\}$, which means $f(b) \in A \Leftrightarrow f(b) \in A'$, which means exactly $A = A'$.

claim 2: F^{-1} is surjective
$B \in \mathcal{B} \Rightarrow f(B) = A$, some $A \in \mathcal{A}$, by the definition of $f^{-1}(\mathcal{A})$, and preimages;
$f(B) = \{x \in X : x = f(y), y \in B\}$, so F^{-1} is surjective.

Thus F^{-1} is bijective.

Lemma 1: $F^{-1}(A^c) = (F^{-1}(A))^c$
$x \in (F^{-1}(A))^c \Leftrightarrow (x \notin F^{-1}(A)\ \&\ x \in Y) \Leftrightarrow (x \in Y\ \&\ x \notin \{y \in Y : f(y) \in A\}) \Leftrightarrow (x \in Y\ \&\ f(x) \notin A)$
$\Leftrightarrow (x \in Y, f(x) \in X, f(x) \notin A) \Leftrightarrow (f(x) \in A^c) \Leftrightarrow (x \in \{y \in Y : f(y) \in A^c\}) \Leftrightarrow (x \in F^{-1}(A^c))$

1a) $\phi \in \mathcal{A}$, $\nexists f(y) \in \phi \Rightarrow f^{-1}(\phi) = \phi \Rightarrow \phi \in \mathcal{B}$.

1b) $B \in \mathcal{B} \Rightarrow f(B) = A$, some $A \in \mathcal{A}$. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \Rightarrow (f(B))^c \in \mathcal{A}$. Then by the definition of $\mathcal{B}$, $F^{-1}(f(B))^c) \in \mathcal{B}$, and by Lemma 1, $F^{-1}(f(B))^c) = F^{-1}(f(B)))^c \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$. We've used that $F^{-1}(f(B)) = B \Leftrightarrow F^{-1}(\{f(x) : x \in B\}) = B \Leftrightarrow \{y \in Y : f(y) \in \{f(x) : x \in B\}\} = B$.

1c) B_k a sequence in $\mathcal{B}$. $x \in f(\cup_{k\in\mathbb{N}} B_k) \Leftrightarrow (x \in f(B_k),$ some $k \in \mathbb{N}) \Leftrightarrow x \in \cup_{k\in\mathbb{N}} f(B_k)$. Thus $f(\cup_{k\in\mathbb{N}} B_k) = \cup_{k\in\mathbb{N}} f(B_k)$. Now, $f(B_k) \in \mathcal{A}$, which is a sigma algebra, so $f(\cup_{k\in\mathbb{N}} B_k) = \cup_{k\in\mathbb{N}} f(B_k) \in \mathcal{A}$, so $f(\cup_{k\in\mathbb{N}} B_k) \in \mathcal{B}$ by the definition of $\mathcal{B}$

Thus, $\mathcal{B}$ is a sigma algebra