$(\Omega, \mathcal{U}, P)$ a probability space.

Problem 1

$Z : \Omega \to \{0,1\}$, measurable, $P(Z^{-1}(\{1\})) = P(Z^{-1}(\{0\})) = 0.5$, so let $A = Z^{-1}(\{1\}), B = A^c$, then $\mathcal{A} := \{\phi, A, B, A \cup B\}$ is a σ-algebra, $\Omega = A \cup B, A \cap B = \phi$, and Z is $\mathcal{A}$-measurable, so we have $Z = \mathbf{1}_A$.

Let $X_n : \Omega^n \to \{1, 2, ..., n\}$, for $\omega \in \Omega^n$, write $\omega = (\omega_k)$, then set $X_n((\omega_k)) = \sum_{k=1}^n Z(\omega_k)$.

Then $X_n^{-1}(\{n\}) = A \times A \times ... \times A$, n-times. $X_n^{-1}(\{n-1\}) = B \times A \times ... \times A \cup A \times B \times A \times ... \times A \cup ... \cup A \times A \times ... \times A \times B$, ..., $X_n^{-1}(\{0\}) = B \times B \times ... \times B$. So for any $0 \leq k \leq n$, $X_n^{-1}(\{k\})$ is a union of all permutations of cartesian products of k many sets A, and $n-k$ many sets B, and so is a union of measurable rectangles of sets from $\mathcal{A}$, and so $X_n^{-1}(\{k\}) \in \mathcal{A} \times \mathcal{A} \times ... \times \mathcal{A} =: \mathcal{A}^n$, the product σ-algebra.

Let P^n be the product measure on $\mathcal{A}^n$, i.e., $P^n = P \times P \times ... \times P$, n-times. Then $E_{P^n}[f] = \int_{\Omega^n} f \, dP^n = \int$. By Fubini's theorem, $E_{P^n}[\mathbf{1}_A] = \int_\Omega \left(... \left(\int_\Omega \mathbf{1}_A \, dP \right) ... \right) dP = \int_\Omega \left(... \int_\Omega \left(P(A) \right) dP ... \right) dP = P(A) \int_\Omega \left(... \int_\Omega (1) \, dP ... \right) dP$ $= P(A) \cdot 1 = P(A)$. Thus $E_{P^n}[X_n] = \sum_{k=1}^n E_{P^n}[Z] = \sum_{k=1}^n E_{P^n}[\mathbf{1}_A] = \sum_{k=1}^n P(A) = \frac{1}{2} n$

So, with $X(\omega', (\omega_k)) = \sum_{k=1}^{N(\omega')} Z(\omega_k)$, where $N(\omega') = \sum_{k=1}^4 k \mathbf{1}_{A_k}(\omega')$, $P(A_1) = 0.5, P(A_2) = 0.1, P(A_3) = 0.2, P(A_4) = 0.2$, and $\{A_k\}$ are independent events, $\Omega = \cup_k A_k$. So $E[X|N] = \sum_{k=1}^4 \frac{1}{P(A_k)} E[\mathbf{1}_{A_k} X] \mathbf{1}_{A_k}$. Now $E[\mathbf{1}_{A_k} X] = \int_{\Omega \times \Omega^4} \mathbf{1}_{A_k} X \, d(P \times P^4) = \int_{\Omega \times \Omega^4} \mathbf{1}_{A_k}(\omega') \sum_{k=1}^{N(\omega')} \mathbf{1}_A(\omega_k) \, d(P \times P^4)$

Problem 2

Consider the discrete stochastic process, $X : \mathbb{N} \times \Omega \to \mathbb{Z}$, where $X_0 > 0$, $X_{n+1} = 0$ if $X_n = 0$, and if $X_n > 0$, then $X_{n+1} = X_n \pm 1$ with each half probability. X_0 is a parameter of the process; it is not a random variable.

1) X is a non-negative martingale. First, we already have that X_0 is positive, suppose that $X_n > 0$, then by definition, $X_{n+1} = X_n \pm 1 > 0$, then and if $X_n = 0$ then $X_{n+1} = 0 \geq 0$, so by induction, $X_n(\omega) \geq 0$ for all $x \in \mathbb{N}$, $X_n(\omega) = |X_n(\omega)|$.

Problem 3

$X : \mathbb{N} \times \Omega \to \mathbb{N}$, let $\mathbb{P}_k = (P_{k,i,j}), P_{k,i,j} = P(\{X_{k+1} = i\} | \{X_k = j\})$, $x_k = (x_{k,i}), x_{k,i} = P(\{X_k = i\})$. Claim: $x_{k+1} = \mathbb{P}_k x_k$, so $x_{k+1,i} = \sum_j P_{k,i,j} x_{k,j}$. Proof: $P_{k,i,j} = P(\{X_{k+1} = i\} | \{X_k = j\}) = \frac{P(\{X_{k+1}=i\} \cap \{X_k=j\})}{P(\{X_k=i\})}$, so $\sum_j P_{k,i,j} x_{k,j} = \sum_j \frac{P(\{X_{k+1}=i\} \cap \{X_k=j\})}{P(\{X_k=j\})} \cdot P(\{X_k = j\}) = \sum_j P(\{X_{k+1} = i\} \cap \{X_k = j\})$, now $\{X_k = j_1\} \cap \{X_k = j_2\} = \phi$ when $i_1 \neq j_2$, becasue inverse images commute with intersections, then $\sum_j P_{k,i,j} x_{k,i} = P\left(\cup_j \left(\{X_{k+1} = i\} \cap \{X_k = j\}\right)\right) = P\left(\{X_{k+1} = i\} \cap \cup_j \{X_k = j\}\right)$, by countable additivity. Now, $\cup_{j \in \mathbb{N}} \{X_k = j\} = \Omega$, so $\sum_j P_{k,i,j} x_{k,i} = P\left(\{X_{k+1} = i\}\right) = x_{k+1}$, and the claim is proved.

If $\Omega = \cup_k E_k, \{E_k\}$ is disjoint, $A, E_k \in \mathcal{U}$, then $\sum_k P(E_k|A) = \sum_k \frac{(P(E_k \cap A)}{P(A)} = \frac{1}{P(A)} P\left(\cup_k (E_k \cap A)\right) = \frac{1}{P(A)} P\left(A \cap \cup_k (E_k)\right) = \frac{1}{P(A)} P\left(A \cap \Omega\right) = 1$. Then, $X_k^{-1}(\{i\}), i \in \mathbb{N}$ generates a measurable disjoint partition of Ω, thus $\sum_{i \in \mathbb{N}} P_{k,i,j} = \sum_{i \in \mathbb{N}} P(\{X_{k+1} = i\} | \{X_k = j\}) = 1$. So the columns of $\mathbb{P}$ are are normalized, i.e. they sum to one. Then, if x_k is normalized in the same sense, so $\sum_i x_i = 1$, and all $x_i \geq 0$, then $x_{k+1} = \mathbb{P} x_k$, $\sum_i x_{k+1,i} = \sum_i \sum_j P_{k,i,j} x_{k,j} = \sum_j \sum_i P_{k,i,j} x_{k,j} = \sum_j x_{k,j} \sum_i P_{k,i,j} = \sum_j x_{k,j} \cdot 1 = 1$, because all pobabilities are non-negative, as are the entries in x_k, and clearly $x_{k+1,i} \geq 0$ for all i. This shows that $\mathbb{P}$ preserves normalization, as we'd expect.

Let $\{Z_k\} : \Omega \to \mathbb{N}$ be i.i.d random variables with $P(\{Z = 0\}) = 0.1$, $P(\{Z = 1\}) = 0.3$, $P(\{Z = 2\}) = 0.2$, $P(\{Z = 3\}) = 0.4$. Define $X : \mathbb{N} \times \Omega \to \mathbb{N}$ by $X_0 = 0$, and $X_k = \max(\{Z_1, Z_2, ..., Z_k\})$. Then $X_{k+1} = \max(\{Z_1, Z_2, ..., Z_k, Z_{k+1}\}) = \max(X_k, Z_{k+1})$.

If $f, g : X \to S$, and $\max(f, g)(x) = \max(f(x), g(x))$, then $\max(f, g)^{-1}(\{s\}) = \{x \in X; \max(f(x), g(x)) = s\} = \max(f, g)^{-1}(\{s\}) = \{x \in X; f(x) \geq g(x), f(x) = s\} \cup \{x \in X; f(x) < g(x), g(x) = s\} = \{f \geq g\} \cap \{f = s\} \cup \{g > f\} \cap \{g = s\}$.

$P_{i,j} := P(\{X_{k+1} = i\}|\{X_k = j\}) = P(\{\max(X_k, Z_{k+1}) = i\}|\{X_k = j\}) = P(\{X_k \geq Z_{k+1}\} \cap \{X_k = i\}|\{X_k = j\}) + P(\{X_k < Z_{k+1}\} \cap \{Z_{k+1} = i\}|\{X_k = j\})$, becausue $\{X_k \geq Z_{k+1}\}$ and $\{X_k < Z_{k+1}\}$ are disjoint events.

Now, $P(\{X_k = j\} \cap \{Z_{k+1} \leq X_k\}) = P(\{X_k = j \text{ and } Z_{k+1} \leq X_k\}) = P(\{X_k = j\} \cap \{Z_{k+1} \leq j\}) = P(\{X_k = j\})P(\{Z_{k+1} \leq j\})$, because Z_k are iid. When $i \neq j$, $P(\{X_k = i\} \cap \{X_k = j\}) = 0$. So $P(\{X_k \geq Z_{k+1}\} \cap \{X_k = i\}|\{X_k = j\}) = \delta_{i,j}P(\{Z_{k+1} \leq j\})$

$P(\{X_k < Z_{k+1}\} \cap \{Z_{k+1} = i\}|\{X_k = j\}) = P(\{X_k < Z_{k+1}\} \cap \{Z_{k+1} = i\} \cap \{X_k = j\}) \div P(\{X_k = j\}) = P(\{X_k < i, X_k = j\} \cap \{Z_{k+1} = i\}) \div P(\{X_k = j\}) = P(\{X_k < i, X_k = j\})P(\{Z_{k+1} = i\}) \div P(\{X_k = j\})$, by iid.

Now $X : \Omega \to \mathbb{R}, a, b \in \mathbb{R}$, then $P(\{X < a\}|\{X = b\}) = P(X^{-1}((\infty, a)) \cap X^{-1}(\{b\})) \div P(\{X = b\}) = P(X^{-1}((\infty, a) \cap \{b\})) \div P(\{X = b\})$. So if $b > a$, then $(\infty, a) \cap \{b\}) = \phi$ and then $P(\{X < a\}|\{X = b\}) = 0$. if $b \leq a$, then $(\infty, a) \cap \{b\}) = \{b\}$, and then $P(\{X < a\}|\{X = b\}) = 1$.n

So, $P(\{X_k < Z_{k+1}\} \cap \{Z_{k+1} = i\}|\{X_k = j\}) = P(\{Z_{k+1} = i\})$ if $j < i$, and 0 else.

$P_{i,j} = P(\{Z_{k+1} \leq j\})$ when $i = j$ and $P_{i,j} = P(\{Z_{k+1} = i\})$ if $j < i$, and $P_{i,j} = 0$ if $i < j$. Computing:

$$\mathbb{P} = [P_{i,j}] = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.4 & 0.4 & 0.4 & 1.0 \end{bmatrix}.$$

Problem 4

$X : \mathbb{N} \times \Omega \to \mathbb{N}$, X_k is the number of infected at step k. Pick two individuals evenly at random, then with probability X_k/N one will already be infected, and $1 - X_k/N$ the other won't be. So with probability $(X_k/N)(1 - X_k/N)$ transmission can occur, then apply a $\alpha = 0.1$ probability that transmission will occur. So,

$$X_0 = 1, X_{k+1} = X_k + \mathbf{1}_{E_k},$$

where $P(E_k) = (X_k/N)(1 - X_k/N)\alpha$. $P_{i,j} := P(\{X_{k+1} = i\}|\{X_k = j\}) = P(\{X_k + \mathbf{1}_{E_k} = i\}|\{X_k = j\}) = P(\{j + \mathbf{1}_{E_k} = i\}|\{X_k = j\}) = P(\{\mathbf{1}_{E_k} = i - j\}|\{X_k = j\})$. So $P_{i,j} = 0$ unless $\mathbf{1}_{E_k} = i - j$ which occurs when $i - j = 0$ or $i - j = 1$. If $i = j$, then $P_{i,j} = P(\{X_k + \mathbf{1}_{E_k} = i\}|\{X_k = j\}) = P(\{X_k + \mathbf{1}_{E_k} = X_k\}|\{X_k = j\}) = P(\{\mathbf{1}_{E_k} = 0\}|\{X_k = j\}) = 1 - (j/N)(1 - j/N)\alpha$. If $i = j + 1$ then

$$P_{i,j} = P(\{j + \mathbf{1}_{E_k} = j+1\}|\{X_k = j\}) = P(\{\mathbf{1}_{E_k} = 1\}|\{X_k = j\}) = (j/N)(1 - j/N)\alpha$$

So the transition matrix is $1 - (j/N)(1 - j/N)\alpha$ on the diagonal, and $(j/N)(1 - j/N)\alpha$ on the subdiagonal, and zeros otherwise.

$$\mathbb{P} = [P_{i,j}] = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.4 & 0.4 & 0.4 & 1.0 \end{bmatrix}.$$

Problem 5

$X : \mathbb{N} \times \Omega \to \mathbb{N}$ a markov chain such that $\begin{bmatrix} P(\{X_{k+1} = 0\}) \\ P(\{X_{k+1} = 1\}) \end{bmatrix} = \mathbb{P}_X \begin{bmatrix} P(\{X_k = 0\}) \\ P(\{X_k = 1\}) \end{bmatrix}$ for all $k \in \mathbb{N}$, where $\mathbb{P}_X := \begin{bmatrix} \alpha & 1-\beta \\ 1-\alpha & \beta \end{bmatrix}$, $\alpha, \beta \in [0,1]$. Then let $n_0 = (0,0), n_1 = (1,0), n_2 = (0,1), n_3 = (1,1)$, and $n_i = (n_{i,1}, n_{i,2})$, and define Z_k as

$$Z_k = \begin{cases} 0; & \text{if } (X_{k-1}, X_k) = (0,0) = n_0 \\ 1; & \text{if } (X_{k-1}, X_k) = (1,0) = n_1 \\ 2; & \text{if } (X_{k-1}, X_k) = (0,1) = n_2 \\ 3; & \text{if } (X_{k-1}, X_k) = (1,1) = n_3 \end{cases}$$

Then let $\mathbb{P}_Z = [P_{i,j}]$, $P_{i,j} = Pr(\{Z_{k+1} = i\}|\{Z_k = j\})$, then

$$P_{i,j} = \frac{P(\{X_k = n_{i,1}\} \cap \{X_{k+1} = n_{i,2}\} \cap \{X_{k-1} = n_{j,1}\} \cap \{X_k = n_{j,2}\})}{P(\{X_{k-1} = n_{j,1}\} \cap \{X_k = n_{j,2}\})}.$$

If $n_{i,1} \neq n_{j,2}$ then $\{X_k = n_{i,1}\} \cap \{X_k = n_{j,2}\} = \phi$ and then the above numerator is zero, becasue $P(\phi) = 0$, so this forces $P_{0,2} = P_{0,3} = P_{1,0} = P_{1,1} = P_{2,2} = P_{2,3} = P_{3,0} = P_{3,1} = 0$. Assuming $n_{i,1} = n_{j,2}$, then $\{X_k = n_{i,1}\} = \{X_k = n_{j,2}\}$, and so

$$P_{i,j} = \frac{P(\{X_{k+1} = n_{i,2}\} \cap \{X_{k-1} = n_{j,1}\} \cap \{X_k = n_{j,2}\})}{P(\{X_{k-1} = n_{j,1}\} \cap \{X_k = n_{j,2}\})} = P(\{X_{k+1} = n_{i,2}\}|\{X_{k-1} = n_{j,1}\} \cap \{X_k = n_{j,2}\}).$$

Now applying the Markov property of X,

$$P_{i,j} = P(\{X_{k+1} = n_{i,2}\}|\{X_k = n_{j,2}\}).$$

So,

$$\begin{aligned} P_{0,0} &= P(\{X_{k+1} = 0\}|\{X_k = 0\}) = (\mathbb{P}_X)_{0,0} = \alpha \\ P_{0,1} &= P(\{X_{k+1} = 0\}|\{X_k = 0\}) = (\mathbb{P}_X)_{0,0} = \alpha \\ P_{1,2} &= P(\{X_{k+1} = 0\}|\{X_k = 1\}) = (\mathbb{P}_X)_{0,1} = 1-\beta \\ P_{1,3} &= P(\{X_{k+1} = 0\}|\{X_k = 1\}) = (\mathbb{P}_X)_{0,1} = 1-\beta \\ P_{2,0} &= P(\{X_{k+1} = 1\}|\{X_k = 0\}) = (\mathbb{P}_X)_{1,0} = 1-\alpha \\ P_{2,1} &= P(\{X_{k+1} = 1\}|\{X_k = 0\}) = (\mathbb{P}_X)_{1,0} = 1-\alpha \\ P_{3,2} &= P(\{X_{k+1} = 1\}|\{X_k = 1\}) = (\mathbb{P}_X)_{1,1} = \beta \\ P_{3,3} &= P(\{X_{k+1} = 1\}|\{X_k = 1\}) = (\mathbb{P}_X)_{1,1} = \beta, \end{aligned}$$

and all together,

$$\mathbb{P}_Z = \begin{bmatrix} \alpha & \alpha & 0 & 0 \\ 0 & 0 & 1-\beta & 1-\beta \\ 1-\alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \beta & \beta \end{bmatrix}.$$