Overview of: "The Wavelet Transform, Time-Frequency Localization and Signal Analysis" By Ingrid Daubechies, 1990.

Nick Maxwell

December 2, 2009

Outline

- Section I: Introduction
- Section II: Frame Questions
  Will not cover this section, due to its length.
- Section III: Phase Space Localization

Introduction

This paper discusses several aspects of the representation (analysis and reconstruciton) of functions in the framework of frames, in contrast to orthonormal bases.

The frames consist of functions, $\phi_{m,n}$ indexed by two parameters, $(m, n) \in \mathbb{Z}^2$. Two major cases are discussed, one in which $\phi_{m,n}$ is the discretized version of the function $g^{(p,q)}(t)$ used in the windowed Fourier transform, and one in which $\phi_{m,n}$ are translated and dilated versions of a wavelet $h(t)$.

These are discussed in therms of phase space.

The windowed FT case:

$$c_{p,q} = \int_{\mathbb{R}} e^{ipx} g(x-q) f(x) dt$$

g is a windowing function, $g(x) \approx 1$ when $x \approx 0$. So $c_{p,q}$ are the Fourier coefficients of f times g, translated by q. Here $c : \mathbb{R}^2 \to \mathbb{C}$, we refer to $\{p \in \mathbb{R}\} \times \{q \in \mathbb{R}\}$ as phase space.

In the discrete case, $p_m = mp_0$, and $q_n = nq_0$, and then $\{p_m : m \in \mathbb{Z}\} \times \{q_n : n \in \mathbb{Z}\}$ form a lattice in phase space.

The wavelet case: we start with the general scaling equation

$$h^{(a,b)}(x) = |a|^{-\frac{1}{2}} h\left(\frac{x-b}{a}\right)$$

for a suitable wavelet h. In analogy with the windowed FT case, we construct a discrete lattice of wavelets, by letting $a = a_0^m$, $b = nb_0 a_0^m$.

Since these lattices of $g_{m,n}$ and $g_{m,n}$ constitute frames, Section II discusses much of the relevant analysis. It mostly answeres these two questions,

- find a range R for the parameters (p_0, q_0) and (a_0, b_0) so that the associated $g_{m,n}$ and $g_{m,n}$ constitute a frame under various circumstances.
- for the range of parameters in R, compute estimates on the frame bounts A, B.

A: localization, the windowed Fourer transform case.

assumtions on g

- $\int |g(x)|^2 dx = 1$ (normalization)
- $\int x|h(x)|^2 dx = 0$
  expected position is 0, can shift h to achieve this
- $\int k|\hat{h}(k)|^2 dk = 0$
  expected momentum is 1, can shift $\hat{h}$ to achieve this.

These assure that if g is localized in phase space, it will be so at (0,0).

- Then $g_{m,n}$ will be localized around $(mp_0, nq_0,)$

$$g_{m,n}(t) := e^{-imp_0 t} g(t - nq_0)$$

assumtions on f

- f is localized in phase space, so essentially limited to $[-T, T]$ in time, and essentially band limmited to $[-\Omega, \Omega]$.
- $f \in L^2(\mathbb{R})$

frame considerations

- $(g_{m,n})$ a frame, with frame bounds A, B and dual $(\tilde{g_{m,n}})$

$$f = \sum_{m,n \in \mathbb{Z}} (\tilde{g_{m,n}}) < g_{m,n}, f >$$

We're interested in a discrete subset of phase space:

$$B_\epsilon = [-(\Omega + \omega_\epsilon), (\Omega + \omega_\epsilon)] \times [-(T + t_\epsilon), (T + t_\epsilon)]$$

so B_ϵ is an extension of the essential support of f in phase space.

Theorem 3.1:
suppose:

$$|\hat{g}(k)| \leq C(1+k^2)^{-\alpha}$$

$$|g(t)| \leq C(1+t^2)^{-\alpha}$$

for some $C > 0, \alpha > \frac{1}{2}$,
then there exists $t_\epsilon, \omega_\epsilon > 0$ such that

$$||f - \sum_{(m,n)\in B_\epsilon} \tilde{(}g_m, n) < g_{m,n}, f > || \leq$$

$$(B/A)^{\frac{1}{2}} \left[||(1-\chi_{[-T,T]})f|| + ||(1-\chi_{[-\Omega,\Omega]})f|| + \epsilon||f|| \right]$$

Theorem 3.1 remarks:

- The important point here is that $t_\epsilon, \omega_\epsilon$ are independent of T, Ω, so the extension of the essential support of f depends only on ϵ.
- For a fixed ϵ, $N_eps(T, \Omega) := 4(T + t_\epsilon)(\Omega + \omega_\epsilon)/(p_0 q_0)$, so taking the limit as $T, \Omega \to \infty$, of $N_\epsilon/(4T\Omega)$ gives $(p_0 q_0)^{-1}$, independent of ϵ. prolate spheroidal functions are obtained as eigen functions of $P_\Omega Q_T P_\Omega$, which is the by operator that time limits to $[-T, T]$ then band limits to $[-\Omega, \Omega]$, and then again time limits. If we expend f in these functions, then the above limit becomes $1/(2\pi)$, which is the nyquist density. continued...

Theorem 3.1 remarks continued:
By results from section II, all frames have a density $p_0 q_0 \leq 2\pi$. The oversampling rate $2\pi(p_0 q_0)^{-1} > 1$, is due to the frame not being orthonormal, so there are 'too many vectors', but the upshot is better phase space localization.

B: localization, the wavelet case.

assumtions on h

- $\int |h(x)|^2 dx = 1$ (normalization)
- $\int x|h(x)|^2 dx = 0$
  expected position is 0, can shift h to achieve this
- $\int k|\hat{h}(k)|^2 dk = 1$
  expected momentum is 1, can dilate h to achieve this.
- $|\hat{h}|$ is even
  see notes

These assure that if h is localized in phase space, it will be so at (0,0).

- Then $h_{m,n}$ will be localized around $(\pm a_0^{-m}, a_0^{-m} n b_0,)$

$$h_{m,n}(t) := a_0^{-m/2}\, h(a_0^{-m}\, t - n b_0)$$

assumtions on f

- f is localized in phase space, so essentially limited to $[-T, T]$ in time, and $[\Omega_0, \Omega_1]$, $[-\Omega_0, -\Omega_1]$ in frequency.
- $f \in L^2(\mathbb{R})$

frame considerations

- $(h_{m,n})$ a frame, with frame bounds A, B and dual $(\tilde{h_{m,n}})$

$$f = \sum_{m,n \in \mathbb{Z}} (\tilde{h_{m,n}}) < h_{m,n}, f >$$

We're interested in a discrete subset of phase space:

$$B_\epsilon \supset \{(m,n) \in \mathbb{Z}^2 : a_0^{-m} \in [\Omega_0, \Omega_1], nb_0 a_0^{-m} \in [-T, T]\}$$

Theorem 3.2:
suppose:

$$|\hat{h}(k)| \leq C|k|^{\beta}(1+k^2)^{-(\alpha+\beta)/2}$$

$$\int (1+t^2)^{\gamma}|h(t)|^2 < \infty$$

for some $C > 0, \beta > 0, \alpha > 1, \gamma > \frac{1}{2}$,
then there exists a finite subset $B_\epsilon \subset \mathbb{Z}^2$ such that

$$||f - \sum_{(m,n)\in B_\epsilon} \tilde{(}h_m, n) < h_{m,n}, f > || \leq$$

$$(B/A)^{\frac{1}{2}} \left[||(1-\chi_{[-T,T]})f|| + ||(1-\chi_{[\Omega_0,\Omega_1]\cup[-\Omega_0,-\Omega_1]})f|| + \epsilon||f|| \right]$$

Theorem 3.2 remarks:

- The construction of B_ϵ from $[-T, T]$ and $[\Omega_1, \Omega_2]$ is more complicated than in WH case; dependence is on Ω_1, Ω_2 in addition to just ϵ.
- The shape of B_ϵ is non rectangular in phase space, as the number 'n' points required increases as the dilation parameter, 'm' increases.
- Limits on estimates of B_ϵ as $\Omega_0 \to 0, \Omega_1 \to \infty$, $T \to \infty$ are not independent on ϵ. This leads to the concept that "phase space denisty" is not well suited to wavelet representation.

B: Extension of Reconstruction Precision.

Reconstructing a function from a wavelet representation (or short windowed FT) can be done at a greater precision than that to which the coefficients were computed. This is due to phase space localization and oversampling.

Phase space locallization is nescesary so that we're only dealing with a finite number of coefficients (N_ϵ, from the previous slide).

As for oversampling, we write $T : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$, $(Tf)_{m,n} =< \phi_{m,n}, f >$. T is bounded and has bounded inverse on its closed range. The range of T is the whole of $\ell^2(\mathbb{Z}^2)$ if and only if $(\phi_{m,n})$ is a basis.

For arbitrary $c_{m,n}$ the reconstrution,

$$\sum_{m,n} \tilde{\phi}_{m,n} c_{m,n}$$

consists of first a projection of $c_{m,n}$ onto the range of T, and then an inversion T on its range (see beginning of secion II).

We model numerical error as noise added to the coefficients: $c_{m,n} = (Tf)_{m,n} +$ noise. So by the previous reslut, this noise will be reduced in norm in the reconstruciton step.

If we assume a model of the noise as $\gamma_{m,n}$, identically distributed random variables, mean zero and variance α^2, then can show that the term in the reconstruction error involving this noise is $A^{-2}\alpha^2 N_\epsilon$.

In the windowed FT case, if the frame is snug, then by a result from section II, $A \approx \frac{2\pi}{p_0 q_0}$, yielding $N_\epsilon \approx \frac{T\Omega}{2\pi p_0 q_0}$. If instead we used an orthonormal basis, $N_\epsilon \approx \frac{2T\Omega}{\pi}$, so the frame yields a net gain of $\frac{2\pi}{p_0 q_0}$.

In the wavelet case, similar results can be obtained, but we lack a simple expression for N_ϵ, by previous remarks.

These results suggest that a frame based reconstruction will be more numerically stable.