Compound matrices for dispersion function

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1 P-SV problem settings in x-z plane

Equations satisfied by the P-wave potential scalar field ϕ and the SV-wave potential vector field (0,ψ,0)'.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\omega^2}{\alpha^2}\phi = 0 \tag{1}$$
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{\omega^2}{\beta^2}\psi = 0 \tag{2}$$

The state variables of interest are expressed as following:

$$u_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \tag{3}$$
$$u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{4}$$
$$\sigma_{xz} = \rho\beta^2(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}) \tag{5}$$
$$\sigma_{zz} = \rho\alpha^2\frac{\partial^2\phi}{\partial z^2} + \rho(\alpha^2 - 2\beta^2)\frac{\partial^2\phi}{\partial x^2} + 2\rho\beta^2\frac{\partial^2\psi}{\partial x\partial z} \tag{6}$$

We consider layered medium with z-axis oriented positively downwards. We seek plane wave solution for the potential fields in a given homogeneous layer:

$$\phi = e^{ik(x-ct)}(A^+e^{ikr_a z} + A^-e^{-ikr_a z}) \tag{7}$$
$$\psi = e^{ik(x-ct)}(B^+e^{ikr_b z} + B^-e^{-ikr_b z}) \tag{8}$$

We use the following notation:

$$\begin{aligned}
A^+, B^+ &\quad \text{Amplitudes of downgoing waves}\\
A^-, B^- &\quad \text{Amplitudes of upgoing waves}\\
\omega &\quad \text{circular frequency, common to all layers}\\
k &\quad \text{horizontal wavenumber, common to all layers}\\
c = \frac{\omega}{k} &\quad \text{phase velocity, common to all layers}\\
r_a = \sqrt{\frac{c^2}{\alpha^2} - 1} &\quad \text{can be complex number}\\
r_b = \sqrt{\frac{c^2}{\beta^2} - 1} &\quad \text{can be complex number}\\
C_a = \cos(k r_a z) &\quad \text{could become hyperbolic function for complex numbers}\\
C_b = \cos(k r_b z) &\\
S_a = \sin(k r_a z) &\\
S_b = \sin(k r_b z) &\\
\mu = \rho\beta^2 &\\
\gamma = \frac{\beta^2}{c^2} &\\
t = 2 - \frac{c^2}{\beta^2} &
\end{aligned}$$

After some algebraic manipulation, the state variables can be expressed as a function of the amplitudes vector:

$$\begin{bmatrix} -iku_x \\ -iku_z \\ \sigma_{xz} \\ \sigma_{zz} \end{bmatrix} = k^2 e^{ik(x-ct)} \cdot M \cdot P \cdot E \cdot \begin{bmatrix} A^+ \\ A^- \\ B^+ \\ B^- \end{bmatrix} \tag{9}$$

where

$$M = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \mu & \\ & & & \mu \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -r_b & r_b \\ r_a & -r_a & 1 & 1 \\ -2r_a & 2r_a & -t & -t \\ t & t & -2r_b & 2r_b \end{bmatrix}$$

$$E = \begin{bmatrix} e^{ikr_a z} & & & \\ & e^{-ikr_a z} & & \\ & & e^{ikr_b z} & \\ & & & e^{-ikr_b z} \end{bmatrix}$$

We write:

$$D = Q \cdot E$$

$$Q = M \cdot P$$

$$Q^{-1} = P^{-1} \cdot M^{-1} = \begin{bmatrix} \gamma & -\frac{2\gamma - 1}{2r_a} & -\frac{\gamma}{2\mu r_a} & -\frac{\gamma}{2\mu} \\ \gamma & \frac{2\gamma - 1}{2r_a} & \frac{\gamma}{2\mu r_a} & -\frac{\gamma}{2\mu} \\ \frac{2\gamma - 1}{2r_b} & \gamma & \frac{\gamma}{2\mu} & -\frac{\gamma}{2\mu r_b} \\ -\frac{2\gamma - 1}{2r_b} & \gamma & \frac{\gamma}{2\mu} & \frac{\gamma}{2\mu r_b} \end{bmatrix}$$

and $T = Q \cdot E \cdot Q^{-1}$ with

$$T_{11} = \gamma(2C_a - tC_b)$$

$$T_{12} = -i\gamma(\frac{t}{r_a}S_a + 2r_bS_b)$$

$$T_{13} = -i\frac{\gamma}{\mu}(\frac{1}{r_a}S_a + r_bS_b)$$

$$T_{14} = -\frac{\gamma}{\mu}(C_a - C_b)$$

$$T_{21} = i\gamma(2r_aS_a + \frac{t}{r_b}S_b)$$

$$T_{22} = -\gamma(tC_a - 2C_b)$$

$$T_{23} = T_{14}$$

$$T_{24} = -i\frac{\gamma}{\mu}(r_aS_a + \frac{1}{r_b}S_b)$$

$$T_{31} = -i\gamma\mu(4r_aS_a + \frac{t^2}{r_b}S_b)$$

$$T_{32} = 2\gamma\mu t(C_a - C_b)$$

$$T_{33} = T_{11}$$

$$T_{34} = T_{21}$$

$$T_{41} = T_{32}$$

$$T_{42} = -i\gamma\mu(\frac{t^2}{r_a}S_a + 4r_bS_b)$$

$$T_{43} = T_{12}$$

$$T_{44} = T_{22}$$

The matrix D is the field matrix relating the amplitudes to the state variables. The matrix T is the propagator matrix relating the state variables at two different depths in the same layer. This matrix satisfies $T^{-1}(z) = T(-z)$ and $T(z_1 + z_2) = T(z_1) \cdot T(z_2)$.

We introduce the delta matrices, or compound matrices of order 2, which are formed from a given matrix by taking the different combinations of order-2 determinant. If X is a 4×4 matrix, its compound matrix $\overline{X}$ is 6×6. A matrix entry $\overline{X}_{ij}$ is itself an order-2 determinant obtained from the original matrix by selecting 2 rows and 2 columns. The order of pairs (of rows or columns) is $\{(12), (13), (14), (23), (24), (34)\}$.

With this we obtain the compound matrices for the needed matrices E, Q, Q^{-1} and T.

$$\overline{E} = \begin{bmatrix} 1 & & & & & \\ & e^{ik(r_a+r_b)z} & & & & \\ & & e^{ik(r_a-r_b)z} & & & \\ & & & e^{-ik(r_a-r_b)z} & & \\ & & & & e^{-ik(r_a+r_b)z} & \\ & & & & & 1 \end{bmatrix}$$

$$
\begin{aligned}
&\overline{Q}_{11} = -2r_a && \overline{Q}_{31} = 0 && \overline{Q}_{51} = 2\mu t r_a \\
&\overline{Q}_{12} = 1 + r_a r_b && \overline{Q}_{32} = -2\mu r_b + \mu t r_b && \overline{Q}_{52} = -2\mu r_a r_b - \mu t \\
&\overline{Q}_{13} = 1 - r_a r_b && \overline{Q}_{33} = -\overline{Q}_{32} && \overline{Q}_{53} = 2\mu r_a r_b - \mu t \\
&\overline{Q}_{14} = \overline{Q}_{13} && \overline{Q}_{34} = \overline{Q}_{32} && \overline{Q}_{54} = \overline{Q}_{53} \\
&\overline{Q}_{15} = \overline{Q}_{12} && \overline{Q}_{35} = -\overline{Q}_{32} && \overline{Q}_{55} = \overline{Q}_{52} \\
&\overline{Q}_{16} = -2r_b && \overline{Q}_{36} = 0 && \overline{Q}_{56} = 4\mu r_b \\
&\overline{Q}_{21} = 4\mu r_a && \overline{Q}_{41} = 0 && \overline{Q}_{61} = -4\mu^2 t r_a \\
&\overline{Q}_{22} = -\mu t - 2\mu r_a r_b && \overline{Q}_{42} = -\mu t r_a + 2\mu r_a && \overline{Q}_{62} = 4\mu^2 r_a r_b + (\mu t)^2 \\
&\overline{Q}_{23} = -\mu t + 2\mu r_a r_b && \overline{Q}_{43} = \overline{Q}_{42} && \overline{Q}_{63} = -4\mu^2 r_a r_b + (\mu t)^2 \\
&\overline{Q}_{24} = \overline{Q}_{23} && \overline{Q}_{44} = -\overline{Q}_{42} && \overline{Q}_{64} = \overline{Q}_{63} \\
&\overline{Q}_{25} = \overline{Q}_{22} && \overline{Q}_{45} = -\overline{Q}_{42} && \overline{Q}_{65} = \overline{Q}_{62} \\
&\overline{Q}_{26} = 2\mu t r_b && \overline{Q}_{46} = 0 && \overline{Q}_{66} = -4\mu^2 t r_b
\end{aligned}
$$

$$
\begin{aligned}
\overline{Q^{-1}}_{11} &= \frac{\gamma(2\gamma-1)}{r_a} \\
\overline{Q^{-1}}_{12} &= \frac{\gamma^2}{\mu r_a} \\
\overline{Q^{-1}}_{13} &= 0 \\
\overline{Q^{-1}}_{14} &= 0 \\
\overline{Q^{-1}}_{15} &= \frac{\overline{Q^{-1}}_{11}}{2\mu} \\
\overline{Q^{-1}}_{16} &= \frac{\overline{Q^{-1}}_{12}}{2\mu} \\
\overline{Q^{-1}}_{21} &= \gamma^2 + \frac{(2\gamma-1)^2}{4r_a r_b} \\
\overline{Q^{-1}}_{22} &= \frac{\gamma^2}{2\mu} + \frac{\gamma(2\gamma-1)}{4\mu r_a r_b} \\
\overline{Q^{-1}}_{23} &= -\frac{\gamma}{4\mu r_b} \\
\overline{Q^{-1}}_{24} &= \frac{\gamma}{4\mu r_a} \\
\overline{Q^{-1}}_{25} &= \overline{Q^{-1}}_{22} \\
\overline{Q^{-1}}_{26} &= \frac{\gamma^2}{4\mu^2}(1 + \frac{1}{r_a r_b})
\end{aligned}
$$

$$
\begin{aligned}
\overline{Q^{-1}}_{31} &= 2\gamma^2 - \overline{Q^{-1}}_{21} \\
\overline{Q^{-1}}_{32} &= \frac{\gamma^2}{\mu} - \overline{Q^{-1}}_{22} \\
\overline{Q^{-1}}_{33} &= -\overline{Q^{-1}}_{23} \\
\overline{Q^{-1}}_{34} &= \overline{Q^{-1}}_{24} \\
\overline{Q^{-1}}_{35} &= \overline{Q^{-1}}_{32} \\
\overline{Q^{-1}}_{36} &= \frac{\gamma^2}{4\mu^2}(1 - \frac{1}{r_a r_b}) \\
\overline{Q^{-1}}_{41} &= \overline{Q^{-1}}_{31} \\
\overline{Q^{-1}}_{42} &= \overline{Q^{-1}}_{32} \\
\overline{Q^{-1}}_{43} &= \overline{Q^{-1}}_{23} \\
\overline{Q^{-1}}_{44} &= -\overline{Q^{-1}}_{24} \\
\overline{Q^{-1}}_{45} &= \overline{Q^{-1}}_{32} \\
\overline{Q^{-1}}_{46} &= \overline{Q^{-1}}_{36}
\end{aligned}
$$

$$
\begin{aligned}
\overline{Q^{-1}}_{51} &= \overline{Q^{-1}}_{21} \\
\overline{Q^{-1}}_{52} &= \overline{Q^{-1}}_{22} \\
\overline{Q^{-1}}_{53} &= -\overline{Q^{-1}}_{23} \\
\overline{Q^{-1}}_{54} &= -\overline{Q^{-1}}_{24} \\
\overline{Q^{-1}}_{55} &= \overline{Q^{-1}}_{22} \\
\overline{Q^{-1}}_{56} &= \overline{Q^{-1}}_{26} \\
\overline{Q^{-1}}_{61} &= \frac{\gamma(2\gamma-1)}{r_b} \\
\overline{Q^{-1}}_{62} &= \frac{\overline{Q^{-1}}_{61}}{2\mu} \\
\overline{Q^{-1}}_{63} &= 0 \\
\overline{Q^{-1}}_{64} &= 0 \\
\overline{Q^{-1}}_{65} &= \frac{\gamma^2}{\mu r_b} \\
\overline{Q^{-1}}_{66} &= \frac{\overline{Q^{-1}}_{65}}{2\mu}
\end{aligned}
$$

$$\overline{T}_{11} = -\gamma^2\left(4t-(t^2+4)C_aC_b+(4r_ar_b+\frac{t^2}{r_ar_b})S_aS_b\right)$$
$$\overline{T}_{12} = -\frac{\gamma^2}{\mu}\left(2+t-(2+t)C_aC_b+(2r_ar_b+\frac{t}{r_ar_b})S_aS_b\right)$$
$$\overline{T}_{13} = -i\frac{\gamma}{\mu}\left(\frac{C_aS_b}{r_b}+r_aS_aC_b\right)$$
$$\overline{T}_{14} = i\frac{\gamma}{\mu}\left(\frac{S_aC_b}{r_a}+r_bC_aS_b\right)$$
$$\overline{T}_{15} = \overline{T}_{12}$$
$$\overline{T}_{16} = -\frac{\gamma^2}{\mu^2}\left(2+(\frac{1}{r_ar_b}+r_ar_b)S_aS_b-2C_aC_b\right)$$
$$\overline{T}_{21} = \gamma^2\mu\left(2t(2+t)(1-C_aC_b)+(8r_ar_b+\frac{t^3}{r_ar_b})S_aS_b\right)$$
$$\overline{T}_{22} = 1+C_aC_b-\overline{T}_{11}$$
$$\overline{T}_{23} = i\gamma\left(\frac{t}{r_b}C_aS_b+2r_aS_aC_b\right)$$
$$\overline{T}_{24} = -i\gamma\left(\frac{t}{r_a}S_aC_b+2r_bC_aS_b\right)$$
$$\overline{T}_{25} = \overline{T}_{22}-1$$
$$\overline{T}_{26} = -\overline{T}_{12}$$

$$\overline{T}_{31} = -i\gamma\mu\left(4r_bC_aS_b+\frac{t^2}{r_a}S_aC_b\right)$$
$$\overline{T}_{32} = \overline{T}_{24}$$
$$\overline{T}_{33} = C_aC_b$$
$$\overline{T}_{34} = \frac{r_b}{r_a}S_aS_b$$
$$\overline{T}_{35} = \overline{T}_{24}$$
$$\overline{T}_{36} = -\overline{T}_{14}$$
$$\overline{T}_{41} = i\gamma\mu\left(4r_aS_aC_b+\frac{t^2}{r_b}C_aS_b\right)$$
$$\overline{T}_{42} = \overline{T}_{23}$$
$$\overline{T}_{43} = \frac{r_a}{r_b}S_aS_b$$
$$\overline{T}_{44} = \overline{T}_{33}$$
$$\overline{T}_{45} = \overline{T}_{23}$$
$$\overline{T}_{46} = -\overline{T}_{13}$$

$$\overline{T}_{51} = \overline{T}_{21}$$
$$\overline{T}_{52} = \overline{T}_{25}$$
$$\overline{T}_{53} = \overline{T}_{23}$$
$$\overline{T}_{54} = \overline{T}_{24}$$
$$\overline{T}_{55} = \overline{T}_{22}$$
$$\overline{T}_{56} = -\overline{T}_{12}$$
$$\overline{T}_{61} = -\gamma^2\mu^2\left(8t^2(1-C_aC_b)+(16r_ar_b+\frac{t^4}{r_ar_b})S_aS_b\right)$$
$$\overline{T}_{62} = -\overline{T}_{21}$$
$$\overline{T}_{63} = -\overline{T}_{41}$$
$$\overline{T}_{64} = -\overline{T}_{31}$$
$$\overline{T}_{65} = -\overline{T}_{21}$$
$$\overline{T}_{66} = \overline{T}_{11}$$