2. Counting Problems

You must show work supporting your answer to receive credit.

Place your answers in a file `counting.pdf`.

1. Consider the word `unusual`. How many unique subsets of 5 letters (of the 7) exist? How many different strings could be made from 5 of those 7 letters?

① UNUSUAL
$\;\;1\;\;2\;\;3$

4 non u letters that are U
so there are no 5-letter subsets with zero u's

1u's ① $5! = 120$
2u's ② $\frac{5!}{2!} \times 4 = 240$
3u's ③ $\frac{5!}{3!} \times 6 = 120$

$120+240+120 = \boxed{480 \text{ different strings}}$

$\binom{4}{4} = 1$ subset $= \frac{1}{5}$ letters u → other 4 open spots = 1 subset
$\binom{4}{3} = 4$ subsets (2u's) → 3 open spots for 4 non-u letters = 4 subsets
$\binom{4}{2} = 6$ subsets (3u's) → 2 open spots for 4 non-u letters = 6 subsets
total $= 1+4+6 = \boxed{11 \text{ subsets}}$

② 2. Using a standard deck of playing cards, how many ways are to form a 5-card hand with 2 pairs (i.e. pair of one value a pair of a different value, and a fifth card of some other value)?

$nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$nCr = \binom{13}{2} = \frac{13!}{2!(13-2)!} = 78$
n=13 ranks
r=2 cards (1 value for each pair)
↳ choose 2 values from 13 values

$nCr = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{4} = 6$
↳ choose 2 suits (from 4) for each pair

52-4-4=44
1 card from remaining cards

$\overset{78}{\binom{13}{2}}\overset{6}{\binom{4}{2}}\overset{6}{\binom{4}{2}}\overset{44}{\binom{44}{1}}$

$= \boxed{123{,}552 \text{ ways}}$

③ 3. A violinist serenades couples at a romantic restaurant. She will play 16 songs in an hour and there are 7 couples. One couple is having a fight and will allow at most 1 song to be played to them before they ask the violist not to return to their table. If we care only about the number of songs each couple receives, how many ways can the songs be distributed amongst the couples?

n = distinguishable = couples
r = indistinguishable = songs

$\binom{n+k-1}{k}$

n = 6 couples
ignore troubled couple
k = 16 songs

$\binom{6+16-1}{16} = \binom{21}{16} \quad \frac{20!}{15!(20-15)!} = 15503.999$

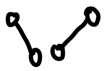
at most 1 song → $\binom{6+15-1}{15} = \binom{20}{15} \quad \frac{21!}{16!(21-16)!} = 20349$

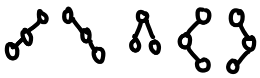
$\binom{20}{15} + \binom{21}{16} = 20349 + 15503.999 = 35852.999$

$\approx \boxed{35853 \text{ ways}}$

4. There is a Binary Search Tree with 12 nodes. Each node has a distinct value between 1 and 12. The root has value 3, and its right child has value 9. How many possible Binary Search Trees could this be? Tip: *Try to define how many ways there are to form a BST of 2 nodes. Then try to define how many ways there are to form a BST of 3 nodes (think about the possible insertion order based on rank: smallest, medium, largest)* ***in terms of 2 node trees*** *for certain cases. Continue to do this for 4 node trees (in terms of 3- and 2-node trees for various cases of insertion ordering based on rank) and 5 node trees.*

Creating BST for 2|3|4 nodes

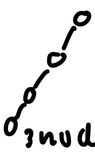
2 nodes:  (2 ways)

3 nodes:  (5 ways)

4 nodes: 

2 nodes 2 node

2 ways + 2 ways = 4 ways



3 nodes 3 nodes

5 ways + 5 ways = 10 ways

4 nodes = 10 + 4 ways = 14 ways total

5 nodes: 

to left

4 nodes = 14 ways

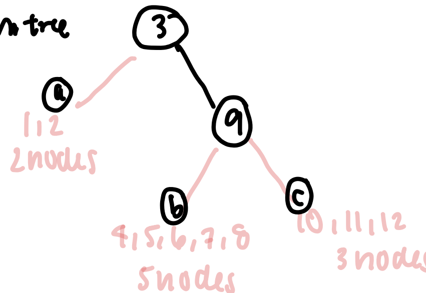
to right 14 ways

4 nodes =

2 node 2 nodes = 14 ways

5 nodes = 14 × 3 = 42 ways

Problem tree



Check

5 node case Nodes = a, b, c, d, e

CASE 1: a = root → 14 ways

- subtree of 4 nodes to right

CASE 2: b = root

- subtree (left) = 1 node → 5 ways
- subtree (right): 3 nodes 5 ways (1) = 5

CASE 3: c = root 2 subtrees

- subtree (left) = 2 node → 4 ways
- subtree (right) = 2 nodes 2(2) = 4

CASE 4: d = root → 5 ways

CASE 5: e = root → 14 ways

14 + 5 + 4 + 5 + 14 = 42 ways for 5 node BST

Subtree a: 2 ways (have node 1 & 2) → 2 nodes = 2 ways

Subtree b: 42 ways (nodes 4, 5, 6, 7, 8) → 5 nodes = 42 ways (2)(42)(5) = 420

Subtree c: 5 ways (nodes 10, 11, 12) → 3 nodes = 5 ways ways to

total = 42 · 2 · 5 = $\boxed{\text{420 ways}}$ build BST

5. 10 friends arrive to get their COVID vaccine during a particular time slot. During that time slot there are 4 identical nurses administering shots, but 1 of the nurses **may** (or **may not**) be scheduled for a break during the time slot in which the friends arrive. Also, how long it takes the nurses to administer a shot varies wildly, so the nurses working during the time slot are guaranteed to serve at least 1 person, but how many additional people they are able to serve is arbitrary. How many different combinations are there for the number of patients served by the nurses?

10 friends covid vaccine

4 nurses

Case 1: 4 nurses working

$$9\left[\begin{array}{cccc} 7 & 1 & 1 & 1 \\ 6 & 2 & 1 & 1 \\ 5 & 3 & 1 & 1 \\ 5 & 2 & 2 & 1 \\ 4 & 4 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 2 & 2 & 4 \end{array}\right.$$

Case 2: 3 nurses working

$$8\left[\begin{array}{ccc} 8 & 1 & 1 \\ 7 & 2 & 1 \\ 6 & 3 & 1 \\ 6 & 2 & 2 \\ 5 & 4 & 1 \\ 5 & 3 & 2 \\ 4 & 4 & 2 \\ 4 & 3 & 3 \end{array}\right.$$

total = 8+9 = $\boxed{\text{17 combinations}}$