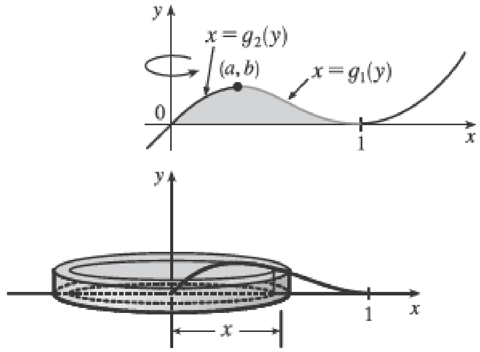
1.

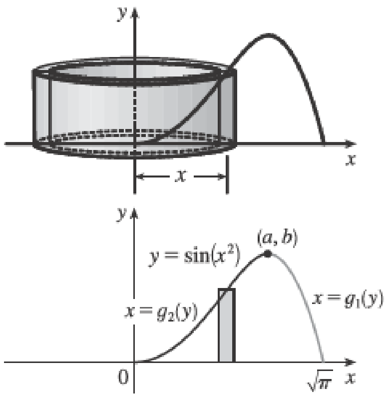


If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x\left[x(x-1)^2\right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

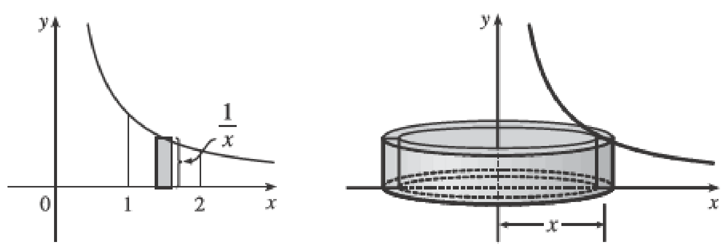
2.



A typical cylindrical shell has circumference $2\pi x$ and height $\sin(x^2)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx$. Let $u = x^2$. Then $du = 2x\, dx$, so $V = \pi \int_0^{\pi} \sin u\, du = \pi[-\cos u]_0^{\pi} = \pi[1 - (-1)] = 2\pi$. For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$. Using shells is definitely preferable to slicing.

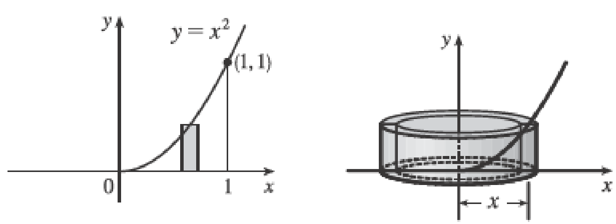
3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 1\, dx$

$= 2\pi\left[\,x\,\right]_1^2 = 2\pi(2-1) = 2\pi$



4. $V = \int_0^1 2\pi x \cdot x^2\, dx = 2\pi \int_0^1 x^3\, dx$

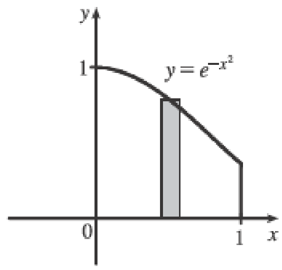
$= 2\pi\left[\frac{1}{4}x^4\right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$

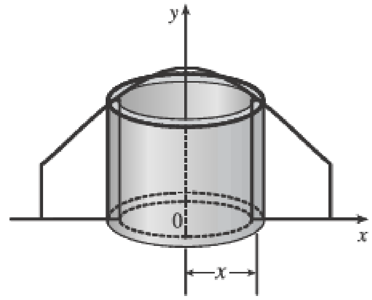


5. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$.

Thus, $du = 2x\,dx$, so

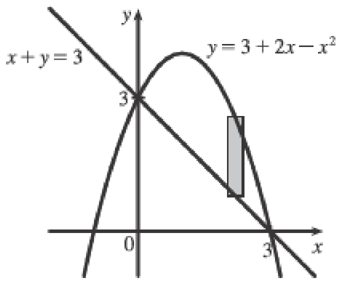
$V = \pi \int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

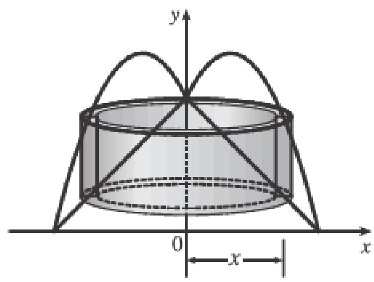




6. $V = 2\pi \int_0^3 \{x[(3 + 2x - x^2) - (3 - x)]\}\,dx = 2\pi \int_0^3 [x(3x - x^2)]\,dx$

$= 2\pi \int_0^3 (3x^2 - x^3)\,dx = 2\pi\left[x^3 - \frac{1}{4}x^4\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27}{2}\pi$





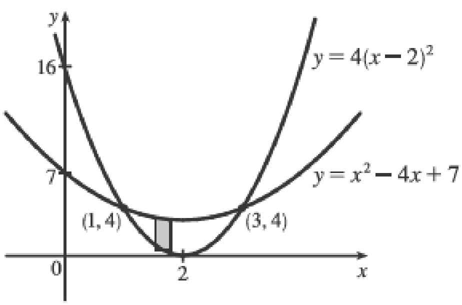
7. The curves intersect when $4(x - 2)^2 = x^2 - 4x + 7 \quad\Leftrightarrow\quad 4x^2 - 16x + 16 = x^2 - 4x + 7 \quad\Leftrightarrow$

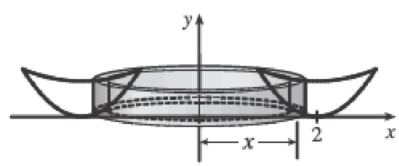
$3x^2 - 12x + 9 = 0 \quad\Leftrightarrow\quad 3(x^2 - 4x + 3) = 0 \quad\Leftrightarrow\quad 3(x - 1)(x - 3) = 0$, so $x = 1$ or 3.

$V = 2\pi \int_1^3 \{x[(x^2 - 4x + 7) - 4(x - 2)^2]\}\,dx = 2\pi \int_1^3 [x(x^2 - 4x + 7 - 4x^2 + 16x - 16)]\,dx$

$= 2\pi \int_1^3 [x(-3x^2 + 12x - 9)]\,dx = 2\pi(-3)\int_1^3 (x^3 - 4x^2 + 3x)\,dx = -6\pi\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$

$= -6\pi\left[\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \frac{4}{3}\right) = -6\pi\left(-\frac{8}{3}\right) = 16\pi$

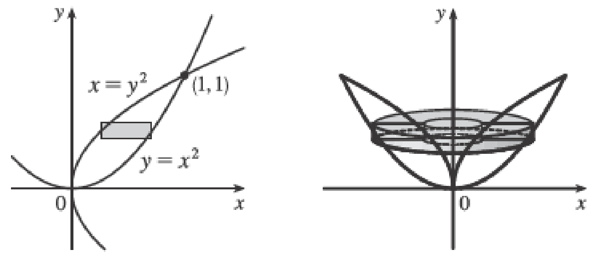




8. By slicing:

$$V = \int_0^1 \pi \left[\left(\sqrt{y} \right)^2 - (y^2)^2 \right] dy = \pi \int_0^1 (y - y^4)\, dy$$

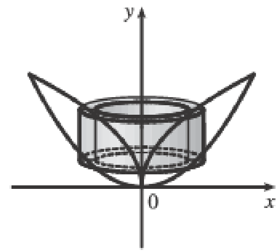
$$= \pi \left[\tfrac{1}{2} y^2 - \tfrac{1}{5} y^5 \right]_0^1 = \pi \left(\tfrac{1}{2} - \tfrac{1}{5} \right) = \tfrac{3}{10} \pi$$



By cylindrical shells:

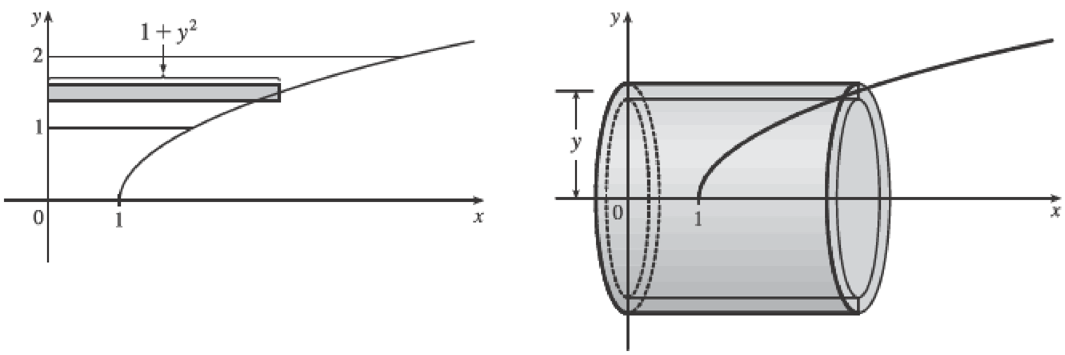
$$V = \int_0^1 2\pi x \left(\sqrt{x} - x^2 \right) dx = 2\pi \int_0^1 (x^{3/2} - x^3)\, dx = 2\pi \left[\tfrac{2}{5} x^{5/2} - \tfrac{1}{4} x^4 \right]_0^1$$

$$= 2\pi \left(\tfrac{2}{5} - \tfrac{1}{4} \right) = 2\pi \left(\tfrac{3}{20} \right) = \tfrac{3}{10} \pi$$



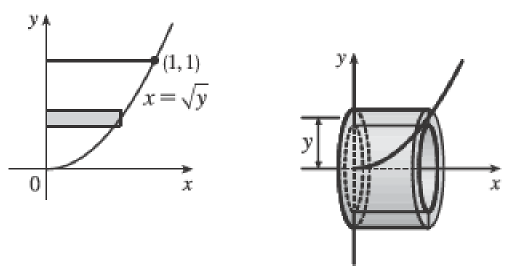
9. $V = \int_1^2 2\pi y (1 + y^2)\, dy = 2\pi \int_1^2 (y + y^3)\, dy = 2\pi \left[\tfrac{1}{2} y^2 + \tfrac{1}{4} y^4 \right]_1^2$

$$= 2\pi \left[(2 + 4) - \left(\tfrac{1}{2} + \tfrac{1}{4} \right) \right] = 2\pi \left(\tfrac{21}{4} \right) = \tfrac{21}{2} \pi$$



10. $V = \int_0^1 2\pi y \sqrt{y}\, dy = 2\pi \int_0^1 y^{3/2}\, dy$

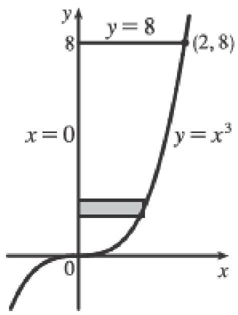
$$= 2\pi \left[\tfrac{2}{5} y^{5/2} \right]_0^1 = \tfrac{4}{5} \pi$$

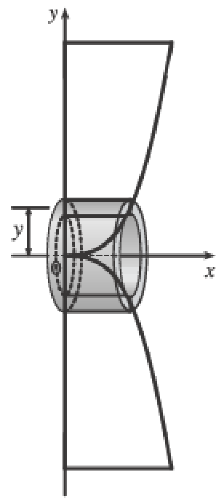


11. $V = 2\pi \int_0^8 \left[y(\sqrt[3]{y} - 0) \right] dy$

$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \left[\frac{3}{7} y^{7/3} \right]_0^8$

$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768}{7}\pi$



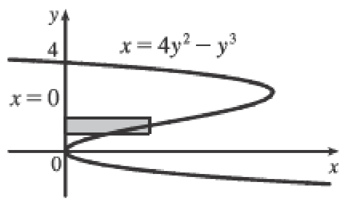


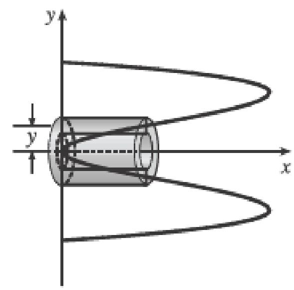
12. $V = 2\pi \int_0^4 \left[y(4y^2 - y^3) \right] dy$

$= 2\pi \int_0^4 (4y^3 - y^4)\, dy$

$= 2\pi \left[y^4 - \frac{1}{5} y^5 \right]_0^4 = 2\pi \left(256 - \frac{1024}{5} \right)$

$= 2\pi \left(\frac{256}{5} \right) = \frac{512}{5}\pi$





13. The height of the shell is $2 - \left[1 + (y-2)^2\right] = 1 - (y-2)^2 = 1 - (y^2 - 4y + 4) = -y^2 + 4y - 3$.

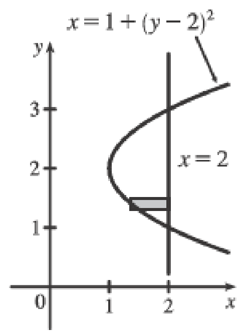
$V = 2\pi \int_1^3 y(-y^2 + 4y - 3)\, dy$

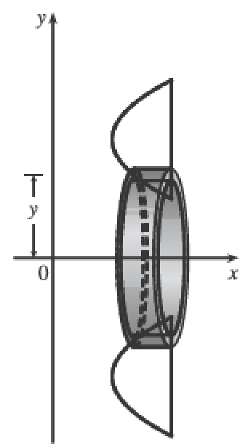
$= 2\pi \int_1^3 (-y^3 + 4y^2 - 3y)\, dy$

$= 2\pi \left[-\frac{1}{4} y^4 + \frac{4}{3} y^3 - \frac{3}{2} y^2 \right]_1^3$

$= 2\pi \left[\left(-\frac{81}{4} + 36 - \frac{27}{2} \right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2} \right) \right]$

$= 2\pi \left(\frac{8}{3} \right) = \frac{16}{3}\pi$



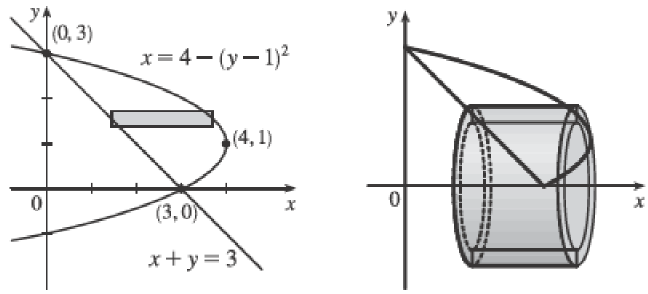


14. $V = \int_0^3 2\pi y\left[4 - (y-1)^2 - (3-y)\right] dy$

$= 2\pi \int_0^3 y(-y^2 + 3y)\, dy$

$= 2\pi \int_0^3 (-y^3 + 3y^2)\, dy = 2\pi\left[-\frac{1}{4}y^4 + y^3\right]_0^3$

$= 2\pi\left(-\frac{81}{4} + 27\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27}{2}\pi$



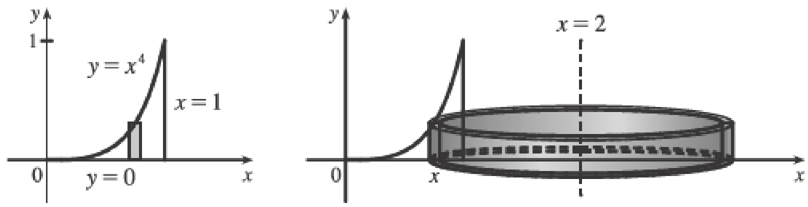
15. The shell has radius $2 - x$, circumference $2\pi(2 - x)$, and height x^4.

$V = \int_0^1 2\pi(2-x)x^4\, dx$

$= 2\pi \int_0^1 (2x^4 - x^5)\, dx$

$= 2\pi\left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right]_0^1$

$= 2\pi\left[\left(\frac{2}{5} - \frac{1}{6}\right) - 0\right] = 2\pi\left(\frac{7}{30}\right) = \frac{7}{15}\pi$



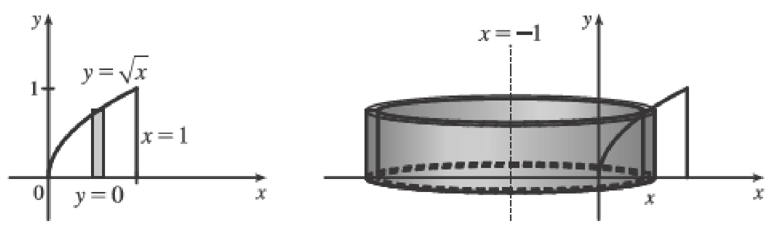
16. The shell has radius $x - (-1) = x + 1$, circumference $2\pi(x + 1)$, and height $\sqrt{x}$.

$V = \int_0^1 2\pi(x+1)\sqrt{x}\, dx$

$= 2\pi \int_0^1 (x^{3/2} + x^{1/2})\, dx$

$= 2\pi\left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right]_0^1$

$= 2\pi\left[\left(\frac{2}{5} + \frac{2}{3}\right) - 0\right] = 2\pi\left(\frac{16}{15}\right) = \frac{32}{15}\pi$



17. The shell has radius $x - 1$, circumference $2\pi(x - 1)$, and height $(4x - x^2) - 3 = -x^2 + 4x - 3$.

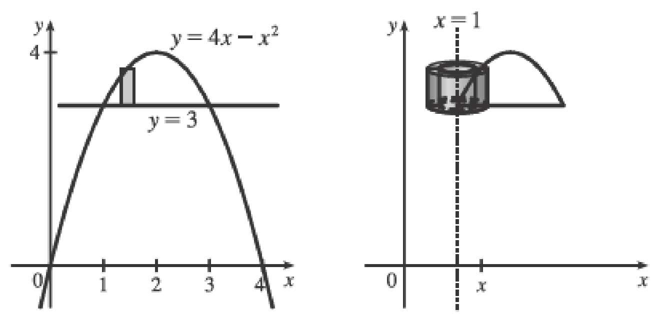
$V = \int_1^3 2\pi(x-1)(-x^2 + 4x - 3)\, dx$

$= 2\pi \int_1^3 (-x^3 + 5x^2 - 7x + 3)\, dx$

$= 2\pi\left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - \frac{7}{2}x^2 + 3x\right]_1^3$

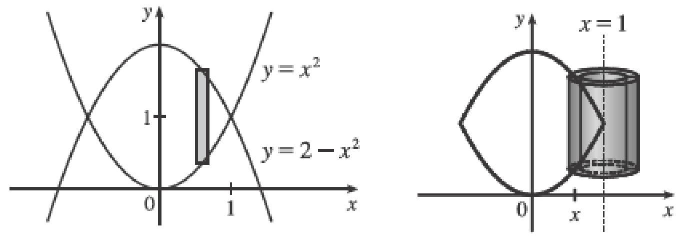
$= 2\pi\left[\left(-\frac{81}{4} + 45 - \frac{63}{2} + 9\right) - \left(-\frac{1}{4} + \frac{5}{3} - \frac{7}{2} + 3\right)\right]$

$= 2\pi\left(\frac{4}{3}\right) = \frac{8}{3}\pi$



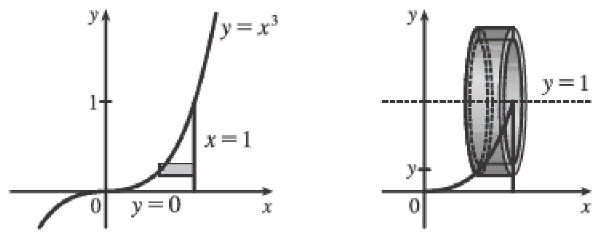
18. The shell has radius $1 - x$, circumference $2\pi(1 - x)$, and height $(2 - x^2) - x^2 = 2 - 2x^2$.

$$\begin{aligned} V &= \int_{-1}^{1} 2\pi(1-x)(2-2x^2)\,dx \\ &= 2\pi(2)\int_{-1}^{1}(1-x)(1-x^2)\,dx \\ &= 4\pi\int_{-1}^{1}(1-x-x^2+x^3)\,dx \\ &= 4\pi(2)\int_{0}^{1}(1-x^2)\,dx \qquad \text{[by Theorem 5.5.7]} \\ &= 8\pi\left[x-\tfrac{1}{3}x^3\right]_0^1 = 8\pi\left[\left(1-\tfrac{1}{3}\right)-0\right] = 8\pi\left(\tfrac{2}{3}\right) = \tfrac{16}{3}\pi \end{aligned}$$



19. The shell has radius $1 - y$, circumference $2\pi(1 - y)$, and height $1 - \sqrt[3]{y}$ $\left[y = x^3 \quad \Leftrightarrow \quad x = \sqrt[3]{y}\right]$.

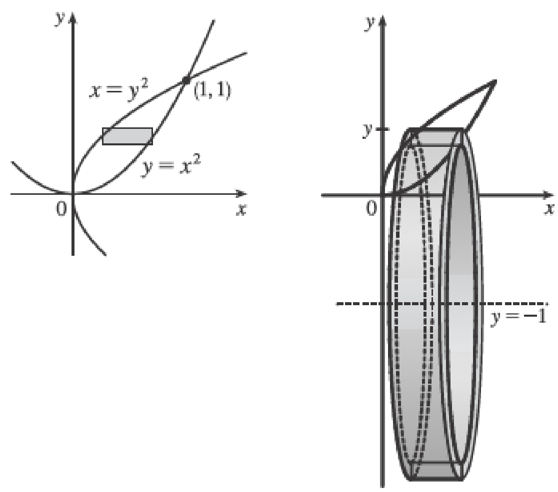
$$\begin{aligned} V &= \int_{0}^{1} 2\pi(1-y)(1-y^{1/3})\,dy \\ &= 2\pi\int_{0}^{1}(1-y-y^{1/3}+y^{4/3})\,dy \\ &= 2\pi\left[y-\tfrac{1}{2}y^2-\tfrac{3}{4}y^{4/3}+\tfrac{3}{7}y^{7/3}\right]_0^1 \\ &= 2\pi\left[\left(1-\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{3}{7}\right)-0\right] \\ &= 2\pi\left(\tfrac{5}{28}\right) = \tfrac{5}{14}\pi \end{aligned}$$



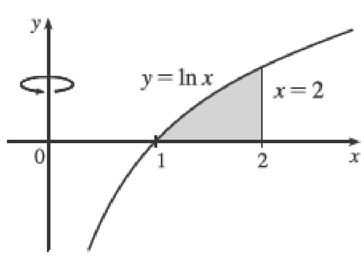
20. The shell has radius $y - (-1) = y + 1$,

circumference $2\pi(y + 1)$, and height $\sqrt{y} - y^2$.

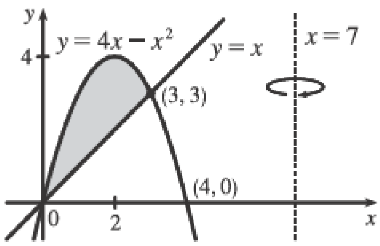
$$\begin{aligned} V &= \int_{0}^{1} 2\pi(y+1)\left(\sqrt{y}-y^2\right)dy \\ &= 2\pi\int_{0}^{1}(y^{3/2}+y^{1/2}-y^3-y^2)\,dy \\ &= 2\pi\left[\tfrac{2}{5}y^{5/2}+\tfrac{2}{3}y^{3/2}-\tfrac{1}{4}y^4-\tfrac{1}{3}y^3\right]_0^1 \\ &= 2\pi\left(\tfrac{2}{5}+\tfrac{2}{3}-\tfrac{1}{4}-\tfrac{1}{3}\right) = 2\pi\left(\tfrac{29}{60}\right) = \tfrac{29}{30}\pi \end{aligned}$$



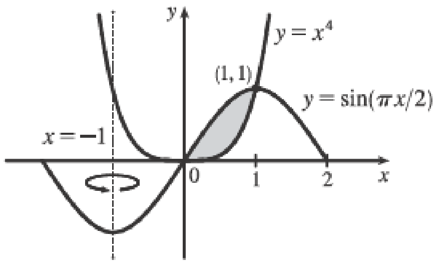
21. $V = \int_{1}^{2} 2\pi x \ln x\,dx$



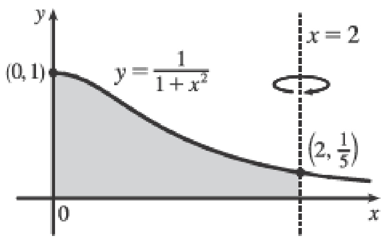
22. $V = \int_0^3 2\pi(7-x)[(4x-x^2)-x]\,dx$



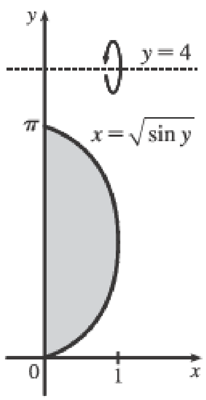
23. $V = \int_0^1 2\pi[x-(-1)]\left(\sin\frac{\pi}{2}x - x^4\right) dx$



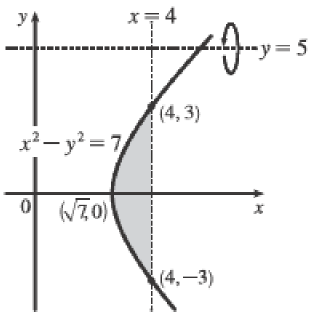
24. $V = \displaystyle\int_0^2 2\pi(2-x)\left(\frac{1}{1+x^2}\right) dx$



25. $V = \int_0^\pi 2\pi(4-y)\sqrt{\sin y}\,dy$



26. $V = \int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{y^2+7}\right) dy$

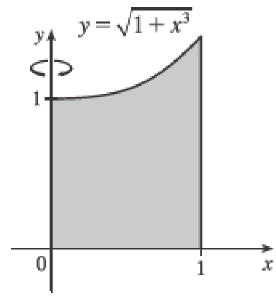


27. $V = \int_0^1 2\pi x \sqrt{1+x^3}\, dx$. Let $f(x) = x\sqrt{1+x^3}$.

Then the Midpoint Rule with $n = 5$ gives

$$\begin{aligned}\int_0^1 f(x)\,dx &\approx \tfrac{1-0}{5}\left[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)\right]\\ &\approx 0.2(2.9290)\end{aligned}$$

Multiplying by 2π gives $V \approx 3.68$.



28. $\Delta x = \frac{12-2}{5} = 2$, $n = 5$ and $x_i^* = 2 + (2i+1)$, where $i = 0, 1, 2, 3, 4$. The values of $f(x)$ are taken directly from the diagram.

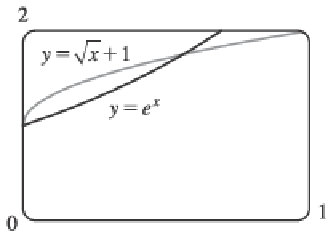
$$\begin{aligned}V &= \int_2^{12} 2\pi x f(x)\,dx \approx 2\pi[3f(3)+5f(5)+7f(7)+9f(9)+11f(11)]\cdot 2\\ &\approx 2\pi[3(2)+5(4)+7(4)+9(2)+11(1)]2 = 332\pi\end{aligned}$$

29. $\int_0^3 2\pi x^5\,dx = 2\pi\int_0^3 x(x^4)\,dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

30. $2\pi \displaystyle\int_0^2 \frac{y}{1+y^2}\,dy = 2\pi\int_0^2 y\left(\frac{1}{1+y^2}\right) dy$. The solid is obtained by rotating the region $0 \le x \le \dfrac{1}{1+y^2}$, $0 \le y \le 2$ about the x-axis using cylindrical shells.

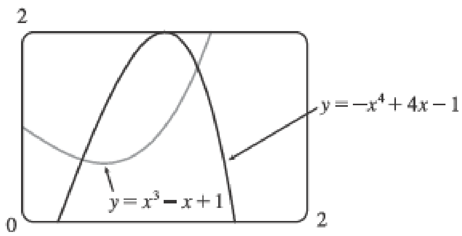
31. $\int_0^1 2\pi(3-y)(1-y^2)\,dy$. The solid is obtained by rotating the region bounded by (i) $x = 1-y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\,dx$. The solid is obtained by rotating the region bounded by (i) $0 \le y \le \cos x - \sin x$, $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$, $0 \le x \le \frac{\pi}{4}$ about the line $x = \pi$ using cylindrical shells.

33. 

From the graph, the curves intersect at $x = 0$ and $x = a \approx 0.56$, with $\sqrt{x} + 1 > e^x$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a x\left[\left(\sqrt{x} + 1\right) - e^x\right]\, dx \approx 0.13.$$

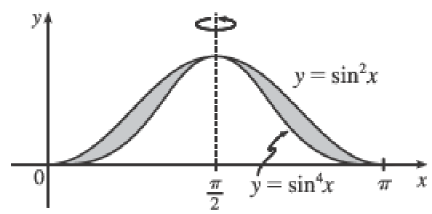
34. 

From the graph, the curves intersect at $x = a \approx 0.42$ and $x = b \approx 1.23$, with $-x^4 + 4x - 1 > x^3 - x + 1$ on the interval (a, b). So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_a^b x\left[(-x^4 + 4x - 1) - (x^3 - x + 1)\right]\, dx$$
$$= 2\pi \int_a^b x(-x^4 - x^3 + 5x - 2)\, dx \approx 3.17$$

35. $V = 2\pi \displaystyle\int_0^{\pi/2} \left[\left(\tfrac{\pi}{2} - x\right)\left(\sin^2 x - \sin^4 x\right)\right] dx$

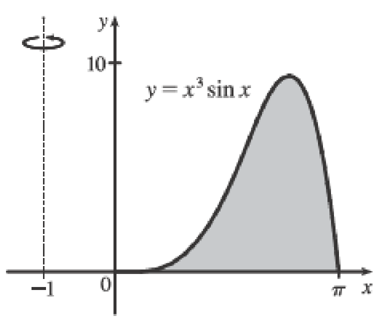
$\overset{\text{CAS}}{=} \frac{1}{32}\pi^3$



36. $V = 2\pi \int_0^{\pi} \{[x - (-1)](x^3 \sin x)\}\, dx$

$\overset{\text{CAS}}{=} 2\pi(\pi^4 + \pi^3 - 12\pi^2 - 6\pi + 48)$

$= 2\pi^5 + 2\pi^4 - 24\pi^3 - 12\pi^2 + 96\pi$



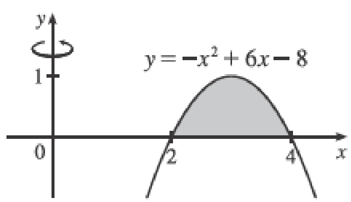
37. Use shells:

$V = \int_2^4 2\pi x(-x^2 + 6x - 8)\, dx = 2\pi \int_2^4 (-x^3 + 6x^2 - 8x)\, dx$

$= 2\pi \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4$

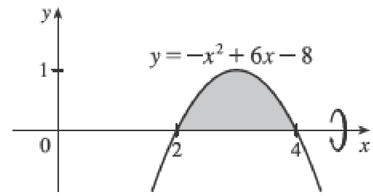
$= 2\pi[(-64 + 128 - 64) - (-4 + 16 - 16)]$

$= 2\pi(4) = 8\pi$



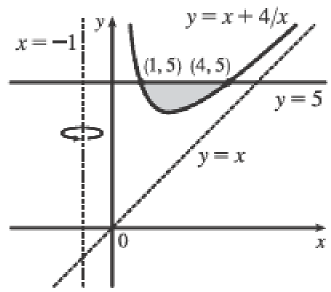
38. Use disks:

$$V = \int_2^4 \pi(-x^2 + 6x - 8)^2\, dx$$
$$= \pi \int_2^4 (x^4 - 12x^3 + 52x^2 - 96x + 64)\, dx$$
$$= \pi \left[\tfrac{1}{5}x^5 - 3x^4 + \tfrac{52}{3}x^3 - 48x^2 + 64x\right]_2^4$$
$$= \pi \left(\tfrac{512}{15} - \tfrac{496}{15}\right) = \tfrac{16}{15}\pi$$



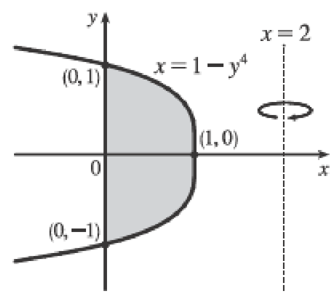
39. Use shells:

$$V = \int_1^4 2\pi[x - (-1)][5 - (x + 4/x)]\, dx$$
$$= 2\pi \int_1^4 (x + 1)(5 - x - 4/x)\, dx$$
$$= 2\pi \int_1^4 (5x - x^2 - 4 + 5 - x - 4/x)\, dx$$
$$= 2\pi \int_1^4 (-x^2 + 4x + 1 - 4/x)\, dx = 2\pi\left[-\tfrac{1}{3}x^3 + 2x^2 + x - 4\ln x\right]_1^4$$
$$= 2\pi\left[\left(-\tfrac{64}{3} + 32 + 4 - 4\ln 4\right) - \left(-\tfrac{1}{3} + 2 + 1 - 0\right)\right]$$
$$= 2\pi(12 - 4\ln 4) = 8\pi(3 - \ln 4)$$



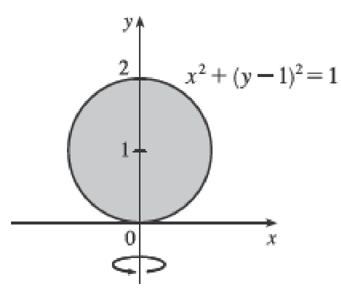
40. Use washers:

$$V = \int_{-1}^1 \pi\left\{[2 - 0]^2 - [2 - (1 - y^4)]^2\right\} dy$$
$$= 2\pi \int_0^1 \left[4 - (1 + y^4)^2\right] dy \qquad \text{[by symmetry]}$$
$$= 2\pi \int_0^1 \left[4 - (1 + 2y^4 + y^8)\right] dy = 2\pi \int_0^1 (3 - 2y^4 - y^8)\, dy$$
$$= 2\pi\left[3y - \tfrac{2}{5}y^5 - \tfrac{1}{9}y^9\right]_0^1 = 2\pi\left(3 - \tfrac{2}{5} - \tfrac{1}{9}\right)$$
$$= 2\pi\left(\tfrac{112}{45}\right) = \tfrac{224}{45}\pi$$



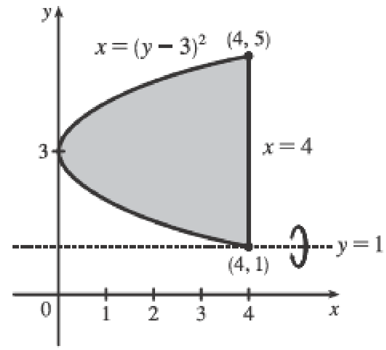
41. Use disks: $x^2 + (y - 1)^2 = 1 \quad \Leftrightarrow \quad x = \pm\sqrt{1 - (y - 1)^2}$

$$V = \pi \int_0^2 \left[\sqrt{1 - (y - 1)^2}\right]^2 dy = \pi \int_0^2 (2y - y^2)\, dy$$
$$= \pi\left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \pi\left(4 - \tfrac{8}{3}\right) = \tfrac{4}{3}\pi$$



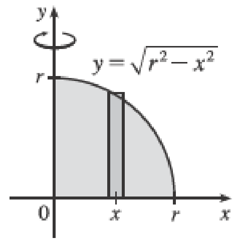
42. Use shells:

$$V = \int_1^5 2\pi(y-1)[4-(y-3)^2]\,dy$$
$$= 2\pi \int_1^5 (y-1)(-y^2+6y-5)\,dy$$
$$= 2\pi \int_1^5 (-y^3+7y^2-11y+5)\,dy$$
$$= 2\pi\left[-\tfrac{1}{4}y^4+\tfrac{7}{3}y^3-\tfrac{11}{2}y^2+5y\right]_1^5$$
$$= 2\pi\left(\tfrac{275}{12}-\tfrac{19}{12}\right) = \tfrac{128}{3}\pi$$



43. Use shells:

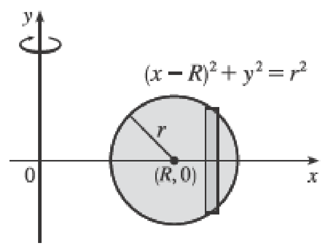
$$V = 2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx$$
$$= -2\pi \int_0^r (r^2-x^2)^{1/2}(-2x)\,dx$$
$$= \left[-2\pi\cdot\tfrac{2}{3}(r^2-x^2)^{3/2}\right]_0^r$$
$$= -\tfrac{4}{3}\pi(0-r^3) = \tfrac{4}{3}\pi r^3$$



44. $V = \int_{R-r}^{R+r} 2\pi x\cdot 2\sqrt{r^2-(x-R)^2}\,dx$

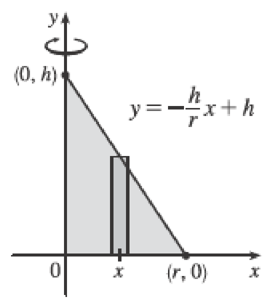
$$= \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \qquad [\text{let } u = x-R]$$
$$= 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du + 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du$$

The first integral is the area of a semicircle of radius r, that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus, $V = 4\pi R\left(\frac{1}{2}\pi r^2\right) + 4\pi\cdot 0 = 2\pi R r^2$.

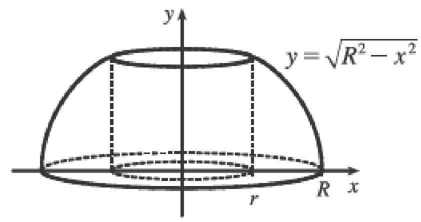


45. $V = 2\pi\displaystyle\int_0^r x\left(-\frac{h}{r}x+h\right)dx = 2\pi h\int_0^r\left(-\frac{x^2}{r}+x\right)dx$

$$= 2\pi h\left[-\frac{x^3}{3r}+\frac{x^2}{2}\right]_0^r = 2\pi h\,\frac{r^2}{6} = \frac{\pi r^2 h}{3}$$



46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x-axis and below the curve $y = \sqrt{R^2 - x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y-axis. This volume is equal to



$$2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi rh\,dx = 2 \cdot 2\pi \int_r^R x\sqrt{R^2 - x^2}\,dx = 4\pi\left[-\tfrac{1}{3}\left(R^2 - x^2\right)^{3/2}\right]_r^R = \tfrac{4}{3}\pi(R^2 - r^2)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = \left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.70.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 6.2.51,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \tfrac{4}{3}\pi R^3 - \pi r^2 h - 2 \cdot \tfrac{\pi}{3}\left(R - \tfrac{1}{2}h\right)^2\left[3R - \left(R - \tfrac{1}{2}h\right)\right] = \tfrac{1}{6}\pi h^3$$