

$$\begin{aligned} 1. Q &= \iint_D \sigma(x, y) dA = \int_1^3 \int_0^2 (2xy + y^2) dy dx = \int_1^3 [xy^2 + \frac{1}{3}y^3]_{y=0}^{y=2} dx \\ &= \int_1^3 (4x + \frac{8}{3}) dx = [2x^2 + \frac{8}{3}x]_1^3 = 16 + \frac{16}{3} = \frac{64}{3} \text{ C} \end{aligned}$$

$$\begin{aligned} 2. Q &= \iint_D \sigma(x, y) dA = \iint_D (x + y + x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 (r \cos \theta + r \sin \theta + r^2) r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 [r^2(\cos \theta + \sin \theta) + r^3] dr d\theta = \int_0^{2\pi} [\frac{1}{3}r^3(\cos \theta + \sin \theta) + \frac{1}{4}r^4]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} [\frac{8}{3}(\cos \theta + \sin \theta) + 4] d\theta = [\frac{8}{3}(\sin \theta - \cos \theta) + 4\theta]_0^{2\pi} = 8\pi \text{ C} \end{aligned}$$

$$\begin{aligned} 3. m &= \iint_D \rho(x, y) dA = \int_0^2 \int_{-1}^1 xy^2 dy dx = \int_0^2 x dx \int_{-1}^1 y^2 dy = [\frac{1}{2}x^2]_0^2 [\frac{1}{3}y^3]_{-1}^1 = 2 \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{x} &= \frac{1}{m} \iint_D x \rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 x^2 y^2 dy dx = \frac{3}{4} \int_0^2 x^2 dx \int_{-1}^1 y^2 dy = \frac{3}{4} [\frac{1}{3}x^3]_0^2 [\frac{1}{3}y^3]_{-1}^1 = \frac{3}{4} \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 xy^3 dy dx = \frac{3}{4} \int_0^2 x dx \int_{-1}^1 y^3 dy = \frac{3}{4} [\frac{1}{2}x^2]_0^2 [\frac{1}{4}y^4]_{-1}^1 = \frac{3}{4} \cdot 2 \cdot 0 = 0. \end{aligned}$$

Hence,  $(\bar{x}, \bar{y}) = (\frac{4}{3}, 0)$ .

$$\begin{aligned} 4. m &= \iint_D \rho(x, y) dA = \int_0^a \int_0^b cxy dy dx = c \int_0^a x dx \int_0^b y dy = c [\frac{1}{2}x^2]_0^a [\frac{1}{2}y^2]_0^b = \frac{1}{4}a^2b^2c, \\ M_y &= \iint_D x \rho(x, y) dA = \int_0^a \int_0^b cx^2y dy dx = c \int_0^a x^2 dx \int_0^b y dy = c [\frac{1}{3}x^3]_0^a [\frac{1}{2}y^2]_0^b = \frac{1}{6}a^3b^2c, \text{ and} \\ M_x &= \iint_D y \rho(x, y) dA = \int_0^a \int_0^b cxy^2 dy dx = c \int_0^a x dx \int_0^b y^2 dy = c [\frac{1}{2}x^2]_0^a [\frac{1}{3}y^3]_0^b = \frac{1}{6}a^2b^3c. \end{aligned}$$

Hence,  $(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) = \left( \frac{2}{3}a, \frac{2}{3}b \right)$ .

$$\begin{aligned} 5. m &= \int_0^2 \int_{x/2}^{3-x} (x+y) dy dx = \int_0^2 [xy + \frac{1}{2}y^2]_{y=x/2}^{y=3-x} dx = \int_0^2 [x(3 - \frac{3}{2}x) + \frac{1}{2}(3-x)^2 - \frac{1}{8}x^2] dx \\ &= \int_0^2 (-\frac{9}{8}x^2 + \frac{9}{2}) dx = [-\frac{9}{8}(\frac{1}{3}x^3) + \frac{9}{2}x]_0^2 = 6, \\ M_y &= \int_0^2 \int_{x/2}^{3-x} (x^2 + xy) dy dx = \int_0^2 [x^2y + \frac{1}{2}xy^2]_{y=x/2}^{y=3-x} dx = \int_0^2 (\frac{9}{2}x - \frac{9}{8}x^3) dx = \frac{9}{2}, \\ M_x &= \int_0^2 \int_{x/2}^{3-x} (xy + y^2) dy dx = \int_0^2 [\frac{1}{2}xy^2 + \frac{1}{3}y^3]_{y=x/2}^{y=3-x} dx = \int_0^2 (9 - \frac{9}{2}x) dx = 9. \end{aligned}$$

Hence  $m = 6$ ,  $(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) = \left( \frac{3}{4}, \frac{3}{2} \right)$ .

6. Here  $D = \{(x, y) \mid 0 \leq x \leq 2, x \leq y \leq 6 - 2x\}$ .

$$\begin{aligned} m &= \int_0^2 \int_x^{6-2x} x^2 dy dx = \int_0^2 x^2 (6 - 2x - x) dx = \int_0^2 (6x^2 - 3x^3) dx = [2x^3 - \frac{3}{4}x^4]_0^2 = 4, \\ M_y &= \int_0^2 \int_x^{6-2x} x \cdot x^2 dy dx = \int_0^2 x^3 (6 - 2x - x) dx = \int_0^2 (6x^3 - 3x^4) dx = [\frac{3}{2}x^4 - \frac{3}{5}x^5]_0^2 = \frac{24}{5}, \\ M_x &= \int_0^2 \int_x^{6-2x} y \cdot x^2 dy dx = \int_0^2 x^2 [\frac{1}{2}(6 - 2x)^2 - \frac{1}{2}x^2] dx = \frac{1}{2} \int_0^2 (3x^4 - 24x^3 + 36x^2) dx \\ &= \frac{1}{2} [\frac{3}{5}x^5 - 6x^4 + 12x^3]_0^2 = \frac{48}{5}. \end{aligned}$$

Hence  $m = 4$ ,  $(\bar{x}, \bar{y}) = \left( \frac{24/5}{4}, \frac{48/5}{4} \right) = \left( \frac{6}{5}, \frac{12}{5} \right)$ .

$$7. m = \int_0^1 \int_0^{e^x} y \, dy \, dx = \int_0^1 \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 e^{2x} dx = \frac{1}{4} e^{2x} \Big|_0^1 = \frac{1}{4}(e^2 - 1),$$

$$M_y = \int_0^1 \int_0^{e^x} xy \, dy \, dx = \frac{1}{2} \int_0^1 x e^{2x} dx = \frac{1}{2} \left[ \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right]_0^1 = \frac{1}{8}(e^2 + 1),$$

$$M_x = \int_0^1 \int_0^{e^x} y^2 \, dy \, dx = \int_0^1 \left[ \frac{1}{3} y^3 \right]_{y=0}^{y=e^x} dx = \frac{1}{3} \int_0^1 e^{3x} dx = \frac{1}{3} \left[ \frac{1}{3} e^{3x} \right]_0^1 = \frac{1}{9}(e^3 - 1).$$

$$\text{Hence } m = \frac{1}{4}(e^2 - 1), (\bar{x}, \bar{y}) = \left( \frac{\frac{1}{8}(e^2 + 1)}{\frac{1}{4}(e^2 - 1)}, \frac{\frac{1}{9}(e^3 - 1)}{\frac{1}{4}(e^2 - 1)} \right) = \left( \frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)} \right).$$

$$8. m = \int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx = \int_0^1 x \left[ y \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{3/2} dx = \frac{2}{5} x^{5/2} \Big|_0^1 = \frac{2}{5},$$

$$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2 \, dy \, dx = \int_0^1 x \left[ y \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{5/2} dx = \frac{2}{7} x^{7/2} \Big|_0^1 = \frac{2}{7},$$

$$M_x = \int_0^1 \int_0^{\sqrt{x}} yx \, dy \, dx = \int_0^1 x \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2} \int_0^1 x^2 dx = \frac{1}{2} \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{6}.$$

$$\text{Hence } m = \frac{2}{5}, (\bar{x}, \bar{y}) = \left( \frac{2/7}{2/5}, \frac{1/6}{2/5} \right) = \left( \frac{5}{7}, \frac{5}{12} \right).$$

9. Note that  $\sin(\pi x/L) \geq 0$  for  $0 \leq x \leq L$ .

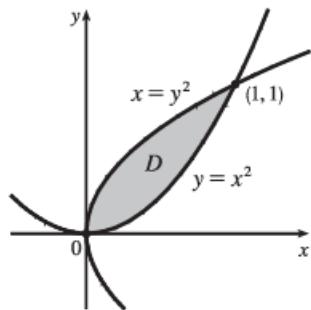
$$m = \int_0^L \int_0^{\sin(\pi x/L)} y \, dy \, dx = \int_0^L \frac{1}{2} \sin^2(\pi x/L) dx = \frac{1}{2} \left[ \frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right]_0^L = \frac{1}{4}L,$$

$$\begin{aligned} M_y &= \int_0^L \int_0^{\sin(\pi x/L)} x \cdot y \, dy \, dx = \frac{1}{2} \int_0^L x \sin^2(\pi x/L) dx \quad \left[ \begin{array}{l} \text{integrate by parts with} \\ u = x, dv = \sin^2(\pi x/L) dx \end{array} \right] \\ &= \frac{1}{2} \cdot x \left( \frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right) \Big|_0^L - \frac{1}{2} \int_0^L \left[ \frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right] dx \\ &= \frac{1}{4}L^2 - \frac{1}{2} \left[ \frac{1}{4}x^2 + \frac{L^2}{4\pi^2} \cos(2\pi x/L) \right]_0^L = \frac{1}{4}L^2 - \frac{1}{2} \left( \frac{1}{4}L^2 + \frac{L^2}{4\pi^2} - \frac{L^2}{4\pi^2} \right) = \frac{1}{8}L^2, \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^L \int_0^{\sin(\pi x/L)} y \cdot y \, dy \, dx = \int_0^L \frac{1}{3} \sin^3(\pi x/L) dx = \frac{1}{3} \int_0^L [1 - \cos^2(\pi x/L)] \sin(\pi x/L) dx \\ &\quad [\text{substitute } u = \cos(\pi x/L)] \Rightarrow du = -\frac{\pi}{L} \sin(\pi x/L) \\ &= \frac{1}{3} \left( -\frac{L}{\pi} \right) [\cos(\pi x/L) - \frac{1}{3} \cos^3(\pi x/L)] \Big|_0^L = -\frac{L}{3\pi} \left( -1 + \frac{1}{3} - 1 + \frac{1}{3} \right) = \frac{4}{9\pi}L. \end{aligned}$$

$$\text{Hence } m = \frac{L}{4}, (\bar{x}, \bar{y}) = \left( \frac{L^2/8}{L/4}, \frac{4L/(9\pi)}{L/4} \right) = \left( \frac{L}{2}, \frac{16}{9\pi} \right).$$

10.



$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x} dy dx = \int_0^1 \sqrt{x}(\sqrt{x} - x^2) dx \\ = \int_0^1 (x - x^{5/2}) dx = \left[ \frac{1}{2}x^2 - \frac{2}{7}x^{7/2} \right]_0^1 = \frac{3}{14},$$

$$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x \sqrt{x} dy dx = \int_0^1 x \sqrt{x}(\sqrt{x} - x^2) dx = \int_0^1 (x^2 - x^{7/2}) dx = \left[ \frac{1}{3}x^3 - \frac{2}{9}x^{9/2} \right]_0^1 = \frac{1}{9}$$

$$M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} y \sqrt{x} dy dx = \int_0^1 \sqrt{x} \cdot \frac{1}{2}(x - x^4) dx = \frac{1}{2} \int_0^1 (x^{3/2} - x^{9/2}) dx \\ = \frac{1}{2} \left[ \frac{2}{5}x^{5/2} - \frac{2}{11}x^{11/2} \right]_0^1 = \frac{1}{2} \cdot \frac{12}{55} = \frac{6}{55}.$$

$$\text{Hence } m = \frac{3}{14}, (\bar{x}, \bar{y}) = \left( \frac{1/9}{3/14}, \frac{6/55}{3/14} \right) = \left( \frac{14}{27}, \frac{28}{55} \right).$$

$$11. \rho(x, y) = ky = kr \sin \theta, m = \int_0^{\pi/2} \int_0^1 kr^2 \sin \theta dr d\theta = \frac{1}{3}k \int_0^{\pi/2} \sin \theta d\theta = \frac{1}{3}k[-\cos \theta]_0^{\pi/2} = \frac{1}{3}k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^3 \sin \theta \cos \theta dr d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \frac{1}{8}k[-\cos 2\theta]_0^{\pi/2} = \frac{1}{8}k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^3 \sin^2 \theta dr d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{8}k[\theta + \sin 2\theta]_0^{\pi/2} = \frac{\pi}{16}k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left( \frac{2}{8}, \frac{3\pi}{16} \right).$$

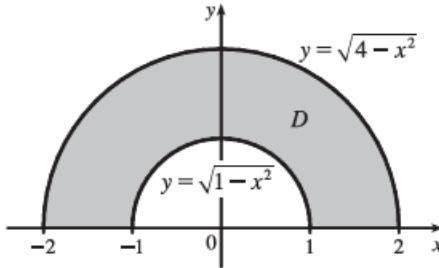
$$12. \rho(x, y) = k(x^2 + y^2) = kr^2, m = \int_0^{\pi/2} \int_0^1 kr^3 dr d\theta = \frac{\pi}{8}k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^4 \cos \theta dr d\theta = \frac{1}{5}k \int_0^{\pi/2} \cos \theta d\theta = \frac{1}{5}k[\sin \theta]_0^{\pi/2} = \frac{1}{5}k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^4 \sin \theta dr d\theta = \frac{1}{5}k \int_0^{\pi/2} \sin \theta d\theta = \frac{1}{5}k[-\cos \theta]_0^{\pi/2} = \frac{1}{5}k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left( \frac{8}{5\pi}, \frac{8}{5\pi} \right).$$

13.



$$\rho(x, y) = k \sqrt{x^2 + y^2} = kr,$$

$$m = \iint_D \rho(x, y) dA = \int_0^\pi \int_1^2 kr \cdot r dr d\theta \\ = k \int_0^\pi d\theta \int_1^2 r^2 dr = k(\pi) \left[ \frac{1}{3} r^3 \right]_1^2 = \frac{7}{3} \pi k,$$

$$M_y = \iint_D x \rho(x, y) dA = \int_0^\pi \int_1^2 (r \cos \theta)(kr) r dr d\theta = k \int_0^\pi \cos \theta d\theta \int_1^2 r^3 dr$$

$$= k [\sin \theta]_0^\pi \left[ \frac{1}{4} r^4 \right]_1^2 = k(0) \left( \frac{15}{4} \right) = 0 \quad [\text{this is to be expected as the region and density function are symmetric about the } y\text{-axis}]$$

$$M_x = \iint_D y \rho(x, y) dA = \int_0^\pi \int_1^2 (r \sin \theta)(kr) r dr d\theta = k \int_0^\pi \sin \theta d\theta \int_1^2 r^3 dr \\ = k [-\cos \theta]_0^\pi \left[ \frac{1}{4} r^4 \right]_1^2 = k(1+1) \left( \frac{15}{4} \right) = \frac{15}{2} k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left( 0, \frac{15k/2}{7\pi k/3} \right) = \left( 0, \frac{45}{14\pi} \right).$$

14. Now  $\rho(x, y) = k / \sqrt{x^2 + y^2} = k/r$ , so

$$m = \iint_D \rho(x, y) dA = \int_0^\pi \int_1^2 (k/r) r dr d\theta = k \int_0^\pi d\theta \int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x \rho(x, y) dA = \int_0^\pi \int_1^2 (r \cos \theta)(k/r) r dr d\theta = k \int_0^\pi \cos \theta d\theta \int_1^2 r dr \\ = k [\sin \theta]_0^\pi \left[ \frac{1}{2} r^2 \right]_1^2 = k(0) \left( \frac{3}{2} \right) = 0,$$

$$M_x = \iint_D y \rho(x, y) dA = \int_0^\pi \int_1^2 (r \sin \theta)(k/r) r dr d\theta = k \int_0^\pi \sin \theta d\theta \int_1^2 r dr \\ = k [-\cos \theta]_0^\pi \left[ \frac{1}{2} r^2 \right]_1^2 = k(1+1) \left( \frac{3}{2} \right) = 3k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left( 0, \frac{3k}{\pi k} \right) = \left( 0, \frac{3}{\pi} \right).$$

15. Placing the vertex opposite the hypotenuse at  $(0, 0)$ ,  $\rho(x, y) = k(x^2 + y^2)$ . Then

$$m = \int_0^a \int_0^{a-x} k(x^2 + y^2) dy dx = k \int_0^a [ax^2 - x^3 + \frac{1}{3}(a-x)^3] dx = k \left[ \frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4 \right]_0^a = \frac{1}{6}ka^4.$$

By symmetry,

$$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2 + y^2) dy dx = k \int_0^a \left[ \frac{1}{2}(a-x)^2 x^2 + \frac{1}{4}(a-x)^4 \right] dx \\ = k \left[ \frac{1}{6}a^2 x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5 \right]_0^a = \frac{1}{15}ka^5$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left( \frac{2}{5}a, \frac{2}{5}a \right).$$

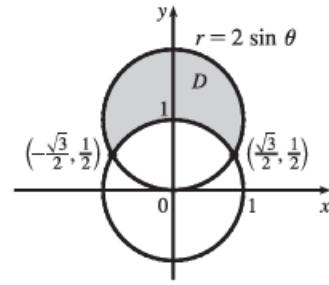
16.  $\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$ .

$$\begin{aligned} m &= \int_{\pi/6}^{5\pi/6} \int_1^{2 \sin \theta} \frac{k}{r} r dr d\theta = k \int_{\pi/6}^{5\pi/6} [(2 \sin \theta) - 1] d\theta \\ &= k \left[ -2 \cos \theta - \theta \right]_{\pi/6}^{5\pi/6} = 2k \left( \sqrt{3} - \frac{\pi}{3} \right) \end{aligned}$$

By symmetry of  $D$  and  $f(x) = x$ ,  $M_y = 0$ , and

$$\begin{aligned} M_x &= \int_{\pi/6}^{5\pi/6} \int_1^{2 \sin \theta} kr \sin \theta dr d\theta = \frac{1}{2} k \int_{\pi/6}^{5\pi/6} (4 \sin^3 \theta - \sin \theta) d\theta \\ &= \frac{1}{2} k \left[ -3 \cos \theta + \frac{4}{3} \cos^3 \theta \right]_{\pi/6}^{5\pi/6} = \sqrt{3} k \end{aligned}$$

Hence  $(\bar{x}, \bar{y}) = \left( 0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)} \right)$ .



17.  $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} y^2 \cdot y dy dx = \int_0^1 \left[ \frac{1}{4} y^4 \right]_{y=0}^{y=e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx = \frac{1}{4} \left[ \frac{1}{4} e^{4x} \right]_0^1 = \frac{1}{16}(e^4 - 1)$ ,

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} x^2 y dy dx = \int_0^1 x^2 \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx \\ &= \frac{1}{2} \left[ \left( \frac{1}{2} x^2 - \frac{1}{2} x + \frac{1}{4} \right) e^{2x} \right]_0^1 \quad [\text{integrate by parts twice}] = \frac{1}{8}(e^2 - 1), \end{aligned}$$

and  $I_0 = I_x + I_y = \frac{1}{16}(e^4 - 1) + \frac{1}{8}(e^2 - 1) = \frac{1}{16}(e^4 + 2e^2 - 3)$ .

18.  $I_x = \int_0^{\pi/2} \int_0^1 (r^2 \sin^2 \theta)(kr^2) r dr d\theta = \frac{1}{6} k \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{6} k \left[ \frac{1}{4} (2\theta - \sin 2\theta) \right]_0^{\pi/2} = \frac{\pi}{24} k$ ,

$$I_y = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2 \theta)(kr^2) r dr d\theta = \frac{1}{6} k \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{1}{6} k \left[ \frac{1}{4} (2\theta + \sin 2\theta) \right]_0^{\pi/2} = \frac{\pi}{24} k,$$

and  $I_0 = I_x + I_y = \frac{\pi}{12} k$ .

19. As in Exercise 15, we place the vertex opposite the hypotenuse at  $(0, 0)$  and the equal sides along the positive axes.

$$\begin{aligned} I_x &= \int_0^a \int_0^{a-x} y^2 k(x^2 + y^2) dy dx = k \int_0^a \int_0^{a-x} (x^2 y^2 + y^4) dy dx = k \int_0^a \left[ \frac{1}{3} x^2 y^3 + \frac{1}{5} y^5 \right]_{y=0}^{y=a-x} dx \\ &= k \int_0^a \left[ \frac{1}{3} x^2 (a-x)^3 + \frac{1}{5} (a-x)^5 \right] dx = k \left[ \frac{1}{3} \left( \frac{1}{3} a^3 x^3 - \frac{3}{4} a^2 x^4 + \frac{3}{5} a x^5 - \frac{1}{6} x^6 \right) - \frac{1}{30} (a-x)^6 \right]_0^a = \frac{7}{180} k a^6, \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^a \int_0^{a-x} x^2 k(x^2 + y^2) dy dx = k \int_0^a \int_0^{a-x} (x^4 + x^2 y^2) dy dx = k \int_0^a \left[ x^4 y + \frac{1}{3} x^2 y^3 \right]_{y=0}^{y=a-x} dx \\ &= k \int_0^a \left[ x^4 (a-x) + \frac{1}{3} x^2 (a-x)^3 \right] dx = k \left[ \frac{1}{5} a x^5 - \frac{1}{6} x^6 + \frac{1}{3} \left( \frac{1}{3} a^3 x^3 - \frac{3}{4} a^2 x^4 + \frac{3}{5} a x^5 - \frac{1}{6} x^6 \right) \right]_0^a = \frac{7}{180} k a^6, \end{aligned}$$

and  $I_0 = I_x + I_y = \frac{7}{90} k a^6$ .

20. If we find the moments of inertia about the  $x$ - and  $y$ -axes, we can determine in which direction rotation will be more difficult.

(See the explanation following Example 4.) The moment of inertia about the  $x$ -axis is given by

$$\begin{aligned} I_x &= \iint_D y^2 \rho(x, y) dA = \int_0^2 \int_0^2 y^2(1 + 0.1x) dy dx = \int_0^2 (1 + 0.1x) \left[ \frac{1}{3}y^3 \right]_{y=0}^{y=2} dx \\ &= \frac{8}{3} \int_0^2 (1 + 0.1x) dx = \frac{8}{3} [x + 0.1 \cdot \frac{1}{2}x^2]_0^2 = \frac{8}{3}(2.2) \approx 5.87 \end{aligned}$$

Similarly, the moment of inertia about the  $y$ -axis is given by

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) dA = \int_0^2 \int_0^2 x^2(1 + 0.1x) dy dx = \int_0^2 x^2(1 + 0.1x) [y]_{y=0}^{y=2} dx \\ &= 2 \int_0^2 (x^2 + 0.1x^3) dx = 2 \left[ \frac{1}{3}x^3 + 0.1 \cdot \frac{1}{4}x^4 \right]_0^2 = 2\left(\frac{8}{3} + 0.4\right) \approx 6.13 \end{aligned}$$

Since  $I_y > I_x$ , more force is required to rotate the fan blade about the  $y$ -axis.

21. Using a CAS, we find  $m = \iint_D \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy dy dx = \frac{\pi^2}{8}$ . Then

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} x^2 y dy dx = \frac{2\pi}{3} - \frac{1}{\pi} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} xy^2 dy dx = \frac{16}{9\pi}, \text{ so } (\bar{x}, \bar{y}) = \left( \frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi} \right).$$

The moments of inertia are  $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy^3 dy dx = \frac{3\pi^2}{64}$ ,

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} x^3 y dy dx = \frac{\pi^2}{16}(\pi^2 - 3), \text{ and } I_0 = I_x + I_y = \frac{\pi^2}{64}(4\pi^2 - 9).$$

22. Using a CAS, we find  $m = \iint_D \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 dr d\theta = \frac{5}{3}\pi$ ,

$$\bar{x} = \frac{1}{m} \iint_D x \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta dr d\theta = \frac{21}{20} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta dr d\theta = 0, \text{ so } (\bar{x}, \bar{y}) = \left( \frac{21}{20}, 0 \right).$$

The moments of inertia are  $I_x = \iint_D y^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \sin^2\theta dr d\theta = \frac{33}{40}\pi$ ,

$$I_y = \iint_D x^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \cos^2\theta dr d\theta = \frac{93}{40}\pi, \text{ and } I_0 = I_x + I_y = \frac{63}{20}\pi.$$

23.  $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^h \int_0^b \rho y^2 dx dy = \rho \int_0^b dx \int_0^h y^2 dy = \rho [x]_0^b \left[ \frac{1}{3}y^3 \right]_0^h = \rho b \left( \frac{1}{3}h^3 \right) = \frac{1}{3}\rho b h^3$ ,

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^h \int_0^b \rho x^2 dx dy = \rho \int_0^b x^2 dx \int_0^h dy = \rho \left[ \frac{1}{3}x^3 \right]_0^b [y]_0^h = \frac{1}{3}\rho b^3 h,$$

and  $m = \rho(\text{area of rectangle}) = \rho b h$  since the lamina is homogeneous. Hence  $\bar{x}^2 = \frac{I_y}{m} = \frac{1}{3}\frac{\rho b^3 h}{\rho b h} = \frac{b^2}{3} \Rightarrow \bar{x} = \frac{b}{\sqrt{3}}$

$$\text{and } \bar{y}^2 = \frac{I_x}{m} = \frac{\frac{1}{3}\rho b h^3}{\rho b h} = \frac{h^2}{3} \Rightarrow \bar{y} = \frac{h}{\sqrt{3}}.$$

24. Here we assume  $b > 0, h > 0$  but note that we arrive at the same results if  $b < 0$  or  $h < 0$ . We have

$$D = \{(x, y) \mid 0 \leq x \leq b, 0 \leq y \leq h - \frac{h}{b}x\}, \text{ so}$$

$$\begin{aligned} I_x &= \int_0^b \int_0^{h-hx/b} y^2 \rho dy dx = \rho \int_0^b \left[ \frac{1}{3}y^3 \right]_{y=0}^{y=h-hx/b} dx = \frac{1}{3}\rho \int_0^b (h - \frac{h}{b}x)^3 dx \\ &= \frac{1}{3}\rho \left[ -\frac{b}{h} \left( \frac{1}{4} \right) (h - \frac{h}{b}x)^4 \right]_0^b = -\frac{b}{12h}\rho(0 - h^4) = \frac{1}{12}\rho b h^3, \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^b \int_0^{h-hx/b} x^2 \rho dy dx = \rho \int_0^b x^2 (h - \frac{h}{b}x) dx = \rho \int_0^b (hx^2 - \frac{h}{b}x^3) dx \\ &= \rho \left[ \frac{h}{3}x^3 - \frac{h}{4b}x^4 \right]_0^b = \rho \left( \frac{hb^3}{3} - \frac{hb^3}{4} \right) = \frac{1}{12}\rho b^3 h, \end{aligned}$$

$$\text{and } m = \int_0^b \int_0^{h-hx/b} \rho dy dx = \rho \int_0^b (h - \frac{h}{b}x) dx = \rho \left[ hx - \frac{h}{2b}x^2 \right]_0^b = \frac{1}{2}\rho b h. \text{ Hence } \bar{x}^2 = \frac{I_y}{m} = \frac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho b h} = \frac{b^2}{6} \Rightarrow$$

$$\bar{x} = \frac{b}{\sqrt{6}} \text{ and } \bar{y}^2 = \frac{I_x}{m} = \frac{\frac{1}{12}\rho b h^3}{\frac{1}{2}\rho b h} = \frac{h^2}{6} \Rightarrow \bar{y} = \frac{h}{\sqrt{6}}.$$

25. In polar coordinates, the region is  $D = \{(r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{2}\}$ , so

$$\begin{aligned} I_x &= \iint_D y^2 \rho dA = \int_0^{\pi/2} \int_0^a \rho(r \sin \theta)^2 r dr d\theta = \rho \int_0^{\pi/2} \sin^2 \theta \int_0^a r^3 dr \\ &= \rho \left[ \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} \left[ \frac{1}{4}r^4 \right]_0^a = \rho \left( \frac{\pi}{4} \right) \left( \frac{1}{4}a^4 \right) = \frac{1}{16}\rho a^4 \pi, \end{aligned}$$

$$\begin{aligned} I_y &= \iint_D x^2 \rho dA = \int_0^{\pi/2} \int_0^a \rho(r \cos \theta)^2 r dr d\theta = \rho \int_0^{\pi/2} \cos^2 \theta \int_0^a r^3 dr \\ &= \rho \left[ \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} \left[ \frac{1}{4}r^4 \right]_0^a = \rho \left( \frac{\pi}{4} \right) \left( \frac{1}{4}a^4 \right) = \frac{1}{16}\rho a^4 \pi, \end{aligned}$$

$$\text{and } m = \rho \cdot A(D) = \rho \cdot \frac{1}{4}\pi a^2 \text{ since the lamina is homogeneous. Hence } \bar{x}^2 = \bar{y}^2 = \frac{\frac{1}{16}\rho a^4 \pi}{\frac{1}{4}\rho a^2 \pi} = \frac{a^2}{4} \Rightarrow \bar{x} = \bar{y} = \frac{a}{2}.$$

$$26. m = \int_0^\pi \int_0^{\sin x} \rho dy dx = \rho \int_0^\pi \sin x dx = \rho [-\cos x]_0^\pi = 2\rho,$$

$$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 dy dx = \frac{1}{3}\rho \int_0^\pi \sin^3 x dx = \frac{1}{3}\rho \int_0^\pi (1 - \cos^2 x) \sin x dx = \frac{1}{3}\rho [-\cos \theta + \frac{1}{3}\cos^3 \theta]_0^\pi = \frac{4}{9}\rho,$$

$$\begin{aligned} I_y &= \int_0^\pi \int_0^{\sin x} \rho x^2 dy dx = \rho \int_0^\pi x^2 \sin x dx = \rho [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^\pi \quad [\text{by integrating by parts twice}] \\ &= \rho(\pi^2 - 4). \end{aligned}$$

$$\text{Then } \bar{y}^2 = \frac{I_x}{m} = \frac{2}{9}, \text{ so } \bar{y} = \frac{\sqrt{2}}{3} \text{ and } \bar{x}^2 = \frac{I_y}{m} = \frac{\pi^2 - 4}{2}, \text{ so } \bar{x} = \sqrt{\frac{\pi^2 - 4}{2}}.$$

27. (a)  $f(x, y)$  is a joint density function, so we know  $\iint_{\mathbb{R}^2} f(x, y) dA = 1$ . Since  $f(x, y) = 0$  outside the rectangle  $[0, 1] \times [0, 2]$ , we can say

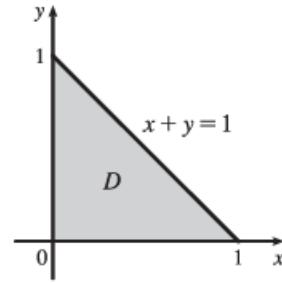
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_0^1 \int_0^2 Cx(1+y) dy dx \\ &= C \int_0^1 x[y + \frac{1}{2}y^2]_{y=0}^{y=2} dx = C \int_0^1 4x dx = C[2x^2]_0^1 = 2C\end{aligned}$$

Then  $2C = 1 \Rightarrow C = \frac{1}{2}$ .

$$\begin{aligned}(b) P(X \leq 1, Y \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 f(x, y) dy dx = \int_0^1 \int_0^1 \frac{1}{2}x(1+y) dy dx \\ &= \int_0^1 \frac{1}{2}x[y + \frac{1}{2}y^2]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{2}x(\frac{3}{2}) dx = \frac{3}{4}[\frac{1}{2}x^2]_0^1 = \frac{3}{8} \text{ or } 0.375\end{aligned}$$

(c)  $P(X + Y \leq 1) = P((X, Y) \in D)$  where  $D$  is the triangular region shown in the figure. Thus

$$\begin{aligned}P(X + Y \leq 1) &= \iint_D f(x, y) dA = \int_0^1 \int_0^{1-x} \frac{1}{2}x(1+y) dy dx \\ &= \int_0^1 \frac{1}{2}x[y + \frac{1}{2}y^2]_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{2}x(\frac{1}{2}x^2 - 2x + \frac{3}{2}) dx \\ &= \frac{1}{4} \int_0^1 (x^3 - 4x^2 + 3x) dx = \frac{1}{4} \left[ \frac{x^4}{4} - 4 \frac{x^3}{3} + 3 \frac{x^2}{2} \right]_0^1 \\ &= \frac{5}{48} \approx 0.1042\end{aligned}$$



28. (a)  $f(x, y) \geq 0$ , so  $f$  is a joint density function if  $\iint_{\mathbb{R}^2} f(x, y) dA = 1$ . Here,  $f(x, y) = 0$  outside the square  $[0, 1] \times [0, 1]$ , so  $\iint_{\mathbb{R}^2} f(x, y) dA = \int_0^1 \int_0^1 4xy dy dx = \int_0^1 [2xy^2]_{y=0}^{y=1} dx = \int_0^1 2x dx = x^2]_0^1 = 1$ .

Thus,  $f(x, y)$  is a joint density function.

(b) (i) No restriction is placed on  $Y$ , so

$$P(X \geq \frac{1}{2}) = \int_{1/2}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_{1/2}^1 \int_0^1 4xy dy dx = \int_{1/2}^1 [2xy^2]_{y=0}^{y=1} dx = \int_{1/2}^1 2x dx = x^2]_{1/2}^1 = \frac{3}{4}.$$

$$\begin{aligned}(ii) P(X \geq \frac{1}{2}, Y \leq \frac{1}{2}) &= \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f(x, y) dy dx = \int_{1/2}^1 \int_0^{1/2} 4xy dy dx \\ &= \int_{1/2}^1 [2xy^2]_{y=0}^{y=1/2} dx = \int_{1/2}^1 \frac{1}{2}x dx = \frac{1}{2} \cdot \frac{1}{2}x^2]_{1/2}^1 = \frac{3}{16}\end{aligned}$$

(c) The expected value of  $X$  is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^1 \int_0^1 x(4xy) dy dx = \int_0^1 2x^2[y^2]_{y=0}^{y=1} dx = 2 \int_0^1 x^2 dx = 2[\frac{1}{3}x^3]_0^1 = \frac{2}{3}$$

The expected value of  $Y$  is

$$\mu_2 = \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^1 \int_0^1 y(4xy) dy dx = \int_0^1 4x[\frac{1}{3}y^3]_{y=0}^{y=1} dx = \frac{4}{3} \int_0^1 x dx = \frac{4}{3}[\frac{1}{2}x^2]_0^1 = \frac{2}{3}$$

29. (a)  $f(x, y) \geq 0$ , so  $f$  is a joint density function if  $\iint_{\mathbb{R}^2} f(x, y) dA = 1$ . Here,  $f(x, y) = 0$  outside the first quadrant, so

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_0^\infty \int_0^\infty 0.1 e^{-(0.5x+0.2y)} dy dx = 0.1 \int_0^\infty \int_0^\infty e^{-0.5x} e^{-0.2y} dy dx = 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - 1)] = (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - 1) = 1\end{aligned}$$

Thus  $f(x, y)$  is a joint density function.

(b) (i) No restriction is placed on  $X$ , so

$$\begin{aligned}P(Y \geq 1) &= \int_{-\infty}^\infty \int_1^\infty f(x, y) dy dx = \int_0^\infty \int_1^\infty 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_1^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_1^t e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_1^t = 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - e^{-0.2})] \\ &\quad (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - e^{-0.2}) = e^{-0.2} \approx 0.8187\end{aligned}$$

$$\begin{aligned}(ii) P(X \leq 2, Y \leq 4) &= \int_{-\infty}^2 \int_{-\infty}^4 f(x, y) dy dx = \int_0^2 \int_0^4 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^2 e^{-0.5x} dx \int_0^4 e^{-0.2y} dy = 0.1 [-2e^{-0.5x}]_0^2 [-5e^{-0.2y}]_0^4 \\ &= (0.1) \cdot (-2)(e^{-1} - 1) \cdot (-5)(e^{-0.8} - 1) \\ &= (e^{-1} - 1)(e^{-0.8} - 1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481\end{aligned}$$

(c) The expected value of  $X$  is given by

$$\begin{aligned}\mu_1 &= \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^\infty \int_0^\infty x [0.1 e^{-(0.5x+0.2y)}] dy dx \\ &= 0.1 \int_0^\infty x e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t x e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy\end{aligned}$$

To evaluate the first integral, we integrate by parts with  $u = x$  and  $dv = e^{-0.5x} dx$  (or we can use Formula 96

in the Table of Integrals):  $\int x e^{-0.5x} dx = -2xe^{-0.5x} - \int -2e^{-0.5x} dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$ .

Thus

$$\begin{aligned}\mu_1 &= 0.1 \lim_{t \rightarrow \infty} [-2(x+2)e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} (-2)[(t+2)e^{-0.5t} - 2] \lim_{t \rightarrow \infty} (-5)[e^{-0.2t} - 1] \\ &= 0.1(-2) \left( \lim_{t \rightarrow \infty} \frac{t+2}{e^{0.5t}} - 2 \right) (-5)(-1) = 2 \quad [\text{by l'Hospital's Rule}]\end{aligned}$$

The expected value of  $Y$  is given by

$$\begin{aligned}\mu_2 &= \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^\infty \int_0^\infty y [0.1 e^{-(0.5x+0.2y)}] dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty y e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t y e^{-0.2y} dy\end{aligned}$$

To evaluate the second integral, we integrate by parts with  $u = y$  and  $dv = e^{-0.2y} dy$  (or again we can use Formula 96 in

the Table of Integrals) which gives  $\int y e^{-0.2y} dy = -5ye^{-0.2y} + \int 5e^{-0.2y} dy = -5(y+5)e^{-0.2y}$ . Then

$$\begin{aligned}\mu_2 &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5(y+5)e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} (-5[(t+5)e^{-0.2t} - 5]) \\ &= 0.1(-2)(-1) \cdot (-5) \left( \lim_{t \rightarrow \infty} \frac{t+5}{e^{0.2t}} - 5 \right) = 5 \quad [\text{by l'Hospital's Rule}]\end{aligned}$$

30. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

If  $X$  and  $Y$  are the lifetimes of the individual bulbs, then  $X$  and  $Y$  are independent, so the joint density function is the product of the individual density functions:

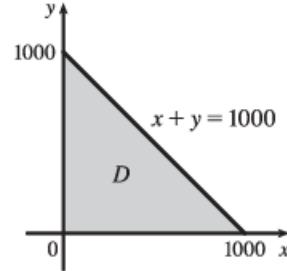
$$f(x, y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned} P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x, y) dy dx = \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000} dx \int_0^{1000} e^{-y/1000} dy \\ &= 10^{-6} \left[ -1000e^{-x/1000} \right]_0^{1000} \left[ -1000e^{-y/1000} \right]_0^{1000} \\ &= (e^{-1} - 1)^2 \approx 0.3996 \end{aligned}$$

- (b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find  $P(X + Y \leq 1000)$ , or equivalently  $P((X, Y) \in D)$  where  $D$  is the triangular region shown in the figure. Then

$$\begin{aligned} P(X + Y \leq 1000) &= \iint_D f(x, y) dA \\ &= \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} \left[ -1000e^{-(x+y)/1000} \right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} (e^{-1} - e^{-x/1000}) dx \\ &= -10^{-3} \left[ e^{-1}x + 1000e^{-x/1000} \right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$



31. (a) The random variables  $X$  and  $Y$  are normally distributed with  $\mu_1 = 45$ ,  $\mu_2 = 20$ ,  $\sigma_1 = 0.5$ , and  $\sigma_2 = 0.1$ .

The individual density functions for  $X$  and  $Y$ , then, are  $f_1(x) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5}$  and

$f_2(y) = \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02}$ . Since  $X$  and  $Y$  are independent, the joint density function is the product

$$f(x, y) = f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5} \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02} = \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}.$$

$$\text{Then } P(40 \leq X \leq 50, 20 \leq Y \leq 25) = \int_{40}^{50} \int_{20}^{25} f(x, y) dy dx = \frac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2} dy dx.$$

Using a CAS or calculator to evaluate the integral, we get  $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$ .

- (b)  $P(4(X - 45)^2 + 100(Y - 20)^2 \leq 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} dA$ , where  $D$  is the region enclosed by the ellipse  $4(x - 45)^2 + 100(y - 20)^2 = 2$ . Solving for  $y$  gives  $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x - 45)^2}$ , the upper and lower halves of the ellipse, and these two halves meet where  $y = 20$  [since the ellipse is centered at  $(45, 20)$ ]  $\Rightarrow 4(x - 45)^2 = 2 \Rightarrow x = 45 \pm \frac{1}{\sqrt{2}}$ . Thus

$$\iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} dA = \frac{10}{\pi} \int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2 - 50(y-20)^2} dy dx.$$

Using a CAS or calculator to evaluate the integral, we get  $P(4(X - 45)^2 + 100(Y - 20)^2 \leq 2) \approx 0.632$ .

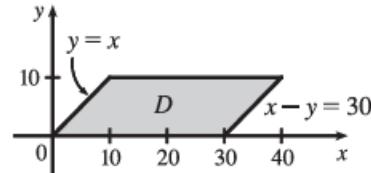
32. Because  $X$  and  $Y$  are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless  $X \geq Y$ .

Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless  $X - Y \leq 30$ . Thus the probability that they meet is  $P((X, Y) \in D)$  where  $D$  is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider  $D$  as a type II region, so

$$\begin{aligned} P((X, Y) \in D) &= \iint_D f(x, y) dx dy = \int_0^{10} \int_y^{y+30} \frac{1}{50}e^{-x}y dx dy = \frac{1}{50} \int_0^{10} y [-e^{-x}]_{x=y}^{x=y+30} dy \\ &= \frac{1}{50} \int_0^{10} y (-e^{-(y+30)} + e^{-y}) dy = \frac{1}{50} (1 - e^{-30}) \int_0^{10} y e^{-y} dy \end{aligned}$$



By integration by parts (or Formula 96 in the Table of Integrals), this is

$$\frac{1}{50}(1 - e^{-30})[-(y+1)e^{-y}]_0^{10} = \frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020. \text{ Thus there is only about a 2\% chance they will meet.}$$

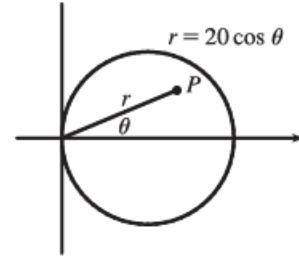
Such is student life!

33. (a) If  $f(P, A)$  is the probability that an individual at  $A$  will be infected by an individual at  $P$ , and  $k dA$  is the number of infected individuals in an element of area  $dA$ , then  $f(P, A)k dA$  is the number of infections that should result from exposure of the individual at  $A$  to infected people in the element of area  $dA$ . Integration over  $D$  gives the number of infections of the person at  $A$  due to all the infected people in  $D$ . In rectangular coordinates (with the origin at the city's center), the exposure of a person at  $A$  is

$$E = \iint_D k f(P, A) dA = k \iint_D \frac{20 - d(P, A)}{20} dA = k \iint_D \left[ 1 - \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{20} \right] dx dy$$

(b) If  $A = (0, 0)$ , then

$$\begin{aligned} E &= k \iint_D \left[ 1 - \frac{1}{20} \sqrt{x^2 + y^2} \right] dx dy \\ &= k \int_0^{2\pi} \int_0^{10} \left( 1 - \frac{r}{20} \right) r dr d\theta = 2\pi k \left[ \frac{r^2}{2} - \frac{r^3}{60} \right]_0^{10} \\ &= 2\pi k \left( 50 - \frac{50}{3} \right) = \frac{200}{3}\pi k \approx 209k \end{aligned}$$



For  $A$  at the edge of the city, it is convenient to use a polar coordinate system centered at  $A$ . Then the polar equation for the circular boundary of the city becomes  $r = 20 \cos \theta$  instead of  $r = 10$ , and the distance from  $A$  to a point  $P$  in the city is again  $r$  (see the figure). So

$$\begin{aligned} E &= k \int_{-\pi/2}^{\pi/2} \int_0^{20 \cos \theta} \left( 1 - \frac{r}{20} \right) r dr d\theta = k \int_{-\pi/2}^{\pi/2} \left[ \frac{r^2}{2} - \frac{r^3}{60} \right]_{r=0}^{r=20 \cos \theta} d\theta \\ &= k \int_{-\pi/2}^{\pi/2} (200 \cos^2 \theta - \frac{400}{3} \cos^3 \theta) d\theta = 200k \int_{-\pi/2}^{\pi/2} [\frac{1}{2} + \frac{1}{2} \cos 2\theta - \frac{2}{3} (1 - \sin^2 \theta) \cos \theta] d\theta \\ &= 200k [\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta - \frac{2}{3} \sin \theta + \frac{2}{3} \cdot \frac{1}{3} \sin^3 \theta]_{-\pi/2}^{\pi/2} = 200k [\frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} + \frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9}] \\ &= 200k (\frac{\pi}{2} - \frac{8}{9}) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.