51.
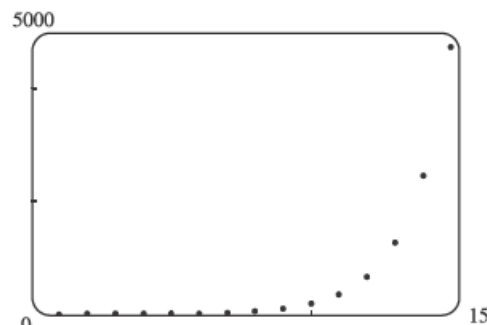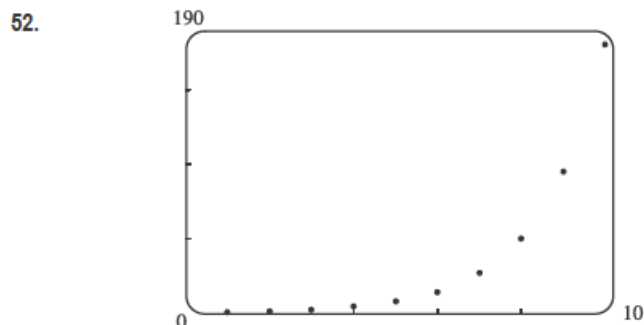

From the graph, it appears that the sequence $\{a_n\} = \left\{\dfrac{n^2 \cos n}{1+n^2}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately). To prove this, suppose that $\{a_n\}$ converges to L. If $b_n = \dfrac{n^2}{1+n^2}$, then $\{b_n\}$ converges to 1, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{1} = L$. But $\dfrac{a_n}{b_n} = \cos n$, so $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist. This contradiction shows that $\{a_n\}$ diverges.

52.


From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \Leftrightarrow 4n+2 \geq 3n+3 \Leftrightarrow n \geq 1$], and so we get that $a_{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{3}{2} = \left(\dfrac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\dfrac{3}{2}\right)^{n-1}\right\}$ diverges [by (9)], so does the given sequence $\{a_n\}$.

53.
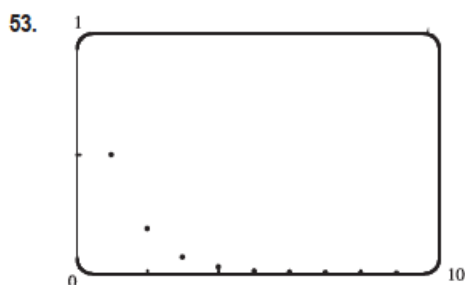
From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n}$$

$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdots (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}\right\}$ converges to 0.

54. (a) $a_1 = 1$, $a_{n+1} = 4 - a_n$ for $n \geq 1$. $a_1 = 1$, $a_2 = 4 - a_1 = 4 - 1 = 3$, $a_3 = 4 - a_2 = 4 - 3 = 1$, $a_4 = 4 - a_3 = 4 - 1 = 3$, $a_5 = 4 - a_4 = 4 - 3 = 1$. Since the terms of the sequence alternate between 1 and 3, the sequence is divergent.

(b) $a_1 = 2$, $a_2 = 4 - a_1 = 4 - 2 = 2$, $a_3 = 4 - a_2 = 4 - 2 = 2$. Since all of the terms are 2, $\displaystyle\lim_{n\to\infty} a_n = 2$ and hence, the sequence is convergent.

55. (a) $a_n = 1000(1.06)^n$ $\Rightarrow$ $a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n \to \infty} a_n = 1000 \lim_{n \to \infty} (1.06)^n$, so the sequence diverges by (9) with $r = 1.06 > 1$.

56. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5,

16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38,

19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4.

The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

57. If $|r| \geq 1$, then $\{r^n\}$ diverges by (9), so $\{nr^n\}$ diverges also, since $|nr^n| = n|r^n| \geq |r^n|$. If $|r| < 1$ then

$$\lim_{x \to \infty} xr^x = \lim_{x \to \infty} \frac{x}{r^{-x}} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{1}{(-\ln r) r^{-x}} = \lim_{x \to \infty} \frac{r^x}{-\ln r} = 0, \text{ so } \lim_{n \to \infty} nr^n = 0, \text{ and hence } \{nr^n\} \text{ converges}$$

whenever $|r| < 1$.

58. (a) Let $\lim_{n \to \infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$

whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N$ $\Leftrightarrow$ $n > N - 1$. It follows that $\lim_{n \to \infty} a_{n+1} = L$ and so

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}$.

(b) If $L = \lim_{n \to \infty} a_n$ then $\lim_{n \to \infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1 + L)$ $\Rightarrow$ $L^2 + L - 1 = 0$ $\Rightarrow$ $L = \frac{-1 + \sqrt{5}}{2}$

(since L has to be nonnegative if it exists).

59. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a

bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than

8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

60. The terms of $a_n = (-2)^{n+1}$ alternate in sign, so the sequence is not monotonic. The first five terms are 4, -8, 16, -32,

and 64. Since $\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} 2^{n+1} = \infty$, the sequence is not bounded.

61. $a_n = \dfrac{1}{2n + 3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1) + 3} = \dfrac{1}{2n + 5} < \dfrac{1}{2n + 3} = a_n$ for each $n \geq 1$. The sequence is

bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

62. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$,

and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

63. The terms of $a_n = n(-1)^n$ alternate in sign, so the sequence is not monotonic. The first five terms are $-1, 2, -3, 4$, and -5.

Since $\lim\limits_{n \to \infty} |a_n| = \lim\limits_{n \to \infty} n = \infty$, the sequence is not bounded.

64. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$.

$[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$ for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

65. $a_n = \dfrac{n}{n^2 + 1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2 + 1}$, $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$

for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

66. $a_n = n + \dfrac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \dfrac{1}{x}$ is increasing for $x > 1$. $[g'(x) = 1 - 1/x^2 > 0$

for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

67. For $\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \right\}$, $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, $\ldots$, so $a_n = 2^{(2^n - 1)/2^n} = 2^{1 - (1/2^n)}$.

$\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate solution: Let $L = \lim\limits_{n \to \infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.)

Then L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

68. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and

then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \Leftrightarrow$

$2 + a_{n+1} \geq 2 + a_n \Leftrightarrow a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \Leftrightarrow \sqrt{2 + a_n} \leq 3 \Leftrightarrow$

$2 + a_n \leq 9 \Leftrightarrow a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so

$a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim\limits_{n \to \infty} a_n$ exists.

(b) If $L = \lim\limits_{n \to \infty} a_n$, then $\lim\limits_{n \to \infty} a_{n+1} = L$ also, so $L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L \Leftrightarrow L^2 - L - 2 = 0 \Leftrightarrow$

$(L+1)(L-2) = 0 \Leftrightarrow L = 2$ [since L can't be negative].

69. $a_1 = 1$, $a_{n+1} = 3 - \dfrac{1}{a_n}$. We show by induction that $\{a_n\}$ is increasing and bounded above by 3. Let P_n be the proposition

that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \;\Rightarrow\; \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \;\Rightarrow$

$-\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$. Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \;\Leftrightarrow\; P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded

above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.

If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \;\Rightarrow\; L^2 - 3L + 1 = 0 \;\Rightarrow\; L = \dfrac{3 \pm \sqrt{5}}{2}$.

But $L > 1$, so $L = \dfrac{3 + \sqrt{5}}{2}$.

70. $a_1 = 2$, $a_{n+1} = \dfrac{1}{3 - a_n}$. We use induction. Let P_n be the statement that $0 < a_{n+1} \le a_n \le 2$. Clearly P_1 is true, since

$a_2 = 1/(3 - 2) = 1$. Now assume that P_n is true. Then $a_{n+1} \le a_n \;\Rightarrow\; -a_{n+1} \ge -a_n \;\Rightarrow\; 3 - a_{n+1} \ge 3 - a_n \;\Rightarrow$

$a_{n+2} = \dfrac{1}{3 - a_{n+1}} \le \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ [since $3 - a_{n+1}$ is positive] and $a_{n+1} \le 2$ by the induction

hypothesis, so P_{n+1} is true. To find the limit, we use the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \;\Rightarrow\; L = \dfrac{1}{3 - L} \;\Rightarrow$

$L^2 - 3L + 1 = 0 \;\Rightarrow\; L = \dfrac{3 \pm \sqrt{5}}{2}$. But $L \le 2$, so we must have $L = \dfrac{3 - \sqrt{5}}{2}$.

71. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is

2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus,

$a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \;\Rightarrow\; a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$,

then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \;\Rightarrow\; L^2 - L - 1 = 0 \;\Rightarrow\; L = \dfrac{1 + \sqrt{5}}{2}$
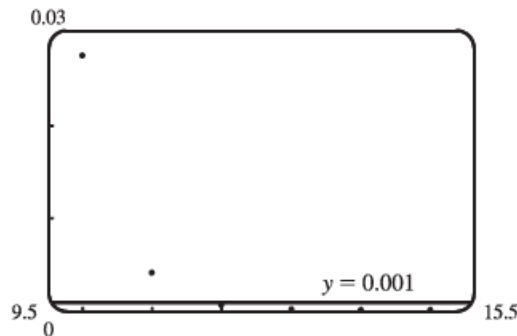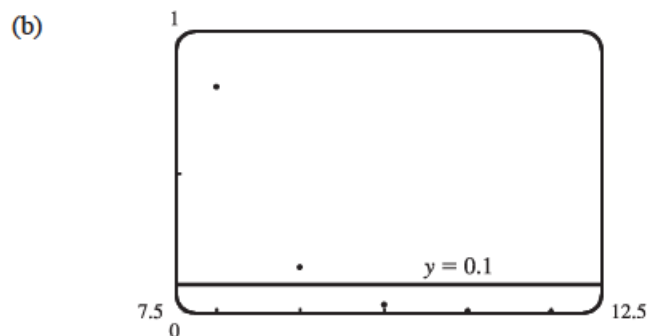
[since L must be positive].

72. (a) If f is continuous, then $f(L) = f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$ by Exercise 58(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed

value stabilizes, we see that $L \approx 0.73909$.

73. (a)
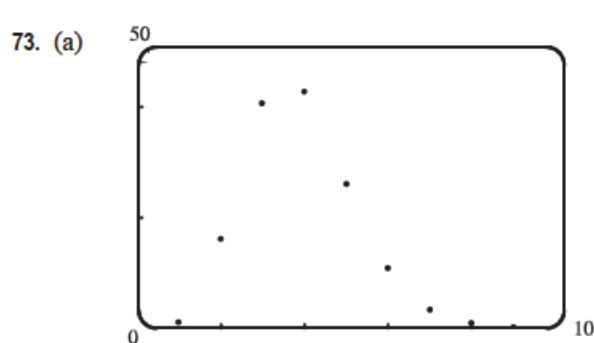
From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$

converges to 0, that is, $\lim\limits_{n\to\infty}\dfrac{n^5}{n!}=0.$

(b)

From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon=0.1$ is 9, since $n^5/n!<0.1$ whenever $n\geq 10$, but $9^5/9!>0.1$. From the second graph, it seems that for $\varepsilon=0.001$, the smallest possible value for N is 11 since $n^5/n!<0.001$ whenever $n\geq 12$.

74. Let $\varepsilon>0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n>N$, then $n>\ln(\varepsilon)/\ln|r|\ \Rightarrow\ n\ln|r|<\ln\varepsilon$ [since $|r|<1\Rightarrow\ln|r|<0$] $\Rightarrow\ \ln(|r|^n)<\ln\varepsilon\ \Rightarrow\ |r|^n<\varepsilon\ \Rightarrow\ |r^n-0|<\varepsilon$, and so by Definition 2, $\lim\limits_{n\to\infty} r^n=0.$

75. Theorem 6: If $\lim\limits_{n\to\infty}|a_n|=0$ then $\lim\limits_{n\to\infty}-|a_n|=0$, and since $-|a_n|\leq a_n\leq|a_n|$, we have that $\lim\limits_{n\to\infty}a_n=0$ by the Squeeze Theorem.

76. Theorem 7: If $\lim\limits_{n\to\infty}a_n=L$ and the function f is continuous at L, then $\lim\limits_{n\to\infty}f(a_n)=f(L)$.

Proof: We must show that, given a number $\varepsilon>0$, there is an integer N such that $|f(a_n)-f(L)|<\varepsilon$ whenever $n>N$. Suppose $\varepsilon>0$. Since f is continuous at L, there is a number $\delta>0$ such that $|f(x)-f(L)|<\varepsilon$ if $|x-L|<\delta$. Since $\lim\limits_{n\to\infty}a_n=L$, there is an integer N such that $|a_n-L|<\delta$ if $n>N$. Suppose $n>N$. Then $0<|a_n-L|<\delta$, so $|f(a_n)-f(L)|<\varepsilon$.

77. To Prove: If $\lim\limits_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim\limits_{n\to\infty} (a_n b_n) = 0$.

Proof: Since $\{b_n\}$ is bounded, there is a positive number M such that $|b_n| \le M$ and hence, $|a_n| |b_n| \le |a_n| M$ for

all $n \ge 1$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = 0$, there is an integer N such that $|a_n - 0| < \dfrac{\varepsilon}{M}$ if $n > N$. Then

$|a_n b_n - 0| = |a_n b_n| = |a_n| |b_n| \le |a_n| M = |a_n - 0| M < \dfrac{\varepsilon}{M} \cdot M = \varepsilon$ for all $n > N$. Since ε was arbitrary,

$\lim\limits_{n\to\infty} (a_n b_n) = 0$.

78. (a) $\dfrac{b^{n+1} - a^{n+1}}{b - a} = b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n$

$< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)b^n$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \;\Rightarrow\; b^{n+1} - (n+1)b^n(b-a) < a^{n+1} \;\Rightarrow\;$

$b^n[(n+1)a - nb] < a^{n+1}$.

(c) With this substitution, $(n+1)a - nb = 1$, and so $b^n = \left(1 + \dfrac{1}{n}\right)^n < a^{n+1} = \left(1 + \dfrac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1 + \dfrac{1}{2n}\right)^n \left(\dfrac{1}{2}\right) < 1 \;\Rightarrow\; \left(1 + \dfrac{1}{2n}\right)^n < 2 \;\Rightarrow\; \left(1 + \dfrac{1}{2n}\right)^{2n} < 4$.

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \le a_n \le 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by the Monotonic Sequence Theorem.

79. (a) First we show that $a > a_1 > b_1 > b$.

$$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0 \quad [\text{since } a > b] \quad \Rightarrow \quad a_1 > b_1. \text{ Also}$$

$a - a_1 = a - \frac{1}{2}(a + b) = \frac{1}{2}(a - b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$. In the same

way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is,

$a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \frac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \frac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right) = \frac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0,$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \frac{1}{2}(a_{k+1} + b_{k+1}) = \frac{1}{2}(a_{k+1} - b_{k+1}) > 0, \text{ and}$$

$$b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0 \quad \Rightarrow \quad a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1},$$

so the assertion is true for $n = k + 1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are

monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2} \quad \Rightarrow \quad \alpha = \dfrac{\alpha + \beta}{2} \quad \Rightarrow$

$2\alpha = \alpha + \beta \quad \Rightarrow \quad \alpha = \beta$.

80. (a) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim\limits_{n\to\infty} a_{2n+1} = L$, there

exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then

$n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so

$|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim\limits_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$,

$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$,

$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the

odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by

mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k} \quad \Rightarrow \quad \dfrac{1}{1 + a_{2k-2}} < \dfrac{1}{1 + a_{2k}} \quad \Rightarrow$

$1 + \dfrac{1}{1 + a_{2k-2}} < 1 + \dfrac{1}{1 + a_{2k}} \quad \Rightarrow \quad a_{2k-1} < a_{2k+1} \quad \Rightarrow \quad 1 + a_{2k-1} < 1 + a_{2k+1} \quad \Rightarrow$

$\dfrac{1}{1 + a_{2k-1}} > \dfrac{1}{1 + a_{2k+1}} \quad \Rightarrow \quad 1 + \dfrac{1}{1 + a_{2k-1}} > 1 + \dfrac{1}{1 + a_{2k+1}} \quad \Rightarrow \quad a_{2k} > a_{2k+2}$. We have thus shown, by

induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both

$\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by the Monotonic Sequence Theorem. Let

$\lim\limits_{n\to\infty} a_{2n} = L$. Then $\lim\limits_{n\to\infty} a_{2n+2} = L$ also. We have

$$a_{n+2} = 1 + \frac{1}{1 + 1 + 1/(1 + a_n)} = 1 + \frac{1}{(3 + 2a_n)/(1 + a_n)} = \frac{4 + 3a_n}{3 + 2a_n}$$

so $a_{2n+2} = \dfrac{4 + 3a_{2n}}{3 + 2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4 + 3L}{3 + 2L} \quad \Rightarrow \quad 3L + 2L^2 = 4 + 3L \quad \Rightarrow \quad L^2 = 2 \quad \Rightarrow$

$L = \sqrt{2}$ [since $L > 0$]. Thus, $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a),

$\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

81. (a) Suppose $\{p_n\}$ converges to p. Then $p_{n+1} = \dfrac{bp_n}{a+p_n}$ $\Rightarrow$ $\displaystyle\lim_{n\to\infty} p_{n+1} = \dfrac{b\,\lim\limits_{n\to\infty} p_n}{a + \lim\limits_{n\to\infty} p_n}$ $\Rightarrow$ $p = \dfrac{bp}{a+p}$ $\Rightarrow$

$p^2 + ap = bp$ $\Rightarrow$ $p(p + a - b) = 0$ $\Rightarrow$ $p = 0$ or $p = b - a$.

(b) $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{\left(\dfrac{b}{a}\right)p_n}{1 + \dfrac{p_n}{a}} < \left(\dfrac{b}{a}\right)p_n$ since $1 + \dfrac{p_n}{a} > 1$.

(c) By part (b), $p_1 < \left(\dfrac{b}{a}\right)p_0$, $p_2 < \left(\dfrac{b}{a}\right)p_1 < \left(\dfrac{b}{a}\right)^2 p_0$, $p_3 < \left(\dfrac{b}{a}\right)p_2 < \left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n < \left(\dfrac{b}{a}\right)^n p_0$,

so $\displaystyle\lim_{n\to\infty} p_n \leq \lim_{n\to\infty} \left(\dfrac{b}{a}\right)^n \cdot p_0 = 0$ since $b < a$. $\left[\text{By result 9, } \displaystyle\lim_{n\to\infty} r^n = 0 \text{ if } -1 < r < 1. \text{ Here } r = \dfrac{b}{a} \in (0, 1).\right]$

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

For $n = 0$, we have $p_1 - p_0 = \dfrac{bp_0}{a+p_0} - p_0 = \dfrac{p_0(b - a - p_0)}{a + p_0} > 0$ since $p_0 < b - a$. So $p_1 > p_0$.

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$b - a - p_{k+1} = b - a - \dfrac{bp_k}{a+p_k} = \dfrac{a(b-a) + bp_k - ap_k - bp_k}{a + p_k} = \dfrac{a(b - a - p_k)}{a + p_k} > 0$ because $p_k < b - a$. So

$p_{k+1} < b - a$. And $p_{k+2} - p_{k+1} = \dfrac{bp_{k+1}}{a+p_{k+1}} - p_{k+1} = \dfrac{p_{k+1}(b - a - p_{k+1})}{a + p_{k+1}} > 0$ since $p_{k+1} < b - a$. Therefore,

$p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k + 1$. It is therefore true for all n by mathematical induction.

A similar proof by induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem.

It then follows from part (a) that $\displaystyle\lim_{n\to\infty} p_n = b - a$.