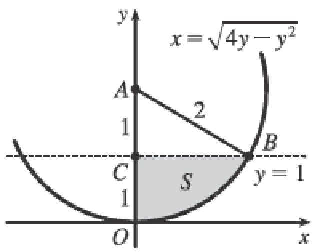
1. $x^2+y^2 \leq 4y \quad \Leftrightarrow \quad x^2+(y-2)^2 \leq 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\int_0^1 \sqrt{4y-y^2}\,dy \overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y-y^2}+2\cos^{-1}\left(\tfrac{2-y}{2}\right)\right]_0^1 \quad [a=2]$$

$$= -\tfrac{1}{2}\sqrt{3}+2\cos^{-1}\left(\tfrac{1}{2}\right)-2\cos^{-1}1$$

$$= -\tfrac{\sqrt{3}}{2}+2\left(\tfrac{\pi}{3}\right)-2(0)=\tfrac{2\pi}{3}-\tfrac{\sqrt{3}}{2}$$

Another method (without calculus): Note that $\theta=\angle CAB=\frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB)-(\text{area of } \triangle ABC)=\tfrac{1}{2}(2^2)\tfrac{\pi}{3}-\tfrac{1}{2}(1)\sqrt{3}=\tfrac{2\pi}{3}-\tfrac{\sqrt{3}}{2}$$

2. $y=\pm\sqrt{x^3-x^4} \quad \Rightarrow \quad$ The loop of the curve is symmetric about $y=0$, and therefore $\overline{y}=0$. At each point x where $0 \leq x \leq 1$, the lamina has a vertical length of $\sqrt{x^3-x^4}-\left(-\sqrt{x^3-x^4}\right)=2\sqrt{x^3-x^4}$. Therefore,

$\overline{x}=\dfrac{\int_0^1 x\cdot 2\sqrt{x^3-x^4}\,dx}{\int_0^1 2\sqrt{x^3-x^4}\,dx}=\dfrac{\int_0^1 x\sqrt{x^3-x^4}\,dx}{\int_0^1 \sqrt{x^3-x^4}\,dx}$. We evaluate the integrals separately:

$$\int_0^1 x\sqrt{x^3-x^4}\,dx=\int_0^1 x^{5/2}\sqrt{1-x}\,dx$$

$$=\int_0^{\pi/2} 2\sin^6\theta\cos\theta\sqrt{1-\sin^2\theta}\,d\theta \qquad \left[\begin{array}{c}\sin\theta=\sqrt{x},\ \cos\theta\,d\theta=dx/(2\sqrt{x}),\\ 2\sin\theta\ \cos\theta\ d\theta=dx\end{array}\right]$$

$$=\int_0^{\pi/2} 2\sin^6\theta\cos^2\theta\,d\theta=\int_0^{\pi/2} 2\left[\tfrac{1}{2}(1-\cos 2\theta)\right]^3\tfrac{1}{2}(1+\cos 2\theta)d\theta$$

$$=\int_0^{\pi/2}\tfrac{1}{8}(1-2\cos 2\theta+2\cos^3 2\theta-\cos^4 2\theta)\,d\theta$$

$$=\int_0^{\pi/2}\tfrac{1}{8}\left[1-2\cos 2\theta+2\cos 2\theta(1-\sin^2 2\theta)-\tfrac{1}{4}(1+\cos 4\theta)^2\right]d\theta$$

$$=\tfrac{1}{8}\left[\theta-\tfrac{1}{3}\sin^3 2\theta\right]_0^{\pi/2}-\tfrac{1}{32}\int_0^{\pi/2}\left(1+2\cos 4\theta+\cos^2 4\theta\right)d\theta$$

$$=\tfrac{\pi}{16}-\tfrac{1}{32}\left[\theta+\tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2}-\tfrac{1}{64}\int_0^{\pi/2}(1+\cos 8\theta)\,d\theta$$

$$=\tfrac{\pi}{16}-\tfrac{1}{32}\left[\theta+\tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2}-\tfrac{1}{64}\int_0^{\pi/2}(1+\cos 8\theta)\,d\theta$$

$$=\tfrac{3\pi}{64}-\tfrac{1}{64}\left[\theta+\tfrac{1}{8}\sin 8\theta\right]_0^{\pi/2}=\tfrac{5\pi}{128}$$

$$\int_0^1\sqrt{x^3-x^4}\,dx=\int_0^1 x^{3/2}\sqrt{1-x}\,dx=\int_0^{\pi/2}2\sin^4\theta\cos\theta\sqrt{1-\sin^2\theta}\,d\theta \qquad [\sin\theta=\sqrt{x}]$$

$$=\int_0^{\pi/2}2\sin^4\theta\cos^2\theta\,d\theta=\int_0^{\pi/2}2\cdot\tfrac{1}{4}(1-\cos 2\theta)^2\cdot\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

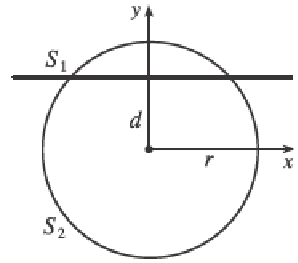
$$=\int_0^{\pi/2}\tfrac{1}{4}(1-\cos 2\theta-\cos^2 2\theta+\cos^3 2\theta)\,d\theta$$

$$=\int_0^{\pi/2}\tfrac{1}{4}\left[1-\cos 2\theta-\tfrac{1}{2}(1+\cos 4\theta)+\cos 2\theta(1-\sin^2 2\theta)\right]d\theta$$

$$=\tfrac{1}{4}\left[\tfrac{\theta}{2}-\tfrac{1}{8}\sin 4\theta-\tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/2}=\tfrac{\pi}{16}$$

Therefore, $\overline{x}=\dfrac{5\pi/128}{\pi/16}=\dfrac{5}{8}$, and $(\overline{x},\overline{y})=\left(\frac{5}{8},0\right)$.

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



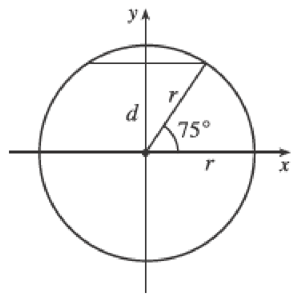
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy = \sqrt{\frac{r^2}{r^2 - y^2}}\,dy = \frac{r\,dy}{\sqrt{r^2 - y^2}}$$

From Formula 8.2.8 we have

$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_d^r 2\pi\sqrt{r^2 - y^2}\frac{r\,dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\,dy = 2\pi r(r - d)$$

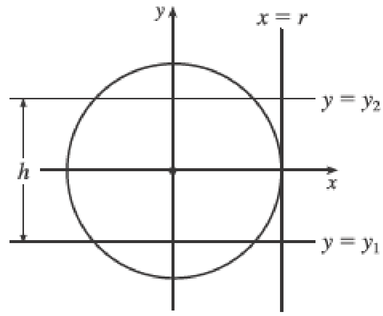
Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x\sqrt{1 + (dx/dy)^2}\,dy = \int_{-r}^{d} 2\pi r\,dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 3960$ mi and $d = r\,(\sin 75^\circ) \approx 3825$ mi, so the surface area of the Arctic Ocean is about $2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6$ mi^2.



(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r\, dy = 2\pi r(y_2 - y_1) = 2\pi rh.$

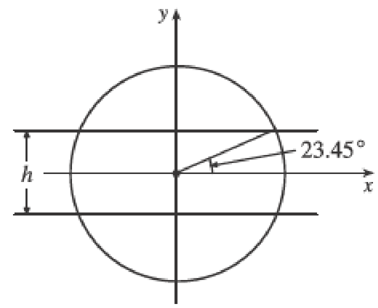
This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is



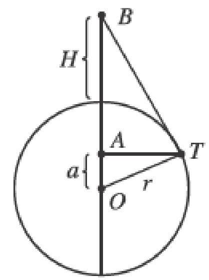
$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\, dy$$

$$= 2\pi r y\Big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi rh$$

(d) $h = 2r \sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7$ mi^2.



4. (a) Since the right triangles OAT and OTB are similar, we have $\dfrac{r+H}{r} = \dfrac{r}{a} \Rightarrow$



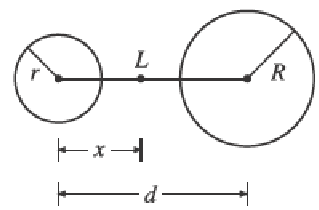
$a = \dfrac{r^2}{r+H}$. The surface area visible from B is $S = \int_a^r 2\pi x\sqrt{1 + (dx/dy)^2}\,dy$.

From $x^2 + y^2 = r^2$, we get $\dfrac{d}{dy}(x^2+y^2) = \dfrac{d}{dy}(r^2) \Rightarrow 2x\dfrac{dx}{dy} + 2y = 0 \Rightarrow$

$\dfrac{dx}{dy} = -\dfrac{y}{x}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{x^2+y^2}{x^2} = \dfrac{r^2}{x^2}$. Thus,

$$S = \int_a^r 2\pi x \cdot \frac{r}{x}\,dy = 2\pi r(r-a) = 2\pi r\left(r - \frac{r^2}{r+H}\right) = 2\pi r^2\left(1 - \frac{r}{r+H}\right) = 2\pi r^2 \cdot \frac{H}{r+H} = \frac{2\pi r^2 H}{r+H}.$$

(b) Assume $R \geq r$. If a light is placed at point L, at a distance x from the center of the sphere of radius r, then from part (a) we find that the total illuminated area A on the two spheres is [with $r + H = x$ and $r + H = d - x$].



$$A(x) = \frac{2\pi r^2(x-r)}{x} + \frac{2\pi R^2(d-x-R)}{d-x} \quad [r \leq x \leq d-R]. \quad \frac{A(x)}{2\pi} = r^2\left(1 - \frac{r}{x}\right) + R^2\left(1 - \frac{R}{d-x}\right),$$

so $A'(x) = 0 \Leftrightarrow 0 = r^2 \cdot \dfrac{r}{x^2} + R^2 \cdot \dfrac{-R}{(d-x)^2} \Leftrightarrow \dfrac{r^3}{x^2} = \dfrac{R^3}{(d-x)^2} \Leftrightarrow \dfrac{(d-x)^2}{x^2} = \dfrac{R^3}{r^3} \Leftrightarrow$

$$\left(\frac{d}{x} - 1\right)^2 = \left(\frac{R}{r}\right)^3 \Rightarrow \frac{d}{x} - 1 = \left(\frac{R}{r}\right)^{3/2} \Leftrightarrow \frac{d}{x} = 1 + \left(\frac{R}{r}\right)^{3/2} \Leftrightarrow x = x^* = \frac{d}{1 + (R/r)^{3/2}}.$$

Now $A'(x) = 2\pi\left(\dfrac{r^3}{x^2} - \dfrac{R^3}{(d-x)^2}\right) \Rightarrow A''(x) = 2\pi\left(-\dfrac{2r^3}{x^3} - \dfrac{2R^3}{(d-x)^3}\right)$ and $A''(x^*) < 0$, so we have a local maximum at $x = x^*$.

However, x^* may not be an allowable value of x—we must show that x^* is between r and $d - R$.

(1) $x^* \geq r \quad \Leftrightarrow \quad \dfrac{d}{1 + (R/r)^{3/2}} \geq r \quad \Leftrightarrow \quad d \geq r + R\sqrt{R/r}$

(2) $x^* \leq d - R \quad \Leftrightarrow \quad \dfrac{d}{1 + (R/r)^{3/2}} \leq d - R \quad \Leftrightarrow \quad d \leq d - R + d\left(\dfrac{R}{r}\right)^{3/2} - R\left(\dfrac{R}{r}\right)^{3/2} \quad \Leftrightarrow$

$R + R\left(\dfrac{R}{r}\right)^{3/2} \leq d\left(\dfrac{R}{r}\right)^{3/2} \quad \Leftrightarrow \quad d \geq \dfrac{R}{(R/r)^{3/2}} + R = R + r\sqrt{r/R}$, but

$R + r\sqrt{r/R} \leq R + r$, and since $d > r + R$ [given], we conclude that $x^* \leq d - R$.

Thus, from (1) and (2), x^* is not an allowable value of x if $d < r + R\sqrt{R/r}$.

So A may have a maximum at $x = r$, x^*, or $d - R$.

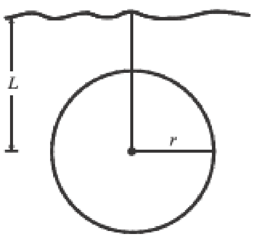
$$A(r) = \frac{2\pi R^2(d - r - R)}{d - r} \qquad \text{and} \qquad A(d - R) = \frac{2\pi r^2(d - r - R)}{d - R}$$

$A(r) > A(d - R) \quad \Leftrightarrow \quad \dfrac{R^2}{d - r} > \dfrac{r^2}{d - R} \quad \Leftrightarrow \quad R^2(d - R) > r^2(d - r) \quad \Leftrightarrow \quad R^2 d - R^3 > r^2 d - r^3 \quad \Leftrightarrow$
$R^2 d - r^2 d > R^3 - r^3 \quad \Leftrightarrow \quad d(R - r)(R + r) > (R - r)(R^2 + Rr + r^2) \quad \Leftrightarrow \quad d > (R^2 + Rr + r^2)/(R + r) \quad \Leftrightarrow$
$d > [(R + r)^2 - Rr]/(R + r) \quad \Leftrightarrow \quad d > R + r - Rr/(R + r)$. Now $R + r - Rr/(R + r) < R + r$, and we know that $d > R + r$, so we conclude that $A(r) > A(d - R)$.

In conclusion, A has an absolute maximum at $x = x^*$ provided $d \geq r + R\sqrt{R/r}$; otherwise, A has its maximum at $x = r$.

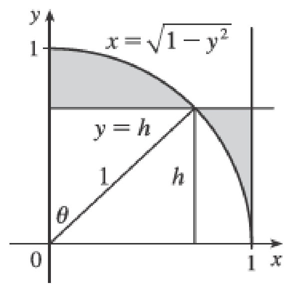
5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

(b)



$$\begin{aligned}
F &= \textstyle\int_{-r}^{r} P(L+x)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= \textstyle\int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\,dz\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= P_0 \textstyle\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 g H \int_{-r}^{r}\left(e^{(L+x)/H}-1\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0 - \rho_0 g H)\textstyle\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 g H \int_{-r}^{r} e^{(L+x)/H}\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \textstyle\int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2-x^2}\,dx
\end{aligned}$$

6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. An equation of the circle in the first quadrant is $x = \sqrt{1-y^2}$. So the shaded area is



$$\begin{aligned}
A(h) &= \int_0^h \left(1-\sqrt{1-y^2}\right)dy + \int_h^1 \sqrt{1-y^2}\,dy \\
&= \int_0^h \left(1-\sqrt{1-y^2}\right)dy - \int_1^h \sqrt{1-y^2}\,dy
\end{aligned}$$

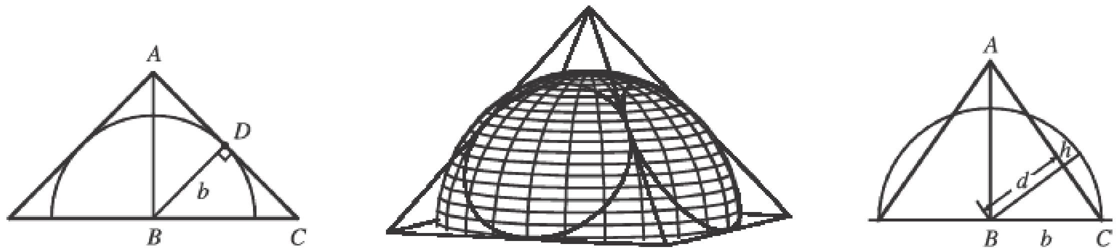
$$A'(h) = 1 - \sqrt{1-h^2} - \sqrt{1-h^2} \quad \text{[by FTC]} \; = 1 - 2\sqrt{1-h^2}$$

$$A' = 0 \;\Leftrightarrow\; \sqrt{1-h^2} = \tfrac{1}{2} \;\Rightarrow\; 1-h^2 = \tfrac{1}{4} \;\Rightarrow\; h^2 = \tfrac{3}{4} \;\Rightarrow\; h = \tfrac{\sqrt{3}}{2}.$$

$A''(h) = -2\cdot\frac{1}{2}(1-h^2)^{-1/2}(-2h) = \dfrac{2h}{\sqrt{1-h^2}} > 0$, so $h = \dfrac{\sqrt{3}}{2}$ gives a minimum value of A.

Note: Another strategy is to use the angle θ as the variable (see the diagram above) and show that $A = \theta + \cos\theta - \frac{\pi}{4} - \frac{1}{2}\sin 2\theta$, which is minimized when $\theta = \frac{\pi}{6}$.

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}\,b$.

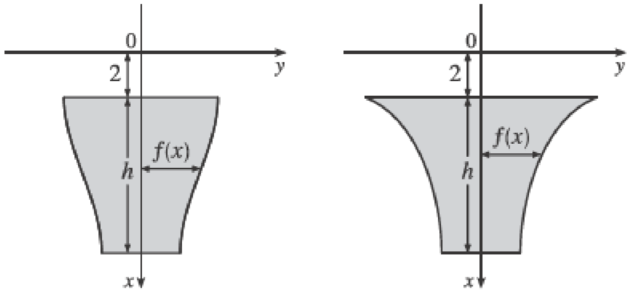


We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.51 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\,b}{\sqrt{b^2 + \left(\sqrt{2}\,b\right)^2}} \quad \Rightarrow \quad d = \frac{\sqrt{2}\,b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\,b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 6.2.51 with $r = b$, we find that the volume of each of the caps is $\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3\cdot 3}b\right) = \frac{15-6\sqrt{6}}{9}\cdot\frac{6+\sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

8. Orient the positive x-axis as in the figure.



Suppose that the plate has height h and is symmetric about the x-axis. At depth x below the water ($2 \le x \le 2+h$), let the width of the plate be $2f(x)$. Now each of the n horizontal strips has height h/n and the ith strip ($1 \le i \le n$) goes from

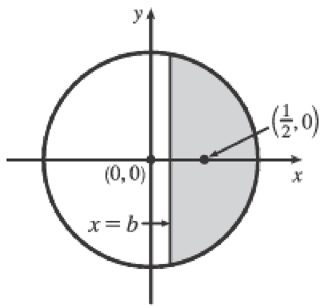
$x = 2 + \left(\frac{i-1}{n}\right)h$ to $x = 2 + \left(\frac{i}{n}\right)h$. The hydrostatic force on the ith strip is $F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5x[2f(x)]\,dx$.

If we now let $x[2f(x)] = k$ (a constant) so that $f(x) = k/(2x)$, then

$$F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5k\,dx = 62.5k\left[x\right]_{2+[(i-1)/n]h}^{2+(i/n)h} = 62.5k\left[\left(2+\frac{i}{n}h\right) - \left(2+\frac{i-1}{n}h\right)\right] = 62.5k\left(\frac{h}{n}\right)$$

So the hydrostatic force on the ith strip is independent of i, that is, the force on each strip is the same. So the plate can be shaped as shown in the figure. (In fact, the required condition is satisfied whenever the plate has width C/x at depth x, for some constant C. Many shapes are possible.)

9. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have $\frac{1}{2} = \overline{x} = \dfrac{\int_b^1 x \cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the



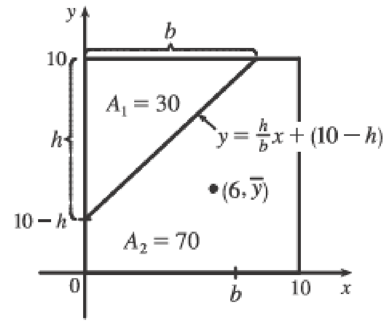
numerator gives us $-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0 - \left(1-b^2\right)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$.

Using Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_b^1 = \left(0 + \frac{\pi}{2}\right) - \left(b\sqrt{1-b^2} + \sin^{-1}b\right)$. Thus, we have $\frac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}$, or,

equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

10. $A_1 = 30 \quad \Rightarrow \quad \frac{1}{2}bh = 30 \quad \Rightarrow \quad bh = 60.$



$$\overline{x} = 6 \quad \Rightarrow \quad \frac{1}{A_2}\int_0^{10} x f(x)\, dx = 6 \quad \Rightarrow$$

$$\int_0^b x\left(\frac{h}{b}x + 10 - h\right) dx + \int_b^{10} x(10)\, dx = 6(70) \quad \Rightarrow$$

$$\int_0^b \left(\frac{h}{b}x^2 + 10x - hx\right) dx + 10 \cdot \tfrac{1}{2}\left[x^2\right]_b^{10} = 420 \quad \Rightarrow$$

$$\left[\frac{h}{3b}x^3 + 5x^2 - \frac{h}{2}x^2\right]_0^b + 5(100 - b^2) = 420 \quad \Rightarrow \quad \tfrac{1}{3}hb^2 + 5b^2 - \tfrac{1}{2}hb^2 + 500 - 5b^2 = 420 \quad \Rightarrow \quad 80 = \tfrac{1}{6}hb^2 \quad \Rightarrow$$

$480 = (hb)b \quad \Rightarrow \quad 480 = 60b \quad \Rightarrow \quad b = 8$. So $h = \frac{60}{8} = \frac{15}{2}$ and an equation of the line is

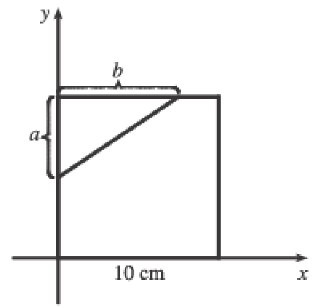
$y = \dfrac{15/2}{8}x + \left(10 - \dfrac{15}{2}\right) = \dfrac{15}{16}x + \dfrac{5}{2}$. Now

$$\overline{y} = \frac{1}{A_2}\int_0^{10} \tfrac{1}{2}[f(x)]^2\, dx = \frac{1}{70 \cdot 2}\left[\int_0^8 \left(\frac{15}{16}x + \frac{5}{2}\right)^2 dx + \int_8^{10} (10)^2\, dx\right]$$

$$= \tfrac{1}{140}\left[\int_0^8 \left(\tfrac{225}{256}x^2 + \tfrac{75}{16}x + \tfrac{25}{4}\right) dx + 100(10 - 8)\right] = \tfrac{1}{140}\left(\left[\tfrac{225}{768}x^3 + \tfrac{75}{32}x^2 + \tfrac{25}{4}x\right]_0^8 + 200\right)$$

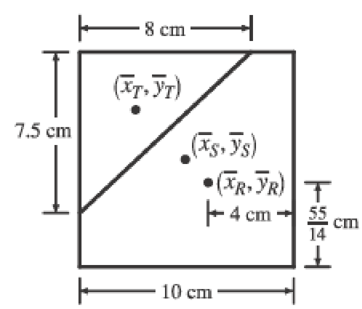
$$= \tfrac{1}{140}(150 + 150 + 50 + 200) = \tfrac{550}{140} = \tfrac{55}{14}$$

Another solution: Assume that the right triangle cut from the square has legs a cm and b cm long as shown. The triangle has area 30 cm^2, so $\frac{1}{2}ab = 30$ and $ab = 60$. We place the square in the first quadrant of the xy-plane as shown, and we let T, R, and S denote the triangle, the remaining portion of the square, and the full square, respectively. By symmetry, the centroid of S is $(5, 5)$. By Exercise 8.3.39, the centroid of T is $\left(\dfrac{b}{3}, 10 - \dfrac{a}{3}\right)$.



We are given that the centroid of R is $(6, c)$, where c is to be determined. We take the density of the square to be 1, so that areas can be used as masses. Then T has mass $m_T = 30$, S has mass $m_S = 100$, and R has mass $m_R = m_S - m_T = 70$. As in Exercises 40 and 41 of Section 8.3, we view S as consisting of a mass m_T at the centroid $(\overline{x}_T, \overline{y}_T)$ of T and a mass R at the centroid $(\overline{x}_R, \overline{y}_R)$ of R. Then $\overline{x}_S = \dfrac{m_T\overline{x}_T + m_R\overline{x}_R}{m_T + m_R}$ and $\overline{y}_S = \dfrac{m_T\overline{y}_T + m_R\overline{y}_R}{m_T + m_R}$; that is, $5 = \dfrac{30(b/3) + 70(6)}{100}$ and $5 = \dfrac{30(10 - a/3) + 70c}{100}$.

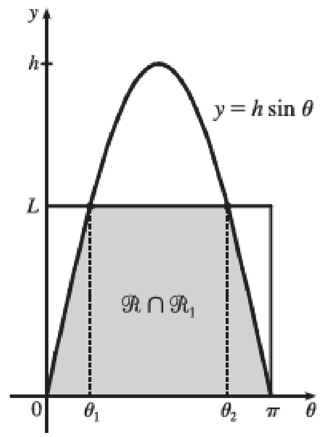
Solving the first equation for b, we get $b = 8$ cm. Since $ab = 60$ cm^2, it follows that $a = \frac{60}{8} = 7.5$ cm. Now the second equation says that $70c = 200 + 10a$, so $7c = 20 + a = \frac{55}{2}$ and $c = \frac{55}{14} = 3.9\overline{285714}$ cm. The solution is depicted in the figure.



11. If $h = L$, then $P = \dfrac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \dfrac{\int_0^\pi L\sin\theta\, d\theta}{\pi L} = \dfrac{[-\cos\theta]_0^\pi}{\pi} = \dfrac{-(-1)+1}{\pi} = \dfrac{2}{\pi}$.

If $h = L/2$, then $P = \dfrac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}} = \dfrac{\int_0^\pi \frac{1}{2}L\sin\theta\, d\theta}{\pi L} = \dfrac{[-\cos\theta]_0^\pi}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$.

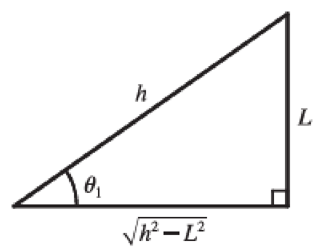
12. (a) The total set of possibilities can be identified with the rectangular region $\mathscr{R} = \{(\theta, y) \mid 0 \le y < L, 0 \le \theta < \pi\}$. Even when $h > L$, the needle intersects at least one line if and only if $y \le h\sin\theta$. Let $\mathscr{R}_1 = \{(\theta, y) \mid 0 \le y \le h\sin\theta, 0 \le \theta < \pi\}$. When $h \le L$, $\mathscr{R}_1$ is contained in $\mathscr{R}$, but that is no longer true when $h > L$. Thus, the probability that the needle intersects a line becomes



$$P = \frac{\text{area}(\mathscr{R} \cap \mathscr{R}_1)}{\text{area}(\mathscr{R})} = \frac{\text{area}(\mathscr{R} \cap \mathscr{R}_1)}{\pi L}$$

When $h > L$, the curve $y = h\sin\theta$ intersects the line $y = L$ twice—at $\left(\sin^{-1}(L/h), L\right)$ and at $\left(\pi - \sin^{-1}(L/h), L\right)$. Set $\theta_1 = \sin^{-1}(L/h)$ and $\theta_2 = \pi - \theta_1$. Then

$$\begin{aligned}
\text{area}(\mathscr{R} \cap \mathscr{R}_1) &= \int_0^{\theta_1} h\sin\theta\, d\theta + \int_{\theta_1}^{\theta_2} L\, d\theta + \int_{\theta_2}^{\pi} h\sin\theta\, d\theta \\
&= 2\int_0^{\theta_1} h\sin\theta\, d\theta + L(\theta_2 - \theta_1) = 2h\left[-\cos\theta\right]_0^{\theta_1} + L(\pi - 2\theta_1) \\
&= 2h(1 - \cos\theta_1) + L(\pi - 2\theta_1) \\
&= 2h\left(1 - \frac{\sqrt{h^2 - L^2}}{h}\right) + L\left[\pi - 2\sin^{-1}\left(\frac{L}{h}\right)\right] \\
&= 2h - 2\sqrt{h^2 - L^2} + \pi L - 2L\sin^{-1}\left(\frac{L}{h}\right)
\end{aligned}$$

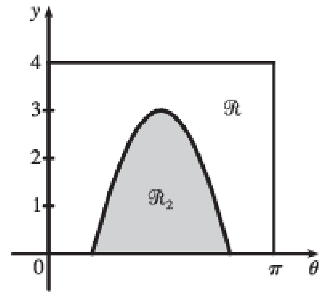


We are told that $L = 4$ and $h = 7$, so $\text{area}(\mathscr{R} \cap \mathscr{R}_1) = 14 - 2\sqrt{33} + 4\pi - 8\sin^{-1}\left(\frac{4}{7}\right) \approx 10.21128$ and $P = \frac{1}{4\pi}\text{area}(\mathscr{R} \cap \mathscr{R}_1) \approx 0.812588$. (By comparison, $P = \frac{2}{\pi} \approx 0.636620$ when $h = L$, as shown in the solution to Problem 11.)

(b) The needle intersects at least two lines when $y + L \le h\sin\theta$; that is, when $y \le h\sin\theta - L$. Set $\mathfrak{R}_2 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - L, 0 \le \theta < \pi\}$.

Then the probability that the needle intersects at least two lines is

$$P_2 = \frac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_2)}{\text{area}(\mathfrak{R})} = \frac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_2)}{\pi L}$$



When $L = 4$ and $h = 7$, $\mathfrak{R}_2$ is contained in $\mathfrak{R}$ (see the figure). Thus,

$$P_2 = \frac{1}{4\pi}\,\text{area}(\mathfrak{R}_2) = \frac{1}{4\pi}\int_{\sin^{-1}(4/7)}^{\pi - \sin^{-1}(4/7)} (7\sin\theta - 4)\,d\theta = \tfrac{1}{4\pi}\cdot 2\int_{\sin^{-1}(4/7)}^{\pi/2} (7\sin\theta - 4)\,d\theta$$

$$= \frac{1}{2\pi}\left[-7\cos\theta - 4\theta\right]_{\sin^{-1}(4/7)}^{\pi/2} = \frac{1}{2\pi}\left[0 - 2\pi + 7\frac{\sqrt{33}}{7} + 4\sin^{-1}\left(\tfrac{4}{7}\right)\right] = \frac{\sqrt{33} + 4\sin^{-1}\left(\frac{4}{7}\right) - 2\pi}{2\pi}$$

$$\approx 0.301497$$

(c) The needle intersects at least three lines when $y + 2L \le h\sin\theta$: that is, when $y \le h\sin\theta - 2L$. Set $\mathfrak{R}_3 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - 2L, 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least three lines is $P_3 = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_3)}{\text{area}(\mathfrak{R})} = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_3)}{\pi L}$. (At this point, the generalization to P_n, n any positive integer, should be clear.)

Under the given assumption,

$$P_3 = \frac{1}{\pi L}\,\text{area}(\mathfrak{R}_3) = \frac{1}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi - \sin^{-1}(2L/h)} (h\sin\theta - 2L)\,d\theta = \frac{2}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi/2} (h\sin\theta - 2L)\,d\theta$$

$$= \frac{2}{\pi L}\left[-h\cos\theta - 2L\theta\right]_{\sin^{-1}(2L/h)}^{\pi/2} = \frac{2}{\pi L}\left[-\pi L + \sqrt{h^2 - 4L^2} + 2L\sin^{-1}(2L/h)\right]$$

Note that the probability that a needle touches exactly one line is $P_1 - P_2$, the probability that it touches exactly two lines is $P_2 - P_3$, and so on.