

$$1. Q = \iint_D \sigma(x, y) dA = \int_1^3 \int_0^2 (2xy + y^2) dy dx = \int_1^3 \left[xy^2 + \frac{1}{3}y^3 \right]_{y=0}^{y=2} dx \\ = \int_1^3 \left(4x + \frac{8}{3} \right) dx = \left[2x^2 + \frac{8}{3}x \right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3} \text{ C}$$

$$2. Q = \iint_D \sigma(x, y) dA = \iint_D (x + y + x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 (r \cos \theta + r \sin \theta + r^2) r dr d\theta \\ = \int_0^{2\pi} \int_0^2 [r^2(\cos \theta + \sin \theta) + r^3] dr d\theta = \int_0^{2\pi} \left[\frac{1}{3}r^3(\cos \theta + \sin \theta) + \frac{1}{4}r^4 \right]_{r=0}^{r=2} d\theta \\ = \int_0^{2\pi} \left[\frac{8}{3}(\cos \theta + \sin \theta) + 4 \right] d\theta = \left[\frac{8}{3}(\sin \theta - \cos \theta) + 4\theta \right]_0^{2\pi} = 8\pi \text{ C}$$

$$3. m = \iint_D \rho(x, y) dA = \int_0^2 \int_{-1}^1 xy^2 dy dx = \int_0^2 x dx \int_{-1}^1 y^2 dy = \left[\frac{1}{2}x^2 \right]_0^2 \left[\frac{1}{3}y^3 \right]_{-1}^1 = 2 \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{x} = \frac{1}{m} \iint_D x\rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 x^2 y^2 dy dx = \frac{3}{4} \int_0^2 x^2 dx \int_{-1}^1 y^2 dy = \frac{3}{4} \left[\frac{1}{3}x^3 \right]_0^2 \left[\frac{1}{3}y^3 \right]_{-1}^1 = \frac{3}{4} \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{y} = \frac{1}{m} \iint_D y\rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 xy^3 dy dx = \frac{3}{4} \int_0^2 x dx \int_{-1}^1 y^3 dy = \frac{3}{4} \left[\frac{1}{2}x^2 \right]_0^2 \left[\frac{1}{4}y^4 \right]_{-1}^1 = \frac{3}{4} \cdot 2 \cdot 0 = 0. \\ \text{Hence, } (\bar{x}, \bar{y}) = \left(\frac{4}{3}, 0 \right).$$

$$4. m = \iint_D \rho(x, y) dA = \int_0^a \int_0^b cxy dy dx = c \int_0^a x dx \int_0^b y dy = c \left[\frac{1}{2}x^2 \right]_0^a \left[\frac{1}{2}y^2 \right]_0^b = \frac{1}{4}a^2b^2c, \\ M_y = \iint_D x\rho(x, y) dA = \int_0^a \int_0^b cx^2y dy dx = c \int_0^a x^2 dx \int_0^b y dy = c \left[\frac{1}{3}x^3 \right]_0^a \left[\frac{1}{2}y^2 \right]_0^b = \frac{1}{6}a^3b^2c, \text{ and} \\ M_x = \iint_D y\rho(x, y) dA = \int_0^a \int_0^b cxy^2 dy dx = c \int_0^a x dx \int_0^b y^2 dy = c \left[\frac{1}{2}x^2 \right]_0^a \left[\frac{1}{3}y^3 \right]_0^b = \frac{1}{6}a^2b^3c. \\ \text{Hence, } (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{2}{3}a, \frac{2}{3}b \right).$$

$$5. m = \int_0^2 \int_{x/2}^{3-x} (x + y) dy dx = \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x(3 - \frac{x}{2}) + \frac{1}{2}(3 - x)^2 - \frac{1}{8}x^2 \right] dx \\ = \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2} \right) dx = \left[-\frac{9}{8}(\frac{1}{3}x^3) + \frac{9}{2}x \right]_0^2 = 6, \\ M_y = \int_0^2 \int_{x/2}^{3-x} (x^2 + xy) dy dx = \int_0^2 \left[x^2y + \frac{1}{2}xy^2 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3 \right) dx = \frac{9}{2}, \\ M_x = \int_0^2 \int_{x/2}^{3-x} (xy + y^2) dy dx = \int_0^2 \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9 - \frac{9}{2}x \right) dx = 9. \\ \text{Hence } m = 6, (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{3}{4}, \frac{3}{2} \right).$$

$$6. \text{ Here } D = \{(x, y) \mid 0 \leq x \leq 2, x \leq y \leq 6 - 2x\}. \\ m = \int_0^2 \int_x^{6-2x} x^2 dy dx = \int_0^2 x^2 (6 - 2x - x) dx = \int_0^2 (6x^2 - 3x^3) dx = \left[2x^3 - \frac{3}{4}x^4 \right]_0^2 = 4, \\ M_y = \int_0^2 \int_x^{6-2x} x \cdot x^2 dy dx = \int_0^2 x^3 (6 - 2x - x) dx = \int_0^2 (6x^3 - 3x^4) dx = \left[\frac{3}{2}x^4 - \frac{3}{5}x^5 \right]_0^2 = \frac{24}{5}, \\ M_x = \int_0^2 \int_x^{6-2x} y \cdot x^2 dy dx = \int_0^2 x^2 \left[\frac{1}{2}(6 - 2x)^2 - \frac{1}{2}x^2 \right] dx = \frac{1}{2} \int_0^2 (3x^4 - 24x^3 + 36x^2) dx \\ = \frac{1}{2} \left[\frac{3}{5}x^5 - 6x^4 + 12x^3 \right]_0^2 = \frac{48}{5}. \\ \text{Hence } m = 4, (\bar{x}, \bar{y}) = \left(\frac{24/5}{4}, \frac{48/5}{4} \right) = \left(\frac{6}{5}, \frac{12}{5} \right).$$

$$7. m = \int_0^1 \int_0^{e^x} y \, dy \, dx = \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 e^{2x} \, dx = \frac{1}{4} e^{2x} \Big|_0^1 = \frac{1}{4}(e^2 - 1),$$

$$M_y = \int_0^1 \int_0^{e^x} xy \, dy \, dx = \frac{1}{2} \int_0^1 x e^{2x} \, dx = \frac{1}{2} \left[\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right]_0^1 = \frac{1}{8}(e^2 + 1),$$

$$M_x = \int_0^1 \int_0^{e^x} y^2 \, dy \, dx = \int_0^1 \left[\frac{1}{3} y^3 \right]_{y=0}^{y=e^x} dx = \frac{1}{3} \int_0^1 e^{3x} \, dx = \frac{1}{3} \left[\frac{1}{3} e^{3x} \right]_0^1 = \frac{1}{9}(e^3 - 1).$$

$$\text{Hence } m = \frac{1}{4}(e^2 - 1), (\bar{x}, \bar{y}) = \left(\frac{\frac{1}{8}(e^2 + 1)}{\frac{1}{4}(e^2 - 1)}, \frac{\frac{1}{9}(e^3 - 1)}{\frac{1}{4}(e^2 - 1)} \right) = \left(\frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)} \right).$$

$$8. m = \int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx = \int_0^1 x [y]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{3/2} \, dx = \frac{2}{5} x^{5/2} \Big|_0^1 = \frac{2}{5},$$

$$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2 \, dy \, dx = \int_0^1 x [y]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{5/2} \, dx = \frac{2}{7} x^{7/2} \Big|_0^1 = \frac{2}{7},$$

$$M_x = \int_0^1 \int_0^{\sqrt{x}} yx \, dy \, dx = \int_0^1 x \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2} \int_0^1 x^2 \, dx = \frac{1}{2} \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{6}.$$

$$\text{Hence } m = \frac{2}{5}, (\bar{x}, \bar{y}) = \left(\frac{2/7}{2/5}, \frac{1/6}{2/5} \right) = \left(\frac{5}{7}, \frac{5}{12} \right).$$

9. Note that $\sin(\pi x/L) \geq 0$ for $0 \leq x \leq L$.

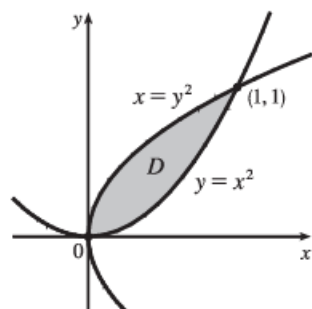
$$m = \int_0^L \int_0^{\sin(\pi x/L)} y \, dy \, dx = \int_0^L \frac{1}{2} \sin^2(\pi x/L) \, dx = \frac{1}{2} \left[\frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right]_0^L = \frac{1}{4} L,$$

$$\begin{aligned} M_y &= \int_0^L \int_0^{\sin(\pi x/L)} x \cdot y \, dy \, dx = \frac{1}{2} \int_0^L x \sin^2(\pi x/L) \, dx \quad \left[\begin{array}{l} \text{integrate by parts with} \\ u = x, dv = \sin^2(\pi x/L) \, dx \end{array} \right] \\ &= \frac{1}{2} \cdot x \left(\frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right) \Big|_0^L - \frac{1}{2} \int_0^L \left[\frac{1}{2} x - \frac{L}{4\pi} \sin(2\pi x/L) \right] dx \\ &= \frac{1}{4} L^2 - \frac{1}{2} \left[\frac{1}{4} x^2 + \frac{L^2}{4\pi^2} \cos(2\pi x/L) \right]_0^L = \frac{1}{4} L^2 - \frac{1}{2} \left(\frac{1}{4} L^2 + \frac{L^2}{4\pi^2} - \frac{L^2}{4\pi^2} \right) = \frac{1}{8} L^2, \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^L \int_0^{\sin(\pi x/L)} y \cdot y \, dy \, dx = \int_0^L \frac{1}{3} \sin^3(\pi x/L) \, dx = \frac{1}{3} \int_0^L [1 - \cos^2(\pi x/L)] \sin(\pi x/L) \, dx \\ &\quad [\text{substitute } u = \cos(\pi x/L)] \Rightarrow du = -\frac{\pi}{L} \sin(\pi x/L) \, dx \\ &= \frac{1}{3} \left(-\frac{L}{\pi} \right) [\cos(\pi x/L) - \frac{1}{3} \cos^3(\pi x/L)]_0^L = -\frac{L}{3\pi} \left(-1 + \frac{1}{3} - 1 + \frac{1}{3} \right) = \frac{4}{9\pi} L. \end{aligned}$$

$$\text{Hence } m = \frac{L}{4}, (\bar{x}, \bar{y}) = \left(\frac{L^2/8}{L/4}, \frac{4L/(9\pi)}{L/4} \right) = \left(\frac{L}{2}, \frac{16}{9\pi} \right).$$

10.



$$\begin{aligned}
 m &= \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x} \, dy \, dx = \int_0^1 \sqrt{x}(\sqrt{x} - x^2) \, dx \\
 &= \int_0^1 (x - x^{5/2}) \, dx = \left[\frac{1}{2}x^2 - \frac{2}{7}x^{7/2} \right]_0^1 = \frac{3}{14},
 \end{aligned}$$

$$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x \sqrt{x} \, dy \, dx = \int_0^1 x \sqrt{x}(\sqrt{x} - x^2) \, dx = \int_0^1 (x^2 - x^{7/2}) \, dx = \left[\frac{1}{3}x^3 - \frac{2}{9}x^{9/2} \right]_0^1 = \frac{1}{9}$$

$$\begin{aligned}
 M_x &= \int_0^1 \int_{x^2}^{\sqrt{x}} y \sqrt{x} \, dy \, dx = \int_0^1 \sqrt{x} \cdot \frac{1}{2}(x - x^4) \, dx = \frac{1}{2} \int_0^1 (x^{3/2} - x^{9/2}) \, dx \\
 &= \frac{1}{2} \left[\frac{2}{5}x^{5/2} - \frac{2}{11}x^{11/2} \right]_0^1 = \frac{1}{2} \cdot \frac{12}{55} = \frac{6}{55}.
 \end{aligned}$$

$$\text{Hence } m = \frac{3}{14}, (\bar{x}, \bar{y}) = \left(\frac{1/9}{3/14}, \frac{6/55}{3/14} \right) = \left(\frac{14}{27}, \frac{28}{55} \right).$$

$$11. \rho(x, y) = ky = kr \sin \theta, m = \int_0^{\pi/2} \int_0^1 kr^2 \sin \theta \, dr \, d\theta = \frac{1}{3}k \int_0^{\pi/2} \sin \theta \, d\theta = \frac{1}{3}k [-\cos \theta]_0^{\pi/2} = \frac{1}{3}k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^3 \sin \theta \cos \theta \, dr \, d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{1}{8}k [-\cos 2\theta]_0^{\pi/2} = \frac{1}{8}k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^3 \sin^2 \theta \, dr \, d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin^2 \theta \, d\theta = \frac{1}{8}k [\theta + \sin 2\theta]_0^{\pi/2} = \frac{\pi}{16}k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16} \right).$$

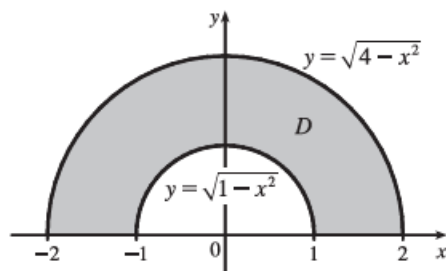
$$12. \rho(x, y) = k(x^2 + y^2) = kr^2, m = \int_0^{\pi/2} \int_0^1 kr^3 \, dr \, d\theta = \frac{\pi}{8}k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^4 \cos \theta \, dr \, d\theta = \frac{1}{5}k \int_0^{\pi/2} \cos \theta \, d\theta = \frac{1}{5}k [\sin \theta]_0^{\pi/2} = \frac{1}{5}k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^4 \sin \theta \, dr \, d\theta = \frac{1}{5}k \int_0^{\pi/2} \sin \theta \, d\theta = \frac{1}{5}k [-\cos \theta]_0^{\pi/2} = \frac{1}{5}k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi} \right).$$

13.



$$\rho(x, y) = k \sqrt{x^2 + y^2} = kr,$$

$$m = \iint_D \rho(x, y) dA = \int_0^\pi \int_1^2 kr \cdot r dr d\theta$$

$$= k \int_0^\pi d\theta \int_1^2 r^2 dr = k(\pi) \left[\frac{1}{3} r^3 \right]_1^2 = \frac{7}{3} \pi k,$$

$$M_y = \iint_D x \rho(x, y) dA = \int_0^\pi \int_1^2 (r \cos \theta) (kr) r dr d\theta = k \int_0^\pi \cos \theta d\theta \int_1^2 r^3 dr$$

$$= k [\sin \theta]_0^\pi \left[\frac{1}{4} r^4 \right]_1^2 = k(0) \left(\frac{15}{4} \right) = 0 \quad \text{[this is to be expected as the region and density function are symmetric about the y-axis]}$$

$$M_x = \iint_D y \rho(x, y) dA = \int_0^\pi \int_1^2 (r \sin \theta) (kr) r dr d\theta = k \int_0^\pi \sin \theta d\theta \int_1^2 r^3 dr$$

$$= k [-\cos \theta]_0^\pi \left[\frac{1}{4} r^4 \right]_1^2 = k(1 + 1) \left(\frac{15}{4} \right) = \frac{15}{2} k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(0, \frac{15k/2}{7\pi k/3} \right) = \left(0, \frac{45}{14\pi} \right).$$

14. Now $\rho(x, y) = k / \sqrt{x^2 + y^2} = k/r$, so

$$m = \iint_D \rho(x, y) dA = \int_0^\pi \int_1^2 (k/r) r dr d\theta = k \int_0^\pi d\theta \int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x \rho(x, y) dA = \int_0^\pi \int_1^2 (r \cos \theta) (k/r) r dr d\theta = k \int_0^\pi \cos \theta d\theta \int_1^2 r dr$$

$$= k [\sin \theta]_0^\pi \left[\frac{1}{2} r^2 \right]_1^2 = k(0) \left(\frac{3}{2} \right) = 0,$$

$$M_x = \iint_D y \rho(x, y) dA = \int_0^\pi \int_1^2 (r \sin \theta) (k/r) r dr d\theta = k \int_0^\pi \sin \theta d\theta \int_1^2 r dr$$

$$= k [-\cos \theta]_0^\pi \left[\frac{1}{2} r^2 \right]_1^2 = k(1 + 1) \left(\frac{3}{2} \right) = 3k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(0, \frac{3k}{\pi k} \right) = \left(0, \frac{3}{\pi} \right).$$

15. Placing the vertex opposite the hypotenuse at $(0, 0)$, $\rho(x, y) = k(x^2 + y^2)$. Then

$$m = \int_0^a \int_0^{a-x} k(x^2 + y^2) dy dx = k \int_0^a \left[ax^2 - x^3 + \frac{1}{3} (a-x)^3 \right] dx = k \left[\frac{1}{3} ax^3 - \frac{1}{4} x^4 - \frac{1}{12} (a-x)^4 \right]_0^a = \frac{1}{6} ka^4.$$

By symmetry,

$$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2 + y^2) dy dx = k \int_0^a \left[\frac{1}{2} (a-x)^2 x^2 + \frac{1}{4} (a-x)^4 \right] dx$$

$$= k \left[\frac{1}{6} a^2 x^3 - \frac{1}{4} ax^4 + \frac{1}{10} x^5 - \frac{1}{20} (a-x)^5 \right]_0^a = \frac{1}{15} ka^5$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{2}{5} a, \frac{2}{5} a \right).$$

16. $\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$.

$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r} r dr d\theta = k \int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1] d\theta$$

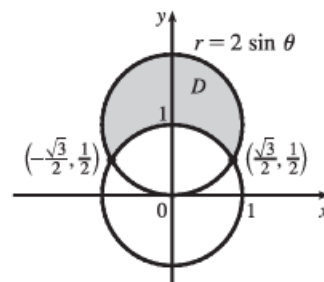
$$= k[-2\cos\theta - \theta]_{\pi/6}^{5\pi/6} = 2k(\sqrt{3} - \frac{\pi}{3})$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} kr \sin\theta dr d\theta = \frac{1}{2}k \int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta) d\theta$$

$$= \frac{1}{2}k[-3\cos\theta + \frac{4}{3}\cos^3\theta]_{\pi/6}^{5\pi/6} = \sqrt{3}k$$

Hence $(\bar{x}, \bar{y}) = (0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)})$.



17. $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} y^2 \cdot y dy dx = \int_0^1 [\frac{1}{4}y^4]_{y=0}^{y=e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx = \frac{1}{4} [\frac{1}{4}e^{4x}]_0^1 = \frac{1}{16}(e^4 - 1),$

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} x^2 y dy dx = \int_0^1 x^2 [\frac{1}{2}y^2]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx$$

$$= \frac{1}{2} [(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4})e^{2x}]_0^1 \quad [\text{integrate by parts twice}] = \frac{1}{8}(e^2 - 1),$$

and $I_0 = I_x + I_y = \frac{1}{16}(e^4 - 1) + \frac{1}{8}(e^2 - 1) = \frac{1}{16}(e^4 + 2e^2 - 3).$

18. $I_x = \int_0^{\pi/2} \int_0^1 (r^2 \sin^2\theta)(kr^2) r dr d\theta = \frac{1}{6}k \int_0^{\pi/2} \sin^2\theta d\theta = \frac{1}{6}k [\frac{1}{4}(2\theta - \sin 2\theta)]_0^{\pi/2} = \frac{\pi}{24}k,$

$$I_y = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2\theta)(kr^2) r dr d\theta = \frac{1}{6}k \int_0^{\pi/2} \cos^2\theta d\theta = \frac{1}{6}k [\frac{1}{4}(2\theta + \sin 2\theta)]_0^{\pi/2} = \frac{\pi}{24}k,$$

and $I_0 = I_x + I_y = \frac{\pi}{12}k.$

19. As in Exercise 15, we place the vertex opposite the hypotenuse at $(0, 0)$ and the equal sides along the positive axes.

$$I_x = \int_0^a \int_0^{a-x} y^2 k(x^2 + y^2) dy dx = k \int_0^a \int_0^{a-x} (x^2 y^2 + y^4) dy dx = k \int_0^a [\frac{1}{3}x^2 y^3 + \frac{1}{5}y^5]_{y=0}^{y=a-x} dx$$

$$= k \int_0^a [\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5] dx = k [\frac{1}{3}(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6) - \frac{1}{30}(a-x)^6]_0^a = \frac{7}{180}ka^6,$$

$$I_y = \int_0^a \int_0^{a-x} x^2 k(x^2 + y^2) dy dx = k \int_0^a \int_0^{a-x} (x^4 + x^2 y^2) dy dx = k \int_0^a [x^4 y + \frac{1}{3}x^2 y^3]_{y=0}^{y=a-x} dx$$

$$= k \int_0^a [x^4(a-x) + \frac{1}{3}x^2(a-x)^3] dx = k [\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6)]_0^a = \frac{7}{180}ka^6,$$

and $I_0 = I_x + I_y = \frac{7}{90}ka^6.$

20. If we find the moments of inertia about the x - and y -axes, we can determine in which direction rotation will be more difficult.

(See the explanation following Example 4.) The moment of inertia about the x -axis is given by

$$\begin{aligned} I_x &= \iint_D y^2 \rho(x, y) dA = \int_0^2 \int_0^2 y^2 (1 + 0.1x) dy dx = \int_0^2 (1 + 0.1x) \left[\frac{1}{3} y^3 \right]_{y=0}^{y=2} dx \\ &= \frac{8}{3} \int_0^2 (1 + 0.1x) dx = \frac{8}{3} \left[x + 0.1 \cdot \frac{1}{2} x^2 \right]_0^2 = \frac{8}{3} (2.2) \approx 5.87 \end{aligned}$$

Similarly, the moment of inertia about the y -axis is given by

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) dA = \int_0^2 \int_0^2 x^2 (1 + 0.1x) dy dx = \int_0^2 x^2 (1 + 0.1x) \left[y \right]_{y=0}^{y=2} dx \\ &= 2 \int_0^2 (x^2 + 0.1x^3) dx = 2 \left[\frac{1}{3} x^3 + 0.1 \cdot \frac{1}{4} x^4 \right]_0^2 = 2 \left(\frac{8}{3} + 0.4 \right) \approx 6.13 \end{aligned}$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y -axis.

21. Using a CAS, we find $m = \iint_D \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy dy dx = \frac{\pi^2}{8}$. Then

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} x^2 y dy dx = \frac{2\pi}{3} - \frac{1}{\pi} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} xy^2 dy dx = \frac{16}{9\pi}, \text{ so } (\bar{x}, \bar{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi} \right).$$

$$\text{The moments of inertia are } I_x = \iint_D y^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy^3 dy dx = \frac{3\pi^2}{64},$$

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} x^3 y dy dx = \frac{\pi^2}{16} (\pi^2 - 3), \text{ and } I_0 = I_x + I_y = \frac{\pi^2}{64} (4\pi^2 - 9).$$

22. Using a CAS, we find $m = \iint_D \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^2 dr d\theta = \frac{5}{3}\pi$,

$$\bar{x} = \frac{1}{m} \iint_D x \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos \theta} r^3 \cos \theta dr d\theta = \frac{21}{20} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos \theta} r^3 \sin \theta dr d\theta = 0, \text{ so } (\bar{x}, \bar{y}) = \left(\frac{21}{20}, 0 \right).$$

$$\text{The moments of inertia are } I_x = \iint_D y^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^4 \sin^2 \theta dr d\theta = \frac{33}{40}\pi,$$

$$I_y = \iint_D x^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^4 \cos^2 \theta dr d\theta = \frac{93}{40}\pi, \text{ and } I_0 = I_x + I_y = \frac{63}{20}\pi.$$

23. $I_x = \iint_D y^2 \rho(x, y) dA = \int_0^h \int_0^b \rho y^2 dx dy = \rho \int_0^b dx \int_0^h y^2 dy = \rho [x]_0^b \left[\frac{1}{3} y^3 \right]_0^h = \rho b \left(\frac{1}{3} h^3 \right) = \frac{1}{3} \rho b h^3,$

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^h \int_0^b \rho x^2 dx dy = \rho \int_0^b x^2 dx \int_0^h dy = \rho \left[\frac{1}{3} x^3 \right]_0^b [y]_0^h = \frac{1}{3} \rho b^3 h,$$

$$\text{and } m = \rho (\text{area of rectangle}) = \rho b h \text{ since the lamina is homogeneous. Hence } \bar{\bar{x}}^2 = \frac{I_y}{m} = \frac{\frac{1}{3} \rho b^3 h}{\rho b h} = \frac{b^2}{3} \Rightarrow \bar{\bar{x}} = \frac{b}{\sqrt{3}}$$

$$\text{and } \bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{\frac{1}{3} \rho b h^3}{\rho b h} = \frac{h^2}{3} \Rightarrow \bar{\bar{y}} = \frac{h}{\sqrt{3}}.$$

24. Here we assume $b > 0$, $h > 0$ but note that we arrive at the same results if $b < 0$ or $h < 0$. We have

$$D = \{(x, y) \mid 0 \leq x \leq b, 0 \leq y \leq h - \frac{h}{b}x\}, \text{ so}$$

$$I_x = \int_0^b \int_0^{h-hx/b} y^2 \rho \, dy \, dx = \rho \int_0^b \left[\frac{1}{3} y^3 \right]_{y=0}^{y=h-hx/b} dx = \frac{1}{3} \rho \int_0^b \left(h - \frac{h}{b}x \right)^3 dx$$

$$= \frac{1}{3} \rho \left[-\frac{b}{h} \left(\frac{1}{4} \right) \left(h - \frac{h}{b}x \right)^4 \right]_0^b = -\frac{b}{12h} \rho (0 - h^4) = \frac{1}{12} \rho b h^3,$$

$$I_y = \int_0^b \int_0^{h-hx/b} x^2 \rho \, dy \, dx = \rho \int_0^b x^2 \left(h - \frac{h}{b}x \right) dx = \rho \int_0^b \left(hx^2 - \frac{h}{b}x^3 \right) dx$$

$$= \rho \left[\frac{h}{3} x^3 - \frac{h}{4b} x^4 \right]_0^b = \rho \left(\frac{hb^3}{3} - \frac{hb^4}{4} \right) = \frac{1}{12} \rho b^3 h,$$

$$\text{and } m = \int_0^b \int_0^{h-hx/b} \rho \, dy \, dx = \rho \int_0^b \left(h - \frac{h}{b}x \right) dx = \rho \left[hx - \frac{h}{2b}x^2 \right]_0^b = \frac{1}{2} \rho b h. \text{ Hence } \bar{\bar{x}}^2 = \frac{I_y}{m} = \frac{\frac{1}{12} \rho b^3 h}{\frac{1}{2} \rho b h} = \frac{b^2}{6} \Rightarrow$$

$$\bar{\bar{x}} = \frac{b}{\sqrt{6}} \text{ and } \bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{\frac{1}{12} \rho b h^3}{\frac{1}{2} \rho b h} = \frac{h^2}{6} \Rightarrow \bar{\bar{y}} = \frac{h}{\sqrt{6}}.$$

25. In polar coordinates, the region is $D = \{(r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{2}\}$, so

$$I_x = \iint_D y^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho (r \sin \theta)^2 r \, dr \, d\theta = \rho \int_0^{\pi/2} \sin^2 \theta \, d\theta \int_0^a r^3 \, dr$$

$$= \rho \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} \left[\frac{1}{4} r^4 \right]_0^a = \rho \left(\frac{\pi}{4} \right) \left(\frac{1}{4} a^4 \right) = \frac{1}{16} \rho a^4 \pi,$$

$$I_y = \iint_D x^2 \rho \, dA = \int_0^{\pi/2} \int_0^a \rho (r \cos \theta)^2 r \, dr \, d\theta = \rho \int_0^{\pi/2} \cos^2 \theta \, d\theta \int_0^a r^3 \, dr$$

$$= \rho \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} \left[\frac{1}{4} r^4 \right]_0^a = \rho \left(\frac{\pi}{4} \right) \left(\frac{1}{4} a^4 \right) = \frac{1}{16} \rho a^4 \pi,$$

$$\text{and } m = \rho \cdot A(D) = \rho \cdot \frac{1}{4} \pi a^2 \text{ since the lamina is homogeneous. Hence } \bar{\bar{x}}^2 = \bar{\bar{y}}^2 = \frac{\frac{1}{16} \rho a^4 \pi}{\frac{1}{4} \rho a^2 \pi} = \frac{a^2}{4} \Rightarrow \bar{\bar{x}} = \bar{\bar{y}} = \frac{a}{2}.$$

$$26. m = \int_0^\pi \int_0^{\sin x} \rho \, dy \, dx = \rho \int_0^\pi \sin x \, dx = \rho [-\cos x]_0^\pi = 2\rho,$$

$$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 \, dy \, dx = \frac{1}{3} \rho \int_0^\pi \sin^3 x \, dx = \frac{1}{3} \rho \int_0^\pi (1 - \cos^2 x) \sin x \, dx = \frac{1}{3} \rho [-\cos \theta + \frac{1}{3} \cos^3 \theta]_0^\pi = \frac{4}{9} \rho,$$

$$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2 \, dy \, dx = \rho \int_0^\pi x^2 \sin x \, dx = \rho [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^\pi \quad [\text{by integrating by parts twice}]$$

$$= \rho (\pi^2 - 4).$$

$$\text{Then } \bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{2}{9}, \text{ so } \bar{\bar{y}} = \frac{\sqrt{2}}{3} \text{ and } \bar{\bar{x}}^2 = \frac{I_y}{m} = \frac{\pi^2 - 4}{2}, \text{ so } \bar{\bar{x}} = \sqrt{\frac{\pi^2 - 4}{2}}.$$

27. (a) $f(x, y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 1] \times [0, 2]$, we can say

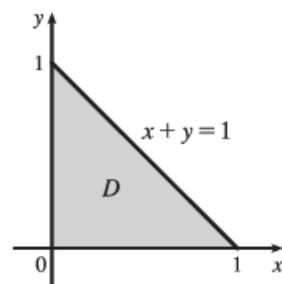
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_0^1 \int_0^2 Cx(1+y) dy dx \\ &= C \int_0^1 x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=2} dx = C \int_0^1 4x dx = C [2x^2]_0^1 = 2C\end{aligned}$$

$$\text{Then } 2C = 1 \Rightarrow C = \frac{1}{2}.$$

$$\begin{aligned}\text{(b) } P(X \leq 1, Y \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 f(x, y) dy dx = \int_0^1 \int_0^1 \frac{1}{2}x(1+y) dy dx \\ &= \int_0^1 \frac{1}{2}x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{2}x \left(\frac{3}{2} \right) dx = \frac{3}{4} \left[\frac{1}{2}x^2 \right]_0^1 = \frac{3}{8} \text{ or } 0.375\end{aligned}$$

- (c) $P(X + Y \leq 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus

$$\begin{aligned}P(X + Y \leq 1) &= \iint_D f(x, y) dA = \int_0^1 \int_0^{1-x} \frac{1}{2}x(1+y) dy dx \\ &= \int_0^1 \frac{1}{2}x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{2}x \left(\frac{1}{2}x^2 - 2x + \frac{3}{2} \right) dx \\ &= \frac{1}{4} \int_0^1 (x^3 - 4x^2 + 3x) dx = \frac{1}{4} \left[\frac{x^4}{4} - 4\frac{x^3}{3} + 3\frac{x^2}{2} \right]_0^1 \\ &= \frac{5}{48} \approx 0.1042\end{aligned}$$



28. (a) $f(x, y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Here, $f(x, y) = 0$ outside the square $[0, 1] \times [0, 1]$, so $\iint_{\mathbb{R}^2} f(x, y) dA = \int_0^1 \int_0^1 4xy dy dx = \int_0^1 [2xy^2]_{y=0}^{y=1} dx = \int_0^1 2x dx = x^2 \Big|_0^1 = 1$. Thus, $f(x, y)$ is a joint density function.

- (b) (i) No restriction is placed on Y , so

$$P\left(X \geq \frac{1}{2}\right) = \int_{1/2}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_{1/2}^1 \int_0^1 4xy dy dx = \int_{1/2}^1 [2xy^2]_{y=0}^{y=1} dx = \int_{1/2}^1 2x dx = x^2 \Big|_{1/2}^1 = \frac{3}{4}.$$

$$\begin{aligned}\text{(ii) } P\left(X \geq \frac{1}{2}, Y \leq \frac{1}{2}\right) &= \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f(x, y) dy dx = \int_{1/2}^1 \int_0^{1/2} 4xy dy dx \\ &= \int_{1/2}^1 [2xy^2]_{y=0}^{y=1/2} dx = \int_{1/2}^1 \frac{1}{2}x dx = \frac{1}{2} \cdot \frac{1}{2}x^2 \Big|_{1/2}^1 = \frac{3}{16}\end{aligned}$$

- (c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^1 \int_0^1 x(4xy) dy dx = \int_0^1 2x^2 [y^2]_{y=0}^{y=1} dx = 2 \int_0^1 x^2 dx = 2 \left[\frac{1}{3}x^3 \right]_0^1 = \frac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^1 \int_0^1 y(4xy) dy dx = \int_0^1 4x \left[\frac{1}{3}y^3 \right]_{y=0}^{y=1} dx = \frac{4}{3} \int_0^1 x dx = \frac{4}{3} \left[\frac{1}{2}x^2 \right]_0^1 = \frac{2}{3}$$

29. (a) $f(x, y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Here, $f(x, y) = 0$ outside the first quadrant, so

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_0^\infty \int_0^\infty 0.1 e^{-(0.5x+0.2y)} dy dx = 0.1 \int_0^\infty \int_0^\infty e^{-0.5x} e^{-0.2y} dy dx = 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - 1)] = (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - 1) = 1\end{aligned}$$

Thus $f(x, y)$ is a joint density function.

(b) (i) No restriction is placed on X , so

$$\begin{aligned}P(Y \geq 1) &= \int_{-\infty}^\infty \int_1^\infty f(x, y) dy dx = \int_0^\infty \int_1^\infty 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_1^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_1^t e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_1^t = 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - e^{-0.2})] \\ &= (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - e^{-0.2}) = e^{-0.2} \approx 0.8187\end{aligned}$$

$$\begin{aligned}\text{(ii)} P(X \leq 2, Y \leq 4) &= \int_{-\infty}^2 \int_{-\infty}^4 f(x, y) dy dx = \int_0^2 \int_0^4 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^2 e^{-0.5x} dx \int_0^4 e^{-0.2y} dy = 0.1 [-2e^{-0.5x}]_0^2 [-5e^{-0.2y}]_0^4 \\ &= (0.1) \cdot (-2)(e^{-1} - 1) \cdot (-5)(e^{-0.8} - 1) \\ &= (e^{-1} - 1)(e^{-0.8} - 1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481\end{aligned}$$

(c) The expected value of X is given by

$$\begin{aligned}\mu_1 &= \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^\infty \int_0^\infty x \left[0.1 e^{-(0.5x+0.2y)} \right] dy dx \\ &= 0.1 \int_0^\infty x e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t x e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x} dx$ (or we can use Formula 96 in the Table of Integrals): $\int x e^{-0.5x} dx = -2x e^{-0.5x} - \int -2e^{-0.5x} dx = -2x e^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$.

Thus

$$\begin{aligned}\mu_1 &= 0.1 \lim_{t \rightarrow \infty} \left[-2(x+2)e^{-0.5x} \right]_0^t \lim_{t \rightarrow \infty} \left[-5e^{-0.2y} \right]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} (-2) \left[(t+2)e^{-0.5t} - 2 \right] \lim_{t \rightarrow \infty} (-5) \left[e^{-0.2t} - 1 \right] \\ &= 0.1(-2) \left(\lim_{t \rightarrow \infty} \frac{t+2}{e^{0.5t}} - 2 \right) (-5)(-1) = 2 \quad [\text{by l'Hospital's Rule}]\end{aligned}$$

The expected value of Y is given by

$$\begin{aligned}\mu_2 &= \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^\infty \int_0^\infty y \left[0.1 e^{-(0.5x+0.2y)} \right] dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty y e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t y e^{-0.2y} dy\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y} dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int y e^{-0.2y} dy = -5y e^{-0.2y} + \int 5e^{-0.2y} dy = -5(y+5)e^{-0.2y}$. Then

$$\begin{aligned}\mu_2 &= 0.1 \lim_{t \rightarrow \infty} \left[-2e^{-0.5x} \right]_0^t \lim_{t \rightarrow \infty} \left[-5(y+5)e^{-0.2y} \right]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} \left[-2(e^{-0.5t} - 1) \right] \lim_{t \rightarrow \infty} (-5) \left[(t+5)e^{-0.2t} - 5 \right] \\ &= 0.1(-2)(-1) \cdot (-5) \left(\lim_{t \rightarrow \infty} \frac{t+5}{e^{0.2t}} - 5 \right) = 5 \quad [\text{by l'Hospital's Rule}]\end{aligned}$$

30. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

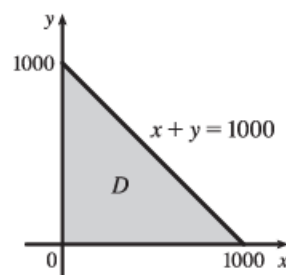
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x, y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned} P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x, y) dy dx = \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000} dx \int_0^{1000} e^{-y/1000} dy \\ &= 10^{-6} \left[-1000 e^{-x/1000} \right]_0^{1000} \left[-1000 e^{-y/1000} \right]_0^{1000} \\ &= (e^{-1} - 1)^2 \approx 0.3996 \end{aligned}$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X, Y) \in D)$ where D is the triangular region shown in the figure. Then



$$\begin{aligned} P(X + Y \leq 1000) &= \iint_D f(x, y) dA \\ &= \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} \left[-1000 e^{-(x+y)/1000} \right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} (e^{-1} - e^{-x/1000}) dx \\ &= -10^{-3} \left[e^{-1} x + 1000 e^{-x/1000} \right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$

31. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y , then, are $f_1(x) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5}$ and

$f_2(y) = \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$f(x, y) = f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5} \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02} = \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}.$$

$$\text{Then } P(40 \leq X \leq 50, 20 \leq Y \leq 25) = \int_{40}^{50} \int_{20}^{25} f(x, y) dy dx = \frac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2} dy dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$.

- (b) $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10} \sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow 4(x-45)^2 = 2 \Rightarrow x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

$$\iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} dA = \frac{10}{\pi} \int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2 - 50(y-20)^2} dy dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) \approx 0.632$.

32. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

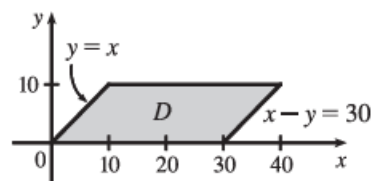
Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$.

Additionally, Yolanda will wait up to half an hour but no longer, so they

won't meet unless $X - Y \leq 30$. Thus the probability that they meet is

$P((X, Y) \in D)$ where D is the parallelogram shown in the figure. The

integral is simpler to evaluate if we consider D as a type II region, so



$$\begin{aligned} P((X, Y) \in D) &= \iint_D f(x, y) dx dy = \int_0^{10} \int_y^{y+30} \frac{1}{50} e^{-x} y dx dy = \frac{1}{50} \int_0^{10} y [-e^{-x}]_{x=y}^{x=y+30} dy \\ &= \frac{1}{50} \int_0^{10} y (-e^{-(y+30)} + e^{-y}) dy = \frac{1}{50} (1 - e^{-30}) \int_0^{10} y e^{-y} dy \end{aligned}$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

$$\frac{1}{50} (1 - e^{-30}) [-(y+1)e^{-y}]_0^{10} = \frac{1}{50} (1 - e^{-30}) (1 - 11e^{-10}) \approx 0.020. \text{ Thus there is only about a 2\% chance they will meet.}$$

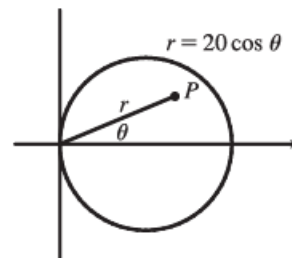
Such is student life!

33. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P , and $k dA$ is the number of infected individuals in an element of area dA , then $f(P, A)k dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA . Integration over D gives the number of infections of the person at A due to all the infected people in D . In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P, A) dA = k \iint_D \frac{20 - d(P, A)}{20} dA = k \iint_D \left[1 - \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{20} \right] dx dy$$

- (b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k \iint_D \left[1 - \frac{1}{20} \sqrt{x^2 + y^2} \right] dx dy \\ &= k \int_0^{2\pi} \int_0^{10} \left(1 - \frac{r}{20} \right) r dr d\theta = 2\pi k \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_0^{10} \\ &= 2\pi k \left(50 - \frac{50}{3} \right) = \frac{200}{3} \pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A . Then the polar equation for the circular boundary of the city becomes $r = 20 \cos \theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k \int_{-\pi/2}^{\pi/2} \int_0^{20 \cos \theta} \left(1 - \frac{r}{20} \right) r dr d\theta = k \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_{r=0}^{r=20 \cos \theta} d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \left(200 \cos^2 \theta - \frac{400}{3} \cos^3 \theta \right) d\theta = 200k \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2} \cos 2\theta - \frac{2}{3} (1 - \sin^2 \theta) \cos \theta \right] d\theta \\ &= 200k \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta - \frac{2}{3} \sin \theta + \frac{2}{3} \cdot \frac{1}{3} \sin^3 \theta \right]_{-\pi/2}^{\pi/2} = 200k \left[\frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} + \frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} \right] \\ &= 200k \left(\frac{\pi}{2} - \frac{8}{9} \right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.