1. The interior of the circle is the set of points $\left\{(x,y) \mid -r \le y \le r,\ -\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 to evaluate the integral, we get

$$V_2 = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dy = \int_{-r}^{r} 2\sqrt{r^2-y^2}\,dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1-\sin^2\theta}\,(r\cos\theta\,d\theta)$$

$$= 2r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = 2r^2\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2$$

2. The region of integration is $\left\{(x,y,z) \mid -r \le z \le r, -\sqrt{r^2-z^2} \le y \le \sqrt{r^2-z^2}, -\sqrt{r^2-z^2-y^2} \le x \le \sqrt{r^2-z^2-y^2}\right\}$. Substituting $y = \sqrt{r^2-z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$V_3 = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}}\int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\,dy\,dz = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} 2\sqrt{r^2-z^2-y^2}\,dy\,dz$$

$$= \int_{-r}^{r}\int_{-\pi/2}^{\pi/2} 2\sqrt{r^2-z^2}\sqrt{1-\sin^2\theta}\left(\sqrt{r^2-z^2}\cos\theta\,d\theta\right)dz$$

$$= 2\left[\int_{-r}^{r}(r^2-z^2)\,dz\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3}$$

3. Here we substitute $y = \sqrt{r^2 - w^2 - z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2}\cos^p\theta\, d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 45 and 46 in Section 8.1 [ET 7.1], we have

$$\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\dots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\dots\cdot 2k}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\dots\cdot 2k}{1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2}\cos^{2k}x\,dx = 2\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\dots\cdot(2k-1)\pi}{2\cdot 4\cdot 6\cdot\dots\cdot 2k} \tag{1}$$

$$\int_{-\pi/2}^{\pi/2}\cos^{2k+1}x\,dx = 2\int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 2\cdot 4\cdot 6\cdot\dots\cdot 2k}{1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)} \tag{2}$$

Thus

$$\begin{aligned}
V_4 &= \int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\pi/2}^{\pi/2}(r^2-w^2-z^2)\cos^2\theta\,d\theta\,dz\,dw \\
&= 2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}(r^2-w^2-z^2)\,dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] \\
&= 2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}(r^2-w^2)^{3/2}\,dw\right] = \pi\left(\tfrac{4}{3}\right)\int_{-\pi/2}^{\pi/2}r^4\cos^4\phi\,d\phi = \frac{4\pi}{3}r^4\cdot\frac{1\cdot 3\cdot\pi}{2\cdot 4} = \frac{\pi^2 r^4}{2}
\end{aligned}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2} \cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$V_n = \int_{-r}^{r} \int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}} \cdots \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}} \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\, dx_2 \cdots dx_{n-1}\, dx_n$$

$$= 2\left[\int_{-\pi/2}^{\pi/2} \cos^2\theta_2\, d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2} \cos^3\theta_3\, d\theta_3\right] \cdots \left[\int_{-\pi/2}^{\pi/2} \cos^{n-1}\theta_{n-1}\, d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2} \cos^n\theta_n\, d\theta_n\right] r^n$$

$$= \begin{cases} \left[2 \cdot \frac{\pi}{2}\right]\left[\frac{2\cdot 2}{1\cdot 3}\cdot\frac{1\cdot 3\pi}{2\cdot 4}\right]\left[\frac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\frac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\frac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\frac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right] r^n & n \text{ even} \\ 2\left[\frac{\pi}{2}\cdot\frac{2\cdot 2}{1\cdot 3}\right]\left[\frac{1\cdot 3\pi}{2\cdot 4}\cdot\frac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\frac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\frac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right] r^n & n \text{ odd} \end{cases}$$

By canceling within each set of brackets, we find that

$$V_n = \begin{cases} \frac{2\pi}{2}\cdot\frac{2\pi}{4}\cdot\frac{2\pi}{6}\cdot\cdots\cdot\frac{2\pi}{n} r^n = \frac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n} r^n = \frac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!} r^n & n \text{ even} \\ 2\cdot\frac{2\pi}{3}\cdot\frac{2\pi}{5}\cdot\frac{2\pi}{7}\cdot\cdots\cdot\frac{2\pi}{n} r^n = \frac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n} r^n = \frac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!} r^n & n \text{ odd} \end{cases}$$