1. $r=\theta^2,\ 0\le\theta\le\frac{\pi}{4}$. $A=\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta=\int_0^{\pi/4}\frac{1}{2}(\theta^2)^2\,d\theta=\int_0^{\pi/4}\frac{1}{2}\theta^4\,d\theta=\left[\frac{1}{10}\theta^5\right]_0^{\pi/4}=\frac{1}{10}\left(\frac{\pi}{4}\right)^5=\frac{1}{10{,}240}\pi^5$

2. $r=e^{\theta/2},\ \pi\le\theta\le 2\pi$. $A=\int_\pi^{2\pi}\frac{1}{2}(e^{\theta/2})^2 d\theta=\int_\pi^{2\pi}\frac{1}{2}e^\theta\,d\theta=\frac{1}{2}\left[e^\theta\right]_\pi^{2\pi}=\frac{1}{2}(e^{2\pi}-e^\pi)$

3. $r=\sin\theta,\ \frac{\pi}{3}\le\theta\le\frac{2\pi}{3}$.

$$A=\int_{\pi/3}^{2\pi/3}\frac{1}{2}\sin^2\theta\,d\theta=\frac{1}{4}\int_{\pi/3}^{2\pi/3}(1-\cos 2\theta)\,d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3}=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]=\frac{1}{4}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$$

4. $r=\sqrt{\sin\theta},\ 0\le\theta\le\pi$. $A=\int_0^\pi\frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta=\int_0^\pi\frac{1}{2}\sin\theta\,d\theta=\left[-\frac{1}{2}\cos\theta\right]_0^\pi=\frac{1}{2}+\frac{1}{2}=1$

5. $r=\sqrt{\theta},\ 0\le\theta\le 2\pi$. $A=\int_0^{2\pi}\frac{1}{2}r^2\,d\theta=\int_0^{2\pi}\frac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta=\int_0^{2\pi}\frac{1}{2}\theta\,d\theta=\left[\frac{1}{4}\theta^2\right]_0^{2\pi}=\pi^2$

6. $r=1+\cos\theta,\ 0\le\theta\le\pi$.

$$A=\int_0^\pi\frac{1}{2}(1+\cos\theta)^2\,d\theta=\frac{1}{2}\int_0^\pi(1+2\cos\theta+\cos^2\theta)\,d\theta=\frac{1}{2}\int_0^\pi\left[1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$=\frac{1}{2}\int_0^\pi\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta=\frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^\pi=\frac{1}{2}\left(\frac{3}{2}\pi+0+0\right)-\frac{1}{2}(0)=\frac{3\pi}{4}$$

7. $r=4+3\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

$$A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}((4+3\sin\theta)^2\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(16+24\sin\theta+9\sin^2\theta)\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(16+9\sin^2\theta)\,d\theta\quad\text{[by Theorem 5.5.7(b)]}$$

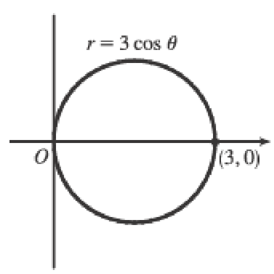
$$=\frac{1}{2}\cdot 2\int_0^{\pi/2}\left[16+9\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta\qquad\text{[by Theorem 5.5.7(a)]}$$

$$=\int_0^{\pi/2}\left(\frac{41}{2}-\frac{9}{2}\cos 2\theta\right)d\theta=\left[\frac{41}{2}\theta-\frac{9}{4}\sin 2\theta\right]_0^{\pi/2}=\left(\frac{41\pi}{4}-0\right)-(0-0)=\frac{41\pi}{4}$$

8. $r=\sin 2\theta,\ 0\le\theta\le\frac{\pi}{2}$.

$$A=\int_0^{\pi/2}\frac{1}{2}\sin^2 2\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}=\frac{1}{4}\left(\frac{\pi}{2}\right)=\frac{\pi}{8}$$

9. The area above the polar axis is bounded by $r=3\cos\theta$ for $\theta=0$ to $\theta=\pi/2$ [*not* π]. By symmetry,



$$A=2\int_0^{\pi/2}\frac{1}{2}r^2\,d\theta=\int_0^{\pi/2}(3\cos\theta)^2\,d\theta=3^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=9\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta=\frac{9}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}=\frac{9}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]=\frac{9\pi}{4}$$

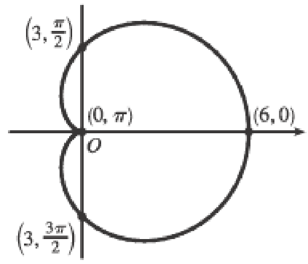
Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi\left(\frac{3}{2}\right)^2=\frac{9\pi}{4}$.

10. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}[3(1+\cos\theta)]^2\,d\theta$

$= \frac{9}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$

$= \frac{9}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right]d\theta$

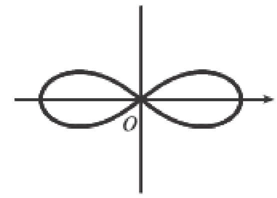
$= \frac{9}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{27}{2}\pi$



11. The curve goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4.

$A = 4\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta = 2\int_0^{\pi/4}(4\cos 2\theta)\,d\theta$

$= 8\int_0^{\pi/4}\cos 2\theta\,d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4$

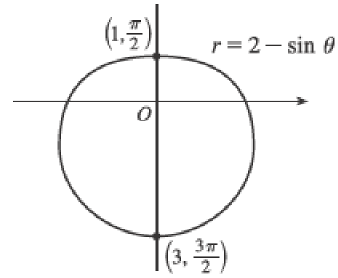


12. To find the area that the curve encloses, we'll double the area to the left of the vertical axis.

$A = 2\int_{\pi/2}^{3\pi/2}\frac{1}{2}(2-\sin\theta)^2\,d\theta = \int_{\pi/2}^{3\pi/2}(4 - 4\sin\theta + \sin^2\theta)\,d\theta$

$= \int_{\pi/2}^{3\pi/2}\left[4 - 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right]d\theta = \int_{\pi/2}^{3\pi/2}\left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$

$= \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{3\pi/2} = \left(\frac{27\pi}{4}\right) - \left(\frac{9\pi}{4}\right) = \frac{9\pi}{2}$



Or: We could have doubled the area to the right of the vertical axis and integrated from $-\pi/2$ to $\pi/2$.

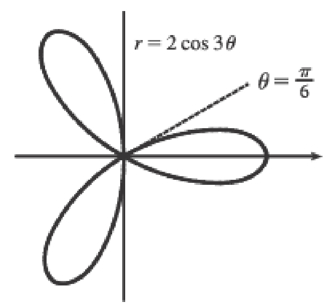
Or: We could have integrated from 0 to 2π [simpler arithmetic].

13. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

$$A = 6\int_0^{\pi/6}\frac{1}{2}(2\cos 3\theta)^2\,d\theta = 12\int_0^{\pi/6}\cos^2 3\theta\,d\theta$$

$$= \frac{12}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta$$

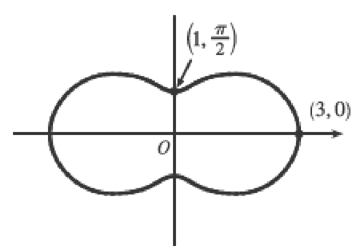
$$= 6\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\frac{\pi}{6}\right) = \pi$$



14. $A = \int_0^{2\pi}\frac{1}{2}(2+\cos 2\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4 + 4\cos 2\theta + \cos^2 2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(4 + 4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)d\theta$

$= \frac{1}{2}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi} = \frac{1}{2}(9\pi) = \frac{9\pi}{2}$



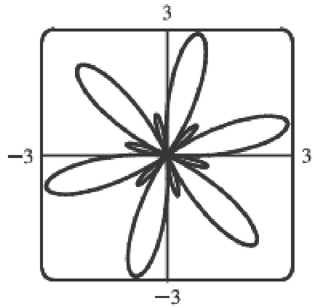
15. $A = \int_0^{2\pi} \frac{1}{2}(1+2\sin 6\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta + 4\sin^2 6\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta + 4\cdot\frac{1}{2}(1-\cos 12\theta)\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta - 2\cos 12\theta)\,d\theta$

$= \frac{1}{2}\left[3\theta - \frac{2}{3}\cos 6\theta - \frac{1}{6}\sin 12\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[\left(6\pi - \frac{2}{3} - 0\right) - \left(0 - \frac{2}{3} - 0\right)\right] = 3\pi$



16. $A = \int_0^{\pi} \frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta = 2\int_0^{\pi/2}\frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta$

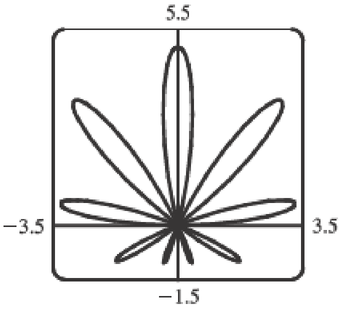
$= \int_0^{\pi/2}(4\sin^2\theta + 12\sin\theta\sin 9\theta + 9\sin^2 9\theta)\,d\theta$

$= \int_0^{\pi/2}\left[2(1-\cos 2\theta) + 12\cdot\frac{1}{2}(\cos(\theta - 9\theta) - \cos(\theta+9\theta)) + \frac{9}{2}(1-\cos 18\theta)\right]d\theta$

[integration by parts could be used for $\int \sin\theta\,\sin 9\theta\,d\theta$]

$= \int_0^{\pi/2}\left(2 - 2\cos 2\theta + 6\cos 8\theta - 6\cos 10\theta + \frac{9}{2} - \frac{9}{2}\cos 18\theta\right)d\theta$

$= \left[\frac{13}{2}\theta - \sin 2\theta + \frac{3}{4}\sin 8\theta - \frac{3}{5}\sin 10\theta - \frac{1}{4}\sin 18\theta\right]_0^{\pi/2} = \frac{13\pi}{4}$

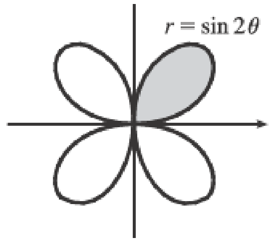


17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

$A = \int_0^{\pi/2}\frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$

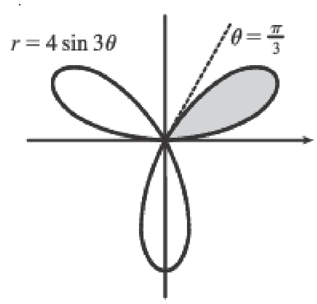
$= \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$



18. $A = \int_0^{\pi/3}\frac{1}{2}(4\sin 3\theta)^2\,d\theta = 8\int_0^{\pi/3}\sin^2 3\theta\,d\theta$

$= 4\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta$

$= 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{4\pi}{3}$



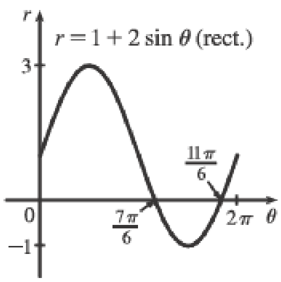
19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{10}$.

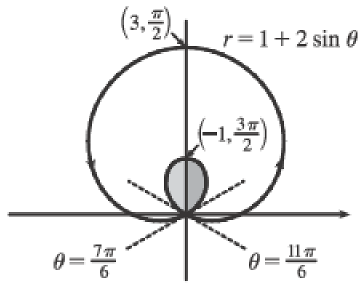
$A = \int_{-\pi/10}^{\pi/10}\frac{1}{2}(3\cos 5\theta)^2\,d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\,d\theta = \frac{9}{2}\int_0^{\pi/10}(1+\cos 10\theta)\,d\theta = \frac{9}{2}\left[\theta + \frac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \frac{9\pi}{20}$

20. $r = 0 \;\Rightarrow\; 2\sin 6\theta = 0 \;\Rightarrow\; 6\theta = 0$ or $\pi \;\Rightarrow\; \theta = 0$ or $\frac{\pi}{6}$.

$A = \int_0^{\pi/6}\frac{1}{2}(2\sin 6\theta)^2\,d\theta = \int_0^{\pi/6}2\sin^2 6\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}(1-\cos 12\theta)\,d\theta = \left[\theta - \frac{1}{12}\sin 12\theta\right]_0^{\pi/6} = \frac{\pi}{6}$

21.





This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

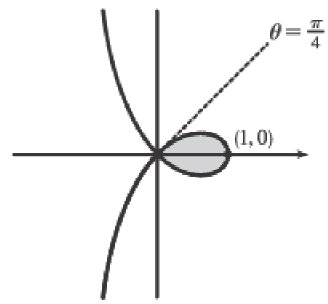
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1+2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta = \int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve

$$r = 0 \quad\Rightarrow\quad 2\cos\theta - \frac{1}{\cos\theta} = 0 \quad\Rightarrow\quad 2\cos^2\theta - 1 = 0 \quad\Rightarrow\quad \cos^2\theta = \frac{1}{2} \quad\Rightarrow$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \tfrac{\pi}{4} \text{ or } \theta = \tfrac{3\pi}{4} \text{ for } 0 \le \theta \le \pi \text{ with } \theta \ne \tfrac{\pi}{2}.$$



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\,d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\,d\theta$$

$$= \int_0^{\pi/4}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 4 + \sec^2\theta\right]d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\,d\theta$$

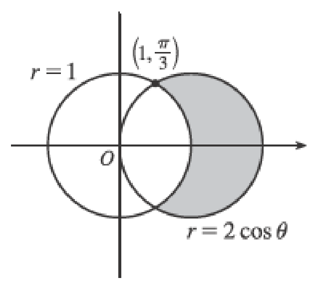
$$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

23. $2\cos\theta = 1 \quad\Rightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left[(2\cos\theta)^2 - 1^2\right]d\theta = \int_0^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\pi/3}\left\{4\left[\tfrac{1}{2}(1+\cos 2\theta)\right] - 1\right\}d\theta = \int_0^{\pi/3}(1 + 2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

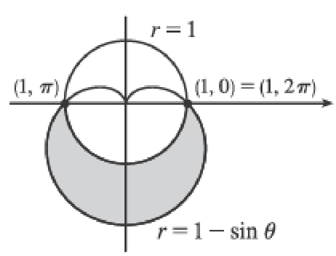


24. $1 - \sin\theta = 1 \quad\Rightarrow\quad \sin\theta = 0 \quad\Rightarrow\quad \theta = 0$ or $\pi \quad\Rightarrow$

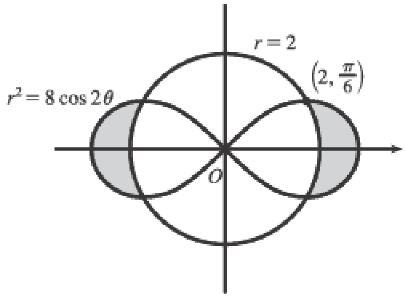
$$A = \int_\pi^{2\pi} \tfrac{1}{2}\left[(1-\sin\theta)^2 - 1\right]d\theta = \tfrac{1}{2}\int_\pi^{2\pi}(\sin^2\theta - 2\sin\theta)\,d\theta$$

$$= \tfrac{1}{4}\int_\pi^{2\pi}(1 - \cos 2\theta - 4\sin\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi}$$

$$= \tfrac{1}{4}\pi + 2$$

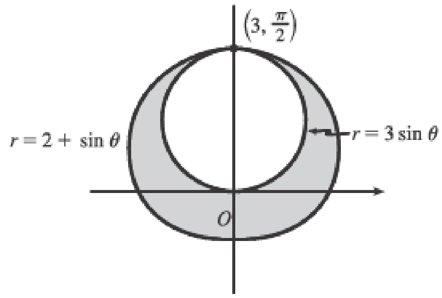


25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \quad \Leftrightarrow \quad 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \quad \Leftrightarrow \quad \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,



$$A = 4\int_0^{\pi/6} \left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right] d\theta = 8\int_0^{\pi/6} (2\cos 2\theta - 1)\, d\theta$$
$$= 8\left[\sin 2\theta - \theta\right]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

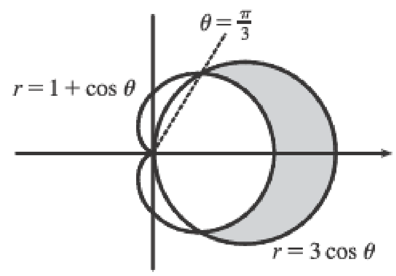
26. To find the shaded area A, we'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi\left(\frac{3}{2}\right)^2$ since $r = 3\sin\theta$ is a circle with radius $\frac{3}{2}$.



$$A_1 = \int_0^{2\pi} \tfrac{1}{2}(2 + \sin\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\, d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left[4 + 4\sin\theta + \tfrac{1}{2}\cdot(1 - \cos 2\theta)\right] d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos 2\theta\right) d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{9}{2}\theta - 4\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}[(9\pi - 4) - (-4)] = \tfrac{9\pi}{2}$$

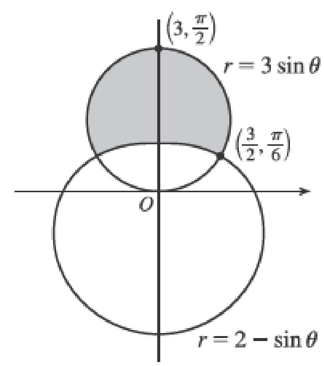
So $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$.

27. $3\cos\theta = 1 + \cos\theta \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.



$$A = 2\int_0^{\pi/3} \tfrac{1}{2}[(3\cos\theta)^2 - (1 + \cos\theta)^2]\, d\theta$$
$$= \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta = \int_0^{\pi/3} [4(1 + \cos 2\theta) - 2\cos\theta - 1]\, d\theta$$
$$= \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

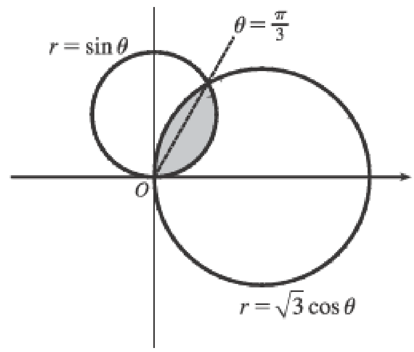
28. $3\sin\theta = 2 - \sin\theta \quad \Rightarrow \quad 4\sin\theta = 2 \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



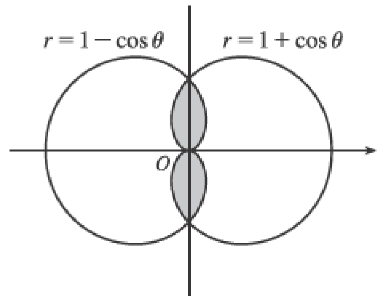
$$A = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}[(3\sin\theta)^2 - (2 - \sin\theta)^2]\, d\theta$$
$$= \int_{\pi/6}^{\pi/2} (9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta]\, d\theta$$
$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta + 4\sin\theta - 4)\, d\theta$$
$$= 4\int_{\pi/6}^{\pi/2} \left[2\cdot\tfrac{1}{2}(1 - \cos 2\theta) + \sin\theta - 1\right] d\theta$$
$$= 4\int_{\pi/6}^{\pi/2} (\sin\theta - \cos 2\theta)\, d\theta = 4\left[-\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$
$$= 4\left[(0 - 0) - \left(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3}$$

29. $\sqrt{3}\cos\theta = \sin\theta \quad\Rightarrow\quad \sqrt{3} = \dfrac{\sin\theta}{\cos\theta} \quad\Rightarrow\quad \tan\theta = \sqrt{3} \quad\Rightarrow\quad \theta = \frac{\pi}{3}.$

$$\begin{aligned}
A &= \int_0^{\pi/3} \tfrac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}\left(\sqrt{3}\cos\theta\right)^2\,d\theta \\
&= \int_0^{\pi/3} \tfrac{1}{2}\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}\cdot 3\cdot\tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{3}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} \\
&= \tfrac{1}{4}\left[\left(\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{4}\right) - 0\right] + \tfrac{3}{4}\left[\left(\tfrac{\pi}{2}+0\right) - \left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right)\right] \\
&= \tfrac{\pi}{12} - \tfrac{\sqrt{3}}{16} + \tfrac{\pi}{8} - \tfrac{3\sqrt{3}}{16} = \tfrac{5\pi}{24} - \tfrac{\sqrt{3}}{4}
\end{aligned}$$

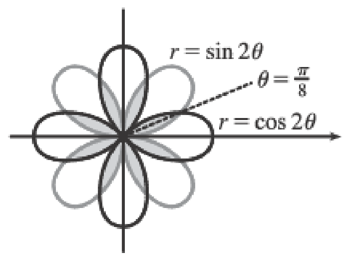


30.
$$\begin{aligned}
A &= 4\int_0^{\pi/2} \tfrac{1}{2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta \\
&= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\
&= 2\int_0^{\pi/2}\left(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\pi/2}(3 - 4\cos\theta + \cos 2\theta)\,d\theta \\
&= \left[3\theta - 4\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{3\pi}{2} - 4
\end{aligned}$$



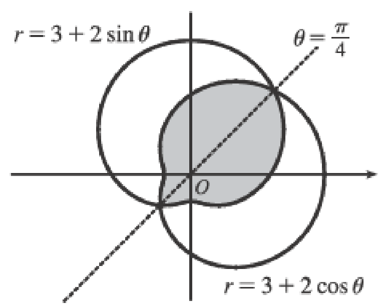
31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$
$\theta = \frac{\pi}{8} \quad\Rightarrow$

$$\begin{aligned}
A &= 8\cdot 2\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8} \tfrac{1}{2}(1-\cos 4\theta)\,d\theta \\
&= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1
\end{aligned}$$



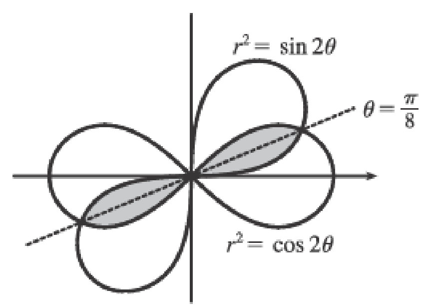
32. $3 + 2\cos\theta = 3 + 2\sin\theta \quad\Rightarrow\quad \cos\theta = \sin\theta \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$$\begin{aligned}
A &= 2\int_{\pi/4}^{5\pi/4} \tfrac{1}{2}(3+2\cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(9 + 12\cos\theta + 4\cos^2\theta)\,d\theta \\
&= \int_{\pi/4}^{5\pi/4}\left[9 + 12\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\
&= \int_{\pi/4}^{5\pi/4}(11 + 12\cos\theta + 2\cos 2\theta)\,d\theta = \left[11\theta + 12\sin\theta + \sin 2\theta\right]_{\pi/4}^{5\pi/4} \\
&= \left(\tfrac{55\pi}{4} - 6\sqrt{2} + 1\right) - \left(\tfrac{11\pi}{4} + 6\sqrt{2} + 1\right) = 11\pi - 12\sqrt{2}
\end{aligned}$$



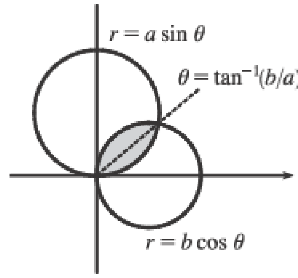
33. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow\quad \theta = \frac{\pi}{8}$

$$\begin{aligned}
A &= 4\int_0^{\pi/8} \tfrac{1}{2}\sin 2\theta\,d\theta \quad [\text{since } r^2 = \sin 2\theta] \\
&= \int_0^{\pi/8} 2\sin 2\theta\,d\theta = \left[-\cos 2\theta\right]_0^{\pi/8} \\
&= -\tfrac{1}{2}\sqrt{2} - (-1) = 1 - \tfrac{1}{2}\sqrt{2}
\end{aligned}$$



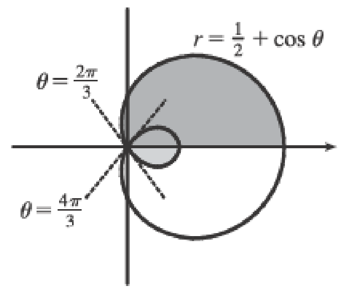
34. Let $\alpha = \tan^{-1}(b/a)$. Then

$$A = \int_0^{\alpha} \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2} \tfrac{1}{2}(b\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}a^2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\alpha} + \tfrac{1}{4}b^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$
$$= \tfrac{1}{4}\alpha(a^2 - b^2) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}(a^2 + b^2)(\sin\alpha\,\cos\alpha)$$
$$= \tfrac{1}{4}(a^2 - b^2)\,\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab$$

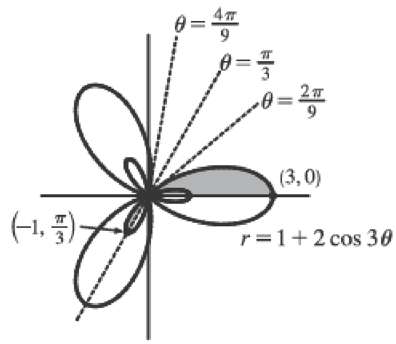


35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.

$$A = 2\left[\int_0^{2\pi/3} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2\,d\theta - \int_{2\pi/3}^{\pi} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2\,d\theta\right]$$
$$= \int_0^{2\pi/3} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta - \int_{2\pi/3}^{\pi} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta$$
$$= \int_0^{2\pi/3} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]d\theta - \int_{2\pi/3}^{\pi} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]d\theta$$
$$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right)$$
$$= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)$$



36. $r = 0 \Rightarrow 1 + 2\cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \le 3\theta \le 2\pi$] $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



$A = 2\left[\int_0^{2\pi/9} \frac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta\right]$

Now

$$\begin{aligned} r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4 \cdot \tfrac{1}{2}(1 + \cos 6\theta) \\ &= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta \end{aligned}$$

and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

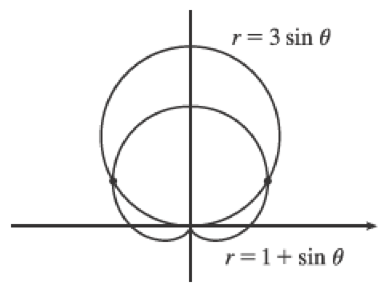
$$\begin{aligned} A &= \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3} \\ &= \left[\left(\tfrac{2\pi}{3} + \tfrac{4}{3} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{3} \cdot \tfrac{-\sqrt{3}}{2}\right) - 0\right] - \left[(\pi + 0 + 0) - \left(\tfrac{2\pi}{3} + \tfrac{4}{3} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{3} \cdot \tfrac{-\sqrt{3}}{2}\right)\right] \\ &= \tfrac{4\pi}{3} + \tfrac{4}{3}\sqrt{3} - \tfrac{1}{3}\sqrt{3} - \pi = \tfrac{\pi}{3} + \sqrt{3} \end{aligned}$$

37. The pole is a point of intersection.

$1 + \sin\theta = 3\sin\theta \Rightarrow 1 = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

The other two points of intersection are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

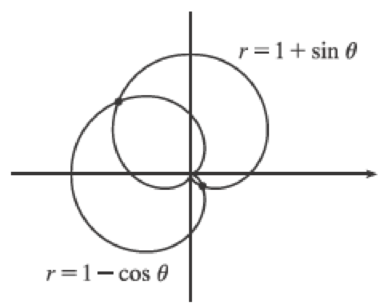


38. The pole is a point of intersection.

$1 - \cos\theta = 1 + \sin\theta \Rightarrow -\cos\theta = \sin\theta \Rightarrow -1 = \tan\theta \Rightarrow$

$\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

The other two points of intersection are $\left(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$ and $\left(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$.



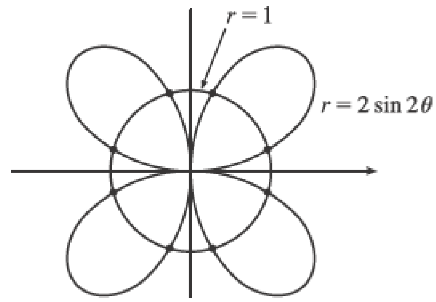
39. $2\sin 2\theta = 1 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

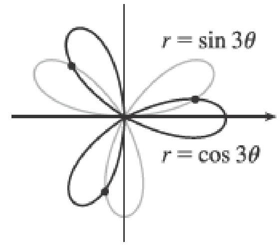
$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

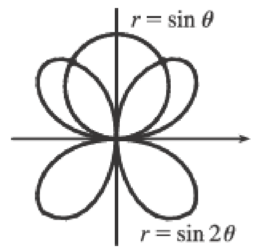
[There are many ways to describe these points.]



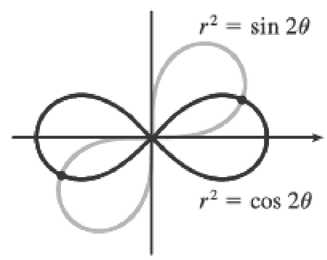
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \quad\Rightarrow\quad \tan 3\theta = 1 \quad\Rightarrow$ $3\theta = \frac{\pi}{4} + n\pi$ [n any integer] $\quad\Rightarrow\quad \theta = \frac{\pi}{12} + \frac{\pi}{3}n \quad\Rightarrow$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



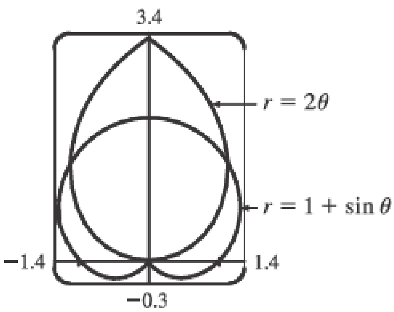
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\,\cos\theta \quad\Leftrightarrow$ $\sin\theta\,(1 - 2\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \quad\Rightarrow$ $\theta = 0, \pi, \frac{\pi}{3}$, or $-\frac{\pi}{3} \quad\Rightarrow\quad$ the other intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ [by symmetry].

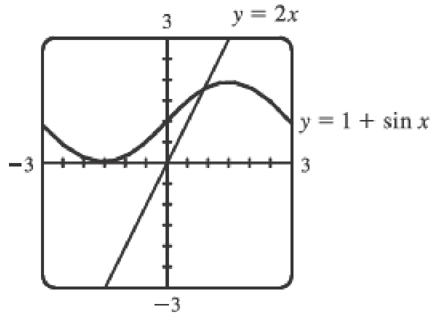


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \quad\Rightarrow$ $\tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\quad\Rightarrow\quad \theta = \frac{\pi}{8} + n\pi \quad\Rightarrow\quad \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.

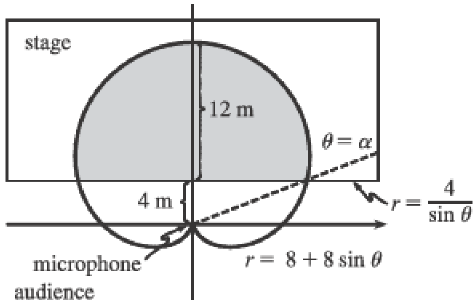




From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^{\alpha} \tfrac{1}{2}(2\theta)^2\,d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\,d\theta = \int_0^{\alpha} 4\theta^2\,d\theta + \int_{\alpha}^{\pi/2}\left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2} + \tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44. 

We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \quad \Leftrightarrow$ $r\sin\theta = 4 \quad \Rightarrow \quad r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \quad \Rightarrow$ $8\sin\theta + 8\sin^2\theta = 4 \quad \Rightarrow \quad 2\sin^2\theta + 2\sin\theta - 1 = 0 \quad \Rightarrow$

$$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1+\sqrt{3}}{2} \quad \text{[the other value is less than } -1\text{]} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

This angle is about $21.5°$ and is denoted by α in the figure.

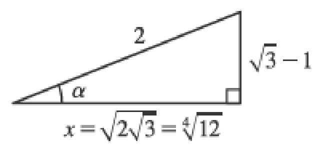
$$A = 2\int_\alpha^{\pi/2} \tfrac{1}{2}(8+8\sin\theta)^2\,d\theta - 2\int_\alpha^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2\,d\theta = 64\int_\alpha^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\,d\theta - 16\int_\alpha^{\pi/2}\csc^2\theta\,d\theta$$

$$= 64\int_\alpha^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)\,d\theta + 16\int_\alpha^{\pi/2}(-\csc^2\theta)\,d\theta = 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_\alpha^{\pi/2} + 16\left[\cot\theta\right]_\alpha^{\pi/2}$$

$$= 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_\alpha^{\pi/a} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)]$$

$$= 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

From the figure, $x^2 + \left(\sqrt{3}-1\right)^2 = 2^2 \quad \Rightarrow \quad x^2 = 4 - \left(3 - 2\sqrt{3} + 1\right) \quad \Rightarrow$ $x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\,\cos\alpha$, we continue:



$$A = 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\approx 204.16\ \text{m}^2$$

45. $$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{\pi/3}\sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\,d\theta = \int_0^{\pi/3}\sqrt{9(\sin^2\theta + \cos^2\theta)}\,d\theta$$

$$= 3\int_0^{\pi/3} d\theta = 3\left[\,\theta\,\right]_0^{\pi/3} = 3\left(\tfrac{\pi}{3}\right) = \pi.$$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right) = 3\pi$, and since $\theta = 0$ to $\pi = \frac{\pi}{3}$ traces out $\frac{1}{3}$ of the circle (from $\theta = 0$ to $\theta = \pi$), $\frac{1}{3}(3\pi) = \pi$.

46. $$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi}\sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\,d\theta = \int_0^{2\pi}\sqrt{e^{4\theta} + 4e^{4\theta}}\,d\theta = \int_0^{2\pi}\sqrt{5e^{4\theta}}\,d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\,d\theta = \tfrac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi} = \tfrac{\sqrt{5}}{2}(e^{4\pi} - 1)$$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}] = \tfrac{8}{3}[(\pi^2+1)^{3/2} - 1]$$

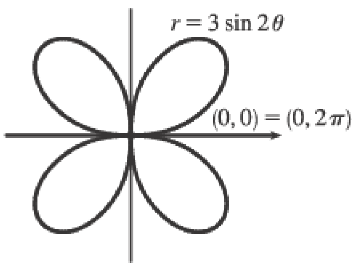
48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^2+1}\, d\theta \overset{21}{=} \left[\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_0^2$

$= \pi\sqrt{4\pi^2+1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$

49. The curve $r = 3\sin 2\theta$ is completely traced with $0 \le \theta \le 2\pi$.

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (3\sin 2\theta)^2 + (6\cos 2\theta)^2 \quad\Rightarrow$

$L = \int_0^{2\pi} \sqrt{9\sin^2 2\theta + 36\cos^2 2\theta}\, d\theta \approx 29.0653$



50. The curve $r = 4\sin 3\theta$ is completely traced with $0 \le \theta \le \pi$.

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (4\sin 3\theta)^2 + (12\cos 3\theta)^2 \quad\Rightarrow$

$L = \int_0^{\pi} \sqrt{16\sin^2 3\theta + 144\cos^2 3\theta}\, d\theta \approx 26.7298$