1. Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \cos x(1+\sin^2 x)\,dx = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$.

2. $$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}(-du) \qquad \left[\begin{array}{c} u = \cos x \\ du = -\sin x\,dx \end{array}\right]$$
$$= \int \left(u - \tfrac{1}{u}\right)du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

3. $$\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right)dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$$

4. $$\int \tan^3\theta\,d\theta = \int (\sec^2\theta - 1)\tan\theta\,d\theta = \int \tan\theta\,\sec^2\theta\,d\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta$$
$$= \int u\,du + \int \frac{dv}{v} \quad \left[\begin{array}{cc} u = \tan\theta, & v = \cos\theta, \\ du = \sec^2\theta\,d\theta & dv = -\sin\theta\,d\theta \end{array}\right]$$
$$= \tfrac{1}{2}u^2 + \ln|v| + C = \tfrac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$$

5. $$\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \quad \left[\begin{array}{c} u = t-3, \\ du = dt \end{array}\right] \quad = \int_{-3}^{-1} \left(\frac{2}{u} + \frac{6}{u^2}\right)du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$$
$$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$$

6. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow \displaystyle\int \frac{x\,dx}{\sqrt{3-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}} + C$.

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \Rightarrow \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

8. $$\int x\csc x\cot x\,dx \quad \left[\begin{array}{cc} u = x, & dv = \csc x\cot x\,dx, \\ du = dx & v = -\csc x \end{array}\right] \quad = -x\csc x - \int(-\csc x)\,dx = -x\csc x + \ln|\csc x - \cot x| + C$$

9. $$\int_1^3 r^4 \ln r\,dr \quad \left[\begin{array}{cc} u = \ln r, & dv = r^4\,dr, \\ du = \frac{dr}{r} & v = \frac{1}{5}r^5 \end{array}\right] \quad = \left[\tfrac{1}{5}r^5\ln r\right]_1^3 - \int_1^3 \tfrac{1}{5}r^4\,dr = \tfrac{243}{5}\ln 3 - 0 - \left[\tfrac{1}{25}r^5\right]_1^3$$
$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

10. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \Rightarrow x - 1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right)dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$
$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

11. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \qquad [u = x-2,\, du = dx]$

$\displaystyle = \tfrac{1}{2}\ln(u^2+1) + \tan^{-1} u + C = \tfrac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$

12. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad \begin{bmatrix} u = x^2, \\ du = 2x\,dx \end{bmatrix} \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$\displaystyle = \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}\,(v^2+1)} \quad \begin{bmatrix} u + \frac{1}{2} = \frac{\sqrt{3}}{2}v, \\ du = \frac{\sqrt{3}}{2}\,dv \end{bmatrix} \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$

$\displaystyle = \tfrac{1}{\sqrt{3}}\tan^{-1} v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\left(x^2+\tfrac{1}{2}\right)\right) + C$

13. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta\,\sin^2\theta\,\sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$= -\int u^5(1-u^2)\,du \quad \begin{bmatrix} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{bmatrix}$

$= \int (u^7 - u^5)\,du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$

Another solution:

$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2\cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

$= \int u^3(1-u^2)^2\,du \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix} \quad = \int u^3(1-2u^2+u^4)\,du$

$= \int (u^3 - 2u^5 + u^7)\,du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$

14. Let $u = 1 + x^2$, so that $du = 2x\,dx$. Then

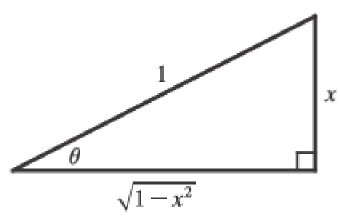
$\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{\sqrt{1+x^2}}\,(x\,dx) = \int \frac{u-1}{u^{1/2}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$

$= \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C \quad \left[\text{or } \frac{1}{3}(x^2-2)\sqrt{1+x^2} + C\right]$

15. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$,

so

$\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C.$



16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$

$= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right] = \frac{\pi}{8} - \frac{1}{4}$

17. $\int x \sin^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{bmatrix}$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sin x\cos x - \int \left(\tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x\right) dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sin x\cos x - \tfrac{1}{4}x^2 + \tfrac{1}{4}\sin^2 x + C = \tfrac{1}{4}x^2 - \tfrac{1}{2}x\sin x\cos x + \tfrac{1}{4}\sin^2 x + C$$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C \quad$ [where $s = \sin x$, $ds = \cos x\,dx$].

A slightly different method is to write $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

18. Let $u = e^{2t}$, $du = 2e^{2t}\,dt$. Then $\displaystyle\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(e^{2t}) + C.$

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

20. Since e^2 is a constant, $\int e^2\,dx = e^2 x + C$.

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with

$u = \arctan t$, $dv = 2t\,dt \quad\Rightarrow\quad du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int \left(1 - \frac{1}{1+t^2}\right) dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

22. Let $u = 1 + (\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \quad\Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2 = \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}.$$

24. Let $u = \ln(x^2-1)$, $dv = dx \quad\Leftrightarrow\quad du = \dfrac{2x}{x^2-1}$, $v = x$. Then

$$\int \ln(x^2-1)\,dx = x\ln(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx = x\ln(x^2-1) - \int \left[2 + \frac{2}{(x-1)(x+1)}\right] dx$$

$$= x\ln(x^2-1) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right] dx = x\ln(x^2-1) - 2x - \ln|x-1| + \ln|x+1| + C$$

25. $\dfrac{3x^2-2}{x^2-2x-8}=3+\dfrac{6x+22}{(x-4)(x+2)}=3+\dfrac{A}{x-4}+\dfrac{B}{x+2} \quad\Rightarrow\quad 6x+22=A(x+2)+B(x-4)$. Setting $x=4$ gives $46=6A$, so $A=\frac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B=-\frac{5}{3}$. Now

$$\int\frac{3x^2-2}{x^2-2x-8}\,dx=\int\left(3+\frac{23/3}{x-4}-\frac{5/3}{x+2}\right)dx=3x+\tfrac{23}{3}\ln|x-4|-\tfrac{5}{3}\ln|x+2|+C.$$

26. $\displaystyle\int\frac{3x^2-2}{x^3-2x-8}\,dx=\int\frac{du}{u}\quad\begin{bmatrix}u=x^3-2x-8,\\ du=(3x^2-2)\,dx\end{bmatrix}\quad=\ln|u|+C=\ln\left|x^3-2x-8\right|+C$

27. Let $u=1+e^x$, so that $du=e^x\,dx=(u-1)\,dx$. Then $\displaystyle\int\frac{1}{1+e^x}\,dx=\int\frac{1}{u}\cdot\frac{du}{u-1}=\int\frac{1}{u(u-1)}\,du=I$. Now

$\dfrac{1}{u(u-1)}=\dfrac{A}{u}+\dfrac{B}{u-1}\quad\Rightarrow\quad 1=A(u-1)+Bu$. Set $u=1$ to get $1=B$. Set $u=0$ to get $1=-A$, so $A=-1$.

Thus, $\displaystyle I=\int\left(\frac{-1}{u}+\frac{1}{u-1}\right)du=-\ln|u|+\ln|u-1|+C=-\ln(1+e^x)+\ln e^x+C=x-\ln(1+e^x)+C.$

Another method: Multiply numerator and denominator by e^{-x} and let $u=e^{-x}+1$. This gives the answer in the form $-\ln(e^{-x}+1)+C$.

28. $\int\sin\sqrt{at}\,dt=\int\sin u\cdot\frac{2}{a}u\,du\quad[u=\sqrt{at},\ u^2=at,\ 2u\,du=a\,dt]\quad=\frac{2}{a}\int u\sin u\,du$

$$=\tfrac{2}{a}[-u\cos u+\sin u]+C\quad\text{[integration by parts]}\quad=-\tfrac{2}{a}\sqrt{at}\,\cos\sqrt{at}+\tfrac{2}{a}\,\sin\sqrt{at}+C$$

$$=-2\sqrt{\tfrac{t}{a}}\,\cos\sqrt{at}+\tfrac{2}{a}\,\sin\sqrt{at}+C$$

29. $\displaystyle\int_0^5\frac{3w-1}{w+2}\,dw=\int_0^5\left(3-\frac{7}{w+2}\right)dw=\Big[3w-7\ln|w+2|\Big]_0^5=15-7\ln 7+7\ln 2$

$$=15+7(\ln 2-\ln 7)=15+7\ln\tfrac{2}{7}$$

30. $x^2-4x<0$ on $[0,4]$, so

$$\int_{-2}^2\left|x^2-4x\right|\,dx=\int_{-2}^0(x^2-4x)\,dx+\int_0^2(4x-x^2)\,dx=\left[\tfrac{1}{3}x^3-2x^2\right]_{-2}^0+\left[2x^2-\tfrac{1}{3}x^3\right]_0^2$$

$$=0-\left(-\tfrac{8}{3}-8\right)+\left(8-\tfrac{8}{3}\right)-0=16$$

31. As in Example 5,

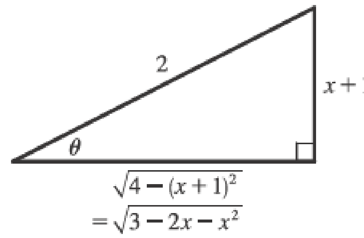
$$\int\sqrt{\frac{1+x}{1-x}}\,dx=\int\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx=\int\frac{1+x}{\sqrt{1-x^2}}\,dx=\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{x\,dx}{\sqrt{1-x^2}}=\sin^{-1}x-\sqrt{1-x^2}+C.$$

Another method: Substitute $u=\sqrt{(1+x)/(1-x)}$.

32. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \quad \begin{bmatrix} u=\sqrt{2x-1},\, 2x+3=u^2+4, \\ u^2=2x-1,\, u\,du=dx \end{bmatrix} \quad = \int \left(1-\frac{4}{u^2+4}\right) du$

$\displaystyle = u - 4\cdot\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\tfrac{1}{2}\sqrt{2x-1}\right) + C$

33. $3-2x-x^2 = -(x^2+2x+1)+4 = 4-(x+1)^2$. Let $x+1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$$\begin{aligned}
\int \sqrt{3-2x-x^2}\,dx &= \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta \\
&= 4\int \cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)\,d\theta \\
&= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2} + C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C
\end{aligned}$$

34. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)}\cdot\frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$

$\displaystyle = \int_{3/\sqrt{2}}^{4} \frac{1}{u}\,du \quad \begin{bmatrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\,dx \end{bmatrix}$

$\displaystyle = \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$

35. Because $f(x) = x^8\sin x$ is the product of an even function and an odd function, it is odd.

Therefore, $\int_{-1}^{1} x^8 \sin x\,dx = 0$ [by (5.5.7)(b)].

36. $\sin 4x\cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 7.2.2(a), so

$\int \sin 4x\cos 3x\,dx = \frac{1}{2}\int(\sin x + \sin 7x)\,dx = \frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C = -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.$

37. $\int_0^{\pi/4} \cos^2\theta\,\tan^2\theta\,d\theta = \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\frac{\pi}{8}-\frac{1}{4}\right) - (0-0) = \frac{\pi}{8} - \frac{1}{4}$

38. $\int_0^{\pi/4} \tan^5\theta\,\sec^3\theta\,d\theta = \int_0^{\pi/4}\left(\tan^2\theta\right)^2\sec^2\theta\cdot\sec\theta\,\tan\theta\,d\theta = \int_1^{\sqrt{2}}(u^2-1)^2u^2\,du \quad \begin{bmatrix} u=\sec\theta, \\ du = \sec\theta\tan\theta\,d\theta \end{bmatrix}$

$= \int_1^{\sqrt{2}}(u^6 - 2u^4 + u^2)\,du = \left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_1^{\sqrt{2}}$

$= \left(\frac{8}{7}\sqrt{2} - \frac{8}{5}\sqrt{2} + \frac{2}{3}\sqrt{2}\right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) = \frac{22}{105}\sqrt{2} - \frac{8}{105} = \frac{2}{105}\left(11\sqrt{2} - 4\right)$

39. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta = \int \frac{1}{u^2 - u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\displaystyle\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \quad\Rightarrow\quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int\left(\frac{-1}{u} + \frac{1}{u-1}\right)du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1 - \cos\theta| + C$].

40. $4y^2 - 4y - 3 = (2y-1)^2 - 2^2$, so let $u = 2y - 1 \quad\Rightarrow\quad du = 2\,dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y-1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}}$$

$$= \tfrac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\,\right| \qquad \text{[by Formula 20 in the table in this section]}$$

$$= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\right| + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \quad\Rightarrow\quad du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$

$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. Let $u = \tan^{-1}x$, $dv = \dfrac{1}{x^2}\,dx \quad\Rightarrow\quad du = \dfrac{1}{1+x^2}\,dx$, $v = -\dfrac{1}{x}$. Then

$$I = \int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{1}{x}\tan^{-1}x - \int\left(-\frac{1}{x(1+x^2)}\right)dx = -\frac{1}{x}\tan^{-1}x + \int\left(\frac{A}{x} + \frac{Bx + C}{1+x^2}\right)dx$$

$\displaystyle\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad\Rightarrow\quad 1 = A(1+x^2) + (Bx+C)x \quad\Rightarrow\quad 1 = (A+B)x^2 + Cx + A$, so $C = 0$, $A = 1$,

and $A + B = 0 \quad\Rightarrow\quad B = -1$. Thus,

$$I = -\frac{1}{x}\tan^{-1}x + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C$$

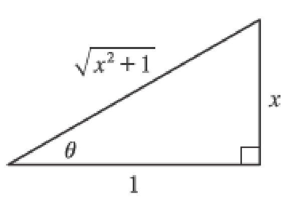
$$= -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

Or: Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\,\sec^2\theta\,d\theta = \int \theta\csc^2\theta\,d\theta = I$. Now use parts

with $u = \theta$, $dv = \csc^2\theta\,d\theta \quad\Rightarrow\quad du = d\theta$, $v = -\cot\theta$. Thus,

$$I = -\theta\cot\theta - \int(-\cot\theta)\,d\theta = -\theta\cot\theta + \ln|\sin\theta| + C$$

$$= -\tan^{-1}x\cdot\frac{1}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$



43. Let $u = 1 + e^x$, so that $du = e^x\,dx$. Then $\int e^x\sqrt{1+e^x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then

$\int e^x\sqrt{1+e^x}\,dx = \int u \cdot 2u\,du = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}\,du$, so

$$\int \sqrt{1+e^x}\,dx = \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right)du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x} - 1\right) - \ln\left(\sqrt{1+e^x} + 1\right) + C$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \quad\Rightarrow\quad I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C$.

46. $$\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \frac{1+2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1+2\sin x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$
$$= \sec^2 x + 2\sec x\,\tan x + \tan^2 x = \sec^2 x + 2\sec x\,\tan x + \sec^2 x - 1 = 2\sec^2 x + 2\sec x\,\tan x - 1$$

Thus, $$\int \frac{1+\sin x}{1-\sin x}\,dx = \int (2\sec^2 x + 2\sec x\,\tan x - 1)\,dx = 2\tan x + 2\sec x - x + C$$

47. Let $u = x - 1$, so that $du = dx$. Then

$$\int x^3(x-1)^{-4}\,dx = \int (u+1)^3u^{-4}\,du = \int (u^3+3u^2+3u+1)u^{-4}\,du = \int (u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du$$
$$= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C$$

48. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow\quad \displaystyle\int \frac{x\,dx}{x^4-a^4} = \int \frac{\frac{1}{2}\,du}{u^2-(a^2)^2} = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right| + C = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C.$

49. Let $u = \sqrt{4x+1} \quad\Rightarrow\quad u^2 = 4x+1 \quad\Rightarrow\quad 2u\,du = 4\,dx \quad\Rightarrow\quad dx = \frac{1}{2}u\,du$. So

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]}$$
$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$.

Now $\dfrac{1}{(u^2-1)^2} = \dfrac{1}{(u+1)^2(u-1)^2} = \dfrac{A}{u+1} + \dfrac{B}{(u+1)^2} + \dfrac{C}{u-1} + \dfrac{D}{(u-1)^2} \quad \Rightarrow$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \quad \Rightarrow \quad D = \frac{1}{4}$, $u = -1 \quad \Rightarrow \quad B = \frac{1}{4}$.

Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D \quad \Rightarrow$

$1 = A + \frac{1}{4} - C + \frac{1}{4} \quad \Rightarrow \quad \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right] du \\
&= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right] du \\
&= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\
&= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C
\end{aligned}$$