1. $P(7,10,4)$ lies on the parametric surface $\mathbf{r}(u,v) = \langle 2u+3v, 1+5u-v, 2+u+v\rangle$ if and only if there are values for u and v where $2u+3v=7$, $1+5u-v=10$, and $2+u+v=4$. But solving the first two equations simultaneously gives $u=2$, $v=1$ and these values do not satisfy the third equation, so P does not lie on the surface.

   $Q(5,22,5)$ lies on the surface if $2u+3v=5$, $1+5u-v=22$, and $2+u+v=5$ for some values of u and v. Solving the first two equations simultaneously gives $u=4$, $v=-1$ and these values satisfy the third equation, so Q lies on the surface.

2. $P(3,-1,5)$ lies on the parametric surface $\mathbf{r}(u,v) = \langle u+v, u^2-v, u+v^2\rangle$ if and only if there are values for u and v where $u+v=3$, $u^2-v=-1$, and $u+v^2=5$. From the first equation we have $v=3-u$ and substituting into the second equation gives $u^2-3+u=-1 \quad\Leftrightarrow\quad u^2+u-2=0 \quad\Leftrightarrow\quad (u+2)(u-1)=0$, so $u=-2 \quad\Rightarrow\quad v=5$ or $u=1 \quad\Rightarrow$ $v=2$. The third equation is satisified by $u=1$, $v=2$ so P does lie on the surface.

   $Q(-1,3,4)$ lies on $\mathbf{r}(u,v)$ if and only if $u+v=-1$, $u^2-v=3$, and $u+v^2=4$, but substituting the first equation into the second gives $u=-2$, $v=1$ or $u=1$, $v=-2$, and neither of these pairs satisfies the third equation. Thus, Q does not lie on the surface.

3. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (3-v)\,\mathbf{j} + (1+4u+5v)\,\mathbf{k} = \langle 0,3,1\rangle + u\,\langle 1,0,4\rangle + v\,\langle 1,-1,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0,3,1)$ and containing vectors $\mathbf{a} = \langle 1,0,4\rangle$ and $\mathbf{b} = \langle 1,-1,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x-0)-(y-3)-(z-1)=0$ or $4x-y-z=-4$.

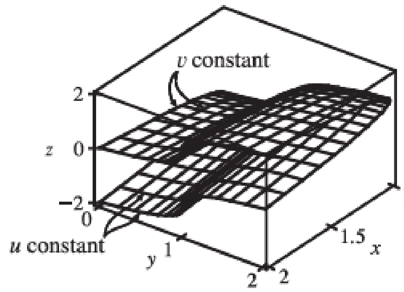
4. $\mathbf{r}(u,v) = 2\sin u\,\mathbf{i} + 3\cos u\,\mathbf{j} + v\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x=2\sin u$, $y=3\cos u$, $z=v$. For any point (x,y,z) on the surface, we have $(x/2)^2+(y/3)^2 = \sin^2 u + \cos^2 u = 1$, so cross-sections parallel to the yz-plane are all ellipses. Since $z=v$ with $0\le v\le 2$, the surface is the portion of the elliptical cylinder $x^2/4+y^2/9=1$ for $0\le z\le 2$.

5. $\mathbf{r}(s,t) = \langle s, t, t^2-s^2\rangle$, so the corresponding parametric equations for the surface are $x=s$, $y=t$, $z=t^2-s^2$. For any point (x,y,z) on the surface, we have $z=y^2-x^2$. With no restrictions on the parameters, the surface is $z=y^2-x^2$, which we recognize as a hyperbolic paraboloid.

6. $\mathbf{r}(s,t) = s\sin 2t\,\mathbf{i} + s^2\,\mathbf{j} + s\cos 2t\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x=s\sin 2t$, $y=s^2$, $z=s\cos 2t$. For any point (x,y,z) on the surface, we have $x^2+z^2 = s^2\sin^2 2t + s^2\cos^2 2t = s^2 = y$. Since no restrictions are placed on the parameters, the surface is $y=x^2+z^2$, which we recognize as a circular paraboloid whose axis is the y-axis.

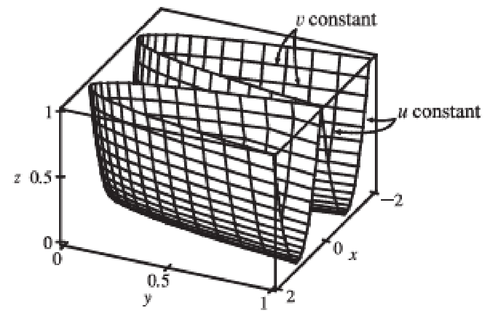
7. $\mathbf{r}(u,v) = \langle u^2+1, v^3+1, u+v \rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$, $-1 \le u \le 1$, $-1 \le v \le 1$. In Maple, the surface can be graphed by entering `plot3d([u^2+1,v^3+1,u+v],u=-1..1,v=-1..1);`. In Mathematica we use the `ParametricPlot3D` command. If we keep u constant at u_0, $x = u_0^2 + 1$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^3 + 1$, a constant, so these grid curves are the curves parallel to the xz-plane.



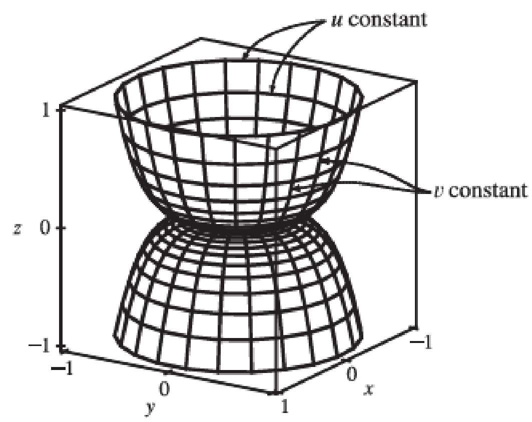
8. $\mathbf{r}(u,v) = \langle u+v, u^2, v^2 \rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u + v$, $y = u^2$, $z = v^2$, $-1 \le u \le 1$, $-1 \le v \le 1$. If $u = u_0$ is constant, $y = u_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane. If $v = v_0$ is constant, $z = v_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



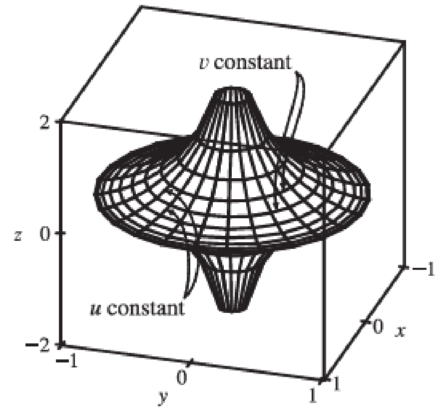
9. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^5 \rangle$.

The surface has parametric equations $x = u\cos v$, $y = u\sin v$, $z = u^5$, $-1 \le u \le 1$, $0 \le v \le 2\pi$. Note that if $u = u_0$ is constant then $z = u_0^5$ is constant and $x = u_0 \cos v$, $y = u_0 \sin v$ describe a circle in x, y of radius $|u_0|$, so the corresponding grid curves are circles parallel to the xy-plane. If $v = v_0$, a constant, the parametric equations become $x = u\cos v_0$, $y = u\sin v_0$, $z = u^5$. Then $y = (\tan v_0)x$, so these are the grid curves we see that lie in vertical planes $y = kx$ through the z-axis.



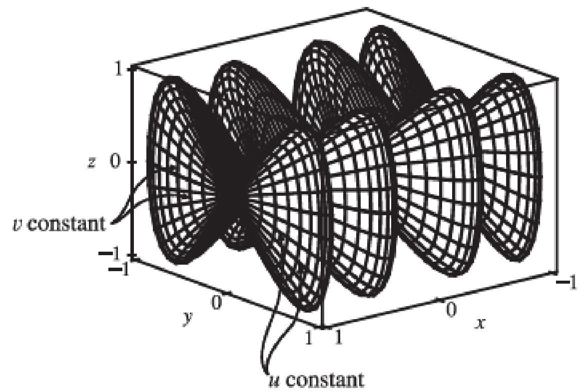
10. $\mathbf{r}(u,v) = \langle \cos u \, \sin v, \sin u \, \sin v, \cos v + \ln \tan(v/2) \rangle$.

The surface has parametric equations $x = \cos u \, \sin v$, $y = \sin u \, \sin v$, $z = \cos v + \ln \tan(v/2)$, $0 \le u \le 2\pi$, $0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \, \sin v_0$, $y = \sin u \, \sin v_0$, $z = \cos v_0 + \ln \tan(v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln \tan(v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \, \sin v$, $y = \sin u_0 \, \sin v$, $z = \cos v + \ln \tan(v/2)$. These curves lie in vertical planes that contain the z-axis.



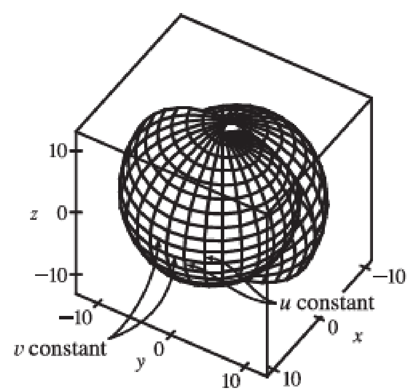
11. $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v, \quad 0 \le u \le 2\pi, \quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}$.

Note that if $v = v_0$ is constant, then $x = \sin v_0$ is constant, so the corresponding grid curves must be parallel to the yz-plane. These are the vertically oriented grid curves we see, each shaped like a "figure-eight." When $u = u_0$ is held constant, the parametric equations become $x = \sin v$, $y = \cos u_0 \sin 4v$, $z = \sin 2u_0 \sin 4v$. Since z is a constant multiple of y, the corresponding grid curves are the curves contained in planes $z = ky$ that pass through the x-axis.



12. $x = u \sin u \, \cos v$, $y = u \cos u \, \cos v$, $\quad z = u \sin v$.

We graph the portion of the surface with parametric domain $0 \le u \le 4\pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \, \cos v_0$, $y = u \cos u \, \cos v_0$, $z = u \sin v_0$. The equations for x and y show that the projections onto the xy-plane give a spiral shape, so the corresponding grid curves are the almost-horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \, \cos v$, $y = u_0 \cos u_0 \, \cos v$, $z = u_0 \sin v$.



13. $\mathbf{r}(u,v) = u \cos v \, \mathbf{i} + u \sin v \, \mathbf{j} + v \, \mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + \sin u\,\mathbf{k}$. The corresponding parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = \sin u$, $-\pi \le u \le \pi$. If $u = u_0$ is held constant, then $x = u_0\cos v$, $y = u_0\sin v$ so each grid curve is a circle of radius $|u_0|$ in the horizontal plane $z = \sin u_0$. If $v = v_0$ is constant, then $x = u\cos v_0$, $y = u\sin v_0 \quad\Rightarrow\quad y = (\tan v_0)x$, so the grid curves lie in vertical planes $y = kx$ through the z-axis. In fact, since x and y are constant multiples of u and $z = \sin u$, each of these traces is a sine wave. The surface is graph I.

15. $\mathbf{r}(u,v) = \sin v\,\mathbf{i} + \cos u\,\sin 2v\,\mathbf{j} + \sin u\,\sin 2v\,\mathbf{k}$. Parametric equations for the surface are $x = \sin v$, $y = \cos u\,\sin 2v$, $z = \sin u\,\sin 2v$. If $v = v_0$ is fixed, then $x = \sin v_0$ is constant, and $y = (\sin 2v_0)\cos u$ and $z = (\sin 2v_0)\sin u$ describe a circle of radius $|\sin 2v_0|$, so each corresponding grid curve is a circle contained in the vertical plane $x = \sin v_0$ parallel to the yz-plane. The only possible surface is graph II. The grid curves we see running lengthwise along the surface correspond to holding u constant, in which case $y = (\cos u_0)\sin 2v$, $z = (\sin u_0)\sin 2v \quad\Rightarrow\quad z = (\tan u_0)y$, so each grid curve lies in a plane $z = ky$ that includes the x-axis.

16. $x = (1-u)(3+\cos v)\cos 4\pi u$, $y = (1-u)(3+\cos v)\sin 4\pi u$, $z = 3u + (1-u)\sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3,0,0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2}\cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. $x = \cos^3 u\,\cos^3 v$, $y = \sin^3 u\,\cos^3 v$, $z = \sin^3 v$. If $v = v_0$ is held constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curve lies in a horizontal plane. Several of the graphs exhibit horizontal grid curves, but the curves for this surface are neither circles nor straight lines, so graph III is the only possibility. (In fact, the horizontal grid curves here are members of the family $x = a\cos^3 u$, $y = a\sin^3 u$ and are called astroids.) The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, as then we have $x = \cos^3 u_0\,\cos^3 v$, $y = \sin^3 u_0\,\cos^3 v$ so the corresponding grid curve lies in the vertical plane $y = (\tan^3 u_0)x$ through the z-axis.

18. $x = (1-|u|)\cos v$, $y = (1-|u|)\sin v$, $z = u$. Then $x^2 + y^2 = (1-|u|)^2\cos^2 v + (1-|u|)^2\sin^2 v = (1-|u|)^2$, so if u is held constant, each grid curve is a circle of radius $(1-|u|)$ in the horizontal plane $z = u$. The graph then must be graph VI. If v is held constant, so $v = v_0$, we have $x = (1-|u|)\cos v_0$ and $y = (1-|u|)\sin v_0$. Then $y = (\tan v_0)x$, so the grid curves we see running vertically along the surface in the planes $y = kx$ correspond to keeping v constant.

19. From Example 3, parametric equations for the plane through the point $(1,2,-3)$ that contains the vectors $\mathbf{a} = \langle 1,1,-1\rangle$ and $\mathbf{b} = \langle 1,-1,1\rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$, $z = -3 + u(-1) + v(1) = -3 - u + v$.

20. Solving the equation for z gives $z^2 = 1 - 2x^2 - 4y^2 \quad \Rightarrow \quad z = -\sqrt{1 - 2x^2 - 4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 - 2x^2 - 4y^2}$.

Alternate solution: The equation can be rewritten as $\dfrac{x^2}{(1/\sqrt{2})^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, and if we let $x = \dfrac{1}{\sqrt{2}} u \cos v$ and $y = \frac{1}{2} u \sin v$, then $z = -\sqrt{1 - 2x^2 - 4y^2} = -\sqrt{1 - u^2 \cos^2 v - u^2 \sin^2 v} = -\sqrt{1 - u^2}$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

21. Solving the equation for y gives $y^2 = 1 - x^2 + z^2 \quad \Rightarrow \quad y = \sqrt{1 - x^2 + z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y \ge 0$.) If we let x and z be the parameters, parametric equations are $x = x$, $z = z$, $y = \sqrt{1 - x^2 + z^2}$.

22. $x = 4 - y^2 - 2z^2$, $y = y$, $z = z$ where $y^2 + 2z^2 \le 4$ since $x \ge 0$. Then the associated vector equation is $\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

23. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \le 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.

24. In spherical coordinates, parametric equations are $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$, $z = 4\cos\phi$. The intersection of the sphere with the plane $z = 2$ corresponds to $z = 4\cos\phi = 2 \quad \Rightarrow \quad \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z = -2$ corresponds to $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Thus the surface is described by $0 \le \theta \le 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.

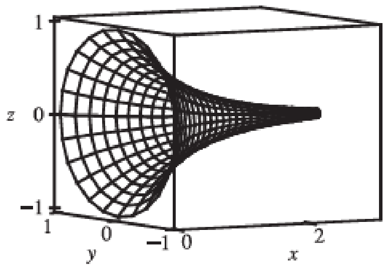
25. Parametric equations are $x = x$, $y = 4\cos\theta$, $z = 4\sin\theta$, $0 \le x \le 5$, $0 \le \theta \le 2\pi$.

26. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \le x^2 + y^2 \le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \le s \le 1$ and $0 \le \theta \le 2\pi$.

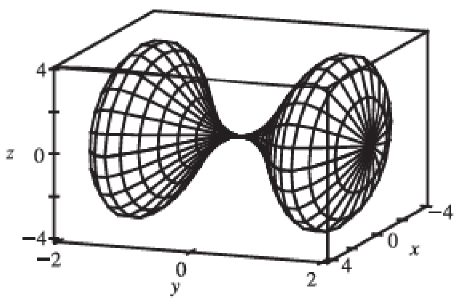
27. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \le x \le 5$, $y \le 0$ to obtain the portion shown. To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter domain $0 \le u \le 5$, $\frac{\pi}{2} \le v \le \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \le x \le 5$ and $-3 \le z \le 3$.

28. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$.

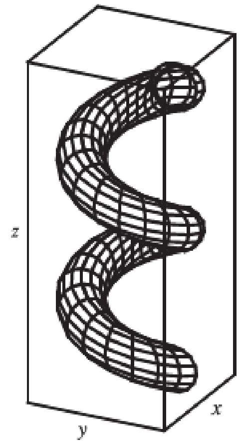
29. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $0 \leq x \leq 3$, $0 \leq \theta \leq 2\pi$.



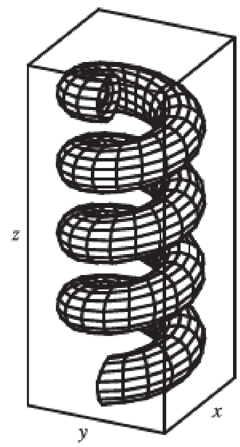
30. Letting θ be the angle of rotation about the y-axis, we have the parametrization $x = (4y^2 - y^4)\cos\theta$, $y = y$, $z = (4y^2 - y^4)\sin\theta$, $-2 \leq y \leq 2$, $0 \leq \theta \leq 2\pi$.



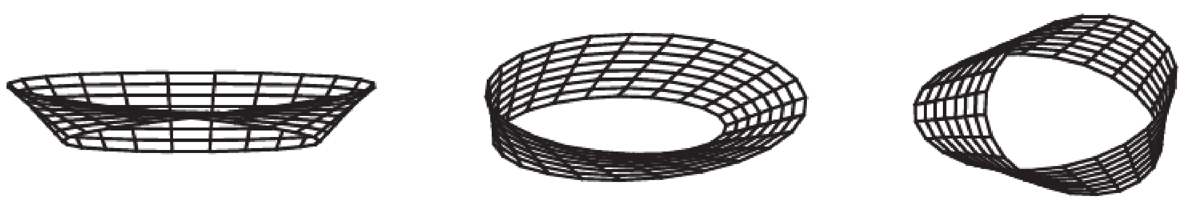
31. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.



32. First we graph the surface as viewed from the front, then from two additional viewpoints.

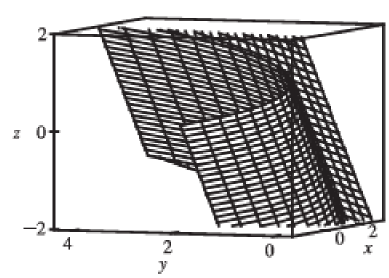


The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 17.7 [ET 16.7].)

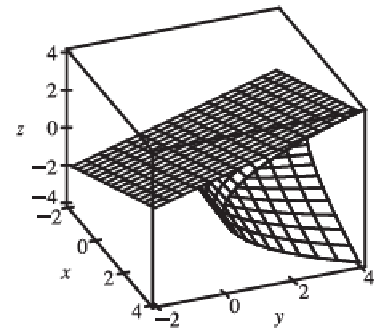
33. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$.

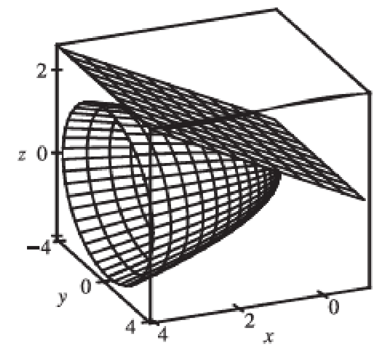
Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



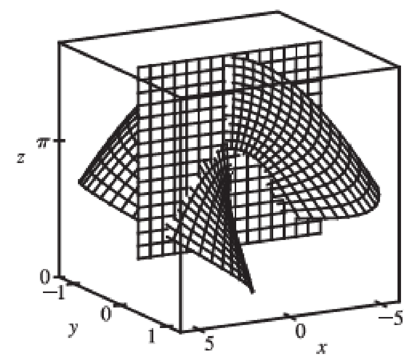
34. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(1,1) = (1,1,1)$.
$\mathbf{r}_u = 2u\,\mathbf{i} + v\,\mathbf{k}$ and $\mathbf{r}_v = 2v\,\mathbf{j} + u\,\mathbf{k}$, so a normal vector to the surface at the point $(1,1,1)$ is
$\mathbf{r}_u(1,1) \times \mathbf{r}_v(1,1) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j} + \mathbf{k}) = -2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$.
Thus an equation of the tangent plane at the point $(1,1,1)$ is
$-2(x-1) - 2(y-1) + 4(z-1) = 0$ or $x + y - 2z = 0$.



35. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(1,0) = (1,0,1)$.
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,
so a normal vector to the surface at the point $(1,0,1)$ is
$\mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.
Thus an equation of the tangent plane at $(1,0,1)$ is
$-2(x-1) + 0(y-0) + 4(z-1) = 0$ or $-x + 2z = 1$.



36. $\mathbf{r}(u,v) = uv\,\mathbf{i} + u\sin v\,\mathbf{j} + v\cos u\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(0,\pi) = (0,0,\pi)$.
$\mathbf{r}_u = v\,\mathbf{i} + \sin v\,\mathbf{j} - v\sin u\,\mathbf{k}$ and $\mathbf{r}_v = u\,\mathbf{i} + u\cos v\,\mathbf{j} + \cos u\;\mathbf{k}$, so a normal vector to the surface at the point $(0,0,\pi)$ is
$\mathbf{r}_u(0,\pi) \times \mathbf{r}_v(0,\pi) = (\pi\,\mathbf{i}) \times (\mathbf{k}) = -\pi\,\mathbf{j}$. Thus an equation of the tangent plane is $-\pi(y-0) = 0$ or $y = 0$.



37. The surface S is given by $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\,dA = \sqrt{14}\iint_D dA = \sqrt{14}\,A(D) = \sqrt{14}\left(\tfrac{1}{2}\cdot 2\cdot 3\right) = 3\sqrt{14}. \end{aligned}$$

38. $z = f(x,y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \le 9$, so by Formula 9,

$$A(S) = \iint_D \sqrt{1 + (-2)^2 + (-5)^2}\,dA = \sqrt{30}\iint_D dA = \sqrt{30}\,A(D) = \sqrt{30}\,(\pi\cdot 3^2) = 9\sqrt{30}\,\pi.$$

39. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$A(S) = \iint_D \sqrt{1 + (\sqrt{x}\,)^2 + (\sqrt{y}\,)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx$$
$$= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3} \int_0^1 \left[(x + 2)^{3/2} - (x + 1)^{3/2}\right] dx$$
$$= \tfrac{2}{3}\left[\tfrac{2}{5}(x + 2)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1)$$

40. $\mathbf{r}_u = \langle 0, 1, -5\rangle$, $\mathbf{r}_v = \langle 1, -2, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle -9, -5, -1\rangle$. Then by Definition 6,

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\, dA = \int_0^1 \int_0^1 |\,\langle -9, -5, -1\rangle\,|\, du\, dv = \sqrt{107} \int_0^1 du \int_0^1 dv = \sqrt{107}$$

41. $z = f(x, y) = xy$ with $0 \le x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$A(S) = \iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}\,(r^2 + 1)^{3/2}\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}(2\sqrt{2} - 1)\, d\theta = \tfrac{2\pi}{3}(2\sqrt{2} - 1)$$

42. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + 3^2 + (4y)^2}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 2y\sqrt{10 + 16y^2}\, dy$$
$$= \tfrac{1}{16} \cdot \tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

43. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr$$
$$= \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

44. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y, z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dA$. Then

$$A(S) = \iint\limits_{0 \le y^2 + z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \tfrac{\pi}{6}(37\sqrt{37} - 1)$$

45. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \le x \le 1$, $0 \le z \le 1$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x, z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$ and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \int_0^1 \int_0^1 \sqrt{17 + 4z^2}\,dx\,dz = \int_0^1 \sqrt{17 + 4z^2}\,dz$$
$$= \tfrac{1}{2}\left(z\sqrt{17 + 4z^2} + \tfrac{17}{2}\ln\left|2z + \sqrt{4z^2 + 17}\,\right|\right)\Big]_0^1 = \tfrac{\sqrt{21}}{2} + \tfrac{17}{4}\left[\ln\left(2 + \sqrt{21}\,\right) - \ln\sqrt{17}\,\right]$$

46. $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^\pi \int_0^1 \sqrt{1 + u^2}\,du\,dv = \int_0^\pi dv \int_0^1 \sqrt{1 + u^2}\,du$$
$$= \pi\left[\tfrac{u}{2}\sqrt{u^2 + 1} + \tfrac{1}{2}\ln\left|u + \sqrt{u^2 + 1}\,\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

47. $\mathbf{r}_u = \langle 2u, v, 0\rangle$, $\mathbf{r}_v = \langle 0, u, v\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle v^2, -2uv, 2u^2\rangle$. Then

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dv\,du = \int_0^1 \int_0^2 \sqrt{(v^2 + 2u^2)^2}\,dv\,du$$
$$= \int_0^1 \int_0^2 (v^2 + 2u^2)\,dv\,du = \int_0^1 \left[\tfrac{1}{3}v^3 + 2u^2 v\right]_{v=0}^{v=2} du = \int_0^1 \left(\tfrac{8}{3} + 4u^2\right) du = \left[\tfrac{8}{3}u + \tfrac{4}{3}u^3\right]_0^1 = 4$$

48. $z = f(x, y) = \cos(x^2 + y^2)$ with $x^2 + y^2 \le 1$.

$$A(S) = \iint_D \sqrt{1 + (-2x\sin(x^2 + y^2))^2 + (-2y\sin(x^2 + y^2))^2}\,dA$$
$$= \iint_D \sqrt{1 + 4x^2\sin^2(x^2 + y^2) + 4y^2\sin^2(x^2 + y^2)}\,dA = \iint_D \sqrt{1 + 4(x^2 + y^2)\sin^2(x^2 + y^2)}\,dA$$
$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2\sin^2(r^2)}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr$$
$$= 2\pi \int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr \approx 4.1073$$

49. $z = f(x, y) = e^{-x^2 - y^2}$ with $x^2 + y^2 \le 4$.

$$A(S) = \iint_D \sqrt{1 + \left(-2xe^{-x^2-y^2}\right)^2 + \left(-2ye^{-x^2-y^2}\right)^2}\,dA = \iint_D \sqrt{1 + 4(x^2 + y^2)e^{-2(x^2+y^2)}}\,dA$$
$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2e^{-2r^2}}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{1 + 4r^2e^{-2r^2}}\,dr = 2\pi \int_0^2 r\sqrt{1 + 4r^2e^{-2r^2}}\,dr \approx 13.9783$$

50. Let $f(x, y) = \dfrac{1 + x^2}{1 + y^2}$. Then $f_x = \dfrac{2x}{1 + y^2}$,

$$f_y = (1 + x^2)\left[-\frac{2y}{(1 + y^2)^2}\right] = -\frac{2y(1 + x^2)}{(1 + y^2)^2}.$$

We use a CAS to estimate

$$\int_{-1}^{1} \int_{-(1-|x|)}^{1-|x|} \sqrt{1 + f_x^2 + f_y^2}\,dy\,dx \approx 2.6959.$$

In order to graph only the part of the surface above the square, we use $-(1 - |x|) \le y \le 1 - |x|$ as the y-range in our plot command.