1. (a) $ay'' + by' + cy = 0$ where a, b, and c are constants.

   (b) $ar^2 + br + c = 0$

   (c) If the auxiliary equation has two distinct real roots r_1 and r_2, the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where r is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

2. (a) An initial-value problem consists of finding a solution y of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

   (b) A boundary-value problem consists of finding a solution y of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) $ay'' + by' + cy = G(x)$ where a, b, and c are constants and G is a continuous function.

   (b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

   (c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

   (d) See the discussion on pages 1158–1160 [ET 1122–1124].

4. Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].

5. See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

1. True. See Theorem 18.1.3 [ET 17.1.3].

2. False. The differential equation is not homogeneous.

3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.

4. False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

1. The auxiliary equation is $r^2 - 2r - 15 = 0 \quad \Rightarrow \quad (r-5)(r+3) = 0 \quad \Rightarrow \quad r = 5, r = -3$. Then the general solution is $y = c_1 e^{5x} + c_2 e^{-3x}$.

2. The auxiliary equation is $r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i$, so $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

3. The auxiliary equation is $r^2 + 3 = 0 \quad \Rightarrow \quad r = \pm\sqrt{3}\, i$. Then the general solution is $y = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)$.

4. The auxiliary equation is $4r^2 + 4r + 1 = 0 \quad \Rightarrow \quad (2r+1)^2 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}$, so the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.

5. $r^2 - 4r + 5 = 0 \quad \Rightarrow \quad r = 2 \pm i$, so $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{2x} \quad \Rightarrow \quad y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \quad \Rightarrow \quad A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.

6. $r^2 + r - 2 = 0 \quad \Rightarrow \quad r = 1, r = -2$ and $y_c(x) = c_1 e^x + c_2 e^{-2x}$. Try $y_p(x) = Ax^2 + Bx + C \quad \Rightarrow \quad y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitution gives $2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 \quad \Rightarrow \quad A = B = -\frac{1}{2}, C = -\frac{3}{4}$ so the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.

7. $r^2 - 2r + 1 = 0 \quad \Rightarrow \quad r = 1$ and $y_c(x) = c_1 e^x + c_2 x e^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \quad \Rightarrow$ $y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \quad \Rightarrow \quad A = 0, B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2}\cos x - \frac{1}{2}(x + 1)\sin x$.

8. $r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$ and $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p(x) = Ax \cos 2x + Bx \sin 2x$ so that no term of y_p is a solution of the complementary equation. Then $y_p' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$ and $y_p'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$. Substitution gives $4B\cos 2x - 4A\sin 2x = \sin 2x \quad \Rightarrow$ $A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x\cos 2x$.

9. $r^2 - r - 6 = 0 \quad \Rightarrow \quad r = -2, r = 3$ and $y_c(x) = c_1 e^{-2x} + c_2 e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p_1}(x) = A$. Then $y_{p_1}'(x) = y_{p_1}''(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p_2}(x) = Bxe^{-2x}$ [since $y = Be^{-2x}$ satisfies the complementary equation]. Then $y_{p_2}' = (B - 2Bx)e^{-2x}$ and $y_{p_2}'' = (4Bx - 4B)e^{-2x}$, and substitution gives $-5Be^{-2x} = e^{-2x} \quad \Rightarrow \quad B = -\frac{1}{5}$. The general solution then is $y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_{p_1}(x) + y_{p_2}(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$.

10. Using variation of parameters, $y_c(x) = c_1 \cos x + c_2 \sin x$, $u_1'(x) = -\csc x \sin x = -1 \quad \Rightarrow \quad u_1(x) = -x$, and $u_2'(x) = \dfrac{\csc x \cos x}{x} = \cot x \quad \Rightarrow \quad u_2(x) = \ln|\sin x| \quad \Rightarrow \quad y_p = -x\cos x + \sin x \ln|\sin x|$. The solution is $y(x) = (c_1 - x)\cos x + (c_2 + \ln|\sin x|)\sin x$.

11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right)$.

13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.

14. $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p_2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p_2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9\cos(x/3) - 27\sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.

15. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3 \cdot 5}$, $c_7 = \dfrac{(-1)^3}{3 \cdot 5 \cdot 7} = \dfrac{(-1)^3 2^3 3!}{7!}, \ldots, c_{2n+1} = \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}\, x^{2n+1}$.

16. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, we have $c_2 = \dfrac{c_0}{1}$, $c_4 = \dfrac{c_2}{3} = \dfrac{c_0}{1 \cdot 3}$, $c_6 = \dfrac{c_4}{5} = \dfrac{c_0}{1 \cdot 3 \cdot 5}, \ldots,$
$c_{2n} = \dfrac{c_0}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = c_0 \dfrac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \dfrac{c_1}{2}$, $c_5 = \dfrac{c_3}{4} = \dfrac{c_1}{2 \cdot 4}$, $c_7 = \dfrac{c_5}{6} = \dfrac{c_1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n+1} = \dfrac{c_1}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} = \dfrac{c_1}{2^n\, n!}$. Thus the general solution is
$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\, x^{2n}}{(2n-1)!} + c \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n\, n!}$. But $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n\, n!} = x \sum_{n=0}^{\infty} \dfrac{\left(\frac{1}{2}x^2\right)^n}{n!} = xe^{x^2/2}$,
so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \sum_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\, x^{2n}}{(2n-1)!}$.

17. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}$. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

18. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$, $x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \quad \Rightarrow \quad c_1 = 0,\ c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

19. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3} r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion, $m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3} y$, so $y''(t) = -k^2 y\,(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so $g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 18.1 [ET 17.1], not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y\,(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s^2, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3}\ \text{s}^{-1}$ and $T = 2\pi/k \approx 5079\ \text{s} \approx 85\ \text{min}$.

(d) $y(t) = 0 \quad \Leftrightarrow \quad \cos kt = 0 \quad \Leftrightarrow \quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad \Rightarrow \quad y'(t) = -kR \sin\!\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899\ \text{mi/s} \approx 17{,}600\ \text{mi/h}$.