51. $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1) = 0 \quad \Leftrightarrow \quad x = -1, 0, 1$.

$f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. So $f(3) = 66$ is the absolute maximum value and $f(\pm 1) = 2$ is the absolute minimum value.

52. $f(x) = (x^2 - 1)^3$, $[-1, 2]$. $f'(x) = 3(x^2 - 1)^2(2x) = 6x(x+1)^2(x-1)^2 = 0 \quad \Leftrightarrow \quad x = -1, 0, 1$. $f(\pm 1) = 0$, $f(0) = -1$, and $f(2) = 27$. So $f(2) = 27$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.

53. $f(x) = \dfrac{x}{x^2 + 1}$, $[0, 2]$. $f'(x) = \dfrac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2} = 0 \quad \Leftrightarrow \quad x = \pm 1$, but -1 is not in $[0, 2]$. $f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

54. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$. $f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \dfrac{16x}{(x^2 + 4)^2} = 0 \quad \Leftrightarrow \quad x = 0$. $f(\pm 4) = \frac{12}{20} = \frac{3}{5}$ and $f(0) = -1$. So $f(\pm 4) = \frac{3}{5}$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.

55. $f(t) = t\sqrt{4 - t^2}$, $[-1, 2]$.

$$f'(t) = t \cdot \tfrac{1}{2}(4 - t^2)^{-1/2}(-2t) + (4 - t^2)^{1/2} \cdot 1 = \frac{-t^2}{\sqrt{4 - t^2}} + \sqrt{4 - t^2} = \frac{-t^2 + (4 - t^2)}{\sqrt{4 - t^2}} = \frac{4 - 2t^2}{\sqrt{4 - t^2}}.$$

$f'(t) = 0 \quad \Rightarrow \quad 4 - 2t^2 = 0 \quad \Rightarrow \quad t^2 = 2 \quad \Rightarrow \quad t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1, 2]$.

$f'(t)$ does not exist if $4 - t^2 = 0 \quad \Rightarrow \quad t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So $f(\sqrt{2}) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.

56. $f(t) = \sqrt[3]{t}\,(8 - t)$, $[0, 8]$. $f(t) = 8t^{1/3} - t^{4/3} \quad \Rightarrow \quad f'(t) = \frac{8}{3}t^{-2/3} - \frac{4}{3}t^{1/3} = \frac{4}{3}t^{-2/3}(2 - t) = \dfrac{4(2 - t)}{3\sqrt[3]{t^2}}$.

$f'(t) = 0 \quad \Rightarrow \quad t = 2$. $f'(t)$ does not exist if $t = 0$. $f(0) = 0$, $f(2) = 6\sqrt[3]{2} \approx 7.56$, and $f(8) = 0$.

So $f(2) = 6\sqrt[3]{2}$ is the absolute maximum value and $f(0) = f(8) = 0$ is the absolute minimum value.

57. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$.

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1)$.

$f'(t) = 0 \quad \Rightarrow \quad \sin t = \frac{1}{2}$ or $\sin t = -1 \quad \Rightarrow \quad t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

58. $f(t)=t+\cot(t/2)$, $[\pi/4,7\pi/4]$. $f'(t)=1-\csc^2(t/2)\cdot\frac{1}{2}$.

$f'(t)=0 \;\Rightarrow\; \frac{1}{2}\csc^2(t/2)=1 \;\Rightarrow\; \csc^2(t/2)=2 \;\Rightarrow\; \csc(t/2)=\pm\sqrt{2} \;\Rightarrow\; \frac{1}{2}t=\frac{\pi}{4}$ or $\frac{1}{2}t=\frac{3\pi}{4}$

$\left[\frac{\pi}{4}\le t\le\frac{7\pi}{4} \;\Rightarrow\; \frac{\pi}{8}\le\frac{1}{2}t\le\frac{7\pi}{8}\right.$ and $\csc(t/2)\neq-\sqrt{2}$ in the last interval$\Big]$ $\Rightarrow$ $t=\frac{\pi}{2}$ or $t=\frac{3\pi}{2}$.

$f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}+\cot\frac{\pi}{8}\approx 3.20$, $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+\cot\frac{\pi}{4}=\frac{\pi}{2}+1\approx 2.57$, $f\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2}+\cot\frac{3\pi}{4}=\frac{3\pi}{2}-1\approx 3.71$, and

$f\left(\frac{7\pi}{4}\right)=\frac{7\pi}{4}+\cot\frac{7\pi}{8}\approx 3.08$. So $f\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2}-1$ is the absolute maximum value and $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+1$ is the absolute minimum value.

59. $f(x)=xe^{-x^2/8}$, $[-1,4]$. $f'(x)=x\cdot e^{-x^2/8}\cdot\left(-\frac{x}{4}\right)+e^{-x^2/8}\cdot 1=e^{-x^2/8}\left(-\frac{x^2}{4}+1\right)$. Since $e^{-x^2/8}$ is never 0,

$f'(x)=0 \;\Rightarrow\; -x^2/4+1=0 \;\Rightarrow\; 1=x^2/4 \;\Rightarrow\; x^2=4 \;\Rightarrow\; x=\pm 2$, but -2 is not in the given interval, $[-1,4]$.

$f(-1)=-e^{-1/8}\approx-0.88$, $f(2)=2e^{-1/2}\approx 1.21$, and $f(4)=4e^{-2}\approx 0.54$. So $f(2)=2e^{-1/2}$ is the absolute maximum value and $f(-1)=-e^{-1/8}$ is the absolute minimum value.

60. $f(x)=x-\ln x$, $\left[\frac{1}{2},2\right]$. $f'(x)=1-\frac{1}{x}=\frac{x-1}{x}$. $f'(x)=0 \;\Rightarrow\; x=1$. [Note that 0 is not in the domain of f.]

$f\left(\frac{1}{2}\right)=\frac{1}{2}-\ln\frac{1}{2}\approx 1.19$, $f(1)=1$, and $f(2)=2-\ln 2\approx 1.31$. So $f(2)=2-\ln 2$ is the absolute maximum value and $f(1)=1$ is the absolute minimum value.

61. $f(x)=\ln(x^2+x+1)$, $[-1,1]$. $f'(x)=\dfrac{1}{x^2+x+1}\cdot(2x+1)=0 \;\Leftrightarrow\; x=-\frac{1}{2}$. Since $x^2+x+1>0$ for all x, the domain of f and f' is $\mathbb{R}$. $f(-1)=\ln 1=0$, $f\left(-\frac{1}{2}\right)=\ln\frac{3}{4}\approx-0.29$, and $f(1)=\ln 3\approx 1.10$. So $f(1)=\ln 3\approx 1.10$ is the absolute maximum value and $f\left(-\frac{1}{2}\right)=\ln\frac{3}{4}\approx-0.29$ is the absolute minimum value.

62. $f(x)=e^{-x}-e^{-2x}$, $[0,1]$. $f'(x)=e^{-x}(-1)-e^{-2x}(-2)=\dfrac{2}{e^{2x}}-\dfrac{1}{e^x}=\dfrac{2-e^x}{e^{2x}}=0 \;\Leftrightarrow\; e^x=2 \;\Leftrightarrow$

$x=\ln 2\approx 0.69$. $f(0)=0$, $f(\ln 2)=e^{-\ln 2}-e^{-2\ln 2}=(e^{\ln 2})^{-1}-(e^{\ln 2})^{-2}=2^{-1}-2^{-2}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$,

$f(1)=e^{-1}-e^{-2}\approx 0.233$. So $f(\ln 2)=\frac{1}{4}$ is the absolute maximum value and $f(0)=0$ is the absolute minimum value.

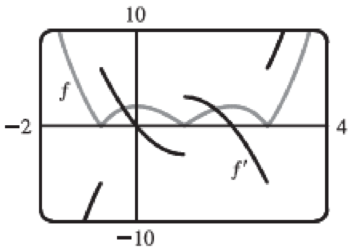
63. $f(x)=x^a(1-x)^b$, $0\le x\le 1$, $a>0$, $b>0$.

$$\begin{aligned}f'(x)&=x^a\cdot b(1-x)^{b-1}(-1)+(1-x)^b\cdot ax^{a-1}=x^{a-1}(1-x)^{b-1}[x\cdot b(-1)+(1-x)\cdot a]\\&=x^{a-1}(1-x)^{b-1}(a-ax-bx)\end{aligned}$$

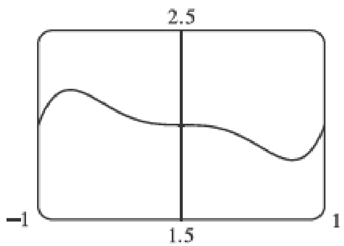
At the endpoints, we have $f(0)=f(1)=0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x)=0 \;\Leftrightarrow\; x=\dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right)=\left(\frac{a}{a+b}\right)^a\left(1-\frac{a}{a+b}\right)^b=\frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b=\frac{a^a}{(a+b)^a}\cdot\frac{b^b}{(a+b)^b}=\frac{a^ab^b}{(a+b)^{a+b}}.$$

So $f\left(\dfrac{a}{a+b}\right)=\dfrac{a^ab^b}{(a+b)^{a+b}}$ is the absolute maximum value.

64. 

We see that $f'(x) = 0$ at about $x = 0.0$ and 2.0, and that $f'(x)$ does not exist at about $x = -0.7$, 1.0, and 2.7, so the critical numbers of f are about -0.7, 0.0, 1.0, 2.0, and 2.7.

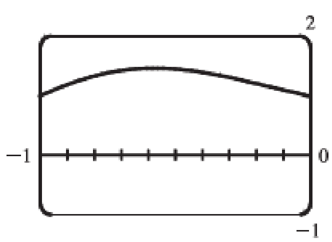
65. (a) 

From the graph, it appears that the absolute maximum value is about $f(-0.77) = 2.19$, and the absolute minimum value is about $f(0.77) = 1.81$.

(b) $f(x) = x^5 - x^3 + 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. So $f'(x) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{\frac{3}{5}}$.

$f\left(-\sqrt{\frac{3}{5}}\right) = \left(-\sqrt{\frac{3}{5}}\right)^5 - \left(-\sqrt{\frac{3}{5}}\right)^3 + 2 = -\left(\frac{3}{5}\right)^2\sqrt{\frac{3}{5}} + \frac{3}{5}\sqrt{\frac{3}{5}} + 2 = \left(\frac{3}{5} - \frac{9}{25}\right)\sqrt{\frac{3}{5}} + 2 = \frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (maximum)

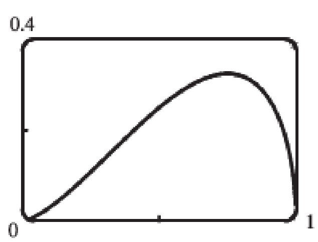
and similarly, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (minimum).

66. (a) 

From the graph, it appears that the absolute maximum value is about $f(-0.58) = 1.47$, and the absolute minimum value is about $f(-1) = f(0) = 1.00$; that is, at both endpoints.

(b) $f(x) = e^{x^3 - x} \quad \Rightarrow \quad f'(x) = e^{x^3 - x}(3x^2 - 1)$. So $f'(x) = 0$ on $[-1, 0] \quad \Rightarrow \quad x = -\sqrt{1/3}$.

$f(-1) = f(0) = 1$ (minima) and $f\left(-\sqrt{1/3}\right) = e^{-\sqrt{3}/9 + \sqrt{3}/3} = e^{2\sqrt{3}/9}$ (maximum).

67. (a) 

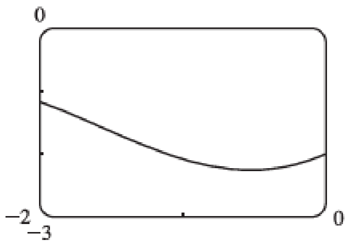
From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3 - 4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$.

$f(0) = f(1) = 0$ (minimum), and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$ (maximum).

68. (a)



From the graph , it appears that the absolute maximum value is about $f(-2) = -1.17$, and the absolute minimum value is about $f(-0.52) = -2.26$.

(b) $f(x) = x - 2\cos x \quad \Rightarrow \quad f'(x) = 1 + 2\sin x$. So $f'(x) = 0 \quad \Rightarrow \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = -\frac{\pi}{6}$ on $[-2, 0]$.

$f(-2) = -2 - 2\cos(-2)$ (maximum) and $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \sqrt{3}$ (minimum).

69. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V

$\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT}\right.$ and V is never $\left.0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow \quad V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$.

Setting this equal to 0 and using the quadratic formula to find T, we get

$T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested

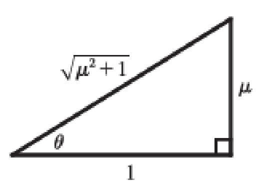
in the region $0°\text{C} \leq T \leq 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$;

$\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

70. $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \mu\cos\theta - \sin\theta = 0 \quad \Rightarrow \quad \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$



If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$.

We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.

Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \leq 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \leq \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$.

Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

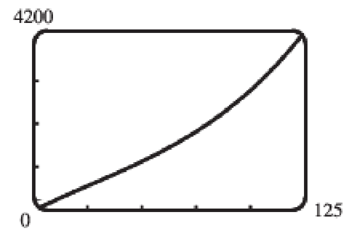
71. Let $a = -0.000\,032\,37$, $b = 0.000\,903\,7$, $c = -0.008\,956$, $d = 0.03629$, $e = -0.04458$, and $f = 0.4074$.

Then $S(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f$ and $S'(t) = 5at^4 + 4bt^3 + 3ct^2 + 2dt + e$.

We now apply the Closed Interval Method to the continuous function S on the interval $0 \le t \le 10$. Since S' exists for all t, the only critical numbers of S occur when $S'(t) = 0$. We use a rootfinder on a CAS (or a graphing device) to find that $S'(t) = 0$ when $t_1 \approx 0.855$, $t_2 \approx 4.618$, $t_3 \approx 7.292$, and $t_4 \approx 9.570$. The values of S at these critical numbers are $S(t_1) \approx 0.39$, $S(t_2) \approx 0.43645$, $S(t_3) \approx 0.427$, and $S(t_4) \approx 0.43641$. The values of S at the endpoints of the interval are $S(0) \approx 0.41$ and $S(10) \approx 0.435$. Comparing the six numbers, we see that sugar was most expensive at $t_2 \approx 4.618$ (corresponding roughly to March 1998) and cheapest at $t_1 \approx 0.855$ (June 1994).

72. (a) The equation of the graph in the figure is

$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.



(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \quad \Rightarrow$

$a'(t) = 0.00876t - 0.23106. \quad a'(t) = 0 \quad \Rightarrow \quad t_1 = \frac{0.23106}{0.00876} \approx 26.4.$

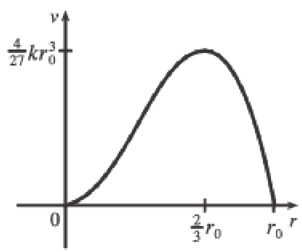
$a(0) \approx 24.98$, $a(t_1) \approx 21.93$, and $a(125) \approx 64.54$.

The maximum acceleration is about $64.5\ \text{ft/s}^2$ and the minimum acceleration is about $21.93\ \text{ft/s}^2$.

73. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \quad \Rightarrow \quad v'(r) = 2kr_0r - 3kr^2. \quad v'(r) = 0 \quad \Rightarrow \quad kr(2r_0 - 3r) = 0 \quad \Rightarrow$ $r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



74. $g(x) = 2 + (x-5)^3 \quad \Rightarrow \quad g'(x) = 3(x-5)^2 \quad \Rightarrow \quad g'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.

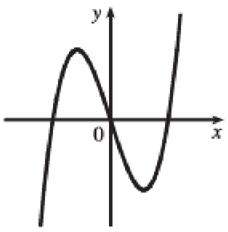
75. $f(x) = x^{101} + x^{51} + x + 1 \quad \Rightarrow \quad f'(x) = 101x^{100} + 51x^{50} + 1 \ge 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

76. Suppose that f has a minimum value at c, so $f(x) \ge f(c)$ for all x near c. Then $g(x) = -f(x) \le -f(c) = g(c)$ for all x near c, so $g(x)$ has a maximum value at c.

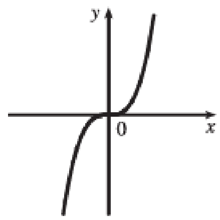
77. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

78. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

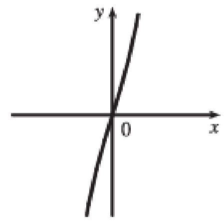
Case (i) [2 critical numbers]:
$f(x) = x^3 - 3x \quad \Rightarrow$
$f'(x) = 3x^2 - 3$, so $x = -1, 1$
are critical numbers.



Case (ii) [1 critical number]:
$f(x) = x^3 \quad \Rightarrow$
$f'(x) = 3x^2$, so $x = 0$
is the only critical number.



Case (iii) [no critical number]:
$f(x) = x^3 + 3x \quad \Rightarrow$
$f'(x) = 3x^2 + 3$,
so there is no critical number.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.