THE INVERSE OF A MATRIX

DEFINITION:

The identity matrix I is the $n \times n$ matrix of the form

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

MAIN PROPERTY:

$$AI = IA = A$$

DEFINITION:

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I.$$

In this case, C is an inverse of A and is denoted by A^{-1}. So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

THEOREM:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

PROBLEM:

Solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$$

SOLUTION:

We have:

$$A\bar{x} = B \Rightarrow \bar{x} = A^{-1}B,$$

therefore

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

PROPERTIES:

Let A and B be invertible $n \times n$ matrices. Then

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$

ALGORITHM FOR FINDING A^{-1}:

1. Row reduce the augmented matrix $[A\ I]$.

2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.

3. Otherwise, A does not have an inverse.

EXAMPLE: Let

$$A=\left[\begin{array}{rrr}3 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{array}\right]$$

Find A^{-1}.

Solution: We have

$$\left[\begin{array}{rrr|rrr}3 & -1 & 1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{rrr|rrr}1 & 1 & 1 & 1 & 2 & 0\\ -1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{rrr|rrr}1 & 1 & 1 & 1 & 2 & 0\\ 0 & 2 & 1 & 1 & 3 & 0\\ 0 & -1 & 0 & -1 & -2 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{rrr|rrr}1 & 1 & 1 & 1 & 2 & 0\\ 0 & 1 & 1 & 0 & 1 & 1\\ 0 & -1 & 0 & -1 & -2 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{rrr|rrr}1 & 1 & 1 & 1 & 2 & 0\\ 0 & 1 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & -1 & 2\end{array}\right]$$

$$\sim\left[\begin{array}{rrr|rrr}1 & 0 & 0 & 1 & 1 & -1\\ 0 & 1 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & -1 & 2\end{array}\right]$$

$$\sim\left[\begin{array}{rrr|rrr}1 & 0 & 0 & 1 & 1 & -1\\ 0 & 1 & 0 & 1 & 2 & -1\\ 0 & 0 & 1 & -1 & -1 & 2\end{array}\right]$$

therefore

$$A^{-1}=\left[\begin{array}{rrr}1 & 1 & -1\\ 1 & 2 & -1\\ -1 & -1 & 2\end{array}\right]$$