# Triple Integrals

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Let's first deal with the simplest case where $f$ is defined on a rectangular box:

$\boxed{1}$    $B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$

The first step is to divide $B$ into sub-boxes. We do this by dividing the interval $[a, b]$ into $l$ subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x$, dividing $[c, d]$ into $m$ subintervals of width $\Delta y$, and dividing $[r, s]$ into $n$ subintervals of width $\Delta z$.

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# Triple Integrals

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box $B$ into $lmn$ sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1.

Each sub-box has volume
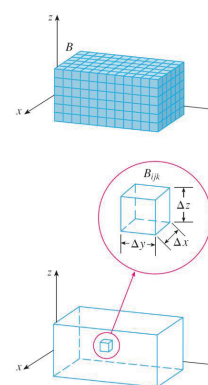$\Delta V = \Delta x \, \Delta y \, \Delta z$.



**Figure 1**    4

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# Triple Integrals

Then we form the **triple Riemann sum**

$\boxed{2}$    $\displaystyle \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in $B_{ijk}$.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in $\boxed{2}$.

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# Triple Integrals

$\boxed{3}$ **Definition**   The **triple integral** of $f$ over the box $B$ is

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V$$

if this limit exists.

Again, the triple integral always exists if $f$ is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point $(x_i, y_j, z_k)$ we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k) \, \Delta V$$

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# Triple Integrals

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

$\boxed{4}$ **Fubini's Theorem for Triple Integrals**   If $f$ is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to $x$ (keeping $y$ and $z$ fixed), then we integrate with respect to $y$ (keeping $z$ fixed), and finally we integrate with respect to $z$.

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# Triple Integrals

There are five other possible orders in which we can integrate, all of which give the same value.

For instance, if we integrate with respect to $y$, then $z$, and then $x$, we have

$$\iiint_B f(x, y, z) \, dV = \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx$$

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## Example 1

Evaluate the triple integral $\iiint_B xyz^2\, dV$, where $B$ is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

Solution:

We could use any of the six possible orders of integration.

If we choose to integrate with respect to $x$, then $y$, and then $z$, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz$$

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## Example 1 – *Solution*  cont'd

$$= \int_0^3 \int_{-1}^2 \left[ \frac{x^2 yz^2}{2} \right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz$$

$$= \int_0^3 \left[ \frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz$$

$$= \frac{z^3}{4} \bigg] = \frac{27}{4}$$

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## Triple Integrals

Now we define the **triple integral over a general bounded region $E$** in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose $E$ in a box $B$ of the type given by Equation 1. Then we define $F$ so that it agrees with $f$ on $E$ but is 0 for points in $B$ that are outside $E$.

By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if $f$ is continuous and the boundary of $E$ is "reasonably smooth."

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## Triple Integrals

The triple integral has essentially the same properties as the double integral.

We restrict our attention to continuous functions $f$ and to certain simple types of regions.
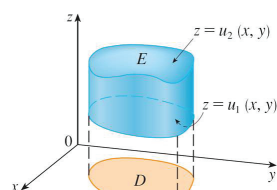
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## Triple Integrals

A solid region $E$ is said to be of **type 1** if it lies between the graphs of two continuous functions of $x$ and $y$, that is,

5    $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where $D$ is the projection of $E$ onto the $xy$-plane as shown in Figure 2.



A type 1 solid region
**Figure 2**

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## Triple Integrals

Notice that the upper boundary of the solid $E$ is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument, it can be shown that if $E$ is a type 1 region given by Equation 5, then

6    $$\iiint_E f(x, y, z)\, dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that $x$ and $y$ are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to $z$.

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# Triple Integrals

In particular, if the projection $D$ of $E$ onto the $xy$-plane is a type I plane region (as in Figure 3),
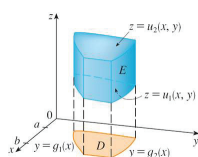


A type 1 solid region where the projection $D$ is a type I plane region

**Figure 3**

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# Triple Integrals

In particular, if the projection $D$ of $E$ onto the $xy$-plane is a type I plane region (as in Figure 3), then
$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$
and Equation 6 becomes

$\boxed{7}$

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$
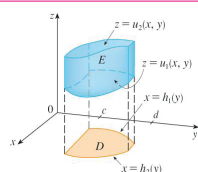
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# Triple Integrals

If, on the other hand, $D$ is a type II plane region (as in Figure 4), then
$E = \{(x, y, z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x, y) \le z \le u_2(x, y)\}$
and Equation 6 becomes

$\boxed{8}$

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$



A type 1 solid region with a type II projection
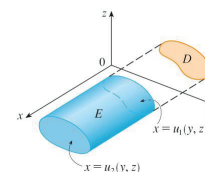
**Figure 4**

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# Triple Integrals

A solid region $E$ is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

where, this time, $D$ is the projection of $E$ onto the $yz$-plane (see Figure 7).



The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

A type 2 region
**Figure 7**

$\boxed{10}$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[ \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$
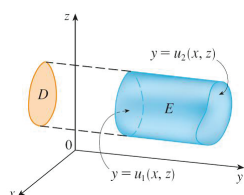
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# Triple Integrals

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

where $D$ is the projection of $E$ onto the $xz$-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8).



A type 3 region
**Figure 8**

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# Triple Integrals

For this type of region we have

$\boxed{11}$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[ \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether $D$ is a type I or type II plane region (and corresponding to Equations 7 and 8).

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## Applications of Triple Integrals

## Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from $a$ to $b$, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\, dA$ represents the volume under the surface $z = f(x, y)$ and above $D$.

The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\, dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that $E$ is just the *domain* of the function $f$; the graph of $f$ lies in four-dimensional space.)

## Applications of Triple Integrals

Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of $x$, $y$, $z$ and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in $E$. Then the triple integral does represent the volume of $E$:

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$$V(E) = \iiint_E dV$$

## Applications of Triple Integrals

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} dz \right] dA = \iint_D \left[ u_2(x, y) - u_1(x, y) \right] dA$$

and we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

## Example 5

Use a triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution:
The tetrahedron $T$ and its projection $D$ onto the $xy$-plane are shown in Figures 14 and 15.
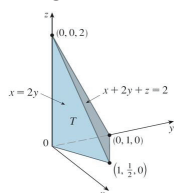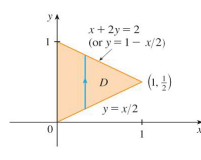


Figure 14        Figure 15

## Example 5 – *Solution*
cont'd

The lower boundary of $T$ is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

Therefore we have

$$V(T) = \iiint_T dV$$

$$= \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.)