

This print-out should have 12 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

MatrixVecProd04
001 10.0 points

Determine \mathbf{vu}^T when

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

1. $\mathbf{vu}^T = -5a + 2b - 3c$

2. $\mathbf{vu}^T = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix}$

3. $\mathbf{vu}^T = -3a + 2b - 5c$

4. $\mathbf{vu}^T = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}$ **correct**

Explanation:

Since

$$\mathbf{u}^T = [-3 \quad 2 \quad -5],$$

we see that

$$\mathbf{vu}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [-3 \quad 2 \quad -5].$$

Consequently,

$$\mathbf{vu}^T = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}.$$

InverseMatrix01a
002 10.0 points

Solve for X when $AX + B = C$,

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}.$$

1. $X = \begin{bmatrix} 4 & -1 \\ -13 & -1 \end{bmatrix}$

2. $X = \begin{bmatrix} 8 & -1 \\ -5 & -1 \end{bmatrix}$

3. $X = \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}$ **correct**

4. $X = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$

5. $X = \begin{bmatrix} 8 & -1 \\ 5 & 2 \end{bmatrix}$

Explanation:

By the algebra of matrices,

$$X = A^{-1}(C - B).$$

But the inverse of any 2×2 matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

is given by

$$D^{-1} = \begin{bmatrix} \frac{d_{22}}{\Delta} & -\frac{d_{12}}{\Delta} \\ -\frac{d_{21}}{\Delta} & \frac{d_{11}}{\Delta} \end{bmatrix}$$

with $\Delta = d_{11}d_{22} - d_{12}d_{21}$.

Thus

$$\begin{aligned} X &= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \left(\begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

Consequently,

$$X = \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}.$$

LUDecomp06h
003 10.0 points

Find L in an LU decomposition of

$$A = \begin{bmatrix} -1 & 3 & 0 & -5 \\ 1 & -3 & -3 & 5 \\ 4 & -12 & -9 & 25 \end{bmatrix}.$$

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$

2. $L = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -4 & 3 & -1 \end{bmatrix}$

3. $L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -4 & 3 & 2 \end{bmatrix}$

4. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}$ **correct**

5. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$

Explanation:

Recall that in a factorization $A = LU$ of an $m \times n$ matrix A , then L is an $m \times m$ lower triangular matrix with ones on the diagonal and U is an $m \times n$ echelon form of A .

We begin by computing U . Now $U = M_0 A$ where j is the number of row operations on A needed to transform A into its echelon form U and M_i is a product of $j - i$ elementary

matrices that represent these row operations:

$$\begin{aligned} U &= M_0 A = M_1 E_1 A \\ &= M_1 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 & -5 \\ 1 & -3 & -3 & 5 \\ 4 & -12 & -9 & 25 \end{bmatrix} \\ &= M_2 E_2 (E_1 A) \\ &= M_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 & -5 \\ 0 & 0 & -3 & 0 \\ 4 & -12 & -9 & 25 \end{bmatrix} \\ &= E_3 (E_2 E_1 A) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 & -5 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -9 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & 0 & -5 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \end{aligned}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change $U = M_0 A$ to $M_0^{-1} U = A$. This shows that $L = M_0^{-1}$. Hence

$$\begin{aligned} L &= M_0^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \\ &= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix} \end{aligned}$$

Consequently,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}.$$

Let H be the set of all vectors

$$\begin{bmatrix} a - 2b \\ ab + 3a \\ b \end{bmatrix}$$

where a and b are real. Determine if H is a subspace of \mathbb{R}^3 , and then check the correct answer below.

1. H is a subspace of \mathbb{R}^3 because it can be written as $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1, \mathbf{v}_2$ in \mathbb{R}^3 .

2. H is not a subspace of \mathbb{R}^3 because it is not closed under vector addition. **correct**

3. H is a subspace of \mathbb{R}^3 because it can be written as $\text{Nul}(A)$ for some matrix A .

4. H is not a subspace of \mathbb{R}^3 because it does not contain $\mathbf{0}$.

Explanation:

To check if the set H of all vectors

$$\begin{bmatrix} a - 2b \\ ab + 3a \\ b \end{bmatrix}$$

is a subspace of \mathbb{R}^3 we check the properties defining a subspace:

1. the zero vector $\mathbf{0}$ is in H : set $a = b = 0$. Then

$$\begin{bmatrix} 0 - 0 \\ 0 + 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so H contains $\mathbf{0}$.

2. for each \mathbf{u}, \mathbf{v} in H the sum $\mathbf{u} + \mathbf{v}$ is in H : set

$$\mathbf{v}_1 = \begin{bmatrix} a_1 - 2b_1 \\ a_1b_1 + 3a_1 \\ b_1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_2 - 2b_2 \\ a_2b_2 + 3a_2 \\ b_2 \end{bmatrix},$$

in H . Then

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= \begin{bmatrix} a_1 - 2b_1 \\ a_1b_1 + 3a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 - 2b_2 \\ a_2b_2 + 3a_2 \\ b_2 \end{bmatrix} \\ &= \begin{bmatrix} (a_1 + a_2) - 2(b_1 + b_2) \\ a_1b_1 + a_2b_2 + 3(a_1 + a_2) \\ (b_1 + b_2) \end{bmatrix}. \end{aligned}$$

But in general,

$$a_1b_1 + a_2b_2 \neq (a_1 + a_2)(b_1 + b_2),$$

in which case $\mathbf{u} + \mathbf{v}$ is not in H .

Consequently, H is not a subspace of \mathbb{R}^3 because it is

not closed under vector addition.

Invertible01/02
005 10.0 points

A is an $n \times n$ matrix. Which of the following statements are equivalent to A being invertible?

(i) The equation $A\mathbf{x} = \mathbf{b}$ has multiple solutions for each \mathbf{b} in \mathbb{R}^n .

(ii) A is not row equivalent to the $n \times n$ identity matrix.

(iii) $\dim(\text{Col } A) = n$.

1. None of these

2. iii **correct**

3. ii and iii

4. i and iii

5. All of these

6. i and ii

Explanation:

(i) Because A is invertible, there is an $n \times n$ D such that $AD = I$. It follows that any \mathbf{b} in \mathbb{R}^n will satisfy the equation $AD\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$. If we let $\mathbf{x} = D\mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$. Hence there

is at least one solution, $D\mathbf{b}$, that satisfies the equation $A\mathbf{x} = \mathbf{b}$.

(ii) Since A is invertible, A has n pivot positions. With n pivot positions, the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n .

(iii) Since A is invertible, $\text{Col } A$ is a basis for \mathbb{R}^n . If $\text{Col } A$ is a basis for \mathbb{R}^n , it must have exactly n vectors. Hence the dimension of $\text{Col } A$ is n .

Rank02c
006 10.0 points

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 7 \\ -3 & 6 & -2 \end{bmatrix}.$$

1. $\text{rank}(A) = 3$
2. $\text{rank}(A) = 4$
3. $\text{rank}(A) = 2$ **correct**
4. $\text{rank}(A) = 1$
5. $\text{rank}(A) = 5$

Explanation:

Since

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the first two rows of $\text{rref}(A)$ contain leading 1's, so

$\text{Rank}(A) = 2$

DetVolume01a
007 10.0 points

Compute the volume of the parallelepiped with adjacent edges \overline{OP} , \overline{OQ} , and \overline{OR} determined by vertices

$$P(3, 2, -3), \quad Q(3, 3, -3), \quad R(3, 4, 4),$$

where O is the origin in 3-space.

1. volume = 18
2. volume = 17
3. volume = 20
4. volume = 21 **correct**
5. volume = 19

Explanation:

The parallelepiped is determined by the vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}.$$

Thus its volume is the absolute value of

$$\begin{aligned} \det \begin{bmatrix} 3 & 2 & -3 \\ 3 & 3 & -3 \\ 3 & 4 & 4 \end{bmatrix} \\ = 3 \begin{vmatrix} 3 & -3 \\ 4 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & -3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}. \end{aligned}$$

Consequently, the parallelepiped has

volume = 21

BasisNul01a
008 10.0 points

Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 2 & 6 & 2 & 4 \\ -1 & -3 & 1 & -8 \\ -1 & -3 & -2 & -2 \end{bmatrix}.$$

1. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
2. $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ **correct**

6. $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix} \right\}$

Explanation:

We first row reduce $[A \ \mathbf{0}]$:

$$\text{rref}([A \ \mathbf{0}]) = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

to identify the free variables for \mathbf{x} in the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Thus x_1, x_3 , and x_4 are basic variables, while x_2 is a free variable. So set $x_2 = s$. Then

$$x_1 = -3s, \ x_2 = s, \ x_3 = 0, \ x_4 = 0,$$

and

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\text{Nul}(A)$.

Basis03a
009 10.0 points

In the vector space V of all real-valued functions, find a basis for the subspace

$$H = \text{Span}\{\sin t, \sin 2t, \sin t \cos t\}.$$

1. $\{\cos t, \sin 2t\}$

2. $\{\cos t, \sin 2t, \sin t \cos t\}$

3. $\{\sin 2t, \sin t \cos t\}$

4. $\{\sin t, \sin 2t, \sin t \cos t\}$

5. $\{\sin t, \sin 2t\}$ **correct**

Explanation:

By double angle formula,

$$\sin 2t = 2 \sin t \cos t,$$

so the functions

$$\{\sin t, \sin 2t, \sin t \cos t\}$$

are linearly dependent, while

$$\{\sin t, \sin 2t\}$$

are linearly independent. Consequently,

$$\boxed{\{\sin t, \sin 2t\}}$$

is a basis for H .

PolyCoordVec01a
010 10.0 points

Find the coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ in \mathbb{R}^3 for the polynomial

$$\mathbf{p}(t) = 1 + 4t + 7t^2$$

with respect to the basis

$$\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$$

for \mathbb{P}_2 .

1. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ **correct**

$$2. [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$$

$$3. [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -6 \\ 1 \end{bmatrix}$$

$$4. [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 1 \\ 2 \end{bmatrix}$$

$$5. [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$$

$$6. [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$$

Explanation:

The coordinate mapping from \mathbb{P}_2 to \mathbb{R}^3 maps

$$\mathbf{p} = a + bt + ct^2 \longrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

When

$$\mathbf{p}(t) = 1 + 4t + 7t^2$$

and

$$\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\},$$

therefore, the entries c_1, c_2, c_3 in $[\mathbf{p}]_{\mathcal{B}}$ are the solutions of the polynomial equation

$$\begin{aligned} c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) \\ = \mathbf{p}(t) = 1 + 4t + 7t^2. \end{aligned}$$

Equating coefficients thus shows that c_1, c_2, c_3 satisfy the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix},$$

which in augmented matrix form becomes

$$A = \left[\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \right].$$

But then

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

ChangeBasis01b**011 (part 1 of 2) 10.0 points**

Determine the change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ of a vector space V when

$$\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2, \quad \mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2.$$

$$1. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 9 & -6 \\ 4 & -2 \end{bmatrix}$$

$$2. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}$$

$$3. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -6 & 9 \\ -2 & 4 \end{bmatrix}$$

$$4. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \text{ correct}$$

$$5. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 9 & 6 \\ -4 & -2 \end{bmatrix}$$

$$6. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & -9 \\ 2 & -4 \end{bmatrix}$$

Explanation:

The change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is the 2×2 matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}].$$

Now

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}.$$

Consequently,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}.$$

012 (part 2 of 2) 10.0 points

Determine $[\mathbf{x}]_{\mathcal{C}}$ when

$$\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2.$$

1. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$
2. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **correct**
3. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$
4. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
5. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
6. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

Explanation:

Now

$$\begin{aligned} \mathbf{x} &= -3\mathbf{b}_1 + 2\mathbf{b}_2 \\ &= -3(6\mathbf{c}_1 - 2\mathbf{c}_2) + 2(9\mathbf{c}_1 - 4\mathbf{c}_2) \\ &= 0\mathbf{c}_1 - 2\mathbf{c}_2. \end{aligned}$$

Consequently,

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$