<u>THEOREM</u> (The Gram-Schmidt Process):

Given an arbitrary basis $\{\bar{x}_1, \ldots, \bar{x}_p\}$ for a subspace W of R^n, define

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2$$

$\ldots$

$$\bar{v}_p = \bar{x}_p - \frac{\bar{x}_p \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \ldots - \frac{\bar{x}_p \cdot \bar{v}_{p-1}}{\bar{v}_{p-1} \cdot \bar{v}_{p-1}}\bar{v}_{p-1}$$

Then $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is an orthogonal basis for W.

<u>EXAMPLE</u>:

Let

$$\bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

be a basis for a subspace W of R^4. Find an orthogonal basis for W.

<u>SOLUTION</u>:

Step 1: Put

$$\bar{v}_1 = \bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 2: Put

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Step 3: Put

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$