Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1.

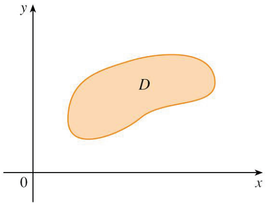


Figure 1

Double Integrals over General Regions

We suppose that *D* is a bounded region, which means that *D* can be enclosed in a rectangular region *R* as in Figure 2.

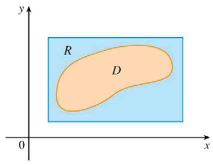


Figure 2

Then we define a new function *F* with domain *R* by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Double Integrals over General Regions

If *F* is integrable over *R*, then we define the **double integral of *f* over *D*** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because *R* is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

Double Integrals over General Regions

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside *D* and so they contribute nothing to the integral.

This means that it doesn't matter what rectangle *R* we use as long as it contains *D*.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above *D* and under the surface $z = f(x, y)$ (the graph of *f*).

Double Integrals over General Regions

You can see that this is reasonable by comparing the graphs of *f* and *F* in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of *F*.

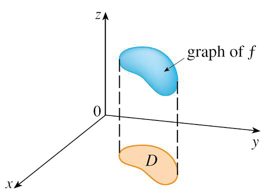


Figure 3

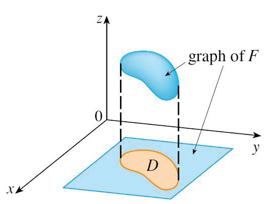


Figure 4

Double Integrals over General Regions

Figure 4 also shows that *F* is likely to have discontinuities at the boundary points of *D*.

Nonetheless, if *f* is continuous on *D* and the boundary curve of *D* is "well behaved", then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

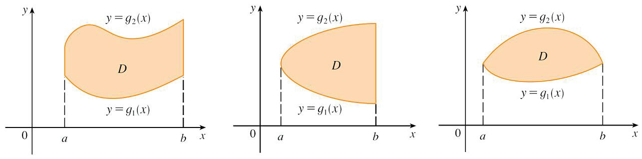


Figure 5
Some type I regions

Double Integrals over General Regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

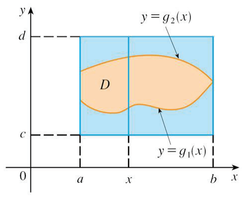


Figure 6

Double Integrals over General Regions

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

Double Integrals over General Regions

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of 3 is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Double Integrals over General Regions

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

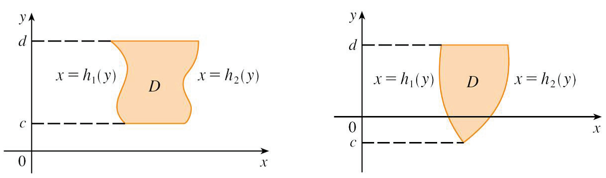


Figure 7
Some type II regions

Double Integrals over General Regions

Using the same methods that were used in establishing 3, we can show that

5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Example 1

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution:

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$

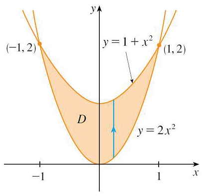


Figure 8

Example 1 – *Solution*

cont'd

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

Example 1 – *Solution*

cont'd

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1}$$

$$= \frac{32}{15}$$

Properties of Double Integrals

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region D follow immediately from Definition 2.

$$\boxed{6} \quad \iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

$$\boxed{7} \quad \iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \ge g(x, y)$ for all (x, y) in D, then

$$\boxed{8} \quad \iint_D f(x, y)\, dA \ge \iint_D g(x, y)\, dA$$

Properties of Double Integrals

The next property of double integrals is similar to the property of single integrals given by the equation

$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

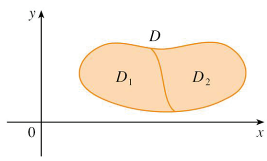
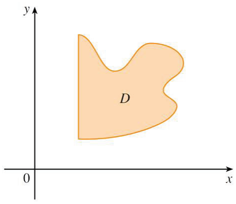


Figure 17

$$\boxed{9} \quad \boxed{\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA}$$

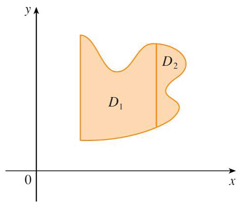
Properties of Double Integrals

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



D is neither type I nor type II.

Figure 18(a)



$D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Figure 18(b)

Properties of Double Integrals

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10 $$\iint_D 1\, dA = A(D)$$

Properties of Double Integrals

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

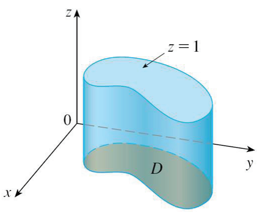


Figure 19

Cylinder with base D and height 1

Properties of Double Integrals

Finally, we can combine Properties 7, 8, and 10 to prove the following property.

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

Example 6

Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

Solution:

Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos y \leq 1$, we have $-1 \leq \sin x \cos y \leq 1$ and therefore

$$e^{-1} \leq e^{\sin x \cos y} \leq e^1 = e$$

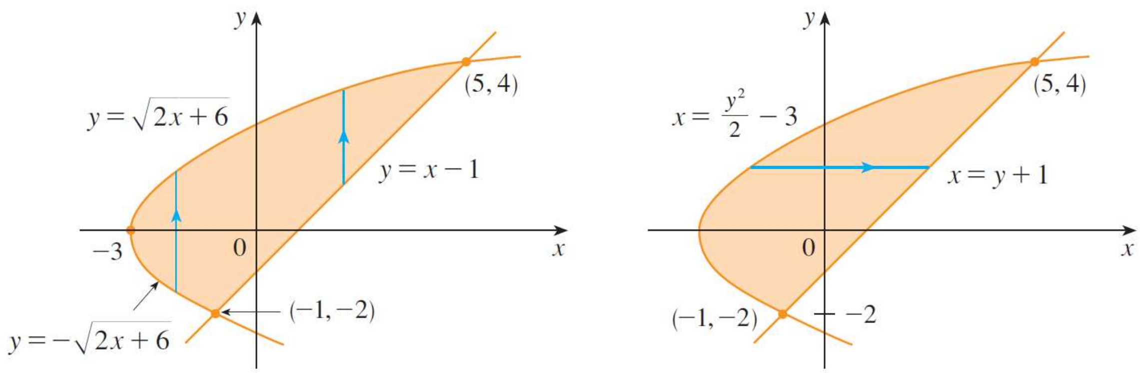
Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

V EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$



(a) D as a type I region

(b) D as a type II region

Then (5) gives

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

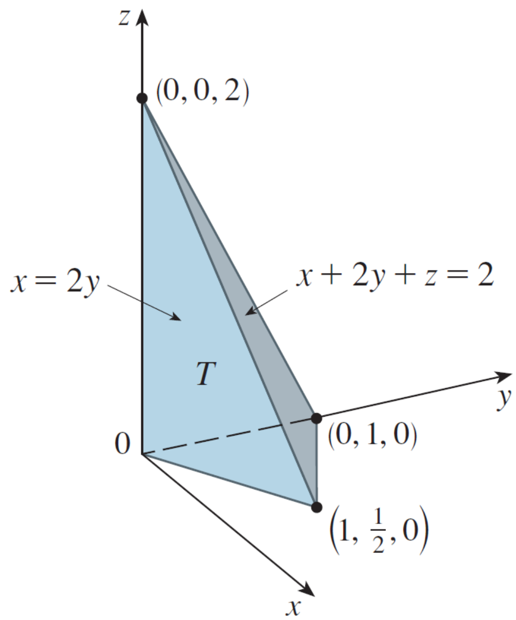
$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method.

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.



SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$\begin{aligned} V &= \iint_D (2 - x - 2y)\, dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\ &= \int_0^1 \Big[2y - xy - y^2\Big]_{y=x/2}^{y=1-x/2} dx \\ &= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx \\ &= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x \Big]_0^1 = \frac{1}{3} \end{aligned}$$