Section 7.2 Line Integrals

Work Done by Force Fields

If $\mathbf{F}$ is a force field in space, then a test particle (for example, a small unit charge in an electric force field or a unit mass in a gravitational field) will experience the force $\mathbf{F}$. Suppose the particle moves along the image of a path $\mathbf{c}$ while being acted upon by $\mathbf{F}$. A fundamental concept is the *work done* by $\mathbf{F}$ on the particle as it traces out the path $\mathbf{c}$. If $\mathbf{c}$ is a straight-line displacement given by the vector $\mathbf{d}$ and if $\mathbf{F}$ is a constant force, then the work done by $\mathbf{F}$ in moving the particle along the path is the dot product $\mathbf{F} \cdot \mathbf{d}$:

$\mathbf{F} \cdot \mathbf{d} =$ (magnitude of force) $\times$ (displacement in direction of force) $= \|\mathbf{F}\| \|\mathbf{d}\| \cos\theta$.

If the path is curved, we can imagine that it is made up of a succession of infinitesimal straight-line displacements or that it is *approximated* by a finite number of straight-line displacements. Then (as in our derivation of the formulas for the path integral in the preceding section) we are led to the following formula for the work done by the force field $\mathbf{F}$ on a particle moving along a path $\mathbf{c}$: $[a, b] \to \mathbb{R}^3$:

$$\text{work done by } \mathbf{F} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

We can further justify this derivation as follows. As t ranges over a small interval t to $t + \Delta t$, the particle moves from $\mathbf{c}(t)$ to $\mathbf{c}(t + \Delta t)$, a vector displacement of $\Delta \mathbf{s} = \mathbf{c}(t + \Delta t) - \mathbf{c}(t)$ (see Figure 7.2.1).

From the definition of the derivative, we get the approximation $\Delta \mathbf{s} \approx \mathbf{c}'(t)\Delta t$. The work done in going from $\mathbf{c}(t)$ to $\mathbf{c}(t + \Delta t)$ is therefore approximately

$$\mathbf{F}(\mathbf{c}(t)) \cdot \Delta \mathbf{s} \approx \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\Delta t.$$

If we subdivide the interval $[a, b]$ into n equal parts $a = t_0 < t_1 < \cdots < t_n = b$, with $\Delta t = t_{i+1} - t_i$, then the work done by $\mathbf{F}$ is approximately

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{c}(t_i)) \cdot \Delta \mathbf{s} \approx \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{c}(t_i)) \cdot \mathbf{c}'(t_i)\Delta t.$$

As $n \to \infty$, this approximation becomes better and better, and so it is reasonable to take as our definition of work to be the limit of the sum just given as $n \to \infty$. This limit is given by the integral

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

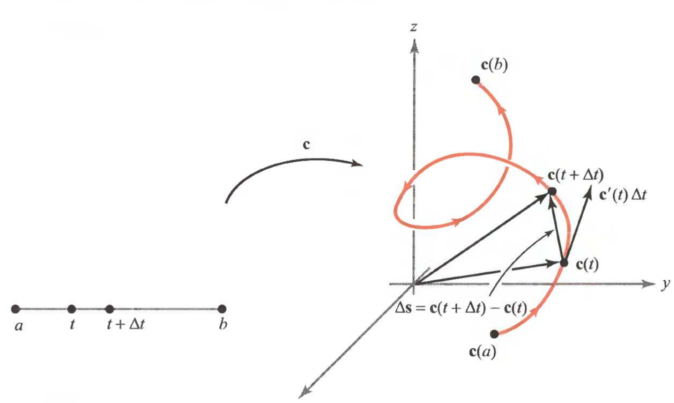

Figure 7.2.1 For small Δt, $\Delta \mathbf{s} = \mathbf{c}(t + \Delta t) - \mathbf{c}(t) \approx \mathbf{c}'(t)\Delta t$.

DEFINITION: Line Integrals Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt;$$

that is, we integrate the dot product of $\mathbf{F}$ with $\mathbf{c}'$ over the interval $[a,b]$.

As is the case with scalar functions, we can also define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ if $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$ is only piecewise continuous.

For paths $\mathbf{c}$ that satisfy $\mathbf{c}'(t) \neq \mathbf{0}$, there is another useful formula for the line integral: Namely, if $\mathbf{T}(t) = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ denotes the unit tangent vector, we have

$$\begin{aligned}
\int \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt && \text{(by definition)} \\
&= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \right] \|\mathbf{c}'(t)\|\, dt && \text{(canceling } \|\mathbf{c}'(t)\|\text{)} \qquad (1) \\
&= \int_a^b [\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)] \|\mathbf{c}'(t)\|\, dt.
\end{aligned}$$

Another common way of writing line integrals is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz,$$

where F_1, F_2, and F_3 are the components of the vector field $\mathbf{F}$. We call the expression $F_1\, dx + F_2\, dy + F_3\, dz$ a ***differential form***.[4] By *definition*, the integral of a differential form along a path $\mathbf{c}$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$, is

$$\int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

Note that we may think of $d\mathbf{s}$ as the differential form $d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. Thus, the differential form $F_1\, dx + F_2\, dy + F_3\, dz$ may be written as the dot product $\mathbf{F} \cdot d\mathbf{s}$.

EXAMPLE 1 Let $\mathbf{c}(t) = (\sin t, \cos t, t)$ with $0 \le t \le 2\pi$. Let the vector field $\mathbf{F}$ be defined by $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

DEFINITION: Line Integrals Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt;$$

that is, we integrate the dot product of $\mathbf{F}$ with $\mathbf{c}'$ over the interval $[a, b]$.

As is the case with scalar functions, we can also define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ if $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$ is only piecewise continuous.

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SOLUTION Here, $\mathbf{F}(\mathbf{c}(t)) = \mathbf{F}(\sin t, \cos t, t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, and $\mathbf{c}'(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}$. Therefore,

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \sin t \cos t - \cos t \sin t + t = t,$$

and so

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} t\, dt = 2\pi^2. \quad \blacktriangle$$

EXAMPLE 2 Evaluate the line integral

$$\int_{\mathbf{c}} x^2\, dx + xy\, dy + dz,$$

where $\mathbf{c}\colon [0, 1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1) = (x(t), y(t), z(t))$.

DEFINITION: Line Integrals Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c}: [a, b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

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SOLUTION We compute $dx/dt = 1,\ dy/dt = 2t,\ dz/dt = 0$; therefore,

$$\int_{\mathbf{c}} x^2\, dx + xy\, dy + dz = \int_0^1 \left([x(t)]^2 \frac{dx}{dt} + [x(t)y(t)] \frac{dy}{dt}\right) dt$$

$$= \int_0^1 (t^2 + 2t^4)\, dt = \left[\frac{1}{3}t^3 + \frac{2}{5}t^5\right]_0^1 = \frac{11}{15}. \quad \blacktriangle$$

EXAMPLE 3 Evaluate the line integral

$$\int_{\mathbf{c}} \cos z\, dx + e^x\, dy + e^y\, dz,$$

where the path $\mathbf{c}$ is defined by $\mathbf{c}(t) = (1, t, e^t)$ and $0 \le t \le 2$.

DEFINITION: Line Integrals Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c}$: $[a, b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

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SOLUTION We compute $dx/dt = 0$, $dy/dt = 1$, $dz/dt = e^t$, and so

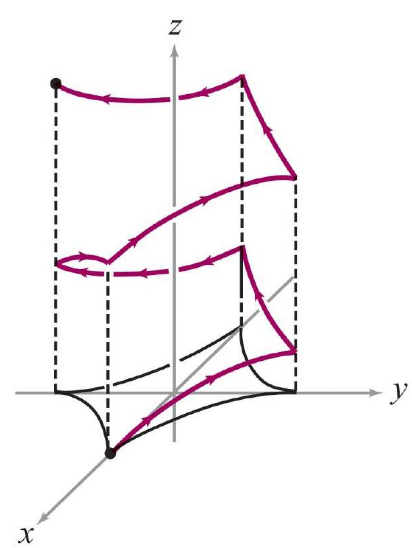
$$\int_{\mathbf{c}} \cos z\, dx + e^x\, dy + e^y\, dz = \int_0^2 (0 + e + e^{2t})\, dt$$

$$= \left[et + \frac{1}{2} e^{2t} \right]_0^2 = 2e + \frac{1}{2} e^4 - \frac{1}{2}. \quad \blacktriangle$$

EXAMPLE 4 Let $\mathbf{c}$ be the path

$$x = \cos^3 \theta, \qquad y = \sin^3 \theta, \qquad z = \theta, \qquad 0 \leq \theta \leq \frac{7\pi}{2}$$

(see Figure 7.2.2). Evaluate the integral $\int_{\mathbf{c}} (\sin z\, dx + \cos z\, dy - (xy)^{1/3}\, dz)$.



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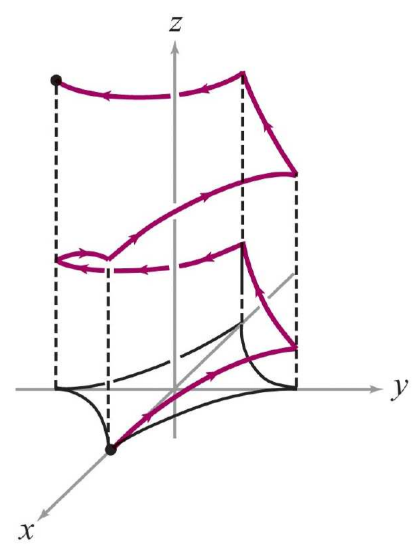


SOLUTION In this case, we have

$$\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta, \qquad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta, \qquad \frac{dz}{d\theta} = 1,$$

so the integral is

$$\int_{\mathbf{c}} \sin z\, dx + \cos z\, dy - (xy)^{1/3}\, dz$$

$$= \int_0^{7\pi/2} (-3\cos^2\theta\sin^2\theta + 3\sin^2\theta\cos^2\theta - \cos\theta\sin\theta)\, d\theta.$$

The first two terms cancel, and so we get

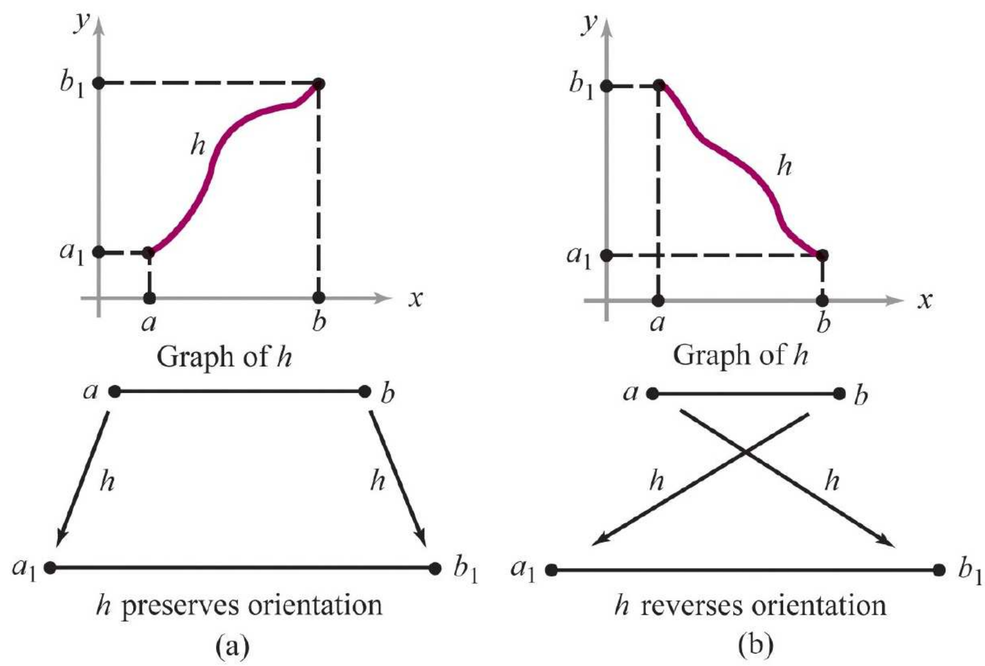
$$-\int_0^{7\pi/2} \cos\theta\sin\theta\, d\theta = -\left[\frac{1}{2}\sin^2\theta\right]_0^{7\pi/2} = -\frac{1}{2}. \quad \blacktriangle$$

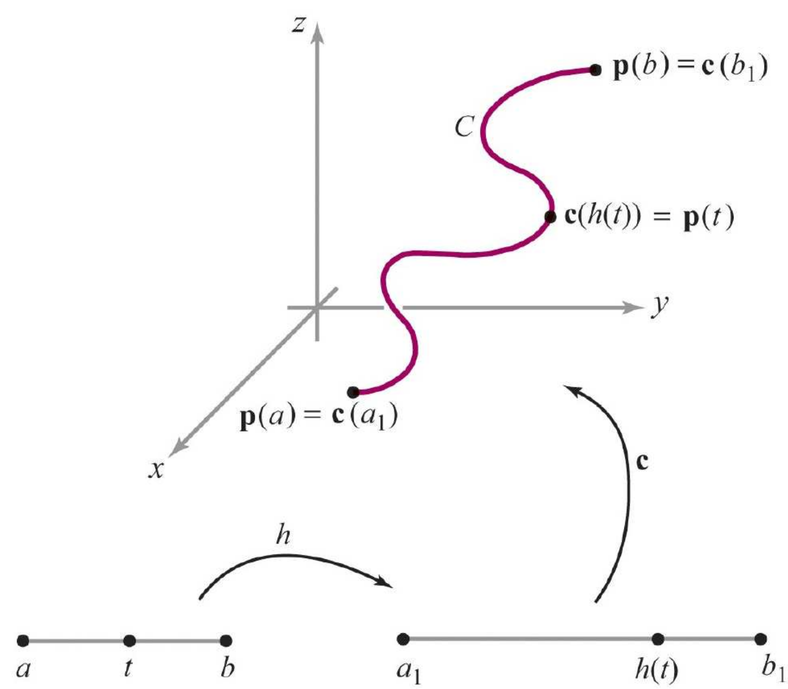
Parametrizations

DEFINITION Let $h: I \to I_1$ be a C^1 real-valued function that is a one-to-one map of an interval $I = [a, b]$ onto another interval $I_1 = [a_1, b_1]$. Let $\mathbf{c}: I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition

$$\mathbf{p} = \mathbf{c} \circ h: I \to \mathbb{R}^3$$

a ***reparametrization*** of $\mathbf{c}$.





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$$\mathbf{p} = \mathbf{c} \circ h: I \to \mathbb{R}^3$$

a ***reparametrization*** of $\mathbf{c}$.

EXAMPLE 7 Let $\mathbf{c}: [a, b] \to \mathbb{R}^3$ be a piecewise C^1 path. Then:

(a) The path $\mathbf{c}_{\text{op}}: [a, b] \to \mathbb{R}^3, t \mapsto \mathbf{c}(a + b - t)$, is reparametrization of $\mathbf{c}$ corresponding to the map $h: [a, b] \to [a, b], t \mapsto a + b - t$; we call $\mathbf{c}_{\text{op}}$ the ***opposite path*** to $\mathbf{c}$. This reparametrization is orientation-reversing.

(b) The path $\mathbf{p}: [0, 1] \to \mathbb{R}^3, t \mapsto \mathbf{c}(a + (b - a)t)$, is an orientation-preserving reparametrization of $\mathbf{c}$ corresponding to a change of coordinates $h: [0, 1] \to [a, b], t \mapsto a + (b - a)t$. ▲

THEOREM 1: Change of Parametrization for Line Integrals
Let $\mathbf{F}$ be a vector field continuous on the C^1 path $\mathbf{c}: [a_1, b_1] \to \mathbb{R}^3$, and let $\mathbf{p}: [a, b] \to \mathbb{R}^3$ be a reparametrization of $\mathbf{c}$. If $\mathbf{p}$ is orientation-preserving, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s},$$

and if $\mathbf{p}$ is orientation-reversing, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

PROOF By hypothesis, we have a map h such that $\mathbf{p} = \mathbf{c} \circ h$. By the chain rule,

$$\mathbf{p}'(t) = \mathbf{c}'(h(t))h'(t),$$

and so

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b [\mathbf{F}(\mathbf{c}(h(t))) \cdot \mathbf{c}'(h(t))]h'(t)\, dt.$$

Changing variables with $s = h(t)$, this becomes

$$\int_{h(a)}^{h(b)} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds$$

$$= \begin{cases} \displaystyle\int_{a_1}^{b_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{p} \text{ is orientation-preserving} \\ \displaystyle\int_{b_1}^{a_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{p} \text{ is orientation-reversing.} \end{cases}$$

■

THEOREM 1: Change of Parametrization for Line Integrals Let $\mathbf{F}$ be a vector field continuous on the C^1 path $\mathbf{c}\colon [a_1, b_1] \to \mathbb{R}^3$, and let $\mathbf{p}\colon [a, b] \to \mathbb{R}^3$ be a reparametrization of $\mathbf{c}$. If $\mathbf{p}$ is orientation-preserving, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s},$$

and if $\mathbf{p}$ is orientation-reversing, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

EXAMPLE 8 Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\mathbf{c}\colon [-5, 10] \to \mathbb{R}^3$ be defined by $t \mapsto (t, t^2, t^3)$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{\mathbf{c}_{\mathrm{op}}} \mathbf{F} \cdot d\mathbf{s}$.

THEOREM 1: Change of Parametrization for Line Integrals Let $\mathbf{F}$ be a vector field continuous on the C^1 path $\mathbf{c}$: $[a_1, b_1] \to \mathbb{R}^3$, and let $\mathbf{p}$: $[a, b] \to \mathbb{R}^3$ be a reparametrization of $\mathbf{c}$. If $\mathbf{p}$ is orientation-preserving, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s},$$

and if $\mathbf{p}$ is orientation-reversing, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

EXAMPLE 8 Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\mathbf{c}$: $[-5, 10] \to \mathbb{R}^3$ be defined by $t \mapsto (t, t^2, t^3)$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{\mathbf{c}_{\mathrm{op}}} \mathbf{F} \cdot d\mathbf{s}$.

SOLUTION For the path $\mathbf{c}$, we have $dx/dt = 1$, $dy/dt = 2t$, $dz/dt = 3t^2$, and $\mathbf{F}(\mathbf{c}(t)) = t^5\mathbf{i} + t^4\mathbf{j} + t^3\mathbf{k}$. Therefore,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{-5}^{10} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_{-5}^{10} (t^5 + 2t^5 + 3t^5)\, dt = [t^6]_{-5}^{10} = 984,375.$$

On the other hand, for

$$\mathbf{c}_{\mathrm{op}}: [-5, 10] \to \mathbb{R}^3, t \mapsto \mathbf{c}(5 - t) = (5 - t, (5 - t)^2, (5 - t)^3),$$

we have $dx/dt = -1$, $dy/dt = -10 + 2t = -2(5 - t)$, $dz/dt = -75 + 30t - 3t^2 = -3(5 - t)^2$, and $\mathbf{F}(\mathbf{c}_{\mathrm{op}}(t)) = (5 - t)^5\mathbf{i} + (5 - t)^4\mathbf{j} + (5 - t)^3\mathbf{k}$. Therefore,

$$\int_{\mathbf{c}_{\mathrm{op}}} \mathbf{F} \cdot d\mathbf{s} = \int_{-5}^{10} [-(5 - t)^5 - 2(5 - t)^5 - 3(5 - t)^5]\, dt = [(5 - t)^6]_{-5}^{10} = -984,375. \quad \blacktriangle$$

THEOREM 2: Change of Parametrization for Path Integrals Let $\mathbf{c}$ be piecewise C^1, let f be a continuous (real-valued) function on the image of $\mathbf{c}$, and let $\mathbf{p}$ be any reparametrization of $\mathbf{c}$. Then

$$\int_{\mathbf{c}} f(x, y, z)\, ds = \int_{\mathbf{p}} f(x, y, z)\, ds. \tag{2}$$

Line Integrals of Gradient Fields

THEOREM 3: Line Integrals of Gradient Vector Fields Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and that $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

PROOF Apply the chain rule to the composite function

$$F\colon t \mapsto f(\mathbf{c}(t))$$

to obtain

$$F'(t) = (f \circ \mathbf{c})'(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

The function F is a real-valued function of the variable t, and so, by the fundamental theorem of single-variable calculus,

$$\int_a^b F'(t)\, dt = F(b) - F(a) = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

Therefore,

$$\begin{aligned}\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt &= \int_a^b F'(t)\, dt = F(b) - F(a) \\ &= f(\mathbf{c}(b)) - f(\mathbf{c}(a)). \quad \blacksquare\end{aligned}$$

EXAMPLE 9 Let $\mathbf{c}$ be the path $\mathbf{c}(t) = (t^4/4, \sin^3(t\pi/2), 0)$, $t \in [0, 1]$. Evaluate

$$\int_{\mathbf{c}} y\, dx + x\, dy$$

$\big($which means $\int_{\mathbf{c}} y\, dx + x\, dy + 0\, dz\big)$.

Line Integrals of Gradient Fields

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$$\int_{\mathbf{c}} y\,dx + x\,dy$$

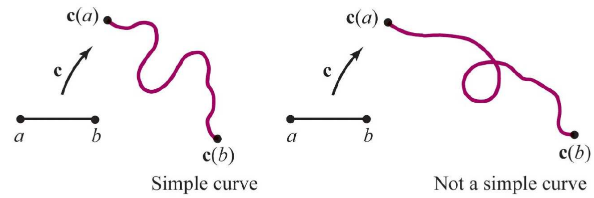
$\left(\text{which means } \int_{\mathbf{c}} y\,dx + x\,dy + 0\,dz\right)$.

SOLUTION We recognize $y\,dx + x\,dy$, or equivalently, the vector field $y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$, as the gradient of the function $f(x, y, z) = xy$. Thus,

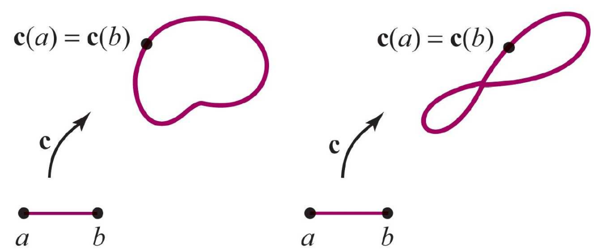
$$\int_{\mathbf{c}} y\,dx + x\,dy = f(\mathbf{c}(1)) - f(\mathbf{c}(0)) = \frac{1}{4} \cdot 1 - 0 = \frac{1}{4}. \quad ▲$$

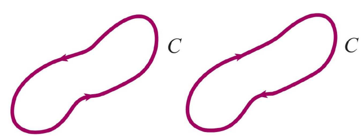
Line Integrals Over Geometric Curves

DEFINITION We define a ***simple curve*** C to be the image of a piecewise C^1 map $\mathbf{c}\colon I \to \mathbb{R}^3$ that is one-to-one on an interval I; $\mathbf{c}$ is called a ***parametrization*** of C. Thus, a simple curve is one that does not intersect itself (Figure 7.2.6). If $I = [a, b]$, we call $\mathbf{c}(a)$ and $\mathbf{c}(b)$ ***endpoints*** of the curve.



DEFINITION: Simple Closed Curves By a ***simple closed curve*** we mean the image of a piecewise C^1 map $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ that is one-to-one on $[a, b)$ and satisfies $\mathbf{c}(a) = \mathbf{c}(b)$ (Figure 7.2.8). If $\mathbf{c}$ satisfies the condition $\mathbf{c}(a) = \mathbf{c}(b)$, but is not necessarily one-to-one on $[a, b)$, we call its image a ***closed curve***. Simple closed curves have two orientations, corresponding to the two possible directions of motion along the curve (Figure 7.2.9).





Line Integrals Over Oriented Simple Curves and Simple Closed Curves C:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \quad \text{and} \quad \int_C f\, ds = \int_{\mathbf{c}} f\, ds, \tag{3}$$

where $\mathbf{c}$ is any *orientation-preserving* parametrization of C.

EXAMPLE 10 If $I = [a, b]$ is a closed interval on the x axis, then I, as a curve, has two orientations: one corresponding to motion from a to b (left to right) and the other corresponding to motion from b to a (right to left). If f is a real-valued function continuous on I, then denoting I with the first orientation by I^+ and I with the second orientation by I^-, we have

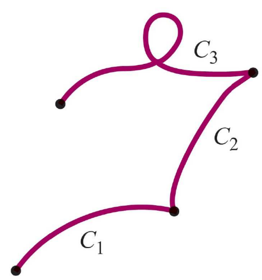
$$\int_{I^+} f(x)\, dx = \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx = -\int_{I^-} f(x)\, dx. \quad \blacktriangle$$

Line Integrals Over Curves with Opposite Orientations Let C^- be the same curve as C, but with the opposite orientation. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_{C^-} \mathbf{F} \cdot d\mathbf{s}.$$

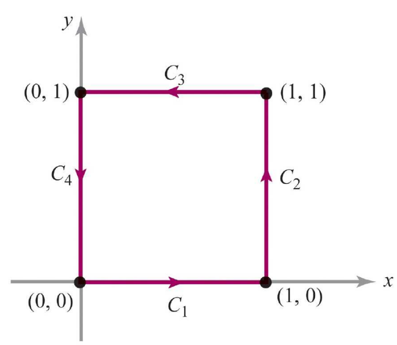
Line Integrals Over Curves Consisting of Several Components Let C be an oriented curve that is made up of several oriented component curves C_i, $i = 1, \ldots, k$, as in Figure 7.2.11. Then we shall write $C = C_1 + C_2 + \cdots + C_k$. Because we can parametrize C by parametrizing the pieces $C_1, \ldots, C_k$ separately, one can prove that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{C_k} \mathbf{F} \cdot d\mathbf{s}. \tag{4}$$



EXAMPLE 11 Consider C, the perimeter of the unit square in $\mathbb{R}^2$, oriented in the counterclockwise sense (see Figure 7.2.12). Evaluate the line integral

$$\int_C x^2\,dx + xy\,dy.$$



EXAMPLE 11 Consider C, the perimeter of the unit square in $\mathbb{R}^2$, oriented in the counterclockwise sense (see Figure 7.2.12). Evaluate the line integral

$$\int_C x^2\,dx + xy\,dy.$$

SOLUTION We evaluate the integral using a convenient parametrization of C that induces the given orientation. For example:

$$\mathbf{c}\colon [0,4] \to \mathbb{R}^2, \qquad t \mapsto \begin{cases} (t,0) & 0 \le t \le 1 \\ (1, t-1) & 1 \le t \le 2 \\ (3-t, 1) & 2 \le t \le 3 \\ (0, 4-t) & 3 \le t \le 4. \end{cases}$$

Then

$$\begin{aligned} \int_C x^2\,dx + xy\,dy &= \int_0^1 (t^2 + 0)\,dt + \int_1^2 [0 + (t-1)]\,dt \\ &\quad + \int_2^3 [-(3-t)^2 + 0]\,dt + \int_3^4 (0+0)\,dt \\ &= \frac{1}{3} + \frac{1}{2} + \left(-\frac{1}{3}\right) + 0 = \frac{1}{2}. \end{aligned}$$

Now let us reevaluate this line integral, using formula (4) and parametrizing the C_i separately. Notice that $C = C_1 + C_2 + C_3 + C_4$, where C_i are the oriented curves pictured in Figure 7.2.12. These can be parametrized as follows:

$$C_1\colon\ \mathbf{c}_1(t) = (t, 0),\ 0 \le t \le 1$$
$$C_2\colon\ \mathbf{c}_2(t) = (1, t),\ 0 \le t \le 1$$
$$C_3\colon\ \mathbf{c}_3(t) = (1-t, 1),\ 0 \le t \le 1$$
$$C_4\colon\ \mathbf{c}_4(t) = (0, 1-t),\ 0 \le t \le 1,$$

and so

$$\int_{C_1} x^2\,dx + xy\,dy = \int_0^1 t^2\,dt = \frac{1}{3}$$
$$\int_{C_2} x^2\,dx + xy\,dy = \int_0^1 t\,dt = \frac{1}{2}$$
$$\int_{C_3} x^2\,dx + xy\,dy = \int_0^1 -(1-t)^2\,dt = -\frac{1}{3}$$
$$\int_{C_4} x^2\,dx + xy\,dy = \int_0^1 0\,dt = 0.$$

Thus, again,

$$\int_C x^2\,dx + xy\,dy = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{2}. \quad \blacktriangle$$