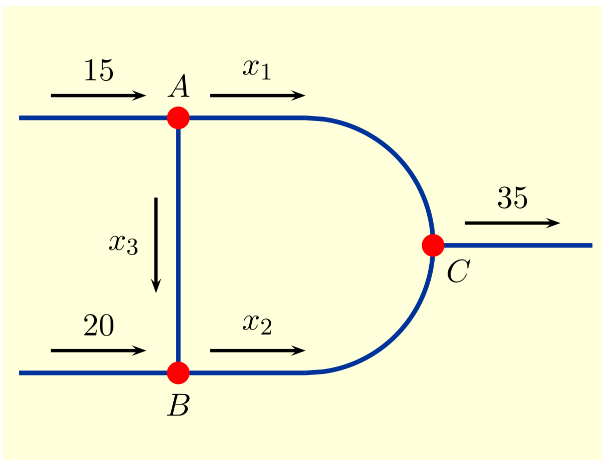
This print-out should have 35 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

Network01a
001 10.0 points

The volume of traffic (in average number of vehicles per minute) through three intersections is shown in



Find all possible values for x_2 in terms of a free variable s.

1. $x_2 = 15 + s$

2. $x_2 = 20 + s$ **correct**

3. $x_2 = 70 + s$

4. $x_2 = 5 + s$

5. $x_2 = 35 + s$

Explanation:

At each intersection flow in equals flow out. Thus

$$x_1 + 0x_2 + x_3 = 15,$$

$$0x_1 + x_2 - x_3 = 20,$$

$$x_1 + x_2 + 0x_3 = 35.$$

But as an augmented matrix,

$$\text{rref}\begin{bmatrix} 1 & 0 & 1 & 15 \\ 0 & 1 & -1 & 20 \\ 1 & 1 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 15 \\ 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus x_3 is a free variable, say $x_3 = s$, and

$$\boxed{x_1 = 15 - s, \qquad x_2 = 20 + s}.$$

Span02a
002 10.0 points

For each of the following pairs of vectors $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$ determine whether

$$H = \text{Span}\{\mathbf{u}, \mathbf{v}\}$$

is a line in $\mathbb{R}^3$.

$$\text{I: } \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix},$$

$$\text{II: } \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix},$$

$$\text{III: } \mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1. I and III **correct**

2. II only

3. I only

4. II and III

5. I and II

6. III only

Explanation:

For vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$, Span$\{\mathbf{u}, \mathbf{v}\}$ consists of all linear combinations

$$s\mathbf{u} + t\mathbf{v}, \quad -\infty < s, t < \infty.$$

Now if $\mathbf{u}$, $\mathbf{v}$ are linearly dependent, then $\mathbf{u}$, $\mathbf{v}$ are scalar multiples of each other, in which case

$$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \{t\mathbf{v} : -\infty < t < \infty\},$$

and Span$\{\mathbf{u},\, \mathbf{v}\}$ is the line Span$\{\mathbf{v}\}$ through the origin.

On the other hand, if $\mathbf{u}$, $\mathbf{v}$ are linearly independent, then $\mathbf{u}$, $\mathbf{v}$ are not scalar multiples of each other and Span$\{\mathbf{u},\, \mathbf{v}\}$ contains the *distinct* lines Span$\{\mathbf{u}\}$ and Span$\{\mathbf{v}\}$. In this case Span$\{\mathbf{u},\, \mathbf{v}\}$ is a plane through the origin.

For the given pairs $\{\mathbf{u},\, \mathbf{v}\}$:

I: $\mathbf{v} = 2\mathbf{u}$,

II: $\mathbf{v}$ is not a scalar multiple of $\mathbf{u}$,

III $\{\mathbf{u},\, \mathbf{0}\}$ always is a linearly dependent set.

Consequently, only

$$\boxed{\text{I and III are lines}}\,.$$

LinTrans02a
003 10.0 points

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix},$$

determine $T(\mathbf{x})$ when $\mathbf{x} = \begin{bmatrix}1\\3\end{bmatrix}$.

1. $T(\mathbf{x}) = \begin{bmatrix}6\\-6\end{bmatrix}$

2. $T(\mathbf{x}) = \begin{bmatrix}1\\3\end{bmatrix}$

3. $T(\mathbf{x}) = \begin{bmatrix}6\\-5\end{bmatrix}$

4. $T(\mathbf{x}) = \begin{bmatrix}5\\-5\end{bmatrix}$ **correct**

5. $T(\mathbf{x}) = \begin{bmatrix}5\\-6\end{bmatrix}$

Explanation:
Since

$$\mathbf{x} = \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$

and T is a linear transformation,

$$\begin{aligned} T(\mathbf{x}) &= T\left(\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 3\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= \begin{bmatrix}2\\-2\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix}. \end{aligned}$$

Consequently

$$\boxed{T(\mathbf{x}) = \begin{bmatrix}2\\-2\end{bmatrix} + \begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}5\\-5\end{bmatrix}}\,.$$

LinTrans03b
004 10.0 points

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$T(x_1,\, x_2) \;=\; (4x_1 + x_2,\, -3x_1 + 2x_2)\,.$$

Determine A so that T can be written as the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

1. $A = \begin{bmatrix}4 & 0 & -3\\0 & 1 & 1\end{bmatrix}$

2. $A = \begin{bmatrix}4 & 0 & 1\\0 & 1 & -3\end{bmatrix}$

3. $A = \begin{bmatrix}4 & -3\\1 & 2\end{bmatrix}$

4. $A = \begin{bmatrix}4 & 1\\-3 & 2\end{bmatrix}$ **correct**

Explanation:
We can write $\mathbb{R}^2$ both as rows and column vectors

$$\text{(i)} \quad (x_1,\, x_2)\,, \qquad \text{(ii)} \quad \begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

In row form

$$T(x_1,\, x_2) \;=\; (4x_1 + x_2,\, -3x_1 + 2x_2)\,,$$

while in column vector form

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where A is the 2×2 matrix standard matrix

$$[T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)]$$

of T. Now

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1, 0)\,, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0, 1)\,,$$

so that

$$T(1, 0) = (4, -3) = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = T(\mathbf{e}_1)\,,$$

and

$$T(0, 1) = (1, 2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = T(\mathbf{e}_2)\,.$$

Consequently,

$$\boxed{A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}}\,.$$

InverseMatrix03a
005 10.0 points

Determine the product AB^{-1} when

$$A = [\,2\ \ 1\ \ -3\,]\,, \quad B = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -5 & 2 \\ -2 & 3 & -1 \end{bmatrix}.$$

1. $AB^{-1} = [\,12\ \ -10\ \ -3\,]$

2. $AB^{-1} = [\,8\ \ -8\ \ -3\,]$

3. $AB^{-1} = [\,12\ \ -8\ \ -3\,]$

4. $AB^{-1} = [\,8\ \ -10\ \ -5\,]$

5. $AB^{-1} = [\,12\ \ -8\ \ -5\,]$

6. $AB^{-1} = [\,8\ \ -8\ \ -5\,]$ **correct**

Explanation:

The inverse matrix B^{-1} can be computed by reducing the augmented matrix $[\,B \quad I_3\,]$ to row-reduced echelon form $[\,I_3 \quad B^{-1}\,]$.

Now after row reduction downwards we see that

$$[\,B \quad I_3\,] = \begin{bmatrix} 1 & -3 & 1 & 1 & 0 & 0 \\ 2 & -5 & 2 & 0 & 1 & 0 \\ -2 & 3 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim [\,B \quad I_3\,] = \begin{bmatrix} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim [\,B \quad I_3\,] = \begin{bmatrix} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{bmatrix}.$$

But then after row reduction upwards we see that

$$[\,B \quad I_3\,] \sim \begin{bmatrix} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{bmatrix}.$$

Thus

$$B^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}.$$

Consequently,

$$AB^{-1} = [\,2\ \ 1\ \ -3\,]\begin{bmatrix} -1 & 0 & -1 \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}$$

$$\boxed{= [\,8\ \ -8\ \ -5\,]}\,.$$

InvertibleTF01c
006 10.0 points

If A is an $n \times n$ matrix, when does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$?

1. ALWAYS

2. NEVER

3. SOMETIMES **correct**

Explanation:

By the Invertible Matrix Theorem, an $n \times n$ matrix A will have the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$ if and only if A is invertible.

Consequently, the equation $A\mathbf{x} = \mathbf{b}$ will

$$\boxed{\text{SOMETIMES}}$$

have at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, but not always.

LUDecomp05b
007 10.0 points

Determine the unique solution x_2 of the matrix equation

$$A\mathbf{x} \;=\; A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;=\; \begin{bmatrix} -9 \\ 24 \\ 35 \end{bmatrix}$$

when A has an LU-decomposition

$$A \;=\; \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

1. $x_2 \;=\; 3$

2. $x_2 \;=\; 0$

3. $x_2 \;=\; 2$

4. $x_2 \;=\; 1$ **correct**

5. $x_2 \;=\; 4$

Explanation:

Set $\mathbf{y} = U\mathbf{x}$. Then $A\mathbf{x} = L\mathbf{y} = \mathbf{b}$, and so $\mathbf{y} = L^{-1}\mathbf{b}$. Now

$$L^{-1} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix},$$

in which case $A\mathbf{x} = \mathbf{b}$ reduces to

$$U\mathbf{x} = L^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ 24 \\ 35 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ -4 \end{bmatrix}.$$

But then,

$$U\mathbf{x} = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ -4 \end{bmatrix}.$$

which is equivalent to the system

$$2x_3 \;=\; -4, \qquad 4x_2 - x_3 \;=\; 6,$$

and

$$-3x_1 + x_2 + 2x_3 \;=\; -9\,.$$

So by back substitution, $x_3 = -2$ and

$$\boxed{x_2 \;=\; 1}\,.$$

NullSpace01a
008 10.0 points

Find a matrix A so that $\mathrm{Nul}(A)$ is the set of all vectors

$$H \;=\; \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \quad \begin{array}{rcl} a + 4b &=& 2c\,, \\ 3a &=& c + d\,, \end{array} \right\}$$

in $\mathbb{R}^4$.

1. $A \;=\; \begin{bmatrix} 1 & -4 & -2 & 0 \\ 3 & 0 & 1 & 1 \end{bmatrix}$

2. $A \;=\; \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & 0 & -1 & -1 \end{bmatrix}$ **correct**

3. $A \;=\; \begin{bmatrix} 1 & 4 & 2 & 0 \\ 3 & 0 & 1 & 1 \end{bmatrix}$

4. $A \;=\; \begin{bmatrix} 1 & -4 & 2 & 0 \\ 3 & 0 & -1 & -1 \end{bmatrix}$

5. $A \;=\; \begin{bmatrix} 1 & -4 & 2 & 0 \\ 3 & 0 & 1 & 1 \end{bmatrix}$

6. $A \;=\; \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & 0 & -1 & 1 \end{bmatrix}$

Explanation:

Rewrite the conditions

$$a + 4b = 2c, \quad 3a = c + d$$

as

$$a + 4b - 2c = 0,$$

$$3a - c - d = 0\,,$$

and set

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & 0 & -1 & -1 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} a + 4b - 2c \\ 3a - c + d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if and only if

$$a + 4b - 2c = 0,$$

$$3a - c - d = 0\,.$$

Consequently,

$$\boxed{\text{Nul}(A) = H}\,.$$

Rank02c
009 10.0 points

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -2 & -6 \\ -2 & 2 & 3 \end{bmatrix}.$$

1. $\text{rank}(A) = 5$

2. $\text{rank}(A) = 3$

3. $\text{rank}(A) = 4$

4. $\text{rank}(A) = 1$

5. $\text{rank}(A) = 2$ **correct**

Explanation:

Since

$$\text{rref}(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the first two rows of rref(A) contain leading 1's, so

$$\boxed{\text{Rank}(A) = 2}\,.$$

SpanningT/F01a
010 10.0 points

Three vectors in $\mathbb{R}^5$ always span $\mathbb{R}^5$. True or False?

1. TRUE

2. FALSE **correct**

Explanation:

The space $\mathbb{R}^5$ is 5-dimensional, so at least *five* vectors are needed to span $\mathbb{R}^5$.

Consequently, the statement is

$$\boxed{\text{FALSE}}\,.$$

ComputeDeterminant01
011 10.0 points

Compute the determinant of the following elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}.$$

1. $1 + k$

2. 1 **correct**

3. $1 - k$

4. 0

5. k

Explanation:

A cofactor expansion along row 1 gives

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} = 1$$

Also since the matrix is triangular, the determinant is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = (1)(1)(1) = 1$$

DetPropTF01c
012 10.0 points

If the columns of an $n \times n$ matrix A are linearly dependent, then $\det A = 0$.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

A square matrix A is invertible if and only if $\det A \neq 0$. Thus if $\det A = 0$, A is not invertible. On the other hand, a square matrix is not invertible if and only if its columns are linearly dependent.

Consequently, $\det A = 0$ when the columns of A are linearly dependent, so the statement is

$$\boxed{\text{TRUE}}.$$

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SubspaceTF01
013 10.0 points

Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$.

H is a subspace of $\mathbb{R}^2$. True or false?

1. FALSE **correct**

2. TRUE

Explanation:

If $\mathbf{u} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and $c = 4$, then $\mathbf{u}$ is in H but $c\mathbf{u}$ is not in H. Since H is not closed under scalar multiplication, H is not a subspace of $\mathbb{R}^2$. Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

VectorSubSpaceTF01f
014 10.0 points

The set

$$H = \left\{ \begin{bmatrix} a + 2b \\ a - b \\ 3b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

By matrix algebra,

$$\begin{bmatrix} a + 2b \\ a - b \\ 3b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},$$

with a, b in $\mathbb{R}$. Thus H consists of all linear combinations of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

in $\mathbb{R}^3$, and so H is the span of $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$. On the other hand, the span of any set of vectors is a vector space.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

BasisNull02a
015 10.0 points

Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 2 & 4 & -10 & -14 \\ -1 & 1 & 2 & 1 \\ 1 & 3 & -6 & -9 \end{bmatrix}.$$

1. $\left\{ \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ **correct**

Explanation:

We first row reduce $[A \;\; \mathbf{0}]$:

$$\text{rref}([A \;\; \mathbf{0}]) = \begin{bmatrix} 1 & 0 & -3 & -3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to identify the free variables for $\mathbf{x}$ in the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

This shows that x_1, x_2 are basic variables, while x_3, x_4 are free variables. So set $x_3 = s$, $x_4 = t$. Then

$$x_1 = 3s + 3t, \quad x_2 = s + 2t,$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 3t \\ s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ s \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\text{Nul}(A)$.

BasisCol01b
016 10.0 points

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 2 & -2 & 8 & 2 \\ 1 & -2 & 5 & 3 \\ -2 & 0 & -6 & -1 \end{bmatrix}.$$

1. $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$

6. $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$ **correct**

Explanation:

We first row reduce A:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to identify the pivot columns of A. These are the first, second and fourth columns of A.

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

is a basis for Col(A).

LinIndSetsTF01b
017 10.0 points

When $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ are vectors in $\mathbb{R}^n$ and

$$H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\},$$

then $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ is a basis for H.

True or False?

1. TRUE

2. FALSE **correct**

Explanation:

For the set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ to be a basis for H, BOTH of the conditions

(i) $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$,

(ii) *the set is linearly independent*,

have to be satisfied. Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

CoordVec02a
018 10.0 points

Find the vector $\mathbf{x}$ in $\mathbb{R}^3$ having coordinate vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$$

with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$.

1. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$ **correct**

2. no such $\mathbf{x}$ exists

3. $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$

4. $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$

5. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$

Explanation:

The coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of a vector $\mathbf{x}$ in $\mathbb{R}^3$ with respect to a basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$$

for $\mathbb{R}^3$ satisfies the matrix equation

$$A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \qquad A = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,].$$

Consequently, when

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\},$$

and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix},$$

$$\boxed{\mathbf{x} = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}}.$$

DimensionTF04d
019 10.0 points

Let V be a vector space. If $\dim V = n$ and if S spans V, then S is a basis for V.

True or False?

1. FALSE **correct**

2. TRUE

Explanation:

Any basis for V must span V and have *exactly* n elements. Consequently, the answer is

$$\boxed{\text{FALSE}}\,.$$

RankTF03
020 10.0 points

When A is a 5×7 matrix, the largest possible dimension of the row space of A is 5.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

The dimension of the row space A is the number of pivot positions in A. But when A is a 5×7 matrix , each column will have 5 entries. So A will have at most 5 pivot positions because there is only one pivot position in each pivot column and each row. Thus the dimension of the row space of A is at most 5.

Consequently, the answer is

$$\boxed{\text{TRUE}}\,.$$

ChangeBasis01b
021 (part 1 of 2) 10.0 points

Determine the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ of a vector space V when

$$\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2, \qquad \mathbf{b}_2 = 9\mathbf{c}_1 - 4c_2\,.$$

1. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 9 & -6 \\ 4 & -2 \end{bmatrix}$

2. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}$

3. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$ **correct**

4. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -6 & 9 \\ -2 & 4 \end{bmatrix}$

5. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & -9 \\ 2 & -4 \end{bmatrix}$

6. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 9 & 6 \\ -4 & -2 \end{bmatrix}$

Explanation:

The change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the 2×2 matrix

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [\,[\mathbf{b}_1]_{\mathcal{C}}\ \ [\mathbf{b}_2]_{\mathcal{C}}\,]\,.$$

Now

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}.$$

Consequently,

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}}\,.$$

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022 (part 2 of 2) 10.0 points

Determine $[\mathbf{x}]_{\mathcal{C}}$ when

$$\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2\,.$$

1. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

2. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **correct**

3. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

4. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

5. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

6. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

Explanation:

Now

$$\begin{aligned} \mathbf{x} &= -3\mathbf{b}_1 + 2\mathbf{b}_2 \\ &= -3(6\mathbf{c}_1 - 2\mathbf{c}_2) + 2(9\mathbf{c}_1 - 4\mathbf{c}_2) \\ &= 0\mathbf{c}_1 - 2\mathbf{c}_2 \,. \end{aligned}$$

Consequently,

$$\boxed{[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}}.$$

Eigenspace02a
023 10.0 points

Find a basis for the eigenspace of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix},$$

corresponding to the eigenvalue $\lambda = -2$.

1. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ **correct**

2. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

5. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Explanation:

The eigenspace corresponding to an eigenvalue λ of A is the Null Space

$$\text{Nul}(A - \lambda I)$$

of all solutions of $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$.

To determine a basis for $\text{Nul}(A - \lambda I)$ we row reduce $A - \lambda I$ with $\lambda = -2$:

$$\text{rref}(A + 2I) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so x_2, x_3 are the free variables. Thus the eigenspace $\text{Nul}(A + 2I)$ has dimension two and

$$\text{Nul}(A + 2I) = \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : \, s, \, t \text{ in } \mathbb{R} \right\}.$$

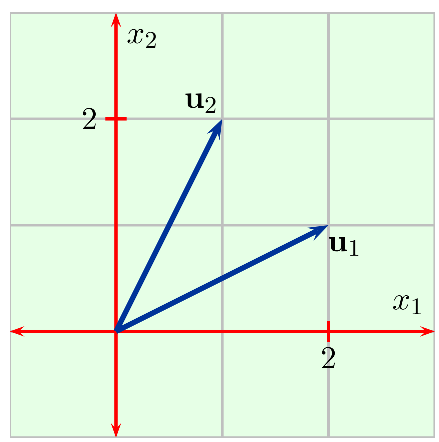
Consequently,

$$\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

is a basis for the eigenspace of A corresponding to $\lambda = -2$.

EigenTrans01a
024 10.0 points

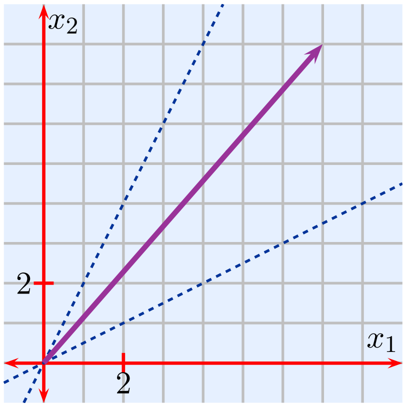
The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ shown in



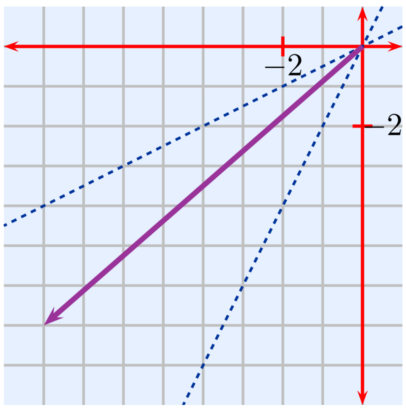
are eigenvectors corresponding to eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$ respectively for a 2×2 matrix A.

Which of the following graphs contains the vector $A(\mathbf{u}_1 + \mathbf{u}_2)$?

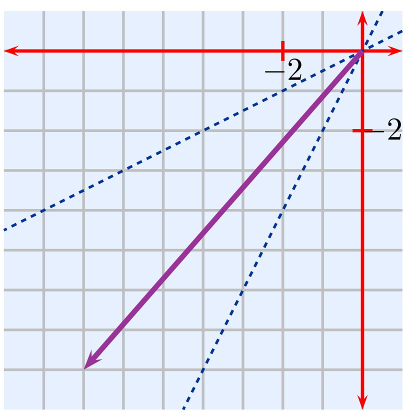
1.



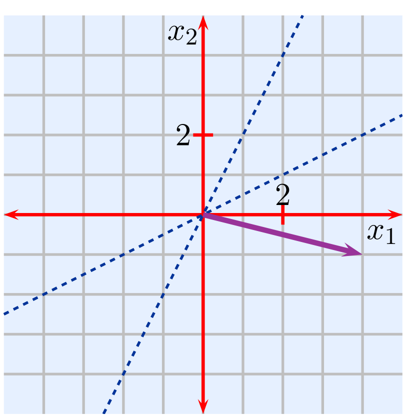
2.



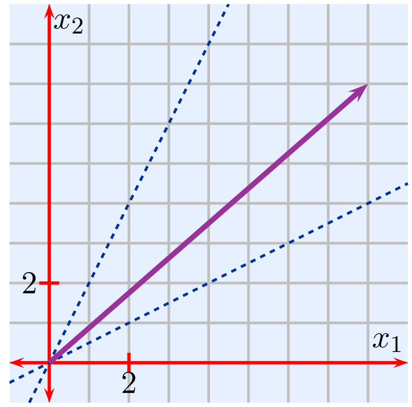
3.



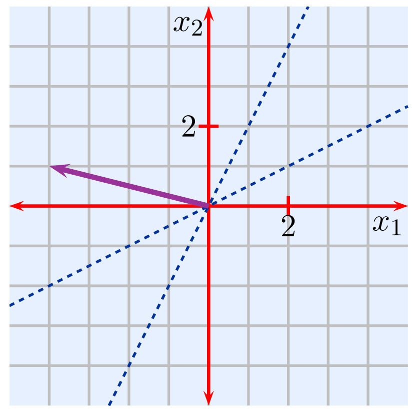
4.



5. **correct**



6.



Explanation:

As the graph of $\mathbf{u}_1$, $\mathbf{u}_2$ shows,

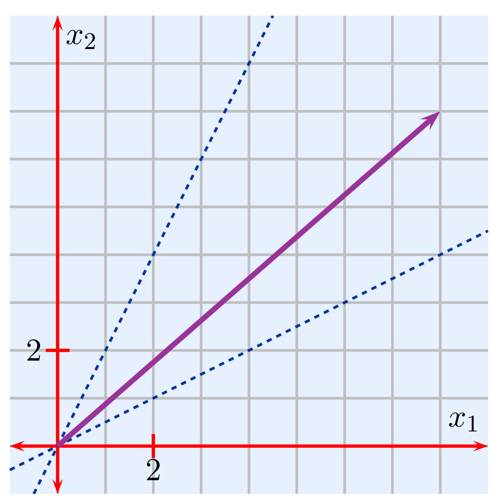
$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

But then

$$A(\mathbf{u}_1 + \mathbf{u}_2) = A\,\mathbf{u}_1 + A\,\mathbf{u}_2 = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2$$

$$= 3\mathbf{u}_1 + 2\mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

Consequently, $A(\mathbf{u}_1 + \mathbf{u}_2)$ is contained in the graph



EigenvalueTF02a
025 10.0 points

If A is an $n \times n$ matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

True or False?

1. FALSE **correct**

2. TRUE

Explanation:

The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero* for λ to be an eigenvalue because by definition an eigenvector must be nonzero.

Consequently, the statement is

$$\boxed{\text{FALSE}}\,.$$

Eigenvalue04a
026 (part 1 of 2) 10.0 points

Determine the Characteristic Polynomial of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

1. $6 - 10\lambda + 6\lambda^2 + \lambda^3$

2. $4 - 10\lambda + 6\lambda^2 - \lambda^3$ **correct**

3. $6 + 4\lambda - 10\lambda^2 + \lambda^3$

4. $4 - 4\lambda + 10\lambda^2 - \lambda^3$

5. $6 + 10\lambda - 6\lambda^2 + \lambda^3$

6. $4 + 4\lambda - 10\lambda^2 - \lambda^3$

Explanation:

The Characteristic polynomial of a matrix A is

$$\det(A - \lambda I).$$

But when

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

then

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 2-\lambda \end{vmatrix}.$$

But

$$\begin{aligned} (2-\lambda)&\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)((2-\lambda)^2 - 1) \\ &= (2-\lambda)(3 - 4\lambda + \lambda^2) \\ &= 6 - 11\lambda + 6\lambda^2 - \lambda^3, \end{aligned}$$

while

$$\begin{vmatrix} -1 & 1 \\ 0 & 2-\lambda \end{vmatrix} = \lambda - 2.$$

Consequently, A has Characteristic polynomial

$$\boxed{4 - 10\lambda + 6\lambda^2 - \lambda^3}\,.$$

027 (part 2 of 2) 10.0 points

One eigenvalue of the matrix A in part (i) is $\lambda = 2$. Determine all the other eigenvalues.

1. $\lambda = 1 \pm \sqrt{2}$

2. $\lambda = 2\sqrt{2} \pm 2$

3. $\lambda = 2 \pm \sqrt{2}$ **correct**

4. $\lambda = 1 \pm 2\sqrt{2}$

5. $\lambda = 2\sqrt{2} \pm 1$

6. $\lambda = 2 \pm 2\sqrt{2}$

Explanation:

The eigenvalues of A are the solutions of

$$\det[A - \lambda I] = 4 - 10\lambda + 6\lambda^2 - \lambda^3 = 0.$$

Given that $\lambda = 2$ is one solution, then there exist constants b, c such that

$$4 - 10\lambda + 6\lambda^2 - \lambda^3 \; = \; (2-\lambda)(\lambda^2 + b\lambda + c) \, .$$

Thus

$$2c \, = \, 4 \, , \quad -b \, = \, 4 \, ,$$

so the eigenvalues of A are the solutions of

$$\det[A - \lambda I] \; = \; (2-\lambda)(\lambda^2 - 4\lambda + 2) \; = \; 0 \, ,$$

i.e., $\lambda = 2, \, 2 \pm \sqrt{2}$. Consequently, the other eigenvalues of A are

$$\boxed{\lambda \; = \; 2 - \sqrt{2}, \; 2 + \sqrt{2}} \, .$$

Diagonalize03a
028 10.0 points

Find a matrix P so that

$$P \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} P^{-1}, \quad d_1 \geq d_2$$

is a diagonalization of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

1. $P = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

2. $P = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

3. $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

4. $P = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$

5. $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ **correct**

6. $P = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

Explanation:

To begin, we must find the eigenvectors and eigenvalues of A. To do this, we will use the characteristic equation, $\det(A - \lambda I) = 0$. That is, we will look for the zeros of the characteristic polynomial.

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-1-\lambda) + 2 \\ &= \lambda^2 - \lambda \\ &= (\lambda - 1)(\lambda + 0) = 0. \end{aligned}$$

So

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now to find the eigenvectors of A, we will solve for the nontrivial solution of the characteristic equation by row reducing the related augmented matrices:

$$\begin{aligned} [\, A - \lambda_1 I \quad \mathbf{0} \,] &= \begin{bmatrix} 2-1 & 2 & 0 \\ -1 & -1-1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\Longrightarrow \mathbf{u}_1 \; = \; \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \end{aligned}$$

while

$$\begin{aligned} [\, A - \lambda_2 I \quad \mathbf{0} \,] &= \begin{bmatrix} 2+0 & 2 & 0 \\ -1 & -1+0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\Longrightarrow \mathbf{u}_2 \; = \; \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

So, $P = [\mathbf{u}_1 \;\; \mathbf{u}_2]$ and

$$A \; = \; PDP^{-1}$$

is a diagonalization of A.

Consequently,

$$\boxed{D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}.$$

DotProdOrthoTF01b
029 10.0 points

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and any scalar c,

$$\mathbf{u}\cdot(c\mathbf{v}) \;=\; c(\mathbf{u}\cdot\mathbf{v})$$

True or False?

1. FALSE

2. TRUE **correct**

Explanation:

By scalar multiplication and matrix multiplication,

$$\mathbf{u}\cdot(c\mathbf{v}) \;=\; \mathbf{u}^T(c\mathbf{v}) = c(\mathbf{u}^T\mathbf{v}) \;=\; c(\mathbf{u}\cdot\mathbf{v})\,.$$

Consequently, the statement is

$$\boxed{\text{ TRUE }}\,.$$

OrthoProj04a
030 10.0 points

Determine the vector $\mathbf{z}$ in $\mathbb{R}^3$ such that $\mathbf{y}-\mathbf{z}$ is the projection of $\mathbf{y}$ in Span($\mathbf{u}$) when

$$\mathbf{y} \;=\; \begin{bmatrix} 9 \\ -5 \\ 2 \end{bmatrix}, \qquad \mathbf{u} \;=\; \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

1. $\mathbf{z} \;=\; \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}$ **correct**

2. $\mathbf{z} \;=\; \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$

3. $\mathbf{z} \;=\; \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix}$

4. $\mathbf{z} \;=\; \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix}$

Explanation:

By definition,

$$\mathbf{y} \;=\; \text{proj}_{\mathbf{u}}\mathbf{y} + \mathbf{z} \;=\; \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \mathbf{z}\,.$$

But

$$\begin{aligned} &\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \left(\frac{(9)(1)+(-5)(-1)+(2)(2)}{1+1+4}\right)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \end{aligned}$$

so after rearrangement, we see that

$$\begin{aligned} \mathbf{z} \;&=\; \mathbf{y} - \text{proj}_{\mathbf{u}}\mathbf{y} \\ &= \begin{bmatrix} 9 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}. \end{aligned}$$

Consequently,

$$\boxed{\mathbf{z} \;=\; \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}}.$$

OrthogProj01a
031 10.0 points

Determine the orthogonal projection of

$$\mathbf{y} \;=\; \begin{bmatrix} -2 \\ -1 \\ -11 \end{bmatrix}$$

onto the subspace W of $\mathbb{R}^3$ spanned by

$$\mathbf{u}_1 \;=\; \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 \;=\; \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

1. $\text{proj}_W\,\mathbf{y} \;=\; \begin{bmatrix} -1 \\ -8 \\ 5 \end{bmatrix}$

2. $\text{proj}_W\,\mathbf{y} \;=\; \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}$

3. $\text{proj}_W\,\mathbf{y} = \begin{bmatrix} 4 \\ -8 \\ -5 \end{bmatrix}$

4. $\text{proj}_W\,\mathbf{y} = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ **correct**

Explanation:
Since $\mathbf{u}_1$, $\mathbf{u}_2$ are non-zero othogonal vectors, they form a basis for W. Thus

$$\text{proj}_W\,\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2.$$

But when

$$\mathbf{y} = \begin{bmatrix} -2 \\ -1 \\ -11 \end{bmatrix}, \ \mathbf{u}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix},$$

we see that

$$\left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 = -\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

while

$$\left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 = 2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Consequently,

$$\boxed{\text{proj}_W\,\mathbf{y} = -\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}}.$$

GramSchmidt04a
032 10.0 points

Find an orthogonal basis for the column space of A when

$$A = \begin{bmatrix} 1 & -1 & -3 \\ -1 & 2 & 1 \\ -3 & 4 & 7 \end{bmatrix}$$

1. $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}.$

3. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ **correct**

4. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$

5. $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

Explanation:
We begin by row reducing A.

$$A \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $A = [\,\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3\,]$. Because columns one and two both contain pivot positions, $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for the column space of A. From this information, we will follow the Gram-Schmidt process to make an orthogonal basis from these two vectors. So

$$\mathbf{v}_1 = \mathbf{x}_1$$
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1.$$

Hence,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 + \frac{15}{11}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} \frac{15}{11} \\ -\frac{15}{11} \\ -\frac{45}{11} \end{bmatrix} = \begin{bmatrix} \frac{4}{11} \\ \frac{7}{11} \\ -\frac{1}{11} \end{bmatrix}.$$

Note that because scalar multiplication does not affect the orthogonality of vectors, the basis vectors can be simplified by scaling them.

Consequently, an orthogonal basis for the column space of A is

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}}.$$

LeastSquares02a
033 10.0 points

Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -4 \end{bmatrix}.$$

1. $\begin{bmatrix} 7 \\ 4 \\ -4 \end{bmatrix}$

2. $\begin{bmatrix} 15 \\ 4 \\ 0 \end{bmatrix}$

3. $\begin{bmatrix} 7 \\ -6 \\ -3 \end{bmatrix}$ **correct**

4. $\begin{bmatrix} 25 \\ 19 \\ 2 \end{bmatrix}$

5. $\begin{bmatrix} 23 \\ -3 \\ -5 \end{bmatrix}$

Explanation:

The normal equations for a least-squares solution of $A\mathbf{x} = \mathbf{b}$ are by definition

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

Now,

$$A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \\ 0 \end{bmatrix}.$$

Hence the least squares solution of $A\mathbf{x} = \mathbf{b}$ is the solution $\mathbf{x}$ to the equation

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 15 \\ 4 \\ 0 \end{bmatrix}.$$

This can be solved with row reduction or inverse matrices to determine that the solution is

$$(A^T A)^{-1}(A^T\mathbf{b}) = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 6 & 3 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ -3 \end{bmatrix}.$$

Consequently, the least squares solution to $A\mathbf{x} = \mathbf{b}$ is

$$\boxed{\begin{bmatrix} 7 \\ -6 \\ -3 \end{bmatrix}}.$$

RegressionLine01a
034 10.0 points

Find the x-intercept of the Least Squares Regression line $y = mx + b$ that best fits the data points

$$(-1,\, 1), \quad (0,\, -2), \quad (1,\, 3)\,.$$

1. x-intercept $= 0$

2. x-intercept $= -\dfrac{4}{3}$

3. $x\text{-intercept} = -\dfrac{1}{3}$

4. $x\text{-intercept} = -\dfrac{2}{3}$ **correct**

5. $x\text{-intercept} = -1$

Explanation:
The design matrix and list of observed values for the data

$$(-1,\, 1), \quad (0,\, -2), \quad (1,\, 3)$$

are given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

The least squares regression line for this data is $y = mx + b$ where $\hat{\mathbf{x}}$ is the solution of the normal equation

$$A^T A\, \hat{\mathbf{x}} = A^T \mathbf{b}, \qquad \hat{\mathbf{x}} = \begin{bmatrix} b \\ m \end{bmatrix}.$$

Now

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

while

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus the normal equation is

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

So

$$\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Consequently, the Least Squares Regression line is

$$y = x + \frac{2}{3},$$

and it has

$$\boxed{x\text{-intercept} = -\frac{2}{3}}.$$

OrthogDiag01b
035 10.0 points

When

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

are eigenvectors of a symmetric 2×2 matrix A corresponding to eigenvalues

$$\lambda_1 = 5, \qquad \lambda_2 = -15,$$

find matrices D and P in an orthogonal diagonalization of A.

1. $D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

2. $D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, \quad P = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

3. $D = \begin{bmatrix} -15 & 0 \\ 0 & 5 \end{bmatrix}, \quad P = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

4. $D = \begin{bmatrix} -15 & 0 \\ 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

5. $D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

6. $D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, \quad P = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$
correct

Explanation:
When

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad Q = [\mathbf{u}_1 \;\; \mathbf{u}_2],$$

then Q has orthogonal columns and $A = QDQ^{-1}$ is a diagonalization of A, but it is not an orthogonal diagonalization because Q is not an orthogonal matrix. We have to normalize $\mathbf{u}_1$ and $\mathbf{u}_2$: set

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then $P = [\mathbf{v}_1 \;\; \mathbf{v}_2]$ is an orthogonal matrix and so

$$A = PDP^{-1}$$

is an orthogonal diagonalization of A when

$$\boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, \quad P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}.$$