Partial Derivatives

In general, if *f* is a function of two variables *x* and *y,* suppose we let only *x* vary while keeping *y* fixed, say *y = b,* where *b* is a constant.

Then we are really considering a function of a single variable *x,* namely, $g(x) = f(x, b)$. If *g* has a derivative at *a,* then we call it the **partial derivative of *f* with respect to *x* at (*a, b*)** and denote it by $f_x(a, b)$. Thus

$$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

Partial Derivatives

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Partial Derivatives

Similarly, the **partial derivative of *f* with respect to *y* at (*a, b*)**, denoted by $f_y(a, b)$, is obtained by keeping *x* fixed (*x = a*) and finding the ordinary derivative at *b* of the function $G(y) = f(a, y)$:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index *I* with respect to the actual temperature *T* and relative humidity *H* when *T* = 96°F and *H* = 70% as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

Partial Derivatives

If we now let the point (*a, b*) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

If *f* is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial Derivatives

There are many alternative notations for partial derivatives.

For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$.

But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to *x* is just the *ordinary* derivative of the function *g* of a single variable that we get by keeping *y* fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$
1. To find f_x, regard *y* as a constant and differentiate $f(x, y)$ with respect to *x*.
2. To find f_y, regard *x* as a constant and differentiate $f(x, y)$ with respect to *y*.

Example 1

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:
Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $\quad f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

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Interpretations of Partial Derivatives

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Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.)
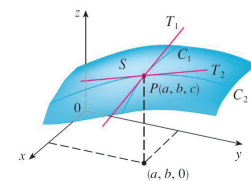
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Interpretations of Partial Derivatives

Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Figure 1

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Interpretations of Partial Derivatives

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

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Example 2

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:
We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

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Example 2 – *Solution*   cont'd

The graph of *f* is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.)

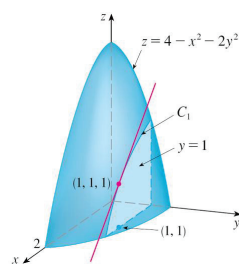The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$.

Figure 2

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Example 2 – *Solution*   cont'd

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)
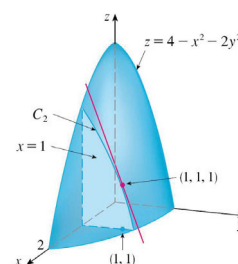

Figure 3

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Functions of More Than Two Variables

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Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating $f(x, y, z)$ with respect to *x*.

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Functions of More Than Two Variables

If $w = f(x, y, z)$, then $f_x = \partial w / \partial x$ can be interpreted as the rate of change of *w* with respect to *x* when *y* and *z* are held fixed. But we can't interpret it geometrically because the graph of *f* lies in four-dimensional space.

In general, if *u* is a function of *n* variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the *i*th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

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Example 5

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution:
Holding *y* and *z* constant and differentiating with respect to *x*, we have

$$f_x = y e^{xy} \ln z$$

Similarly,

$$f_y = x e^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}$$

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Higher Derivatives

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.
If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

Higher Derivatives

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f / \partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 6

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

Solution:
In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2 y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}\,(3x^2 + 2xy^3)$$

$$= 6x + 2y^3$$

Example 6 – *Solution*

$$f_{xy} = \frac{\partial}{\partial y}\,(3x^2 + 2xy^3)$$

$$= 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\,(3x^2 y^2 - 4y)$$

$$= 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}\,(3x^2 y^2 - 4y)$$

$$= 6x^2 y - 4$$

Higher Derivatives

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Higher Derivatives

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Tangent Planes

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.
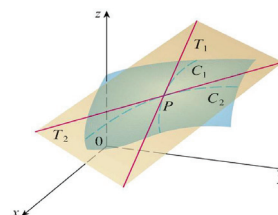
Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

The tangent plane contains the tangent lines T_1 and T_2.

Figure 1

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P. We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$\boxed{1}$ $z - z_0 = a(x - x_0) + b(y - y_0)$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Tangent Planes

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get
$z - z_0 = b(y - y_0)$, which must represent the tangent line T_2,
so $b = f_y(x_0, y_0)$.

> **2** Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is
>
> $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Solution:
Let $f(x, y) = 2x^2 + y^2$.
Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then **2** gives the equation of the tangent plane at (1, 1, 3) as
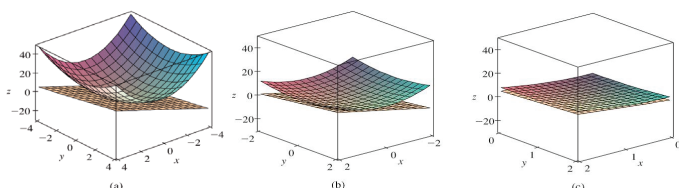
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or
$$z = 4x + 2y - 3$$

Tangent Planes

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.

(a)   (b)   (c)

The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Figure 2

Linear Approximations

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

Linear Approximations

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of

$$f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225.$$

But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations

In general, we know from $\boxed{2}$ that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$\boxed{3}$ $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

is called the **linearization** of f at (a, b)

Linear Approximations

The approximation

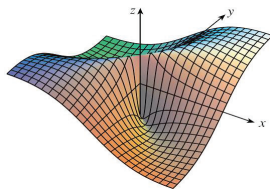$\boxed{4}$ $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Linear Approximations

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$f(x, y) = \frac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

Figure 4

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous.

Linear Approximations

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist.

To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations

If f is differentiable at a, then

$\boxed{5}$ $\Delta y = f'(a)\, \Delta x + \varepsilon\, \Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$\boxed{6}$ $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations

By analogy with $\boxed{5}$ we define the differentiability of a function of two variables as follows.

> $\boxed{7}$ **Definition**   If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form
> $$\Delta z = f_x(a, b)\, \Delta x + f_y(a, b)\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y$$
> where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation $\boxed{4}$ is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

> **8 Theorem** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Functions of Three or More Variables

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

The Gradient Vectors

The Gradient Vectors

> **8 Definition** If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by
>
> $$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

(a) The gradient of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$= \langle \sin yz, xz \cos yz, xy \cos yz \rangle$$