This print-out should have 35 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

FitParabola01a
001 10.0 points

The graph of the function

$$y = ax^2 + bx + c$$

is a parabola passing through the points

$$(1,\, 12), \quad (-1,\, 2), \quad (-3,\, 0).$$

Find the y-intercept of this parabola.

1. y-intercept $= 7$

2. y-intercept $= 5$

3. y-intercept $= 8$

4. y-intercept $= 9$

5. y-intercept $= 6$ **correct**

Explanation:

The y-intercept of the parabola is the value of y at $x = 0$ *i.e.*,

$$y\text{-intercept} = y(0) = c.$$

Hence the task is to find c.

Since the parabola passes through the points

$$(1,\, 12), \quad (-1,\, 2), \quad (-3,\, 0),$$

the coefficients a, b and c must satisfy the equations

$$\begin{aligned} a + b + c &= 12 \\ a - b + c &= 2 \\ 9a - 3b + c &= 0 \end{aligned}$$

To solve these equations for c we reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & -1 & 1 & 2 \\ 9 & -3 & 1 & 0 \end{array}\right]$$

to echelon form by successive row operations:

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & -2 & 0 & -10 \\ 9 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 9R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & -2 & 0 & -10 \\ 0 & -12 & -8 & -108 \end{array}\right]$$

$$\xrightarrow{R_3 - 6R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & -2 & 0 & -10 \\ 0 & 0 & -8 & -48 \end{array}\right]$$

Thus

$$\boxed{y\text{-intercept} = 6}.$$

EchelonForm01e
002 10.0 points

If the augmented matrix for a system of linear equations in variables x_1, x_2, and x_3 is row equivalent to the matrix

$$B = \begin{bmatrix} 2 & -4 & -4 & 0 \\ 3 & -6 & -4 & -2 \\ 1 & -2 & 1 & -3 \end{bmatrix},$$

determine x_1.

1. $x_1 = -1$

2. $x_1 = -2 + 2t\,, \quad t$ arbitrary **correct**

3. $x_1 = -3$

4. $x_1 = -2$

5. $x_1 = -1 + 2t\,, \quad t$ arbitrary

6. system inconsistent

Explanation:

By row reduction

$$B = \begin{bmatrix} 2 & -4 & -4 & 0 \\ 3 & -6 & -4 & -2 \\ 1 & -2 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -4 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -4 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is now in echelon form. But the system

$$2x_1 - 4x_2 - 4x_3 = 0$$

$$2x_3 = -2$$

$$0x_1 + 0x_2 + 0x_3 = 0,$$

associated with this matrix has a free variable $x_2 = t$, say, and by back substitution, we see that

$$x_3 = -1, \quad x_1 = -2 + 2t,$$

Consequently,

$$\boxed{x_1 = -2 + 2t \quad t \text{ arbitrary}}.$$

M340LSpanM02
003 10.0 points

Given

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

determine all values of λ for which

$$\mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ \lambda \end{bmatrix}$$

is a vector in Span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$?

1. $\lambda = 2$ **correct**

2. $\lambda = -2$

3. $\lambda = -4$

4. $\lambda = 2, -4$

5. $\lambda = -2, -4$

6. $\lambda = 2, -2$

Explanation:

The vector $\mathbf{w}$ is in Span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ if there exist weights x_1, x_2, x_3 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}.$$

Such weights exist when the rightmost column in the augmented matrix

$$[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{w}] = \begin{bmatrix} 1 & 2 & 1 & -3 \\ 1 & 4 & 0 & -1 \\ 0 & 2 & -1 & \lambda \end{bmatrix}$$

is not a pivot column. But

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 1 & 4 & 0 & -1 \\ 0 & 2 & -1 & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & \lambda \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & \lambda - 2 \end{bmatrix}$$

Thus the rightmost column is not a pivot column when $\lambda - 2 = 0$. Consequently, $\mathbf{w}$ lies in Span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ when

$$\boxed{\lambda = 2}.$$

MatEquTF03
004 10.0 points

If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then the columns of A span $\mathbb{R}^m$.

True or False?

1. FALSE **correct**

2. TRUE

Explanation:

When A is $m \times n$, then the columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *every* $\mathbf{b}$ in $\mathbb{R}^m$.

It is not enough to say the equation is consistent for *some* $\mathbf{b}$ in $\mathbb{R}^m$. For example, the columns of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

are scalar multiples of each other, so the columns cannot span $\mathbb{R}^2$. But the matrix equation

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

has the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

On the other hand, when

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix},$$

then

$$x_1 + 2x_2 = 3, \qquad 2x_1 + 4x_2 = 3,$$

which is never true. So

$$A\mathbf{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

is inconsistent.

Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

BalChemEqt02a
005 10.0 points

During photosynthesis green plants convert carbon dioxide CO_2 and water H_2O into glucose $C_6H_{12}O_6$ and oxygen O_2, represented chemically by

$$CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2.$$

If 10 molecules of glucose were produced in one particular conversion, how many molecules of carbon dioxide were used?

1. # molecules = 54
2. # molecules = 63
3. # molecules = 57
4. # molecules = 51
5. # molecules = 60 **correct**

Explanation:

We need to solve first for the relative numbers $x_1, \ldots, x_4$ of molecules in the balanced chemical equation

$$x_1CO_2 + x_2H_2O \longrightarrow x_3C_6H_{12}O_6 + x_4O_2.$$

Now the fundamental rule governing this reaction is that the left and right hand sides contain the same number of the respective carbon, oxygen and hydrogen atoms. Thus

$$\begin{aligned} x_1 + 0x_2 &= 6x_3 + 0x_4, \\ 2x_1 + x_2 &= 6x_3 + 2x_4, \\ 0x_1 + 2x_2 &= 12x_3 + 0x_4, \end{aligned}$$

which as a homogeneous system can be written in augmented matrix form

$$[A\ \ \mathbf{0}] = \begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{bmatrix}.$$

But

$$\text{rref}([A\ \ \mathbf{0}]) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix}.$$

So x_4 is a free variable, say $x_4 = s$, and

$$x_1 = s, \quad x_2 = s, \quad x_3 = \frac{1}{6}s,$$

give the respective proportions of the other molecules in the reaction with respect to x_4.

Consequently, if 10 molecules of glucose were produced, then

$$\boxed{\text{60 molecules}}$$

of carbon dioxide were used.

SpanTF04
006 10.0 points

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^2$ and $\mathbf{u}$ is not a multiple of $\mathbf{v}$, is $\mathbf{w}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

1. SOMETIMES

2. ALWAYS **correct**

3. NEVER

Explanation:

When $\mathbf{u}$, $\mathbf{v}$ are nonzero vectors and $\mathbf{u}$ is not a multiple of $\mathbf{v}$, they *are linearly independent.* But then $\text{Span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$, so every vector $\mathbf{w}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$.

Consequently, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^2$ and $\mathbf{u}$ is not a multiple of $\mathbf{v}$, then $\mathbf{w}$

$$\boxed{\text{ALWAYS}}$$

is a linear combination of $\mathbf{u}$, $\mathbf{v}$.

LinTransform02a
007 10.0 points

If A is an $m \times n$ matrix, then the range of the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T_A : \mathbf{x} \to A\,\mathbf{x},$$

is the set of all linear combinations of the columns of A.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

By definition, the range of $T_A : \mathbf{x} \to A\,\mathbf{x}$ is the set

$$\{A\mathbf{x} : \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

But when

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

is a linear combination of the columns of A with weights being the entries in $\mathbf{x}$. Conversely, any linear combination

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$$

of the columns of A can be written as $A\mathbf{x}$ with

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Thus the range of T_A consists of *all* linear combinations of the columns of A.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

MatrixTrans02a
008 10.0 points

If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

and $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, determine $T(\mathbf{u})$ when

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

1. $T(\mathbf{u}) = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

2. $T(\mathbf{u}) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

3. $T(\mathbf{u}) = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

4. $T(\mathbf{u}) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

5. $T(\mathbf{u}) = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ **correct**

6. $T(\mathbf{u}) = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

Explanation:

But the Fundamental Theorem, T is given by the matrix mapping

$$T : \mathbf{x} \to [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ T(\mathbf{e}_3)]\,\mathbf{x}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus

$$T(\mathbf{u}) = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Consequently,

$$\boxed{T(\mathbf{u}) = \begin{bmatrix} 8 \\ 7 \end{bmatrix}}.$$

MatrixOpsTF02c
009 10.0 points

If A is an $n \times n$ matrix, then

$$(A^2)^T = (A^T)^2$$

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

The transpose of the product of two matrices has the property

$$(AB)^T = B^T A^T.$$

But then

$$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2.$$

Thus, $(A^2)^T = (A^T)^2$.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

InverseMatrix05b
010 10.0 points

Evaluate the matrix product $B^{-1}A^T$ when

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

1. $B^{-1}A^T = \begin{bmatrix} 0 & 3 & 5 \\ -1 & 8 & 13 \end{bmatrix}$ **correct**

2. $B^{-1}A^T = \begin{bmatrix} -8 & -5 \\ 7 & 4 \\ 9 & 5 \end{bmatrix}$

3. $B^{-1}A^T = \begin{bmatrix} -8 & 7 & 9 \\ -1 & 8 & 13 \end{bmatrix}$

4. $B^{-1}A^T = \begin{bmatrix} -8 & -1 \\ 7 & 8 \\ 9 & 13 \end{bmatrix}$

5. $B^{-1}A^T = \begin{bmatrix} 0 & 3 & 5 \\ -5 & 4 & 5 \end{bmatrix}$

6. $B^{-1}A^T = \begin{bmatrix} 0 & -5 \\ 3 & 4 \\ 5 & 5 \end{bmatrix}$

Explanation:

The inverse of a 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$\Delta = ad - bc\,.$$

Thus

$$B^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

since $\Delta(B) = 1$. But then

$$B^{-1}A^T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

Consequently,

$$\boxed{B^{-1}A^T = \begin{bmatrix} 0 & 3 & 5 \\ -1 & 8 & 13 \end{bmatrix}}\,.$$

InvertibleTF02a
011 10.0 points

If A and D are $n \times n$ matrices such that $AD = I$, then $DA = I$

True or False?

1. FALSE

2. TRUE **correct**

Explanation:

Because A and D are square matrices and $AD = I$, then A and D are both invertible, with $D = A^{-1}$ and $A = D^{-1}$. So using this substitution, the first equation can be rewritten as $AA^{-1} = I$, and the second as $DD^{-1} = I$. Both of these statements are true by the definition of inverse matrices.

Consequently, the statement is

$$\boxed{\text{TRUE}}\,.$$

LUDecomp06g
012 10.0 points

Find U in an LU decomposition of

$$A = \begin{bmatrix} 4 & 1 & 2 & 2 \\ 8 & 2 & 8 & -1 \\ 8 & 2 & -4 & 16 \end{bmatrix}.$$

1. $U = \begin{bmatrix} 4 & -1 & -2 & -2 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. $U = \begin{bmatrix} 1 & -1 & -2 & -2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $U = \begin{bmatrix} 4 & 2 & 4 & 2 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

4. $U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. $U = \begin{bmatrix} 4 & 1 & 2 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ **correct**

6. $U = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Explanation:

Recall that in a factorization $A = LU$ of an $m \times n$ matrix A, then L is an $m \times m$ lower triangular matrix with ones on the diagonal and U is an $m \times n$ echelon form of A.

We begin by computing U. Now $U = M_0A$ where j is the number of row operations on A needed to transform A into its echelon form U and M_i is a product of $j - i$ elementary matrices that represent these row operations.

$$\begin{aligned} U &= M_0A = M_1E_1A \\ &= M_1\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 & 2 \\ 8 & 2 & 8 & -1 \\ 8 & 2 & -4 & 16 \end{bmatrix} \\ &= M_2E_2(E_1A) \\ &= M_2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 & 2 \\ 0 & 0 & 4 & -5 \\ 8 & 2 & -4 & 16 \end{bmatrix} \\ &= E_3(E_2E_1A) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & -8 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 & 2 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change $U = M_0 A$ to $M_0^{-1}U = A$. This shows that $M_0^{-1} = L$. Hence

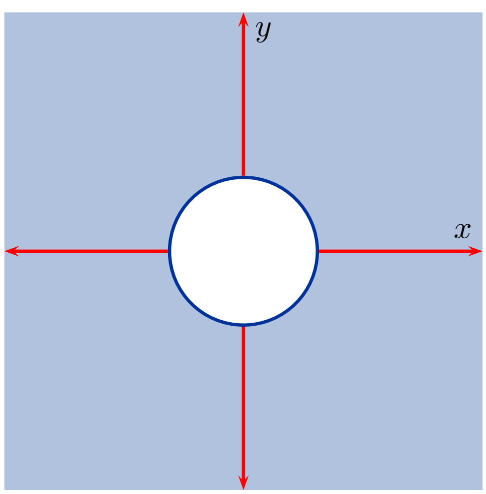
$$\begin{aligned} L = M_0^{-1} &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= E_1^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \end{aligned}$$

Consequently,

$$\boxed{U = \begin{bmatrix} 4 & 1 & 2 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}}.$$

Subspace01cT/F
013 10.0 points

The set of points in the shaded region (including the bounding lines and assumed to stretch to $\pm\infty$ in all directions) shown in



is a subspace of $\mathbb{R}^2$.

True or False?

1. FALSE **correct**

2. TRUE

Explanation:

The shaded region excludes the origin, so the set of points does not contain the zero vector.

Consequently, the set is

$$\boxed{\text{NOT a subspace of } \mathbb{R}^2}.$$

ColNulDimTF01a
014 10.0 points

If A is a 4×5 matrix, then

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = 5\,.$$

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

By Fundamental Theorem of Linear Algebra, for an $m \times n$ matrix A,

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = n\,.$$

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

Determinant02e
015 10.0 points

Compute the determinant of the matrix

$$A = \begin{bmatrix} -3 & -3 & 3 \\ 6 & 4 & -9 \\ -9 & -11 & 5 \end{bmatrix}$$

1. $\det(A) = -6$ **correct**

2. $\det(A) = -9$

3. $\det(A) = -7$

4. $\det(A) = -5$

5. $\det(A) = -8$

Explanation:
Expanding by co-factors of the first row we see that

$$\det(A) = -3\begin{vmatrix} 4 & -9 \\ -11 & 5 \end{vmatrix}$$

$$+3\begin{vmatrix} 6 & -9 \\ -9 & 5 \end{vmatrix} + 3\begin{vmatrix} 6 & 4 \\ -9 & -11 \end{vmatrix}$$

$$= (-3 \times (-79)) + ((3) \times (-51)) + ((3) \times (-30)).$$

Consequently,

$$\boxed{\det(A) = -6}.$$

DetMult05
016 10.0 points

Evaluate $\det\left[B^5\right]$ when

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

1. $\det\left[B^5\right] = 32$

2. $\det\left[B^5\right] = -32$ **correct**

3. $\det\left[B^5\right] = -2$

4. $\det\left[B^5\right] = -10$

5. $\det\left[B^5\right] = 10$

Explanation:
Since

$$\det[CD] = \det[C]\det[D],$$

for all $n \times n$ matrices C and D,

$$\det\left[B^5\right] = (\det[B])^5.$$

But

$$\det[B] = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (1)(1-4) + (1)(2-1) = -2.$$

Consequently,

$$\boxed{\det[B^5] = (-2)^5 = -32}.$$

VectorSpaceT/F04a
017 10.0 points

The set H of all polynomials

$$\mathbf{p}(x) = a + x^3, \quad a \text{ in } \mathbb{R},$$

is a subspace of the vector space $\mathbb{P}_6$ of all polynomials of degree at most 6.

True or False?

1. TRUE

2. FALSE **correct**

Explanation:
The zero polynomial $\mathbf{p}(x) = 0 + 0x^3$ does not belong to H.

Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

BasisNull02b
018 10.0 points

Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 2 & -6 & 2 & -4 \\ -3 & 9 & -5 & 8 \\ 1 & -3 & -1 & 0 \end{bmatrix}.$$

1. $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ **correct**

2. $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$

3. $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

6. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Explanation:
We first row reduce $[A\ \mathbf{0}]$:

$$\mathrm{rref}([A\ \mathbf{0}]) = \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to identify the free variables for $\mathbf{x}$ in the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Thus x_1 and x_3 are basic variables, while x_2 and x_4 are free variables. So set $x_2 = s$ and $x_4 = t$. Then

$$x_1 = 3s + t, \quad x_3 = t,$$

and

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathrm{Nul}(A)$.

BasisCol02a

019 10.0 points

First find a basis for $\mathrm{Col}(A)$ when

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -3 & 15 \end{bmatrix} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3],$$

and then select *all* the correct statements from among the following:

I: $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ *is a linearly dependent set.*
II: $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ *is a basis for* $\mathbb{R}^3$.
III: $\mathrm{rank}(A) = 2$.
IV: $\mathrm{nullity}(A) = 1$.
V: $\mathrm{rank}(A) = 3$.

1. I, II, and V

2. II only

3. II and V

4. I, III, and IV **correct**

5. I and III

Explanation:
We first row reduce A:

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

to identify the pivot columns of A. These are the first and second columns of A. So $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis for $\mathrm{Col}(A)$. Thus

$$\dim(\mathrm{Col}(A)) = 2 = \mathrm{rank}(A),$$

and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ cannot be linearly independent, hence not a basis for $\mathbb{R}^3$.

On the other hand, by the Fundamental Theorem of Linear Algebra,

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = 3,$$

showing that $\mathrm{nullity}(A) = 1$.

Consequently, only

$$\boxed{\text{I, III, and IV are correct.}}.$$

Basis02
020 10.0 points

Find a basis for the space spanned by the following vectors.

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}$$

1. $\left\{\begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}\right\}$

2. $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}\right\}$

3. $\left\{\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}\right\}$

4. $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}\right\}$ **correct**

5. $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}\right\}$

Explanation:

When

$$A = [\,\mathbf{v_1}\quad \mathbf{v_2}\quad \mathbf{v_3}\quad \mathbf{v_4}\quad \mathbf{v_5}\,]$$

is the 4×5 matrix whose columns are the five given vectors, this problem is equivalent to finding a basis for Col$A$. Since the reduced echelon form of A is

$$\begin{bmatrix}1 & -2 & 3 & 5 & 2\\ 0 & 0 & -1 & -3 & -1\\ 0 & 0 & 1 & 3 & 1\\ 1 & 2 & -1 & -4 & 0\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 0 & 0 & -5/2 & 0\\ 0 & 1 & 0 & 3/4 & 1/2\\ 0 & 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}\right\}$$

CoordVec03a
021 10.0 points

Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ in $\mathbb{R}^3$ for the vector

$$\mathbf{x} = \begin{bmatrix}3\\-5\\4\end{bmatrix}$$

with respect to the basis

$$\mathcal{B} = \left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}2\\1\\8\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$$

for $\mathbb{R}^3$.

1. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\0\\-5\end{bmatrix}$

2. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}5\\2\\0\end{bmatrix}$

3. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-5\\-2\\0\end{bmatrix}$

4. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}0\\2\\5\end{bmatrix}$

5. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}0\\-2\\5\end{bmatrix}$

6. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\0\\5\end{bmatrix}$ **correct**

Explanation:

The coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of a vector $\mathbf{x}$ in $\mathbb{R}^3$ with respect to a basis

$$\mathcal{B} = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\}$$

for $\mathbb{R}^3$ satisfies the matrix equation

$$A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \qquad A = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]\,.$$

Consequently, when

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\},$$

and

$$\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix},$$

the associated augmented matrix is

$$[\,A \ \ \mathbf{x}\,] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix}.$$

But then

$$\text{rref}[A \ \ \mathbf{x}] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Thus

$$\boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}}.$$

PolySpanVecTF01a
022 10.0 points

The polynomials

$$\mathbf{p}_1 = 1 - 3t + 5t^2, \ \ \mathbf{p}_2 = -3 + 5t - 7t^2,$$

and

$$\mathbf{p}_3 = -4 + 5t - 6t^2, \ \ \mathbf{p}_4 = 1 - t^2,$$

span $\mathbb{P}_2$.

True or False? (Hint: use coordinate vectors.)

1. FALSE **correct**

2. TRUE

Explanation:

The coordinate mapping $\mathbf{p} \to [\mathbf{p}]_{\mathcal{B}}$ from $\mathbb{P}_2$ to $\mathbb{R}^3$ with respect to the standard monomial basis $\mathcal{B}$ maps

$$\mathbf{p} = a + bt + ct^2 \longrightarrow [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Thus

$$[\mathbf{p}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad [\mathbf{p}_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix},$$

and

$$[\mathbf{p}_3]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \quad [\mathbf{p}_4]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now $\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,\ \mathbf{p}_4\}$ span $\mathbb{P}_2$ if and only if

$$\text{Span}\,\{\,[\mathbf{p}_1]_{\mathcal{B}},\ [\mathbf{p}_2]_{\mathcal{B}},\ [\mathbf{p}_3]_{\mathcal{B}},\ [\mathbf{p}_4]_{\mathcal{B}}\,\}$$

has dimension 3 *i.e.*, if and only if the 3×4 matrix

$$\begin{aligned} A &= [\,[\mathbf{p}_1]_{\mathcal{B}} \ \ [\mathbf{p}_2]_{\mathcal{B}} \ \ [\mathbf{p}_3]_{\mathcal{B}} \ \ [\mathbf{p}_4]_{\mathcal{B}}\,] \\ &= \begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \end{aligned}$$

has 3 pivot columns. But

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{5}{4} & -\frac{5}{4} \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so A has only 2 pivot columns.

Consequently, the statement is

$$\boxed{\text{FALSE}}.$$

RankTF06c
023 10.0 points

The dimensions of the row space and column space of an $m \times n$ matrix A are the same, even if $m \neq n$.

True or False?

1. TRUE **correct**

2. FALSE

Explanation:

Recall that the rank A is the number of pivot columns in A. Equivalently, rank A is the number of pivot positions in an echelon form B of A. Furthermore, since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A, rank A is also the dimension of the row space.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

ChangeBasis04b
024 (part 1 of 2) 10.0 points

In $\mathbb{P}_2$ determine the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from basis $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ to the standard monomial basis $\mathcal{C} = \{1, t, t^2\}$ when

$$\mathbf{p}_1 = 1 - 3t^2, \quad \mathbf{p}_2 = 2 + t - 5t^2$$

and

$$\mathbf{p}_3 = 1 + 2t.$$

1. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -5 & 2 & 1 \\ 0 & 1 & -2 \\ -3 & -5 & 0 \end{bmatrix}$

2. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

3. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -2 & -1 & -5 \\ 0 & -1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$

4. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

5. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -3 & 5 & 0 \end{bmatrix}$

6. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$ **correct**

Explanation:

The $\mathcal{B}$-coordinate vectors of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ with respect to $\mathcal{C}$ are

$$[\mathbf{p}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad [\mathbf{p}_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix},$$

and

$$[\mathbf{p}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

while those for $\mathcal{C}$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} &[\, I_3 \quad P_{\mathcal{C}\leftarrow\mathcal{B}} \,] \\ &\qquad = \text{rref}\,[\, I_3 \quad [\mathbf{p}_1]_{\mathcal{C}} \quad [\mathbf{p}_2]_{\mathcal{C}} \quad [\mathbf{p}_3]_{\mathcal{C}} \,]. \end{aligned}$$

Consequently,

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}}.$$

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025 (part 2 of 2) 10.0 points

Express $\mathbf{q}(t) = t^2$ as a linear combination of the polynomials in the basis $\mathcal{B}$.

1. $\mathbf{q} = 2\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3$

2. $\mathbf{q} = 3\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$ **correct**

3. $\mathbf{q} = 2\mathbf{p}_1 + 3\mathbf{p}_2 - \mathbf{p}_3$

4. $\mathbf{q} = 2\mathbf{p}_1 - 3\mathbf{p}_2 - \mathbf{p}_3$

5. $\mathbf{q} = 3\mathbf{p}_1 + 2\mathbf{p}_2 + \mathbf{p}_3$

6. $\mathbf{q} = 3\mathbf{p}_1 + 2\mathbf{p}_2 - \mathbf{p}_3$

Explanation:

By definition,

$$P_{\mathcal{B}}[\mathbf{q}]_{\mathcal{B}} = [\mathbf{q}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} [\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

while

$$[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

As an augmented matrix this becomes

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

so

$$\boxed{\mathbf{q}(t) = 3\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3}.$$

Eigenspace02a
026 10.0 points

Find a basis for the eigenspace of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix},$$

corresponding to the eigenvalue $\lambda = -2$.

1. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ **correct**

5. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Explanation:

The eigenspace corresponding to an eigenvalue λ of A is the Null Space

$$\text{Nul}(A - \lambda I)$$

of all solutions of $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$.

To determine a basis for $\text{Nul}(A - \lambda I)$ we row reduce $A - \lambda I$ with $\lambda = -2$:

$$\text{rref}(A + 2I) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so x_2, x_3 are the free variables. Thus the eigenspace $\text{Nul}(A + 2I)$ has dimension two and

$\text{Nul}(A + 2I)$

$$= \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}.$$

Consequently,

$$\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

is a basis for the eigenspace of A corresponding to $\lambda = -2$.

CharPoly05a
027 10.0 points

Determine the Characteristic Polynomial of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1. $4 + 4\lambda - 10\lambda^2 - \lambda^3$

2. $4 - 4\lambda + 10\lambda^2 - \lambda^3$

3. $4 - 10\lambda + 4\lambda^2 - \lambda^3$ **correct**

4. $6 - 10\lambda + 4\lambda^2 + \lambda^3$

5. $6 + 4\lambda - 10\lambda^2 + \lambda^3$

6. $6 + 10\lambda - 4\lambda^2 + \lambda^3$

Explanation:

The Characteristic Polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 2-\lambda \end{vmatrix}.$$

But

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)((2-\lambda)^2 - 1)$$

$$= (2-\lambda)(3 - 4\lambda + \lambda^2)$$

$$= 6 - 11\lambda + 6\lambda^2 - \lambda^3,$$

while

$$\begin{vmatrix} -1 & 1 \\ 0 & 2-\lambda \end{vmatrix} = \lambda - 2.$$

Consequently, A has Characteristic Polynomial

$$\boxed{4 - 10\lambda + 6\lambda^2 - \lambda^3}.$$

Diagonalize02a
028 10.0 points

Find a matrix P and d_2, d_3 so that

$$P\begin{bmatrix} 3 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}P^{-1}, \quad d_1 \geq d_2 \geq d_3,$$

is a diagonalization of the matrix

$$A = \begin{bmatrix} 3 & 0 & 15 \\ 2 & 1 & 15 \\ 0 & 0 & -2 \end{bmatrix}.$$

1. $d_2 = 2,\ d_3 = -1,$

$$P = \begin{bmatrix} -3 & 0 & 1 \\ -3 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2. $d_2 = 1,\ d_3 = -2,$

$$P = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

correct

3. $d_2 = 1,\ d_3 = -2,$

$$P = \begin{bmatrix} -3 & 0 & 1 \\ -3 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

4. $d_2 = 2,\ d_3 = -1,$

$$P = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

5. $d_2 = 1,\ d_3 = -2,$

$$P = \begin{bmatrix} 1 & -3 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

6. $d_2 = 2,\ d_3 = -1,$

$$P = \begin{bmatrix} 1 & -3 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Explanation:

The entries 3, d_2, d_3 in the diagonal matrix are the respective eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ of A. But

$$\det[A - \lambda I] = \begin{vmatrix} 3-\lambda & 0 & 15 \\ 2 & 1-\lambda & 15 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

$$= -(\lambda - 3)(\lambda - 1)(\lambda + 2)\,.$$

So $\lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = -2$.

Now let $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3$ be eigenvectors of A corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3$ respectively. Since the eigenvalues are distinct,

$$P = [\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3]$$

has orthogonal columns.

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P^{-1}$$

is a diagonalization of A.

To determine $\mathbf{u}_1$ we row reduce $A - \lambda I$ with $\lambda_1 = 3$:

$$\text{rref}(A - 3I) = \text{rref} \begin{bmatrix} 0 & 0 & 15 \\ 2 & -2 & 15 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To determine $\mathbf{u}_2$ we row reduce $A - \lambda I$ with $\lambda_2 = 1$:

$$\text{rref}(A - I) = \text{rref} \begin{bmatrix} 2 & 0 & 15 \\ 2 & 0 & 15 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

To determine $\mathbf{u}_3$ we row reduce $A - \lambda I$ with $\lambda_3 = -2$:

$$\text{rref}(A + 2I) = \text{rref} \begin{bmatrix} 5 & 0 & 15 \\ 2 & 3 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -3 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, finally,

$$\mathbf{u}_3 = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}.$$

Consequently, $d_2 = 1,\ d_3 = -2$ and

$$\boxed{P = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}.$$

CalC13c03a
029 10.0 points

Which of the following statements are true for all vectors $\mathbf{a}$, $\mathbf{b}$?

A. $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|,\ \mathbf{a} \neq 0,\ \mathbf{b} \neq 0 \implies$ $\mathbf{a}$ parallel to $\mathbf{b}$,

B. $\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\,\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2$,

C. $\mathbf{a}\cdot\mathbf{b} = 0 \implies \mathbf{a} = 0$ or $\mathbf{b} = 0$.

1. all of them

2. A only

3. A and B only **correct**

4. B only

5. A and C only

6. B and C only

7. none of them

8. C only

Explanation:

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} \; = \; \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \, .$$

A. TRUE: when

$$|\mathbf{a} \cdot \mathbf{b}| \; = \; \|\mathbf{a}\| \, \|\mathbf{b}\| \, , \quad \mathbf{a} \neq 0, \quad \mathbf{b} \neq 0 \, ,$$

then $|\cos\theta| = 1$, *i.e.*, $\theta = 0$ or π. In this case $\mathbf{a}$ is parallel to $\mathbf{b}$.

B. TRUE: since $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$,

$$\begin{aligned} \|\mathbf{a}+\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) \\ &= \|\mathbf{a}\|^2 + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \|\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 + 2\,\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 \end{aligned}$$

because $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

C. FALSE: if $\mathbf{a} \perp \mathbf{b}$, then $\theta = \pi/2$. But then $\cos\theta = 0$. So $\mathbf{a} \cdot \mathbf{b} = 0$ when $\mathbf{a} \perp \mathbf{b}$, as well as when $\mathbf{a} = 0$ or $\mathbf{b} = 0$.

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keywords:

OrthoBasis01b
030 10.0 points

Determine c_2 so that

$$\mathbf{y} \; = \; c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

when

$$\mathbf{y} \; = \; \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

1. $c_2 \; = \; \dfrac{1}{2}$

2. $c_2 \; = \; -\dfrac{1}{2}$ **correct**

3. No value of c_2 exists.

4. $c_2 \; = \; 0$

5. $c_2 \; = \; \dfrac{3}{2}$

6. $c_2 \; = \; -\dfrac{3}{2}$

Explanation:

Since

$$\mathbf{u}_1 \cdot \mathbf{u}_2 \; = \; \mathbf{u}_1 \cdot \mathbf{u}_3 \; = \; \mathbf{u}_2 \cdot \mathbf{u}_3 \; = \; 0 \, ,$$

the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are mutually orthogonal in $\mathbb{R}^3$. As they are also non-zero, they thus form a basis for the three-dimensional space $\mathbb{R}^3$. So there exist unique c_1, c_2, and c_3 such that

$$\mathbf{y} \; = \; c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

for any $\mathbf{y}$ in $\mathbb{R}^3$. But by orthogonality,

$$\begin{aligned} \mathbf{y} \cdot \mathbf{u}_k \; &= \; c_1\mathbf{u}_1 \cdot \mathbf{u}_k + c_2\mathbf{u}_2 \cdot \mathbf{u}_k + c_3\mathbf{u}_3 \cdot \mathbf{u}_k \\ &= \; c_k\mathbf{u}_k \cdot \mathbf{u}_k, \qquad 1 \leq k \leq 3 \, , \end{aligned}$$

in particular,

$$c_2 \; = \; \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \, .$$

When

$$\mathbf{y} \; = \; \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix},$$

therefore,

$$c_2 = \frac{(-4) + (0) + (-6)}{(16) + (0) + (4)} = -\frac{1}{2}$$

Consequently,

$$\boxed{c_2 \;=\; -\frac{1}{2}}.$$

DistanceMC01
031 10.0 points

Find the distance from $\mathbf{y}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ when

$$\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

1. dist $= \sqrt{6}$

2. dist $= 2\sqrt{10}$ **correct**

3. dist $= 2\sqrt{5}$

4. dist $= 8$

5. dist $= 4$

6. dist $= 6$

Explanation:

The distance from a point $\mathbf{y}$ in $\mathbb{R}^3$ to the subspace $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is the distance

$$\|\mathbf{y} - \text{proj}_W \mathbf{y}\|$$

from $\mathbf{y}$ to the closest point, $\text{proj}_W \mathbf{y}$, in W.

Now $\mathbf{u}_1$, $\mathbf{u}_2$ are orthogonal because

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = [-3 \;\; -5 \;\; 1] \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0,$$

so

$$\text{proj}_W \mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2$$

$$= \frac{35}{35}\mathbf{u}_1 - \frac{28}{14}\mathbf{u}_2 = \mathbf{u}_1 - 2\mathbf{u}_2 = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}.$$

Thus

$$\mathbf{y} - \text{proj}_W \mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}.$$

Consequently, the distance from $\mathbf{y}$ to W is

$$\boxed{\|\mathbf{y} - \text{proj}_W \mathbf{y}\| \;=\; \sqrt{40} \;=\; 2\sqrt{10}}.$$

GramSchmidt01a
032 10.0 points

Use the fact that

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to determine an orthogonal basis for $\text{Col}(A)$.

1. $\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ **correct**

Explanation:

The pivot columns of A provide a basis for Col(A). But by row reduction,

$$A \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the pivot columns of A are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}.$$

We apply Gram-Schmidt to produce an orthogonal basis: set $\mathbf{u}_1 = \mathbf{a}_1$ and

$$\mathbf{u}_2 = \mathbf{a}_2 - \left(\frac{\mathbf{a}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1$$

$$= \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} - \frac{(-36)}{27}\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 20/3 \end{bmatrix} = \begin{bmatrix} -8/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$$

Consequently, the set of vectors

$$\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

is an orthogonal basis for Col(A).

LeastSquares02c
033 10.0 points

Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{bmatrix} -2 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}.$$

1. $\begin{bmatrix} -5 \\ -1 \end{bmatrix}$

2. $\begin{bmatrix} 12 \\ 7 \end{bmatrix}$

3. $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ **correct**

4. $\dfrac{1}{2}\begin{bmatrix} -1 \\ 8 \end{bmatrix}$

5. $\begin{bmatrix} -12 \\ -9 \end{bmatrix}$

Explanation:

The normal equations for a least-squares solution of $A\mathbf{x} = \mathbf{b}$ are by definition

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

Now,

$$A^T A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Hence the least squares solution of $A\mathbf{x} = \mathbf{b}$ is the solution $\mathbf{x}$ to the equation

$$\begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This can be solved with row reduction or inverse matrices to determine that the solution is

$$(A^T A)^{-1}(A^T\mathbf{b}) = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 8 \\ -6 \end{bmatrix}.$$

Consequently, the least squares solution to $A\mathbf{x} = \mathbf{b}$ is

$$\boxed{\begin{bmatrix} 4 \\ -3 \end{bmatrix}}.$$

RegressionLine03c
034 10.0 points

Find the Least Squares Regression line $y = mx + b$ that best fits the data points

$$(-1,\,-2),\quad (0,\,3),\quad (1,\,-1),\quad (2,\,-3)\,.$$

1. $y = -\dfrac{7}{10}x + \dfrac{2}{5}$

2. $y = -\dfrac{2}{5}x - \dfrac{7}{10}$

3. $y = \dfrac{2}{5}x - \dfrac{7}{10}$

4. $y = \dfrac{2}{5}x + \dfrac{7}{10}$

5. $y = \dfrac{7}{10}x + \dfrac{2}{5}$

6. $y = -\dfrac{7}{10}x - \dfrac{2}{5}$ **correct**

Explanation:

The design matrix and list of observed values for the data

$$(-1,\,-2),\quad (0,\,3),\quad (1,\,-1),\quad (2,\,-3)\,.$$

are given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ -3 \end{bmatrix}.$$

The least squares regression line for this data is $y = mx + b$ where $\hat{\mathbf{x}}$ is the solution of the normal equation

$$A^T A\,\hat{\mathbf{x}} = A^T\mathbf{b}\,, \qquad \hat{\mathbf{x}} = \begin{bmatrix} b \\ m \end{bmatrix}.$$

Now

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix},$$

while

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}.$$

Thus the normal equation is

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}.$$

So

$$\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{7}{10} \end{bmatrix}.$$

Consequently, the Least Squares Regression line is

$$\boxed{y = -\frac{7}{10}x - \frac{2}{5}}\,.$$

OrthogDiag02a
035 10.0 points

When

$$A = \begin{bmatrix} -4 & 2 \\ 2 & -7 \end{bmatrix}$$

find matrices D and P in an orthogonal diagonalization of A given that $\lambda_1 > \lambda_2$.

1. $D = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}, \quad P = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

2. $D = \begin{bmatrix} -8 & 0 \\ 0 & -3 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

3. $D = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

4. $D = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}, \quad P = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ **correct**

5. $D = \begin{bmatrix} -8 & 0 \\ 0 & -3 \end{bmatrix}$, $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

6. $D = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Explanation:

To begin, we must find the eigenvectors and eigenvalues of A. To do this, we will use the characteristic equation, $\det(A - \lambda I) = 0$. That is, we will look for the zeros of the characteristic polynomial.

$$\begin{aligned} \det(A - \lambda I) &= (-4 - \lambda)(-7 - \lambda) - 4 \\ &= \lambda^2 + 11\lambda 24 \\ &= (\lambda + 3)(\lambda + 8) = 0. \end{aligned}$$

So $\lambda_1 = -3$, $\lambda_2 = -8$, and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}.$$

Now to find the eigenvectors of A, we will solve for the nontrivial solution of the characteristic equation by row reducing the related augmented matrices:

$$[\, A - \lambda_1 I \quad \mathbf{0} \,] = \begin{bmatrix} -4+3 & 2 & 0 \\ 2 & -7+3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

while

$$[\, A - \lambda_2 I \quad \mathbf{0} \,] = \begin{bmatrix} -4+8 & 2 & 0 \\ 2 & -7+8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Now, when

$$Q = [\mathbf{u}_1 \ \ \mathbf{u}_2],$$

then Q has orthogonal columns and

$$A = QDQ^{-1}$$

is a diagonalization of A, but it is not an orthogonal diagonalization because Q is not an orthogonal matrix. We have to normalize $\mathbf{u}_1$ and $\mathbf{u}_2$: set

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Then $P = [\mathbf{v}_1 \ \ \mathbf{v}_2]$ is an orthogonal matrix and so

$$A = PDP^{-1}$$

is an orthogonal diagonalization of A when

$$\boxed{D = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}, \ P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}.$$