

This print-out should have 35 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

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**FitParabola01a**  
**001 10.0 points**

The graph of the function

$$y = ax^2 + bx + c$$

is a parabola passing through the points

$$(1, 10), \quad (-1, 8), \quad (-3, 22).$$

Find the  $y$ -intercept of this parabola.

1.  $y$ -intercept = 9
2.  $y$ -intercept = 10
3.  $y$ -intercept = 8
4.  $y$ -intercept = 6
5.  $y$ -intercept = 7 **correct**

**Explanation:**

The  $y$ -intercept of the parabola is the value of  $y$  at  $x = 0$  *i.e.*,

$$y\text{-intercept} = y(0) = c.$$

Hence the task is to find  $c$ .

Since the parabola passes through the points

$$(1, 10), \quad (-1, 8), \quad (-3, 22),$$

the coefficients  $a$ ,  $b$  and  $c$  must satisfy the equations

$$a + b + c = 10$$

$$a - b + c = 8$$

$$9a - 3b + c = 22$$

To solve these equations for  $c$  we reduce the augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 1 & -1 & 1 & 8 \\ 9 & -3 & 1 & 22 \end{array} \right]$$

to echelon form by successive row operations:

$$\xrightarrow{R_2 - R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -2 & 0 & -2 \\ 9 & -3 & 1 & 22 \end{array} \right]$$

$$\xrightarrow{R_3 - 9R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -2 & 0 & -2 \\ 0 & -12 & -8 & -68 \end{array} \right]$$

$$\xrightarrow{R_3 - 6R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -8 & -56 \end{array} \right]$$

Thus

$y\text{-intercept} = 7$

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**EchelonForm01e**  
**002 10.0 points**

If the augmented matrix for a system of linear equations in variables  $x_1$ ,  $x_2$ , and  $x_3$  is row equivalent to the matrix

$$B = \left[ \begin{array}{cccc} 3 & 9 & 3 & 6 \\ -1 & -3 & -2 & -1 \\ -1 & -3 & -4 & 1 \end{array} \right],$$

determine  $x_1$ .

1. system inconsistent
2.  $x_1 = 2$
3.  $x_1 = -1$
4.  $x_1 = 3 - 3t$ ,  $t$  arbitrary **correct**
5.  $x_1 = -1 - 3t$ ,  $t$  arbitrary
6.  $x_1 = 3$

**Explanation:**

By row reduction

$$\begin{aligned}
 B &= \begin{bmatrix} 3 & 9 & 3 & 6 \\ -1 & -3 & -2 & -1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \\
 &\sim \begin{bmatrix} 3 & 9 & 3 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \\
 &\sim \begin{bmatrix} 3 & 9 & 3 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},
 \end{aligned}$$

which is now in echelon form. But the system

$$\begin{aligned}
 3x_1 + 9x_2 + 3x_3 &= 6 \\
 -x_3 &= 1 \\
 0x_1 + 0x_2 + 0x_3 &= 0,
 \end{aligned}$$

associated with this matrix has a free variable  $x_2 = t$ , say, and by back substitution, we see that

$$x_3 = -1, \quad x_1 = 3 - 3t,$$

Consequently,

$$\boxed{x_1 = 3 - 3t \quad t \text{ arbitrary}}.$$

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**M340LSpanM02**  
**003 10.0 points**

Given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

determine all values of  $\lambda$  for which

$$\mathbf{w} = \begin{bmatrix} -2 \\ 3 \\ \lambda \end{bmatrix}$$

is a vector in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ ?

1.  $\lambda = 5, 1$
2.  $\lambda = -5, 1$
3.  $\lambda = 1$

4.  $\lambda = -5$

5.  $\lambda = 5, -5$

6.  $\lambda = 5$  **correct**

**Explanation:**

The vector  $\mathbf{w}$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  if there exist weights  $x_1, x_2, x_3$  such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}.$$

Such weights exist when the rightmost column in the augmented matrix

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{w}] = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 1 & 4 & 0 & 3 \\ 0 & 2 & -1 & \lambda \end{bmatrix}$$

is not a pivot column. But

$$\begin{aligned}
 \begin{bmatrix} 1 & 2 & 1 & -2 \\ 1 & 4 & 0 & 3 \\ 0 & 2 & -1 & \lambda \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 2 & -1 & 5 \\ 0 & 2 & -1 & \lambda \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 0 & \lambda - 5 \end{bmatrix}
 \end{aligned}$$

Thus the rightmost column is not a pivot column when  $\lambda - 5 = 0$ . Consequently,  $\mathbf{w}$  lies in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  when

$$\boxed{\lambda = 5}.$$

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**MatEquTF03**  
**004 10.0 points**

If  $A$  is an  $m \times n$  matrix and the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for some  $\mathbf{b}$  in  $\mathbb{R}^m$ , then the columns of  $A$  span  $\mathbb{R}^m$ .

True or False?

1. TRUE
2. FALSE **correct**

**Explanation:**

When  $A$  is  $m \times n$ , then the columns of  $A$  span  $\mathbb{R}^m$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for *every*  $\mathbf{b}$  in  $\mathbb{R}^m$ .

It is not enough to say the equation is consistent for *some*  $\mathbf{b}$  in  $\mathbb{R}^m$ . For example, the columns of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

are scalar multiples of each other, so the columns cannot span  $\mathbb{R}^2$ . But the matrix equation

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

has the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

On the other hand, when

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix},$$

then

$$x_1 + 2x_2 = 3, \quad 2x_1 + 4x_2 = 3,$$

which is never true. So

$$A\mathbf{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

is inconsistent.

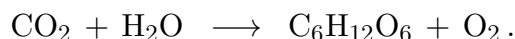
Consequently, the statement is

FALSE

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**BalChemEq02a**  
**005 10.0 points**

During photosynthesis green plants convert carbon dioxide  $\text{CO}_2$  and water  $\text{H}_2\text{O}$  into glucose  $\text{C}_6\text{H}_{12}\text{O}_6$  and oxygen  $\text{O}_2$ , represented chemically by

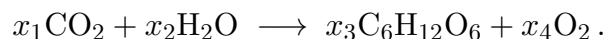


If 11 molecules of glucose were produced in one particular conversion, how many molecules of carbon dioxide were used?

1. # molecules = 60
2. # molecules = 57
3. # molecules = 66 **correct**
4. # molecules = 63
5. # molecules = 69

**Explanation:**

We need to solve first for the relative numbers  $x_1, \dots, x_4$  of molecules in the balanced chemical equation



Now the fundamental rule governing this reaction is that the left and right hand sides contain the same number of the respective carbon, oxygen and hydrogen atoms. Thus

$$x_1 + 0x_2 = 6x_3 + 0x_4,$$

$$2x_1 + x_2 = 6x_3 + 2x_4,$$

$$0x_1 + 2x_2 = 12x_3 + 0x_4,$$

which as a homogeneous system can be written in augmented matrix form

$$[A \ \mathbf{0}] = \begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{bmatrix}.$$

But

$$\text{rref}([A \ \mathbf{0}]) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix}.$$

So  $x_4$  is a free variable, say  $x_4 = s$ , and

$$x_1 = s, \quad x_2 = s, \quad x_3 = \frac{1}{6}s,$$

give the respective proportions of the other molecules in the reaction with respect to  $x_4$ .

Consequently, if 11 molecules of glucose were produced, then

66 molecules

of carbon dioxide were used.

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**SpanTF04**  
**006 10.0 points**

If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are nonzero vectors in  $\mathbb{R}^2$  and  $\mathbf{u}$  is not a multiple of  $\mathbf{v}$ , is  $\mathbf{w}$  a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ ?

1. ALWAYS correct
2. NEVER
3. SOMETIMES

**Explanation:**

When  $\mathbf{u}$ ,  $\mathbf{v}$  are nonzero vectors and  $\mathbf{u}$  is not a multiple of  $\mathbf{v}$ , they *are linearly independent*. But then  $\text{Span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$ , so every vector  $\mathbf{w}$  in  $\mathbb{R}^2$  is a linear combination of  $\mathbf{u}$ ,  $\mathbf{v}$ .

Consequently, if  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are nonzero vectors in  $\mathbb{R}^2$  and  $\mathbf{u}$  is not a multiple of  $\mathbf{v}$ , then  $\mathbf{w}$

ALWAYS

is a linear combination of  $\mathbf{u}$ ,  $\mathbf{v}$ .

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**LinTransform02a**  
**007 10.0 points**

If  $A$  is an  $m \times n$  matrix, then the range of the transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad T_A : \mathbf{x} \rightarrow A\mathbf{x},$$

is the set of all linear combinations of the columns of  $A$ .

True or False?

1. TRUE correct
2. FALSE

**Explanation:**

By definition, the range of  $T_A : \mathbf{x} \rightarrow A\mathbf{x}$  is the set

$$\{A\mathbf{x} : \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

But when

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$$

is a linear combination of the columns of  $A$  with weights being the entries in  $\mathbf{x}$ . Conversely, any linear combination

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n$$

of the columns of  $A$  can be written as  $A\mathbf{x}$  with

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Thus the range of  $T_A$  consists of *all* linear combinations of the columns of  $A$ .

Consequently, the statement is

TRUE

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**MatrixTrans02a**  
**008 10.0 points**

If  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

and  $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , determine  $T(\mathbf{u})$  when

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

1.  $T(\mathbf{u}) = \begin{bmatrix} 5 \\ 18 \end{bmatrix}$  **correct**

2.  $T(\mathbf{u}) = \begin{bmatrix} 6 \\ 19 \end{bmatrix}$

3.  $T(\mathbf{u}) = \begin{bmatrix} 4 \\ 18 \end{bmatrix}$

4.  $T(\mathbf{u}) = \begin{bmatrix} 4 \\ 19 \end{bmatrix}$

5.  $T(\mathbf{u}) = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$

6.  $T(\mathbf{u}) = \begin{bmatrix} 5 \\ 19 \end{bmatrix}$

**Explanation:**

But the Fundamental Theorem,  $T$  is given by the matrix mapping

$$T : \mathbf{x} \rightarrow [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)] \mathbf{x}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus

$$T(\mathbf{u}) = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Consequently,

$$T(\mathbf{u}) = \begin{bmatrix} 5 \\ 18 \end{bmatrix}.$$

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**MatrixOpsTF02c**  
**009 10.0 points**

If  $A$  is an  $n \times n$  matrix, then

$$(A^2)^T = (A^T)^2$$

True or False?

1. FALSE

2. TRUE **correct**

**Explanation:**

The transpose of the product of two matrices has the property

$$(AB)^T = B^T A^T.$$

But then

$$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2.$$

Thus,  $(A^2)^T = (A^T)^2$ .

Consequently, the statement is

TRUE

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**InverseMatrix05b**  
**010 10.0 points**

Evaluate the matrix product  $B^{-1}A^T$  when

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$

1.  $B^{-1}A^T = \begin{bmatrix} 4 & -7 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$

2.  $B^{-1}A^T = \begin{bmatrix} 12 & -7 \\ -3 & 2 \\ -1 & 1 \end{bmatrix}$

3.  $B^{-1}A^T = \begin{bmatrix} 4 & 1 & 3 \\ -7 & 2 & 1 \end{bmatrix}$

4.  $B^{-1}A^T = \begin{bmatrix} 12 & -11 \\ -3 & -2 \\ -1 & -7 \end{bmatrix}$

5.  $B^{-1}A^T = \begin{bmatrix} 12 & -3 & -1 \\ -11 & -2 & -7 \end{bmatrix}$

6.  $B^{-1}A^T = \begin{bmatrix} 4 & 1 & 3 \\ -11 & -2 & -7 \end{bmatrix}$  **correct**

**Explanation:**

The inverse of a  $2 \times 2$  matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$\Delta = ad - bc.$$

Thus

$$B^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

since  $\Delta(B) = 1$ . But then

$$B^{-1}A^T = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

Consequently,

$$B^{-1}A^T = \begin{bmatrix} 4 & 1 & 3 \\ -11 & -2 & -7 \end{bmatrix}.$$

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**InvertibleTF02a**  
**011 10.0 points**

If  $A$  and  $D$  are  $n \times n$  matrices such that  $AD = I$ , then  $DA = I$

True or False?

1. TRUE **correct**

2. FALSE

**Explanation:**

Because  $A$  and  $D$  are square matrices and  $AD = I$ , then  $A$  and  $D$  are both invertible, with  $D = A^{-1}$  and  $A = D^{-1}$ . So using this substitution, the first equation can be rewritten as  $AA^{-1} = I$ , and the second as  $DD^{-1} = I$ . Both of these statements are true by the definition of inverse matrices.

Consequently, the statement is

TRUE

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**LUDecomp06g**  
**012 10.0 points**

Find  $U$  in an  $LU$  decomposition of

$$A = \begin{bmatrix} 3 & -2 & -4 & -2 \\ 9 & -6 & -11 & -11 \\ 12 & -8 & -13 & -20 \end{bmatrix}.$$

$$1. U = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$2. U = \begin{bmatrix} 3 & 2 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$3. U = \begin{bmatrix} 1 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$4. U = \begin{bmatrix} 1 & -1 & 4 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$5. U = \begin{bmatrix} 3 & -1 & 4 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$6. U = \begin{bmatrix} 3 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ correct}$$

**Explanation:**

Recall that in a factorization  $A = LU$  of an  $m \times n$  matrix  $A$ , then  $L$  is an  $m \times m$  lower triangular matrix with ones on the diagonal and  $U$  is an  $m \times n$  echelon form of  $A$ .

We begin by computing  $U$ . Now  $U = M_0A$  where  $j$  is the number of row operations on  $A$  needed to transform  $A$  into its echelon form  $U$  and  $M_i$  is a product of  $j - i$  elementary matrices that represent these row operations.

$$U = M_0A = M_1E_1A$$

$$= M_1 \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & -4 & -2 \\ 9 & -6 & -11 & -11 \\ 12 & -8 & -13 & -20 \end{bmatrix}$$

$$= M_2E_2(E_1A)$$

$$= M_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 12 & -8 & -13 & -20 \end{bmatrix}$$

$$= E_3(E_2E_1A)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 3 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change  $U = M_0 A$  to  $M_0^{-1}U = A$ . This shows that  $M_0^{-1} = L$ . Hence

$$\begin{aligned} L &= M_0^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \\ &= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \end{aligned}$$

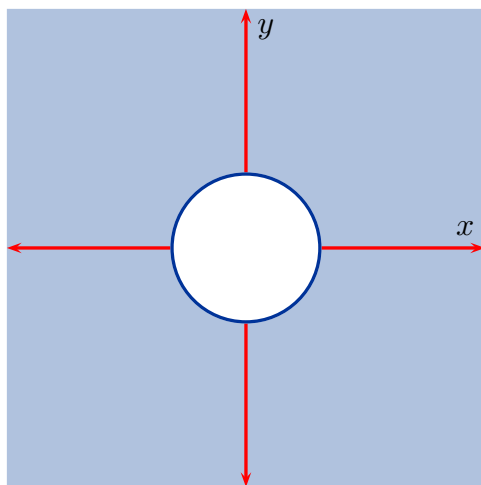
Consequently,

$$U = \begin{bmatrix} 3 & -2 & -4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

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**Subspace01cT/F**  
**013 10.0 points**

The set of points in the shaded region (including the bounding lines and assumed to stretch to  $\pm\infty$  in all directions) shown in



is a subspace of  $\mathbb{R}^2$ .

True or False?

1. TRUE

2. FALSE correct

**Explanation:**

The shaded region excludes the origin, so the set of points does not contain the zero vector.

Consequently, the set is

NOT a subspace of  $\mathbb{R}^2$ .

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**ColNulDimTF01a**  
**014 10.0 points**

If  $A$  is a  $4 \times 5$  matrix, then

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = 5.$$

True or False?

1. FALSE

2. TRUE correct

**Explanation:**

By Fundamental Theorem of Linear Algebra, for an  $m \times n$  matrix  $A$ ,

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = n.$$

Consequently, the statement is

TRUE.

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**Determinant02e**  
**015 10.0 points**

Compute the determinant of the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 \\ 2 & 5 & -2 \\ -2 & 1 & 8 \end{bmatrix}$$

1.  $\det(A) = -7$

2.  $\det(A) = -8$

3.  $\det(A) = -4$

4.  $\det(A) = -6$  correct

5.  $\det(A) = -5$

**Explanation:**

Expanding by co-factors of the first row we see that

$$\begin{aligned}\det(A) &= - \begin{vmatrix} 5 & -2 \\ 1 & 8 \end{vmatrix} \\ &\quad + \begin{vmatrix} 2 & -2 \\ -2 & 8 \end{vmatrix} + 2 \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} \\ &= (-1 \times (42)) + ((1) \times (12)) + ((2) \times (12)).\end{aligned}$$

Consequently,

$$\boxed{\det(A) = -6}.$$

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**DetMult05**  
**016 10.0 points**

Evaluate  $\det[B^5]$  when

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

1.  $\det[B^5] = 32$
2.  $\det[B^5] = -32$  **correct**
3.  $\det[B^5] = -10$
4.  $\det[B^5] = -2$
5.  $\det[B^5] = 10$

**Explanation:**

Since

$$\det[CD] = \det[C] \det[D],$$

for all  $n \times n$  matrices  $C$  and  $D$ ,

$$\det[B^5] = (\det[B])^5.$$

But

$$\begin{aligned}\det[B] &= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \\ &= (1)(1-4) + (1)(2-1) = -2.\end{aligned}$$

Consequently,

$$\boxed{\det[B^5] = (-2)^5 = -32}.$$

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**VectorSpaceT/F04a**  
**017 10.0 points**

The set  $H$  of all polynomials

$$\mathbf{p}(x) = a + bx^4, \quad a, b \text{ in } \mathbb{R},$$

is a subspace of the vector space  $\mathbb{P}_6$  of all polynomials of degree at most 6.

True or False?

1. TRUE **correct**
2. FALSE

**Explanation:**

The zero polynomial  $\mathbf{p}(x) = 0+0x^4$  belongs to  $H$ . So we need to check if the linear combination  $c_1\mathbf{p}_1 + c_2\mathbf{p}_2$  of elements

$$\mathbf{p}_1(x) = a_1 + b_1x^4, \quad \mathbf{p}_2(x) = a_2 + b_2x^4$$

in  $H$  also is a polynomial in  $H$ . But

$$\begin{aligned}(c_1\mathbf{p}_1 + c_2\mathbf{p}_2)(x) &= c_1\mathbf{p}_1(x) + c_2\mathbf{p}_2(x) \\ &= c_1(a_1 + b_1x^4) + c_2(a_2 + b_2x^4) \\ &= (c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)x^4.\end{aligned}$$

Since

$$c_1a_1 + c_2a_2, \quad c_1b_1 + c_2b_2$$

are in  $\mathbb{R}$ , the linear combination  $c_1\mathbf{p}_1 + c_2\mathbf{p}_2$  belongs to  $H$ .

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

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**BasisNull02b**  
**018 10.0 points**



Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 3 & 6 & -6 & 0 \\ 2 & 4 & -5 & 1 \\ -2 & -4 & 3 & 1 \end{bmatrix}.$$

1.  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

2.  $\left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

3.  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

4.  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

5.  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

6.  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$  **correct**

**Explanation:**

We first row reduce  $[A \ 0]$ :

$$\text{rref}([A \ 0]) = \begin{bmatrix} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to identify the free variables for  $\mathbf{x}$  in the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

Thus  $x_1$  and  $x_3$  are basic variables, while  $x_2$  and  $x_4$  are free variables. So set  $x_2 = s$  and  $x_4 = t$ . Then

$$x_1 = -2s + 2t, \quad x_3 = t,$$

and

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for  $\text{Nul}(A)$ .

---

**BasisCol02a**  
**019 10.0 points**

First find a basis for  $\text{Col}(A)$  when

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 2 & -6 & -4 \end{bmatrix} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3],$$

and then select *all* the correct statements from among the following:

I:  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  is a linearly dependent set.

II:  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  is a basis for  $\mathbb{R}^3$ .

III:  $\text{rank}(A) = 2$ .

IV:  $\text{nullity}(A) = 1$ .

V:  $\text{rank}(A) = 3$ .

1. I, II, and V

2. II and V

3. II only

4. I, III, and IV **correct**

5. I and III

**Explanation:**

We first row reduce  $A$ :

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

to identify the pivot columns of  $A$ . These are the first and second columns of  $A$ . So  $\{\mathbf{a}_1, \mathbf{a}_2\}$  is a basis for  $\text{Col}(A)$ . Thus

$$\dim(\text{Col}(A)) = 2 = \text{rank}(A),$$

and  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  cannot be linearly independent, hence not a basis for  $\mathbb{R}^3$ .

On the other hand, by the Fundamental Theorem of Linear Algebra,

$$\text{rank}(A) + \text{nullity}(A) = 3,$$

showing that  $\text{nullity}(A) = 1$ .

Consequently, only

I, III, and IV are correct.

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**Basis02**  
**020 10.0 points**

Find a basis for the space spanned by the following vectors.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

1.  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$  correct

2.  $\left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix} \right\}$

3.  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

4.  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

5.  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

**Explanation:**

When

$$A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$$

is the  $4 \times 5$  matrix whose columns are the five given vectors, this problem is equivalent to finding a basis for  $\text{Col}A$ . Since the reduced echelon form of  $A$  is

$$\begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 2 & -1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5/2 & 0 \\ 0 & 1 & 0 & 3/4 & 1/2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we see that the first, second, and third columns of  $A$  are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

---

**CoordVec03a**  
**021 10.0 points**

Find the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  in  $\mathbb{R}^3$  for the vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

for  $\mathbb{R}^3$ .

1.  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$

2.  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$

$$3. [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -5 \\ -2 \\ 0 \end{bmatrix}$$

$$4. [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

$$5. [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

$$6. [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \text{ correct}$$

**Explanation:**

The coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  of a vector  $\mathbf{x}$  in  $\mathbb{R}^3$  with respect to a basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$$

for  $\mathbb{R}^3$  satisfies the matrix equation

$$A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad A = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3].$$

Consequently, when

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\},$$

and

$$\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix},$$

the associated augmented matrix is

$$[A \quad \mathbf{x}] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix}.$$

But then

$$\text{rref}[A \quad \mathbf{x}] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Thus

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

**PolySpanVecTF01a****022 10.0 points**

The polynomials

$$\mathbf{p}_1 = 1 - 3t + 5t^2, \quad \mathbf{p}_2 = -3 + 5t - 7t^2,$$

and

$$\mathbf{p}_3 = -4 + 5t - 6t^2, \quad \mathbf{p}_4 = 1 - t^2,$$

span  $\mathbb{P}_2$ .

True or False? (Hint: use coordinate vectors.)

1. TRUE

2. FALSE correct

**Explanation:**

The coordinate mapping  $\mathbf{p} \rightarrow [\mathbf{p}]_{\mathcal{B}}$  from  $\mathbb{P}_2$  to  $\mathbb{R}^3$  with respect to the standard monomial basis  $\mathcal{B}$  maps

$$\mathbf{p} = a + bt + ct^2 \longrightarrow [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Thus

$$[\mathbf{p}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad [\mathbf{p}_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix},$$

and

$$[\mathbf{p}_3]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \quad [\mathbf{p}_4]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now  $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$  span  $\mathbb{P}_2$  if and only if

$$\text{Span}\{[\mathbf{p}_1]_{\mathcal{B}}, [\mathbf{p}_2]_{\mathcal{B}}, [\mathbf{p}_3]_{\mathcal{B}}, [\mathbf{p}_4]_{\mathcal{B}}\}$$

has dimension 3 *i.e.*, if and only if the  $3 \times 4$  matrix

$$A = [\mathbf{p}_1]_{\mathcal{B}} \quad [\mathbf{p}_2]_{\mathcal{B}} \quad [\mathbf{p}_3]_{\mathcal{B}} \quad [\mathbf{p}_4]_{\mathcal{B}} \\ = \begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix}$$

has 3 pivot columns. But

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{5}{4} & -\frac{5}{4} \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so  $A$  has only 2 pivot columns.

Consequently, the statement is

FALSE

.

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**RankTF06c**  
**023 10.0 points**

The dimensions of the row space and column space of an  $m \times n$  matrix  $A$  are the same, even if  $m \neq n$ .

True or False?

1. TRUE **correct**

2. FALSE

**Explanation:**

Recall that the rank  $A$  is the number of pivot columns in  $A$ . Equivalently, rank  $A$  is the number of pivot positions in an echelon form  $B$  of  $A$ . Furthermore, since  $B$  has a nonzero row for each pivot, and since these rows form a basis for the row space of  $A$ , rank  $A$  is also the dimension of the row space.

Consequently, the statement is

TRUE

.

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**ChangeBasis04b**  
**024 (part 1 of 2) 10.0 points**

In  $\mathbb{P}_2$  determine the change of coordinates matrix  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  from basis  $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$  to the standard monomial basis  $\mathcal{C} = \{1, t, t^2\}$  when

$$\mathbf{p}_1 = 1 - 3t^2, \quad \mathbf{p}_2 = 2 + t - 5t^2$$

and

$$\mathbf{p}_3 = 1 + 2t.$$

$$1. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

$$2. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -5 & 2 & 1 \\ 0 & 1 & -2 \\ -3 & -5 & 0 \end{bmatrix}$$

$$3. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -2 & -1 & -5 \\ 0 & -1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$$

$$4. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

$$5. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -3 & 5 & 0 \end{bmatrix}$$

$$6. P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} \text{ correct}$$

**Explanation:**

The  $\mathcal{B}$ -coordinate vectors of  $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$  with respect to  $\mathcal{C}$  are

$$[\mathbf{p}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad [\mathbf{p}_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix},$$

and

$$[\mathbf{p}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

while those for  $\mathcal{C}$  are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} [I_3 \quad P_{\mathcal{C} \leftarrow \mathcal{B}}] \\ = \text{rref} [I_3 \quad [\mathbf{p}_1]_{\mathcal{C}} \quad [\mathbf{p}_2]_{\mathcal{C}} \quad [\mathbf{p}_3]_{\mathcal{C}}]. \end{aligned}$$

Consequently,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}.$$

**025 (part 2 of 2) 10.0 points**

Express  $\mathbf{q}(t) = t^2$  as a linear combination of the polynomials in the basis  $\mathcal{B}$ .

1.  $\mathbf{q} = 3\mathbf{p}_1 + 2\mathbf{p}_2 - \mathbf{p}_3$
2.  $\mathbf{q} = 3\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$  **correct**
3.  $\mathbf{q} = 3\mathbf{p}_1 + 2\mathbf{p}_2 + \mathbf{p}_3$
4.  $\mathbf{q} = 2\mathbf{p}_1 + 3\mathbf{p}_2 - \mathbf{p}_3$
5.  $\mathbf{q} = 2\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3$
6.  $\mathbf{q} = 2\mathbf{p}_1 - 3\mathbf{p}_2 - \mathbf{p}_3$

**Explanation:**

By definition,

$$P_{\mathcal{B}}[\mathbf{q}]_{\mathcal{B}} = [\mathbf{q}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} [\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

while

$$[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

As an augmented matrix this becomes

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{q}(t) = 3\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3.$$

**Eigenspace02a****026 10.0 points**

Find a basis for the eigenspace of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix},$$

corresponding to the eigenvalue  $\lambda = -2$ .

1.  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
2.  $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$  **correct**
3.  $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
4.  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
5.  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

**Explanation:**

The eigenspace corresponding to an eigenvalue  $\lambda$  of  $A$  is the Null Space

$$\text{Nul}(A - \lambda I)$$

of all solutions of  $(A - \lambda I) \mathbf{x} = \mathbf{0}$ .

To determine a basis for  $\text{Nul}(A - \lambda I)$  we row reduce  $A - \lambda I$  with  $\lambda = -2$ :

$$\text{rref}(A + 2I) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so  $x_2, x_3$  are the free variables. Thus the eigenspace  $\text{Nul}(A + 2I)$  has dimension two and

$$\text{Nul}(A + 2I) = \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}.$$

Consequently,

$$\left[ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right]$$

is a basis for the eigenspace of  $A$  corresponding to  $\lambda = -2$ .

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**CharPoly05a**  
**027 10.0 points**

Determine the Characteristic Polynomial of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1.  $6 + 4\lambda - 10\lambda^2 + \lambda^3$
2.  $6 + 10\lambda - 4\lambda^2 + \lambda^3$
3.  $4 - 10\lambda + 4\lambda^2 - \lambda^3$  **correct**
4.  $4 + 4\lambda - 10\lambda^2 - \lambda^3$
5.  $4 - 4\lambda + 10\lambda^2 - \lambda^3$
6.  $6 - 10\lambda + 4\lambda^2 + \lambda^3$

**Explanation:**

The Characteristic Polynomial of  $A$  is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda) \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 2 - \lambda \end{vmatrix}. \end{aligned}$$

But

$$\begin{aligned} (2 - \lambda) \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} &= (2 - \lambda)((2 - \lambda)^2 - 1) \\ &= (2 - \lambda)(3 - 4\lambda + \lambda^2) \\ &= 6 - 11\lambda + 6\lambda^2 - \lambda^3, \end{aligned}$$

while

$$\begin{vmatrix} -1 & 1 \\ 0 & 2 - \lambda \end{vmatrix} = \lambda - 2.$$

Consequently,  $A$  has Characteristic Polynomial

$$4 - 10\lambda + 6\lambda^2 - \lambda^3.$$

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**Diagonalize02a**  
**028 10.0 points**

Find a matrix  $P$  and  $d_2, d_3$  so that

$$P \begin{bmatrix} 5 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} P^{-1}, \quad d_1 \geq d_2 \geq d_3,$$

is a diagonalization of the matrix

$$A = \begin{bmatrix} 5 & -9 & -9 \\ 0 & 3 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

1.  $d_2 = -2, d_3 = -3,$

$$P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2.  $d_2 = 3, d_3 = 2,$

$$P = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

3.  $d_2 = -2, d_3 = -3,$

$$P = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

4.  $d_2 = 3, d_3 = 2,$

$$P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

correct

5.  $d_2 = 3, d_3 = 2,$

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

6.  $d_2 = -2, d_3 = -3,$

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

### Explanation:

The entries 5,  $d_2$ ,  $d_3$  in the diagonal matrix are the respective eigenvalues  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  of  $A$ . But

$$\begin{aligned} \det[A - \lambda I] &= \begin{vmatrix} 5 - \lambda & -9 & -9 \\ 0 & 3 - \lambda & 0 \\ 0 & -1 & 2 - \lambda \end{vmatrix} \\ &= -\lambda^3 + 10\lambda^2 - 31\lambda + 30 \\ &= -(\lambda - 5)(\lambda - 3)(\lambda - 2). \end{aligned}$$

So  $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 2$ .

Now let  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  be eigenvectors of  $A$  corresponding to  $\lambda_1, \lambda_2, \lambda_3$  respectively. Since the eigenvalues are distinct,

$$P = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$$

has orthogonal columns.

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P^{-1}$$

is a diagonalization of  $A$ .

To determine  $\mathbf{u}_1$  we row reduce  $A - \lambda I$  with  $\lambda_1 = 5$ :

$$\begin{aligned} \text{rref}(A - 5I) &= \text{rref} \begin{bmatrix} 0 & -9 & -9 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

To determine  $\mathbf{u}_2$  we row reduce  $A - \lambda I$  with  $\lambda_2 = 3$ :

$$\begin{aligned} \text{rref}(A - 3I) &= \text{rref} \begin{bmatrix} 2 & -9 & -9 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

To determine  $\mathbf{u}_3$  we row reduce  $A - \lambda I$  with  $\lambda_3 = 2$ :

$$\begin{aligned} \text{rref}(A - 2I) &= \text{rref} \begin{bmatrix} 3 & -9 & -9 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, finally,

$$\mathbf{u}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently,  $d_2 = 3, d_3 = 2$  and

$$P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

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**CalC13c03a**  
**029 10.0 points**

Which of the following statements are true for all vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ?

- A.  $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$ ,
  - B.  $|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\|$ ,  $\mathbf{a} \neq 0$ ,  $\mathbf{b} \neq 0 \implies \mathbf{a} \perp \mathbf{b}$ ,
  - C.  $\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2$ .
1. none of them
  2. B and C only
  3. A and B only
  4. A only **correct**
  5. C only
  6. all of them
  7. A and C only
  8. B only

**Explanation:**

If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta.$$

A. TRUE: since  $|\cos \theta| \leq 1$ ,

$$|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\| |\cos \theta| = \|\mathbf{a}\| \|\mathbf{b}\|.$$

B. FALSE: when

$$|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\|, \quad \mathbf{a} \neq 0, \quad \mathbf{b} \neq 0,$$

then  $|\cos \theta| = 1$ , *i.e.*,  $\theta = 0$  or  $\pi$ . But then  $\mathbf{a}$  is parallel to  $\mathbf{b}$ .

C. FALSE: since  $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$ ,

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \|\mathbf{b}\|^2 \\ &= \|\mathbf{a}\|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 \end{aligned}$$

because  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ .

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keywords:

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**OrthoBasis01b**  
**030 10.0 points**

Determine  $c_2$  so that

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$$

when

$$\mathbf{y} = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -4 \\ 3 \\ 3 \end{bmatrix}.$$

1.  $c_2 = \frac{16}{17}$
2. No value of  $c_2$  exists.
3.  $c_2 = 0$
4.  $c_2 = -\frac{16}{17}$  **correct**
5.  $c_2 = -\frac{8}{17}$
6.  $c_2 = \frac{8}{17}$

**Explanation:**

Since

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0,$$

the vectors  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ , and  $\mathbf{u}_3$  are mutually orthogonal in  $\mathbb{R}^3$ . As they are also non-zero, they thus form a basis for the three-dimensional space  $\mathbb{R}^3$ . So there exist unique  $c_1$ ,  $c_2$ , and  $c_3$  such that

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$$



for any  $\mathbf{y}$  in  $\mathbb{R}^3$ . But by orthogonality,

$$\begin{aligned}\mathbf{y} \cdot \mathbf{u}_k &= c_1 \mathbf{u}_1 \cdot \mathbf{u}_k + c_2 \mathbf{u}_2 \cdot \mathbf{u}_k + c_3 \mathbf{u}_3 \cdot \mathbf{u}_k \\ &= c_k \mathbf{u}_k \cdot \mathbf{u}_k, \quad 1 \leq k \leq 3,\end{aligned}$$

in particular,

$$c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}.$$

When

$$\mathbf{y} = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -4 \\ 3 \\ 3 \end{bmatrix},$$

therefore,

$$c_2 = \frac{(-12) + (-4) + (0)}{(9) + (4) + (4)} = -\frac{16}{17}$$

Consequently,

$$\boxed{c_2 = -\frac{16}{17}}.$$

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**DistanceMC01**  
**031 10.0 points**

Find the distance from  $\mathbf{y}$  to the plane in  $\mathbb{R}^3$  spanned by  $\mathbf{u}_1$  and  $\mathbf{u}_2$  when

$$\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

1. dist = 8

2. dist = 6

3. dist = 4

4. dist =  $\sqrt{6}$

5. dist =  $2\sqrt{10}$  correct

6. dist =  $2\sqrt{5}$

**Explanation:**

The distance from a point  $\mathbf{y}$  in  $\mathbb{R}^3$  to the subspace  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$  is the distance

$$\|\mathbf{y} - \text{proj}_W \mathbf{y}\|$$

from  $\mathbf{y}$  to the closest point,  $\text{proj}_W \mathbf{y}$ , in  $W$ .

Now  $\mathbf{u}_1, \mathbf{u}_2$  are orthogonal because

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \begin{bmatrix} -3 & -5 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0,$$

so

$$\begin{aligned}\text{proj}_W \mathbf{y} &= \left( \frac{\mathbf{y} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \right) \mathbf{u}_1 + \left( \frac{\mathbf{y} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \right) \mathbf{u}_2 \\ &= \frac{35}{35} \mathbf{u}_1 - \frac{28}{14} \mathbf{u}_2 = \mathbf{u}_1 - 2\mathbf{u}_2 = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}.\end{aligned}$$

Thus

$$\mathbf{y} - \text{proj}_W \mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}.$$

Consequently, the distance from  $\mathbf{y}$  to  $W$  is

$$\boxed{\|\mathbf{y} - \text{proj}_W \mathbf{y}\| = \sqrt{40} = 2\sqrt{10}}.$$

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**GramSchmidt01a**  
**032 10.0 points**

Use the fact that

$$\begin{aligned}A &= \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

to determine an orthogonal basis for  $\text{Col}(A)$ .

$$1. \quad \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

2.  $\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$
3.  $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$  **correct**
4.  $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$

**Explanation:**

The pivot columns of  $A$  provide a basis for  $\text{Col}(A)$ . But by row reduction,

$$\begin{aligned} A &\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus the pivot columns of  $A$  are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}.$$

We apply Gram-Schmidt to produce an orthogonal basis: set  $\mathbf{u}_1 = \mathbf{a}_1$  and

$$\begin{aligned} \mathbf{u}_2 &= \mathbf{a}_2 - \left( \frac{\mathbf{a}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \right) \mathbf{u}_1 \\ &= \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} - \frac{(-36)}{27} \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 20/3 \end{bmatrix} = \begin{bmatrix} -8/3 \\ 2/3 \\ 2/3 \end{bmatrix}. \end{aligned}$$

Consequently, the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for  $\text{Col}(A)$ .

**LeastSquares02c****033 10.0 points**

Find the least-squares solution of  $A\mathbf{x} = \mathbf{b}$  when

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}.$$

1.  $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$
2.  $\begin{bmatrix} -1 \\ 7 \end{bmatrix}$
3.  $\frac{1}{2} \begin{bmatrix} 8 \\ 5 \end{bmatrix}$  **correct**
4.  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
5.  $\frac{1}{2} \begin{bmatrix} 22 \\ 19 \end{bmatrix}$

**Explanation:**

The normal equations for a least-squares solution of  $A\mathbf{x} = \mathbf{b}$  are by definition

$$A^T A \mathbf{x} = A^T \mathbf{b}.$$

Now,

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \end{aligned}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

Hence the least squares solution of  $A\mathbf{x} = \mathbf{b}$  is the solution  $\mathbf{x}$  to the equation

$$\begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

This can be solved with row reduction or inverse matrices to determine that the solution is

$$\begin{aligned}(A^T A)^{-1}(A^T \mathbf{b}) &= \frac{1}{2} \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 8 \\ 5 \end{bmatrix}.\end{aligned}$$

Consequently, the least squares solution to  $A\mathbf{x} = \mathbf{b}$  is

$$\boxed{\frac{1}{2} \begin{bmatrix} 8 \\ 5 \end{bmatrix}}.$$

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**RegressionLine03c**  
**034 10.0 points**

Find the Least Squares Regression line  $y = mx + b$  that best fits the data points

$$(-1, 1), \quad (0, -2), \quad (1, 3), \quad (2, 2).$$

1.  $y = \frac{3}{5}x + \frac{4}{5}$
2.  $y = -\frac{3}{5}x + \frac{4}{5}$
3.  $y = \frac{4}{5}x - \frac{3}{5}$
4.  $y = -\frac{4}{5}x - \frac{3}{5}$
5.  $y = \frac{4}{5}x + \frac{3}{5}$  **correct**
6.  $y = -\frac{3}{5}x - \frac{4}{5}$

**Explanation:**

The design matrix and list of observed values for the data

$$(-1, 1), \quad (0, -2), \quad (1, 3), \quad (2, 2).$$

are given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}.$$

The least squares regression line for this data is  $y = mx + b$  where  $\hat{\mathbf{x}}$  is the solution of the normal equation

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}, \quad \hat{\mathbf{x}} = \begin{bmatrix} b \\ m \end{bmatrix}.$$

Now

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix},$$

while

$$\begin{aligned}A^T A &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}.\end{aligned}$$

Thus the normal equation is

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

So

$$\begin{aligned}\begin{bmatrix} b \\ m \end{bmatrix} &= \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ &= \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.\end{aligned}$$

Consequently, the Least Squares Regression line is

$$\boxed{y = \frac{4}{5}x + \frac{3}{5}}.$$

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**OrthogDiag02a**  
**035 10.0 points**

When

$$A = \begin{bmatrix} 1 & 8 \\ 8 & -11 \end{bmatrix}$$

find matrices  $D$  and  $P$  in an orthogonal diagonalization of  $A$  given that  $\lambda_1 > \lambda_2$ .

$$1. D = \begin{bmatrix} -15 & 0 \\ 0 & 5 \end{bmatrix}, P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$2. D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

correct

$$3. D = \begin{bmatrix} -15 & 0 \\ 0 & 5 \end{bmatrix}, P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$4. D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$5. D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$6. D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

### Explanation:

To begin, we must find the eigenvectors and eigenvalues of  $A$ . To do this, we will use the characteristic equation,  $\det(A - \lambda I) = 0$ . That is, we will look for the zeros of the characteristic polynomial.

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda)(-11 - \lambda) - 64 \\ &= \lambda^2 + 10\lambda - 75 \\ &= (\lambda - 5)(\lambda + 15) = 0. \end{aligned}$$

So  $\lambda_1 = 5$ ,  $\lambda_2 = -15$ , and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}.$$

Now to find the eigenvectors of  $A$ , we will solve for the nontrivial solution of the characteristic equation by row reducing the related augmented matrices:

$$\begin{aligned} [A - \lambda_1 I \quad \mathbf{0}] &= \begin{bmatrix} 1 - 5 & 8 & 0 \\ 8 & -11 - 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 8 & 0 \\ 8 & -16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \implies \mathbf{u}_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \end{aligned}$$

while

$$\begin{aligned} [A - \lambda_2 I \quad \mathbf{0}] &= \begin{bmatrix} 1 + 15 & 8 & 0 \\ 8 & -11 + 15 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 8 & 0 \\ 8 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \implies \mathbf{u}_2 &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}. \end{aligned}$$

Now, when

$$Q = [\mathbf{u}_1 \quad \mathbf{u}_2],$$

then  $Q$  has orthogonal columns and

$$A = QDQ^{-1}$$

is a diagonalization of  $A$ , but it is not an orthogonal diagonalization because  $Q$  is not an orthogonal matrix. We have to normalize  $\mathbf{u}_1$  and  $\mathbf{u}_2$ : set

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Then  $P = [\mathbf{v}_1 \quad \mathbf{v}_2]$  is an orthogonal matrix and so

$$A = PDP^{-1}$$

is an orthogonal diagonalization of  $A$  when

$$\boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & -15 \end{bmatrix}, P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}.$$