

This print-out should have 11 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

---

**QuadApprox02a**  
**001 10.0 points**

Find the quadratic approximation to

$$f(x, y) = \cos(x + y) - 2 \sin(x - y)$$

at  $P(0, 0)$ .

1.  $Q(x, y) = 2 - 2x + 2y + \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$

2.  $Q(x, y) = 1 + 2x - 2y + \frac{1}{2}x^2 - xy + y^2$

3.  $Q(x, y) = 2 - 2x + 2y + \frac{1}{2}x^2 - xy + y^2$

4.  $Q(x, y) = 2 - 2x + 2y - \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$

5.  $Q(x, y) = 1 - 2x + 2y - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$   
**correct**

6.  $Q(x, y) = 1 + 2x - 2y - \frac{1}{2}x^2 + xy + y^2$

**Explanation:**

The Quadratic Approximation to  $f(x, y)$  at  $P(0, 0)$  is given by

$$Q(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2.$$

But when

$$f(x, y) = \cos(x + y) - 2 \sin(x - y)$$

we see that

$$f_x = -\sin(x + y) - 2 \cos(x - y),$$

$$f_y = -\sin(x + y) + 2 \cos(x - y),$$

so that  $f(0, 0) = 1$  and

$$f_x(0, 0) = -2, \quad f_y(0, 0) = 2,$$

while

$$f_{xx} = -\cos(x + y) + 2 \sin(x - y),$$

$$f_{xy} = \cos(x + y) - 2 \sin(x - y),$$

$$f_{yy} = \cos(x + y) + 2 \sin(x - y),$$

so that  $f_{xx}(0, 0) = 1$  and

$$f_{xy}(0, 0) = -1, \quad f_{yy}(0, 0) = -1,$$

Consequently, the Quadratic Approximation to  $f$  at  $P(0, 0)$  is

$$Q(x, y) = 1 - 2x + 2y - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$$

---

keywords: quadratic approximation, partial derivative, second order partial derivative, trig function,

---

**CalC15g19b**  
**002 10.0 points**

Locate and classify the critical point of

$$f(x, y) = \ln(xy) + 2y^2 - y - 2xy + 3,$$

for  $x, y > 0$ .

1. local maximum at  $\left(2, \frac{1}{4}\right)$

2. saddle-point at  $\left(\frac{1}{4}, 2\right)$

3. local minimum at  $\left(\frac{1}{4}, 2\right)$

4. local maximum at  $\left(\frac{1}{4}, 2\right)$

5. local minimum at  $\left(2, \frac{1}{4}\right)$

6. saddle-point at  $\left(2, \frac{1}{4}\right)$  **correct**

**Explanation:**

The critical point of  $f$  is the common solution of the equations

$$\frac{\partial f}{\partial x} = \frac{1}{x} - 2y = 0,$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} + 4y - 1 - 2x = 0.$$

By the first equation,  $2x = 1/y$ . Using this in the second equation, we see that

$$4y - 1 = 0 \quad \text{i.e., } y = \frac{1}{4}.$$

So  $f$  has a critical point at

$$\left(2, \frac{1}{4}\right).$$

Now after differentiation,

$$f_{xx} = -\frac{1}{x^2}, \quad f_{xy} = -2, \quad f_{yy} = 4 - \frac{1}{y^2}.$$

Thus at the critical point  $\left(2, \frac{1}{4}\right)$ ,

$$A = f_{xx}\bigg|_{\left(2, \frac{1}{4}\right)} = -\frac{1}{4} < 0, \quad B = -2,$$

$$C = f_{yy}\bigg|_{\left(2, \frac{1}{4}\right)} = -12 < 0,$$

in which case

$$AC - B^2 = -1 < 0,$$

Consequently, by the second derivative test  $f$  has a

saddle-point at  $\left(2, \frac{1}{4}\right)$ .

---

**CalC14d16s**  
**003    10.0 points**

Determine the position vector,  $\mathbf{r}(t)$ , of a particle having acceleration

$$\mathbf{a}(t) = -6\mathbf{k}$$

when its initial velocity and position are given by

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - 4\mathbf{k}, \quad \mathbf{r}(0) = 3\mathbf{i} + 5\mathbf{j}$$

respectively.

$$\mathbf{1.} \quad \mathbf{r}(t) = (t-3)\mathbf{i} + (t+5)\mathbf{j} - (3t^2-4t)\mathbf{k}$$

$$\mathbf{2.} \quad \mathbf{r}(t) = (t+3)\mathbf{i} + (t+5)\mathbf{j} - (3t^2+4t)\mathbf{k}$$

**correct**

$$\mathbf{3.} \quad \mathbf{r}(t) = (t-3)\mathbf{i} + (t+5)\mathbf{j} - (3t^2+4t)\mathbf{k}$$

$$\mathbf{4.} \quad \mathbf{r}(t) = (t+3)\mathbf{i} - (t-5)\mathbf{j} - (3t^2+4t)\mathbf{k}$$

$$\mathbf{5.} \quad \mathbf{r}(t) = (t+3)\mathbf{i} - (t-5)\mathbf{j} - (3t^2-4t)\mathbf{k}$$

$$\mathbf{6.} \quad \mathbf{r}(t) = (t+3)\mathbf{i} + (t+5)\mathbf{j} - (3t^2-4t)\mathbf{k}$$

**Explanation:**

Since

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -6\mathbf{k},$$

we see that

$$\mathbf{v}(t) = -6t\mathbf{k} + C$$

where  $C$  is a constant vector such that

$$\mathbf{v}(0) = C = \mathbf{i} + \mathbf{j} - 4\mathbf{k}.$$

Thus

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} - (6t+4)\mathbf{k}.$$

But then

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - (3t^2+4t)\mathbf{k} + D$$

where  $D$  is a constant vector such that

$$\mathbf{r}(0) = D = 3\mathbf{i} + 5\mathbf{j}.$$

Consequently,

$\mathbf{r}(t) = (t+3)\mathbf{i} + (t+5)\mathbf{j} - (3t^2+4t)\mathbf{k}.$

---

**CalC14c04a**  
**004    10.0 points**

The curve  $C$  is parametrized by

$$\mathbf{c}(t) = (5 - 2t)\mathbf{i} + \ln(4t)\mathbf{j} + (2 - t^2)\mathbf{k}.$$

Find the arc length of  $C$  between  $\mathbf{c}(1)$  and  $\mathbf{c}(2)$ .

1. arc length =  $3 - 4 \ln 2$
2. arc length =  $3 + \ln 2$  **correct**
3. arc length =  $4 + 4 \ln 2$
4. arc length =  $2 + \ln 8$
5. arc length =  $3 - \ln 2$
6. arc length =  $8 - \ln 2$

**Explanation:**

The arc length of  $C$  between  $\mathbf{c}(t_0)$  and  $\mathbf{c}(t_1)$  is given by the integral

$$L = \int_{t_0}^{t_1} \|\mathbf{c}'(t)\| dt.$$

Now when

$$\mathbf{c}(t) = (5 - 2t)\mathbf{i} + \ln(4t)\mathbf{j} + (2 - t^2)\mathbf{k}$$

we see that

$$\mathbf{c}'(t) = -2\mathbf{i} + \frac{1}{t}\mathbf{j} - 2t\mathbf{k}.$$

But then

$$\|\mathbf{c}'(t)\| = \left(4 + \frac{1}{t^2} + 4t^2\right)^{1/2} = \frac{2t^2 + 1}{t}.$$

Thus

$$L = \int_1^2 \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln t\right]_1^2.$$

Consequently,

$\text{arc length} = L = 3 + \ln 2$

If  $f(x, y)$  is a potential function for the gradient vector field

$$\mathbf{F}(x, y) = (4x + y)\mathbf{i} + (x + 2y)\mathbf{j},$$

evaluate

$$f(1, 2) - f(0, 1).$$

1.  $f(1, 2) - f(0, 1) = \frac{13}{2}$
2.  $f(1, 2) - f(0, 1) = \frac{15}{2}$
3.  $f(1, 2) - f(0, 1) = 6$
4.  $f(1, 2) - f(0, 1) = \frac{11}{2}$
5.  $f(1, 2) - f(0, 1) = 7$  **correct**

**Explanation:**

If  $f(x, y)$  is a potential function for the gradient vector field

$$\mathbf{F}(x, y) = (4x + y)\mathbf{i} + (x + 2y)\mathbf{j},$$

then

$$\frac{\partial f}{\partial x} = 4x + y, \quad \frac{\partial f}{\partial y} = x + 2y.$$

Now by the first equation,

$$f(x, y) = 2x^2 + xy + D(y)$$

for an arbitrary function  $D(y)$ , which by the second equation satisfies

$$x + D'(y) = x + 2y, \quad \text{i.e., } D(y) = y^2 + K,$$

for an arbitrary constant  $K$ . Thus

$$f(x, y) = 2x^2 + xy + y^2 + K.$$

But then

$$f(0, 1) = 1 + K,$$

while

$$f(1, 2) = 2 + 2 + 4 + K = 8 + K.$$

Consequently,

$$\boxed{f(1, 2) - f(0, 1) = 7}.$$

---

**Curl01a**  
**006 10.0 points**

Find the curl of the vector field

$$\mathbf{F}(x, y, z) = zx \mathbf{i} - 2xy \mathbf{j} + 3yz \mathbf{k}.$$

1.  $\text{curl } \mathbf{F} = -2x \mathbf{i} - 3y \mathbf{j} - z \mathbf{k}$
2.  $\text{curl } \mathbf{F} = 3z \mathbf{i} - x \mathbf{j} - 2y \mathbf{k}$
3.  $\text{curl } \mathbf{F} = x \mathbf{i} - 2y \mathbf{j} + 3z \mathbf{k}$
4.  $\text{curl } \mathbf{F} = z \mathbf{i} + 2x \mathbf{j} + 3y \mathbf{k}$
5.  $\text{curl } \mathbf{F} = -2x \mathbf{i} + 3y \mathbf{j} + z \mathbf{k}$
6.  $\text{curl } \mathbf{F} = 3z \mathbf{i} + x \mathbf{j} - 2y \mathbf{k}$  **correct**

**Explanation:**

The curl of  $\mathbf{F}$  is given symbolically by

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & -2xy & 3yz \end{vmatrix} \\ &= X \mathbf{i} - Y \mathbf{j} + Z \mathbf{k} \end{aligned}$$

where

$$\begin{aligned} X &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2xy & 3yz \end{vmatrix} \\ &= \left( \frac{\partial}{\partial y}(3yz) - \frac{\partial}{\partial z}(-2xy) \right) = 3z, \\ Y &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ zx & 3yz \end{vmatrix} \\ &= \left( \frac{\partial}{\partial x}(3yz) - \frac{\partial}{\partial z}(zx) \right) = -x, \end{aligned}$$

and

$$\begin{aligned} Z &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ zx & -2xy \end{vmatrix} \\ &= \left( \frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(zx) \right) = -2y. \end{aligned}$$

Consequently,

$$\boxed{\text{curl } \mathbf{F} = 3z \mathbf{i} + x \mathbf{j} - 2y \mathbf{k}}.$$

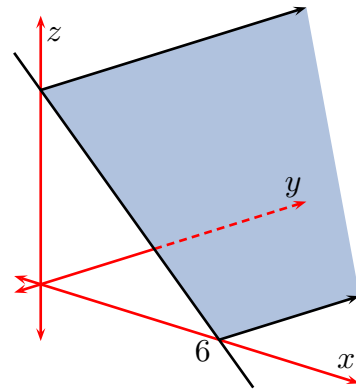
---

**CalC16b01a**  
**007 10.0 points**

The graph of the function

$$z = f(x, y) = 6 - x$$

is the plane shown in



Determine the value of the double integral

$$I = \int \int_A f(x, y) \, dx \, dy$$

over the region

$$A = \{(x, y) : 0 \leq x \leq 4, \ 0 \leq y \leq 4\}$$

in the  $xy$ -plane by first identifying it as the volume of a solid below the graph of  $f$ .

1.  $I = 65$  cu. units
2.  $I = 62$  cu. units
3.  $I = 61$  cu. units

4.  $I = 63$  cu. units

5.  $I = 64$  cu. units **correct**

**Explanation:**

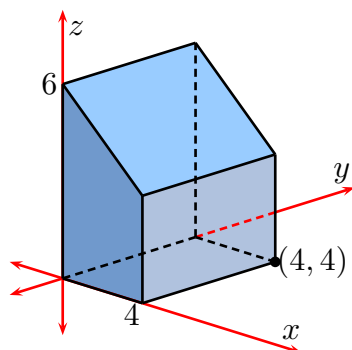
The double integral

$$I = \iint_A f(x, y) \, dx \, dy$$

is the volume of the solid below the graph of  $f$  having the rectangle

$$A = \left\{ (x, y) : 0 \leq x \leq 4, \ 0 \leq y \leq 4 \right\}$$

for its base. Thus the solid is the wedge



and so its volume is the area of trapezoidal face multiplied by the thickness of the wedge. Consequently,

$$I = 64 \text{ cu. units}.$$

keywords:

---

**CalC16c05s**  
**008 10.0 points**

Evaluate the iterated integral

$$I = \int_0^{\pi/2} \int_0^{\cos(\theta)} 4e^{\sin(\theta)} \, dr \, d\theta.$$

1.  $I = 4(e - 1)$  **correct**

2.  $I = e - 4$

3.  $I = \frac{1}{e} - 4$

4.  $I = 4\left(\frac{1}{e} - 1\right)$

5.  $I = 0$

6.  $I = 4e$

**Explanation:**

After simple integration

$$\begin{aligned} \int_0^{\cos(\theta)} 4e^{\sin(\theta)} \, dr &= \left[ 4r e^{\sin(\theta)} \right]_0^{\cos(\theta)} \\ &= 4\cos(\theta) e^{\sin(\theta)}. \end{aligned}$$

In this case,

$$I = \int_0^{\pi/2} 4\cos(\theta) e^{\sin(\theta)} \, d\theta = \left[ 4e^{\sin(\theta)} \right]_0^{\pi/2}.$$

Consequently,

$$I = 4(e - 1).$$

---

**CalC16g01a**  
**009 10.0 points**

Evaluate the triple integral

$$I = \int_0^1 \int_0^x \int_0^{x+y} (3x - 2y) \, dz \, dy \, dx.$$

1.  $I = \frac{5}{8}$

2.  $I = \frac{7}{8}$

3.  $I = \frac{17}{24}$  **correct**

4.  $I = \frac{13}{24}$

5.  $I = \frac{19}{24}$

**Explanation:**

As a repeated integral,

$$I = \int_0^1 \left( \int_0^x \left( \int_0^{x+y} (3x - 2y) dz \right) dy \right) dx.$$

Now

$$\begin{aligned} \int_0^{x+y} (3x - 2y) dz &= \left[ (3x - 2y)z \right]_0^{x+y} \\ &= (3x - 2y)(x + y) = 3x^2 + xy - 2y^2, \end{aligned}$$

while

$$\begin{aligned} \int_0^x (3x^2 + xy - 2y^2) dy \\ = \left[ 3x^2y + \frac{1}{2}xy^2 - \frac{2}{3}y^3 \right]_0^x = \frac{17}{6}x^3. \end{aligned}$$

Consequently,

$$I = \int_0^1 \frac{17}{6}x^3 dx = \frac{17}{24}.$$

---

keywords: integral, triple integral, repeated integral, linear function, polynomial integrand, binomial integrand, evaluation of triple integral

---

**Div01a**

**010 10.0 points**

Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = 3x^2yz \mathbf{i} + 2xy^2z \mathbf{j} - xyz^2 \mathbf{k}.$$

1.  $\text{div } \mathbf{F} = 6xyz$

2.  $\text{div } \mathbf{F} = 10xyz$

3.  $\text{div } \mathbf{F} = 7xyz$

4.  $\text{div } \mathbf{F} = 8xyz$  **correct**

5.  $\text{div } \mathbf{F} = 9xyz$

**Explanation:**

The div of  $\mathbf{F}$  is given symbolically by

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= 3 \frac{\partial}{\partial x}(x^2yz) + 2 \frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(xyz^2). \end{aligned}$$

Thus

$$\text{div } \mathbf{F} = (6 + 4 - 2)xyz = 8xyz.$$

---

**CalC15h04exam**

**011 10.0 points**

Determine the maximum value of

$$f(x, y) = 3x + 4y + 3$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

1. max value = 7

2. max value = 11

3. max value = 9

4. max value = 10

5. max value = 8 **correct**

**Explanation:**

The extreme values of  $f$  subject to the constraint  $g = 0$  occur at solutions of

$$(\nabla f)(x, y) = \lambda(\nabla g)(x, y), \quad g(x, y) = 0.$$

Now

$$(\nabla f)(x, y) = \langle 3, 4 \rangle,$$

while

$$(\nabla g)(x, y) = \langle 2x, 2y \rangle.$$

Thus

$$3 = 2\lambda x, \quad 4 = 2\lambda y,$$

and so

$$\lambda = \frac{3}{2x} = \frac{2}{y}, \quad \text{i.e., } y = \frac{4}{3}x.$$

But

$$g\left(x, \frac{4}{3}x\right) = x^2 + \frac{16}{9}x^2 - 1 = 0,$$

*i.e.*,  $x = \pm 3/5$ . Consequently, the extreme points are

$$\left(\frac{3}{5}, \frac{4}{5}\right), \quad \left(-\frac{3}{5}, -\frac{4}{5}\right).$$

Since

$$f\left(\frac{3}{5}, \frac{4}{5}\right) = 8, \quad f\left(-\frac{3}{5}, -\frac{4}{5}\right) = -2,$$

we thus see that

$$\boxed{\text{max value} = 8}.$$

---

keywords: