This print-out should have 12 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

MatrixProp01a
001 10.0 points

Compute $AA^T - A^TA$ for the matrix

$$A = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}.$$

1. $AA^T - A^TA = \begin{bmatrix} -5 & -1 \\ -1 & 5 \end{bmatrix}$

2. $AA^T - A^TA = \begin{bmatrix} 5 & -1 \\ -1 & -5 \end{bmatrix}$

3. $AA^T - A^TA = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}$

4. $AA^T - A^TA = \begin{bmatrix} 5 & -1 \\ 1 & -5 \end{bmatrix}$

5. $AA^T - A^TA = \begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix}$ **correct**

6. $AA^T - A^TA = \begin{bmatrix} -5 & -1 \\ 1 & 5 \end{bmatrix}$

Explanation:
By matrix multiplication,

$$AA^T = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 13 \\ 13 & 13 \end{bmatrix},$$

while

$$A^TA = \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 12 \\ 12 & 8 \end{bmatrix},$$

Consequently,

$$AA^T - A^TA = \begin{bmatrix} 13 & 13 \\ 13 & 13 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 12 & 8 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix}}.$$

InverseMatrix01b
002 10.0 points

Solve for X when $A(X+B) = C$,

$$A = \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}.$$

1. $X = \begin{bmatrix} -5 & -8 \\ 4 & 23 \end{bmatrix}$

2. $X = \begin{bmatrix} -5 & 8 \\ 4 & 23 \end{bmatrix}$

3. $X = \begin{bmatrix} 0 & 8 \\ 3 & 21 \end{bmatrix}$

4. $X = \begin{bmatrix} -6 & 8 \\ 3 & 21 \end{bmatrix}$

5. $X = \begin{bmatrix} -6 & -8 \\ 3 & 23 \end{bmatrix}$ **correct**

6. $X = \begin{bmatrix} 0 & -8 \\ 4 & 23 \end{bmatrix}$

Explanation:
By the algebra of matrices,

$$X = A^{-1}C - B.$$

But the inverse of any 2×2 matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

is given by

$$D^{-1} = \begin{bmatrix} d_{22}/\Delta & -d_{12}/\Delta \\ -d_{21}/\Delta & d_{11}/\Delta \end{bmatrix}$$

with $\Delta = d_{11}d_{22} - d_{12}d_{21}$, so

$$X = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -10 \\ 4 & 26 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 1 & -3 \end{bmatrix}.$$

Thus

$$\boxed{X = \begin{bmatrix} -6 & -8 \\ 3 & 23 \end{bmatrix}}.$$

LUDecomp06h
003 10.0 points

Find L in an LU decomposition of

$$A = \begin{bmatrix} 3 & -2 & 5 & 2 \\ -6 & 4 & -6 & 1 \\ 12 & -8 & 4 & -17 \end{bmatrix}.$$

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -4 & 1 \end{bmatrix}$

2. $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix}$ **correct**

3. $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$

4. $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & -4 & 2 \end{bmatrix}$

5. $L = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & -4 & -1 \end{bmatrix}$

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix}$

Explanation:

Recall that in a factorization $A = LU$ of an $m \times n$ matrix A, then L is an $m \times m$ lower triangular matrix with ones on the diagonal and U is an $m \times n$ echelon form of A.

We begin by computing U. Now $U = M_0 A$ where j is the number of row operations on A needed to transform A into its echelon form U and M_i is a product of $j - i$ elementary matrices that represent these row operations:

$$\begin{aligned} U &= M_0 A = M_1 E_1 A \\ &= M_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 & 2 \\ -6 & 4 & -6 & 1 \\ 12 & -8 & 4 & -17 \end{bmatrix} \\ &= M_2 E_2 (E_1 A) \\ &= M_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 & 2 \\ 0 & 0 & 4 & 5 \\ 12 & -8 & 4 & -17 \end{bmatrix} \\ &= E_3 (E_2 E_1 A) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 & 2 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & -16 & -25 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 & 5 & 2 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & -5 \end{bmatrix} \end{aligned}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change $U = M_0 A$ to $M_0^{-1} U = A$. This shows that $L = M_0^{-1}$. Hence

$$\begin{aligned} L &= M_0^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \\ &= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \end{aligned}$$

Consequently,

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix}}.$$

Subspace02a
004 10.0 points

Which of the following describes

$$H = \text{Span}\{\,\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\,\}$$

when

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

1. $H = \mathbb{R}^3$

2. H is a plane not through origin

3. H is a plane through origin **correct**

4. H is a line

Explanation:
Since H is a subspace of $\mathbb{R}^3$, H contains the origin. On the other hand, if

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 1 \\ -3 & 5 & 4 \end{bmatrix},$$

then $H = \text{Col}(A)$, and

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are pivot columns of A, so $\{\mathbf{v}_1,\,\mathbf{v}_2\}$ is a basis for $\text{Col}(A)$.

Consequently,

$$\boxed{H \text{ is a plane through origin}}\,.$$

Invertible02
005 10.0 points

A is an $n \times n$ matrix. Which of the following statements are equivalent to A being invertible?

(i) $\text{Col}\,A = \{\mathbf{0}\}$.

(ii) *The columns of A do not form a basis of $\mathbb{R}^n$.*

(iii) $\dim(\text{Col}\,A) = n$.

1. iii **correct**

2. None of these

3. i and ii

4. i

5. ii

6. All of these

Explanation:
(i) Because A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. $\text{Col}\,A$ is the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent. Because there is at least on solution for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation is consistent for all $\mathbf{b}$ in $\mathbb{R}^n$, and hence all of $\mathbb{R}^n$.

(ii) Because A is invertible, the columns of A span $\mathbb{R}^n$ and form a linearly independent set. By definition, a basis of a subspace is a linearly independent set of vectors that span that subspace. Hence the columns of A form a basis of $\mathbb{R}^n$.

(iii) Since A is invertible, $\text{Col}\,A$ is a basis for $\mathbb{R}^n$. If $\text{Col}\,A$ is a basis for $\mathbb{R}^n$, it must have exactly n vectors. Hence the dimension of $\text{Col}\,A$ is n.

Rank02e
006 10.0 points

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 5 \end{bmatrix}.$$

1. $\text{rank}(A) = 1$

2. $\text{rank}(A) = 4$

3. $\text{rank}(A) = 5$

4. $\text{rank}(A) = 2$

5. $\text{rank}(A) = 3$ **correct**

Explanation:

Since

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

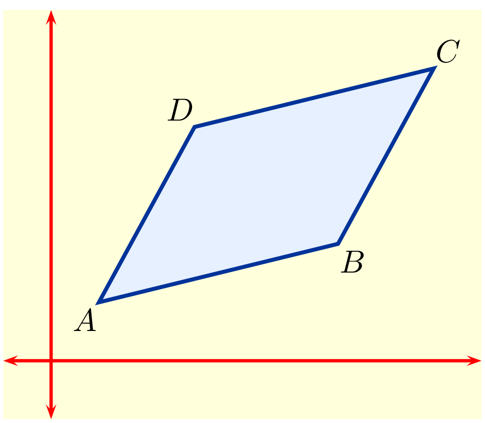
all three rows of rref(A) contain leading 1's, so

$$\boxed{\text{Rank}(A) = 3}.$$

DetArea03a

007 10.0 points

Compute the area of the parallelogram $ABCD$ shown in



having vertices at

$$A = (1, 1), \qquad B = (6, 2),$$

and

$$C = (8, 5), \qquad D = (3, 4).$$

1. area $= 13$ **correct**

2. area $= 10$

3. area $= 11$

4. area $= 12$

5. area $= 14$

Explanation:

After translating $ABCD$ so that A becomes the origin, we obtain a new parallelogram $OB'C'D'$ of equal area with vertices at the origin and

$$B' = (5, 1), \quad C' = (7, 4), \quad D' = (2, 3).$$

Now

$$\text{area}(OB'C'D') = \left|\det \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}\right| = 13.$$

Consequently, $ABCD$ has

$$\boxed{\text{Area} = 13}.$$

BasisNull01b

008 10.0 points

Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 2 & -2 & 10 & 2 \\ -1 & -1 & -1 & -3 \\ 2 & -3 & 12 & -1 \end{bmatrix}.$$

1. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ **correct**

3. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \\ 12 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

6. $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Explanation:

We first row reduce $[A\ \ \mathbf{0}]$:

$$\text{rref}([A\ \ \mathbf{0}]) \;=\; \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

to identify the free variables for $\mathbf{x}$ in the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Thus x_1, x_2, and x_4 are basic variables, while x_3 is a free variable. So set $x_3 = s$. Then

$$x_1 \;=\; -3s, \;\; x_2 \;=\; 2s, \;\; x_3 \;=\; s, \;\; x_4 \;=\; 0\,,$$

and

$$\text{Nul}(A) \;=\; \text{Span}\left\{\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Consequently,

$$\mathcal{B} \;=\; \left\{\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

is a basis for $\text{Nul}(A)$.

Basis03a
009 10.0 points

In the vector space V of all real-valued functions, find a basis for the subspace

$$H \;=\; \text{Span}\{\sin t,\ \sin 2t,\ \sin t\cos t\,\}\,.$$

1. $\{\sin 2t,\ \sin t\cos t\,\}$

2. $\{\sin t,\ \sin 2t,\ \sin t\cos t\,\}$

3. $\{\sin t,\ \sin 2t\,\}$ **correct**

4. $\{\cos t,\ \sin 2t,\ \sin t\cos t\,\}$

5. $\{\cos t,\ \sin 2t\,\}$

Explanation:

By double angle formula,

$$\sin 2t \;=\; 2\sin t\cos t\,,$$

so the functions

$$\{\sin t,\ \sin 2t,\ \sin t\cos t\,\}$$

are linearly dependent, while

$$\{\sin t,\ \sin 2t\,\}$$

are linearly independent. Consequently,

$$\boxed{\{\sin t,\ \sin 2t\,\}}$$

is a basis for H.

PolyCoordVec01b
010 10.0 points

Find the coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ in $\mathbb{R}^3$ for the polynomial

$$\mathbf{p}(t) \;=\; 2 + 3t - 6t^2$$

with respect to the basis

$$\mathcal{B} \;=\; \left\{1 - t^2,\ t - t^2,\ 1 - t + t^2\right\}$$

for $\mathbb{P}_2$.

1. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

2. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

3. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ **correct**

4. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

5. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

6. $[\mathbf{p}]_{\mathcal{B}} \;=\; \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Explanation:

The coordinate mapping from $\mathbb{P}_2$ to $\mathbb{R}^3$ maps

$$\mathbf{p} = a + bt + ct^2 \longrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

When

$$\mathbf{p}(t) = 2 + 3t - 6t^2$$

and

$$\mathcal{B} = \{1 - t^2,\ t - t^2,\ 1 - t + t^2\},$$

therefore, the entries c_1, c_2, c_3 in $[\mathbf{p}]_{\mathcal{B}}$ are the solutions of the polynomial equation

$$c_1(1-t^2) + c_2(t-t^2) + c_3(1-t+t^2)$$
$$= \mathbf{p}(t) = 2 + 3t - 6t^2.$$

Equating coefficients thus shows that c_1, c_2, c_3 satisfy the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix},$$

which in augmented matrix form becomes

$$A = \left[\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{bmatrix}\right].$$

But then

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus

$$\boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}.$$

ChangeBasis01c
011 (part 1 of 2) 10.0 points

Determine the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ of a vector space V when

$$\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2, \qquad \mathbf{b}_2 = 5\mathbf{c}_1 - 3c_2.$$

1. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}$

2. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix}$

3. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ **correct**

4. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -5 \\ -4 & -3 \end{bmatrix}$

5. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -5 \\ 4 & -3 \end{bmatrix}$

6. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 3 & -4 \end{bmatrix}$

Explanation:

The change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the 2×2 matrix

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [\ [\mathbf{b}_1]_{\mathcal{C}}\ \ [\mathbf{b}_2]_{\mathcal{C}}\ ].$$

Now

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Consequently,

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}}.$$

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012 (part 2 of 2) 10.0 points

Determine $[\mathbf{x}]_{\mathcal{C}}$ when

$$\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2.$$

1. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 11 \\ -10 \end{bmatrix}$

2. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -11 \\ 10 \end{bmatrix}$

3. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$ **correct**

4. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

5. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -10 \\ 11 \end{bmatrix}$

6. $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 10 \\ -11 \end{bmatrix}$

Explanation:

When

$$\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2,$$

then

$$\mathbf{x} = 5(-\mathbf{c}_1 + 4\mathbf{c}_2) + 3(5\mathbf{c}_1 - 3c_2).$$

Consequently,

$$\boxed{[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}}.$$