This print-out should have 12 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

MatrixVecProd04
001 10.0 points

Determine $\mathbf{v}\mathbf{u}^T$ when

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

1. $\mathbf{v}\mathbf{u}^T = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix}$

2. $\mathbf{v}\mathbf{u}^T = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}$ **correct**

3. $\mathbf{v}\mathbf{u}^T = -5a + 2b - 3c$

4. $\mathbf{v}\mathbf{u}^T = -3a + 2b - 5c$

Explanation:
Since

$$\mathbf{u}^T = [-3 \quad 2 \quad -5],$$

we see that

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [-3 \quad 2 \quad -5].$$

Consequently,

$$\boxed{\mathbf{v}\mathbf{u}^T = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}}.$$

InverseMatrix01a
002 10.0 points

Solve for X when $AX + B = C$,

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

1. $X = \begin{bmatrix} 0 & 1 \\ -7 & 1 \end{bmatrix}$

2. $X = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ **correct**

3. $X = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$

4. $X = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

5. $X = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$

Explanation:
By the algebra of matrices,

$$X = A^{-1}(C - B).$$

But the inverse of any 2×2 matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

is given by

$$D^{-1} = \begin{bmatrix} \dfrac{d_{22}}{\Delta} & -\dfrac{d_{12}}{\Delta} \\ -\dfrac{d_{21}}{\Delta} & \dfrac{d_{11}}{\Delta} \end{bmatrix}$$

with $\Delta = d_{11}d_{22} - d_{12}d_{21}$.

Thus

$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \left(\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.$$

Consequently,

$$\boxed{X = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}}.$$

LUDecomp06h
003 10.0 points

Find L in an LU decomposition of

$$A = \begin{bmatrix} 5 & 0 & 0 & 1 \\ 20 & 0 & 5 & 2 \\ 20 & 0 & 10 & -2 \end{bmatrix}.$$

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -2 & 1 \end{bmatrix}$

2. $L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$

3. $L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 4 & 2 & 2 \end{bmatrix}$

4. $L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$

5. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$ **correct**

6. $L = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 4 & 2 & -1 \end{bmatrix}$

Explanation:

Recall that in a factorization $A = LU$ of an $m \times n$ matrix A, then L is an $m \times m$ lower triangular matrix with ones on the diagonal and U is an $m \times n$ echelon form of A.

We begin by computing U. Now $U = M_0 A$ where j is the number of row operations on A needed to transform A into its echelon form U and M_i is a product of $j - i$ elementary matrices that represent these row operations:

$$\begin{aligned} U &= M_0 A = M_1 E_1 A \\ &= M_1 \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 1 \\ 20 & 0 & 5 & 2 \\ 20 & 0 & 10 & -2 \end{bmatrix} \\ &= M_2 E_2 (E_1 A) \\ &= M_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 0 & 5 & -2 \\ 20 & 0 & 10 & -2 \end{bmatrix} \\ &= E_3 (E_2 E_1 A) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 10 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \end{aligned}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change $U = M_0 A$ to $M_0^{-1} U = A$. This shows that $L = M_0^{-1}$. Hence

$$\begin{aligned} L = M_0^{-1} &= E_1^{-1} E_2^{-1} E_3^{-1} \\ &= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \end{aligned}$$

Consequently,

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}}.$$

Subspace05a
004 10.0 points

Let H be the set of all vectors

$$\begin{bmatrix} a-2b \\ ab+3a \\ b \end{bmatrix}$$

where a and b are real. Determine if H is a subspace of $\mathbb{R}^3$, and then check the correct answer below.

1. *H is not a subspace of $\mathbb{R}^3$ because it does not contain* $\mathbf{0}$.

2. *H is a subspace of $\mathbb{R}^3$ because it can be written as Nul(A) for some matrix A.*

3. *H is not a subspace of $\mathbb{R}^3$ because it is not closed under vector addition.* **correct**

4. *H is a subspace of $\mathbb{R}^3$ because it can be written as Span$\{\mathbf{v}_1,\, \mathbf{v}_2\}$ with $\mathbf{v}_1,\, \mathbf{v}_2$ in $\mathbb{R}^3$.*

Explanation:

To check if the set H of all vectors

$$\begin{bmatrix} a-2b \\ ab+3a \\ b \end{bmatrix}$$

is a subspace of $\mathbb{R}^3$ we check the properties defining a subspace:

1. *the zero vector* $\mathbf{0}$ *is in* H: set $a=b=0$. Then

$$\begin{bmatrix} 0-0 \\ 0+0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so H contains $\mathbf{0}$.

2. *for each* $\mathbf{u}$, $\mathbf{v}$ *in* H *the sum* $\mathbf{u}+\mathbf{v}$ *is in* H: set

$$\mathbf{v}_1 = \begin{bmatrix} a_1-2b_1 \\ a_1b_1+3a_1 \\ b_1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_2-2b_2 \\ a_2b_2+3a_2 \\ b_2 \end{bmatrix},$$

in H. Then

$$\mathbf{v}_1+\mathbf{v}_2 = \begin{bmatrix} a_1-2b_1 \\ a_1b_1+3a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2-2b_2 \\ a_2b_2+3a_2 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} (a_1+a_2)-2(b_1+b_2) \\ a_1b_1+a_2b_2+3(a_1+a_2) \\ (b_1+b_2) \end{bmatrix}.$$

But in general,

$$a_1b_1+a_2b_2 \neq (a_1+a_2)(b_1+b_2),$$

in which case $\mathbf{u}+\mathbf{v}$ is not in H.

Consequently, H is not a subspace of $\mathbb{R}^3$ because it is

$$\boxed{\textit{not closed under vector addition}}.$$

Invertible01/02

005 10.0 points

A is an $n\times n$ matrix. Which of the following statements are equivalent to A being invertible?

(i) *The linear transformation* $\mathbf{x} \to A\mathbf{x}$ *is not one-to-one.*

(ii) *A is not row equivalent to the $n\times n$ identity matrix.*

(iii) *The columns of A do not form a basis of $\mathbb{R}^n$.*

1. ii and iii

2. i and iii

3. i

4. None of these **correct**

5. All of these

6. i and ii

Explanation:

(i) Since the equation $A\mathbf{x}=\mathbf{0}$ has only the trivial solution, the linear transformation $\mathbf{x}\to A\mathbf{x}$ is one-to-one.

(ii) Since A is invertible, A has n pivot positions. With n pivot positions, the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n.

(iii) Because A is invertible, the columns of A span $\mathbb{R}^n$ and form a linearly independent set. By definition, a basis of a subspace is a linearly independent set of vectors that span that subspace. Hence the columns of A form a basis of $\mathbb{R}^n$.

Rank02c
006 10.0 points

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 6 & 1 \\ -3 & 6 & 0 \end{bmatrix}.$$

1. $\text{rank}(A) = 4$

2. $\text{rank}(A) = 3$

3. $\text{rank}(A) = 1$

4. $\text{rank}(A) = 5$

5. $\text{rank}(A) = 2$ **correct**

Explanation:

Since

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the first two rows of rref(A) contain leading 1's, so

$$\boxed{\text{Rank}(A) = 2}.$$

DetVolume01a
007 10.0 points

Compute the volume of the parallelepiped with adjacent edges $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$ determined by vertices

$$P(4, -4, -4), \quad Q(2, -4, -3), \quad R(2, 2, 1),$$

where O is the origin in 3-space.

1. volume $= 12$

2. volume $= 10$

3. volume $= 9$

4. volume $= 11$

5. volume $= 8$ **correct**

Explanation:

The parallelepiped is determined by the vectors

$$\mathbf{u} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Thus its volume is the absolute value of

$$\det \begin{bmatrix} 4 & -4 & -4 \\ 2 & -4 & -3 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= 4 \begin{vmatrix} -4 & -3 \\ 2 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & -4 \\ 2 & 2 \end{vmatrix}.$$

Consequently, the parallelepiped has

$$\boxed{\text{volume} = 8}.$$

BasisNul01a
008 10.0 points

Find a basis for the Null space of the matrix

$$A = \begin{bmatrix} 3 & 6 & -3 & -6 \\ 1 & 2 & -4 & 4 \\ 1 & 2 & 0 & -6 \end{bmatrix}.$$

1. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ **correct**

5. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

6. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -6 \end{bmatrix} \right\}$

Explanation:

We first row reduce $[A\ \mathbf{0}]$:

$$\mathrm{rref}([A\ \mathbf{0}]) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

to identify the free variables for $\mathbf{x}$ in the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Thus x_1, x_3, and x_4 are basic variables, while x_2 is a free variable. So set $x_2 = s$. Then

$$x_1 = -2s, \quad x_2 = s, \quad x_3 = 0, \quad x_4 = 0,$$

and

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Consequently,

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathrm{Nul}(A)$.

Basis03a
009 10.0 points

In the vector space V of all real-valued functions, find a basis for the subspace

$$H = \mathrm{Span}\{\sin t,\ \sin 2t,\ \sin t \cos t\}.$$

1. $\{\cos t,\ \sin 2t\}$

2. $\{\sin t,\ \sin 2t\}$ **correct**

3. $\{\sin 2t,\ \sin t \cos t\}$

4. $\{\sin t,\ \sin 2t,\ \sin t \cos t\}$

5. $\{\cos t,\ \sin 2t,\ \sin t \cos t\}$

Explanation:

By double angle formula,

$$\sin 2t = 2 \sin t \cos t,$$

so the functions

$$\{\sin t,\ \sin 2t,\ \sin t \cos t\}$$

are linearly dependent, while

$$\{\sin t,\ \sin 2t\}$$

are linearly independent. Consequently,

$$\boxed{\{\sin t,\ \sin 2t\}}$$

is a basis for H.

PolyCoordVec01a
010 10.0 points

Find the coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ in $\mathbb{R}^3$ for the polynomial

$$\mathbf{p}(t) = 1 + 4t + 7t^2$$

with respect to the basis

$$\mathcal{B} = \left\{1 + t^2,\ t + t^2,\ 1 + 2t + t^2\right\}$$

for $\mathbb{P}_2$.

1. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ **correct**

2. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -6 \\ 1 \end{bmatrix}$

3. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$

4. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$

5. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$

6. $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 1 \\ 2 \end{bmatrix}$

Explanation:

The coordinate mapping from $\mathbb{P}_2$ to $\mathbb{R}^3$ maps

$$\mathbf{p} = a + bt + ct^2 \longrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

When

$$\mathbf{p}(t) = 1 + 4t + 7t^2$$

and

$$\mathcal{B} = \{1 + t^2,\ t + t^2,\ 1 + 2t + t^2\},$$

therefore, the entries c_1, c_2, c_3 in $[\mathbf{p}]_{\mathcal{B}}$ are the solutions of the polynomial equation

$$c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2)$$

$$= \mathbf{p}(t) = 1 + 4t + 7t^2.$$

Equating coefficients thus shows that c_1, c_2, c_3 satisfy the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix},$$

which in augmented matrix form becomes

$$A = \left[\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \right].$$

But then

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus

$$\boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}}.$$

ChangeBasis01b
011 (part 1 of 2) 10.0 points

Determine the change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ of a vector space V when

$$\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2, \qquad \mathbf{b}_2 = 9\mathbf{c}_1 - 4c_2.$$

1. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 9 & 6 \\ -4 & -2 \end{bmatrix}$

2. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & -9 \\ 2 & -4 \end{bmatrix}$

3. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$ **correct**

4. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 9 & -6 \\ 4 & -2 \end{bmatrix}$

5. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -6 & 9 \\ -2 & 4 \end{bmatrix}$

6. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}$

Explanation:

The change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is the 2×2 matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [\ [\mathbf{b}_1]_{\mathcal{C}}\ \ [\mathbf{b}_2]_{\mathcal{C}}\ ].$$

Now

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}.$$

Consequently,

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} \;=\; \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}}\,.$$

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012 (part 2 of 2) 10.0 points

Determine $[\mathbf{x}]_{\mathcal{C}}$ when

$$\mathbf{x} \;=\; -3\mathbf{b}_1 + 2\mathbf{b}_2\,.$$

1. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

2. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

3. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **correct**

4. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

5. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

6. $[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Explanation:

Now

$$\begin{aligned} \mathbf{x} \;&=\; -3\mathbf{b}_1 + 2\mathbf{b}_2 \\ &=\; -3(6\mathbf{c}_1 - 2\mathbf{c}_2) + 2(9\mathbf{c}_1 - 4\mathbf{c}_2) \\ &=\; 0\mathbf{c}_1 - 2\mathbf{c}_2\,. \end{aligned}$$

Consequently,

$$\boxed{[\mathbf{x}]_{\mathcal{C}} \;=\; \begin{bmatrix} 0 \\ -2 \end{bmatrix}}\,.$$