# The Matrix of a Linear Transformation

**EXAMPLE 1** The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose $T$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

**SOLUTION** Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \tag{1}$$

Since $T$ is a *linear* transformation,

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \qquad (2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix} \end{aligned}$$

■

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the $j$th column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \tag{3}$$

**PROOF** Write $\mathbf{x} = I_n\mathbf{x} = [\,\mathbf{e}_1 \;\; \cdots \;\; \mathbf{e}_n\,]\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, and use the linearity of $T$ to compute

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

The uniqueness of $A$ is treated in Exercise 33. ■

The matrix $A$ in (3) is called the **standard matrix for the linear transformation** $T$.

**EXAMPLE 2** Find the standard matrix $A$ for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.

**SOLUTION** Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

■

**EXAMPLE 3** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle $\varphi$, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Fig. 6 in Section 1.8.) Find the standard matrix $A$ of this transformation.

**SOLUTION** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$. See Fig. 1. By Theorem 10,

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

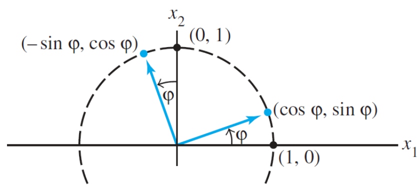


**FIGURE 1** A rotation transformation.

## Existence and Uniqueness Questions

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, $T$ is onto $\mathbb{R}^m$ when the range of $T$ is all of the codomain $\mathbb{R}^m$. That is, $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each $\mathbf{b}$ in the codomain $\mathbb{R}^m$, there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. "Does $T$ map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping $T$ is *not* onto when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution. See Fig. 3.

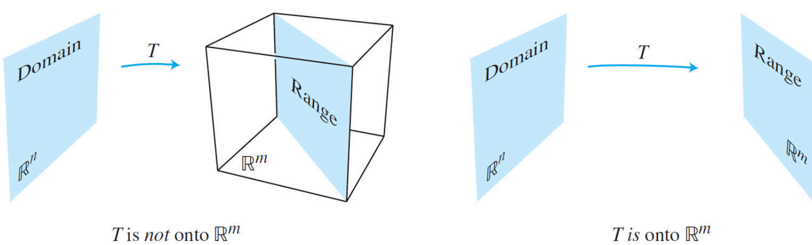


**FIGURE 3** Is the range of $T$ all of $\mathbb{R}^m$?

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, $T$ is one-to-one if, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is $T$ one-to-one?" is a uniqueness question. The mapping $T$ is *not* one-to-one when some $\mathbf{b}$ in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$. If there is no such $\mathbf{b}$, then $T$ is one-to-one. See Fig. 4.

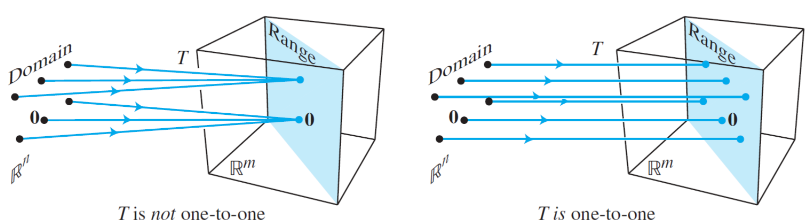


**FIGURE 4** Is every $\mathbf{b}$ the image of at most one vector?

**EXAMPLE 4** Let $T$ be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does $T$ map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is $T$ a one-to-one mapping?

**SOLUTION** Since $A$ happens to be in echelon form, we can see at once that $A$ has a pivot position in each row. By Theorem 4 in Section 1.4, for each $\mathbf{b}$ in $\mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent. In other words, the linear transformation $T$ maps $\mathbb{R}^4$ (its domain) onto $\mathbb{R}^3$. However, since the equation $A\mathbf{x} = \mathbf{b}$ has a free variable (because there are four variables and only three basic variables), each $\mathbf{b}$ is the image of more than one $\mathbf{x}$. That is, $T$ is *not* one-to-one. ■

> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

**PROOF** Since $T$ is linear, $T(\mathbf{0}) = \mathbf{0}$. If $T$ is one-to-one, then the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution and hence only the trivial solution. If $T$ is not one-to-one, then there is a $\mathbf{b}$ that is the image of at least two different vectors in $\mathbb{R}^n$—say, $\mathbf{u}$ and $\mathbf{v}$. That is, $T(\mathbf{u}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{b}$. But then, since $T$ is linear,

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

The vector $\mathbf{u} - \mathbf{v}$ is not zero, since $\mathbf{u} \neq \mathbf{v}$. Hence the equation $T(\mathbf{x}) = \mathbf{0}$ has more than one solution. So, either the two conditions in the theorem are both true or they are both false. ■

> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:
>
> a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$;
>
> b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

**PROOF**

a. By Theorem 4 in Section 1.4, the columns of $A$ span $\mathbb{R}^m$ if and only if for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ is consistent—in other words, if and only if for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has at least one solution. This is true if and only if $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b. The equations $T(\mathbf{x}) = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are the same except for notation. So, by Theorem 11, $T$ is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if the columns of $A$ are linearly independent, as was already noted in the boxed statement (3) in Section 1.7. ■

**EXAMPLE 5** Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that $T$ is a one-to-one linear transformation. Does $T$ map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

**SOLUTION** When $\mathbf{x}$ and $T(\mathbf{x})$ are written as column vectors, you can determine the standard matrix of $T$ by inspection, visualizing the row–vector computation of each entry in $A\mathbf{x}$.

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

So $T$ is indeed a linear transformation, with its standard matrix $A$ shown in (4). The columns of $A$ are linearly independent because they are not multiples. By Theorem 12(b), $T$ is one-to-one. To decide if $T$ is onto $\mathbb{R}^3$, examine the span of the columns of $A$. Since $A$ is $3 \times 2$, the columns of $A$ span $\mathbb{R}^3$ if and only if $A$ has 3 pivot positions, by Theorem 4. This is impossible, since $A$ has only 2 columns. So the columns of $A$ do not span $\mathbb{R}^3$, and the associated linear transformation is not onto $\mathbb{R}^3$. ■