Three-Dimensional Coordinate Systems

To locate a point in a plane, two numbers are necessary.

We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate.

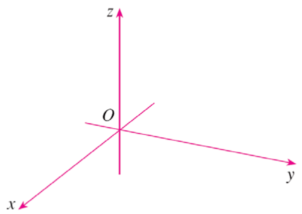
For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required.

We represent any point in space by an ordered triple (a, b, c) of real numbers.

Three-Dimensional Coordinate Systems

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis.

Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1.

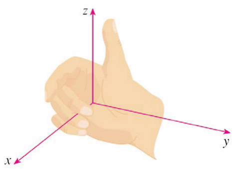


Coordinate axes

Figure 1

Three-Dimensional Coordinate Systems

The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2:



Right-hand rule

Figure 2

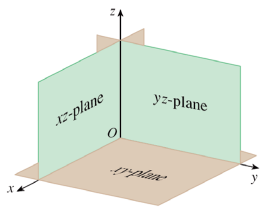
If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

Three-Dimensional Coordinate Systems

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a).

The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes.

These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.



Coordinate planes

Figure 3(a)

Three-Dimensional Coordinate Systems

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)].

Look at any bottom corner of a room and call the corner the origin.

The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane.

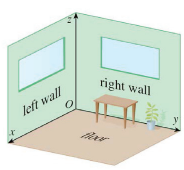


Figure 3(b)

Three-Dimensional Coordinate Systems

The x-axis runs along the intersection of the floor and the left wall.

The y-axis runs along the intersection of the floor and the right wall.

The z-axis runs up from the floor toward the ceiling along the intersection of the two walls.

You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Three-Dimensional Coordinate Systems

Now if *P* is any point in space, let *a* be the (directed) distance from the *yz*-plane to *P*, let *b* be the distance from the *xz*-plane to *P*, and let *c* be the distance from the *xy*-plane to *P*.

We represent the point *P* by the ordered triple (*a*, *b*, *c*) of real numbers and we call *a*, *b*, and *c* the **coordinates** of *P*; *a* is the *x*-coordinate, *b* is the *y*-coordinate, and *c* is the *z*-coordinate.

Three-Dimensional Coordinate Systems

Thus, to locate the point (*a*, *b*, *c*), we can start at the origin *O* and move *a* units along the *x*-axis, then *b* units parallel to the *y*-axis, and then *c* units parallel to the *z*-axis as in Figure 4.

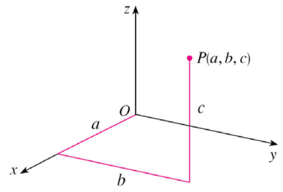


Figure 4

Three-Dimensional Coordinate Systems

The point *P*(*a*, *b*, c) determines a rectangular box as in Figure 5.

If we drop a perpendicular from *P* to the *xy*-plane, we get a point *Q* with coordinates (*a*, *b*, 0) called the **projection** of *P* onto the *xy*-plane.

Similarly, *R*(0, *b*, *c*) and *S*(*a*, 0, *c*) are the projections of *P* onto the *yz*-plane and *xz*-plane, respectively.

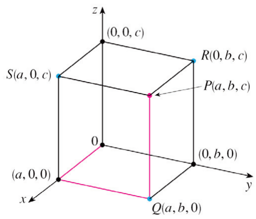


Figure 5

Three-Dimensional Coordinate Systems

As numerical illustrations, the points (–4, 3, –5) and (3, –2, –6) are plotted in Figure 6.

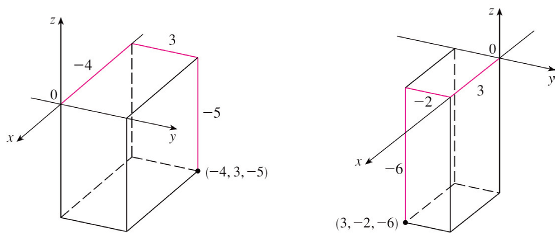


Figure 6

Three-Dimensional Coordinate Systems

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

We have given a one-to-one correspondence between points *P* in space and ordered triples (*a*, *b*, *c*) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**.

Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter $(\vec{v})$.

Vectors

For instance, suppose a particle moves along a line segment from point *A* to point *B*.

The corresponding **displacement vector v**, shown in Figure 1, has **initial point** *A* (the tail) and **terminal point** *B* (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$

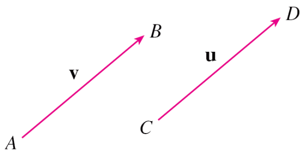


Figure 1
Equivalent vectors

Vectors

Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position.

We say that **u** and **v** are **equivalent** (or **equal**) and we write **u** = **v**.

The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

Combining Vectors

Combining Vectors

Suppose a particle moves from *A* to *B*, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from *B* to *C*, with displacement vector $\overrightarrow{BC}$ as in Figure 2.

The combined effect of these displacements is that the particle has moved from *A* to *C*.

The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

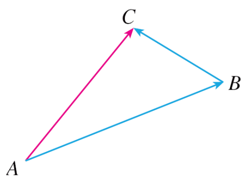


Figure 2

Combining Vectors

In general, if we start with vectors **u** and **v**, we first move **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

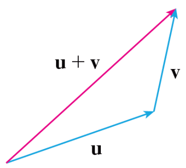


Figure 3
The Triangle Law

Combining Vectors

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**.

Completing the parallelogram, we see that **u** + **v** = **v** + **u**.

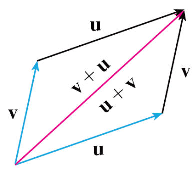


Figure 4
The Parallelogram Law

This also gives another way to construct the sum: If we place **u** and **v** so they start at the same point, then **u** + **v** lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)

Example 1

Draw the sum of the vectors **a** and **b** shown in Figure 5.

Figure 5

Solution:

First we translate **b** and place its tail at the tip of **a**, being careful to draw a copy of **b** that has the same length and direction.

Example 1 – *Solution*

cont'd

Then we draw the vector **a** + **b** [see Figure 6(a)] starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

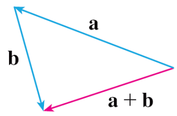


Figure 6(a)

Alternatively, we could place **b** so it starts where **a** starts and construct **a** + **b** by the Parallelogram Law as in Figure 6(b).

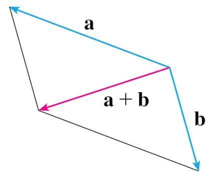


Figure 6(b)

Combining Vectors

It is possible to multiply a vector by a real number *c*. (In this context we call the real number *c* a **scalar** to distinguish it from a vector.)

For instance, we want 2**v** to be the same vector as **v** + **v**, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

> **Definition of Scalar Multiplication** If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Combining Vectors

This definition is illustrated in Figure 7.

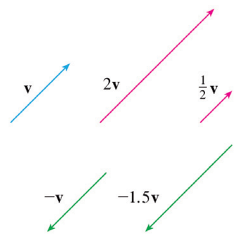


Figure 7

Scalar multiples of **v**

We see that real numbers work like scaling factors here; that's why we call them scalars.

Combining Vectors

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another.

In particular, the vector –**v** = (–1)**v** has the same length as **v** but points in the opposite direction. We call it the **negative** of **v**.

By the **difference u** – **v** of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Combining Vectors

So we can construct **u** – **v** by first drawing the negative of **v**, –**v**, and then adding it to **u** by the Parallelogram Law as in Figure 8(a).

Alternatively, since **v** + (**u** – **v**) = **u**, the vector **u** – **v**, when added to **v**, gives **u**. So we could construct **u** – **v** as in Figure 8(b) by means of the Triangle Law.

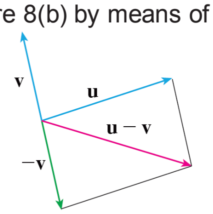


Figure 8(a)

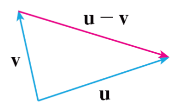


Figure 8(b)

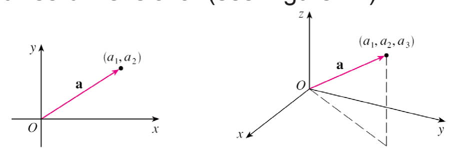
Drawing **u – v**

Components

Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).



$\mathbf{a} = \langle a_1, a_2 \rangle$ $\qquad$ $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Figure 11

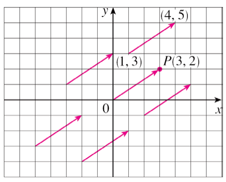
Components

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$.



Representations of the vector $\mathbf{a} = \langle 3, 2 \rangle$

Figure 12

Components

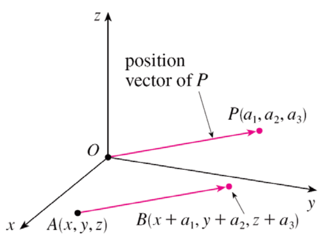
What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward.

We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$.

The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

Components

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.)



Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Figure 13

Components

Let's consider any other representation $\overrightarrow{AB}$ of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$.

Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.

Thus we have the following result.

> **1** Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is
>
> $$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 3

Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Solution:

By $\boxed{1}$, the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle$$

$$= \langle -4, 4, -3 \rangle$$

Components

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive.

In other words, *to add algebraic vectors we add their components*. Similarly, *to subtract vectors we subtract components*.

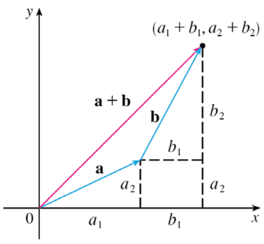


Figure 14

Components

From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2.

So *to multiply a vector by a scalar we multiply each component by that scalar*.

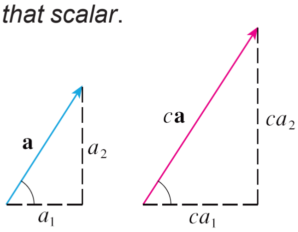


Figure 15

Components

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Components

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors.

More generally, we will consider the set V_n of all n-dimensional vectors.

An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a**.

Components

Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	8. $1\mathbf{a} = \mathbf{a}$

Components

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

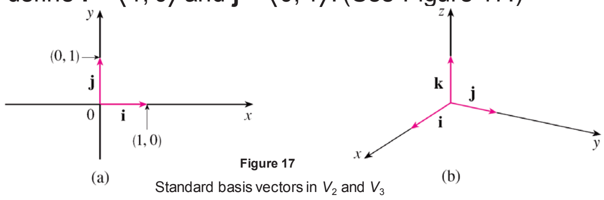


Figure 17
Standard basis vectors in V_2 and V_3

Components

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Thus any vector in V_3 can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For instance,

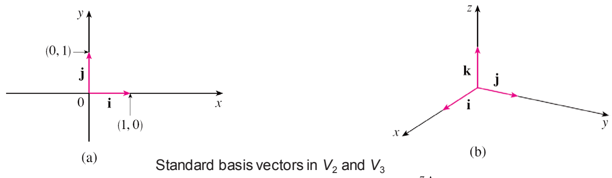
$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

3 $\quad \mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$

Components

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.



Standard basis vectors in V_2 and V_3
Figure 17

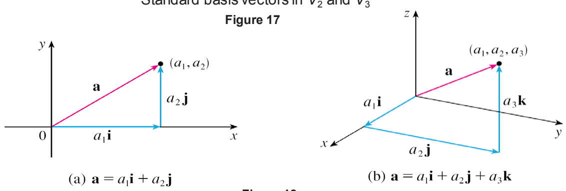


Figure 18

Components

A **unit vector** is a vector whose length is 1. For instance, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

4 $$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Equations of Lines and Planes

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given.

The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to L.

Equations of Lines and Planes

Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$).

If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

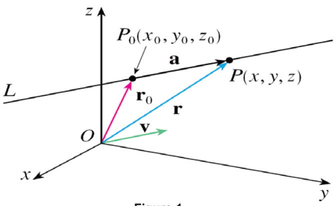


Figure 1

Equations of Lines and Planes

But, since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

$$\boxed{1} \qquad \boxed{\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}}$$

which is a **vector equation** of L.

Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

Equations of Lines and Planes

As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

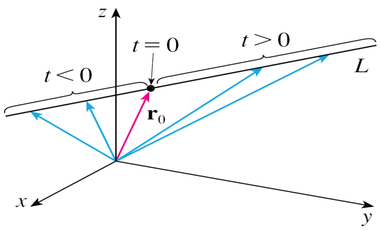


Figure 2

Equations of Lines and Planes

If the vector $\mathbf{v}$ that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c\rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc\rangle$.

We can also write $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, so the vector equation $\boxed{1}$ becomes

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

Two vectors are equal if and only if corresponding components are equal.

Equations of Lines and Planes

Therefore we have the three scalar equations:

$$\boxed{2} \qquad \boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

where $t \in \mathbb{R}$.

These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$.

Each value of the parameter t gives a point (x, y, z) on L.

Example 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

Solution:

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3\rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation $\boxed{1}$ becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or $$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Example 1 – *Solution*

cont'd

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line.

Similarly, $t = -1$ gives the point $(4, -3, 5)$.

Equations of Lines and Planes

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change.

For instance, if, instead of $(5, 1, 3)$, we choose the point $(6, 5, 1)$ in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Equations of Lines and Planes

Or, if we stay with the point $(5, 1, 3)$ but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Equations of Lines and Planes

Another way of describing a line L is to eliminate the parameter t from Equations 2.

If none of a, b, or c is 0, we can solve each of these equations for t, equate the results, and obtain

$$\boxed{3} \qquad \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of L.