Section 8.3 Concervative Fields

When are Vector Fields Gradients?

THEOREM 7: Conservative Fields Let $\mathbf{F}$ be a C^1 vector field defined on $\mathbb{R}^3$ except possibly for a finite number of points. The following conditions on $\mathbf{F}$ are all equivalent:

(i) For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(ii) For any two oriented simple curves C_1 and C_2 that have the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(iii) $\mathbf{F}$ is the gradient of some function f; that is, $\mathbf{F} = \nabla f$ (and if $\mathbf{F}$ has one or more exceptional points where it fails to be defined, f is also undefined there).

(iv) $\nabla \times \mathbf{F} = \mathbf{0}$.

A vector field satisfying one (and, hence, all) of the conditions (i)–(iv) is called a ***conservative vector field***.[6]

EXAMPLE 1 Consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) = y\mathbf{i} + (z \cos yz + x)\mathbf{j} + (y \cos yz)\mathbf{k}.$$

Show that $\mathbf{F}$ is irrotational and find a scalar potential for $\mathbf{F}$.

SOLUTION We compute $\nabla \times \mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x + z\cos yz & y \cos yz \end{vmatrix} \\
&= (\cos yz - yz \sin yz - \cos yz + yz \sin yz)\mathbf{i} + (0 - 0)\mathbf{j} + (1 - 1)\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},
\end{aligned}$$

so $\mathbf{F}$ is irrotational. Thus, a potential exists by Theorem 7. We can find it in several ways.

Method 1. By the technique used to prove that condition (ii) implies condition (iii) in Theorem 7, we can set

$$\begin{aligned}
f(x, y, z) &= \int_0^x F_1(t, 0, 0)\, dt + \int_0^y F_2(x, t, 0)\, dt + \int_0^z F_3(x, y, t)\, dt \\
&= \int_0^x 0\, dt + \int_0^y x\, dt + \int_0^z y \cos yt\, dt \\
&= 0 + xy + \sin yz = xy + \sin yz.
\end{aligned}$$

One easily verifies that $\nabla f = \mathbf{F}$, as required:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = y\mathbf{i} + (x + z\cos yz)\mathbf{j} + (y \cos yz)\mathbf{k}.$$

Method 2. Because we know that f exists, we know that we can solve the system of equations

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x + z\cos yz, \qquad \frac{\partial f}{\partial z} = y\cos yz,$$

for $f(x, y, z)$. These are equivalent to the simultaneous equations

(a) $f(x, y, z) = xy + h_1(y, z)$

(b) $f(x, y, z) = \sin yz + xy + h_2(x, z)$

(c) $f(x, y, z) = \sin yz + h_3(x, y)$

for functions h_1, h_2, h_3 independent of x, y, and z (respectively). When $h_1(y, z) = \sin yz$, $h_2(x, z) = 0$, and $h_3(x, y) = xy$, the three equations agree and so yield a potential for **F**. However, we have only guessed at the values of h_1, h_2, and h_3. To derive the formula for f more systematically, we note that because $f(x, y, z) = xy + h_1(y, z)$ and $\partial f/\partial z = y\cos yz$, we find that

$$\frac{\partial h_1(y, z)}{\partial z} = y\cos yz$$

or

$$h_1(y, z) = \int y\cos yz\, dz + g(y) = \sin yz + g(y).$$

Therefore, substituting this back into equation (a), we get

$$f(x, y, z) = xy + \sin yz + g(y);$$

but by equation (b),

$$g(y) = h_2(x, z).$$

Because the right side of this equation is a function of x and z and the left side is a function of y alone, we conclude that they must equal some constant C. Thus,

$$f(x, y, z) = xy + \sin yz + C$$

and we have determined f up to a constant. ▲

EXAMPLE 2 A mass M at the origin in $\mathbb{R}^3$ exerts a force on a mass m located at $\mathbf{r} = (x, y, z)$ with magnitude GmM/r^2 and directed toward the origin. Here, G is the gravitational constant, which depends on the units of measurement, and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. If we remember that $-\mathbf{r}/r$ is a unit vector directed toward the origin, then we can write the force field as

$$\mathbf{F}(x, y, z) = -\frac{GmM\mathbf{r}}{r^3}.$$

Show that $\mathbf{F}$ is irrotational and find a scalar potential for $\mathbf{F}$. (Notice that $\mathbf{F}$ is not defined at the origin, but Theorem 7 still applies, because it allows an exceptional point.)

SOLUTION First let us verify that $\nabla \times \mathbf{F} = \mathbf{0}$. Referring to formula 10 in the table of vector identities in Section 4.4, we get

$$\nabla \times \mathbf{F} = -GmM\left[\nabla\left(\frac{1}{r^3}\right) \times \mathbf{r} + \frac{1}{r^3}\nabla \times \mathbf{r}\right].$$

But $\nabla(1/r^3) = -3\mathbf{r}/r^5$ (see Exercise 30, Section 4.4), and so the first term vanishes, because $\mathbf{r} \times \mathbf{r} = \mathbf{0}$. The second term vanishes, because

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Hence, $\nabla \times \mathbf{F} = 0$ (for $\mathbf{r} \neq 0$).

If we recall the formula $\nabla(r^n) = nr^{n-2}\mathbf{r}$ (again, see Exercise 30, Section 4.4), then we can read off a scalar potential for $\mathbf{F}$ by inspection. We have $\mathbf{F} = -\nabla V$, where $V(x, y, z) = -GmM/r$ is called the ***gravitational potential energy***.

[We observe in passing that by Theorem 3 of Section 7.2, the work done by $\mathbf{F}$ in moving a particle of mass m from a point P_1 to a point P_2 is given by

$$V(\mathrm{P}_1) - V(\mathrm{P}_2) = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right),$$

where r_1 is the radial distance of P_1 from the origin, with r_2 similarly defined.] ▲

The Planar Case

COROLLARY 1 If $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^2$ of the form $P\mathbf{i} + Q\mathbf{j}$ that satisfies $\partial P/\partial y = \partial Q/\partial x$, then $\mathbf{F} = \nabla f$ for some f on $\mathbb{R}^2$.

EXAMPLE 3 (a) Determine whether the vector field

$$\mathbf{F} = e^{xy}\mathbf{i} + e^{x+y}\mathbf{j}$$

is a gradient field.

(b) Repeat part (a) for

$$\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}.$$

SOLUTION (a) Here $P(x, y) = e^{xy}$ and $Q(x, y) = e^{x+y}$, and so we compute

$$\frac{\partial P}{\partial y} = xe^{xy}, \qquad \frac{\partial Q}{\partial x} = e^{x+y}.$$

These are not equal, and so $\mathbf{F}$ cannot have a potential function.

(b) In this case, we find

$$\frac{\partial P}{\partial y} = -2x \sin y = \frac{\partial Q}{\partial x},$$

and so $\mathbf{F}$ has a potential function f. To compute f we solve the equations

$$\frac{\partial f}{\partial x} = 2x \cos y, \qquad \frac{\partial f}{\partial y} = -x^2 \sin y.$$

Thus, $f(x, y) = x^2 \cos y + h_1(y)$ and $f(x, y) = x^2 \cos y + h_2(x)$. If h_1 and h_2 are the same constant, then both equations are satisfied, and so $f(x, y) = x^2 \cos y$ is a potential for $\mathbf{F}$. ▲

EXAMPLE 4 Let $\mathbf{c}\colon [1, 2] \to \mathbb{R}^2$ be given by $x = e^{t-1}$, $y = \sin(\pi/t)$. Compute the integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} 2x \cos y \, dx - x^2 \sin y \, dy,$$

where $\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}$.

SOLUTION The endpoints are $\mathbf{c}(1) = (1, 0)$ and $\mathbf{c}(2) = (e, 1)$. Because $\partial(2x \cos y)/\partial y = \partial(-x^2 \sin y)/\partial x$, $\mathbf{F}$ is irrotational and hence a gradient vector field (as we saw in Example 3). Thus, by Theorem 7, we can replace $\mathbf{c}$ by any piecewise C^1 curve having the same endpoints, in particular, by the polygonal path from $(1, 0)$ to $(e, 0)$ to $(e, 1)$. Thus, the line integral must be equal to

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_1^e 2t \cos 0 \, dt + \int_0^1 -e^2 \sin t \, dt = (e^2 - 1) + e^2(\cos 1 - 1)$$
$$= e^2 \cos 1 - 1.$$

Alternatively, using Theorem 3 of Section 7.2, we have

$$\int_{\mathbf{c}} 2x \cos y \, dx - x^2 \sin y \, dy = \int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(2)) - f(\mathbf{c}(1)) = e^2 \cos 1 - 1,$$

because $f(x, y) = x^2 \cos y$ is a potential function for $\mathbf{F}$. Evidently, this technique is simpler than computing the integral directly. ▲

THEOREM 8 If $\mathbf{F}$ is a C^1 vector field on all of $\mathbb{R}^3$ with $\operatorname{div} \mathbf{F} = 0$, then there exists a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \operatorname{curl} \mathbf{G}$.