Section 7.5 Integrals of Scalar Functions Over Surfaces

DEFINITION: The Integral of a Scalar Function Over a Surface If $f(x, y, z)$ is a real-valued continuous function defined on a parametrized surface S, we define the ***integral of f over S*** to be

$$\iint_S f(x, y, z)\, dS = \iint_S f\, dS = \iint_D f(\mathbf{\Phi}(u, v))\|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv. \quad (1)$$

Written out, equation (1) becomes

$$\iint_S f\, dS = \iint_D f(x(u, v), y(u, v), z(u, v))\sqrt{\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, z)}{\partial(u, v)}\right]^2}\, du\, dv. \quad (2)$$

EXAMPLE 1 Suppose a helicoid is described as in Example 2, Section 7.4, and let f be given by $f(x, y, z) = \sqrt{x^2 + y^2 + 1}$. Find $\iint_S f\, dS$.

SOLUTION As in Examples 1 and 2 of Section 7.4,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \qquad \frac{\partial(y, z)}{\partial(r, \theta)} = \sin\theta, \qquad \frac{\partial(x, z)}{\partial(r, \theta)} = \cos\theta.$$

Also, $f(r\cos\theta, r\sin\theta, \theta) = \sqrt{r^2 + 1}$. Therefore,

$$\begin{aligned}\iint_S f(x, y, z)\, dS &= \iint_D f(\mathbf{\Phi}(r, \theta))\|\mathbf{T}_r \times \mathbf{T}_\theta\|\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 \sqrt{r^2 + 1}\sqrt{r^2 + 1}\, dr\, d\theta = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8}{3}\pi. \quad \blacktriangle\end{aligned}$$

Surface Integrals Over Graphs

Suppose S is the graph of a C^1 function $z = g(x, y)$. Recall from Section 7.4 that we can parametrize S by

$$x = u, \qquad y = v, \qquad z = g(u, v),$$

and that in this case

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2},$$

so

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy. \quad (4)$$

EXAMPLE 2 Let S be the surface defined by $z = x^2 + y$, where D is the region $0 \le x \le 1$, $-1 \le y \le 1$. Evaluate $\iint_S x\, dS$.

SOLUTION If we let $z = g(x, y) = x^2 + y$, formula (4) gives

$$\begin{aligned}
\iint_S x\, dS &= \iint_D x\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy = \int_{-1}^{1}\int_0^1 x\sqrt{1 + 4x^2 + 1}\, dx\, dy \\
&= \frac{1}{8}\int_{-1}^{1}\left[\int_0^1 (2 + 4x^2)^{1/2}(8x\, dx)\right] dy = \frac{2}{3}\cdot\frac{1}{8}\int_{-1}^{1} [(2 + 4x^2)^{3/2}]|_0^1\, dy \\
&= \frac{1}{12}\int_{-1}^{1}(6^{3/2} - 2^{3/2})\, dy = \frac{1}{6}(6^{3/2} - 2^{3/2}) = \sqrt{6} - \frac{\sqrt{2}}{3} \\
&= \sqrt{2}\left(\sqrt{3} - \frac{1}{3}\right). \quad \blacktriangle
\end{aligned}$$

EXAMPLE 3 Evaluate $\iint_S z^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION For this problem, it is convenient to use spherical coordinates and to represent the sphere parametrically by the equations $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$, $z = \cos\phi$, over the region D in the $\theta\phi$ plane given by the inequalities $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. From equation (1) we get

$$\iint_S z^2\, dS = \iint_D (\cos\phi)^2 \|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\, d\theta\, d\phi.$$

A little computation [use formula (2) of Section 7.4; see Exercise 6] shows that

$$\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \sin\phi.$$

(Note that for $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$). Thus,

$$\begin{aligned}
\iint_S z^2\, dS &= \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi\, d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi} [-\cos^3\phi]_0^{\pi}\, d\theta = \frac{2}{3}\int_0^{2\pi} d\theta = \frac{4\pi}{3}. \quad \blacktriangle
\end{aligned}$$

Integrals Over Graphs

We now develop another formula for surface integrals when the surface can be represented as a graph. To do so, we let S be the graph of $z = g(x, y)$ and consider formula (4). We claim that

$$\iint_S f(x, y, z)\, dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta}\, dx\, dy, \tag{5}$$

where θ is the angle the normal to the surface makes with the unit vector $\mathbf{k}$ at the point $(x, y, g(x, y))$ (see Figure 7.5.2). Describing the surface by the equation $\phi(x, y, z) = z - g(x, y) = 0$, a normal vector $\mathbf{N}$ is $\nabla\phi$; that is,

$$\mathbf{N} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \tag{6}$$

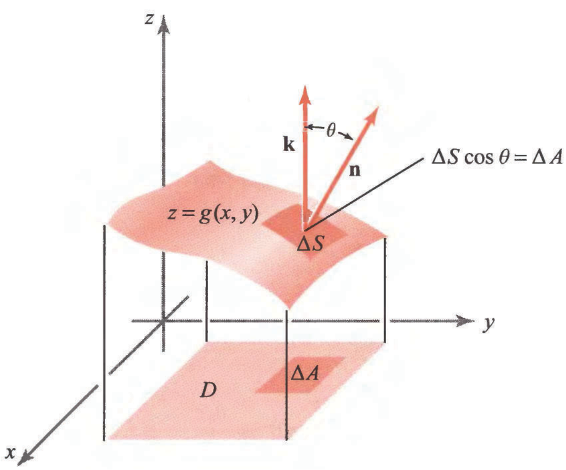

Figure 7.5.2 The area of a patch of area ΔS over a patch ΔA is $\Delta S = \Delta A/\cos\theta$, where θ is the angle the unit normal $\mathbf{n}$ makes with $\mathbf{k}$.

[see Example 4 of Section 7.3, or recall that the normal to a surface with equation $g(x, y, z) =$ constant is given by ∇g]. Thus,

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(\partial g/\partial x)^2 + (\partial g/\partial y)^2 + 1}}.$$

Substitution of this formula into equation (4) gives equation (5). Note that $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, where $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$ is the unit normal. Thus, we can write

$$d\mathbf{S} = \frac{dx\, dy}{\mathbf{n}\cdot\mathbf{k}}$$

The result is, in fact, obvious geometrically, for if a small rectangle in the xy plane has area ΔA, then the area of the portion above it on the surface is $\Delta S = \Delta A/\cos\theta$ (Figure 7.5.2). This intuitive approach can help us to remember formula (5) and to apply it in problems.

EXAMPLE 4 Compute $\iint_S x\, dS$, where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ (see Figure 7.5.3).

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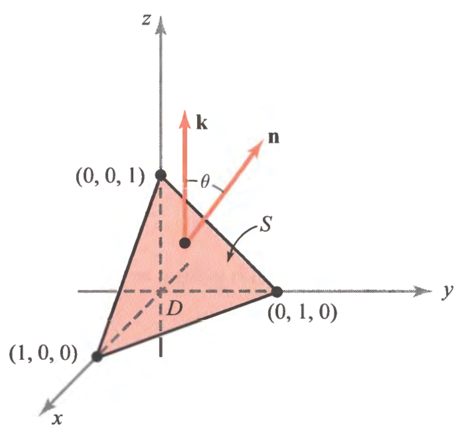

Figure 7.5.3 In computing a specific surface integral, one finds a formula for the unit normal $\mathbf{n}$ and computes the angle θ in preparation for formula (5).

SOLUTION This surface is the plane described by the equation $x + y + z = 1$. Because the surface is a plane, the angle θ is constant and a unit normal vector is $\mathbf{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Thus, $\cos\theta = \mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3}$, and by equation (5),

$$\iint_S x\, dS = \sqrt{3}\iint_D x\, dx\, dy,$$

where D is the domain in the xy plane. But

$$\sqrt{3}\iint_D x\, dx\, dy = \sqrt{3}\int_0^1\int_0^{1-x} x\, dy\, dx = \sqrt{3}\int_0^1 x(1-x)\, dx = \frac{\sqrt{3}}{6}. \quad ▲$$