THEOREM 8 (Section 2.3)

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

$$\begin{tikzcd} & \text{(a)} \arrow[dr, Rightarrow] & \\ \text{(b)} \arrow[ur, Rightarrow] & & \text{(j)} \arrow[d, Rightarrow] \\ \text{(c)} \arrow[u, Rightarrow] & & \text{(d)} \arrow[ll, Rightarrow] \end{tikzcd}$$

$$\begin{tikzcd} & \text{(k)} \arrow[dr, Rightarrow] & \\ \text{(a)} \arrow[ur, Rightarrow] & & \text{(g)} \arrow[ll, Rightarrow] \end{tikzcd}$$

$$\text{(g)} \Leftrightarrow \text{(h)} \Leftrightarrow \text{(i)}$$

$$\text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)}$$

$$\text{(a)} \Leftrightarrow \text{(l)}$$

PROOF If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j). Next, (j) $\Rightarrow$ (d) by Exercise 23 in Section 2.1. (Turn back and read the exercise.) Also, (d) $\Rightarrow$ (c) by Exercise 23 in Section 2.2. If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n. Thus (c) $\Rightarrow$ (b). Also, (b) $\Rightarrow$ (a) by Theorem 7 in Section 2.2. This completes the circle in Fig. 1.

Next, (a) $\Rightarrow$ (k) because A^{-1} works for D. Also, (k) $\Rightarrow$ (g) by Exercise 26 in Section 2.1, and (g) $\Rightarrow$ (a) by Exercise 24 in Section 2.2. So (k) and (g) are linked to the circle. Further, (g), (h), and (i) are equivalent for any matrix, by Theorem 4 in Section 1.4 and Theorem 12(a) in Section 1.9. Thus, (h) and (i) are linked through (g) to the circle.

Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A. (See Section 1.7 and Theorem 12(b) in Section 1.9.) Finally, (a) $\Rightarrow$ (l) by Theorem 6(c) in Section 2.2, and (l) $\Rightarrow$ (a) by the same theorem with A and A^T interchanged. This completes the proof. ■

23. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So every variable is a basic variable and every column of A is a pivot column. (A variation of this argument could be made using linear independence and Exercise 30 in Section 1.7.) Since each pivot is in a different row, A must have at least as many rows as columns.

23. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A has n pivot columns and A is row equivalent to I_n. By Theorem 7, this shows that A must be invertible.

Suppose A is $n\times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then there are no free variables in this equation, and so A has n pivot columns. Since A is *square* and the n pivot positions must be in different rows, the pivots in an echelon form of A must be on the main diagonal. Hence A is row equivalent to the $n\times n$ identity matrix.

THEOREM 7 (Section 2.2)

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

26. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

Take any $\mathbf{b}$ in $\mathbf{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$. By Theorem 4 in Section 1.4, A has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

24. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. [*Hint:* Is A row equivalent to I_n?]

If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

EXAMPLE 1 Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

SOLUTION

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c). ∎

Invertible Linear Transformations

Recall from Section 2.1 that matrix multiplication corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See Fig. 2.

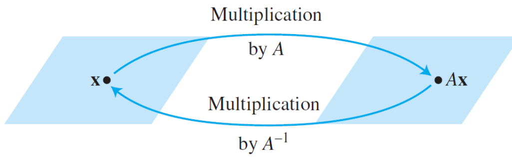

FIGURE 2 A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

The next theorem shows that if such an S exists, it is unique and must be a linear transformation. We call S the **inverse** of T and write it as T^{-1}.

> Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

PROOF Suppose that T is invertible. Then (2) shows that T is onto $\mathbb{R}^n$, for if $\mathbf{b}$ is in $\mathbb{R}^n$ and $\mathbf{x} = S(\mathbf{b})$, then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so each $\mathbf{b}$ is in the range of T. Thus A is invertible, by the Invertible Matrix Theorem, statement (i).

Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S obviously satisfies (1) and (2). For instance,

$$S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$$

Thus T is invertible. The proof that S is unique is outlined in Exercise 38. ■

38. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and let S and U be functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that $U(\mathbf{v}) = S(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$. This will show that T has a unique inverse, as asserted in Theorem 9. [*Hint:* Given any $\mathbf{v}$ in $\mathbb{R}^n$, we can write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$. Why? Compute $S(\mathbf{v})$ and $U(\mathbf{v})$.]

Given any $\mathbf{v}$ in $\mathbf{R}^n$, we may write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$, because T is an onto mapping. Then, the assumed properties of S and U show that $S(\mathbf{v}) = S(T(\mathbf{x})) = \mathbf{x}$ and $U(\mathbf{v}) = U(T(\mathbf{x})) = \mathbf{x}$. So $S(\mathbf{v})$ and $U(\mathbf{v})$ are equal for each $\mathbf{v}$. That is, S and U are the same function from $\mathbf{R}^n$ into $\mathbf{R}^n$.

EXAMPLE 2 What can you say about a one-to-one linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^n$?

SOLUTION The columns of the standard matrix A of T are linearly independent (by Theorem 12 in Section 1.9). So A is invertible, by the Invertible Matrix Theorem, and T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Also, T is invertible, by Theorem 9. ■