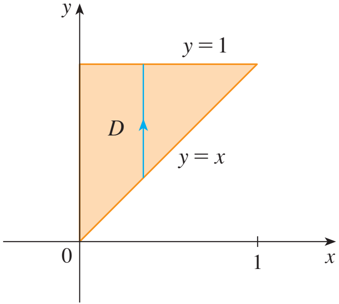

FIGURE 15
D as a type I region

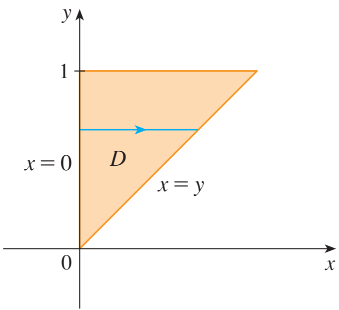

FIGURE 16
D as a type II region

V EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using $\boxed{3}$ backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

where
$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

This enables us to use $\boxed{5}$ to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}
\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx &= \iint_D \sin(y^2)\, dA \\
&= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy \\
&= \int_0^1 y \sin(y^2)\, dy = -\tfrac{1}{2} \cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)
\end{aligned}$$

EXAMPLE 1 By changing the order of integration, evaluate

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2}\, dy\, dx.$$

SOLUTION Note that x varies between 0 and a, and for each such fixed x, we have $0 \le y \le (a^2 - x^2)^{1/2}$. Thus, the iterated integral is equivalent to the double integral

$$\iint_D (a^2 - y^2)^{1/2}\, dy\, dx,$$

where D is the set of points (x, y) such that $0 \le x \le a$ and $0 \le y \le (a^2 - x^2)^{1/2}$. But this is the representation of one quarter (the positive quadrant portion) of the disk of radius a; hence, D can also be described as the set of points (x, y) satisfying

$$0 \le y \le a, \qquad 0 \le x \le (a^2 - y^2)^{1/2}$$

(see Figure 5.4.1). Thus,

$$\begin{aligned}
\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2}\, dy\, dx &= \int_0^a \left[\int_0^{(a^2-y^2)^{1/2}} (a^2 - y^2)^{1/2}\, dx\right] dy \\
&= \int_0^a [x(a^2 - y^2)^{1/2}]_{x=0}^{(a^2-y^2)^{1/2}}\, dy \\
&= \int_0^a (a^2 - y^2)\, dy = \left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{2a^3}{3}. \quad \blacktriangle
\end{aligned}$$

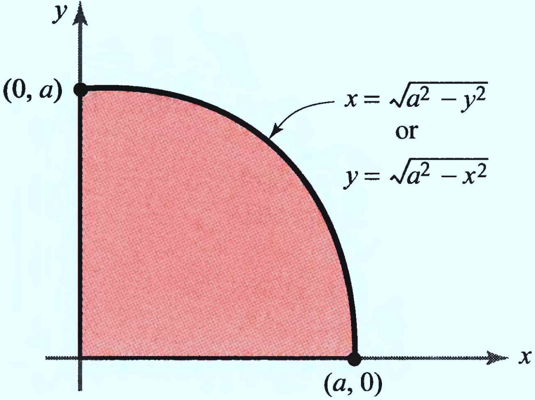

Figure 5.4.1 The positive-quadrant portion of a disk of radius a.

EXAMPLE 2 Evaluate

$$\int_1^2 \int_0^{\log x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx.$$

SOLUTION It will simplify matters if we first interchange the order of integration. First notice that the integral is equal to $\iint_D (x-1)\sqrt{1+e^{2y}}\, dA$, where D is the set of (x, y) such that

$$1 \le x \le 2 \qquad \text{and} \qquad 0 \le y \le \log x.$$

The region D is simple (see Figure 5.4.2) and can also be described by

$$0 \le y \le \log 2 \qquad \text{and} \qquad e^y \le x \le 2.$$

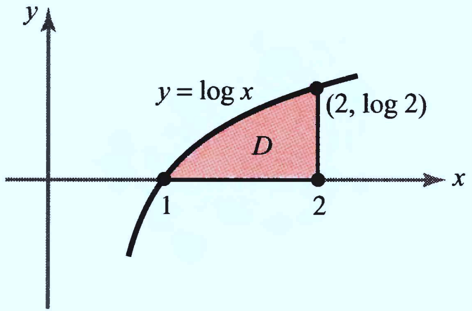

Figure 5.4.2 D is the region of integration for Example 2.

Thus, the given iterated integral is equal to

$$\begin{aligned}
\int_0^{\log 2}\int_{e^y}^{2}(x-1)\sqrt{1+e^{2y}}\,dx\,dy &= \int_0^{\log 2}\sqrt{1+e^{2y}}\left[\int_{e^y}^{2}(x-1)\,dx\right]dy\\
&= \int_0^{\log 2}\sqrt{1+e^{2y}}\left[\frac{x^2}{2}-x\right]_{e^y}^{2}dy\\
&= -\int_0^{\log 2}\left(\frac{e^{2y}}{2}-e^y\right)\sqrt{1+e^{2y}}\,dy\\
&= -\frac{1}{2}\int_0^{\log 2}e^{2y}\sqrt{1+e^{2y}}\,dy+\int_0^{\log 2}e^y\sqrt{1+e^{2y}}\,dy. \qquad (1)
\end{aligned}$$

In the first integral in expression (1), we substitute $u=e^{2y}$, and in the second, $v=e^y$. Hence, we obtain

$$-\frac{1}{4}\int_1^4\sqrt{1+u}\,du+\int_1^2\sqrt{1+v^2}\,dv. \qquad (2)$$

Both integrals in expression (2) are easily found with techniques of one-variable calculus (or by consulting the table of integrals at the back of the book). For the first integral, we get

$$\frac{1}{4}\int_1^4\sqrt{1+u}\,du=\left[\frac{1}{6}(1+u)^{3/2}\right]_1^4=\frac{1}{6}[(1+4)^{3/2}-2^{3/2}]=\frac{1}{6}[5^{3/2}-2^{3/2}]. \quad (3)$$

The second integral is

$$\begin{aligned}
\int_1^2\sqrt{1+v^2}\,dv &= \frac{1}{2}\left[v\sqrt{1+v^2}+\log\left(\sqrt{1+v^2}+v\right)\right]_1^2\\
&= \frac{1}{2}\left[2\sqrt{5}+\log\left(\sqrt{5}+2\right)\right]-\frac{1}{2}\left[\sqrt{2}+\log\left(\sqrt{2}+1\right)\right] \qquad (4)
\end{aligned}$$

(see formula 43 in the table of integrals at the back of the book). Finally, we subtract equation (3) from equation (4) to obtain the answer

$$\frac{1}{2}\left(2\sqrt{5}-\sqrt{2}+\log\frac{\sqrt{5}+2}{\sqrt{2}+1}\right)-\frac{1}{6}[5^{3/2}-2^{3/2}]. \quad \blacktriangle$$