Section 6.2 The Change of Variables Theorem

Given two regions D and D^* in $\mathbb{R}^2$, a differentiable map T on D^* with image D, that is, $T(D^*) = D$, and any real-values integrable function $f : D \to \mathbb{R}$, we would like to express $\iint_D f(x,y)dA$ as an integral over D^* of the composite function $f \circ T$.

Assume that D^* is a region in the uv plane and that D is a region in the xy plane. The map T is given by two coordinate functions:

$$T(u,v) = (x(u,v), y(u,v)) \quad \text{for} \quad (u,v) \in D^*$$

One might conjecture that

$$\iint_D f(x,y)dxdy \stackrel{?}{=} \iint_{D^*} f(x(u,v), y(u,v))dudv \tag{1}$$

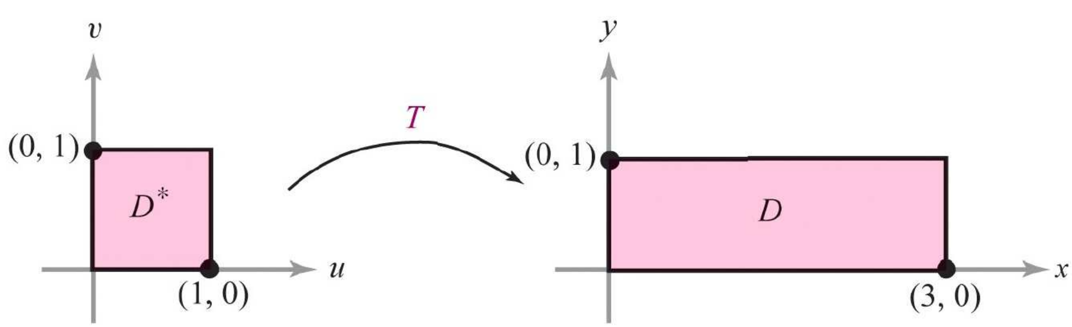
where $f \circ T(u,v) = f(x(u,v), y(u,v))$ is the composite function defined on D^*. However, this formula is wrong. In fact, consider the function $f : D \to \mathbb{R}^2$ where $f(x,y) = 1$, then equation (1) would imply

$$A(D) = \iint_D dxdy \stackrel{?}{=} \iint_{D^*} dudv = A(D^*) \tag{2}$$

But (2) will hold for only a few special cases and not for a general map T. For example, define T by

$$T(u,v) = (-u^2 + 4u, v)$$

Restrict T to the unit square $D^* = [0,1] \times [0,1]$ in the uv plane. Then, as in Exercise 3, Section 6.1, T takes D^* onto $D = [0,3] \times [0,1]$. Clearly, $A(D) \neq A(D^*)$, and so formula (2) is *not valid*.



Jacobian Determinants

DEFINITION: Jacobian Determinant Let $T\colon D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u, v)$ and $y = y(u, v)$. The ***Jacobian determinant*** of T, written $\partial(x, y)/\partial(u, v)$, is the determinant of the derivative matrix $\mathbf{D}T(u, v)$ of T:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

EXAMPLE 1: The function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that transforms polar coordinates into Cartesian coordinates is given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

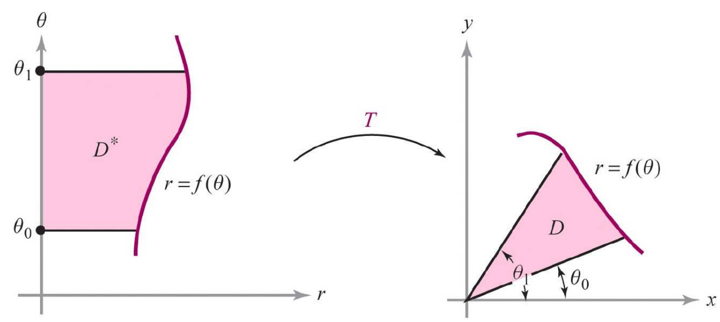
and its Jacobian determinant is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

One can show that under suitable restrictions on the function T the area $A(D)$ of $D = T(D^*)$ can be found by using the formula

$$\boxed{A(D) = \iint_D dxdy = \iint_{D^*} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv} \tag{3}$$

EXAMPLE 2: Let the elementary region D in the xy plane be bounded by the graph of a polar equation $f(\theta)$, where $\theta_0 \le \theta \le \theta_1$ and $f(\theta) \ge 0$.

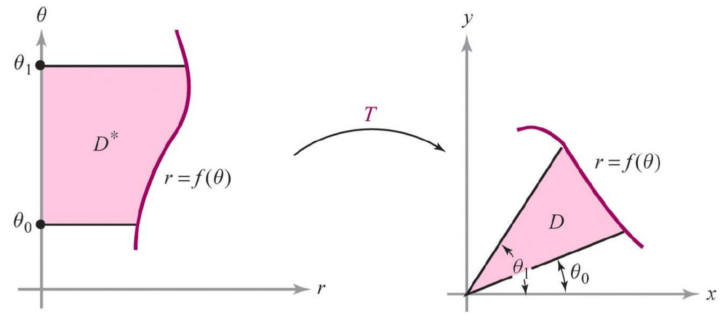


In the $r\theta$ plane we consider the r-simple region D^* where $\theta_0 \le \theta \le \theta_1$ and $0 \le r \le f(\theta)$. Under the transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

the region D^* is carried onto the region D. Use (3) to calculate the area of D.

EXAMPLE 2: Let the elementary region D in the xy plane be bounded by the graph of a polar equation $f(\theta)$, where $\theta_0 \leq \theta \leq \theta_1$ and $f(\theta) \geq 0$.



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$$x = r\cos\theta, \quad y = r\sin\theta$$

the region D^* is carried onto the region D. Use (3) to calculate the area of D.

Solution: From Example 1 it follows that

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

therefore

$$\begin{aligned} A(D) = \iint_D dxdy &= \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \iint_{D^*} rdrd\theta \\ &= \int_{\theta_0}^{\theta_1} \left[\int_0^{f(\theta)} rdr\right] d\theta \\ &= \int_{\theta_0}^{\theta_1} \left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta \\ &= \int_{\theta_0}^{\theta_1} \frac{[f(\theta)]^2}{2} d\theta \end{aligned}$$

Change of Variables Formula

THEOREM 2: Change of Variables: Double Integrals Let D and D^* be elementary regions in the plane and let $T\colon D^* \to D$ be of class C^1; suppose that T is one-to-one on D^*. Furthermore, suppose that $D = T(D^*)$. Then for any integrable function $f\colon D \to \mathbb{R}$, we have

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv. \qquad (6)$$

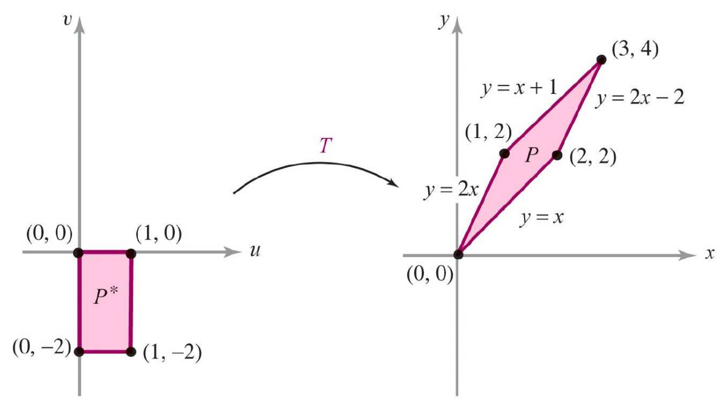
EXAMPLE 3: Let P be the parallelogram bounded by

$$y = 2x, \quad y = 2x - 2, \quad y = x, \quad \text{and} \quad y = x + 1$$

Evaluate $\iint_P xydxdy$ by making the change of variables

$$x = u - v, \quad y = 2u - v$$

that is, $T(u, v) = (u - v, 2u - v)$.



Solution: The transformation T has nonzero determinant and so is one-to-one (see Exercise 8, Section 6.1). It is designed so that it takes the *rectangle* P^* bounded by

$$v = 0, \quad v = -2, \quad u = 0, \quad u = 1$$

onto P. The use of T simplifies the region of integration from P to P^*. Moreover,

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\det \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\right| = 1$$

Therefore, by the change of variables formula we have

$$\iint_P xydxdy = \iint_{P^*} (u - v)(2u - v)dudv = \int_{-2}^{0}\int_0^1 (2u^2 - 2vu + v^2)dudv$$

$$= \int_{-2}^{0} \left[\frac{2}{3}u^3 - \frac{3u^2v}{2} + v^2u\right]_0^1 dv = \int_{-2}^{0} \left[\frac{2}{3} - \frac{3}{2}v + v^2\right] dv$$

$$= \left[\frac{2}{3}v - \frac{3}{4}v^2 + \frac{v^3}{3}\right]_{-2}^{0} = -\left[\frac{2}{3}(-2) - 3 - \frac{8}{3}\right] = -\left[-\frac{12}{3} - 3\right] = 7$$

Integrals in Polar Coordinates

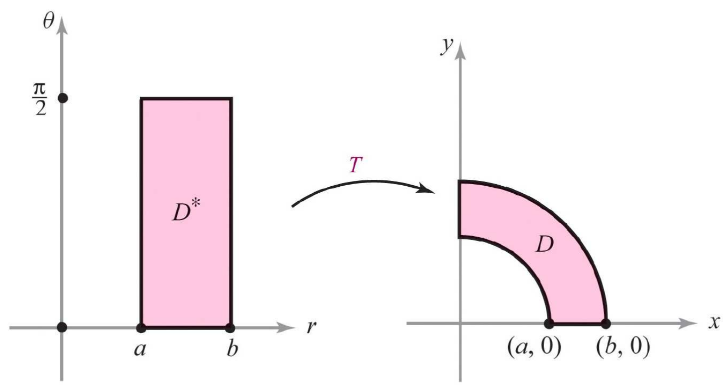
Change of Variables---Polar Coordinates

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{7}$$

EXAMPLE 4: Evaluate $\displaystyle\iint_D \log(x^2+y^2)dxdy$, where D is the region in the first quadrant lying between the arcs of the circles

$$x^2+y^2=a^2 \quad \text{and} \quad x^2+y^2=b^2$$

where $0<a<b$.



Solution: From Example 7, Section 6.1, the polar-coordinate transformation

$$x=r\cos\theta, \quad y=r\sin\theta$$

sends the rectangle D^* given by $a\le r\le b,\ 0\le\theta\le\pi/2$ onto the region D. This transformation is one-to-one on D^* and so, by formula (7), we have

$$\iint_D \log(x^2+y^2)dxdy = \int_a^b\int_0^{\pi/2} r\log r^2 d\theta dr = \frac{\pi}{2}\int_a^b r\log r^2 dr = \frac{\pi}{2}\int_a^b 2r\log r dr$$

Integrating by parts, we get

$$\frac{\pi}{2}\int_a^b 2r\log r dr = \frac{\pi}{2}\left[b^2\log b - a^2\log a - \frac{1}{2}(b^2-a^2)\right]$$

EXAMPLE 1 Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

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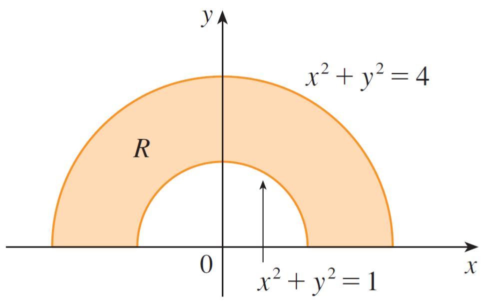

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$

SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by Formula 2,

$$\begin{aligned}
\iint_R (3x + 4y^2)\, dA &= \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta \\
&= \int_0^{\pi} \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta \\
&= \int_0^{\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta = \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta \\
&= \int_0^{\pi} \Big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\Big]\, d\theta \\
&= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}
\end{aligned}$$

V **EXAMPLE 2** Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

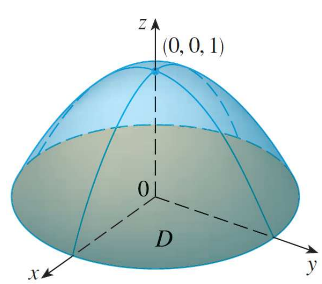

FIGURE 6

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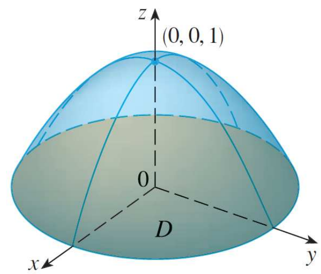

FIGURE 6

SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

which is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$.

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

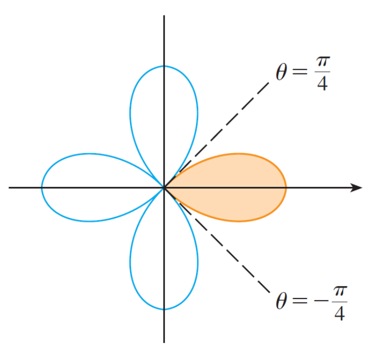

FIGURE 8

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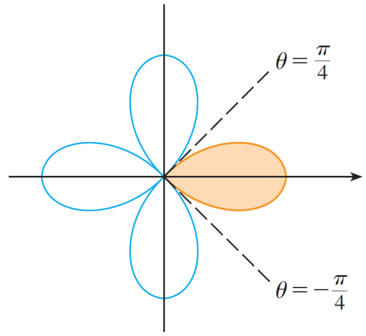

FIGURE 8

SOLUTION From the sketch of the curve in Figure 8, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

So the area is

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \tfrac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \tfrac{1}{4}\left[\theta + \tfrac{1}{4} \sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

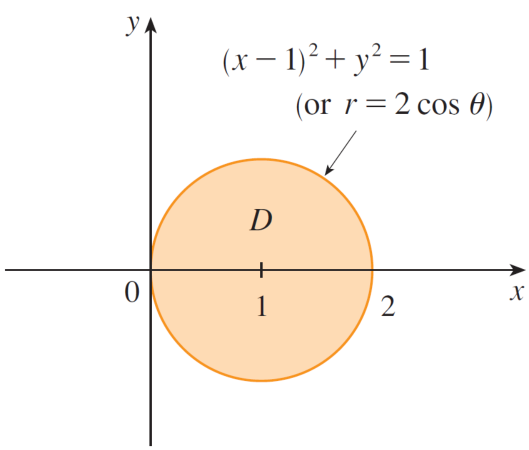

FIGURE 9

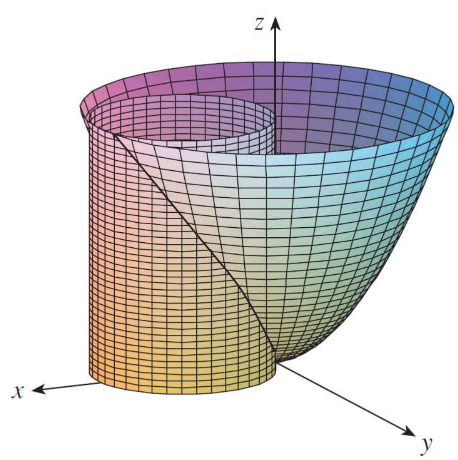

FIGURE 10

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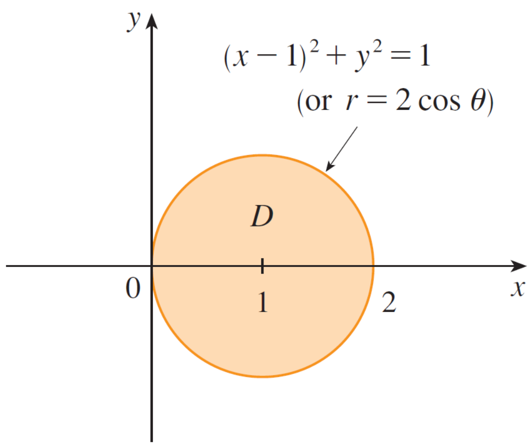

FIGURE 9

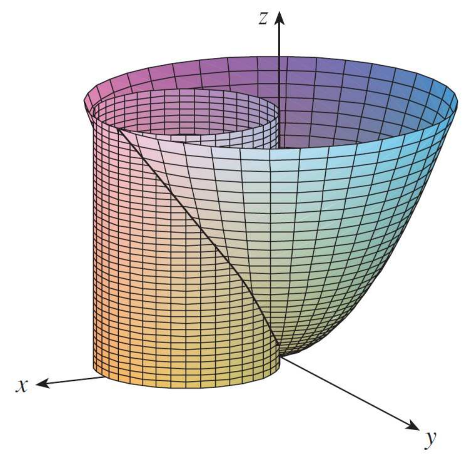

FIGURE 10

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x-1)^2 + y^2 = 1$$

(See Figures 9 and 10.) In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$\begin{aligned} V &= \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta \\ &= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= 2\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \tfrac{1}{2}(1+\cos 4\theta)\right] d\theta \\ &= 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} \end{aligned}$$

EXAMPLE 5 (The Gaussian Integral): Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

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$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Solution: We first evaluate

$$\iint_{D_a} e^{-(x^2+y^2)}dxdy, \quad \text{where} \quad D_a \text{ is the disk } x^2+y^2 \le a^2$$

Because $r^2 = x^2 + y^2$ and $dxdy = rdrd\theta$, the change of variables formula gives

$$\begin{aligned}
\iint_{D_a} e^{-(x^2+y^2)}dxdy &= \int_0^{2\pi}\int_0^a e^{-r^2} r dr d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a d\theta \\
&= -\frac{1}{2}\int_0^{2\pi} (e^{-a^2}-1)d\theta \\
&= \pi(1-e^{-a^2})
\end{aligned}$$

If we let $a \to \infty$ in this expression, we get

$$\iint_{R^2} e^{-(x^2+y^2)}dxdy = \pi$$

Assuming (as shown in the Internet supplement) that we can also evaluate this improper integral as the limit of the integrals over the rectangles $R_a = [-a, a] \times [-a, a]$ as $a \to \infty$, we get

$$\lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)}dxdy = \pi$$

By reduction to iterated integrals, we can write this as

$$\lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}dx \int_{-a}^{a} e^{-y^2}dy\right] = \left[\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}dx\right]^2 = \pi$$

That is,

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 = \pi$$

Thus, taking square roots, we arrive at the desired result.

EXAMPLE 6: Evaluate

$$\int_{-\infty}^{\infty} e^{-2x^2}dx$$

Solution: We will use the change of variables formula $y = \sqrt{2}x$:

$$\int_{-\infty}^{\infty} e^{-2x^2}dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-2x^2}dx = \lim_{a\to\infty}\int_{-\sqrt{2}a}^{\sqrt{2}a} e^{-2y^2}\frac{dy}{\sqrt{2}} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\frac{\pi}{2}}$$

Change of Variables Formula for Triple Integrals

DEFINITION Let $T\colon\ W \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. Then the ***Jacobian*** of T, which is denoted $\partial(x, y, z)/\partial(u, v, w)$, is the determinant

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Change of Variables Formula: Triple Integrals

$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw, \tag{8}$$

where W^* is an elementary region in uvw space corresponding to W in xyz space, under a mapping $T\colon (u, v, w) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w))$, provided T is of class C^1 and is one-to-one, except possibly on a set that is the union of graphs of functions of two variables.

Change of Variables—Cylindrical Coordinates

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz. \tag{9}$$

EXAMPLE 4 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$

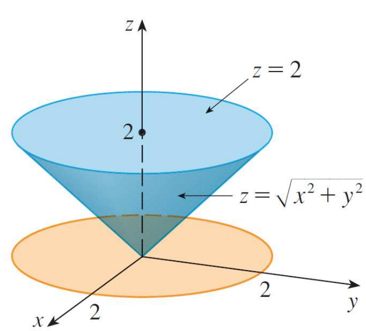

FIGURE 9

EXAMPLE 4 Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.

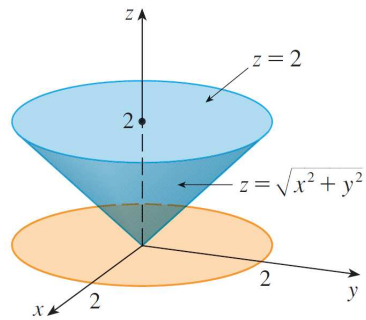

FIGURE 9

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ -\sqrt{4-x^2} \leqslant y \leqslant \sqrt{4-x^2},\ \sqrt{x^2+y^2} \leqslant z \leqslant 2\right\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2+y^2}$ and its upper surface is the plane $z = 2$. (See Figure 9.) This region has a much simpler description in cylindrical coordinates:

$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ r \leqslant z \leqslant 2\right\}$$

Therefore, we have

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx &= \iiint_E (x^2+y^2)\,dV \\
&= \int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)\,dr \\
&= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi
\end{aligned}$$

Change of Variables---Spherical Coordinates

$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_{W^*} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi. \qquad (10)$$

EXAMPLE 6: Evaluate

$$\iiint_W \exp(x^2 + y^2 + z^2)^{3/2} dV$$

where W is the unit ball in $\mathbb{R}^3$.

Solution: If W^* is the region such that

$$0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

we may apply formula (10) and write

$$\iiint_W \exp(x^2 + y^2 + z^2)^{3/2} dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi d\rho d\theta d\phi$$

This integral equals the iterated integral

$$\int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3} \rho^2 \sin\phi d\theta d\phi d\rho = 2\pi \int_0^1 \int_0^\pi e^{\rho^3} \rho^2 \sin\phi d\phi d\rho$$

$$= -2\pi \int_0^1 \rho^2 e^{\rho^3} [\cos\phi]_0^\pi d\rho = 4\pi \int_0^1 e^{\rho^3} \rho^2 d\rho = \frac{4}{3}\pi \int_0^1 e^{\rho^3}(3\rho^2) d\rho$$

$$= \left[\frac{4}{3}\pi e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e - 1)$$

NOTE It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

EXAMPLE: Let W be the ball of radius R and center $(0, 0, 0)$ in $\mathbb{R}^3$. Find the volume of W.

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Solution: The volume of W is $\iiint_W dxdydz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\iiint_W dxdydz = \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2 \sin\phi d\rho d\theta d\phi = \frac{R^3}{3}\int_0^\pi \int_0^{2\pi} \sin\phi d\theta d\phi$$

$$= \frac{2\pi R^3}{3}\int_0^\pi \sin\phi d\phi = \frac{2\pi R^3}{3}(-[\cos(\pi)-\cos(0)]) = \frac{4\pi R^3}{3}$$

which is the standard formula for the volume of a solid sphere.

V EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$. (See Figure 9.)

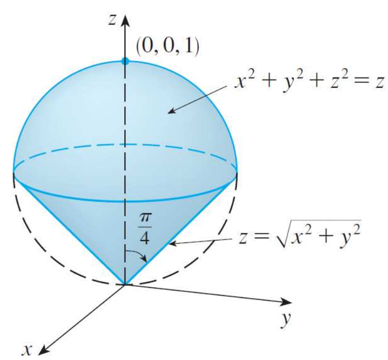


FIGURE 9

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

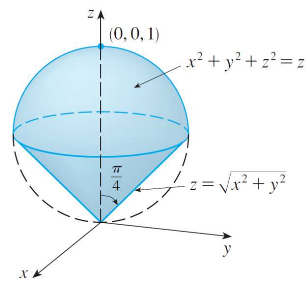


FIGURE 9

SOLUTION Notice that the sphere passes through the origin and has center $\left(0, 0, \frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \cos \phi$$

The equation of the cone can be written as

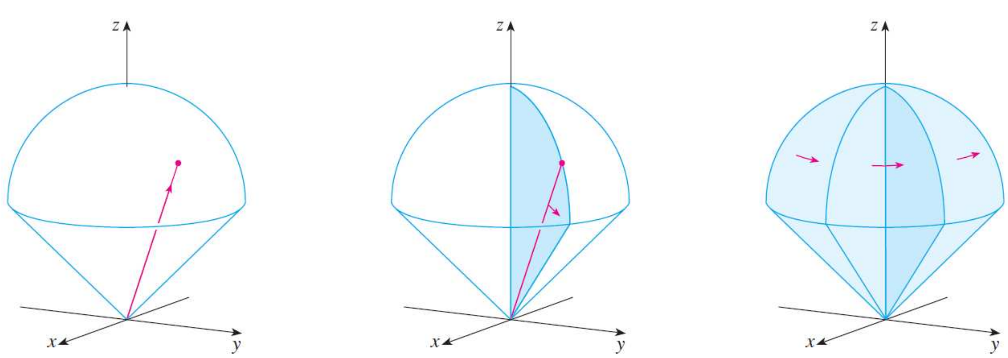
$$\rho \cos \phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} = \rho \sin \phi$$

This gives $\sin \phi = \cos \phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos \phi\}$$

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$\begin{aligned} V(E) &= \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin \phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos \phi} d\phi \\ &= \frac{2\pi}{3} \int_0^{\pi/4} \sin \phi \, \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$



ρ varies from 0 to $\cos \phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Appendix

Under suitable restrictions on the function T, we will argue below that the area of $D = T(D^*)$ is obtained by integrating the absolute value of the Jacobian $\partial(x, y)/\partial(u, v)$ over D^*; that is, we have the equation

$$A(D) = \iint_D dx\, dy = \iint_{D^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \tag{3}$$

To illustrate: From Example 1 in Section 6.1, take $T: D^* \to D$, where $D = T(D^*)$ is the set of (x, y) with $x^2 + y^2 \leq 1$ and $D^* = [0, 1] \times [0, 2\pi]$, and $T(r, \theta) = (r\cos\theta, r\sin\theta)$. By formula (3),

$$A(D) = \iint_{D^*} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta = \iint_{D^*} r\, dr\, d\theta \tag{4}$$

(here r and θ play the role of u and v). From the preceding computation it follows that

$$\iint_{D^*} r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

is the area of the unit disk D, confirming formula (3) in this case. In fact, we may recall from first-year calculus that equation (4) is the correct formula for the area of a region in polar coordinates.

It is not so easy to rigorously prove assertion (3). However, looked at in the proper way, it becomes quite plausible. Recall that $A(D) = \iint_D dx\, dy$ was obtained by dividing up D into little rectangles, summing their areas, and then taking the limit of this sum as the size of the subrectangles tended to zero. The problem is that T may map rectangles into regions whose area is not easy to compute. The solution is to approximate these images by simpler regions whose area we can compute. A useful tool for doing this is the derivative of T, which we know (from Chapter 2) gives the best linear approximation to T.

Consider a small rectangle D^* in the uv plane as shown in Figure 6.2.2. Let T' denote the derivative of T evaluated at (u_0, v_0), so T' is a 2×2 matrix. From our work in Chapter 2, we know that a good approximation to $T(u, v)$ is given by

$$T(u_0, v_0) + T' \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix},$$

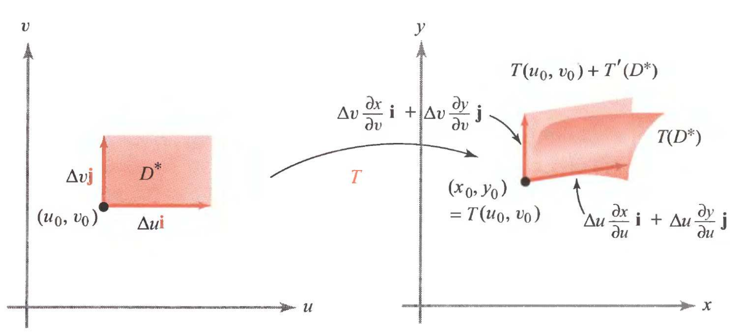


Figure 6.2.2 The effect of the transformation T on a small rectangle D^*.

where $\Delta u = u - u_0$ and $\Delta v = v - v_0$. This mapping T' takes D^* into a parallelogram with vertex at $T(u_0, v_0)$ and with adjacent sides given by the vectors

$$T'(\Delta u\mathbf{i}) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} = \Delta u \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} \end{bmatrix} = \Delta u \mathbf{T}_u$$

and

$$T'(\Delta v\mathbf{j}) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} = \Delta v \begin{bmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \end{bmatrix} = \Delta v \mathbf{T}_v,$$

where

$$\mathbf{T}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} \qquad \text{and} \qquad \mathbf{T}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

are evaluated at (u_0, v_0).

Recall from Section 1.3 that the area of the parallelogram with sides equal to the vectors $a\mathbf{i} + b\mathbf{j}$ and $c\mathbf{i} + d\mathbf{j}$ is equal to the absolute value of the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Thus, the area of $T(D^*)$ is approximately equal to the *absolute value* of

$$\begin{vmatrix} \dfrac{\partial x}{\partial u}\Delta u & \dfrac{\partial x}{\partial v}\Delta v \\ \dfrac{\partial y}{\partial u}\Delta u & \dfrac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \Delta u\, \Delta v = \frac{\partial(x, y)}{\partial(u, v)}\Delta u\, \Delta v$$

evaluated at (u_0, v_0).

This fact and a partitioning argument should make formula (3) plausible. Indeed, if we partition D^* into small rectangles with sides of length Δu and Δv, the images of these rectangles are approximated by parallelograms with sides $\mathbf{T}_u\,\Delta u$ and $\mathbf{T}_v\,\Delta v$, and hence with area $|\partial(x, y)/\partial(u, v)|\,\Delta u\,\Delta v$. Thus, the area of D^* is approximately $\sum \Delta u\,\Delta v$, where the sum is taken over all the rectangles R inside D^* (see Figure 6.2.3). Hence, the area of $T(D^*)$ is approximately the sum

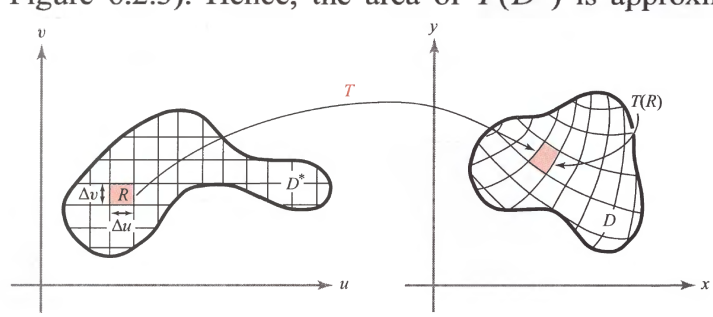


Figure 6.2.3 The area of the little rectangle R is $\Delta u\,\Delta v$. The area of $T(R)$ is approximately $|\partial(x, y)/\partial(u, v)|\Delta u \Delta v$.

$\sum |\partial(x, y)/\partial(u, v)| \Delta u \, \Delta v$. In the limit, this sum becomes

$$\iint_{D^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv.$$

Let us give another informal argument for the special case (4) of formula (3), that is, the case of polar coordinates. Consider a region D in the xy plane and a grid corresponding to a partition of the r and θ variables (Figure 6.2.4). The area of the shaded region shown is approximately $(\Delta r)(r_{jk} \, \Delta\theta)$, because the arc length of a segment of a circle of radius r subtending an angle ϕ is $r\phi$. The total area is then the limit of $\sum r_{jk} \, \Delta r \, \Delta\theta$; that is, $\iint_{D^*} r \, dr \, d\theta$. The key idea is thus that the jkth "polar rectangle" in the grid has area approximately equal to $r_{jk} \, \Delta r \, \Delta\theta$. (For n large, the jkth polar rectangle will look like a rectangle with sides of lengths $r_{jk} \, \Delta\theta$ and Δr). This should provide some insight into why we say *the "area element $dx \, dy$" is transformed into the "area element $r \, dr \, d\theta$."*

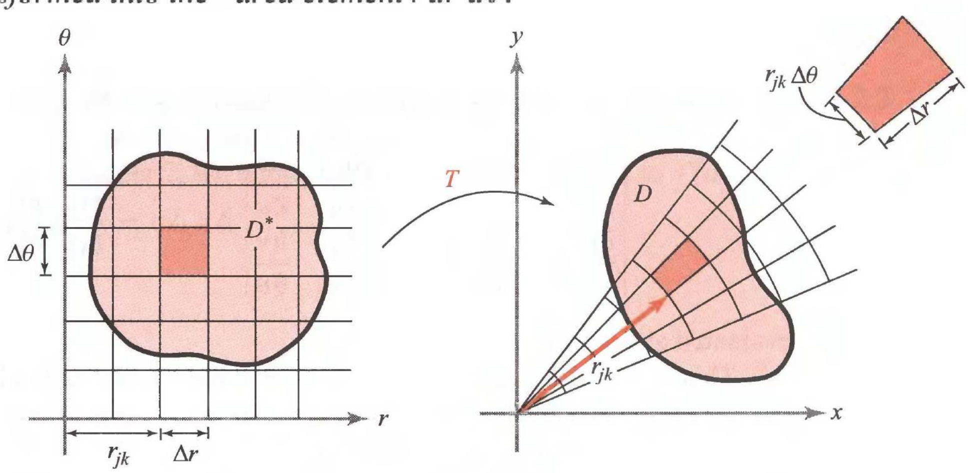


Figure 6.2.4 D^* is mapped to D under the polar-coordinate mapping T.