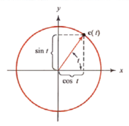
Paths and Curves A ***path*** in $\mathbb{R}^n$ is a map $\mathbf{c} : [a, b] \to \mathbb{R}^n$; it is a ***path in the plane*** if $n = 2$ and a ***path in space*** if $n = 3$. The collection C of points $\mathbf{c}(t)$ as t varies in $[a, b]$ is called a ***curve***, and $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are its ***endpoints***. The path $\mathbf{c}$ is said to ***parametrize*** the curve C. We also say $\mathbf{c}(t)$ ***traces out*** C as t varies.

If $\mathbf{c}$ is a path in $\mathbb{R}^3$, we can write $\mathbf{c}(t) = (x(t), y(t), z(t))$ and we call $x(t)$, $y(t)$, and $z(t)$ the ***component functions*** of $\mathbf{c}$. We form component functions similarly in $\mathbb{R}^2$ or, generally, in $\mathbb{R}^n$.

EXAMPLE 2 The unit circle C: $x^2 + y^2 = 1$ in the plane is the image of the path

$$\mathbf{c} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{c}(t) = (\cos t, \sin t), \qquad 0 \le t \le 2\pi,$$

(see Figure 2.4.3). The unit circle is also the image of the path $\tilde{\mathbf{c}}(t) = (\cos 2t, \sin 2t)$, $0 \le t \le \pi$. Thus, different paths may parametrize the same curve. ▲



Example 4 – *Sketching a helix*

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

Solution:

The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$.

The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane.

Example 4 – *Solution*

cont'd

(The projection of the curve onto the xy-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**.

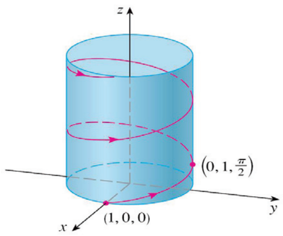


Figure 2

DEFINITION: Velocity Vector If $\mathbf{c}$ is a path and it is differentiable, we say $\mathbf{c}$ is a ***differentiable path***. The ***velocity*** of $\mathbf{c}$ at time t is defined by[3]

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t + h) - \mathbf{c}(t)}{h}.$$

We normally draw the vector $\mathbf{c}'(t)$ with its tail at the point $\mathbf{c}(t)$. The ***speed*** of the path $\mathbf{c}(t)$ is $s = \|\mathbf{c}'(t)\|$, the length of the velocity vector. If $\mathbf{c}(t) = (x(t), y(t))$ in $\mathbb{R}^2$, then

$$\mathbf{c}'(t) = (x'(t), y'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

and if $\mathbf{c}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$, then

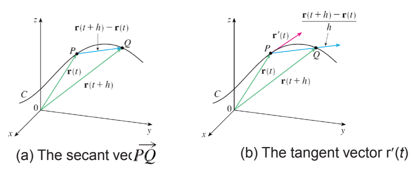
$$\mathbf{c}'(t) = (x'(t), y'(t), z'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

Derivatives

The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real valued functions:

1 $$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\vec{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

Figure 1

Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\ \mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2 \cos 2t\ \mathbf{k}$$

Example 1 – *Solution*

cont'd

(b) Since **r**(0) = **i** and **r**′(0) = **j** + 2**k**, the unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$

$$= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Tangent Vector The velocity $\mathbf{c}'(t)$ is a vector ***tangent*** to the path $\mathbf{c}(t)$ at time t. If C is a curve traced out by $\mathbf{c}$ and if $\mathbf{c}'(t)$ is not equal to $\mathbf{0}$, then $\mathbf{c}'(t)$ is a vector tangent to the curve C at the point $\mathbf{c}(t)$.

Tangent Line to a Path If $\mathbf{c}(t)$ is a path, and if $\mathbf{c}'(t_0) \neq \mathbf{0}$, the equation of its ***tangent line*** at the point $\mathbf{c}(t_0)$ is

$$\mathbf{l}(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0).$$

If C is the curve traced out by $\mathbf{c}$, then the line traced out by $\mathbf{l}$ is the tangent line to the curve C at $\mathbf{c}(t_0)$.

EXAMPLE 8 A path in $\mathbb{R}^3$ goes through the point (3, 6, 5) at $t = 0$ with tangent vector $\mathbf{i} - \mathbf{j}$. Find the equation of the tangent line.

SOLUTION The equation of the tangent line is

$$\mathbf{l}(t) = (3, 6, 5) + t(\mathbf{i} - \mathbf{j}) = (3, 6, 5) + t(1, -1, 0) = (3 + t, 6 - t, 5).$$

In (x, y, z) coordinates, the tangent line is $x = 3 + t$, $y = 6 - t$, $z = 5$. ▲