DEFINITION:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly dependent if there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}.$$

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly independent if the vector equation

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}$$

has only the trivial solution.

DEFINITION:

Let H be a subspace of a vector space V. A set of vectors

$$B = \{\bar{b}_1, \ldots, \bar{b}_p\}$$

in V is a basis for H if

(a) B is a linearly independent set;

(b) $H = \text{Span } \{\bar{b}_1, \ldots, \bar{b}_p\}$.

STANDARD BASIS FOR R^n:

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

STANDARD BASIS FOR P_n:

Vectors

$$\bar{e}_1 = 1,\ \bar{e}_2 = t,\ \bar{e}_3 = t^2, \ldots,\ \bar{e}_{n+1} = t^n$$

form the so-called standard basis for the vector space P_n.

THEOREM:

The set of vectors $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis of R^n if and only if $n = p$ and the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has exactly n pivot positions.

PROBLEM:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Determine if $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since we have 3 vectors and 3 pivots, $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

THEOREM:

The pivot columns of a matrix A form a basis for Col A.

PROBLEM:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Find a basis for Col $[\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3]$.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the first and the second columns are pivot columns, $\{\bar{v}_1, \ \bar{v}_2\}$ is a basis for Col $[\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3]$.

PROBLEM:

It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the bases for Col A and Nul A.

SOLUTION:

(a) By the Theorem above, $\{\bar{v}_1, \ \bar{v}_3, \ \bar{v}_5\}$ is a basis for Col A.

(b) To find the basis for Nul A, we consider a system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0. \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2}$$

so $\{\bar{v}_1, \bar{v}_2\}$ is the basis for Nul A.

<u>THEOREM</u>:

Let V be a p-dimensional vector space, $p \geq 1$. Then

(a) Any linearly independent set of exactly p elements in V is automatically a basis for V.

(b) Any set of exactly p elements that spans V is automatically a basis for V.