Section 7.1 The Path Integral

DEFINITION: Path Integrals The ***path integral***, or the ***integral of $f(x, y, z)$ along the path*** $\mathbf{c}$, is defined when $\mathbf{c}: I = [a, b] \to \mathbb{R}^3$ is of class C^1 and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on I. We define this integral by the equation

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt.$$

Sometimes $\int_{\mathbf{c}} f\, ds$ is denoted

$$\int_{\mathbf{c}} f(x, y, z)\, ds$$

or

$$\int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt.$$

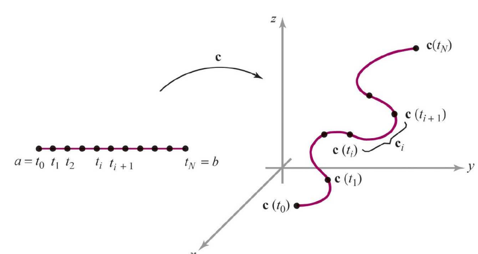
If $\mathbf{c}(t)$ is only piecewise C^1 or $f(\mathbf{c}(t))$ is piecewise continuous, we define $\int_{\mathbf{c}} f\, ds$ by breaking $[a, b]$ into pieces over which $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuous, and summing the integrals over the pieces.

To motivate the definition of the path integral, we shall consider "Riemann-like" sums S_N in the same general way we did to define arc length in Section 4.2. For simplicity, let $\mathbf{c}$ be of class C^1 on I. Subdivide the interval $I = [a, b]$ by means of a partition

$$a = t_0 < t_1 < \cdots < t_N = b.$$

This leads to a decomposition of $\mathbf{c}$ into paths $\mathbf{c}_i$ (Figure 7.1.1) defined on $[t_i, t_{i+1}]$ for $0 \le i \le N - 1$. Denote the arc length of $\mathbf{c}_i$ by Δs_i; thus,

$$\Delta s_i = \int_{t_i}^{t_{i+1}} \|\mathbf{c}'(t)\|\, dt.$$



When N is large, the arc length Δs_i is small and $f(x, y, z)$ is approximately constant for points on $\mathbf{c}_i$. We consider the sums

$$S_N = \sum_{i=0}^{N-1} f(x_i, y_i, z_i)\, \Delta s_i,$$

where $(x_i, y_i, z_i) = \mathbf{c}(t)$ for some $t \in [t_i, t_{i+1}]$. By the mean-value theorem we know that $\Delta s_i = \|\mathbf{c}'(t_i^*)\| \Delta t_i$, where $t_i \le t_i^* \le t_{i+1}$ and $\Delta t_i = t_{i+1} - t_i$. From the theory of Riemann sums, it can be shown that

$$\begin{aligned} \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_i, z_i) \|\mathbf{c}'(t_i^*)\| \Delta t_i &= \int_I f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt \\ &= \int_{\mathbf{c}} f(x, y, z)\, ds. \end{aligned}$$

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EXAMPLE 1 Let $\mathbf{c}$ be the helix $\mathbf{c}\colon [0, 2\pi] \to \mathbb{R}^3$, $t \mapsto (\cos t, \sin t, t)$ (see Figure 2.4.9), and let $f(x, y, z) = x^2 + y^2 + z^2$. Evaluate the integral $\int_{\mathbf{c}} f(x, y, z)\, ds$.

SOLUTION First we compute $\|\mathbf{c}'(t)\|$:

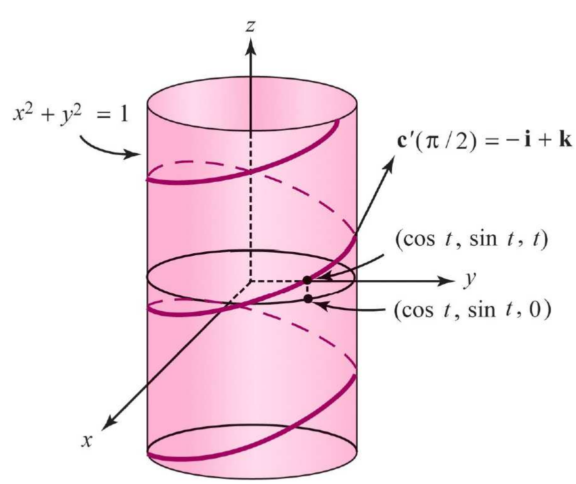
$$\|\mathbf{c}'(t)\| = \sqrt{\left[\frac{d(\cos t)}{dt}\right]^2 + \left[\frac{d(\sin t)}{dt}\right]^2 + \left[\frac{dt}{dt}\right]^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Next, we substitute for x, y, and z in terms of t to obtain

$$f(x, y, z) = x^2 + y^2 + z^2 = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$$

along $\mathbf{c}$. Inserting this information into the definition of the path integral yields

$$\int_{\mathbf{c}} f(x, y, z)\, ds = \int_0^{2\pi} (1 + t^2)\sqrt{2}\, dt = \sqrt{2}\left[t + \frac{t^3}{3}\right]_0^{2\pi} = \frac{2\sqrt{2}\pi}{3}(3 + 4\pi^2). \quad \blacktriangle$$

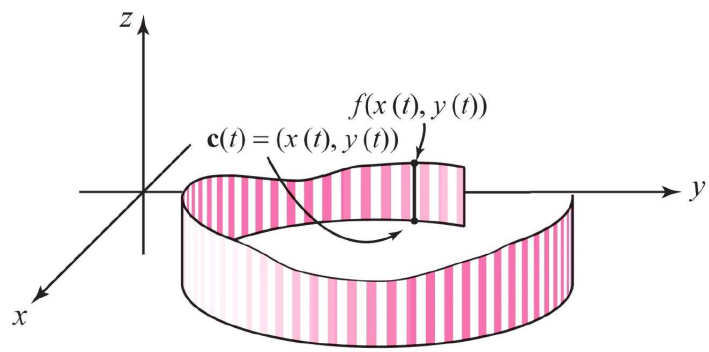


The Path Integral for Planar Curves

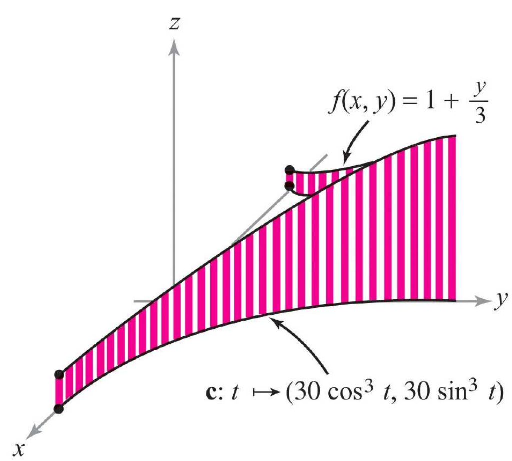
An important special case of the path integral occurs when the path **c** describes a plane curve. Suppose that all points $\mathbf{c}(t)$ lie in the xy plane and f is a real-valued function of two variables. The path integral of f along **c** is

$$\int_{\mathbf{c}} f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

When $f(x, y) \geq 0$, this integral has a geometric interpretation as the "area of a fence." We can construct a "fence" with base the image of **c** and with height $f(x, y)$ at (x, y) (Figure 7.1.2). If **c** moves only once along the image of **c**, the integral $\int_{\mathbf{c}} f(x, y)\, ds$ represents the area of a side of this fence. Readers should try to justify this interpretation for themselves, using an argument like the one used to justify the arc-length formula.



EXAMPLE 2 Tom Sawyer's aunt has asked him to whitewash both sides of the old fence shown in Figure 7.1.3. Tom estimates that for each 25 ft^2 of whitewashing he lets someone do for him, the willing victim will pay 5 cents. How much can Tom hope to earn, assuming his aunt will provide whitewash free of charge?



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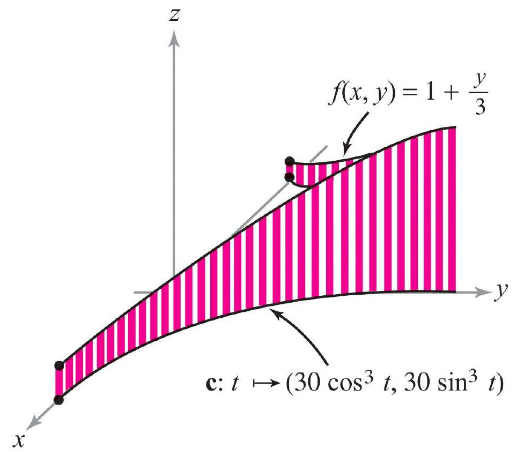

SOLUTION From Figure 7.1.3, the base of the fence in the first quadrant is the path $\mathbf{c}\colon [0, \pi/2] \to \mathbb{R}^2$, $t \mapsto (30\cos^3 t, 30\sin^3 t)$, and the height of the fence at (x, y) is $f(x, y) = 1 + y/3$. The area of one side of the half of the fence is equal to the *integral* $\int_{\mathbf{c}} f(x, y)\, ds = \int_{\mathbf{c}} (1 + y/3)\, ds$. Because $\mathbf{c}'(t) = (-90\cos^2 t \sin t, 90\sin^2 t \cos t)$, we have $\|\mathbf{c}'(t)\| = 90 \sin t \cos t$. Thus, the integral is

$$\begin{aligned}\int_{\mathbf{c}} \left(1 + \frac{y}{3}\right) ds &= \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) 90 \sin t \cos t \, dt \\ &= 90 \int_0^{\pi/2} (\sin t + 10 \sin^4 t) \cos t \, dt \\ &= 90 \left[\frac{\sin^2 t}{2} + 2\sin^5 t\right]_0^{\pi/2} = 90\left(\frac{1}{2} + 2\right) = 225,\end{aligned}$$

which is the area in the first quadrant. Hence, the area of one side of the fence is 450 ft^2. Because both sides are to be whitewashed, we must multiply by 2 to find the total area, which is 900 ft^2. Dividing by 25 and then multiplying by 5, we find that Tom could realize as much as \$1.80 for the job. ▲