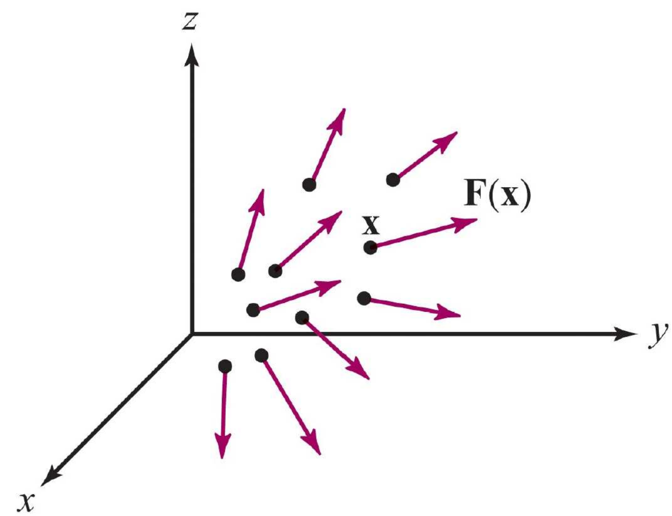
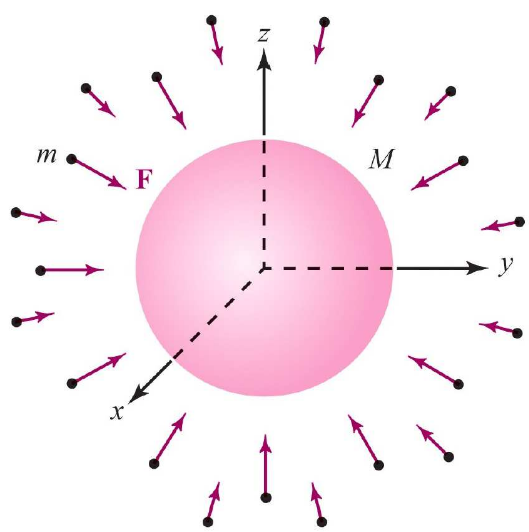
Section 4.3 Vector Fields

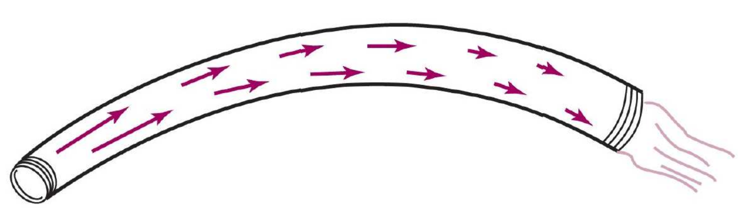
DEFINITION: A *vector field* in $\mathbb{R}^n$ is a map $\mathbf{F} : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\mathbf{x}$ in its domain A a vector $\mathbf{F}(\mathbf{x})$. If $n = 2$, $\mathbf{F}$ is called a *vector field in the plane*, and if $n = 3$, $\mathbf{F}$ is a *vector field in space*.



Graphically, a vector field is illustrated by, from a few points in space $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ drawing the arrow representing the vector $\mathbf{F}(\mathbf{x})$. There are many physical examples of vector fields, e.g. the gravitational field at each point in space by Newton's law tells a mass in which direction to accelerate and how much.

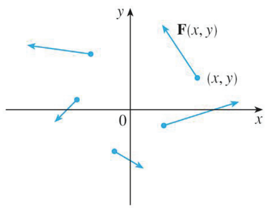


Another example is the velocity vector field of air flow: Sand particles moving in a storm or dust moving in the room, where the vector field at a point in space $\mathbf{F}(\mathbf{x})$ gives the velocity of the particles at the point $\mathbf{x}$. Similarly, the velocity vector field of water flowing through a pipe gives the velocity of the fluid (particle) at each point in space.



Example 1

A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

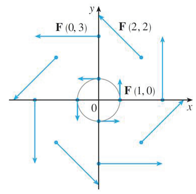


Vector field on $\mathbb{R}^2$

Figure 3

Example 1 – *Solution*

Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1\rangle$ starting at the point (1, 0) in Figure 5.



$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure 5

Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0\rangle$ with starting point (0, 1).

Example 1 – *Solution*

cont'd

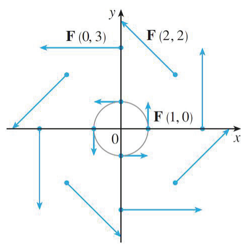
Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 5.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, -2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

Example 1 – *Solution*

cont'd

It appears from Figure 5 that each arrow is tangent to a circle with center the origin.



$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure 5

Example 1 – *Solution*

cont'd

To confirm this, we take the dot product of the position vector $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j})$$

$$= -xy + yx$$

$$= 0$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y\rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$.

Example 1 – *Solution*

cont'd

Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2}$$

$$= \sqrt{x^2 + y^2}$$

$$= |\mathbf{x}|$$

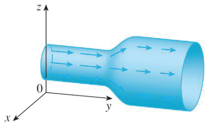
so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle.

Example 3

Imagine a fluid flowing steadily along a pipe and let **V**(x, y, z) be the velocity vector at a point (x, y, z).

Then **V** assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so **V** is a vector field on $\mathbb{R}^3$ called a **velocity field**.

A possible velocity field is illustrated in Figure 13.



Velocity field in fluid flow

Figure 13

Example 3

cont'd

The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics.

For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel.

Example 4

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law.)

Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. (For instance, M could be the mass of the earth and the origin would be at its center.)

Example 4

cont'd

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

Example 4

cont'd

[Physicists often use the notation **r** instead of **x** for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.]

The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force **F**(**x**)] with every point **x** in space.

Example 4

cont'd

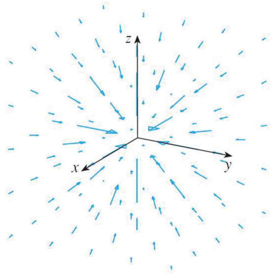
Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathrm{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

Example 4

cont'd

The gravitational field **F** is pictured in Figure 14.



Gravitational force field

Figure 14

Example 5

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

where ε is a constant (that depends on the units used).

For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive.

Example 5

cont'd

Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force **F**, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

Then **E** is a vector field on $\mathbb{R}^3$ called the **electric field** of Q.

Gradient Fields

Gradient Fields

If f is a scalar function of two variables, recall that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**.

Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

Example 6

Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

Solution:

The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$= 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

Example 6 – *Solution*

cont'd

Figure 15 shows a contour map of *f* with the gradient vector field.

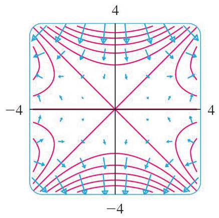


Figure 15

Notice that the gradient vectors are perpendicular to the level curves.

Example 6 – *Solution*

cont'd

Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart.

That's because the length of the gradient vector is the value of the directional derivative of *f* and closely spaced level curves indicate a steep graph.

Gradient Fields

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function *f* such that **F** = ∇*f*.

In this situation *f* is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Gradient Fields

For example, the gravitational field **F** in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

$$= \mathbf{F}(x, y, z)$$

EXAMPLE: Show that the vector field $\mathbf{V}$ on $\mathbb{R}^2$ defined by $\mathbf{V}(x,y) = y\mathbf{i} - x\mathbf{j}$ is not a gradient vector field; that is, there is no C^1 function f such that

$$\mathbf{V}(x,y) = \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Solution: Suppose that such an f exists. Then

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = -x$$

Because these are C^1 functions, f itself must have continuous first- and second-order partial derivatives. But,

$$\frac{\partial^2 f}{\partial x \partial y} = -1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} = 1$$

which violates the equality of mixed partials.

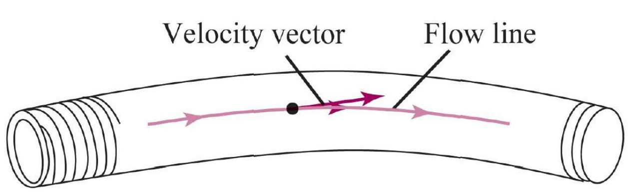
Flow Lines

DEFINITION: If $\mathbf{F}$ is a vector field, a <u>*flow line*</u> for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

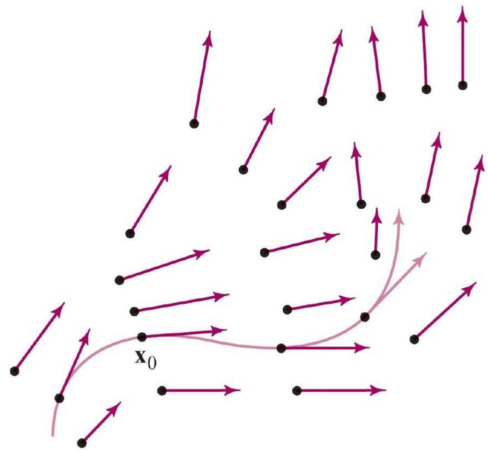
$$\boxed{\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))} \tag{1}$$

That is, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}(t)$.

REMARK: The velocity vector of a fluid is tangent to a flow line:



The flow lines for the velocity vector field of a fluid are just the curves along which the fluid particles travel. From the graphic illustration of a vector field one can approximately draw the flow lines, by going in the direction of the vector field.



The flow lines can not intersect; since the vector field gives the direction there is a unique flow line through each point. In fact, through each initial point there is a unique solution to the system of differential equations (1):

$$\frac{dx}{dt} = F_1, \qquad \frac{dy}{dt} = F_2, \qquad \frac{dz}{dt} = F_3$$

In order to solve this system one can formally eliminate dt and think of one of the other variables as the parameter along the curve:

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3}$$

EXAMPLES:

1. Find the flow lines for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

Solution: We have

$$dx/x = dy/y = dz/0$$

The last equation should be interpreted as $dz = 0$ so $z = C_1$. The second equation $dx/x = dy/y$ gives $\ln x = \ln y - C$ so if we exponentiate, we get

$$y = xe^C = C_2 x$$

The flow lines are therefore half lines going out from the z-axis

2. Find the flow lines for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

Solution: We have

$$-dx/y = dy/x = dz/0$$

Hence $dz = 0$ so $z = C_1$. Also $-dx/y = dy/x$, so $-xdx = ydy$ and $-x^2/2 = y^2/2 - C$, therefore $x^2 + y^2 = C_2$. The flow lines are circles around the z-axis. One can check that the parametrization for the circle $x = r\cos t,\ y = r\sin t,\ z = C$ satisfies the equations of the flow line $dx/dt = -y,\ dy/dt = x,\ dz/dt = 0$.