

The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector \mathbf{c} that is perpendicular to both \mathbf{a} and \mathbf{b} .

If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$\boxed{1} \quad a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$\boxed{2} \quad b_1c_1 + b_2c_2 + b_3c_3 = 0$$

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The Cross Product

To eliminate c_3 we multiply $\boxed{1}$ by b_3 and $\boxed{2}$ by a_3 and subtract:

$$\boxed{3} \quad (a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of $\boxed{3}$ is

$$c_1 = a_2b_3 - a_3b_2 \quad c_2 = a_3b_1 - a_1b_3$$

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The Cross Product

Substituting these values into $\boxed{1}$ and $\boxed{2}$, we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both \mathbf{a} and \mathbf{b} is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the *cross product* of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{a} \times \mathbf{b}$.

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The Cross Product

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} , unlike the dot product, is a vector. For this reason it is also called the **vector product**.

Note that $\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are *three-dimensional* vectors.

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The Cross Product

In order to make Definition 4 easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

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The Cross Product

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\boxed{5} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears.

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The Cross Product

Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0)$$

$$= -38$$

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The Cross Product

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , we see that the cross product of the vectors

$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ is

$$\boxed{6} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

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The Cross Product

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6.

The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

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Example 1

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

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The Cross Product

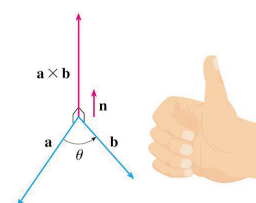
We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both \mathbf{a} and \mathbf{b} . This is one of the most important properties of a cross product.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

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The Cross Product

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through \mathbf{a} and \mathbf{b} .



The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

Figure 1

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The Cross Product

It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*: If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from \mathbf{a} to \mathbf{b} , then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

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The Cross Product

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both \mathbf{a} and \mathbf{b} , whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}| |\mathbf{b}| \sin \theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

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The Cross Product

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2.

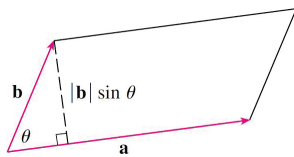


Figure 2

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The Cross Product

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}| \sin \theta$, and area

$$A = |\mathbf{a}| (|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .

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Example 4

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

In Example 3 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\begin{aligned} |\overrightarrow{PQ} \times \overrightarrow{PR}| &= \sqrt{(-40)^2 + (-15)^2 + 15^2} \\ &= 5\sqrt{82} \end{aligned}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

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The Cross Product

If we apply Theorems 8 and 9 to the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

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The Cross Product

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products.

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The Cross Product

The following theorem summarizes the properties of vector products.

11 Theorem If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

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The Cross Product

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\mathbf{12} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

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Triple Products

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Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\mathbf{13} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

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Triple Products

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . (See Figure 3.)

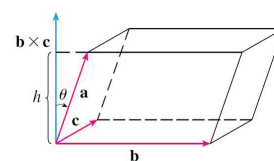


Figure 3

The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$.

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Triple Products

If θ is the angle between \mathbf{a} and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = \|\mathbf{a}\| \cos \theta$. (We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = \|\mathbf{b} \times \mathbf{c}\| \|\mathbf{a}\| \cos \theta = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

[14] The volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

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Triple Products

If we use the formula in [14] and discover that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

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Example 5

Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

Solution:

We use Equation 13 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

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Example 5 – Solution

cont'd

$$\begin{aligned} &= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= 1(18) - 4(36) - 7(-18) \\ &= 0 \end{aligned}$$

Therefore, by [14], the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0. This means that \mathbf{a} , \mathbf{b} , and \mathbf{c} are coplanar.

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Triple Products

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of \mathbf{a} , \mathbf{b} , and \mathbf{c} .

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Planes

Let $P(x, y, z)$ be an arbitrary point in the plane, and let \mathbf{r}_0 and \mathbf{r} be the position vectors of P_0 and P .

Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (Figure 6.)

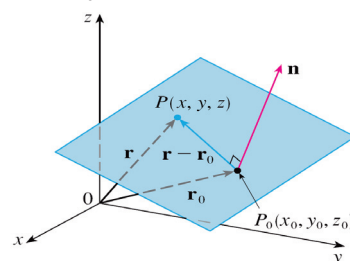


Figure 6

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Planes

The normal vector \mathbf{n} is orthogonal to every vector in the given plane. In particular, \mathbf{n} is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

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Planes

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Then the vector equation [5] becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equation 7 is the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} = \langle a, b, c \rangle$.

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Example 4

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Solution:

Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

$$\text{or} \quad 2x + 3y + 4z = 12$$

To find the x -intercept we set $y = z = 0$ in this equation and obtain $x = 6$.

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Example 4 – Solution

cont'd

Similarly, the y -intercept is 4 and the z -intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

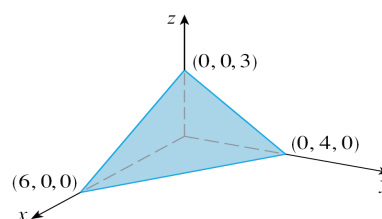


Figure 7

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Planes

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Equation 8 is called a **linear equation** in x , y , and z . Conversely, it can be shown that if a , b , and c are not all 0, then the linear equation [8] represents a plane with normal vector $\langle a, b, c \rangle$.

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