Section 1.1 Systems of Linear Equations

DEFINITION: A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + \ldots + a_nx_n = b,$$

where $a_1, \ldots, a_n$ and b are constants, $x_1, \ldots, x_n$ are variables.

EXAMPLE: The equation

$$2x_1 + x_2 - 7x_3 = \sqrt{5}$$

is linear. The equation

$$3x_1x_2 + 2x_3^2 = 1$$

is NOT linear.

DEFINITION: A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations.

EXAMPLE:

$$\begin{cases} 3x_1 + 2x_2 + 7x_3 - x_4 = 6 \\ x_1 + x_2 - x_3 + x_4 = 1 \\ 4x_1 + 3x_2 + 6x_3 = 8 \end{cases}$$

Solving a Linear System in Two Variables

EXAMPLE: Solve the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

Solution: We have

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_2 = 2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

The Method of Substitution

To use the **method of substitution** to solve a system of two equations in x and y, perform the following steps.

1. Solve one of the equations for one variable in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. Solve the equation obtained in Step 2.
4. Back-substitute the value(s) obtained in Step 3 into the expression obtained in Step 1 to find the value(s) of the other variable.
5. Check that each solution satisfies *both* of the original equations.

EXAMPLE: Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

Solution: Begin by solving for in Equation 1.

$$y = 4 - x \qquad \text{Solve for } y \text{ in Equation 1.}$$

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for by *back-substituting* $x = 3$ into the equation $y = 4 - x$, to obtain

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1. && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair $(3, 1)$. You can check this solution as follows.

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1.} \end{aligned}$$

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2.} \end{aligned}$$

The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y, perform the following steps.

1. Obtain coefficients for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. Add the equations to eliminate one variable; solve the resulting equation.
3. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. Check your solution in both of the original equations.

EXAMPLE: Solve the system of equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

Solution: You can eliminate the y-terms by adding the two equations.

$$\begin{array}{rll} 3x + 2y &= 4 & \text{Write Equation 1.} \\ \underline{5x - 2y} &\underline{= 8} & \text{Write Equation 2.} \\ 8x \qquad &= 12 & \text{Add equations.} \end{array}$$

So, $x = \dfrac{12}{8} = \dfrac{3}{2}$. By back-substituting into Equation 1, you can solve for y.

$$\begin{array}{rll} 3x + 2y &= 4 & \text{Write Equation 1.} \\ 3\left(\tfrac{3}{2}\right) + 2y &= 4 & \text{Substitute } \tfrac{3}{2} \text{ for } x. \\ y &= -\tfrac{1}{4} & \text{Solve for } y. \end{array}$$

The solution is $\left(\dfrac{3}{2}, -\dfrac{1}{4}\right)$. You can check the solution algebraically by substituting into the original system.

Check

$$\begin{array}{rll} 3\left(\tfrac{3}{2}\right) + 2\left(-\tfrac{1}{4}\right) &\stackrel{?}{=} 4 & \text{Substitute into Equation 1.} \\ \tfrac{9}{2} - \tfrac{1}{2} &= 4 & \text{Equation 1 checks. } \checkmark \\ 5\left(\tfrac{3}{2}\right) - 2\left(-\tfrac{1}{4}\right) &\stackrel{?}{=} 8 & \text{Substitute into Equation 2.} \\ \tfrac{15}{2} + \tfrac{1}{2} &= 8 & \text{Equation 2 checks. } \checkmark \end{array}$$

EXAMPLE: Find all solutions of the system

$$\begin{cases} 2x + y = 1 \\ 3x + 4y = 14 \end{cases}$$

Solution 1(Substitution Method): We solve for y in the first equation.

$$2x + y = 1 \quad \Longleftrightarrow \quad y = 1 - 2x$$

Now we substitute for y in the second equation and solve for x:

$$\begin{aligned} 3x + 4y &= 14 \\ 3x + 4(1 - 2x) &= 14 \\ 3x + 4 - 8x &= 14 \\ -5x &= 10 \\ x &= -2 \end{aligned}$$

Finally, we back-substitute $x = -2$ into the equation $y = 1 - 2x$:

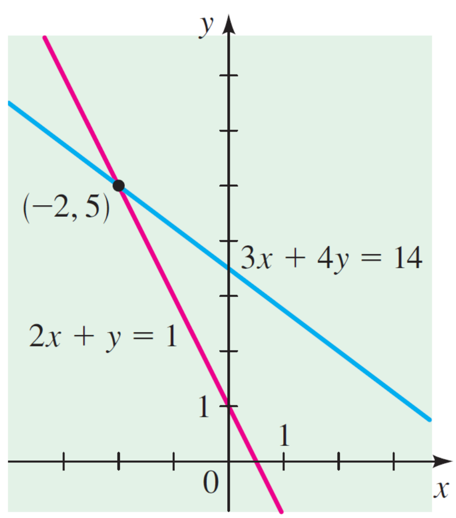
$$y = 1 - 2(-2) = 5$$

Solution 2(Elimination Method): We have

$$\begin{cases} 2x + y = 1 \\ 3x + 4y = 14 \end{cases} \Longleftrightarrow \begin{cases} 8x + 4y = 4 \\ 3x + 4y = 14 \end{cases} \Longleftrightarrow \begin{cases} 5x = -10 \\ 3x + 4y = 14 \end{cases} \Longleftrightarrow \begin{cases} x = -2 \\ 3x + 4y = 14 \end{cases}$$

Next we substitute $x = -2$ into the equation $3x + 4y = 14$:

$$3(-2) + 4y = 14 \quad \Longleftrightarrow \quad -6 + 4y = 14 \quad \Longleftrightarrow \quad 4y = 20 \quad \Longleftrightarrow \quad y = 5$$



EXAMPLE: Find all solutions of the system

$$\begin{cases} 2x + 3y = -4 \\ \\ 5x - 7y = 1 \end{cases}$$

Solution 1(Substitution Method): We solve for x in the first equation.

$$2x + 3y = -4 \quad \iff \quad 2x = -4 - 3y \quad \iff \quad x = -\frac{4 + 3y}{2}$$

Now we substitute for x in the second equation and solve for y:

$$\begin{aligned} 5x - 7y &= 1 \\ -5\frac{4 + 3y}{2} - 7y &= 1 \\ -5(4 + 3y) - 14y &= 2 \\ -20 - 15y - 14y &= 2 \\ -29y &= 22 \\ y &= -\frac{22}{29} \end{aligned}$$

Finally, we back-substitute $y = -\dfrac{22}{29}$ into the equation $x = -\dfrac{4 + 3y}{2}$:

$$x = -\frac{4 + 3\left(-\frac{22}{29}\right)}{2} = -\frac{25}{29}$$

Solution 2(Elimination Method): On the one hand, we have

$$\begin{cases} 2x + 3y = -4 \\ \\ 5x - 7y = 1 \end{cases} \iff \begin{cases} 10x + 15y = -20 \\ \\ 10x - 14y = 2 \end{cases} \implies 29y = -22 \iff y = -\frac{22}{29}$$

On the other hand, we have

$$\begin{cases} 2x + 3y = -4 \\ \\ 5x - 7y = 1 \end{cases} \iff \begin{cases} 14x + 21y = -28 \\ \\ 15x - 21y = 3 \end{cases} \implies 29x = -25 \iff x = -\frac{25}{29}$$

EXAMPLE: Find all solutions of the system

$$\begin{cases} -\dfrac{3}{5}x + \dfrac{1}{2}y = \dfrac{7}{2} \\ \dfrac{1}{3}x + \dfrac{4}{5}y = \dfrac{3}{2} \end{cases}$$

EXAMPLE: Find all solutions of the system

$$\begin{cases} -\dfrac{3}{5}x + \dfrac{1}{2}y = \dfrac{7}{2} \\[2ex] \dfrac{1}{3}x + \dfrac{4}{5}y = \dfrac{3}{2} \end{cases}$$

Solution (Elimination Method): We have

$$\begin{cases} -\dfrac{3}{5}x + \dfrac{1}{2}y = \dfrac{7}{2} \\[2ex] \dfrac{1}{3}x + \dfrac{4}{5}y = \dfrac{3}{2} \end{cases} \iff \begin{cases} -6x + 5y = 35 \\[2ex] 10x + 24y = 45 \end{cases}$$

On the one hand, we have

$$\begin{cases} -6x + 5y = 35 \\[2ex] 10x + 24y = 45 \end{cases} \iff \begin{cases} -30x + 25y = 175 \\[2ex] 30x + 72y = 135 \end{cases} \implies 97y = 310 \iff y = \frac{310}{97}$$

On the other hand, we have

$$\begin{cases} -6x + 5y = 35 \\[2ex] 10x + 24y = 45 \end{cases} \iff \begin{cases} -144x + 120y = 840 \\[2ex] 50x + 120y = 225 \end{cases} \implies -194x = 615 \iff x = -\frac{615}{194}$$

EXAMPLE: Solve the system of linear equations

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

Solution: Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100 to produce a system with integer coefficients.

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients that differ only by sign, multiply revised Equation 1 by 3 and multiply revised equation 2 by -2.

$$\begin{array}{rcrl} 2x - 5y = -38 & \Rightarrow & 6x - 15y = -114 & \text{Multiply revised Equation 1 by 3.} \\ \underline{3x + 4y = 104} & \Rightarrow & \underline{-6x - 8y = -208} & \text{Multiply revised Equation 2 by } -2. \\ & & -23y = -322 & \text{Add equations.} \end{array}$$

So, you can conclude that $y = \dfrac{-322}{-23} = 14$. Back-substitution this value into revised Equation 2 produces the following.

$$\begin{array}{rll} 3x + 4y = 104 & & \text{Write revised Equation 2.} \\ 3x + 4(14) = 104 & & \text{Substitute 14 for } y. \\ 3x = 48 & & \text{Combine like terms.} \\ x = 16 & & \text{Solve for } x. \end{array}$$

EXAMPLE: Find all solutions of the system

$$\begin{cases} x+y=1 \\ x+y=2 \end{cases}$$

Answer: No solution.

EXAMPLE: Find all solutions of the system

$$\begin{cases} 3x-2y=4 \\ -6x+4y=7 \end{cases}$$

Solution: We have

$$\begin{cases} 3x-2y=4 \\ -6x+4y=7 \end{cases} \iff \begin{cases} 3x-2y=4 \\ 3x-2y=-\dfrac{7}{2} \end{cases}$$

It follows that the system has no solution (**inconsistent**).

EXAMPLE: Find all solutions of the system

$$\begin{cases} x+y=1 \\ x+y=1 \end{cases}$$

Answer: The system has infinitely many solutions (**dependent**).

EXAMPLE: Find all solutions of the system

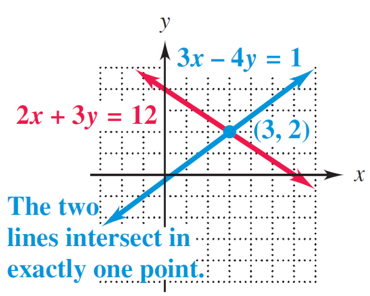
$$\begin{cases} 8x-2y=-4 \\ -4x+y=2 \end{cases}$$

Solution: We have

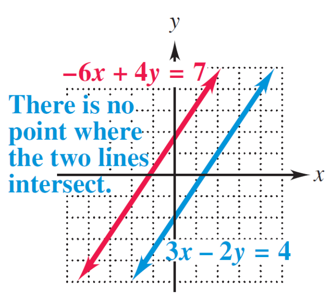
$$\begin{cases} 8x-2y=-4 \\ -4x+y=2 \end{cases} \iff \begin{cases} 8x-2y=-4 \\ 8x-2y=-4 \end{cases}$$

It follows that the system has infinitely many solutions (**dependent**).

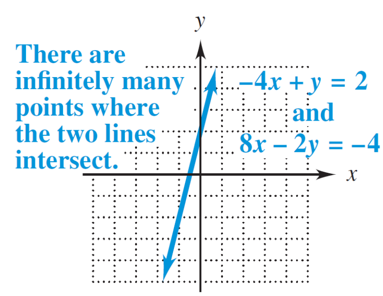
Lines intersect at one point: exactly one solution.



Lines are parallel: no solutions.



Lines coincide: infinitely many solutions.



Larger Systems of Linear Equations

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 4z = 7 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution: From Equation 3, you know the value z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$y + 4(2) = 7 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

The solution is $x = 1$, $y = -1$ and $z = 2$.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y + z = -2 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

Solution: Because the leading coefficient of the first equation is 1, you can begin by saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that all but the first x have been eliminated from the first column, go to work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{3}$ produces a new third equation.

This is the same system that was solved in Example 1. As in that example, you can conclude that the solution is $x = 1$, $y = -1$, and $z = 2$.

EXAMPLE: Find all solutions of the system

$$\begin{cases} 2x - y + 3z = 1 \\ x - 2y + z = 1 \\ 2x - 3y - z = 2 \end{cases}$$

Solution (Elimination Method): We have

$$\begin{cases} 2x - y + 3z = 1 \\ x - 2y + z = 1 \\ 2x - 3y - z = 2 \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ 2x - 4y + 2z = 2 \\ 2x - 3y - z = 2 \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ -3y - z = 1 \\ -2y - 4z = 1 \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ -12y - 4z = 4 \\ -2y - 4z = 1 \end{cases}$$

therefore

$$\begin{cases} 2x - y + 3z = 1 \\ -12y - 4z = 4 \\ 10y = -3 \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ -12y - 4z = 4 \\ y = -\dfrac{3}{10} \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ -12\left(-\dfrac{3}{10}\right) - 4z = 4 \\ y = -\dfrac{3}{10} \end{cases} \iff \begin{cases} 2x - y + 3z = 1 \\ z = -\dfrac{1}{10} \\ y = -\dfrac{3}{10} \end{cases}$$

so

$$\begin{cases} 2x - \left(-\dfrac{3}{10}\right) + 3\left(-\dfrac{1}{10}\right) = 1 \\ z = -\dfrac{1}{10} \\ y = -\dfrac{3}{10} \end{cases} \iff \begin{cases} x = \dfrac{1}{2} \\ z = -\dfrac{1}{10} \\ y = -\dfrac{3}{10} \end{cases}$$

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

Solution:

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding -1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is false statement, you can conclude that this system is inconsistent and so has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

Solution:

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

Adding -3 times the second equation to the third equation produces a new third equation.

This result means that Equation 3 depends on Equations 1 and 2 in the sense that it gives us no additional information about the variables. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}$$

In the last equation, solve for y in terms of z to obtain $y = z$. Back-substituting $y = z$ in the first equation produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions to the given system are all of the form

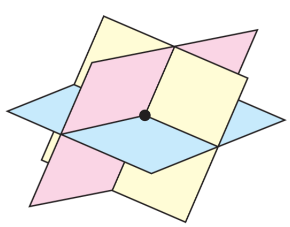
$$x = 2a - 1, \quad y = a, \quad \text{and} \quad z = a$$

So, every ordered triple of the form $(2a - 1, a, a)$ is a solution of the system.

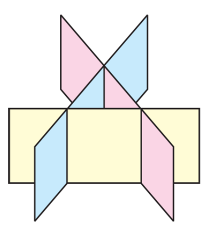
The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

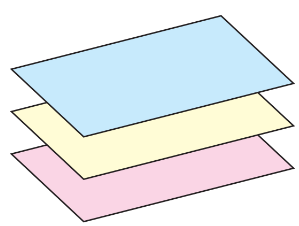
1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.



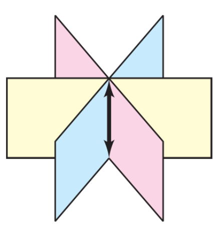
Solution: One point



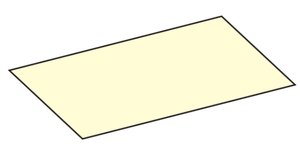
Solution: None



Solution: None



Solution: One line



Solution: One plane

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the augmented matrix of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

The next Example demonstrates the elementary row operations described above.

EXAMPLE:

(a) Add -2 times the first row of the original matrix to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

(b) Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{array}{r} \frac{1}{2}R_1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

(c) Interchange the first and second rows of the original matrix.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{array}{r} R_2 \\ R_1 \\ \\ \end{array} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

Symbol	Description
$R_i + kR_j \rightarrow R_i$	Change the ith row by adding k times row j to it, then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x-y+3z=4 \\ x+2y-2z=10 \\ 3x-y+5z=14 \end{cases}$$

Solution: Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison, we write both the system of equations and its augmented matrix.

	System		**Augmented matrix**
	$\begin{cases} x-\ \ y+3z=\ \ 4 \\ x+2y-2z=10 \\ 3x-\ \ y+5z=14 \end{cases}$		$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
Add $(-1)\times$ Equation 1 to Equation 2. Add $(-3)\times$ Equation 1 to Equation 3.	$\begin{cases} x-\ \ y+3z=4 \\ 3y-5z=6 \\ 2y-4z=2 \end{cases}$	$\xrightarrow[R_3-3R_1\to R_3]{R_2-R_1\to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$
Multiply Equation 3 by $\frac{1}{2}$.	$\begin{cases} x-\ \ y+3z=4 \\ 3y-5z=6 \\ y-2z=1 \end{cases}$	$\xrightarrow{\frac{1}{2}R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Add $(-3)\times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).	$\begin{cases} x-\ \ y+3z=4 \\ z=3 \\ y-2z=1 \end{cases}$	$\xrightarrow{R_2-3R_3\to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Interchange Equations 2 and 3.	$\begin{cases} x-\ \ y+3z=4 \\ y-2z=1 \\ z=3 \end{cases}$	$\xrightarrow{R_2\leftrightarrow R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Now we use back-substitution to find that

$$\begin{array}{ccc} z=3 & y-2z=1 & x-y+3z=4 \\ & y-2(3)=1 & x-7+3(3)=4 \\ & y-6=1 & x+2=4 \\ & y=7 & x=2 \end{array}$$

The solution is $(2,7,3)$.

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x-2y+3z=9 \\ -x+3y+z=-2 \\ 2x-5y+5z=17 \end{cases}$$

EXAMPLE: Solve the system of linear equations.

$$\begin{cases} x-2y+3z=9 \\ -x+3y+z=-2 \\ 2x-5y+5z=17 \end{cases}$$

Solution:

Linear System

$$\begin{cases} x-2y+3z=9 \\ -x+3y+z=-2 \\ 2x-5y+5z=17 \end{cases}$$

Associated Augmented Matrix

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ -1 & 3 & 1 & -2 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Add the first equation to the second equation.

$$\begin{cases} x-2y+3z=9 \\ y+4z=7 \\ 2x-5y+5z=17 \end{cases}$$

Add the first row to the second row (R_1+R_2).

$$\begin{array}{r} \\ R_1+R_2\rightarrow \\ \\ \end{array}\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x-2y+3z=9 \\ y+4z=7 \\ -y-z=-1 \end{cases}$$

Add -2 times the first row to the third row ($-2R_1+R_3$).

$$\begin{array}{r} \\ \\ -2R_1+R_3\rightarrow \end{array}\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

Add the second equation to the third equation.

$$\begin{cases} x-2y+3z=9 \\ y+4z=7 \\ 3z=6 \end{cases}$$

Add the second row to the third row (R_2+R_3).

$$\begin{array}{r} \\ \\ R_2+R_3\rightarrow \end{array}\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Multiply the third equation by $\frac{1}{3}$.

$$\begin{cases} x-2y+3z=9 \\ y+4z=7 \\ z=2 \end{cases}$$

Multiply the third row by $\frac{1}{3}$ ($\frac{1}{3}R_3$).

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3\rightarrow \end{array}\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Now we use back-substitution to find that

$$z=2 \qquad \begin{aligned} y+4z&=7 \\ y+4(2)&=7 \\ y+8&=7 \\ y&=-1 \end{aligned} \qquad \begin{aligned} x-2y+3z&=9 \\ x-2(-1)+3(2)&=9 \\ x+8&=9 \\ x&=1 \end{aligned}$$

The solution is $(1,-1,2)$.

EXAMPLE: Show that the following system has no solutions.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution: We have

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{bmatrix} \xrightarrow{\frac{1}{18}R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means the system *has no solution.*

EXAMPLE: Show that the following system has infinitely many solutions.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution: We have

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. The complete solution of this system will be discussed in the next section.