DERIVATION OF EKMAN SPIRAL
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1 Ekman Spiral

The Ekman spiral is a phenomenon that occurs due to winds interacting with the surface of a fluid. In Earth Sciences, this is more often than not, the ocean. We can begin by using the Navier-Stokes equations (zonal, meridional, and vertical winds...or just u-,v-,and w-winds for short) in cartesian coordinates...

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\delta p}{\delta x} + f \times v + \nu_h \nabla_h^2 u + \nu_z \nabla_z^2 u$$
$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\delta p}{\delta y} - f \times u + \nu_h \nabla_h^2 v + \nu_z \nabla_z^2 v$$
$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\delta p}{\delta z} - g + \nu_h \nabla_h^2 w + \nu_z \nabla_z^2 w$$

Remember, that the ν_z term is the viscosity coefficient in some direction (e.g., in this case vertical), ρ is the fluid denisty, and p is the pressure. Also note that $f \times u$ or a variant of that, is short for the coriolis force caused by being on a rotating sphere. First, we assume steady state (i.e., $\frac{d}{dt} = 0$), horizontally homogeneous flow (i.e., $\frac{d}{dx} = \frac{d}{dy} = 0$)

$$0 = 0 + f \times v + 0 + \nu_z \nabla_z^2 u$$
$$0 = 0 - f \times u + 0 + \nu_z \nabla_z^2 v$$
$$0 = -\frac{1}{\rho}\frac{\delta p}{\delta z} - g + 0 + \nu_z \nabla_z^2 w$$

Now we only need to look at the horizontal components because the vertical components are much much smaller in magnitude.

$$fv = -\nu_z \frac{\delta^2 u}{\delta z^2}$$
$$fu = \nu_z \frac{\delta^2 v}{\delta z^2}$$

Rearranging the terms to isolate the second derivative of the wind speed and collecting other terms to make things notationally simpler.

$$a^2 v = -\frac{\delta^2 u}{\delta z^2}$$
$$a^2 u = \frac{\delta^2 v}{\delta z^2}$$
$$a = \sqrt{\frac{f}{\nu_z}}$$

From here, we can isolate a single variable. It doesn't really matter which, so we will take the u-wind. To do this, we take the derivative of the first equation twice w.r.t. z

$$a^2 \frac{\delta^2 v}{\delta z^2} = -\frac{\delta^4 u}{\delta z^4}$$
$$\frac{\delta^2 v}{\delta z^2} = -\frac{1}{a^2}\frac{\delta^4 u}{\delta z^4}$$

and now we may substitute...

$$\begin{aligned}a^2u &= -\frac{1}{a^2}\frac{\delta^4 u}{\delta z^4}\\ \frac{\delta^4 u}{\delta z^4} + a^4 u &= 0\end{aligned}$$

This equation is a fourth order ordinary homogeneous equation. Fortunately, this means there is a unique solution and all we must do is find the characteristic equation. This is done in the following way.

$$\begin{aligned}\frac{\delta^4 u}{\delta z^4} + a^4 u &= 0\\ m^4 - a^4 &= 0\\ m &= \pm\sqrt{\sqrt{-1}}a\\ &= \pm\sqrt{i}a = \{\frac{1+i}{\sqrt{2}}a, \frac{-1-i}{\sqrt{2}}a\}\end{aligned}$$

The roots presented are complex conjugate solutions to the characteristic equation. The solution to the ODE will take the form of two exponentials.

$$u = C_1 e^{m_1 z} + C_2 e^{m_2 z}$$

Substituting our roots in for m_1 and m_2

$$\begin{aligned}u &= C_1 e^{(1+i)bz+\phi_1} + C_2 e^{(-1-i)bz+\phi_2}\\ &= C_1 e^{(bz+i(bz+\phi_1))} + C_2 e^{(-bz-i(bz+\phi_2))}\\ b &= \frac{a}{\sqrt{2}} = \sqrt{\frac{f}{2\nu_z}}\end{aligned}$$

Now utilizing Euler's Formula to examine the sinusoids comprising the exponentials.

$$\begin{aligned}u &= C_1 e^{bz}\left[\cos(bz+\phi_1) + i\sin(bz+\phi_1)\right] + C_2 e^{-bz}\left[\cos(bz+\phi_2) - i\sin(bz+\phi_2)\right]\\ &= C_1 e^{bz}\cos(bz+\phi_1) + C_2 e^{-bz}\cos(bz+\phi_2) + i\left[C_1 e^{bz}\sin(bz+\phi_1) - C_2 e^{-bz}\sin(bz+\phi_2)\right]\end{aligned}$$

Using the real part of this solution, we can see...

$$u = C_1 e^{bz}\cos(bz+\phi_1) + C_2 e^{-bz}\cos(bz+\phi_2)$$

From here we can use a boundary condition to simplify this equation. Imagine the depth of ocean goes to infinitiy. That implies $z = -\infty$. At this depth all motion stops, such that $u = 0$. From inspection, we can see only one of the two terms does not approach infinity and obeys the boundary condition. That is the first term. Now, we may use Euler's Formula to view the sinusoidal signals of the imaginary exponential.

$$u = C_1 e^{bz}\cos(bz+\phi_1)$$

We can obtain the equaiton for v in the following manner. First take the derivative above twice.

$$\begin{aligned}
u &= C_1 e^{bz}\cos(bz+\phi_1)\\
\frac{\delta u}{\delta z} &= bC_1 e^{bz}\cos(bz+\phi_1) + bC_1 e^{bz}\sin(bz+\phi_1)\\
\frac{\delta^2 u}{\delta z^2} &= b^2 C_1 e^{bz}\cos(bz+\phi_1) + b^2 C_1 e^{bz}\sin(bz+\phi_1) + b^2 C_1 e^{bz}\sin(bz+\phi_1) - b^2 C_1 e^{bz}\cos(bz+\phi_1)\\
&= 2b^2 C_1 e^{bz}\sin(bz+\phi_1)
\end{aligned}$$

Substitute

$$\begin{aligned}
a^2 v &= -2b^2 C_1 e^{bz}\sin(bz+\phi_1)\\
v &= -\frac{2b^2}{a^2} C_1 e^{bz}\sin(bz+\phi_1)\\
&= -C_1 e^{bz}\sin(bz+\phi_1)
\end{aligned}$$

Now we must solve for C_1 and ϕ_1. To do this, we need to take the first derivative of u- and v-winds w.r.t. z.

$$\begin{aligned}
\frac{\delta u}{\delta z} &= bC_1 e^{bz}\cos(bz+\phi_1) - bC_1 e^{bz}\sin(bz+\phi_1)\\
\frac{\delta v}{\delta z} &= -bC_1 e^{bz}\sin(bz+\phi_1) - bC_1 e^{bz}\cos(bz+\phi_1)
\end{aligned}$$

Another boundary condition must be met at the surface. Imagine a purely zonal (i.e., west-to-east flow). That would imply all of the force exerted upon the surface is caused by the u-wind (i.e., v=0, everywhere). This takes the form of surface stress, which follows as [INCLUDE A DERIVATION OF THIS AS WELL]:

$$\tau_w = \rho \nu_z \frac{\delta u}{\delta z}$$

At the surface, we know the following boundary conditions.

$$\begin{aligned}
\frac{\delta u}{\delta z} &= \frac{\tau_w}{\rho K_\nu}\\
\frac{\delta v}{\delta z} &= 0
\end{aligned}$$

Using these and z=0 with our derivatives we can find the coefficients. Starting with $\frac{\delta v}{\delta z}$

$$\begin{aligned}
\frac{\delta v}{\delta z} &= -bC_1 e^{bz}\sin(bz+\phi_1) - bC_1 e^{bz}\cos(bz+\phi_1)\\
0 &= \sin(\phi_1) + \cos(\phi_1)\\
\sin(\phi_1) &= -\cos(\phi_1)\\
\phi_1 &= -\frac{\pi}{4}
\end{aligned}$$

Now that we have ϕ_1, we may solve for C_1 using $\frac{\delta u}{\delta z}$:

$$
\begin{aligned}
\frac{\delta u}{\delta z} = \frac{\tau_w}{\rho \nu_z} &= bC_1 e^{bz}\cos(bz+\phi_1) - bC_1 e^{bz}\sin(bz+\phi_1) \\
&= bC_1\left(\cos(\phi_1) - \sin(\phi_1)\right) \\
&= bC_1\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \\
&= \sqrt{2}bC_1 \\
&= aC_1 \\
C_1 &= \frac{\tau_w}{\rho \nu_z a} \\
&= \frac{\tau_w}{\rho \nu_z \sqrt{\frac{f}{\nu_z}}} \\
&= \frac{\tau_w}{\rho \sqrt{f \nu_z}}
\end{aligned}
$$

Thus, the equations that describe the so-called "Ekman Spiral" are:

$$
\begin{aligned}
u &= \frac{\tau_w}{\rho\sqrt{f\nu_z}} e^{\sqrt{\frac{f}{2\nu_z}}z}\cos(\sqrt{\frac{f}{2\nu_z}}z - \frac{\pi}{4}) \\
v &= -\frac{\tau_w}{\rho\sqrt{f\nu_z}} e^{\sqrt{\frac{f}{2\nu_z}}z}\sin(\sqrt{\frac{f}{2\nu_z}}z - \frac{\pi}{4})
\end{aligned}
$$

Please see the example figure 1

2 Ekman Mass Transport

$$
\begin{aligned}
M_x &= \rho \int_{-d}^{0} u dz \\
M_y &= \rho \int_{-d}^{0} v dz
\end{aligned}
$$

We could integrate the Ekman equations above, but instead we can use the easier forms..

$$
\begin{aligned}
fv &= -\nu_z \frac{\delta^2 u}{\delta z^2} \\
fu &= \nu_z \frac{\delta^2 v}{\delta z^2}
\end{aligned}
$$

So...

$$
\begin{aligned}
M_x &= \rho \int_{-d}^{0} \frac{\nu_z}{f}\frac{\delta^2 v}{\delta z^2} dz \\
M_y &= \rho \int_{-d}^{0} -\frac{\nu_z}{f}\frac{\delta^2 u}{\delta z^2} dz
\end{aligned}
$$

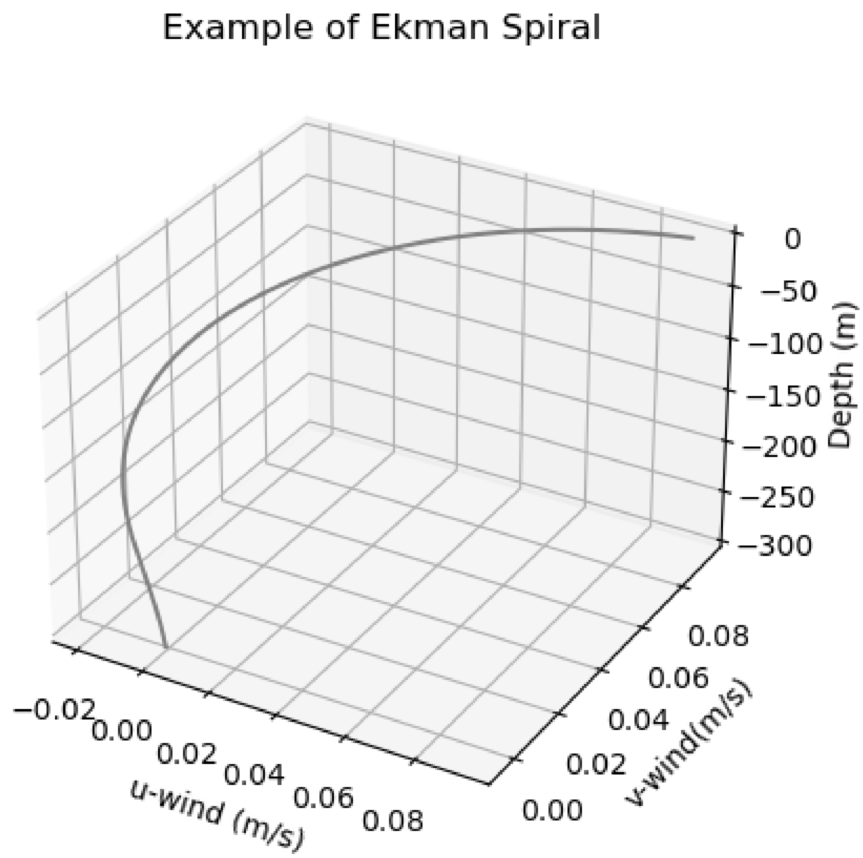

Figure 1: Example of the Ekman Spiral

The term $\frac{\nu_z}{f}$ is a constant, so we may integrate...

$$
\begin{aligned}
M_x &= \frac{\rho\nu_z}{f}\int_{-d}^{0}\frac{\delta^2 v}{\delta z^2}dz \\
&= \frac{\rho\nu_z}{f}\left[\frac{\delta v}{\delta z}\right]_{-d}^{0} \\
M_y &= -\frac{\rho\nu_z}{f}\int_{-d}^{0}\frac{\delta^2 u}{\delta z^2}dz \\
&= -\frac{\rho\nu_z}{f}\left[\frac{\delta u}{\delta z}\right]_{-d}^{0}
\end{aligned}
$$

Now evaluate the derivatives at z=0

$$\begin{aligned}
\frac{\delta u}{\delta z} &= bC_1 e^{bz}\cos(bz+\phi_1) - bC_1 e^{bz}\sin(bz+\phi_1) \\
&= bC_1\cos(\phi_1) - bC_1\sin(\phi_1) \\
&= \sqrt{2}bC_1 = aC_1 \\
\frac{\delta v}{\delta z} &= -bC_1 e^{bz}\sin(bz+\phi_1) - bC_1 e^{bz}\cos(bz+\phi_1) \\
&= -bC_1\sin(\phi_1) - bC_1\cos(\phi_1) \\
&= bC_1\frac{\sqrt{2}}{2} - bC_1\frac{\sqrt{2}}{2} = 0
\end{aligned}$$

Now lets say the depth goes to infinity...(i.e., d=$-\infty$)

$$\begin{aligned}
\frac{\delta u}{\delta z} &= bC_1 e^{bz}\cos(bz+\phi_1) - bC_1 e^{bz}\sin(bz+\phi_1) \\
&= 0 \\
\frac{\delta v}{\delta z} &= -bC_1 e^{bz}\sin(bz+\phi_1) - bC_1 e^{bz}\cos(bz+\phi_1) \\
&= 0
\end{aligned}$$

So plugging these in...

$$\begin{aligned}
M_x &= \frac{\rho\nu_z}{f}\left[0-0\right] \\
&= 0 \\
M_y &= -\frac{\rho\nu_z}{f}\left[aC_1-0\right] \\
&= -\frac{\rho\nu_z}{f}\left[\sqrt{\frac{f}{\nu_z}}\frac{\tau_w}{\rho\sqrt{f\nu_z}}\right] \\
&= -\frac{\tau_w}{f}
\end{aligned}$$

From this, we can see that mass transport is perpendicular to the surface wind stress.

3 Ekman Pumping

Ekman Pumping is mathematical description of how ocean mass flow caused by surface wind stress is conserved. This is done by vertically integrating the continuity equation:

$$\rho \int_{-d}^{0} \left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} \right) dz = 0$$
$$\rho \int_{-d}^{0} \frac{\delta u}{\delta x} dz + \rho \int_{-d}^{0} \frac{\delta v}{\delta y} dz + \rho \int_{-d}^{0} \frac{\delta w}{\delta z} dz = 0$$
$$\frac{\delta}{\delta x} \int_{-d}^{0} \rho u dz + \frac{\delta}{\delta y} \int_{-d}^{0} \rho v dz + \rho \int_{-d}^{0} \frac{\delta w}{\delta z} dz = 0$$
$$\frac{\delta M_x}{\delta x} + \frac{\delta M_y}{\delta y} + \rho \int_{-d}^{0} \frac{\delta w}{\delta z} dz = 0$$
$$\nabla_H \cdot \vec{M} = -\rho \int_{-d}^{0} \frac{\delta w}{\delta z} dz$$
$$\nabla_H \cdot \vec{M} = -\rho \left[w(0) - w(-d) \right] = -\rho w(0)$$

If we observe this last equation, it states that the horizontal divergence of mass is balanced by vertical velocity at the ocean surface. Does this make sense?

$$\nabla_H \cdot \vec{M} = -\rho w(0)$$
$$\frac{\delta}{\delta y} \left(\frac{\tau_w}{f} \right) = \rho w(0)$$

What this shows is that if the gradient of the surface stress is positive, then the vertical velocity will be positive (i.e., upwelling). This occurs when we have increasing wind speed or surface wind divergence because the surface stress increases. The opposite occurs when we have surface wind convergence. This leads to downwelling.

4 Surface Wind Stresses

At the surface of the ocean. we can see that two fluid boundaries are evidently meeting: Sea and Atmosphere. In the event, one moves faster than the other, the drag between the two fluids causes a shear stress to form at the boundary interface. This shear stress vector is in the direction of the flow and causes turbulence to form. The turbulent disruptions flux momentum, heat, and moisture from one fluid into another. Winds can be broken into two pieces using Reynold's Averaging:

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$
$$w = \overline{w} + w'$$

When multiplied together and averaged over a long enough timescale these values become the following:

$$\overline{xy} = \overline{x}\,\overline{y} + \overline{x'y'}$$

Where $\overline{x'y'}$ is the turbulent contribution. Imagining a cube, any turbulent component may interact along a boundary. This allows for 9 possible turbulent fluxes, or a second order tensor:

$$\begin{bmatrix} \overline{u'u'}, \overline{u'v'}, \overline{u'w'} \\ \overline{u'u'}, \overline{v'v'}, \overline{v'w'} \\ \overline{w'u'}, \overline{w'v'}, \overline{w'w'} \end{bmatrix}$$

The nature of deformation allows us to assume this is a symmetric matrix such that $\overline{w'u'} = \overline{u'w'}$. Thus, we need only worry about 6 terms. These turbulent motions are known as "Reynolds Stress." We may think of this as the deformation of one fluid caused by another moving across the fluid boundary. They are not properties of the fluid and only arise when one fluid moves over another. Now when we describe a flux, it is an exchange of something over an area per second. In the case of MOMENTUM, we can state in the vertical that is may be descibed as the following:

$$M_{xz} = \frac{mu}{A_z t}$$
$$M_{yz} = \frac{mv}{A_z t}$$
$$M_{zz} = \frac{mw}{A_z t}$$

Where m is mass, A_z is the area of a boundary, and t is time. The turbulent momentum flux vector component in a particular direction can be described as simply the Reynolds stress across a boundary

$$M_{ij} = \tau_{ij} = \rho\overline{u'_i u'_j}$$

Where i and j correspond to particular directions. This may be simplified by Newton's Law of Viscosity which states:

$$\tau_{ij} \propto \mu_z \frac{\delta u_i}{\delta x_j}$$

The shear stress is proportional to the rate of change of the i velocity component in the j-direction multiplied by some kinematic viscosity. The kinematic viscosity can be further decomposed into a dynamic viscosity multiplied by the density of the fluid:

$$\tau_{ij} \propto \rho\nu_z \frac{\delta u_i}{\delta x_j}$$

When placed into our context, we care concerned with the vertical exchanges of momentum caused by the u- and v- winds...this becomes...

$$\tau_{xz} \propto \rho\nu_z \frac{\delta u}{\delta z}$$
$$\tau_{yz} \propto \rho\nu_z \frac{\delta v}{\delta z}$$

Thus, the turbulent momentum exchange of wind a direction is proportional to the vertical wind gradient.