

LATENT HEAT FLUX FROM FICK'S LAW

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In many text books like Monteith and Unsworth (Principles of Environmental Physics), you will often find an equation for latent heat flux that looks like this:

$$\lambda E = \frac{\rho_a C_p}{\gamma} \frac{e^*(T_s) - e_a}{r_s + r_a} \quad (1)$$

where ρ_a is the total air density, C_p is the specific heat capacity of dry air at constant pressure, γ is the psychrometric constant, e^* is a saturation vapor pressure, e is a vapor pressure, r_s is the surface resistance and r_a is the aerodynamic resistance. Where does this come from? The purpose of this quick study is to begin with Fick's First Law of Diffusion and derive this equation.

Fick's Law and Resistance

To begin, we can define Fick's First Law of Diffusion

$$F = -D \frac{\delta x}{\delta z}$$

This formula defines a flux across an area where D is a diffusion coefficient, x is a mass quantity (like water vapor), and z is some distance. The negative sign is present to signify a transfer downgradient. For water vapor, we can define Fick's Law in terms of a water vapor mass flux or evaporation, E .

$$E = -D \frac{\delta \rho_v}{\delta z}$$

Where evaporation is now a function of the change in vapor density. We continue by moving δz and D to the LHS, integrating both sides, and assuming evaporation is constant over the span of z .

$$\begin{aligned} E \int_{z_1}^{z_2} \frac{\delta z}{D} &= \int_{z_2}^{z_1} \delta \rho_v \\ E r_v &= (\rho_v(z_1) - \rho_v(z_2)) \\ E &= \frac{\rho_v(z_1) - \rho_v(z_2)}{r_v} \\ r_v &= \int_{z_1}^{z_2} \frac{\delta z}{D} \end{aligned}$$

Here r_v is a resistance to mass transfer. It is analogous to how a resistance works in electrical systems. You will find terms like this within heat transfer and electrical systems (e.g., Ohm's Law). Anyway, over a typical height, we can break a canopy into two sections. First, we know that z_1 is the surface and z_2 is the free boundary layer air above the canopy. Over this span, there are two bulk air masses a molecule must pass through, each with its own resistance. First, is from the surface through the canopy (i.e., r_s). The second is from the canopy into the free boundary layer air (i.e., r_a). These resistances are in series with one another and that implies that they can be added with each other to obtain the mass transfer resistance r_v . That is:

$$\begin{aligned} E &= \frac{\rho_v(z_1) - \rho_v(z_2)}{r_v} \\ &= \frac{\rho_v(z_1) - \rho_v(z_2)}{r_s + r_a} \end{aligned}$$

Now we can consider the surface itself. Since the surface is described here as within the stomate or surface touching liquid water itself, the vapor density is representative of saturated air (i.e., relative humidity is 100 percent). This implies:

$$E = \frac{\rho_v^*(T_s) - \rho_v(z_2)}{r_s + r_a}$$

Where $\rho_v^*(T_s)$ is the saturation vapor density at the surface temperature. Now focusing on z_2 . Since z_2 is representative of the free boundary layer air, we can assume its vapor density is representative of the temperature of the air.

$$E = \frac{\rho_v^*(T_s) - \rho_v(T_a)}{r_s + r_a} \quad (2)$$

This equation is a common one for evaporative flux. Now, we can further deconstruct this through the vapor densities. Vapor density can be defined from the total air density, ρ_a , and the specific humidity, q .

$$\rho_a q = \rho_v \quad (3)$$

Now rearrange this equation for the specific humidity. Then, introduce the ideal gas law and the law of partial pressures/densities. From these, we can find another relation between specific humidity, vapor pressure, and total air pressure, P .

$$\begin{aligned} q &= \frac{\rho_v}{\rho_a} \\ &= \frac{\rho_v}{\rho_v + \rho_d} \\ &= \frac{\frac{eM_w}{RT}}{\frac{eM_w}{RT} + \frac{eM_d}{RT}} \\ &= \frac{eM_w}{eM_w + P_d M_d} \\ &= \frac{eM_w}{eM_w + (P - e)M_d} \\ &= \frac{1}{M_d} \left(\frac{eM_w}{e \frac{M_w}{M_d} + (P - e)} \right) \\ &= \frac{\epsilon e}{\epsilon e + (P - e)} \\ \epsilon &= \frac{M_w}{M_d} = 0.622 \end{aligned}$$

Under sufficiently small vapor pressures (i.e., $e \ll P$), we can approximate this relationship as:

$$q = \frac{\epsilon e}{P} \quad (4)$$

Note, that P here is the total air pressure at z_1 or z_2 , much like temperature. Next, we can substitute equations (3) and (4) into equation (2).

$$\begin{aligned} E &= \frac{\rho_v^*(T_s) - \rho_v(T_a)}{r_s + r_a} \\ &= \frac{\rho_a q_v^*(T_s) - \rho_a q(T_a)}{r_s + r_a} \\ &= \rho_a \frac{\frac{\epsilon e_s^*(T_s)}{P_s} - \frac{\epsilon e(T_a)}{P_a}}{r_s + r_a} \end{aligned}$$

Now assume that $P_s \approx P_a = P$. This is reasonable near the surface where a flux tower might be. The height differential may be 3 meters or so, which under the hydrostatic approximation would lead to a pressure difference of about 40 Pa. Thus:

$$E = \frac{\rho_a \epsilon}{P} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \quad (5)$$

Latent Heat Flux

We obtain the latent heat flux by multiplying equation (5) by the latent heat of vaporization, λ .

$$\lambda E = \rho_a \frac{\lambda \epsilon}{P} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a}$$

Lastly, we can introduce the definition of the psychrometric constant and rearrange it.

$$\begin{aligned} \gamma &= \frac{PC_p}{\epsilon \lambda} \\ \frac{\lambda \epsilon}{P} &= \frac{C_p}{\gamma} \end{aligned}$$

Substitute this result in to our latent heat flux equation to find our solution:

$$\boxed{\lambda E = \frac{\rho_a C_p}{\gamma} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a}}$$

An Alternative to the Evaporative Flux

There is an alternative form to the evaporative flux equation. Here is a short section on how to find it. Starting with equation (5), we can use the law of partial densities and the ideal gas law to define the total density of the air:

$$\begin{aligned} \rho_a &= \rho_d + \rho_v \\ &= \frac{P_d M_d}{RT} + \frac{e M_w}{RT} \\ &= \frac{P_d M_d (1 + \epsilon r_v)}{RT} \\ r_v &= \frac{e}{P_d} \end{aligned}$$

where r_v is the water vapor mixing ratio. Substituting this equation into equation (5) and using the definition of total pressure:

$$\begin{aligned}
 E &= \frac{P_d M_d (1 + \epsilon r_v)}{RT} \frac{\epsilon}{P} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 &= \frac{P_d M_d (1 + \epsilon r_v)}{RT} \frac{\epsilon}{P_d + e} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 &= \frac{P_d M_d (1 + \epsilon r_v)}{RT} \frac{\epsilon}{P_d (1 + r_v)} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 &= \frac{(1 + \epsilon r_v)}{(1 + r_v)} \frac{M_v}{RT} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a}
 \end{aligned}$$

The mixing ratio term is the final piece. Recall that we assumed that the vapor pressure is much less than the total air pressure. This implies that the vapor pressure is also much less than the dry air pressure. Therefore, under the definition r_v , it can be assumed that the mixing ratio is small (i.e., $r_v \ll 1$). This assumption is fairly valid since a typical water vapor mixing ratio near the surface in the mid-latitudes is between 6-12 $\frac{g}{kg}$ or 0.006-0.012 $\frac{g}{g}$. Therefore, $(1 + \epsilon r_v) \approx 1$ and $(1 + r_v) \approx 1$. Using this assumption, the mixing ratio term is essentially 1. Therefore, we see:

$$E = \frac{M_v}{RT} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a}$$

This is an alternative form to the evaporative flux equation. Where, T is assumed to be a mean temperature between the z_1 and z_2 . It is common for this to be simply T_a .

Latent Heat Flux from Alternative Equation

Insurance of the latent heat flux in this situation can be found by multiplying the value by latent heat of vaporization and the definition of the psychrometric constant:

$$\begin{aligned}
 \lambda E &= \frac{\lambda M_v}{RT} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 \lambda &= \frac{PC_p}{\epsilon \gamma} \\
 \lambda E &= \frac{PC_p M_v}{RT \epsilon \gamma} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 &= \frac{PC_p M_d}{RT \gamma} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a} \\
 &= \frac{\rho_a C_p}{\gamma} \frac{e_s^*(T_s) - e(T_a)}{r_s + r_a}
 \end{aligned}$$

In the equation above, ρ_a again is the total air density. However, since the molecular weight of dry air is used in the ideal gas law and not total molecular mass, it can be assumed that this density will be representative of dry air. We can see it is functionally the same as the one above, but it will yield a slightly different answer.