1 Cartesian to Spherical

We begin by defining the coordinates of a vector in the cartesian plane (i.e., x,y,z), but in terms of its angles from the various axes and vector magnitude. Since, the Earth is a spheroid and I happen to be an Earth scientist, I've chosen to represent the spherical coordinates in a (λ,ϕ,R_e) system. Where λ is longitude, or the angle from a axis line of the meridion of th spherioid (e.g., 0 deg longitude), ϕ is the latitude, or the vertical angle created from the horizontal axis plane, and R_e is the radius of the Earth. For simplification, I am assuming R_e is constant, but it can represent any distance away from the spheroid center.

INSERT DRAWING

The conversion into each of these coordinates is straightforward. All we need to do is decompose the vector onto the respective axial planes. They are:

$$x = R_e \sin\theta \cos\lambda$$
$$y = R_e \sin\theta \sin\lambda$$
$$z = R_e \cos\theta$$

Since latitude is measured from the equator we can see that a similarity among using a simple trig identity

$$\sin\theta = \cos(\frac{\pi}{2} - \theta)$$
$$\frac{\pi}{2} - \theta = \phi$$

ϕ is what is considered the latitude angle. We have measured from the angle from the xy plane versus the z axis. Thus, the cartesian forms of this can be alternatively written as...

$$x = R_e \cos\phi \cos\lambda$$
$$y = R_e \cos\phi \sin\lambda$$
$$z = R_e \sin\phi$$

2 Unit Vectors and Back

2.1 via Normal Physics Definition

The unit vectors in cartesian coordinates are:

$$\hat{x} = \{1, 0, 0\}$$
$$\hat{y} = \{0, 1, 0\}$$
$$\hat{z} = \{0, 0, 1\}$$

Any vector $\hat{r}$ can be decomposed into the following in cartesian coordinates and unit vectors:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Using the definitions of the cartesian mapping we can find the following:

$$
\begin{aligned}
\vec{r} &= R_e \sin\theta\cos\lambda\hat{x} + R_e \sin\theta\sin\lambda\hat{y} + R_e\cos\theta\hat{z} \\
&= R_e\left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right] \\
&= R_e\hat{r}
\end{aligned}
$$

From this we can see the vector $\vec{r}$ is some combination of scaling factor R_e and a unit vector $\hat{r}$

To create a new basis vector that points in increasing r we can define the following

$$
\begin{aligned}
e_r &= \frac{\delta\vec{r}}{\delta r} \\
&= \hat{r} \\
\hat{e}_r &= \frac{\hat{r}}{|\hat{r}|} = \hat{r} \\
&= \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}
\end{aligned}
$$

Dividing by the magnitude of a unit vector is 1 thus. Now if we define the longitudinal basis vector...

$$
\begin{aligned}
e_\lambda &= \frac{\delta\vec{r}}{\delta\lambda} \\
&= \frac{\delta}{\delta\lambda}R_e\left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right] \\
&= R_e\left[-\sin\theta\sin\lambda\hat{x} + \sin\theta\cos\lambda\hat{y}\right] \\
&= R_e\sin\theta\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right] \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
|e_\lambda| &= \sqrt{e_\lambda\cdot e_\lambda} \\
&= \sqrt{(R_e\sin\theta)^2\left[\sin^2\lambda + \cos^2\lambda\right]} \\
&= R_e\sin\theta \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
&= \frac{R_e\sin\theta\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right]}{R_e\sin\theta} \\
&= -\sin\lambda\hat{x} + \cos\lambda\hat{y}
\end{aligned}
$$

And for the latitudinal vector

$$
\begin{aligned}
e_\theta &= \frac{\delta \vec{r}}{\delta \theta} \\
&= \frac{\delta}{\delta \theta} R_e \left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right] \\
&= R_e \left[\cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}\right] \\
\hat{e}_\theta &= \frac{e_\theta}{|e_\theta|} \\
|e_\theta| &= \sqrt{e_\theta \cdot e_\theta} \\
&= \sqrt{R_e^2 \left[\cos^2\theta\cos^2\lambda + \cos^2\theta\sin^2\lambda + \sin^2\theta\right]} \\
&= \sqrt{R_e^2 \left[\cos^2\theta(\cos^2\lambda + \sin^2\lambda) + \sin^2\theta\right]} \\
&= \sqrt{R_e^2 \left[\cos^2\theta + \sin^2\theta\right]} \\
&= \sqrt{R_e^2} \\
&= R_e \\
\hat{e}_\theta &= \frac{e_\theta}{|e_\theta|} \\
&= \frac{R_e \left[\cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}\right]}{R_e} \\
&= \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}
\end{aligned}
$$

Thus the unit vectors in spherical coordinates are:

$$
\begin{aligned}
\hat{e}_r &= \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
\hat{e}_\theta &= \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}
\end{aligned}
$$

To solve for the cartesian unit vectors...

$$
\begin{aligned}
\hat{e}_r &= \sin\phi\cos\lambda\hat{x} + \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
\hat{e}_\phi &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} - \sin\phi\hat{z}
\end{aligned}
$$

STEP 1.

$$
\begin{aligned}
\cos\phi\hat{e}_r &= \cos\phi\sin\phi\cos\lambda\hat{x} + \cos\phi\sin\phi\sin\lambda\hat{y} + \cos^2\phi\hat{z} \\
\sin\phi\hat{e}_\phi &= \sin\phi\cos\phi\cos\lambda\hat{x} + \sin\phi\cos\phi\sin\lambda\hat{y} - \sin^2\phi\hat{z} \\
\cos\phi\hat{e}_r - \sin\phi\hat{e}_\phi &= (\cos^2\phi + \sin^2\phi)\hat{z} \\
\hat{z} &= \cos\phi\hat{e}_r - \sin\phi\hat{e}_\phi
\end{aligned}
$$

STEP 2.

$$
\begin{aligned}
\sin\phi\cos\lambda\hat{e}_r &= \sin^2\phi\cos^2\lambda\hat{x} + \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} + \sin\phi\cos\lambda\cos\phi\hat{z} \\
\sin\lambda\hat{e}_\lambda &= -\sin^2\lambda\hat{x} + \sin\lambda\cos\lambda\hat{y} \\
\cos\phi\cos\lambda\hat{e}_\phi &= \cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} - \cos\phi\cos\lambda\sin\phi\hat{z}
\end{aligned}
$$

Add 1 and 3

$$\begin{aligned}\sin\phi\cos\lambda\hat{e}_r + \cos\phi\cos\lambda\hat{e}_\phi &= \sin^2\phi\cos^2\lambda\hat{x} + \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} + \sin\phi\cos\lambda\cos\phi\hat{z} \\ &\cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} - \cos\phi\cos\lambda\sin\phi\hat{z} \\ &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y}\end{aligned}$$

Subtract 2 from result...

$$\begin{aligned}\sin\phi\cos\lambda\hat{e}_r + \cos\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y} + \sin^2\lambda\hat{x} - \sin\lambda\cos\lambda\hat{y} \\ \hat{x} &= \sin\phi\cos\lambda\hat{e}_r + \cos\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda\end{aligned}$$

STEP 3.

$$\begin{aligned}\sin\phi\sin\lambda\hat{e}_r &= \sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} + \sin^2\phi\sin^2\lambda\hat{y} + \sin\phi\sin\lambda\cos\phi\hat{z} \\ \cos\lambda\hat{e}_\lambda &= -\cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y} \\ \cos\phi\sin\lambda\hat{e}_\phi &= \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} - \cos\phi\sin\lambda\sin\phi\hat{z}\end{aligned}$$

Add 1 and 3...

$$\begin{aligned}\sin\phi\sin\lambda\hat{e}_r + \cos\phi\sin\lambda\hat{e}_\phi &= \sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} + \sin^2\phi\sin^2\lambda\hat{y} + \sin\phi\sin\lambda\cos\phi\hat{z} \\ &+ \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} - \cos\phi\sin\lambda\sin\phi\hat{z} \\ &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y}\end{aligned}$$

Add 2.

$$\begin{aligned}\sin\phi\sin\lambda\hat{e}_r + \cos\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y} - \cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y} \\ \hat{y} &= \sin\phi\sin\lambda\hat{e}_r + \cos\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda\end{aligned}$$

Thus...

$$\begin{aligned}\hat{x} &= \sin\phi\cos\lambda\hat{e}_r + \cos\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda \\ \hat{y} &= \sin\phi\sin\lambda\hat{e}_r + \cos\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda \\ \hat{z} &= \cos\phi\hat{e}_r - \sin\phi\hat{e}_\phi\end{aligned}$$

2.2 via Meteorology Definition

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Using the definitions of the cartesian mapping we can find the following:

$$\begin{aligned}\vec{r} &= R_e\cos\phi\cos\lambda\hat{x} + R_e\cos\phi\sin\lambda\hat{y} + R_e\sin\phi\hat{z} \\ &= R_e\left[\cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\right] \\ &= R_e\hat{r}\end{aligned}$$

From this we can see the vector $\vec{r}$ is some combination of scaling factor R_e and a unit vector $\hat{r}$

To create a new basis vector that points in increasing r we can define the following

$$\begin{aligned}
e_r &= \frac{\delta \vec{r}}{\delta r} \\
&= \hat{r} \\
\hat{e}_r &= \frac{\hat{r}}{|\hat{r}|} = \hat{r} \\
&= \cos\phi \cos\lambda \hat{x} + \cos\phi \sin\lambda \hat{y} + \sin\phi \hat{z}
\end{aligned}$$

Dividing by the magnitude of a unit vector is 1 thus. Now if we define the longitudinal basis vector...

$$\begin{aligned}
e_\lambda &= \frac{\delta \vec{r}}{\delta \lambda} \\
&= \frac{\delta}{\delta \lambda} R_e \left[\cos\phi \cos\lambda \hat{x} + \cos\phi \sin\lambda \hat{y} + \sin\phi \hat{z}\right] \\
&= R_e \left[-\cos\phi \sin\lambda \hat{x} + \cos\phi \cos\lambda \hat{y}\right] \\
&= R_e \cos\phi \left[-\sin\lambda \hat{x} + \cos\lambda \hat{y}\right] \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
|e_\lambda| &= \sqrt{e_\lambda \cdot e_\lambda} \\
&= \sqrt{(R_e \cos\phi)^2 \left[\sin^2\lambda + \cos^2\lambda\right]} \\
&= R_e \cos\phi \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
&= \frac{R_e \cos\phi \left[-\sin\lambda \hat{x} + \cos\lambda \hat{y}\right]}{R_e \cos\phi} \\
&= -\sin\lambda \hat{x} + \cos\lambda \hat{y}
\end{aligned}$$

And for the latitudinal vector

$$\begin{aligned}
e_\phi &= \frac{\delta \vec{r}}{\delta \phi} \\
&= \frac{\delta}{\delta \phi} R_e \left[\cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\right] \\
&= R_e \left[-\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}\right] \\
\hat{e}_\phi &= \frac{e_\phi}{|e_\phi|} \\
|e_\phi| &= \sqrt{e_\phi \cdot e_\phi} \\
&= \sqrt{R_e^2\left[\sin^2\phi\cos^2\lambda + \sin^2\phi\sin^2\lambda + \cos^2\phi\right]} \\
&= \sqrt{R_e^2\left[\sin^2\phi(\cos^2\lambda + \sin^2\lambda) + \cos^2\phi\right]} \\
&= \sqrt{R_e^2\left[\sin^2\phi + \cos^2\phi\right]} \\
&= \sqrt{R_e^2} \\
&= R_e \\
\hat{e}_\phi &= \frac{e_\phi}{|e_\phi|} \\
&= \frac{R_e\left[-\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}\right]}{R_e} \\
&= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}
\end{aligned}$$

Thus the unit vectors in spherical coordinates are:

$$\begin{aligned}
\hat{e}_r &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
\hat{e}_\phi &= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}
\end{aligned}$$

Are they equivalent to the normal methods? Let's say $\theta = 0$, $\phi = \frac{\pi}{2}$ and $\lambda = \frac{\pi}{2}$ Meteo

$$\begin{aligned}
\hat{e}_r &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z} \\
&= \hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
&= -\hat{x} \\
\hat{e}_\phi &= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z} \\
&= -\hat{y}
\end{aligned}$$

Normal

$$\begin{aligned}
\hat{e}_r &= \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z} \\
&= \hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
&= -\hat{x} \\
\hat{e}_\theta &= \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z} \\
&= \hat{y}
\end{aligned}$$

As can be seen, the X and Z unit vectors are identical in both systems. The reversal from measuring the angle from the pole versus the equator in meteorology has reversed the direction of the Y unit vector. Thus, the Y unit vector in meteorology is the opposite of that found in the normal method. To make these consistent with the right hand rule, we must multiply the unit vector, $\hat{e}_\phi$, by -1.

$$\begin{aligned}
\hat{e}_r &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
\hat{e}_\phi &= \sin\phi\cos\lambda\hat{x} + \sin\phi\sin\lambda\hat{y} - \cos\phi\hat{z}
\end{aligned}$$

The meteorology and normal methods are now consistent with one another...

TO SOLVE FOR THE CARTESIAN MAPPING...

$$\begin{aligned}
\hat{e}_r &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z} \\
\hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y} \\
\hat{e}_\phi &= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}
\end{aligned}$$

STEP 1. (ADD 1 and 3)

$$\begin{aligned}
\sin\phi\hat{e}_r &= \sin\phi\cos\phi\cos\lambda\hat{x} + \sin\phi\cos\phi\sin\lambda\hat{y} + \sin^2\phi\hat{z} \\
\cos\phi\hat{e}_\phi &= -\cos\phi\sin\phi\cos\lambda\hat{x} - \cos\phi\sin\phi\sin\lambda\hat{y} + \cos^2\phi\hat{z} \\
\sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi &= (\cos^2\phi + \sin^2\phi)\hat{z} \\
\hat{z} &= \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi
\end{aligned}$$

STEP 2. isolate x

$$\begin{aligned}
\cos\phi\cos\lambda\hat{e}_r &= \cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} + \cos\phi\cos\lambda\sin\phi\hat{z} \\
\sin\lambda\hat{e}_\lambda &= -\sin^2\lambda\hat{x} + \sin\lambda\cos\lambda\hat{y} \\
\sin\phi\cos\lambda\hat{e}_\phi &= -\sin^2\phi\cos^2\lambda\hat{x} - \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} + \sin\phi\cos\lambda\cos\phi\hat{z}
\end{aligned}$$

Subtract 1 and 3

$$\begin{aligned}
\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi &= \cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} + \cos\phi\cos\lambda\sin\phi\hat{z} \\
&\quad + \sin^2\phi\cos^2\lambda\hat{x} + \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} - \sin\phi\cos\lambda\cos\phi\hat{z} \\
&= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y}
\end{aligned}$$

Subtract 2 from result...

$$\begin{aligned}
\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y} + \sin^2\lambda\hat{x} - \sin\lambda\cos\lambda\hat{y} \\
\hat{x} &= \cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda
\end{aligned}$$

STEP 3. Isolate Y

$$\begin{aligned}
\cos\phi\sin\lambda\hat{e}_r &= \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} + \cos\phi\sin\lambda\sin\phi\hat{z} \\
\cos\lambda\hat{e}_\lambda &= -\cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y} \\
\sin\phi\sin\lambda\hat{e}_\phi &= -\sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} - \sin^2\phi\sin^2\lambda\hat{y} + \sin\phi\sin\lambda\cos\phi\hat{z}
\end{aligned}$$

Subtract 1 and 3...

$$\begin{aligned}\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi &= \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} + \cos\phi\sin\lambda\sin\phi\hat{z}\\ &+ \sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} + \sin^2\phi\sin^2\lambda\hat{y} - \sin\phi\sin\lambda\cos\phi\hat{z}\\ &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y}\end{aligned}$$

Add 2.

$$\begin{aligned}\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y} - \cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y}\\ \hat{y} &= \cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda\end{aligned}$$

Thus...NORMAL

$$\begin{aligned}\hat{x} &= \sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda\\ \hat{y} &= \sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda\\ \hat{z} &= \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\end{aligned}$$

METEO

$$\begin{aligned}\hat{x} &= \cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda\\ \hat{y} &= \cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda\\ \hat{z} &= \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi\end{aligned}$$

3 ∇, or Gradient Operator

3.1 Gradient via Normal Physics Definition

Recall the definition of the gradient in cartesian coordinates

$$\nabla = \frac{\delta}{\delta x}\hat{x} + \frac{\delta}{\delta y}\hat{y} + \frac{\delta}{\delta z}\hat{z}$$

The mapping in normal physics for cartesian unit vectors is...

$$\begin{aligned}\hat{x} &= \sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda\\ \hat{y} &= \sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda\\ \hat{z} &= \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\end{aligned}$$

Now place these into the equation...

$$\begin{aligned}\nabla = &\left[\sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta x}\\ &+ \left[\sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta y}\\ &+ \left[\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right]\frac{\delta}{\delta z}\end{aligned}$$

Reorganizing terms such that unit vectors are consistent...

$$\begin{aligned}\nabla = &\left[\sin\theta\cos\lambda\frac{\delta}{\delta x} + \sin\theta\sin\lambda\frac{\delta}{\delta y} + \cos\theta\frac{\delta}{\delta z}\right]\hat{e}_r \\ &+ \left[\cos\theta\cos\lambda\frac{\delta}{\delta x} + \cos\theta\sin\lambda\frac{\delta}{\delta y} - \sin\phi\frac{\delta}{\delta z}\right]\hat{e}_\phi \\ &+ \left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]\hat{e}_\lambda\end{aligned}$$

Now remember our definition of cartesian coordinates for normal physics...

$$\begin{aligned}x &= R_e\sin\theta\cos\lambda \\ y &= R_e\sin\theta\sin\lambda \\ z &= R_e\cos\theta\end{aligned}$$

To transform the partial derivatives these into those corresponding with spherical coordinates, we can use the chain rule...

$$\begin{aligned}\frac{\delta}{\delta R_e} &= \frac{\delta x}{\delta R_e}\frac{\delta}{\delta x} + \frac{\delta y}{\delta R_e}\frac{\delta}{\delta y} + \frac{\delta z}{\delta R_e}\frac{\delta}{\delta z} \\ \frac{\delta}{\delta\theta} &= \frac{\delta x}{\delta\theta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\theta}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\theta}\frac{\delta}{\delta z} \\ \frac{\delta}{\delta\lambda} &= \frac{\delta x}{\delta\lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\lambda}\frac{\delta}{\delta z}\end{aligned}$$

Are the above operators consistent with these chain rules? FIRST TERM...

$$\frac{\delta}{\delta R_e} = \frac{\delta x}{\delta R_e}\frac{\delta}{\delta x} + \frac{\delta y}{\delta R_e}\frac{\delta}{\delta y} + \frac{\delta z}{\delta R_e}\frac{\delta}{\delta z} = \sin\theta\cos\lambda\frac{\delta}{\delta x} + \sin\theta\sin\lambda\frac{\delta}{\delta y} + \cos\theta\frac{\delta}{\delta z}$$

By inspection, the first term IS consistent. SECOND TERM

$$\frac{\delta}{\delta\theta} = \frac{\delta x}{\delta\theta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\theta}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\theta}\frac{\delta}{\delta z} = \cos\theta\cos\lambda\frac{\delta}{\delta x} + \cos\theta\sin\lambda\frac{\delta}{\delta y} - \sin\phi\frac{\delta}{\delta z}$$

By examination, these two are not the same. Viewing the third term indicates that we are missing a factor to make them equivalent.

$$\frac{\delta z}{\delta\phi}\frac{\delta}{\delta z} = -R_e\sin\phi\frac{\delta}{\delta z}$$

Thus, we need to multiply the second term by R_e to make these equivalent. So...our second term becomes

$$\frac{\delta}{\delta\theta} = \frac{\delta x}{\delta\theta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\theta}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\theta}\frac{\delta}{\delta z} = R_e\left[\cos\theta\cos\lambda\frac{\delta}{\delta x} + \cos\theta\sin\lambda\frac{\delta}{\delta y} - \sin\phi\frac{\delta}{\delta z}\right]$$

Third term

$$\frac{\delta}{\delta\lambda} = \frac{\delta x}{\delta\lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\lambda}\frac{\delta}{\delta z} = -\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}$$

As with the second term, this is not true. The third term needs a factor of $R_e \sin\theta$ to help equalize this formula. So...our third term becomes

$$\frac{\delta}{\delta\lambda} = \frac{\delta x}{\delta\lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\lambda}\frac{\delta}{\delta z} = R_e \sin\theta\left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]$$

Putting these terms all together yields the gradient usually found in normal physics textbooks.

$$\nabla = \frac{\delta}{\delta R_e}\hat{e}_r + \frac{1}{R_e}\frac{\delta}{\delta\theta}\hat{e}_\theta + \frac{1}{R_e\sin\theta}\frac{\delta}{\delta\lambda}\hat{e}_\lambda$$

3.2 Gradient via Meteo Definition

Recall the definition of the gradient in cartesian coordinates

$$\nabla = \frac{\delta}{\delta x}\hat{x} + \frac{\delta}{\delta y}\hat{y} + \frac{\delta}{\delta z}\hat{z}$$

The mapping in meteorology for cartesian unit vectors is...

$$\begin{aligned}
\hat{x} &= \cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda \\
\hat{y} &= \cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda \\
\hat{z} &= \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi
\end{aligned}$$

Now place these into the equation...

$$\begin{aligned}
\nabla = &\left[\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta x} \\
&+ \left[\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta y} \\
&+ \left[\sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi\right]\frac{\delta}{\delta z}
\end{aligned}$$

Reorganizing terms such that unit vectors are consistent...

$$\begin{aligned}
\nabla = &\left[\cos\phi\cos\lambda\frac{\delta}{\delta x} + \cos\phi\sin\lambda\frac{\delta}{\delta y} + \sin\phi\frac{\delta}{\delta z}\right]\hat{e}_r \\
&+ \left[-\sin\phi\cos\lambda\frac{\delta}{\delta x} - \sin\phi\sin\lambda\frac{\delta}{\delta y} + \cos\phi\frac{\delta}{\delta z}\right]\hat{e}_\phi \\
&+ \left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]\hat{e}_\lambda
\end{aligned}$$

Now remember our definition of cartesian coordinates for meteorology...

$$x = R_e \cos\phi \cos\lambda$$
$$y = R_e \cos\phi \sin\lambda$$
$$z = R_e \sin\phi$$

To transform the partial derivatives these into those corresponding with spherical coordinates, we can use the chain rule...

$$\frac{\delta}{\delta R_e} = \frac{\delta x}{\delta R_e}\frac{\delta}{\delta x} + \frac{\delta y}{\delta R_e}\frac{\delta}{\delta y} + \frac{\delta z}{\delta R_e}\frac{\delta}{\delta z}$$
$$\frac{\delta}{\delta \phi} = \frac{\delta x}{\delta \phi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \phi}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \phi}\frac{\delta}{\delta z}$$
$$\frac{\delta}{\delta \lambda} = \frac{\delta x}{\delta \lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \lambda}\frac{\delta}{\delta z}$$

Are the above operators consistent with these chain rules? FIRST TERM...

$$\frac{\delta}{\delta R_e} = \frac{\delta x}{\delta R_e}\frac{\delta}{\delta x} + \frac{\delta y}{\delta R_e}\frac{\delta}{\delta y} + \frac{\delta z}{\delta R_e}\frac{\delta}{\delta z} = \cos\phi\cos\lambda\frac{\delta}{\delta x} + \cos\phi\sin\lambda\frac{\delta}{\delta y} + \sin\phi\frac{\delta}{\delta z}$$

By inspection, the first term IS consistent. SECOND TERM

$$\frac{\delta}{\delta \phi} = \frac{\delta x}{\delta \phi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \phi}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \phi}\frac{\delta}{\delta z} = -\sin\phi\cos\lambda\frac{\delta}{\delta x} - \sin\phi\sin\lambda\frac{\delta}{\delta y} + \cos\phi\frac{\delta}{\delta z}$$

By examination, these two are not the same. Viewing the third term indicates that we are missing a factor to make them equivalent.

$$\frac{\delta z}{\delta \phi}\frac{\delta}{\delta z} = R_e\cos\phi\frac{\delta}{\delta z}$$

Thus, we need to multiply the second term by R_e to make these equivalent. So...our second term becomes

$$\frac{\delta}{\delta \phi} = \frac{\delta x}{\delta \phi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \phi}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \phi}\frac{\delta}{\delta z} = R_e\left[-\sin\phi\cos\lambda\frac{\delta}{\delta x} - \sin\phi\sin\lambda\frac{\delta}{\delta y} + \cos\phi\frac{\delta}{\delta z}\right]$$

Third term

$$\frac{\delta}{\delta \lambda} = \frac{\delta x}{\delta \lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \lambda}\frac{\delta}{\delta z} = -\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}$$

As with the second term, this is not true. The third term needs a factor of $R_e\cos\phi$ to help equalize this formula. So...our third term becomes

$$\frac{1}{R_e\cos\phi}\frac{\delta}{\delta \lambda} = \frac{\delta x}{\delta \lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \lambda}\frac{\delta}{\delta z} = R_e\cos\phi\left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]$$

Putting these terms all together yields the gradient usually found in meteorology textbooks.

$$\nabla = \frac{\delta}{\delta R_e}\hat{e}_r + \frac{1}{R_e}\frac{\delta}{\delta \phi}\hat{e}_\phi + \frac{1}{R_e\cos\phi}\frac{\delta}{\delta \lambda}\hat{e}_\lambda$$