1 Vertical Integration of Conservation of a scalar quantity in a general coordinate

Conservation of an arbitrary scalar quantity on an arbitrary vertical coordinate is the following

$$\frac{\delta}{\delta t}\left(m_\zeta \psi_\zeta\right) + \nabla_\zeta \cdot \left(m_\zeta \psi V_H\right) - \frac{\delta}{\delta \zeta}\left(m_\zeta \psi \dot{\zeta}\right) = \frac{\delta}{\delta \zeta} S_\psi \tag{1}$$

$$m_\zeta = \rho \frac{\delta z}{\delta \zeta} \equiv -\frac{1}{g}\frac{\delta p}{\delta \zeta} \tag{2}$$

$$\dot{\zeta} = \frac{d\zeta}{dt} = \left(\frac{\delta \zeta}{\delta t}\right)_\zeta + V_H \cdot \nabla_\zeta \zeta + \dot{\zeta} \tag{3}$$

Begin by defining the vertical integral between two ζ surfaces.

$$\int_{\zeta_S}^{\zeta_T} \frac{\delta}{\delta t}\left(m_\zeta \psi_\zeta\right)\delta\zeta + \int_{\zeta_S}^{\zeta_T} \nabla_\zeta \cdot \left(m_\zeta \psi V_H\right)\delta\zeta - \int_{\zeta_S}^{\zeta_T}\left(m_\zeta \psi \dot{\zeta}\right) = \int_{\zeta_S}^{\zeta_T} S_\psi \tag{4}$$

Now evaluating the integrals...

$$\int_{\zeta_S}^{\zeta_T} \frac{\delta}{\delta t}\left(m_\zeta \psi_\zeta\right)\delta\zeta + \int_{\zeta_S}^{\zeta_T} \nabla_\zeta \cdot \left(m_\zeta \psi V_H\right)\delta\zeta - \left[\left(m_\zeta \psi\right)_{\zeta_T}\dot{\zeta}_{\zeta_T} - \left(m_\zeta \psi\right)_{\zeta_S}\dot{\zeta}_{\zeta_S}\right] = S_{\psi,\zeta_T} - S_{\psi,\zeta_S}$$

$$\int_{\zeta_S}^{\zeta_T} \frac{\delta}{\delta t}\left(m_\zeta \psi_\zeta\right)\delta\zeta + \int_{\zeta_S}^{\zeta_T} \nabla_\zeta \cdot \left(m_\zeta \psi V_H\right)\delta\zeta - \left(m_\zeta \psi\right)_{\zeta_T}\dot{\zeta}_{\zeta_T} + \left(m_\zeta \psi\right)_{\zeta_S}\dot{\zeta}_{\zeta_S} = S_{\psi,\zeta_T} - S_{\psi,\zeta_S}$$

Now, assume that at the top of the atmosphere there is no source or sink of ϕ. Furthermore, assume that there is no vertical velocity (i.e., no matter passes through these surfaces) present at either the surface or the top of atmosphere

$$S_{\psi,\zeta_T} = 0$$
$$\dot{\zeta}_{\zeta_S} = \dot{\zeta}_{\zeta_T} = 0$$

Also assume The lower boundary condition at the surface...

$$\int_{\zeta_S}^{\zeta_T} \frac{\delta}{\delta t}\left(m_\zeta \psi_\zeta\right)\delta\zeta + \int_{\zeta_S}^{\zeta_T} \nabla_\zeta \cdot \left(m_\zeta \psi V_H\right)\delta\zeta = -S_{\psi,\zeta_S}$$

Using Leibniz rule, introduce the integrals into the other terms. Additionally, flip the bounds on the remaining integrals and sign on the surface source term...

$$\boxed{\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}\left(m_\zeta \psi_\zeta\right)\delta\zeta\right) + \nabla_\zeta \cdot \left(\int_{\zeta_T}^{\zeta_S} m_\zeta \psi V_H \delta\zeta\right) = S_{\psi,\zeta_S}} \tag{5}$$

This is Parazoo's equation 2.8

2 Eddy Decomposition of Vertical Integration

Now we can expand the horizontal terms of equation 5 into the spherical coordinate system

$$\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}\left(m_\zeta \psi_\zeta\right)\delta\zeta\right) + \frac{1}{acos\phi}\frac{\delta}{\delta\lambda}\left(\int_{\zeta_T}^{\zeta_S} m_\zeta \psi V_H \delta\zeta\right) + \frac{1}{a\cos\phi}\frac{\delta}{\delta\phi}\left(\int_{\zeta_T}^{\zeta_S} m_\zeta \psi V_H \cos\phi\delta\zeta\right) = S_{\psi,\zeta_S}$$

With this, apply a time mean to all components

$$\overline{\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}(m_\zeta\psi_\zeta)\,\delta\zeta\right)} + \overline{\frac{1}{acos\phi}\frac{\delta}{\delta\lambda}\left(\int_{\zeta_T}^{\zeta_S} m_\zeta\psi V_H\delta\zeta\right)} + \overline{\frac{1}{a\cos\phi}\frac{\delta}{\delta\phi}\left(\int_{\zeta_T}^{\zeta_S} m_\zeta\psi V_H\cos\phi\delta\zeta\right)} = \overline{S_{\psi,\zeta_S}}$$

Assuming the surface or top of atmosphere designations do not change in time, we can introduce the time averaging into the integrals

$$\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}\overline{(m_\zeta\psi_\zeta)}\delta\zeta\right) + \frac{1}{acos\phi}\frac{\delta}{\delta\lambda}\left(\int_{\zeta_T}^{\zeta_S}\overline{m_\zeta\psi V_H}\delta\zeta\right) + \frac{1}{a\cos\phi}\frac{\delta}{\delta\phi}\left(\int_{\zeta_T}^{\zeta_S}\overline{m_\zeta\psi V_H}\cos\phi\delta\zeta\right) = \overline{S_{\psi,\zeta_S}} \tag{6}$$

Now we can apply a zonal mean. This will eliminate the second term of equation 6, due to averaging around a latitude circle

$$\left[\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}\overline{(m_\zeta\psi_\zeta)}\delta\zeta\right)\right] + \left[\frac{1}{a\cos\phi}\frac{\delta}{\delta\phi}\left(\int_{\zeta_T}^{\zeta_S}\overline{m_\zeta\psi V_H}\cos\phi\delta\zeta\right)\right] = [\overline{S_{\psi,\zeta_S}}]$$

Following the same logic as the time mean operator, we can introduce the zonal mean into the integrand

$$\frac{\delta}{\delta t}\left(\int_{\zeta_T}^{\zeta_S}\left[\overline{(m_\zeta\psi_\zeta)}\right]\delta\zeta\right) + \frac{1}{a\cos\phi}\frac{\delta}{\delta\phi}\left(\int_{\zeta_T}^{\zeta_S}\left[\overline{m_\zeta\psi V_H}\right]\cos\phi\delta\zeta\right) = [\overline{S_{\psi,\zeta_S}}]$$

Now we can use eddy decomposition by using the following form...

$$[\overline{\phi\eta}] = [\overline{\phi}][\overline{\eta}] + [\overline{\phi}^*\overline{\eta}^*] + \overline{[\phi'][\eta']} + [\overline{\phi'^*\eta'^*}] \tag{7}$$

Using $(m_\zeta V_\zeta)$ and ψ for ϕ and η in equation 8, respectively.

$$\boxed{\left[\overline{(m_\zeta V_\zeta)\,\psi_\zeta}\right] = \overline{[(m_\zeta V_\zeta)][\psi_\zeta]} + \left[\overline{(m_\zeta V_\zeta)}^*\,\overline{\psi_\zeta}^*\right] + \overline{[(m_\zeta V_\zeta)']\,[\psi_\zeta']} + \left[\overline{(m_\zeta V_\zeta)'^*\psi_\zeta'^*}\right]} \tag{8}$$

On the RHS of equation 8 the terms mean the following. The first term is indicative of the mean meridional cirulation. The second term is associated with stationary eddies. The third term is the overturning meridional circulation (like the Hadley Cell). And the fourth term is the transient eddy term (like synoptic waves)