Note, this derivation largely mirrors the derivation for the virtual temperature. The only difference being that now the total density of air has the sum of dry air and all three phases of water instead of only water vapor.

Define the total air density as the sum between dry air, water vapor, liquid water, and ice

$$\rho = \rho_d + \rho_v + \rho_l + \rho_i = \frac{M_d + M_v + M_l + M_i}{V} \tag{1}$$

Furthermore, we will need Dalton's Law of Partial Pressures (use for gases only) and the ideal gas law configured for water vapor and dry air

$$P = P_d + e \tag{2}$$

$$P_d = \rho_d R_d T \tag{3}$$

$$e = \rho_v R_v T \tag{4}$$

where R_d and R_v are the gas constants for dry air and water vapor, respectively. Now, sub in 3 and 4 into 2

$$\begin{aligned} P &= \rho_d R_d T + \rho_v R_v T \\ &= T\left(\rho_d R_d + \rho_v R_v\right) \end{aligned} \tag{5}$$

Now using eqn 1, we can sub in the definitions for density of both gases into the eqn 5

$$\begin{aligned} P &= T\left(\frac{M_d}{V} R_d + \frac{M_v}{V} R_v\right) \\ &= \frac{T}{V}\left(M_d R_d + M_v R_v\right) \end{aligned} \tag{6}$$

Now, using the definition of the gas constants we can remove the universal gas constant term in 6

$$\begin{aligned} P &= \frac{T}{V}\left(M_d \frac{\Re}{m_d} + M_v \frac{\Re}{m_v}\right) \\ &= \frac{T\Re}{V}\left(\frac{M_d}{m_d} + \frac{M_v}{m_v}\right) \end{aligned} \tag{7}$$

Where m_d and m_v are the molecular mass of dry air and water vapor, respectively. Now, reusing eqn 1, we can exchange total density for the total volume.

$$P = \frac{\rho T \Re}{M_d + M_v + M_l + M_i}\left(\frac{M_d}{m_d} + \frac{M_v}{m_v}\right) \tag{8}$$

Now, remove the mass of dry air from the terms in the parenthesis. Then, rearrange the denominator in the fraction to remove the mass of dry air in eqn 8

$$\begin{aligned} P &= \frac{\rho T \Re M_d}{M_d(1 + \frac{M_v}{M_d} + \frac{M_l}{M_d} + \frac{M_i}{M_d})}\left(\frac{1}{m_d} + \frac{M_v}{M_d m_v}\right) \\ &= \frac{\rho T \Re}{(1 + r_v + r_l + r_i)}\left(\frac{1}{m_d} + \frac{r_v}{m_v}\right) \end{aligned} \tag{9}$$

$$r_v = \frac{M_v}{M_d} \tag{10}$$

$$r_l = \frac{M_l}{M_d} \tag{11}$$

$$r_i = \frac{M_l}{M_d} \tag{12}$$

Where r_v is the water vapor mixing ratio. Now multiply eqn 9 by 1 (a.k.a. $\frac{m_d}{m_d}$) and rearrange with the help of the definition of the dry gas constant.

$$\begin{aligned} P &= \frac{m_d \rho T \Re}{m_d(1 + r_v + r_l + r_i)} \left(\frac{1}{m_d} + \frac{r_v}{m_v} \right) \\ &= \frac{\rho T \Re}{m_d(1 + r_v + r_l + r_i)} \left(\frac{m_d}{m_d} + \frac{m_d r_v}{m_v} \right) \\ &= \frac{\rho T R_d}{(1 + r_v + r_l + r_i)} \left(1 + \frac{r_v}{\epsilon} \right) && (13) \\ \epsilon &= \frac{m_v}{m_d} && (14) \end{aligned}$$

You may see ϵ as the ratio between the dry air and water vapor gas constants, but it truly is the ratio of molecular masses of water vapor and dry air. One quick readjustment and the density temperature is derived.

$$\begin{aligned} P &= \rho R_d T_\rho \\ \therefore T_\rho &= T \frac{\left(1 + \frac{r_v}{\epsilon}\right)}{(1 + r_v + r_l + r_i)} && (15) \end{aligned}$$