From the derivation of virtual temperature, we know that contributions to pressure from water vapor and dry air becomes the following...

$$P = P_d + e$$
$$P = \rho R_m T \tag{1}$$
$$R_m = R_d \frac{\left(1 + \frac{r_v}{\epsilon}\right)}{(1 + r_v)} \tag{2}$$
$$r_v = \frac{M_v}{M_d} \tag{3}$$

This is slightly different than the normal virtual temperature equation, but all I've done is rearranged the moisture effect term to the gas constant instead of the temperature.

From the first law and enthalpy, we know that specific heats are additive due to their extensive nature. That is, it is the sum of the contributions of the specific heats scaled by mass fraction of the substance. With some rearranging we find the following.

$$\begin{aligned} C_{pm} &= \frac{M_d}{M_d + M_v} C_{pd} + \frac{M_v}{M_d + M_v} C_{pv} \\ &= \frac{1}{1 + r_v} C_{pd} + \frac{r_v}{1 + r_v} C_{pv} \\ C_{pm}(1 + r_v) &= C_{pd} + r_v C_{pv} \\ &= C_{pd}\left(1 + r_v \frac{C_{pv}}{C_{pd}}\right) \end{aligned} \tag{4}$$

From the first law, we know the following under adiabatic processes...

$$C_{vm} \frac{dT}{dt} + P \frac{d\alpha}{dt} = 0 \tag{5}$$

Taking the derivative with time of eqn 1...

$$\begin{aligned} P &= \rho R_m T \\ P\alpha &= R_m T \\ P \frac{d\alpha}{dt} &= -\alpha \frac{dP}{dT} + R_m \frac{dT}{dt} \end{aligned} \tag{6}$$

Subbing in eqn 6 into 4. Then rearranging...

$$\begin{aligned} C_{vm} \frac{dT}{dt} - \alpha \frac{dV}{dT} + R_m \frac{dT}{dt} &= 0 \\ C_{vm} \frac{dT}{dt} + R_m \frac{dT}{dt} &= \alpha \frac{dP}{dt} \\ C_{pm} \frac{dT}{dt} &= \alpha \frac{dP}{dt} \\ C_{pm} &= C_{vm} + R_m \end{aligned} \tag{7}$$

Now using 1 for the specific volume in 7.

$$C_{pm}\frac{dT}{dt} = \frac{R_m T}{P}\frac{dP}{dt}$$

$$\frac{1}{T}\frac{dT}{dt} = \kappa^*\frac{1}{P}\frac{dP}{dt} \tag{8}$$

$$\kappa^* = \frac{R_m}{C_{pm}} \tag{9}$$

Integrate both sides of 8

$$\int_{T_o}^{T_f}\frac{1}{T}dT = \int_{P_o}^{P_f}\frac{\kappa^*}{P}dP$$

$$ln(T)|_{T_o}^{T_f} = \kappa^* ln(P)|P_o{}^{P_f}$$

$$ln\left(\frac{T_f}{T_o}\right) = \kappa^* ln(\frac{P_f}{P_o})$$

$$ln\left(\frac{T_f}{T_o}\right) = ln\left(\frac{P_f}{P_o}\right)^{\kappa^*}$$

$$\frac{T_f}{T_o} = \left(\frac{P_f}{P_o}\right)^{\kappa^*}$$

$$T_f = T_o\left(\frac{P_f}{P_o}\right)^{\kappa^*} \tag{10}$$

Now taking a closer look at κ^* in eqn 9 and using the relation derived in eqn 4 and the definition of R_m in eqn 2...

$$\begin{aligned}\kappa^* &= \frac{R_m}{C_{pm}}\\ &= \frac{R_d\frac{\left(1+\frac{r_v}{\epsilon}\right)}{(1+r_v)}}{C_{pm}}\\ &= \frac{R_d\left(1+\frac{r_v}{\epsilon}\right)}{C_{pm}(1+r_v)}\\ &= \frac{R_d\left(1+\frac{r_v}{\epsilon}\right)}{C_{pd}\left(1+r_v\frac{C_{pv}}{C_{pd}}\right)}\end{aligned} \tag{11}$$

Subbing the new definition of κ^* into eqn 10

$$T_f = T_o\left(\frac{P_f}{P_o}\right)^{\frac{R_d\left(1+\frac{r_v}{\epsilon}\right)}{C_{pd}\left(1+r_v\frac{C_{pv}}{C_{pd}}\right)}} \tag{12}$$

If we define the initial temperature to be the temperature of an air parcel at some pressure and the final temperature to be the temperature when brought down to a reference pressure (such as the surface) then we have the definition of moist potential temperature

$$\theta_m = T_o\left(\frac{P_{sfc}}{P}\right)^{\frac{R_d\left(1+\frac{r_v}{\epsilon}\right)}{C_{pd}\left(1+r_v\frac{C_{pv}}{C_{pd}}\right)}} \tag{13}$$