NORTHWARD MASS FLUX DERIVATION

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The purpose of this quick study is to begin with the concept of a mass flux and derive the northward mass transport flux in height, pressure and general coordinates:

$$\int_0^{\frac{\pi}{2}} a\Psi_z \delta\phi$$
$$\int_0^{\frac{\pi}{2}} a\Psi_p \delta\phi$$
$$\int_0^{\frac{\pi}{2}} a\Psi_\zeta \delta\phi$$

Where Ψ is mass streamfunction.

To begin, the mass flux is defined as the air density times the velocity vector:

$$\rho\vec{V}$$

If we consider northward mass transport, then the only component of the wind vector needed is the meridional wind. Thus, $\vec{V} = v$. The total mass flux around a latitude band is the following in spherical coordinates:

$$\int_0^{2\pi} a\rho v \delta\lambda$$

The poleward transport of this mass around a latitude band in spherical coordinates:

$$\int_0^{\frac{\pi}{2}} a\cos\phi\delta\phi \int_0^{2\pi} a\rho v \delta\lambda$$

This is poleward transport of mass at a certain height level.

Height Coordinate

The total poleward transport of mass must take into account the entire mass of the atmosphere from the height surface to the top of the atmosphere. This implies:

$$\int_z^{\infty} \delta z \int_0^{\frac{\pi}{2}} a\cos\phi\delta\phi \int_0^{2\pi} a\rho v \delta\lambda = \int_z^{\infty} \int_0^{\frac{\pi}{2}} a\cos\phi \int_0^{2\pi} a\rho v \delta\lambda\delta\phi\delta z \qquad (1)$$

From this, we can use the definition of a zonal mean to eliminate the longitudinal integral:

$$\int_z^{\infty} \int_0^{\frac{\pi}{2}} 2\pi a\cos\phi a\left[\rho v\right]\delta\phi\delta z$$

Now we can use Leibniz's rule to switch the order of integration. We can rearrange the vertical integral before the latitudinal integral because the mass flux will be conserved regardless of order. Thus, we can say:

$$\int_0^{\frac{\pi}{2}} \int_z^{\infty} 2\pi a\cos\phi a\left[\rho v\right]\delta z\delta\phi$$

From here, we can first introduce a mass (a.k.a. Stoke's) streamfunction in a height coordinate system:

$$\Psi_z = \int_z^{\infty} 2\pi a \cos\phi \, [\rho v] \, \delta z$$

Using the mass streamfunction, we find the total northward transport in height coordinates as:

$$\int_0^{\frac{\pi}{2}} a\Psi_z \delta\phi$$

Pressure Coordinate

The extension of this derivation to other coordinate systems is straightforward. For pressure, we can begin by introducing a vertical coordinate transformation. From height to pressure coordinates we can use hydrostatic approximation and its integration over an vertical airmass to transform coordates:

$$\begin{aligned} -\rho g &= \frac{\delta p}{\delta z} \\ \rho \delta z &= -\frac{\delta p}{g} \\ \int_z^{\infty} \rho \delta z &= -\int_p^0 \frac{\delta p}{g} \\ &= \int_0^p \frac{\delta p}{g} \end{aligned}$$

Now stubstituting this into equation (1):

$$\int_z^{\infty} \int_0^{\frac{\pi}{2}} a\cos\phi \int_0^{2\pi} a\rho v \delta\lambda\delta\phi\delta z = \int_0^p \int_0^{\frac{\pi}{2}} a\cos\phi \int_0^{2\pi} av\delta\lambda\delta\phi\delta p$$

Now we can follow a similar proceedure to the height coordinate. Start by using a zonal mean integration.

$$\int_0^p \int_0^{\frac{\pi}{2}} 2\pi a \cos\phi a[v]\delta\phi\delta p$$

Next, switch the order of integration using Leibniz's rule:

$$\int_0^{\frac{\pi}{2}} 2\pi a \cos\phi \int_0^p a[v]\delta p\delta\phi$$

Then, define a mass streamfunction in a pressure coordinate system:

$$\Psi_p = 2\pi a \cos\phi \int_0^p [v]\delta p$$

Thus, the total northward transport in pressure coordinates is:

$$\int_0^{\frac{\pi}{2}} a\Psi_p \delta\phi$$

General Vertical Coordinate

The extension of this derivation to a general coordinate system is nearly as straightforward as a pressure coordinate. First, we can introduce a new generic coordinate system, ζ, into equation (1):

$$\int_z^\infty \int_0^{\frac{\pi}{2}} a\cos\phi \int_0^{2\pi} a\rho v \delta\lambda\delta\phi\delta z \frac{\delta\zeta}{\delta\zeta}$$

Next, introduce the concept of a pseudo-density, m_ζ:

$$m_\zeta = \rho\frac{\delta z}{\delta\zeta}$$

If we employ the hydrostatic approximation the pseudo-density becomes:

$$m_\zeta = -\frac{\delta p}{g}$$

Now vertically integrate these terms from a height surface to the top of the atmosphere:

$$m_\zeta\delta\zeta = \rho\delta z$$
$$\int_{\zeta_s}^{\zeta_T} m_\zeta\delta\zeta = \int_z^\infty \rho\delta z$$

With these definitions, we can change the coordinate system.

$$\int_{\zeta_s}^{\zeta_T} \int_0^{\frac{\pi}{2}} a\cos\phi \int_0^{2\pi} a m_\zeta v \delta\lambda\delta\phi\delta\zeta$$

Next, we can take the zonal mean:

$$\int_{\zeta_s}^{\zeta_T} \int_0^{\frac{\pi}{2}} 2\pi a\cos\phi a[m_\zeta v]\delta\phi\delta\zeta$$

After that, we can switch the limits of integration using Leibniz's rule:

$$\int_0^{\frac{\pi}{2}} 2\pi a\cos\phi \int_{\zeta_s}^{\zeta_T} a[m_\zeta v]\delta\zeta\delta\phi$$

Now introduce a mass streamfunction in a general coordinate system:

$$\Psi_\zeta = 2\pi a\cos\phi \int_{\zeta_s}^{\zeta_T} [m_\zeta v]\delta\zeta$$

Insert this into our derivation:

$$\int_0^{\frac{\pi}{2}} a\Psi_\zeta\delta\phi$$

General Coordinate System and Presure

There is actually an extension that we can make if we use a hydrostatic atmosphere. In a hydrostatic atmosphere, the pseudo-density can be defined in another manner:

$$m_\zeta = -\frac{\delta p}{g}$$

Following the same derivation as in the pressure coordinate, we find the mass streamfunction in a general coordinate system is identical to that as in the pressure coordinate system.

$$\Psi_\zeta = 2\pi a \cos\phi \int_0^p [v]\delta p$$

This equation would suffice if the general coordinate system mimics the pressure system, however is usually not true (e.g., isentropic). This equation can be adjusted extended for usage in a general coordinate system by utilizing the concept of massless layers. The concept states that any coordinate that intercepts the ground could actually follow the surface in a stack of massless layers. As such, no continuity error arise due to a discontinuity. This can be introduced into this equation by using the Heaviside function:

$$\Psi_\zeta = 2\pi a \cos\phi \int_0^p H(\zeta_o - \zeta)[v]\delta p$$

The Heaviside function allows the pressure variant of the mass streamfunction to be calculated only in the atmosphere where $\zeta \geq \zeta_o$.