Northward tracer or thermal transport is partitioning of total transport using Reynold's decomposition (i.e., eddy). This Quick Study will follow how to derive the various components as outlined in the climate physics classic text book by Piexoto and Oort in 1992. This method is a fairly straightforward derivation that is repeated here for brevity and completeness.

1 General Variable Decomposition

Any reynold's averaging decomposition is a method that describes any variable as the sum of a mean and the deviation from the mean. Given the proper assumptions, this variable can be decomposed in any dimension. For instance, this can be applied in both time and space.

$$\psi(\lambda,\phi,\zeta,t) = \overline{\psi}(\lambda,\phi,\zeta) + \psi'(\lambda,\phi,\zeta,t) \tag{1}$$
$$\psi(\lambda,\phi,\zeta,t) = [\psi]\,(\phi,\zeta,t) + \psi^*(\lambda,\phi,\zeta,t) \tag{2}$$

Where ϕ is an 4-dimensional arbitrary variable with typical horizontal coordinates, an arbitrary vertical coordinate system (ζ), and time. Equation 1 decomposes the variable into contributions from the time mean and the "transient" component. Equation 2 decomposes the variable into contributions from a zonal mean and eddies. It is important to note that when a dimensional deviation component is averaged over that dimension, the mean of a deviation is zero.

$$\overline{\phi'} = 0$$
$$[\phi^*] = 0$$

2 Combination of Two General Quantities

Say we now have two variables being multiplied together to create a new quantity, such as in tracer transport from airmass transport and a gas concentration. The new quantity can be the multiplication of both variables as decomposed in either dimension.

$$\begin{aligned}\psi\eta &= \left(\overline{\psi} + \psi'\right)\left(\overline{\eta} + \eta'\right)\\ &= \overline{\psi}\overline{\eta} + \overline{\psi}\eta' + \psi'\overline{\eta} + \psi'\eta' \end{aligned}\tag{3}$$
$$\begin{aligned}\psi\eta &= \left([\psi] + \psi^*\right)\left([\eta] + \eta^*\right)\\ &= [\psi]\,[\eta] + [\psi]\,\eta^* + \psi^*\,[\eta] + \psi^*\eta^* \end{aligned}\tag{4}$$

The instantaneous value created by equations 3 or 4 are usually not what we are interested since most processes described under Reynolds averaging operate over large spatiotemporal scales. This means the mean of this instantaneous value is what we are interested in. For instance, the quantity in equation 3 integrated over time becomes...

$$\begin{aligned}\overline{\psi\eta} &= \overline{\overline{\psi}\overline{\eta} + \overline{\psi}\eta' + \psi'\overline{\eta} + \psi'\eta'}\\ &= \overline{\overline{\psi}\overline{\eta}} + \overline{\overline{\psi}\eta'} + \overline{\psi'\overline{\eta}} + \overline{\psi'\eta'}\\ &= \overline{\psi}\overline{\eta} + \overline{\psi'\eta'} \end{aligned}\tag{5}$$

The first term represents the time mean while the second quantity represents the temporal covariance of the two quantities.

3 Extension Over Two Dimensions

Reynolds decomposition may be applied over two dimensions, so long as they both exist within the scope of the variable. For instance, the example quantity, $\psi\eta$, can be decomposed both temporally and zonally since it has both dimensions. The first step is to decompose and reynolds average the quanity in one of the two dimensions as in equation 5. Next, the time mean term may be decomposed and reynolds averaged over the next dimension as a new quantity. For instance, the time mean $\overline{\psi\eta}$ may be zonally averaged in the following manner

$$\begin{aligned}\overline{\psi}\overline{\eta} &= \left(\left[\overline{\psi}\right] + \overline{\psi}^*\right)\left([\overline{\eta}] + \overline{\eta}^*\right) \\ &= \left[\overline{\psi}\right][\overline{\eta}] + \left[\overline{\psi}\right]\overline{\eta}^* + \overline{\psi}^*[\overline{\eta}] + \overline{\psi}^*\overline{\eta}^* \end{aligned} \tag{6}$$

Substitution of equation 6 into 5 and zonally averaging yields

$$\left[\overline{\psi\eta}\right] = \left[\overline{\psi}\right][\overline{\eta}] + \left[\overline{\psi}^*\overline{\eta}^*\right] + \left[\overline{\psi'\eta'}\right] \tag{7}$$

The terms of equation 7 are more sensible if the dummy variables are replaced with physical quantities, such as those for transport. That is airmass transport for ψ, $\psi = mv$, and CO_2 for η, $\eta = C$

$$\left[\overline{mvC}\right] = [\overline{mv}]\left[\overline{C}\right] + \left[\overline{mv}^*\overline{C}^*\right] + \left[\overline{(mv)'C'}\right] \tag{8}$$

On the right hand side of equation 8, the northward transport of CO_2 these terms can be described as the contributions from the mean meridional circulation, stationary eddies, and transient eddies.

4 Equivalence and Permutability

The same decomposition may be utilized if the spatial dimension is chosen first.

$$[\psi\eta] = [\psi][\eta] + [\psi^*\eta^*] \tag{9}$$

Following the same proceedure producing equations 6 and 7, we find a functionally similar product to 8

$$\overline{[\psi\eta]} = \overline{[\psi][\eta]} + \overline{[\psi]'[\eta]'} + \overline{[\psi^*\eta^*]} \tag{10}$$

Since time and zonal mean operators are both permutable, the right hand side quantities and first term on the left hand side in equations 7 and 10 are the equivalent

$$\begin{aligned}\overline{[\psi\eta]} &= \left[\overline{\psi\eta}\right] \\ \overline{[\psi][\eta]} &= \left[\overline{\psi}\right][\overline{\eta}]\end{aligned}$$

That implies that the remaining two terms in equations 7 and 10 are in equivalent.

$$\overline{[\psi]'[\eta]'} + \overline{[\psi^*\eta^*]} = \left[\overline{\psi}^*\overline{\eta}^*\right] + \left[\overline{\psi'\eta'}\right] \tag{11}$$

This is proven by expanding the second term on both sides of equation 11 into the unaccounted for dimension. The second term on the left hand side is decomposed using the time dimension first.

$$
\begin{aligned}
\psi^*\eta^* &= \overline{\psi^*}\,\overline{\eta^*} + \overline{\psi^*}\eta'^* + \psi'^*\overline{\eta^*} + \psi'^*\eta'^* \\
[\psi^*\eta^*] &= \left[\overline{\psi^*}\,\overline{\eta^*}\right] + \left[\overline{\psi^*}\eta'^*\right] + \left[\psi'^*\overline{\eta^*}\right] + [\psi'^*\eta'^*] \\
\overline{[\psi^*\eta^*]} &= \left[\overline{\psi^*}\,\overline{\eta^*}\right] + \overline{[\psi'^*\eta'^*]}
\end{aligned}
\tag{12}
$$

The second term on the right hand side is decomposed using the spatial dimension first.

$$
\begin{aligned}
\psi'\eta' &= [\psi']\,[\eta'] + [\psi']\,\eta'^* + \psi'^*\,[\eta'] + \psi'^*\eta'^* \\
\overline{\psi'\eta'} &= \overline{[\psi']\,[\eta']} + \overline{[\psi']\,\eta'^*} + \overline{\psi'^*\,[\eta']} + \overline{\psi'^*\eta'^*} \\
\left[\overline{\psi'\eta'}\right] &= \overline{[\psi']\,[\eta']} + \left[\overline{\psi'^*\eta'^*}\right]
\end{aligned}
\tag{13}
$$

When equations 12 and 13 are substituted into equation 11 and when permutability is used, we find both methods of decompositions are identical to one another.

$$
\overline{[\psi]'\,[\eta]'} + \left[\overline{\psi^*}\,\overline{\eta^*}\right] + \overline{[\psi'^*\eta'^*]} = \left[\overline{\psi}^*\,\overline{\eta}^*\right] + \overline{[\psi']\,[\eta']} + \left[\overline{\psi'^*\eta'^*}\right]
\tag{14}
$$

The choice of which method is most useful is often relevant to the physical question being asked and the computational needs.