1 Time Mean Decomposition

Reynolds decomposition is a method of separating a value or set of values from the mean and their deviation from the mean. As such we can do this in a variety of ways. First, think of any particular value in 4-D space.

$$\phi(x, y, \zeta, t) \tag{1}$$

Where we have an arbitrary vertical coordinate system (ζ) and time. We can begin by decomposing this particular value into its time mean and deviation of the mean in time or "transient" component.

$$\phi(x, y, \zeta, t) = \overline{\phi}(x, y, \zeta) + \phi'(x, y, \zeta, t) \tag{2}$$

Such that

$$\overline{\phi'} = 0$$

2 Zonal Mean Decomposition

In a similar manner equation 1 can be decomposed in the zonal direction (east-west). In this way, we have a zonal mean and the deviations are known as eddies rather than transients

$$\phi(x, y, \zeta, t) = [\phi]\,(y, \zeta, t) + \phi^*(x, y, \zeta, t) \tag{3}$$

Such that

$$[\phi^*] = 0$$

NOTE: this is only valid for geostrophically and hydrostatically valid flow...

3 Combination of two quantities

Say we now have 2 values being scaled together such as in tracer transport (wind velocity and concentration). The concepts of reynolds decomposition can be applied to both...

$$\begin{aligned} \phi\eta &= \left(\overline{\phi} + \phi'\right)\left(\overline{\eta} + \eta'\right) \\ &= \overline{\phi}\,\overline{\eta} + \overline{\phi}\eta' + \phi'\overline{\eta} + \phi'\eta' \end{aligned} \tag{4}$$

The value in 4 is not always the value we are interested in. Sometimes we are interested at the time mean of this scalar quantity in which case equation 4 becomes...

$$\begin{aligned} \overline{\phi\eta} &= \overline{\overline{\phi}\,\overline{\eta} + \overline{\phi}\eta' + \phi'\overline{\eta} + \phi'\eta'} \\ &= \overline{\overline{\phi}\,\overline{\eta}} + \overline{\overline{\phi}\eta'} + \overline{\phi'\overline{\eta}} + \overline{\phi'\eta'} \end{aligned}$$

Recall how the mean of the mean is the mean and that the mean of deviations from the mean are 0...

$$\begin{aligned} \overline{\phi\eta} &= \overline{\phi}\,\overline{\eta} + \overline{\phi}\,\overline{\eta'} + \overline{\phi'}\,\overline{\eta} + \overline{\phi'\eta'} \\ &= \overline{\phi}\,\overline{\eta} + \overline{\phi'\eta'} \end{aligned} \tag{5}$$

The first term represents the time mean while the second quantity represents the temporal covariance of the two quantities.

4 Extension to a second dimension

If we apply what have been shown in equations 2 through 5, we can expand this to time and zonal decompositions. Starting with the first term in 5, a zonal decomposition would be...

$$\begin{aligned}
[\overline{\phi\eta}] &= \left[\overline{([\phi] + \phi^*)([\eta] + \eta^*)}\right] \\
&= \left[\left(\overline{[\phi]} + \overline{\phi^*}\right)\left(\overline{[\eta]} + \overline{\eta^*}\right)\right] \\
&= \left[\overline{[\phi]}\,\overline{[\eta]} + \overline{[\phi]}\,\overline{\eta^*} + \overline{\phi^*}\,\overline{[\eta]} + \overline{\phi^*}\,\overline{\eta^*}\right] \\
&= \left[\overline{[\phi]}\,\overline{[\eta]}\right] + \left[\overline{[\phi]}\,\overline{\eta^*}\right] + \left[\overline{\phi^*}\,\overline{[\eta]}\right] + [\overline{\phi^*}\,\overline{\eta^*}] \\
&= \left[\overline{[\phi]}\,\overline{[\eta]}\right] + \overline{[\phi]}\,[\overline{\eta^*}] + [\overline{\phi^*}]\,\overline{[\eta]} + [\overline{\phi^*}\,\overline{\eta^*}] \\
&= \overline{[\phi]}\,\overline{[\eta]} + [\overline{\phi^*}\,\overline{\eta^*}]
\end{aligned} \tag{6}$$

Now for the second term...

$$\begin{aligned}
[\overline{\phi'\eta'}] &= \left[\overline{([\phi'] + \phi'^*)([\eta'] + \eta'^*)}\right] \\
&= \left[\left(\overline{[\phi']} + \overline{\phi'^*}\right)\left(\overline{[\eta']} + \overline{\eta'^*}\right)\right]
\end{aligned} \tag{7}$$

Note that the time deviations, although time averaged, will not go to zero, this is due to the zonal decompositon.

$$\begin{aligned}
[\overline{\phi'\eta'}] &= \left[\overline{[\phi']}\,\overline{[\eta']} + \overline{[\phi']}\,\overline{\eta'^*} + \overline{\phi'^*}\,\overline{[\eta']} + \overline{\phi'^*}\,\overline{\eta'^*}\right] \\
&= \left[\overline{[\phi']}\,\overline{[\eta']}\right] + \left[\overline{[\phi']}\,\overline{\eta'^*}\right] + \left[\overline{\phi'^*}\,\overline{[\eta']}\right] + [\overline{\phi'^*}\,\overline{\eta'^*}] \\
&= \left[\overline{[\phi']}\,\overline{[\eta']}\right] + \overline{[\phi']}\,[\overline{\eta'^*}] + [\overline{\phi'^*}]\,\overline{[\eta']} + [\overline{\phi'^*}\,\overline{\eta'^*}] \\
&= \overline{[\phi']}\,\overline{[\eta']} + [\overline{\phi'^*}\,\overline{\eta'^*}]
\end{aligned} \tag{8}$$

Subbing in equations 6 and 8 into 5, we see the following...

$$\begin{aligned}
[\overline{\phi\eta}] &= [\overline{\phi}\,\overline{\eta}] + [\overline{\phi'\eta'}] \\
&= \overline{[\phi]}\,\overline{[\eta]} + [\overline{\phi^*}\,\overline{\eta^*}] + \overline{[\phi']}\,\overline{[\eta']} + [\overline{\phi'^*}\,\overline{\eta'^*}]
\end{aligned} \tag{9}$$