1 Equations Needed

To obtain the bousinessq approximation we will need the 3-D Navier-Stokes Equations

$$\frac{\delta w}{\delta t} + w \cdot \nabla \vec{w} = -\frac{1}{\rho}\frac{\delta P}{\delta z} - g_z + \nu \nabla^2 w \tag{1}$$

$$\frac{\delta v}{\delta t} + v \cdot \nabla \vec{v} = -\frac{1}{\rho}\frac{\delta P}{\delta y} + \nu \nabla^2 v \tag{2}$$

$$\frac{\delta u}{\delta t} + u \cdot \nabla \vec{u} = -\frac{1}{\rho}\frac{\delta P}{\delta x} + \nu \nabla^2 u \tag{3}$$

$$P = \rho R T \tag{4}$$

$$\theta = T\left(\frac{P_s}{P}\right)^{\frac{R}{C_p}} = T\left(\frac{P_s}{P}\right)^{\kappa} \tag{5}$$

$$\frac{\delta \rho}{\delta t} + \nabla \cdot \rho \vec{v} = 0 \tag{6}$$

2 Decomposition

Assume that the velocity, pressure, temperature, and density are composed of a long term mean and a perturbation from the mean. Additionally, the vertical velocity mean on a synoptic scale is very close to zero. Meaning

$$\zeta = \zeta_o + \zeta'(x, y, z, t) \tag{7}$$

3 Equation of state

Start by applying equation 7 to the density, pressure, and temperature terms of the equation of state (equation 4)

$$\begin{aligned} P_o + P' &= (\rho_o + \rho')R(T_o + T') \\ &= R\left(\rho_o T_o + \rho_o T' + \rho' T_o + \rho' T'\right) \end{aligned} \tag{8}$$

Now take the mean of equation 8

$$\begin{aligned} \overline{P_o + P'} &= \overline{R\left(\rho_o T_o + \rho_o T' + \rho' T_o + \rho' T'\right)} \\ \overline{P_o} + \overline{P'} &= R\left(\overline{\rho_o T_o} + \overline{\rho_o T'} + \overline{\rho' T_o} + \overline{\rho' T'}\right) \\ P_o &= R\left(\rho_o T_o + \overline{\rho' T'}\right) \end{aligned} \tag{9}$$

With this in mind divide equation 8 through by P_o and sub in equation 9

$$\begin{aligned}
P_o + P' &= R\left(\rho_o T_o + \rho_o T' + \rho' T_o + \rho' T'\right) \\
1 + \frac{P'}{P_o} &= \frac{R\left(\rho_o T_o + \rho_o T' + \rho' T_o + \rho' T'\right)}{P_o} \\
&= \frac{R\left(\rho_o T_o + \rho_o T' + \rho' T_o + \rho' T'\right)}{R\left(\rho_o T_o + \overline{\rho' T'}\right)} \\
&= \frac{\rho_o T_o}{\left(\rho_o T_o + \overline{\rho' T'}\right)} + \frac{\rho_o T'}{\left(\rho_o T_o + \overline{\rho' T'}\right)} + \frac{\rho' T_o}{\left(\rho_o T_o + \overline{\rho' T'}\right)} + \frac{\rho' T'}{\left(\rho_o T_o + \overline{\rho' T'}\right)} \\
&= \frac{1}{\left(1 + \frac{\overline{\rho' T'}}{\rho_o T_o}\right)} + \frac{T'}{\left(T_o + \frac{\overline{\rho' T'}}{\rho_o}\right)} + \frac{\rho'}{\left(\rho_o + \frac{\overline{\rho' T'}}{T_o}\right)} + \frac{\rho' T'}{\left(\rho_o T_o + \overline{\rho' T'}\right)}
\end{aligned} \tag{10}$$

If we assume that the covariance of the perturbations is much less than the product of the two mean terms (i.e. $\overline{\rho' T'} << \rho_o T_o$) we can deduce the following...

$$\begin{aligned}
\frac{\overline{\rho' T'}}{T_o} &<< \rho_o \\
\frac{\overline{\rho' T'}}{\rho_o} &<< T_o
\end{aligned}$$

These relations reduce equation 10 to the following...

$$\begin{aligned}
1 + \frac{P'}{P_o} &= 1 + \frac{T'}{T_o} + \frac{\rho'}{\rho_o} + \frac{\rho' T'}{\rho_o T_o} \\
\frac{P'}{P_o} &= \frac{T'}{T_o} + \frac{\rho'}{\rho_o} + \frac{\rho' T'}{\rho_o T_o}
\end{aligned} \tag{11}$$

If we use the shallow convection approximation, which states that pressure perturbations in the vertical are really small in comparison to the mean state, then we can ignore the pressure term in 11

$$\begin{aligned}
\frac{\rho'}{\rho_o} &= -\frac{T'}{T_o} \\
&= -\frac{\theta'_\rho}{\theta_{\rho,o}}
\end{aligned}$$

Vertical Motion

Starting with equation 1, expand the pressure and density terms

$$\begin{aligned}
\frac{\delta w}{\delta t} + w \cdot \nabla \vec{w} &= -\frac{1}{(\rho_o + \rho')}\frac{\delta (P_o + P')}{\delta z} - g + \nu \nabla^2 w \\
&= -\frac{1}{(\rho_o + \rho')}\frac{\delta P_o}{\delta z} - \frac{1}{(\rho_o + \rho')}\frac{\delta P'}{\delta z} - \frac{(\rho_o + \rho')}{(\rho_o + \rho')} g + \nu \nabla^2 w \\
&= \frac{1}{(\rho_o + \rho')}\left(-\frac{\delta P_o}{\delta z} - \frac{\delta P'}{\delta z} - \rho_o g - \rho' g \right) + \nu \nabla^2 w \qquad (12)
\end{aligned}$$

Now in equation 12 assume the hydrostatic approximation using the background pressure and density $\frac{\delta P_o}{\delta z} = -\rho_o g$

$$\begin{aligned}
\frac{\delta w}{\delta t} + w \cdot \nabla \vec{w} &= \frac{1}{(\rho_o + \rho')}\left(-\frac{\delta P_o}{\delta z} - \frac{\delta P'}{\delta z} + \frac{\delta P_o}{\delta z} - \rho' g \right) + \nu \nabla^2 w \\
&= -\frac{1}{(\rho_o + \rho')}\left(\frac{\delta P'}{\delta z} + \rho' g \right) + \nu \nabla^2 w \qquad (13)
\end{aligned}$$

In equation 13, expand the terms again and rearrange...

$$\frac{\delta w}{\delta t} + w \cdot \nabla \vec{w} = -\frac{1}{\rho_o(1 + \frac{\rho'}{\rho_o})}\left(\frac{\delta P'}{\delta z} + \rho' g \right) + \nu \nabla^2 w$$

If we assume that density perturbations are much much smaller then the mean state we can invoke the following,

$$\rho' << \rho_o$$

Therefore, $(1 + \frac{\rho'}{\rho_o}) \approx 1$. Thus,

$$\begin{aligned}
\frac{\delta w}{\delta t} + w \cdot \nabla \vec{w} &= -\frac{1}{\rho_o}\left(\frac{\delta P'}{\delta z} + \rho' g \right) + \nu \nabla^2 w \\
&= -\frac{1}{\rho_o}\frac{\delta P'}{\delta z} - \frac{\rho'}{\rho_o} g + \nu \nabla^2 w \\
&= -\frac{1}{\rho_o}\frac{\delta P'}{\delta z} + B + \nu \nabla^2 w \qquad (14) \\
B &= -\frac{\rho'}{\rho_o} g \qquad (15)
\end{aligned}$$

Horizontal Motion

Combine the horizontal winds in equations 3 and 4 follow the same basic steps as in the Vertical Motion section

$$\begin{aligned}\frac{\delta \vec{V}_H}{\delta t} + \vec{V}_H \cdot \vec{\nabla}\vec{V}_H &= -\frac{1}{\rho}\frac{\delta P}{\delta x} - \frac{1}{\rho}\frac{\delta P}{\delta y} + \nu\nabla^2\vec{V}_H \\ &= -\frac{1}{(\rho_o+\rho')}\frac{\delta(P_o+P')}{\delta x} - \frac{1}{(\rho_o+\rho')}\frac{\delta(P_o+P')}{\delta y} + \nu\nabla^2 V_H \\ &= -\frac{1}{(\rho_o+\rho')}\frac{\delta(P_o)}{\delta x} - \frac{1}{(\rho_o+\rho')}\frac{\delta P'}{\delta x} - \frac{1}{(\rho_o+\rho')}\frac{\delta(P_o)}{\delta y} - \frac{1}{(\rho_o+\rho')}\frac{\delta P'}{\delta y} + \nu\nabla^2\vec{V}_H \end{aligned} \tag{16}$$

Since we are assuming horizontal homogenity then...

$$\frac{\delta(P_o)}{\delta x} = \frac{\delta(P_o)}{\delta y} = 0 \tag{17}$$

Place equation 17 into 16

$$\begin{aligned}\frac{\delta \vec{V}_H}{\delta t} + \vec{V}_H \cdot \vec{\nabla}\vec{V}_H &= -\frac{1}{(\rho_o+\rho')}\frac{\delta P'}{\delta x} - \frac{1}{(\rho_o+\rho')}\frac{\delta P'}{\delta y} + \nu\nabla^2\vec{V}_H \\ &= -\frac{1}{\rho_o(1+\frac{\rho'}{\rho_o})}\frac{\delta P'}{\delta x} - \frac{1}{\rho_o(1+\frac{\rho'}{\rho_o})}\frac{\delta P'}{\delta y} + \nu\nabla^2\vec{V}_H \\ &= -\frac{1}{\rho_o}\frac{\delta P'}{\delta x} - \frac{1}{\rho_o}\frac{\delta P'}{\delta y} + \nu\nabla^2\vec{V}_H \end{aligned} \tag{18}$$

4 OTHER

$$\vec{\nabla}^2 p'_d = -\vec{\nabla}\cdot(\rho_o\vec{v}\cdot\nabla\vec{v}) + \vec{\nabla}\cdot(\rho_o\vec{F})$$

$$\vec{\nabla}^2 p'_d = -\rho_o\left[\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta y}\right)^2 + \left(\frac{\delta w}{\delta z}\right)^2 - \frac{d^2 ln\rho_o}{dz^2}w^2\right] - 2\rho_o\left[\frac{\delta v}{\delta x}\frac{\delta u}{\delta y} + \frac{\delta w}{\delta x}\frac{\delta u}{\delta z} + \frac{\delta w}{\delta y}\frac{\delta v}{\delta z}\right] + \vec{\nabla}\cdot(\rho_o\vec{F})$$

$$\begin{aligned}\vec{v}\cdot\nabla\vec{v} &= \frac{1}{2}\nabla\vec{v}\cdot\vec{v} - \vec{v}\times\nabla\times\vec{v} \\ &= \frac{1}{2}\nabla\vec{v}\cdot\vec{v} - \vec{v}\times\omega\end{aligned}$$

$$\begin{aligned}&\omega = \left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right)i + \left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right)j + \left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)k \\ &\omega = \omega_x i + \omega_y j + \omega_z k\end{aligned}$$

$$\begin{aligned}
\vec{v} \times \omega &= (v\omega_z - w\omega_y)\, i + (u\omega_z - w\omega_x)\, j + (v\omega_x - u\omega_y)\, k \\
(v\omega_z - w\omega_y)\, i &= \left[v\frac{\delta v}{\delta x} - v\frac{\delta u}{\delta y} - w\frac{\delta u}{\delta z} + w\frac{\delta w}{\delta x} \right] i \\
(u\omega_z - w\omega_x)\, j &= \left[u\frac{\delta v}{\delta x} - u\frac{\delta u}{\delta y} - w\frac{\delta w}{\delta y} + w\frac{\delta v}{\delta z} \right] j \\
(v\omega_x - u\omega_y)\, k &= \left[v\frac{\delta w}{\delta y} - v\frac{\delta v}{\delta z} - u\frac{\delta u}{\delta z} + u\frac{\delta w}{\delta x} \right] k
\end{aligned}$$

$$\frac{1}{2}\nabla\vec{v}\cdot\vec{v} = \frac{1}{2}\left[\frac{\delta}{\delta x}(uu+vv+ww)i + \frac{\delta}{\delta y}(uu+vv+ww)j + \frac{\delta}{\delta z}(uu+vv+ww)k \right]$$

$$\begin{aligned}
\nabla\cdot(\vec{v}\times\omega) &= \vec{\omega}\cdot(\nabla\times\vec{v}) - \vec{v}\cdot(\nabla\times\vec{\omega}) \\
\vec{\omega}\cdot\nabla\times\vec{v} &= \vec{\omega}\cdot\vec{\omega} \\
&= \omega_x\omega_x + \omega_y\omega_y + \omega_z\omega_z \\
\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right)\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right) &= \frac{\delta w}{\delta y}\frac{\delta w}{\delta y} + \frac{\delta v}{\delta z}\frac{\delta v}{\delta z} - 2\frac{\delta w}{\delta y}\frac{\delta v}{\delta z} \\
\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right)\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right) &= \frac{\delta u}{\delta z}\frac{\delta u}{\delta z} + \frac{\delta w}{\delta x}\frac{\delta w}{\delta x} - 2\frac{\delta w}{\delta x}\frac{\delta u}{\delta z} \\
\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right) &= \frac{\delta v}{\delta x}\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\frac{\delta u}{\delta y} - 2\frac{\delta v}{\delta x}\frac{\delta u}{\delta y} \\
\vec{v}\cdot(\nabla\times\vec{\omega}) &= (ui + vj + wk)\cdot \\
\nabla\times\vec{\omega} &= \left(\frac{\delta\omega_z}{\delta y} - \frac{\delta\omega_y}{\delta z}\right) i + \left(\frac{\delta\omega_x}{\delta z} - \frac{\delta\omega_z}{\delta x}\right) j + \left(\frac{\delta\omega_y}{\delta x} - \frac{\delta\omega_x}{\delta x}\right) k \\
\left(\frac{\delta\omega_z}{\delta y} - \frac{\delta\omega_y}{\delta z}\right) &= \left(\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right) - \frac{\delta}{\delta z}\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right)\right) \\
&= \left(\frac{\delta}{\delta y}\frac{\delta v}{\delta x} - \frac{\delta}{\delta y}\frac{\delta u}{\delta y} - \frac{\delta}{\delta z}\frac{\delta u}{\delta z} + \frac{\delta}{\delta z}\frac{\delta w}{\delta x}\right) \\
\left(\frac{\delta\omega_x}{\delta z} - \frac{\delta\omega_z}{\delta x}\right) &= \left(\frac{\delta}{\delta z}\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right) - \frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)\right) \\
&= \left(\frac{\delta}{\delta z}\frac{\delta w}{\delta y} - \frac{\delta}{\delta z}\frac{\delta v}{\delta z} - \frac{\delta}{\delta x}\frac{\delta v}{\delta x} + \frac{\delta}{\delta x}\frac{\delta u}{\delta y}\right) \\
\left(\frac{\delta\omega_y}{\delta x} - \frac{\delta\omega_x}{\delta y}\right) &= \left(\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right) - \frac{\delta}{\delta y}\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right)\right) \\
&= \left(\frac{\delta}{\delta x}\frac{\delta u}{\delta z} - \frac{\delta}{\delta x}\frac{\delta w}{\delta x} - \frac{\delta}{\delta y}\frac{\delta w}{\delta y} + \frac{\delta}{\delta y}\frac{\delta v}{\delta z}\right)
\end{aligned}$$

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\begin{aligned}
\vec{v}\cdot(\nabla\times\vec{\omega}) &= (ui+vj+wk)\cdot(\nabla\times(\nabla\times\vec{v}))\\
\nabla\times(\nabla\times\vec{v}) &= \nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}\\
\nabla(\nabla\cdot\vec{v}) &= \left(\frac{\delta}{\delta x}i+\frac{\delta}{\delta y}j+\frac{\delta}{\delta z}k\right)\cdot\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta w}{\delta z}\right)\\
&= \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta w}{\delta z}\right)i+\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta w}{\delta z}\right)j+\frac{\delta}{\delta z}\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta w}{\delta z}\right)k\\
&= \left(\frac{\delta}{\delta x}\frac{\delta u}{\delta x}+\frac{\delta}{\delta x}\frac{\delta v}{\delta y}+\frac{\delta}{\delta x}\frac{\delta w}{\delta z}\right)i+\left(\frac{\delta}{\delta y}\frac{\delta u}{\delta x}+\frac{\delta}{\delta y}\frac{\delta v}{\delta y}+\frac{\delta}{\delta y}\frac{\delta w}{\delta z}\right)j+\left(\frac{\delta}{\delta z}\frac{\delta u}{\delta x}+\frac{\delta}{\delta z}\frac{\delta v}{\delta y}+\frac{\delta}{\delta z}\frac{\delta w}{\delta z}\right)k\\
\nabla^2\vec{v} &= \nabla^2ui+\nabla^2vj+\nabla^2wk\\
&= \left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}u+\frac{\delta}{\delta y}\frac{\delta}{\delta y}u+\frac{\delta}{\delta z}\frac{\delta}{\delta z}u\right)i+\left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}v+\frac{\delta}{\delta y}\frac{\delta}{\delta y}v+\frac{\delta}{\delta z}\frac{\delta}{\delta z}v\right)j\\
&+\left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}w+\frac{\delta}{\delta y}\frac{\delta}{\delta y}w+\frac{\delta}{\delta z}\frac{\delta}{\delta z}w\right)k
\end{aligned}
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\begin{aligned}
\nabla\times(\nabla\times\vec{v}) &= \nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}\\
&= \left(\frac{\delta}{\delta x}\frac{\delta u}{\delta x}+\frac{\delta}{\delta x}\frac{\delta v}{\delta y}+\frac{\delta}{\delta x}\frac{\delta w}{\delta z}\right)i+\left(\frac{\delta}{\delta y}\frac{\delta u}{\delta x}+\frac{\delta}{\delta y}\frac{\delta v}{\delta y}+\frac{\delta}{\delta y}\frac{\delta w}{\delta z}\right)j+\left(\frac{\delta}{\delta z}\frac{\delta u}{\delta x}+\frac{\delta}{\delta z}\frac{\delta v}{\delta y}+\frac{\delta}{\delta z}\frac{\delta w}{\delta z}\right)k\\
&-\left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}u+\frac{\delta}{\delta y}\frac{\delta}{\delta y}u+\frac{\delta}{\delta z}\frac{\delta}{\delta z}u\right)i-\left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}v+\frac{\delta}{\delta y}\frac{\delta}{\delta y}v+\frac{\delta}{\delta z}\frac{\delta}{\delta z}v\right)j\\
&-\left(\frac{\delta}{\delta x}\frac{\delta}{\delta x}w+\frac{\delta}{\delta y}\frac{\delta}{\delta y}w+\frac{\delta}{\delta z}\frac{\delta}{\delta z}w\right)k\\
&= \left(\frac{\delta}{\delta x}\frac{\delta u}{\delta x}+\frac{\delta}{\delta x}\frac{\delta v}{\delta y}+\frac{\delta}{\delta x}\frac{\delta w}{\delta z}-\frac{\delta}{\delta x}\frac{\delta}{\delta x}u-\frac{\delta}{\delta y}\frac{\delta}{\delta y}u-\frac{\delta}{\delta z}\frac{\delta}{\delta z}u\right)i\\
&+\left(\frac{\delta}{\delta y}\frac{\delta u}{\delta x}+\frac{\delta}{\delta y}\frac{\delta v}{\delta y}+\frac{\delta}{\delta y}\frac{\delta w}{\delta z}-\frac{\delta}{\delta x}\frac{\delta}{\delta x}v-\frac{\delta}{\delta y}\frac{\delta}{\delta y}v-\frac{\delta}{\delta z}\frac{\delta}{\delta z}v\right)j\\
&+\left(\frac{\delta}{\delta z}\frac{\delta u}{\delta x}+\frac{\delta}{\delta z}\frac{\delta v}{\delta y}+\frac{\delta}{\delta z}\frac{\delta w}{\delta z}-\frac{\delta}{\delta x}\frac{\delta}{\delta x}w-\frac{\delta}{\delta y}\frac{\delta}{\delta y}w-\frac{\delta}{\delta z}\frac{\delta}{\delta z}w\right)k\\
&= \left(\frac{\delta}{\delta x}\frac{\delta v}{\delta y}+\frac{\delta}{\delta x}\frac{\delta w}{\delta z}-\frac{\delta}{\delta y}\frac{\delta u}{\delta y}-\frac{\delta}{\delta z}\frac{\delta u}{\delta z}\right)i\\
&+\left(\frac{\delta}{\delta y}\frac{\delta u}{\delta x}+\frac{\delta}{\delta y}\frac{\delta w}{\delta z}-\frac{\delta}{\delta x}\frac{\delta v}{\delta x}-\frac{\delta}{\delta z}\frac{\delta v}{\delta z}\right)j\\
&+\left(\frac{\delta}{\delta z}\frac{\delta u}{\delta x}+\frac{\delta}{\delta z}\frac{\delta v}{\delta y}-\frac{\delta}{\delta x}\frac{\delta w}{\delta x}-\frac{\delta}{\delta y}\frac{\delta w}{\delta y}\right)k
\end{aligned}
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\begin{aligned}
\vec{v} \cdot (\nabla \times (\nabla \times \vec{v})) &= \vec{v} \cdot (\nabla \times \vec{\omega}) \\
&= u \left(\frac{\delta}{\delta x} \frac{\delta v}{\delta y} + \frac{\delta}{\delta x} \frac{\delta w}{\delta z} - \frac{\delta}{\delta y} \frac{\delta u}{\delta y} - \frac{\delta}{\delta z} \frac{\delta u}{\delta z} \right) \\
&+ v \left(\frac{\delta}{\delta y} \frac{\delta u}{\delta x} + \frac{\delta}{\delta y} \frac{\delta w}{\delta z} - \frac{\delta}{\delta x} \frac{\delta v}{\delta x} - \frac{\delta}{\delta z} \frac{\delta v}{\delta z} \right) \\
&+ w \left(\frac{\delta}{\delta z} \frac{\delta u}{\delta x} + \frac{\delta}{\delta z} \frac{\delta v}{\delta y} - \frac{\delta}{\delta x} \frac{\delta w}{\delta x} - \frac{\delta}{\delta y} \frac{\delta w}{\delta y} \right)
\end{aligned}
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