We begin by defining the first law of thermodynamics

$$\frac{dU}{dt} = Q + W \tag{1}$$

If we define the rate of work in a closed system and rearrange eqn 1 for Q

$$Q = \frac{dU}{dt} + P\frac{dV}{dt} \tag{2}$$

$$W = -P\frac{dV}{dt}$$

Now redefine the internal energy change as...

$$\frac{dU}{dt} = \frac{\delta U}{\delta T}\frac{dT}{dt} + \frac{\delta U}{\delta V}\frac{dV}{dt} \tag{3}$$

Now sub in eqn 3 into 2 and rearrange

$$Q = \frac{\delta U}{\delta T}\frac{dT}{dt} + \frac{\delta U}{\delta V}\frac{dV}{dt} + P\frac{dV}{dt}$$

$$= C_v\frac{dT}{dt} + \left(\frac{\delta U}{\delta V} + P\right)\frac{dV}{dt} \tag{4}$$

$$C_v = \frac{\delta U}{\delta T} \tag{5}$$

Now in an adiabatic, constant volume process, internal energy is conserved. Additionally, under the adiabatic assumption no heat is added to the system. Therefore eqn 4 will become

$$0 = C_v\frac{dT}{dt} + P\frac{dV}{dt} \tag{6}$$

$$\frac{\delta U}{\delta V} = 0$$

Rearranging eqn 6

$$C_v\frac{dT}{dt} = -P\frac{dV}{dt} \tag{7}$$

Take the derivative of the ideal gas law with respect to time and rearrange

$$PV = NRTP\frac{dV}{dt} + V\frac{dP}{dt} = NR\frac{dT}{dt}$$

$$P\frac{dV}{dt} = -V\frac{dP}{dt} + NR\frac{dT}{dt}$$

$$-P\frac{dV}{dt} = V\frac{dP}{dt} - NR\frac{dT}{dt} \tag{8}$$

Also assume you only have 1 molecule (i.e., N=1) and sub eqn 8 into eqn 7 and rearrange.

$$C_v\frac{dT}{dt} = V\frac{dP}{dt} - R\frac{dT}{dt}$$

$$C_v\frac{dT}{dt} + R\frac{dT}{dt} = V\frac{dP}{dt}$$

$$(C_v + R)\frac{dT}{dt} = V\frac{dP}{dt}$$

$$C_p\frac{dT}{dt} = V\frac{dP}{dt} \tag{9}$$

Where C_p is the specific heat at constant pressure Now again use the ideal gas law rearranged for volume $V = \frac{RT}{P}$ in eqn 9. Then rearrange

$$C_p \frac{dT}{dt} = \frac{RT}{P} \frac{dP}{dt}$$
$$\frac{1}{T} \frac{dT}{dt} = \frac{\kappa}{P} \frac{dP}{dt} \tag{10}$$
$$\kappa = \frac{R}{C_p}$$

Regardless of the change with time, we find the following relationship from eqn 10

$$\frac{1}{T} dT = \frac{\kappa}{P} dP \tag{11}$$

We can now integrate these two sides between two temperatures and their corresponding pressures...

$$\int_{T_o}^{T_f} \frac{1}{T} dT = \int_{P_o}^{P_f} \frac{\kappa}{P} dP$$
$$ln(T)|_{T_o}^{T_f} = \kappa ln(P)|_{P_o}^{P_f}$$
$$ln\left(\frac{T_f}{T_o}\right) = \kappa ln(\frac{P_f}{P_o})$$
$$ln\left(\frac{T_f}{T_o}\right) = ln\left(\frac{P_f}{P_o}\right)^{\kappa}$$
$$\frac{T_f}{T_o} = \left(\frac{P_f}{P_o}\right)^{\kappa}$$
$$T_f = T_o \left(\frac{P_f}{P_o}\right)^{\kappa} \tag{12}$$

If we define the initial temperature to be the temperature of an air parcel at some pressure and the final temperature to be the temperature when brought down to a reference pressure (such as the surface then we have the definition of potential temperature

$$\theta = T \left(\frac{P_{sfc}}{P}\right)^{\kappa} \tag{13}$$