CARTESIAN TO SPHERICAL COORDINATES: TWO VIEWS

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In Earth Sciences and the greater physics community, the cartesian plane (i.e., the typical x,y,z axes) are often insufficient for anything other than a perfectly flat 3-D plane. When a spheroid is involved, such as Earth, the coordinate system must adjust because distances on the sphere do not stay constant as they do in the cartesian framework. Enter: SPHERICAL Coordinates. This quick study will guide you through the derivation of the spherical coordinate system and how to generate common operators found in atmospheric research.

1 Cartesian to Spherical

Although the cartesian framework is insufficient for spheroids, we can relate the position on the sphere in space with respect to the cartesian plane. This is done in terms of the vertical and horizontal angles the position vector makes with respect to the origin. The common physics representation of the position vector in spherical coordinates is (λ,θ,r). Where λ is longitude, or the angle from a axis line of the meridion of the spherioid (e.g., 0 deg longitude), θ is the latitude, or the vertical angle measured from the polar axis, and r is the radius of sphere.

In Atmospheric Science and Atmospheric Sciences, we usually follow a slightly different convention for spherical coordinates. In this system a position on a sphere is (λ,ϕ,r) system. Where ϕ is the latitude, or the vertical angle created from the horizontal axis plane For simplification, we can assume that r is constant. This is often ok for my applications where , and $r = R_e$, or the radius of the Earth. The differences between the common physics and Atmospheric Science framework are illustrated below

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The relation between the angles subtended on the sphere and the cartesian counterparts are straightforward using trigonometry. All we need to do is decompose the vector onto the respective axial planes. In the normal physics frame work they are:

$$x = r\sin\theta\cos\lambda \tag{1}$$
$$y = r\sin\theta\sin\lambda \tag{2}$$
$$z = r\cos\theta \tag{3}$$

And in Atmospheric Science they are:

$$x = r\cos\phi\cos\lambda \tag{4}$$
$$y = r\cos\phi\sin\lambda \tag{5}$$
$$z = r\sin\phi \tag{6}$$

As you can see, there is a slight difference because of the way the 2nd horizontal angle axis is measured. Fortunately, the relationship between the two forms is simply the co-angle.

$$\sin\theta = \cos(\frac{\pi}{2} - \theta)$$
$$\frac{\pi}{2} - \theta = \phi$$

It is important to not get this confused, because many studies in other physical fields will use one or the other definition.

2 Unit Vectors and Back

The first step to understanding how to relate the two the spherical coordinate to the cartesian framework is to define the unit vectors in both frameworks. A unit vectors is simply a normalized vector of magnitude 1 that points in a direction that represents the coordinate space and axes of the parent vector. In cartesian coordinates, the unit vectors are:

$$\begin{aligned}\hat{x} &= \{1, 0, 0\}\\ \hat{y} &= \{0, 1, 0\}\\ \hat{z} &= \{0, 0, 1\}\end{aligned}$$

They can be used to project a vector onto a particular axis. The position of any vector in the cartesian framework can be described as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{7}$$

If we know the definitions of all of these terms, then we may have a mapping function for the position vector with respect to different coordinate systems.This is slightly different in either framework, so both will be presented below.

2.1 Spherical Coordinate Unit Vectors via Normal Physics Definition

We start by substituting equations (1)-(3) into (7):

$$\begin{aligned}\vec{r} &= r\sin\theta\cos\lambda\hat{x} + r\sin\theta\sin\lambda\hat{y} + r\cos\theta\hat{z}\\ &= r\left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right]\\ &= r\hat{r}\end{aligned}$$

From this we can see the position vector, $\vec{r}$, is some combination of scaling factor r and a unit vector $\hat{r}$.

Now to need to redefine a new basis vector that points toward the increasing r axis. This is done by taking the derivative of the position vector with respect to, r.

$$\begin{aligned}e_r &= \frac{\delta\vec{r}}{\delta r}\\ &= \hat{r}\end{aligned}$$

Now the unit vector is simply the basis vector divided by the magnitude of the basis vector.

$$\begin{aligned}\hat{e}_r &= \frac{e_r}{|e_r|}\\ &= \frac{\hat{r}}{|\hat{r}|} = \hat{r}\\ &= \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\end{aligned}$$

Now we have the radial unit vector. Moving onto the longitudinal basis vector. This follows the same procedure as above. First, define the new basis vector along the longitudinal axis:

$$\begin{aligned}
e_\lambda &= \frac{\delta \vec{r}}{\delta \lambda} \\
&= \frac{\delta}{\delta \lambda} r \left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right] \\
&= r\left[-\sin\theta\sin\lambda\hat{x} + \sin\theta\cos\lambda\hat{y}\right] \\
&= r\sin\theta\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right]
\end{aligned}$$

With this result, we can solve for the longitudinal unit vector:

$$\begin{aligned}
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
|e_\lambda| &= \sqrt{e_\lambda \cdot e_\lambda} \\
&= \sqrt{(r\sin\theta)^2\left[\sin^2\lambda + \cos^2\lambda\right]} \\
&= r\sin\theta \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
&= \frac{r\sin\theta\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right]}{r\sin\theta} \\
&= -\sin\lambda\hat{x} + \cos\lambda\hat{y}
\end{aligned}$$

Lastly, we can represent the latitudinal basis vector as:

$$\begin{aligned}
e_\theta &= \frac{\delta \vec{r}}{\delta \theta} \\
&= \frac{\delta}{\delta \theta} r \left[\sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\right] \\
&= r\left[\cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}\right]
\end{aligned}$$

With this result, we can solve for the latitudinal unit vector:

$$\begin{aligned}
\hat{e}_\theta &= \frac{e_\theta}{|e_\theta|} \\
|e_\theta| &= \sqrt{e_\theta \cdot e_\theta} \\
&= \sqrt{r^2\left[\cos^2\theta\cos^2\lambda + \cos^2\theta\sin^2\lambda + \sin^2\theta\right]} \\
&= \sqrt{r^2\left[\cos^2\theta(\cos^2\lambda + \sin^2\lambda) + \sin^2\theta\right]} \\
&= \sqrt{r^2\left[\cos^2\theta + \sin^2\theta\right]} \\
&= \sqrt{r^2} \\
&= r \\
\hat{e}_\theta &= \frac{e_\theta}{|e_\theta|} \\
&= \frac{r\left[\cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}\right]}{r} \\
&= \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}
\end{aligned}$$

Thus, the unit vectors in spherical coordinates for the normal physics framework are:

$$\hat{e}_r = \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z} \tag{8}$$
$$\hat{e}_\lambda = -\sin\lambda\hat{x} + \cos\lambda\hat{y} \tag{9}$$
$$\hat{e}_\theta = \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z} \tag{10}$$

2.2 Cartesian Unit Vectors as Described by Spherical Coordinates via Normal Physics Definition

Now, even though we have solved for the spherical coordinate unit vectors, there is still the issue of defining the cartesian unit vectors in terms of the spherical coordinates. A back-and-forth operation, if you will. We can do this by using linear algebra and trigonometry to isolate each cartesian unit vector.

To obtain the z-unit vector, we begin by multiplying (8) by $\cos\theta$ and (10) by $\sin\theta$ and then subtracting their results.

$$\begin{aligned}
\cos\theta\hat{e}_r &= \cos\theta\sin\theta\cos\lambda\hat{x} + \cos\theta\sin\theta\sin\lambda\hat{y} + \cos^2\theta\hat{z} \\
\sin\theta\hat{e}_\theta &= \sin\theta\cos\theta\cos\lambda\hat{x} + \sin\theta\cos\theta\sin\lambda\hat{y} - \sin^2\theta\hat{z} \\
\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta &= (\cos^2\theta + \sin^2\theta)\hat{z} \\
\hat{z} &= \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta
\end{aligned}$$

To obtain the x-unit vector, we begin by multiplying (8) by $\sin\theta\cos\lambda$, (9) by $\cos\lambda$, and (10) by $\cos\theta\cos\lambda$

$$\sin\theta\cos\lambda\hat{e}_r = \sin^2\theta\cos^2\lambda\hat{x} + \sin\theta\cos\lambda\sin\theta\sin\lambda\hat{y} + \sin\theta\cos\lambda\cos\theta\hat{z} \tag{11}$$
$$\sin\lambda\hat{e}_\lambda = -\sin^2\lambda\hat{x} + \sin\lambda\cos\lambda\hat{y} \tag{12}$$
$$\cos\theta\cos\lambda\hat{e}_\theta = \cos^2\theta\cos^2\lambda\hat{x} + \cos\theta\cos\lambda\cos\theta\sin\lambda\hat{y} - \cos\theta\cos\lambda\sin\theta\hat{z} \tag{13}$$

Now add (11) and (13):

$$\begin{aligned}
\sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta &= \sin^2\theta\cos^2\lambda\hat{x} + \sin\theta\cos\lambda\sin\theta\sin\lambda\hat{y} + \sin\theta\cos\lambda\cos\theta\hat{z} \\
&\cos^2\theta\cos^2\lambda\hat{x} + \cos\theta\cos\lambda\cos\theta\sin\lambda\hat{y} - \cos\theta\cos\lambda\sin\theta\hat{z} \\
&= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y}
\end{aligned}$$

Then subtract (12) from result:

$$\begin{aligned}
\sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y} + \sin^2\lambda\hat{x} - \sin\lambda\cos\lambda\hat{y} \\
\hat{x} &= \sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda
\end{aligned}$$

To obtain the y-unit vector, we begin by multiplying (8) by $\sin\theta\sin\lambda$, (9) by $\cos\lambda$, and (10) by $\cos\theta\sin\lambda$

$$\sin\theta\sin\lambda\hat{e}_r = \sin\theta\sin\lambda\sin\theta\cos\lambda\hat{x} + \sin^2\theta\sin^2\lambda\hat{y} + \sin\theta\sin\lambda\cos\theta\hat{z} \tag{14}$$
$$\cos\lambda\hat{e}_\lambda = -\cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y} \tag{15}$$
$$\cos\theta\sin\lambda\hat{e}_\theta = \cos\theta\sin\lambda\cos\theta\cos\lambda\hat{x} + \cos^2\theta\sin^2\lambda\hat{y} - \cos\theta\sin\lambda\sin\theta\hat{z} \tag{16}$$

Now add (14) and (16):

$$\begin{aligned}\sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta &= \sin\theta\sin\lambda\sin\theta\cos\lambda\hat{x} + \sin^2\theta\sin^2\lambda\hat{y} + \sin\theta\sin\lambda\cos\theta\hat{z}\\ &+ \cos\theta\sin\lambda\cos\theta\cos\lambda\hat{x} + \cos^2\theta\sin^2\lambda\hat{y} - \cos\theta\sin\lambda\sin\theta\hat{z}\\ &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y}\end{aligned}$$

Then add (15) from result:

$$\begin{aligned}\sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y} - \cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y}\\ \hat{y} &= \sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda\end{aligned}$$

With these steps, we now have derived the cartesian unit vectors in terms of the spherical coordinates definition from normal physics

$$\hat{x} = \sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda \tag{17}$$

$$\hat{y} = \sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda \tag{18}$$

$$\hat{z} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \tag{19}$$

2.3 Spherical Coordinate Unit Vectors via Atmospheric Science Definition

Starting from the same point as the normal physics method, we begin by defining the position vector in terms of the spherical coordiantes:

$$\begin{aligned}\vec{r} &= r\left[\cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\right]\\ &= r\hat{r}\end{aligned}$$

Now we can derive define new basis and unit vectors for each of the spherical coordinate axes. Beginning with the r axis, we can see the basis vector is the same. Thus, the r axis unit vector becomes:

$$\begin{aligned}\hat{e}_r &= \frac{\hat{r}}{|\hat{r}|} = \hat{r}\\ &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\end{aligned}$$

Moving onto the longitudinal basis vector. This follows the same procedure as above. First, define the new basis vector along the longitudinal axis:

$$\begin{aligned}e_\lambda &= \frac{\delta\vec{r}}{\delta\lambda}\\ &= \frac{\delta}{\delta\lambda}r\left[\cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\right]\\ &= r\left[-\cos\phi\sin\lambda\hat{x} + \cos\phi\cos\lambda\hat{y}\right]\\ &= r\cos\phi\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right]\end{aligned}$$

With this result, we can solve for the longitudinal unit vector:

$$
\begin{aligned}
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
|e_\lambda| &= \sqrt{e_\lambda \cdot e_\lambda} \\
&= \sqrt{(r\cos\phi)^2\left[\sin^2\lambda + \cos^2\lambda\right]} \\
&= r\cos\phi \\
\hat{e}_\lambda &= \frac{e_\lambda}{|e_\lambda|} \\
&= \frac{r\cos\phi\left[-\sin\lambda\hat{x} + \cos\lambda\hat{y}\right]}{r\cos\phi} \\
&= -\sin\lambda\hat{x} + \cos\lambda\hat{y}
\end{aligned}
$$

Lastly, we can represent the latitudinal basis vector as:

$$
\begin{aligned}
e_\phi &= \frac{\delta\vec{r}}{\delta\phi} \\
&= \frac{\delta}{\delta\phi} r\left[\cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\right] \\
&= r\left[-\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}\right]
\end{aligned}
$$

With this result, we can solve for the latitudinal unit vector:

$$
\begin{aligned}
\hat{e}_\phi &= \frac{e_\phi}{|e_\phi|} \\
|e_\phi| &= \sqrt{e_\phi \cdot e_\phi} \\
&= \sqrt{r^2\left[\sin^2\phi\cos^2\lambda + \sin^2\phi\sin^2\lambda + \cos^2\phi\right]} \\
&= \sqrt{r^2\left[\sin^2\phi(\cos^2\lambda + \sin^2\lambda) + \cos^2\phi\right]} \\
&= \sqrt{r^2\left[\sin^2\phi + \cos^2\phi\right]} \\
&= \sqrt{r^2} \\
&= r \\
\hat{e}_\phi &= \frac{e_\phi}{|e_\phi|} \\
&= \frac{r\left[-\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}\right]}{r} \\
&= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}
\end{aligned}
$$

Thus, the unit vectors in spherical coordinates for the Atmospheric Science framework are:

$$\hat{e}_r = \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z} \tag{20}$$

$$\hat{e}_\lambda = -\sin\lambda\hat{x} + \cos\lambda\hat{y} \tag{21}$$

$$\hat{e}_\phi = -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z} \tag{22}$$

Now, it should be clear that the spherical coordiante unit vectors are NOT identical in the normal physics and Atmospheric Science frameworks. As a demonstration, we can show how these two forms produce slightly different results. Let's say $\theta = 0$, $\phi = \frac{\pi}{2}$ and $\lambda = \frac{\pi}{2}$ Under the definition derived using the Atmospheric Science convention:

$$\begin{aligned}\hat{e}_r &= \cos\phi\cos\lambda\hat{x} + \cos\phi\sin\lambda\hat{y} + \sin\phi\hat{z}\\ &= \hat{z}\\ \hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y}\\ &= -\hat{x}\\ \hat{e}_\phi &= -\sin\phi\cos\lambda\hat{x} - \sin\phi\sin\lambda\hat{y} + \cos\phi\hat{z}\\ &= -\hat{y}\end{aligned}$$

Under the definition derived using the normal physics convention:

$$\begin{aligned}\hat{e}_r &= \sin\theta\cos\lambda\hat{x} + \sin\theta\sin\lambda\hat{y} + \cos\theta\hat{z}\\ &= \hat{z}\\ \hat{e}_\lambda &= -\sin\lambda\hat{x} + \cos\lambda\hat{y}\\ &= -\hat{x}\\ \hat{e}_\theta &= \cos\theta\cos\lambda\hat{x} + \cos\theta\sin\lambda\hat{y} - \sin\theta\hat{z}\\ &= \hat{y}\end{aligned}$$

As can be seen, the X and Z unit vectors are identical in both systems. The reversal from measuring the angle from the pole versus the equator in Atmospheric Science has reversed the direction of the Y unit vector. Thus, the Y unit vector in Atmospheric Science is the opposite of that found in the normal method.

2.4 Cartesian Unit Vectors as Described by Spherical Coordinates via Atmospheric Science Definition

As with the normal physics framework, we must also solve for the cartesian unit vectors as defined by the spherical coordinate unit vectors. To obtain the z-unit vector, we begin by multiplying (20) by $\sin\phi$ and (22) by $\cos\phi$. Then, add their results.

$$\begin{aligned}\sin\phi\hat{e}_r &= \sin\phi\cos\phi\cos\lambda\hat{x} + \sin\phi\cos\phi\sin\lambda\hat{y} + \sin^2\phi\hat{z}\\ \cos\phi\hat{e}_\phi &= -\cos\phi\sin\phi\cos\lambda\hat{x} - \cos\phi\sin\phi\sin\lambda\hat{y} + \cos^2\phi\hat{z}\\ \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi &= (\cos^2\phi + \sin^2\phi)\hat{z}\\ \hat{z} &= \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi\end{aligned}$$

To obtain the x-unit vector, we begin by multiplying (20) by $\cos\phi\cos\lambda$, (21) by $\sin\lambda$, and (22) by $\sin\phi\cos\lambda$

$$\cos\phi\cos\lambda\hat{e}_r = \cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} + \cos\phi\cos\lambda\sin\phi\hat{z} \tag{23}$$

$$\sin\lambda\hat{e}_\lambda = -\sin^2\lambda\hat{x} + \sin\lambda\cos\lambda\hat{y} \tag{24}$$

$$\sin\phi\cos\lambda\hat{e}_\phi = -\sin^2\phi\cos^2\lambda\hat{x} - \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} + \sin\phi\cos\lambda\cos\phi\hat{z} \tag{25}$$

Now subtract (23) and (25):

$$\begin{aligned}\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi &= \cos^2\phi\cos^2\lambda\hat{x} + \cos\phi\cos\lambda\cos\phi\sin\lambda\hat{y} + \cos\phi\cos\lambda\sin\phi\hat{z}\\ &\quad + \sin^2\phi\cos^2\lambda\hat{x} + \sin\phi\cos\lambda\sin\phi\sin\lambda\hat{y} - \sin\phi\cos\lambda\cos\phi\hat{z}\\ &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y}\end{aligned}$$

Then, subtract (24) from the result:

$$\begin{aligned}\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda &= \cos^2\lambda\hat{x} + \cos\lambda\sin\lambda\hat{y} + \sin^2\lambda\hat{x} - \sin\lambda\cos\lambda\hat{y}\\ \hat{x} &= \cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda\end{aligned}$$

To obtain the y-unit vector, we begin by multiplying (20) by $\cos\phi\sin\lambda$, (21) by $\cos\lambda$, and (22) by $\sin\phi\sin\lambda$

$$\cos\phi\sin\lambda\hat{e}_r = \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} + \cos\phi\sin\lambda\sin\phi\hat{z} \tag{26}$$
$$\cos\lambda\hat{e}_\lambda = -\cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y} \tag{27}$$
$$\sin\phi\sin\lambda\hat{e}_\phi = -\sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} - \sin^2\phi\sin^2\lambda\hat{y} + \sin\phi\sin\lambda\cos\phi\hat{z} \tag{28}$$

Now subtract (26) and (28):

$$\begin{aligned}\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi &= \cos\phi\sin\lambda\cos\phi\cos\lambda\hat{x} + \cos^2\phi\sin^2\lambda\hat{y} + \cos\phi\sin\lambda\sin\phi\hat{z}\\ &\quad + \sin\phi\sin\lambda\sin\phi\cos\lambda\hat{x} + \sin^2\phi\sin^2\lambda\hat{y} - \sin\phi\sin\lambda\cos\phi\hat{z}\\ &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y}\end{aligned}$$

Then, add (27) to the result:

$$\begin{aligned}\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda &= \sin\lambda\cos\lambda\hat{x} + \sin^2\lambda\hat{y} - \cos\lambda\sin\lambda\hat{x} + \cos^2\lambda\hat{y}\\ \hat{y} &= \cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda\end{aligned}$$

With these steps, we now have derived the cartesian unit vectors in terms of the spherical coordinates definition from Atmospheric Science

$$\hat{x} = \cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda \tag{29}$$
$$\hat{y} = \cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda \tag{30}$$
$$\hat{z} = \sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi \tag{31}$$

3 ∇, or Gradient Operator

The gradient operator is the rate of change of some field in the dimensions of the coordinate system. The definition of this operator in the cartesian framework is:

$$\nabla = \frac{\delta}{\delta x}\hat{x} + \frac{\delta}{\delta y}\hat{y} + \frac{\delta}{\delta z}\hat{z} \tag{32}$$

Since, we have solved for the cartesian unit vectors in terms of the spherical coordinate unit vectors, we may substitute and reorganize to show how this operator changes in the spherical coordinate system. As with the previous definitions and derivations, the normal physics and Atmospheric Science frameworks produce slightly different results. Both are presented here:

3.1 Gradient via Normal Physics Definition

We begin insertion of the normal physics variants by substituting equations (20) through (22) into equation (32)

$$\begin{aligned}\nabla = &\left[\sin\theta\cos\lambda\hat{e}_r + \cos\theta\cos\lambda\hat{e}_\theta - \sin\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta x}\\ &+ \left[\sin\theta\sin\lambda\hat{e}_r + \cos\theta\sin\lambda\hat{e}_\theta + \cos\lambda\hat{e}_\lambda\right]\frac{\delta}{\delta y}\\ &+ \left[\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right]\frac{\delta}{\delta z}\end{aligned}$$

As can be seen above, we have common terms that coordinate with the spherical coordinate unit vectors. What we may do is carry the partial derivatives through and rearrange to isolate terms with respect to the spherical coordinate unit vectors:

$$\begin{aligned}\nabla = &\left[\sin\theta\cos\lambda\frac{\delta}{\delta x} + \sin\theta\sin\lambda\frac{\delta}{\delta y} + \cos\theta\frac{\delta}{\delta z}\right]\hat{e}_r\\ &+ \left[\cos\theta\cos\lambda\frac{\delta}{\delta x} + \cos\theta\sin\lambda\frac{\delta}{\delta y} - \sin\phi\frac{\delta}{\delta z}\right]\hat{e}_\phi\\ &+ \left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]\hat{e}_\lambda\end{aligned} \tag{33}$$

Now we must deal with the partial derivatives. This is done by utilizing the chain rule to transform the partial derivatives into those corresponding with spherical coordinates:

$$\frac{\delta}{\delta r} = \frac{\delta x}{\delta r}\frac{\delta}{\delta x} + \frac{\delta y}{\delta r}\frac{\delta}{\delta y} + \frac{\delta z}{\delta r}\frac{\delta}{\delta z} \tag{34}$$

$$\frac{\delta}{\delta \theta} = \frac{\delta x}{\delta \theta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \theta}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \theta}\frac{\delta}{\delta z} \tag{35}$$

$$\frac{\delta}{\delta \lambda} = \frac{\delta x}{\delta \lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \lambda}\frac{\delta}{\delta z} \tag{36}$$

Starting with the radial derivative, we may take the derivatives of the cartesian coordinates for normal physics (i.e., equations (1)-(3)) with respect to the radial direction and substitute them into (34):

$$\frac{\delta}{\delta r} = \frac{\delta x}{\delta r}\frac{\delta}{\delta x} + \frac{\delta y}{\delta r}\frac{\delta}{\delta y} + \frac{\delta z}{\delta r}\frac{\delta}{\delta z} = \sin\theta\cos\lambda\frac{\delta}{\delta x} + \sin\theta\sin\lambda\frac{\delta}{\delta y} + \cos\theta\frac{\delta}{\delta z}$$

For substitution into (33), this definition must match one of the terms in that equation. By inspection, the first term IS consistent with the first term of (33), which is along the radial direction. The process for determining the latitudinal derivative is the same as that of the radial derivative. Simply take the latitudinal derivative of equations (1)-(3) and substitute them into (35):

$$\frac{\delta}{\delta \theta} = \frac{\delta x}{\delta \theta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta \theta}\frac{\delta}{\delta y} + \frac{\delta z}{\delta \theta}\frac{\delta}{\delta z} = r\left[\cos\theta\cos\lambda\frac{\delta}{\delta x} + \cos\theta\sin\lambda\frac{\delta}{\delta y} - \sin\phi\frac{\delta}{\delta z}\right]$$

By examination, this term largely matches the second term of equation (33), however the second term of equation (33) is missing the factor of r. All we must do to allow substitution is multiply the second term of equation (33) by 1 (e.g., $\frac{r}{r}$). Like the previous two directions, the longitudinal derivative follows a similar analysis. Simply take the longitudinal derivative of equations (1)-(3) and substitute them into (36):

$$\frac{\delta}{\delta\lambda} = \frac{\delta x}{\delta\lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\lambda}\frac{\delta}{\delta z} = r\sin\theta\left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]$$

As with the latitudinal derivative, the longitudinal derivative does not quite match with any of the terms in equation (33). The third term of (33) needs a factor of $r\sin\theta$ to allow substitution. Thus, we can simply multiply the third term of (33) by 1 (e.g., $\frac{r\sin\theta}{r\sin\theta}$) and substitute. Substitution of all the spherical coordinate derivatives and their necessary factors into equation (33) yields the gradient usually found in normal physics textbooks.

$$\nabla = \frac{\delta}{\delta r}\hat{e}_r + \frac{1}{r}\frac{\delta}{\delta\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\delta}{\delta\lambda}\hat{e}_\lambda \tag{37}$$

3.2 Gradient via Atmospheric Science Definition

We begin insertion of the normal physics variants by substituting equations (29) through (31) into equation (32)

$$\begin{aligned}\nabla = {} & [\cos\phi\cos\lambda\hat{e}_r - \sin\phi\cos\lambda\hat{e}_\phi - \sin\lambda\hat{e}_\lambda]\frac{\delta}{\delta x} \\ & + [\cos\phi\sin\lambda\hat{e}_r - \sin\phi\sin\lambda\hat{e}_\phi + \cos\lambda\hat{e}_\lambda]\frac{\delta}{\delta y} \\ & + [\sin\phi\hat{e}_r + \cos\phi\hat{e}_\phi]\frac{\delta}{\delta z}\end{aligned}$$

As can be seen above, we have common terms that coordinate with the spherical coordinate unit vectors. What we may do is carry the partial derivatives through and rearrange to isolate terms with respect to the spherical coordinate unit vectors:

$$\begin{aligned}\nabla = {} & \left[\cos\phi\cos\lambda\frac{\delta}{\delta x} + \cos\phi\sin\lambda\frac{\delta}{\delta y} + \sin\phi\frac{\delta}{\delta z}\right]\hat{e}_r \\ & + \left[-\sin\phi\cos\lambda\frac{\delta}{\delta x} - \sin\phi\sin\lambda\frac{\delta}{\delta y} + \cos\phi\frac{\delta}{\delta z}\right]\hat{e}_\phi \\ & + \left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]\hat{e}_\lambda\end{aligned} \tag{38}$$

Now we must deal with the partial derivatives. Like the normal physics version, this is done by utilizing the chain rule to transform the partial derivatives into those corresponding with spherical coordinates. The only difference between the two is the definition of the latitudinal derivative:

$$\frac{\delta}{\delta\phi} = \frac{\delta x}{\delta\phi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\phi}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\phi}\frac{\delta}{\delta z} \tag{39}$$

Starting with the radial derivative, we may take the radial derivatives of the cartesian coordinates for Atmospheric Science (i.e., equations (4)-(6)) and substitute them into (34):

$$\frac{\delta}{\delta r} = \frac{\delta x}{\delta r}\frac{\delta}{\delta x} + \frac{\delta y}{\delta r}\frac{\delta}{\delta y} + \frac{\delta z}{\delta r}\frac{\delta}{\delta z} = \cos\phi\cos\lambda\frac{\delta}{\delta x} + \cos\phi\sin\lambda\frac{\delta}{\delta y} + \sin\phi\frac{\delta}{\delta z}$$

For substitution into (38), this definition must match one of the terms in that equation. By inspection, the first term IS consistent with the first term of (38), which is along the radial direction. The process for determining the latitudinal derivative is the same as that of the radial derivative. Simply take the latitudinal derivative of equations (4)-(5) and substitute them into (39):

$$\frac{\delta}{\delta\phi} = \frac{\delta x}{\delta\phi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\phi}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\phi}\frac{\delta}{\delta z} = r\left[-\sin\phi\cos\lambda\frac{\delta}{\delta x} - \sin\phi\sin\lambda\frac{\delta}{\delta y} + \cos\phi\frac{\delta}{\delta z}\right]$$

By examination, this term largely matches the second term of equation (38), but the second term of equation (38) is missing the factor of r. All we must do to allow substitution is multiply the second term of equation (38) by 1 (e.g., $\frac{r}{r}$). Like the previous two directions, the longitudinal derivative follows a similar analysis. Simply take the longitudinal derivative of equations (4)-(6) and substitute them into (36):

$$\frac{\delta}{\delta\lambda} = \frac{\delta x}{\delta\lambda}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\lambda}\frac{\delta}{\delta y} + \frac{\delta z}{\delta\lambda}\frac{\delta}{\delta z} = r\cos\phi\left[-\sin\lambda\frac{\delta}{\delta x} + \cos\lambda\frac{\delta}{\delta y}\right]$$

As with the latitudinal derivative, the longitudinal derivative does not quite match with any of the terms in equation (38). The third term of (38) needs a factor of $r\cos\phi$ to allow substitution. Thus, we can simply multiply the third term of (38) by 1 (e.g., $\frac{r\cos\phi}{r\cos\phi}$) and substitute. Substitution of all the spherical coordinate derivatives and their necessary factors into equation (38) yields the gradient usually found in Atmospheric Science textbooks.

$$\nabla = \frac{\delta}{\delta r}\hat{e}_r + \frac{1}{r}\frac{\delta}{\delta\phi}\hat{e}_\phi + \frac{1}{r\cos\phi}\frac{\delta}{\delta\lambda}\hat{e}_\lambda \tag{40}$$

4 $\nabla\cdot$, or Divergence Operator

The gradient operator is the rate of change of some field in the dimensions of the coordinate system. The definition of this operator in the cartesian framework is:

$$\nabla\cdot = \frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z} \tag{41}$$

5 $\nabla\times$, or Curl Operator

The gradient operator is the rate of change of some field in the dimensions of the coordinate system. The definition of this operator in the cartesian framework is:

$$\nabla\times = \left[\frac{\delta}{\delta y} - \frac{\delta}{\delta z}\right]\hat{x} + \left[\frac{\delta}{\delta z} - \frac{\delta}{\delta x}\right]\hat{y} + \left[\frac{\delta}{\delta x} - \frac{\delta}{\delta y}\right]\hat{z} \tag{42}$$

6 ∇^2, or Laplacian Operator

The gradient operator is the rate of change of some field in the dimensions of the coordinate system. The definition of this operator in the cartesian framework is:

$$\nabla\cdot\nabla = \nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \tag{43}$$