Neural Power Unit
– A Bayesian Approach To Neural Arithmetic

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Abstract

Common Neural Networks can approximate simple arithmetic operations, but fail to generalize well beyond the training manifold. A new class of units called *Neural Arithmetic Units* aim to overcome this difficulty, but are limited either to operate on positive numbers (NALU by Trask et al. [2018]), or can only represent simple addition and multiplication (NAU & NMU by Madsen and Johansen [2020]). We present the first Neural Arithmetic Unit that operates on the full domain of real numbers and is capable of learning arbitrary power functions.

1 Introduction

Neural Networks (NN) have proven to perform exceptionally well at function approximation tasks. The great approximation performance unfortunately come at the cost of generalization beyond the training domain. The inability to generalize to unseen inputs is a fundamental problem that hints at a lack of *understanding* of the given task. The model merely memorizes the seen inputs and fails to abstract the true learning task.

cite universal approx

cite something?

Neural Arithmetic Units aim to overcome this problem with units that are able to represent simple arithmetic operations exactly. By composition of these units it becomes possible to learn e.g. exponentials, sinus functions, logarithms, etc. This promises to improve on the extrapolation capabilities of NNs even on tasks that are beyond artificial arithmetic tasks such as learning multiplication, or a certain power function .

cite some examples

mention ODEs as use case

2 Related Work

Trask et al. [2018] have demonstrated how severe the extrapolation problem is even for the simplest arithmetic operations, such as summing or multiplying two numbers. In order to increase the power of abstraction of NNs they propose a *Neural Arithmetic Logic Unit* (NALU) which is capable of learning arithmetic addition, subtraction, multiplication, and division $\{+, -, \times, \div\}$ with stunning extrapolation accuracy. However, the NALU comes with the severe limitation not being able to handle negative inputs due to the logarithm in the multiplication part of the NALU:

$$\text{Addition: } \boldsymbol{a} = \boldsymbol{W}\boldsymbol{x} \qquad \boldsymbol{W} = \tanh(\hat{\boldsymbol{W}}) \odot \sigma(\hat{\boldsymbol{M}}) \tag{1}$$

$$\text{Multiplication: } \boldsymbol{m} = \exp \boldsymbol{W}(\log(|\boldsymbol{x}| + \epsilon)) \tag{2}$$

$$\text{Output: } \boldsymbol{y} = \boldsymbol{a} \odot \boldsymbol{g} + \boldsymbol{m} \odot (1 - \boldsymbol{g}) \qquad \boldsymbol{g} = \sigma(\boldsymbol{G}\boldsymbol{x}) \tag{3}$$

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Table 1: Comparison of different 2×2 layers on the task $(x, y) \to (xy, \frac{x}{y})$. $(x, y) \in \mathcal{U}^2(-2, 2)$

Model	MultTrain	DivTrain	MultTest	DivTest
NPU	8.211327e-6	0.11842908	0.00011762501,	0.21404411
NMU	0.0	496.00977	0.0,	1454.7388
NALU	1.307899	498.6583	23.914778,	1448.0072

A multiplication layer that can handle negative inputs was introduced by Madsen and Johansen [2020]. The *Neural Multiplication Unit* (NMU) is defined by Eq. 4 and is typically used in conjunction with the so-called *Neural Addition Unit* (NAU) in Eq. 5.

$$\text{NMU: } y_j = \prod_i M_{ij} z_i + 1 - M_{ij} \qquad M_{ij} = \min(\max(M_{ij}, 0), 1) \tag{4}$$

$$\text{NAU: } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \qquad A_{ij} = \min(\max(A_{ij}, 0), 1) \tag{5}$$

In both NMU and NAU the weights are clipped to $[0, 1]$, and typically regularized with $\mathcal{R}$:

$$\mathcal{R} = \sum_{ij} \min(W_{ij}, 1 - W_{ij}) \tag{6}$$

The combination of NAU and NMU can thus learn $\{+, -, \times\}$, but no division.

put this in acknowledgements all the results in this paper were create with the help of the following Julia packages: Rackauckas and Nie [2017] + (Flux.jl)

3 Neural Power Unit

Our *Neural Power Unit* (NPU) can learn arbitrary power functions (which includes division) while still being able to correctly deal with negative inputs. The NPU layer is defined by

$$k_i = \begin{cases} 0 & x_i \leq 0 \\ \pi & x_i > 0 \end{cases} \tag{7}$$

$$\boldsymbol{r} = |\boldsymbol{x}| \tag{8}$$

$$\boldsymbol{m} = \exp(\boldsymbol{M} \log(\boldsymbol{r})) \odot \cos(\boldsymbol{M}\boldsymbol{k}) \tag{9}$$

where $\boldsymbol{k}$ is a vector that is zero where $\boldsymbol{x}$ is positive and π where it is negative, $\boldsymbol{r}$ is the elementwise absolute value, and $\boldsymbol{M}$ the multiplication matrix that we want to learn.

The NPU is inspired by the multiplication part of the NALU Eq. 2, extended to negative inputs by taking advantage of the complex logarithm. In general, for any complex number z[1]

$$\log(z) = \log\left(r \cdot e^{i\varphi}\right) = \log(r) + i\varphi. \tag{10}$$

However, as z is assumed to be real, we can simplify as follows:

$$\log(z) = \log(r) + ik\pi, \tag{11}$$

where $k \in [0, 1]$ is zero for positive inputs and one for negative inputs z.

$$\boldsymbol{m} = \exp(\boldsymbol{M} \log(\boldsymbol{x})) \tag{12}$$

$$= \exp(\boldsymbol{M}(\log(\boldsymbol{r}) + i\pi\boldsymbol{k})) \tag{13}$$

$$\boldsymbol{m}_{\text{re}} = \text{real}(\exp(\boldsymbol{M} \log \boldsymbol{r} + i\pi\boldsymbol{M}\boldsymbol{k})) \tag{14}$$

$$= \text{real}(\exp(\boldsymbol{M} \log \boldsymbol{r}) \odot (\cos(\pi\boldsymbol{M}\boldsymbol{k}) + i \sin(\pi\boldsymbol{M}\boldsymbol{k}))) \tag{15}$$

$$= \exp(\boldsymbol{M} \log \boldsymbol{r}) \odot \cos(\pi\boldsymbol{M}\boldsymbol{k}) \tag{16}$$

[1]Note that complex numbers can be represented in the polar, complex plane where $z = re^{i\varphi}$ with $r > 0$ and $\varphi \in (0, 2\pi)$.

4 Experimetns

4.1 10 param func

4.2 Gradient surfaces

References

Andreas Madsen and Alexander Rosenberg Johansen. Neural Arithmetic Units. page 31, 2020.

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