

# Homework week 5

## Exercise 1

Let  $C_1$  be event that new data belongs to class 1

Let  $C_2$  be event that new data belongs to class 2

$\Rightarrow$  Posterior probability for  $C_1$  is:

$$\begin{aligned} p(C_1|x) &= \frac{p(x|C_1)p(C_1)}{p(x|C_1)p(C_1) + p(x|C_2)p(C_2)} \\ &= \frac{1}{1 + \frac{p(x|C_2)p(C_2)}{p(x|C_1)p(C_1)}} \\ &= \frac{1}{1 + e^{-a}} \end{aligned}$$

$$\text{with } a = \log \frac{p(x|C_1)p(C_1)}{p(x|C_2)p(C_2)}$$

$\Rightarrow \sigma(a) = \frac{1}{1+e^{-a}}$  is the logistic sigmoid function

With  $\hat{y} = w_0 + w_1\phi(x_2) + \dots + w_n\phi(x_n) = w^T\phi$ , the model of logistic regression is defined as

$$p(C_1|\phi) = y(\phi) = \sigma(w^T\phi)$$

$$p(C_2|\phi) = 1 - p(C_1|\phi)$$

For dataset  $\phi_n$  and  $t_n$  where  $t_n \in 0;1$  and  $\phi_n = \phi(x_n)$  the likelihood function can be written:

$$p(t|w) = \prod_{n=1}^N y_n^{t_n} (1 - y_n)^{1-t_n}$$

where  $t = (t_1, \dots, t_n)^T$  and  $y_n = p(C_1|\phi)$

We defined the cross function as:

$$L = -\log p(t|w) = -\sum t_n \log(y_n) - \sum (1 - t_n) \log(1 - y_n) = -t_n \log(y_n) - (1 - t)^T \log(1 - y)$$

Taking gradient of the function with respect to  $w$

Using the Chain rule,

$$\frac{\partial L}{\partial w} = \frac{\partial L}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \quad \text{where } z = w^T \phi$$

With:

- $\frac{\partial L}{\partial y} = \left[ \frac{-t^T}{y} + \frac{(1-t)^T}{1-y} \right] = \frac{y-t}{y(1-y)}$
- $\frac{\partial y}{\partial z} = \frac{\partial \sigma(z)}{\partial z} = \left( \frac{1}{1+e^{-a}} \right)' = \sigma(z)(1 - \sigma(z)) = y(1 - y)$
- $\frac{\partial z}{\partial w} = \phi$

$$\Rightarrow \frac{\partial L}{\partial w} = \frac{y-t}{y(1-y)} y(1-y) \phi = (y-t)\phi$$

## Exercise 2

$$f'(x) = f(x)(1 - f(x))$$

$$\frac{f'(x)}{f(x)(1 - f(x))} = 1$$

$$\int \frac{f'(x)}{f(x)(1 - f(x))} dx = \int 1 dx$$

$$\int \left[ \frac{1}{f(x)} + \frac{1}{1 - f(x)} \right] d(f(x)) = x + C$$

$$\ln|f(x)| - \ln|1 - f(x)| = x + C$$

$$\frac{f(x)}{1 - f(x)} = e^{x+C}$$

$$f(x) = e^{x+C} - f(x)e^{x+C}$$

$$f(x) = \frac{e^{x+C}}{1 + e^{x+C}} = \frac{e^x}{C + e^x}$$