Homework week 1

Exercise 1:
A - event that the person is known to have Hansen's disease.

$$P(A) = 0.05$$

A^c - event that the person is not known to have Hansen's disease.

$$P(A^c) = 0.95$$

B - event that the test is positive.

$$P(B|A) = 0.98$$
$$P(B|A^c) = 0.03$$

The probability that someone testing positive for Hansen's disease under this new test actually has it is:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.98 * 0.05}{0.98 * 0.05 + 0.03 * 0.95} = 0.632$$

Exercise 2:
Univariate Normal Distribution.
The distribution is normalized if the area under the curve equals 1.
Assume

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} exp(\frac{-(x-\mu)^2}{2\sigma^2})\, dx$$

Let

$$z = \frac{x-\mu}{\sigma}$$
$$dx = \sigma du$$

Now integral I becomes:

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(\frac{-z^2}{2})\, dz$$

Let

$$y^2 = \frac{z^2}{2}$$
$$y = \frac{z}{\sqrt{2}}$$
$$dz = \sqrt{2}\, dy$$

Then, I becomes:

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-y^2\sqrt{2})\, dy$$

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} exp(-y^2)\, dy$$

Apply Gaussian Integral:

$$\int_{-\infty}^{\infty} exp(-y^2)\, dy = \sqrt{\pi}$$

Hence,

$$I = \frac{1}{\sqrt{\pi}} * \sqrt{\pi} = 1$$

Mean of Univariate Normal Distribution

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} x\, exp(\frac{-(x-\mu)^2}{2\sigma^2})\, dx$$

Let

$$x = \sigma\, y + \mu$$

$$dx = \sigma\, dy$$

Then,

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (\sigma\, y + \mu)\, exp(\frac{-((\sigma\, y + \mu) - \mu)^2}{2\sigma^2})\, dy$$

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, exp(-y^2/2)\, dy \; + \; \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-y^2/2)\, dy$$

$$E(X) = -\frac{\sigma}{\sqrt{2\pi}} \left[\frac{-y^2}{2}\right]_{-\infty}^{\infty} \; + \; \frac{\mu}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$E(X) = 0 + \mu = \mu$$

Calculating $E(X^2)$

$$E(X^2) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} x^2\, exp(\frac{-(x-\mu)^2}{2\sigma^2})\, dx$$

Let

$$x^2 = \sigma^2\, y^2 + 2\sigma\, y\, \mu + \mu^2$$

$$dx = \sigma\, dy$$

Then,

$$E(X^2) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2\, exp(-y^2/2)\, dy \; + \; \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, exp(-y^2/2)\, dy \; + \; \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-y^2/2)\, dy$$

$$E(X^2) = \frac{\sigma^2}{\sqrt{2\pi}} \sqrt{2\pi} + \frac{2\sigma\mu}{\sqrt{2\pi}} * 0 + \frac{\mu^2}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$E(X^2) = \sigma^2 + \mu^2$$

Standard deviation of Univariate Normal Distribution

$$std(X) = \sqrt{var(X} = \sqrt{E(X^2) - E(X)^2} = \sqrt{\sigma^2 + \mu^2 - \mu^2} = \sqrt{\sigma^2}$$