Rotations in 3D: Review of Rotation Mathematics and Quaternions

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Does Maya use quaternions for rotations of objects? I initially started out before even starting to look into the mathematics of rotations with this question. After a not so great look into whether quaternion use existed in Maya, I unwisely concluded that they were not and hence I started looking into as to why they were not used.

Surprise! They are actually used in Maya and all the 3D softwares I have incorporated quaternions to a nice extent. They are used extensively in fact and their importance cannot be refuted in any way. This review paper looks into how quaternions are used to solve problems in various areas, why quaternions are better in expressing rotations and what are their advantages and disadvantages of using them in context of Autodesk Maya.

1. **A Brief Overview of Rotation Mathematics**

   Euler Rotations: The traditional technique to rotate points, rotate frames of references was done using the Euler angles. These allow rotations with three degrees of freedom in three different axis of rotation, i.e the three Cartesian axes. “Euler rotations allow for rotation about one of the three Cartesian axes, a combination of two rotations about two different axes, or a combination of any three rotations” (Caraballo et al., 2022).

To describe rotations, Euler rotations use rotation matrices. A rotation around the x, y, or z axis can be defined by Rx, Ry, and Rz respectively. These matrices describe the rotation by an angle θ around each respective axis. All three matrices can then be combined to find the general rotation matrix (Caraballo et al., 2022).

$$Rx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad Ry = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad Rz = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 1: Rotation Matrices

Quaternions:

Definition 1.1. *A quaternion is a number of the form:*

$$q = s + a\vec{i} + b\vec{i} + c\vec{k}$$

where $s, a, b, c \in \mathbb{R}$ *and* $\vec{i}, \vec{j}, \vec{k}$ *are imaginary. We call* s *the* ***scalar*** *part and* i, j, k *the* ***vector*** *part. We denote the set of quaternions as* $\mathbb{H}$.

Equation 1: The basic form of the quaternion (Caraballo et al., 2022)

Here, $i^2 = j^2 = k^2 = -1$.

Quaternions are also used to determine rotations and for orienting objects.

Theorem 4.1. *Given a unit vector $\vec{u}$ and some vector $\vec{v}$, where $\vec{u}, \vec{v} \in \mathbb{R}^3$, we want to rotate $\vec{v}$ around $\vec{u}$ by an angle θ. To do this we must first find a quaternion*

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{u}.$$

To find the quaternion that represents the rotation:

$$R_q(\vec{v}) = q\vec{v}q^*$$

Equation 2: How the quaternion rotation works is underlined by these formulae (Caraballo et al., 2022)

Although every other method is mathematically simpler to understand, easier to actually code in the software and inherently more intuitive, still some issues arise while working with the simpler methods.

1.1 Gimbal Lock in 3D Softwares

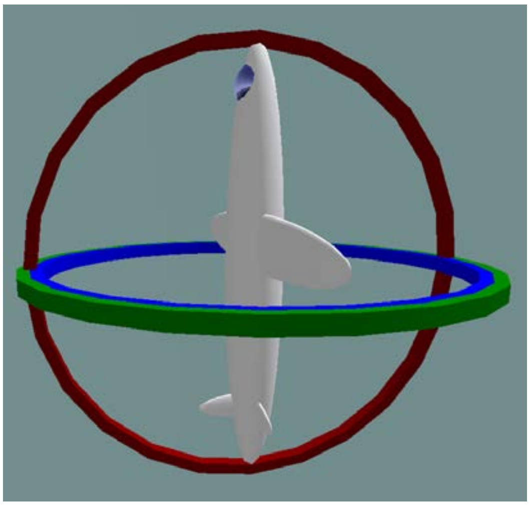

Figure 2: A visual representation of the Gimbal Lock

When trying to rotate things in 3D along three axis of rotation, we have three degrees of freedom to rotate. Along the X, along the Y and along the Z , we are able to rotate our object. Since the axis are distinct, they are independent of each other, hence we can rotate the object around any axis without distortion from the other two.

Gimbal lock occurs when angles of rotation on the axis causes two axis to become parallel, just as the figure above. The blue circle and green circle are parallel here meaning that they are locked. Once this condition is reached is reached , we cannot unlock the axes and one degree of freedom is lost.

Gimbal lock is always an issue while working with Euler rotations. There are ways to mitigate the issue but we cannot completely bypass the problem. This is one of the reasons we use quaternions for precise rotations.

Although matrices are easy to visualize and work with, their calculations are long and tedious, taking a lot of computational power. On the other hand, quaternion calculations are less demanding. “Because calculations with quaternions are not computationally difficult it makes them ideal to use in software that is already computationally heavy, such as those used to model 3D objects. This is because there are fewer processes that needs carrying out by the computer, which leads to less memory and processing power being needed. (Caraballo et al., 2022) ”

2. Benefits of using Quaternions

Researchers have found ways to determine whether what mathematical rotations are more feasible or in more crude terms, better. Researchers (Skublewska-Paszkowska, M ,2013) wanted to find the which rotation method had better efficiency. They created two arm movement animation based on Euler and Quaternion rotation using C++ and Mokka. It was found that for complex movements, the Euler rotation was not reliable, although both results produced adequate animations.

Similarly, (Ariel, Buijs, & Chung, 1970) demonstrated advantages of quaternions by using real time key frame animations. They say that animations become jerky near gimbal lock while using Euler rotations. The researchers also say that using rotation matrices for interpolation is difficult because rotation matrices also represent other transformations such as shear as special conditions needs enforcing.

3. Problem Solving using Quaternions

(Pletincks, 1988), (Pletinckx, 1989) describes a new method for splining quaternions so that they can be used with keyframe animation. The author implements a novel approach that creates new splines called as the cardinal Splines. “Cardinal splines pass through the control points, they don't need additional control points which are not associated with keyframes. Therefore they are most suited for interpolating quaternions in a keyframe animation system. (Pletincks, 1988)”

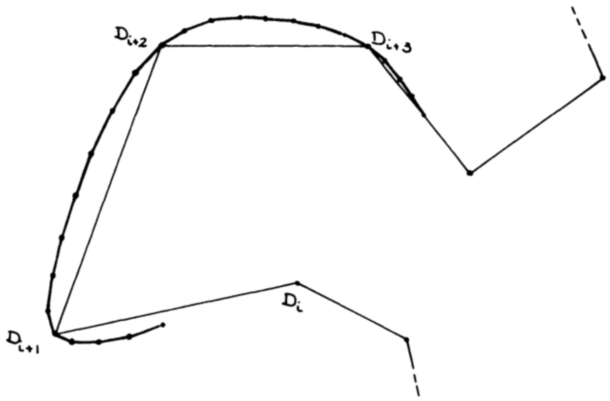

Figure 3: Approximated Cardinal spline (Pletincks, 1988)

The papers also discuss the advantages of using quaternions in rendering and modelling.

(O-larnnithipong, Barreto, Tangnimitchok, & Ratchatanantakit, 2018) outline the orientation correction algorithm for a miniature commercial-grade Inertial Measurement Unit to improve orientation tracking of human hand motion by using the combination of gyroscope, accelerometer and magnetometer measurements. Their algorithm includes use of quaternion to correct orientation using gravity vector and magnetic North vector.

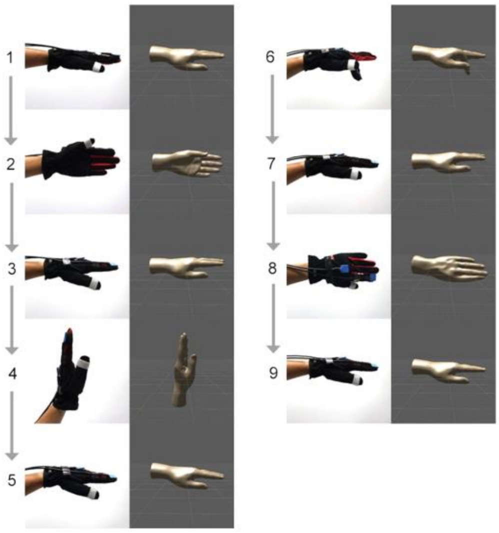

Figure 4: Orientation Tracking of a hand

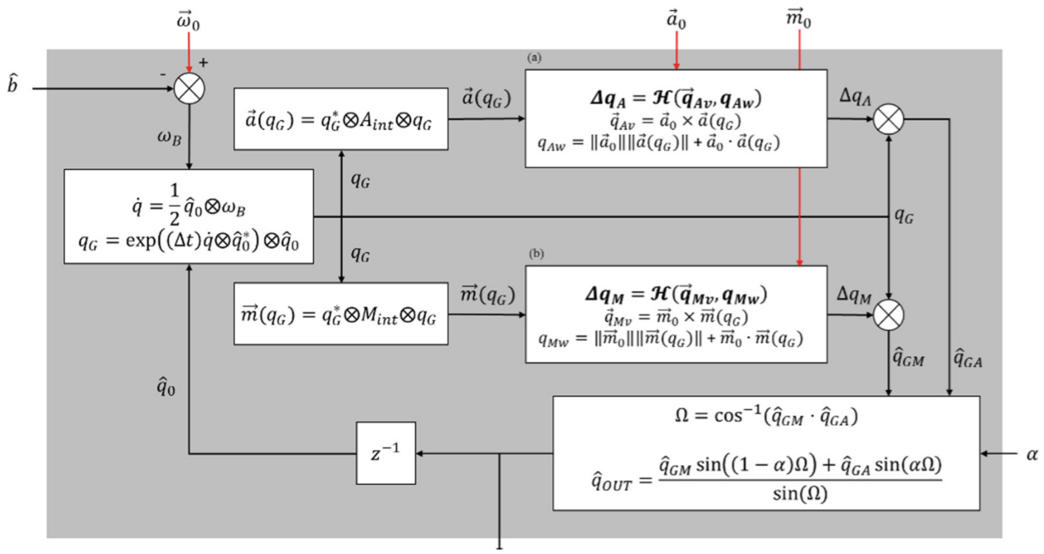

Figure 5: Proposed Drift correction Algorithm

4. Quaternions in other fields

Whenever there are rotations involved, we will see quaternions in action. Incidently, quaternions are not only limited to 3D graphics in their application. (Mansur, Reddy, R, & Sujatha, 2020) present the implementation of complementary filter using quaternion angles to mitigate the gimbal lock occurrence in drones.

Another one, from Chemistry. "The relative superiority of the quaternion method when applied to molecular docking is demonstrated by practical experiment, as is the crucial importance of proper adjustment calculations in search methods. (Skone, Cameron, & Voiculescu, 2013)"

5. Conclusion

Quaternions is an interesting bunch of mathematical field that describes rotation quite well in 3D space. Although good for 3D, they are not well defined for 2D. Even their use for 3D softwares is not perfect, meaning they cannot be used everywhere and it is very context dependent while animating in Maya. Yet they have made life easier for so many people and so many different fields. We can all look upto what the future has for rotations and what new mathematics we will be able to witness in this area.

6. References

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