Markov Chains

Example :

$$\text{Rain} \xrightarrow{.6} \text{Sunny}$$

$$\text{Sunny} \xrightarrow{.5} \text{Rain}$$

today   tomorrow

Sunny = State 1

Rain = State 2

$\boxed{\text{Markov Property}}$: Transition probability b/w states are affected only by <u>Current states</u>

$$P\left(\begin{matrix}\text{Tomorrow is} \\ \text{Sunny}\end{matrix} \,\middle|\, \begin{matrix}\text{all past} \\ \text{weather}\end{matrix}\right) = P\left(\begin{matrix}\text{Tomorrow's} \\ \text{sunny}\end{matrix} \,\middle|\, \begin{matrix}\text{Today} \\ \cancel{\text{Yesterday}} \\ \text{was sunny}\end{matrix}\right)$$

Transition Matrix

$$\text{Rain} \xrightarrow{.6} \text{Sunny}$$

$$\text{Sunny} \xrightarrow{.5} \text{Rain}$$

$$\mathbb{P} = \begin{bmatrix} .5 & .5 \\ .6 & .4 \end{bmatrix}$$

today is sunny

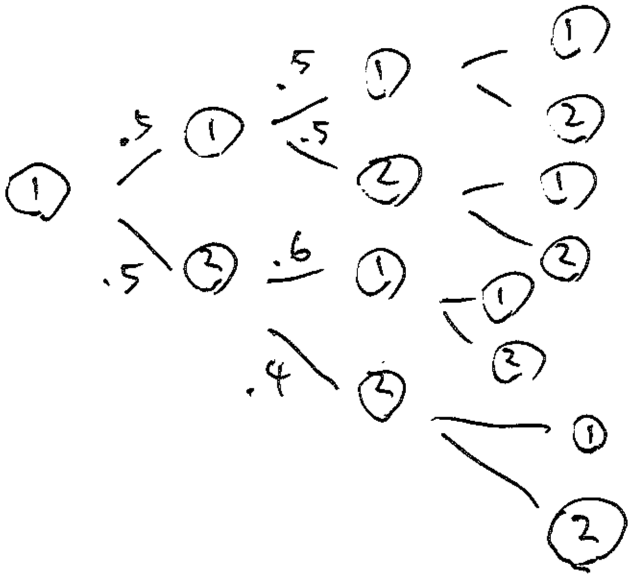
Tomorrow's Prob. Distribution.

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .6 & .4 \end{bmatrix} = \begin{bmatrix} .5 & .5 \end{bmatrix}$$

How can we calculate Prob. Dist. in 3 days?

$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}.5 & .5\\ .6 & .4\end{bmatrix}^3 = \begin{bmatrix}.545 & .455\end{bmatrix}$$

Chapman - Kolmogorov equation



R:

```
p = matrix(c(.5,.5,.6,.4),2,2,
                     byrow = T)

p %*% p %*% p
```

What if that's too unrealistic ...

Example (4.4): ~~W~~Weather depends ~~on~~ history of past 2 days.

Past two days	Prob.	today
sunny, sunny	$\xrightarrow{.8}$	sunny
sunny, Rain	$\xrightarrow{.5}$	Sunny
Rain, sunny	$\xrightarrow{.6}$	sunny
Rain, Rain	$\xrightarrow{.3}$	sunny

Can we use Markov Chain to ~~deat~~ Model this phenomenon?

If past two days were sunny,

SS = state ①

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix} = \begin{bmatrix} .8 & .2 & 0 & 0 \end{bmatrix}$$

Example 4.7 : Auto Insurance

Bonus-Malus system
good - bad

Driver = state

Premium $\propto$ state.

		Next state if			
current state	Premium	0 claims	1 claim	2 claim	3^{+} claim
1	200	1	2	3	4
2	300	1	3	4	4
3	400	2	4	4	4
4	600	3	4	4	4

what is $\mathbb{P} = ?$ if (# of claim) $\sim \text{Poi}(\lambda)$

$\lambda =$ average # of claims per year.

Poisson Distribution

$$X \sim Poi(\lambda)$$

pmf

$$p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$E[X] = \lambda$$

$$V[X] = \lambda$$

R: dpois(x, λ)

ppois(x, λ)

# of claim $\sim$ Poi(λ)

$$P(\text{\# of claims} = 0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-\lambda} =: a_0$$

$$P(\text{\# of claim} = 1) = \frac{e^{-\lambda}\lambda}{1!} =: a_1$$

$$P(\text{\# of claim} = 2) = \frac{e^{-\lambda}\lambda^{2}}{2!} =: a_2$$

$$P(\text{\# of claim} \geq 3) = 1-(a_0+a_1+a_2)$$

Claim

	0	1	2	≥3
1	1	2	3	4
2	1	3	4	4
3	2	4	4	4
4	3	4	4	4

$$\mathbb{P} = \begin{bmatrix} a_0 & a_1 & a_2 & 1-(a_0+a_1+a_2) \\ a_0 & 0 & a_1 & 1-(a_0+a_1) \\ 0 & a_0 & 0 & 1-a_0 \\ 0 & 0 & a_0 & 1-a_0 \end{bmatrix}$$

→ each row adds up to 1.

→ Markov Property : $\mathbb{P}$ same each year.