4.4 Limiting Probabilities

Ex: 2-stage weather.

Sunny - 1

Rain - 2

$$\mathbb{P} = \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

$$\mathbb{P}^4 = \begin{bmatrix} .5789 & .4251 \\ .5668 & .4332 \end{bmatrix}$$

$$\mathbb{P}^8 = \begin{bmatrix} .572 & .428 \\ .570 & .430 \end{bmatrix}$$

$$\mathbb{P}^{20} = ?$$

$$\mathbb{P}^{20} = \begin{bmatrix} .571 & .428 \\ .571 & .428 \end{bmatrix}$$ ← does not depend on i.

$\mathbb{P}^{21}$ will be the same.

$\lim\limits_{n \to \infty} P_{11}^{n} = .571$ since it doesn't depend on i.

$\lim\limits_{n \to \infty} P_{i1}^{n} = .571$. (limiting probability).

$$\mathbb{P}^{20} \approx \lim_{n\to\infty} \mathbb{P}^{n}$$ limiting prob.

all rows will be the same.
in $\mathbb{P}^{20}$

$[.571 \quad .428]$ = long-run distribution of states.
① ②

i.e. In a long-run, 57.1 % of days will be sunny.

This means ...

$$[.571 \quad .428] \cdot \mathbb{P} = [.571 \quad .428]$$

(That's why $\mathbb{P}^{20} = \mathbb{P}^{21}$).

i.e. $[.571 \quad .428]$ is an solution to equation

$$\underline{\pi} \cdot \mathbb{P} = \underline{\pi}$$

row vector.

Notation:

$$\underline{\pi} = [\pi_1 \quad \pi_2]$$

Interpretation (Stationary Probabilities)

π_1 : ① limiting distribution of state 1

: ② long-Run proportion of times that MC will be in stage 1.

: ③ $= \dfrac{1}{E(\text{time it takes from 1 to 1})} := \dfrac{1}{m_{11}}$.

3

Say

m_{11} = E(time until state 1, starting from 1)

= 22.

→ MC Come back to state 1 every 22 time units.
(on Average)

→ Proportion ~~amount~~ of time spent in state 1.

$\frac{1}{22}$.

Example: Two-day Weather.

SS - 1
SR - 2
RS - 3
RR - 4

$$\mathbb{P} = \begin{bmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix}$$

$$\underline{\pi} = [.45 \quad .15 \quad .15 \quad .25]$$

$$\frac{1}{\underline{\pi}} = [\,2.22 \quad 6.67 \quad 6.67 \quad 4\,]$$

mean return time to each state.

Theory:

CASE A

If MC is <u>irredocible</u>, <u>positive recurrent</u>, <u>aperiodic</u>,

Then

$$\boxed{\pi_j = \lim_{n\to\infty} P_{ij}^{n}}$$

exists and unique.

Case B

If MC contains <u>one positive recurrent</u> state that is accessible from all other states.

Then

$$\boxed{\pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{k}}$$

exists and unique.

Ex 4.13

$$\mathbb{P} = \begin{bmatrix} 0 & 0.5 & .5 \\ 1 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .5 & 0 & .5 \end{bmatrix}$$

Should Π exist?

which case?

All states communicates (irreducible)

$\Downarrow$

All states must be pos. recurrent.

Ex

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ .25 & .25 & .25 & .25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Should $\underline{\pi}$ exist?

If so, which case?

state 1, 2 } (positive) recurrent

3 } transient

4 } absorbing (pos recurrent),

but {1,2} nor {4} is accessible

from <u>all</u> states.

$\longrightarrow \underline{\pi}$ DNE.

Solve $\underline{\pi} = \underline{\pi}\,\mathbb{P}$ by hand?

$$\underline{\pi} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

$$\begin{cases} .7\pi_1 + .4\pi_2 = \pi_1 \\ .3\pi_1 + .6\pi_2 = \pi_2 \end{cases}$$

linearly independent? No

use extra condition $\pi_1 + \pi_2 = 1$.

Ex.

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 \\ .5 & .5 & 0 \\ .33 & .33 & .34 \end{bmatrix}$$

should $\underline{\pi}$ exist ?

$1 \leftrightarrow 2$ but you can't go to 3

from 1 or 2.

↳ 3 is transient.

(not irreducible).

1 and 2

are recurrent.

(pos).

and

1, 2 accessible from all states.

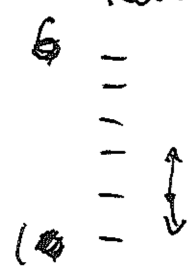
→ Π exist

(case B)

State i has period d if ...

$$p_{ii}^{n} = 0 \quad \text{whenever } n \text{ is not divisible by } d.$$

<u>Ex</u>. Periodic Random Walk

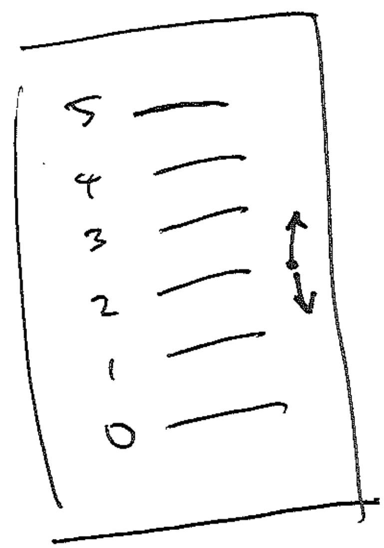


state 5 → down w.p. 1.

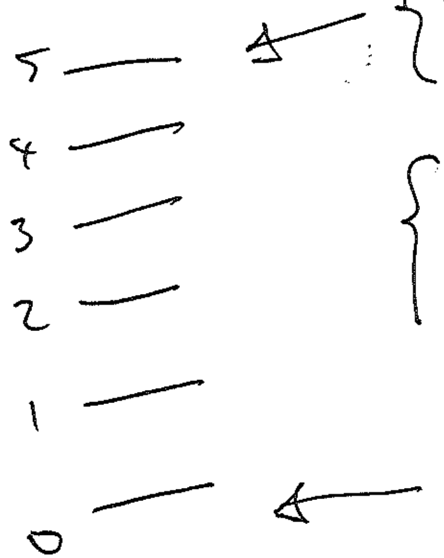
$$1 \sim 4 \begin{cases} \text{up} & \text{with prob. } .5 \\ \text{down} & \text{w.p. } .5. \end{cases}$$

state 0 → up w.p. 1

$$\mathbb{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Ex Allow to stay at end points.



$$\begin{cases} \text{down} & 50\% \\ \text{stay} & 50\% \end{cases}$$

$$\begin{cases} \text{up} & 50\% \\ \text{down} & 50\% \end{cases}$$

$$\begin{cases} \text{up} & 50\% \\ \text{stay} & 50\% \end{cases}.$$

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & .5 & .5 \end{bmatrix}$$

Period is a class property:

if state i has period d,

state i communicates with j.

then state j has period d.

period 1 = aperiodic.