Computing $P_{ij}(t)$ and Uniformization

Suppose $\nu_i = \lambda_i + \mu_i = \nu$. then we can write,

$$P_{ij}(t) = P(X_{(t)} = j \mid X_{(0)} = i)$$

$$= \sum_{n=0}^{\infty} P(X_{(t)} = j \mid X_{(0)} = i, N_{(t)} = n) \cdot P(N_{(t)} = n \mid X_{(0)} = i)$$

$$= \sum_{n=0}^{\infty} P_{ij}^{n} \frac{e^{-\nu t} (\nu t)^n}{n!}$$

$$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{n} \frac{e^{-\nu t} (\nu t)^{n}}{n!}}$$

if $\underline{\nu_i = \nu}$

P_{ij}^{n} from D.t.M.C.

Achieve this by Uniformization.

Uniformization

For ν_i let ν be upper bound such that

$\nu_i \le \nu$ for all i, then

let your D.T.M.C. allow to jump to itself

$$P^*_{ij} = \begin{cases} 1 - \frac{\nu_i}{\nu} & i = j \\ \frac{\nu_i}{\nu} P_{ij} & i \neq j \end{cases}$$

$$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{*n} \frac{e^{-\nu t}(\nu t)^n}{n!}}$$

$$P_{ij}^{*} = \begin{cases} 1 - \dfrac{\nu_i}{\nu} & i = j \\[2ex] \dfrac{\nu_i}{\nu} P_{ij} & i \neq j \end{cases}$$

Ross

Ex 6.21

Machine Breaks $\sim \text{Exp}(\lambda)$.

repairs ~~[illegible]~~ made $\sim \text{Exp}(\mu)$.

States.

$$\begin{cases} 0 = \text{working} \\ 1 = \text{not} \end{cases}$$

$\lambda_0 = \lambda \quad \mu_0 = 0$ $\quad \nu_0 = \lambda$

$\lambda_1 = 0 \quad \mu_1 = \mu$. $\quad \nu_1 = \mu$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$$

$$P_1'(t) = -\mu P_1(t) + \lambda P_0(t)$$

$\lambda P_0 = \mu P_1$

$P_1 + P_0 = 1$

Let $\nu = \lambda + \mu$.

before,

$$\mathbb{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now,

$$\mathbb{P}^* = \begin{bmatrix} 1 - \frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & 1 - \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

$\mathbb{P}^*$ already is limit dist.

$$P_{00}(t) = \sum_{n=0}^{\infty} P_{00}^{n} e^{-(\lambda+\mu)t} \left[(\lambda+\mu)t\right]^{n} \Big/ n!$$

$$= e^{-(\lambda+\mu)t} + \left(\frac{\mu}{\lambda+\mu}\right) \sum_{n=1}^{\infty} e^{-(\lambda+\mu)t} \left[(\lambda+\mu)t\right]^{n} \Big/ n!$$

$$= e^{-(\lambda+\mu)t} + \left(\frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t} \Big[\underbrace{\sum_{n=0}^{\infty}}_{e^{(\lambda+\mu)t}} - 1 \Big]$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

Let $D(t)$ = total ~~down~~ working time of Machine in $[0, t]$, (occupation time).

$$E[D(t)] = ?$$

Let $I(s) = \begin{cases} 1 & X(s) = 0 \text{ (working)} \\ 0 & X(s) = 1 \text{ (not)} \end{cases}$

$$E(D(t)) = E\left[\int_0^t I(s)\,ds\right].$$

$$= \int_0^t E\left(\mathrm{II}(s)\right)\, ds$$

$$= \int_0^t P_{00}(s)\, ds$$

$$= \frac{\mu}{\lambda+\mu}\, t + \frac{\lambda}{(\lambda+\mu)^2}\left[1 - e^{-(\lambda+\mu)t}\right]$$