Estimating Volatility

σ : volatility.

→ historical volatility

e.g. observe 10 weekly price.

compute sample SD. $\hat{\sigma}_{weekly}$

$$\hat{\sigma}_{annual} = \sqrt{52}\ \hat{\sigma}_{weekly}.$$

Table 11, 1   p361.

Black - Scholes Formula

Black - Scholes Formula (1973)

- Derived using binomical option pricing.
- Only for European Option.

Assumptions :

(A1) Cont. Comp. return is Normally distributed. independent over time.

$$e^{(r-\delta)h \pm \sigma\sqrt{h}} \rightarrow e^{N(0,\sigma)}$$

(A2) σ is known and constant.

(A3) dividends are known as \$ or δ.

Assumptions in environment:

(E1) r is known and constant.

(E2) no transaction costs or taxes

(E3) You can short-sale costlessly and borrow at risk-free rate.

Black - Scholes Call Option

$$\boxed{C(S, K, \sigma, r, T, \delta) = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)}$$

S = current stock price

K = strike

T = time to expiration

r = risk-free rate

δ = dividend rate

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$N(\cdot) = \text{CDF of Standard Normal} = \Phi(\cdot)$$

Black-Scholes put option

$$\boxed{P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)}$$

all notation same as

Call op.

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\boxed{d_1 = \frac{\ln(S/K) - (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}$$

$$= \frac{\ln(S/K) - (r - \delta)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

$$d_2 = \frac{\ln(S/K) - (r - \delta)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$$

$$\boxed{= \frac{\ln(S/K) - (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}$$

Binomial Model and B-S formula

If $\delta = 0$, then

$$\begin{array}{c}\text{CRR} \\ \text{Binomial} \\ \text{Pricing}\end{array} \xrightarrow[n \to \infty]{} \begin{array}{c}\text{B-S} \\ \text{pricing}\end{array}.$$

European Call in 1yr 1-step

Example

$r = .08$

$\delta = 0$

$\sigma_{annual} = .2$

$h = 1$ yr

$S = 30$

$K = 30$

$$\begin{array}{ccc} & & 39.694 \\ & & 9.694 \\ 30 & & \\ C_0 & & \\ & & 26.608 \\ & & 0 \end{array}$$

$$C_0 = e^{-.08}\left[p^*(9.694) + (1-p^*)(0)\right]$$

$$= \boxed{4.0287}$$

$p^* = .4502$

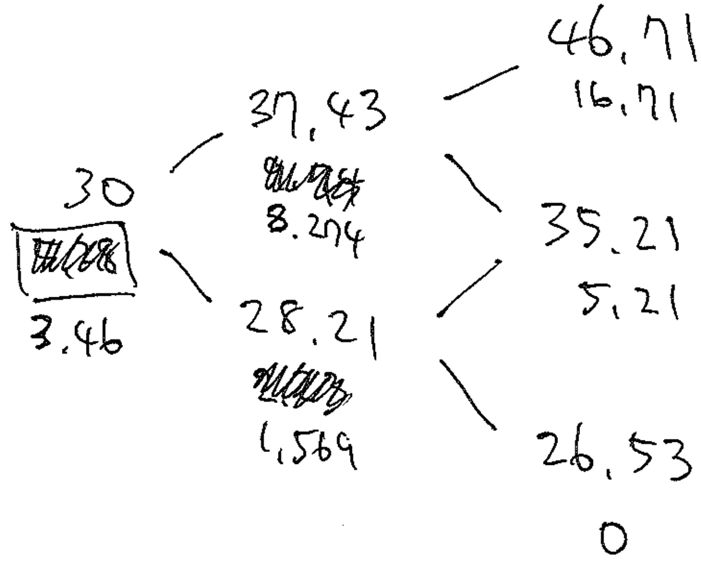
$u = 1.3231$

$d = .8869$

European Call in 1 yr

Bin 2-step

$$\begin{cases} r = .08/2 \\ \delta = 0 \\ \sigma_{semi} = .2/2 \\ h = 2 \quad (1h = \tfrac{1}{2}\text{ year}) \\ S = 30 \\ K = 30 \end{cases}$$



$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = 1.2478$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = .9404$$

$$C = e^{-rh}\left[p^*(uS) + (1-p^*)(dS)\right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = .4647035$$

European Call in 1 yr

B-S

$r = .08$

$\delta = 0$

$\sigma_{ann} = .2$

$T = 1$

$S = 30$

$K = 30$

$$C = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = .5$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) - \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = .3$$

$N(.5) = .6915$

$N(.3) = .6179$

$$C = \boxed{3.632}$$

Option Greeks

		Change in	when
Δ	: delta	Option price ~~change~~	$S \uparrow \$1$
Γ	: gamma	Δ change	$S \uparrow \$1$
Θ	: theta	Option price	$T \downarrow 1$ day
ρ	: rho	Option price	$r \uparrow 1\%$
Ψ	: psi	Option price	$\delta \uparrow 1\%$
Vega		Option price	$\sigma \uparrow 1\%$.

Δ : Delta

Binomial pricing

Op. price $= \Delta S + B$ $\qquad \Delta = \#$ of stock-shares.

$S \uparrow \$1$. $\qquad$ Op. price $\uparrow$ by $\$\Delta$.

"Dollar risk of the option relative to stock".

Delta Δ :

rate of change in C w.r.t. S.

$$\frac{d}{dS} C(B\text{-}S) = \Delta \text{ for Call}$$

$$\frac{d}{dS} P(B\text{-}S) = \Delta \text{ for put}.$$

put-Call parity says.

$$P(\ ) = C(\ ) + Ke^{-rT} - Se^{-\delta T}$$

$$\frac{d}{dS}P = \frac{d}{dS}C - e^{-\delta T}.$$

Delta Δ :

$$\frac{d}{dS} C(B\text{-}S) = \frac{d}{dS}\left[S e^{-\delta T} N(d_1) - K e^{-r(T)} N(d_2) \right]$$

S inside

S inside

$$d_2 = d_1 - \sigma\sqrt{T}.$$

$$= e^{-\delta T} N(d_1) + \underbrace{S e^{-\delta T} N'(d_1)\left(\frac{d}{dS} d_1\right) - K e^{-rT} N'(d_2)\left(\frac{d}{dS} d_2\right)}_{\star}$$

$$\frac{d}{dS} d_2 = \frac{d}{dS} d_1 - 0.$$

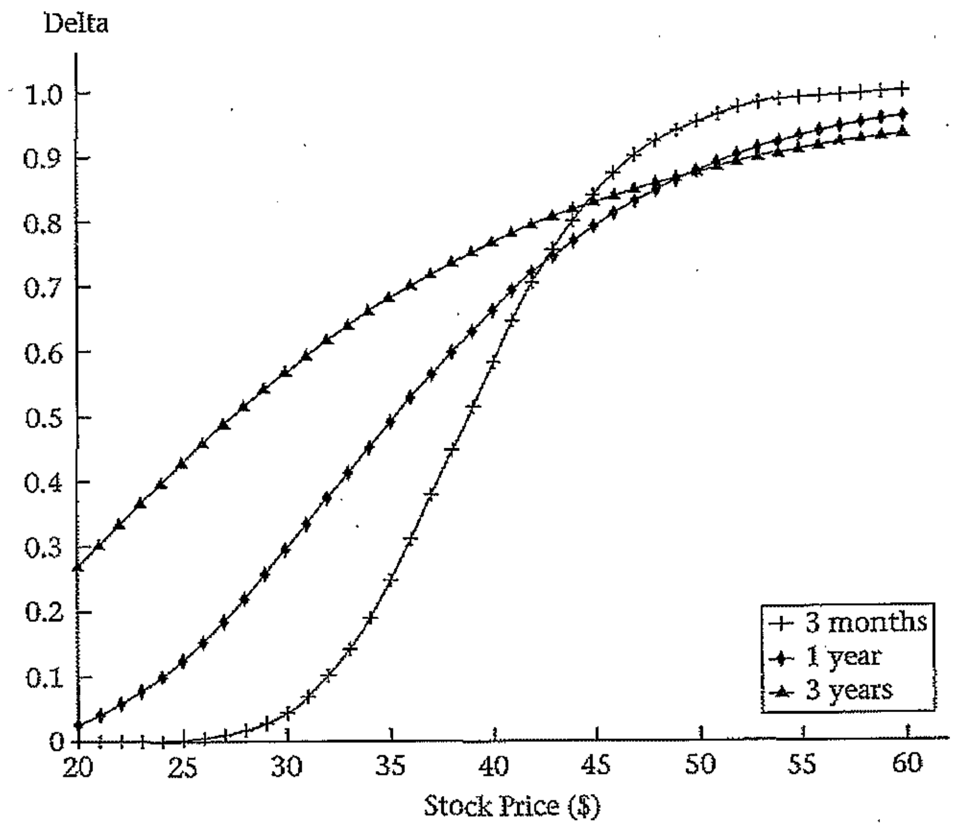
Cancel out after long calculation.

$$= \boxed{e^{-\delta T} N(d_1)}$$ call

Δ_{call} $K = 40$

FIGURE 12.1

Call deltas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



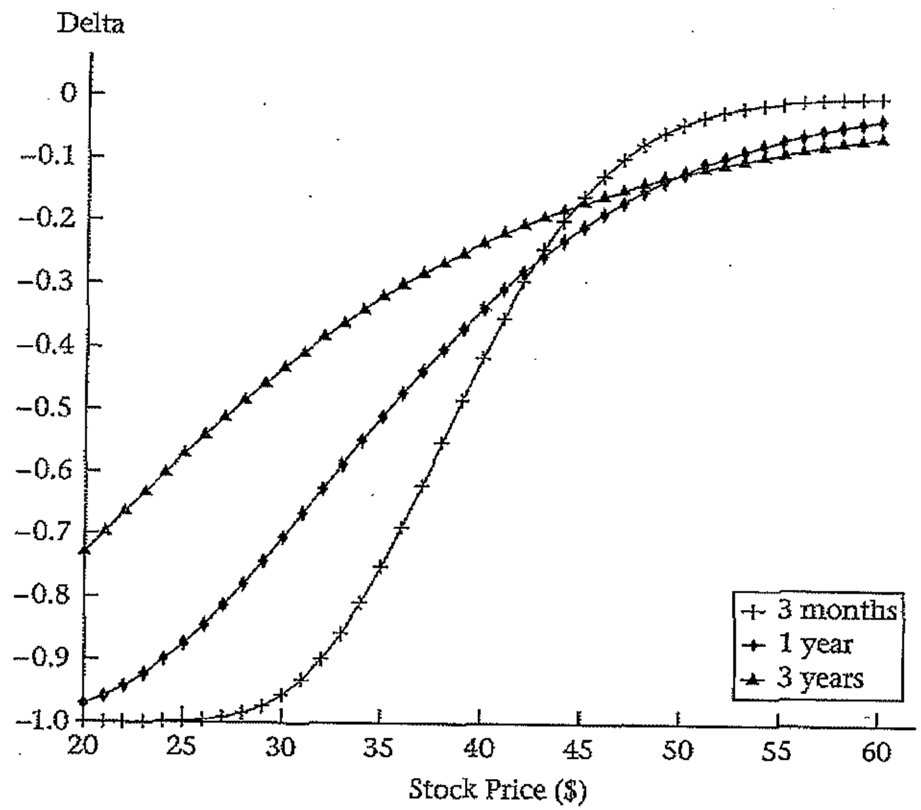
S

$$\Delta_{call} = e^{-\delta T} N(d_1)$$

Δ_{put}

FIGURE 12.2

Put deltas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



$$\Delta_{put} = -e^{\delta T} N(-d_1)$$

Gamma (Γ)

change in Δ as S changes.

$$\frac{\partial}{\partial S}\Delta = \frac{\partial^2}{\partial S^2} C(S, K, \sigma, r, T-t, \delta)$$

$$= \frac{\partial}{\partial S} e^{-\delta(T-t)} N(d_1)$$

$$= e^{-\delta(T-t)} \underbrace{N'(d_1)}_{\text{Normal density.}} \cdot \left(\frac{\partial}{\partial S} d_1\right)$$

$$\frac{\partial}{\partial S} d_1 = \frac{\partial}{\partial S}\left[\frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]$$

$$= \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{S/K} \cdot \frac{1}{K}$$

$$= \frac{1}{S\sigma\sqrt{T-t}}$$

$$\boxed{\Gamma = \frac{e^{-\delta(T-t)} \cdot f_N(d_1)}{S\sigma\sqrt{T-t}}}$$ call

Put-Call Parity says ...

$$P(B-S) = C(B-S) + Ke^{-rT} - Se^{-\delta T}$$

take $\frac{d^2}{dS^2}$ of both sides and,

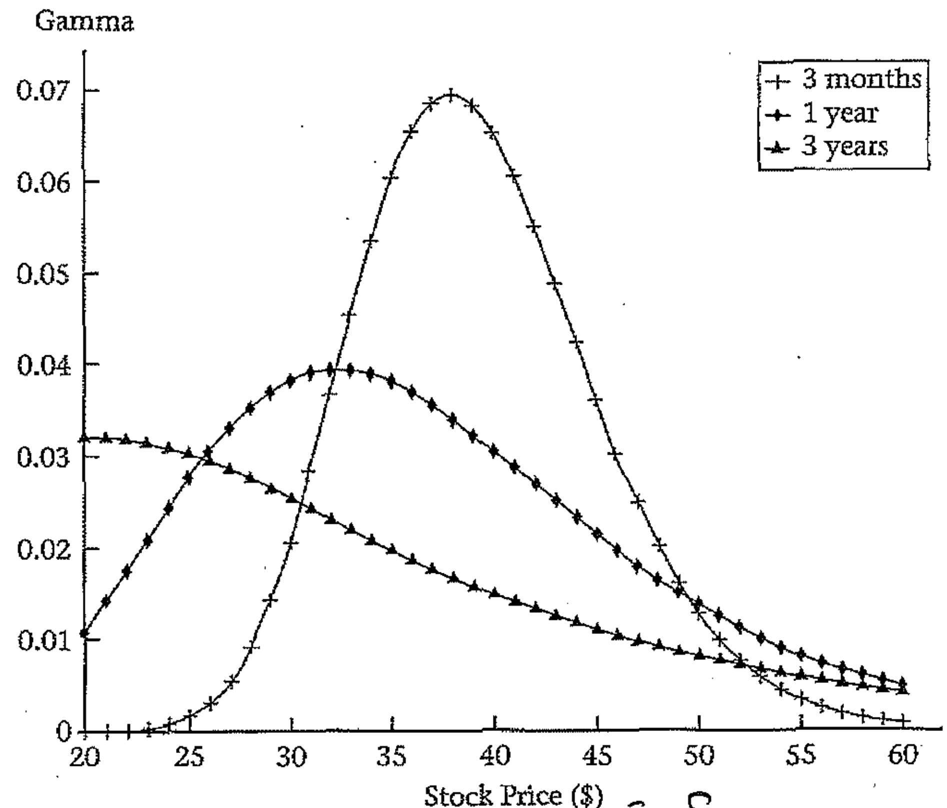
$$\frac{d^2}{dS^2} P(B-S) = \frac{d^2}{dS^2} C(B-S).$$

$$\Gamma_{call} = \Gamma_{put}.$$

Γ_{call} $K = 40$

FIGURE 12.3

Call gammas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



$= T$

$= S$

$$\Gamma_{call} = \frac{e^{-\delta T} N'(d_1)}{S \sigma \sqrt{T}} = \Gamma_{put}$$

Vega

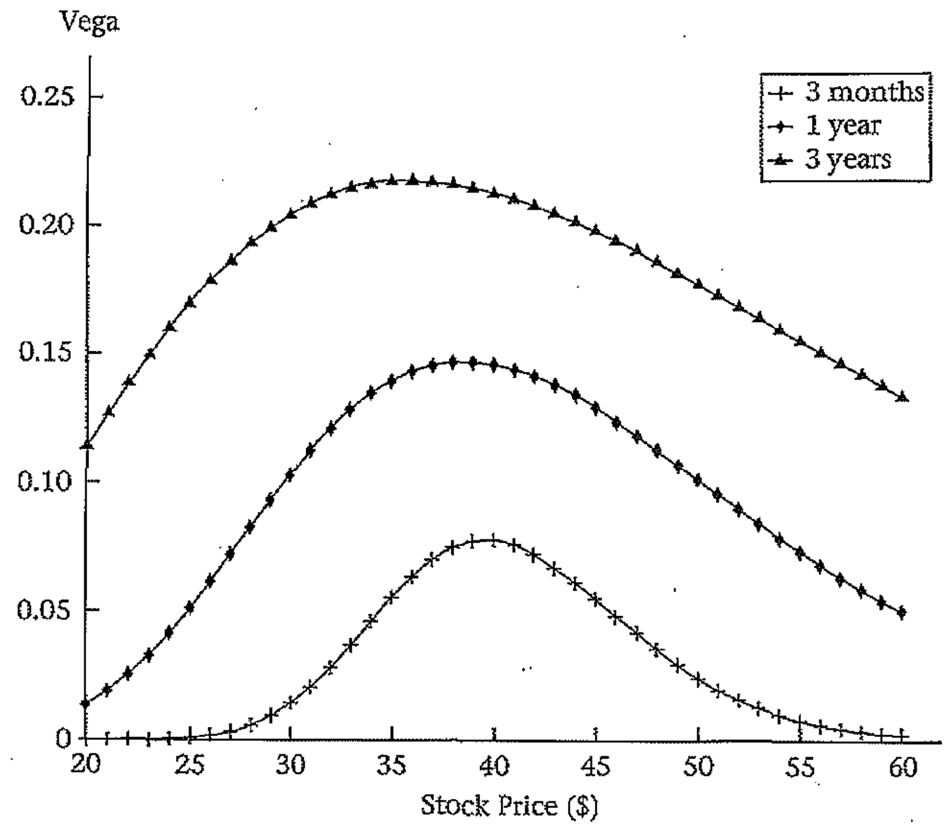
How change in σ by 1 % change

Option price ?

Vega.

FIGURE 12.4

Call vegas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



S

K = 40.

Dollar risk of the Option

Stock Price change by $\$\varepsilon$.

change in Option price

$$= \Delta \varepsilon .$$

$$= e^{-\delta T} N(d_1) \cdot \varepsilon$$

Example 12.7

$S = 41$

$K = 40$

$\sigma = .3$

$r = .08$

$\delta = 0$.

$T = 1$

European Call.

40-strike Call in 1yr

\$ 6.961

$$c(B\text{-}S) = 6.961$$

$$\Delta = .6911$$

If options are to buy 1000 stocks,

$$\underset{\text{Option}}{\Delta} = 1000 \cdot \underset{\text{per stock}}{\Delta} = 691.1$$

Same as $\Delta S + B$

hold 691.1 stock.

if $S \uparrow$ by \$1, $C_{\text{option}} \uparrow$ by 691.1.

Option Elasticity

If Stock price change by $\$\varepsilon$. then

$$\% \text{ change in Stock} = \frac{\varepsilon}{S},$$

But Option price will change by $\Delta\varepsilon$. Then,

$$\% \text{ change in Option price} = \frac{\Delta\varepsilon}{C}.$$

$$\text{Option Elasticity} = \Omega = \frac{\% \text{ ch. in Op.}}{\% \text{ ch. in Stock}} = \frac{\frac{\Delta\varepsilon}{C}}{\frac{\varepsilon}{S}} = \boxed{\frac{\Delta S}{C}}$$

For call $\Omega \geq 1$

For put $\Omega \leq 0$.