Heteroscedasticity

and

GARCH model

Heteroscedastic : variance not constant.

Homoscedastic : constant variance.

Generalized AutoRegressive Conditionally Heteroscedastic model

GARCH

SP500 and logreturn

Stylized facts about return data

- uncorrelated. (Ljung-Box)
- Squares are correlated (McLeod-Li)
- Clustering.
- Asymmetry
- Heavy tailed distribution.

ARCH(1) model

$$\begin{cases} Y_t = \sigma_t e_t & e_t \sim N(0,1) \\ \sigma_t^2 = \omega + \alpha Y_{t-1}^2 . \end{cases}$$

ARCH(p) model

$$\sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2 + \cdots + \alpha_p Y_{t-p}^2 .$$

ARCH is uncorrelated

$$E(Y_t) = E(\sigma_t)\underbrace{E(e_t)}_{0} = 0$$ if $E(\sigma_t)$ is finite.

(or look at conditional mean)

$$\begin{aligned}
Cov(Y_{t+1}, Y_t) &= E(Y_{t+1} Y_t) \\
&= E\left[(\sigma_{t+1} \cdot e_{t+1})(\sigma_t \cdot e_t)\right] \\
&= E(\sigma_{t+1} \cdot \sigma_t) \cdot \underbrace{E(e_{t+1})}_{=0} \cdot \underbrace{E(e_t)}_{=0}. \\
&= 0.
\end{aligned}$$

Squares of ARCH is Correlated

$$\mathrm{Cov}(Y_{t+1}^2, Y_t^2) = E(Y_{t+1}^2, Y_t^2)$$

$$= E(\sigma_{t+1}^2 \cdot \sigma_t^2) \cdot \underbrace{E(e_{t+1}^2)}_{1} \underbrace{E(e_t^2)}_{1}$$

$$= E(\sigma_{t+1}^2 \cdot \sigma_t^2)$$

$$= E\left[(\omega + \alpha Y_t^2)(\omega + \alpha Y_{t-1}^2)\right]$$

$$= \omega^2 + \omega\alpha E(Y_t^2) + \omega\alpha E(Y_{t-1}^2) + \alpha^2 E(Y_t^2 \cdot Y_{t-1}^2)$$

$$> 0.$$

Writing ARCH(1) as AR(1)

$$\begin{cases} y_t = \sigma_t e_t \\ \sigma_t^2 = \omega + \alpha Y_{t-1}^2 \end{cases}$$

$$\begin{cases} y_t^2 = \sigma_t^2 e_t^2 \\ \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 \end{cases}$$

Subtract the two

$$y_t^2 - \alpha_0 - \alpha_1 Y_{t-1}^2 = \sigma_t^2 e_t^2 - \sigma_t^2$$

$$y_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + \underbrace{\sigma_t^2 (e_t^2 - 1)}$$

not i.i.d. $E(\cdot) = 0$.

but uncorrelated (martingale difference)

Var of ARCH(1)

$$Y_t = \sigma_t e_t \qquad \sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2$$

$$E(Y_t) = 0.$$

$$V(Y_t) = E(Y_t^2) = E(\sigma_t^2)\underbrace{E(e_t^2)}_{1}$$

$$= E(\omega + \alpha_1 Y_{t-1}^2)$$

$$= \omega + \alpha \underbrace{E(Y_{t-1}^2)}_{V(Y_t)}$$

$$V(Y_t) = \boxed{\frac{\omega}{1-\alpha}}$$

Conditional Variance for ARCH

$$Y_t = \sigma_t \, \varepsilon_t \quad , \quad \sigma_t^2 = \omega + \alpha Y_{t-1}^2$$

$$\mathrm{Var}(Y_t \mid Y_{t-1}, \sigma_{t-1})$$

$$= \mathrm{Var}(\sigma_t \mid Y_{t-1}, \sigma_{t-1}) \; 1$$

$$= \sigma_t^2 \; 1 .$$

σ_t^2 : Conditional variance.

ARCH(1)

$$E(y_t^4) = \frac{3\omega^2}{(1-\alpha)^2}\left(\frac{1-\alpha^2}{1-3\alpha^2}\right) \qquad 3\alpha^2 < 1$$

Kurtosis

$$K = \frac{E(y_t^4)}{[E(y_t^2)]^2} = 3\left(\frac{1-\alpha_1^2}{1-3\alpha_1^2}\right) > \underset{\text{Normal}}{3}$$

ARCH has heavy tails even though

$e_t \sim N(0,1)$

GARCH(1,1) models

$$Y_t = \sigma_t e_t \qquad e_t \sim IID(0, 1)$$

obs.

Conditional SD

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2$$

Conditional Variance

ARCH(p)

$$\sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2 + \alpha_2 Y_{t-2}^2 + \cdots + \alpha_p Y_{t-p}^2$$

GARCH(p,q)

$$\sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2 + \cdots + \alpha_p Y_{t-p}^2$$

$$+ \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

ARMA - GARCH models

$$\Phi(B) Y_t = \Theta(B) \varepsilon_t$$

$$\varepsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2$$

$Y_t \sim ARMA(p,q)$ with error $\varepsilon_t \neq$ iid.

$\sim WN$.

$\varepsilon_t \sim GARCH(1,1)$

Examples

Cryer : CREF stock values.

Shumway : NYSE, US GNP

Cowpertwait : SP500, Southern hem. Temp.