Spectral Analysis

Spectral Densities

Let $\{X_t\}$ be zero-mean stationary time series with

$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. spectral density

$$\boxed{f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)}$$

$$\underbrace{-\infty < \lambda < \infty}$$

$$-\pi < \lambda \leq \pi$$

$$i = \sqrt{-1}$$

$$e^{i\lambda} = \cos(\lambda) + i \sin(\lambda)$$

Properties of $f(\lambda)$

① f is even $\quad f(\lambda) = f(-\lambda)$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} [\underbrace{\cos(h\lambda)}_{\text{even}} + i \underbrace{\sin(h\lambda)}_{\text{odd}}] \underbrace{\gamma(h)}_{\text{even}}$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(h\lambda) \gamma(h) + 0 .$$

$$= f(-\lambda) .$$

(2) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$

(3) $\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda$ for all k.

$$= \int_{-\pi}^{\pi} e^{ik\lambda} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)\, d\lambda$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \underbrace{\int_{-\pi}^{\pi} e^{i(k-h)\lambda}\, d\lambda}$$

$$\begin{cases} 0 & \text{if } k \neq h \\ 1 & \text{if } k = h \end{cases}.$$

$$= \gamma(k)$$

Spectral Densities are essentially unique

$f(\cdot)$ will determine $\gamma(\cdot)$.

<u>Thm</u>. Real valued f defined on $(-\pi, \pi]$ is spectral density of a <u>stationary</u> time series if and only if

$$\begin{cases} \bullet\ f(\lambda) = f(-\lambda) \\ \bullet\ f(\lambda) \geq 0 \\ \bullet\ \int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty \end{cases}$$

Corollary Absolutely summable function $\gamma(\cdot)$ is ACVF of a stationary time series if and only if

$$\begin{cases} \gamma(h) = \gamma(-h) \\ f(\lambda) \geq 0 \quad \text{for all } \lambda \in (-\pi, \pi] \end{cases}$$

→ Not all ACVF have a spectral density.

Example Consider function

$$K(h) = \begin{cases} 1 & h = 0 \\ \rho & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

Is this a ACVF of a stationary time series ?

$$K(h) = K(-h).$$ ✓

$$f(\lambda) \geq 0$$ ← check this.

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} K(h)$$

$$= \frac{1}{2\pi} \sum_{h=-1}^{1} \underbrace{e^{-ih\lambda}}_{\cos(h\lambda) + i\sin(h\lambda)} K(h)$$

$$= \frac{1}{2\pi} \left[\cos(-\lambda)\,\rho + \cos(0)\cdot 1 + \cos(\lambda)\,\rho \right]$$

$$= \frac{1}{2\pi} \left[1 + 2\rho \underbrace{\cos(\lambda)}_{-1 \sim 1} \right]$$

$\Rightarrow$ $K(h)$ is ACVF if and only if $|\rho| \le \frac{1}{2}$.

Example White Noise $(0, \sigma^2)$

If $X_t \sim WN(0, \sigma^2)$, then $\gamma(h) = \begin{cases} \sigma^2 & \text{if } h = 0, \\ 0 & \text{o/w}. \end{cases}$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi} \cos(0) \cdot \sigma^2$$

$$= \frac{\sigma^2}{2\pi}$$

Example AR(1)

$$X_t = \phi X_{t-1} + e_t \qquad e_t \sim IID(0, \sigma^2)$$

$$\begin{cases} \gamma(0) = \dfrac{\sigma^2}{1-\phi^2} & \\ \gamma(h) = \phi^{|h|}\gamma(0) & h \neq 0. \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

$$= \frac{1}{2\pi} \left[\sum_{h=-\infty}^{-1} e^{-ih\lambda} \phi^{|h|} \gamma(0) + \gamma(0) + \sum_{h=1}^{\infty} e^{-ih\lambda} \phi^{|h|} \gamma(0) \right]$$

$$= \frac{1}{2\pi} \gamma(0) \left[1 + \underbrace{\sum_{h=1}^{\infty} \phi^{|h|} \left(e^{-ih\lambda} + e^{ih\lambda} \right)}_{\text{Geometric Series.}} \right]$$

$$= \frac{1}{2\pi}\gamma(0)\left[1 + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}}\right]$$

$$= \frac{1}{2\pi}\gamma(0)\left[1 + \frac{(1-\phi e^{i\lambda})\phi e^{-i\lambda} + (1-\phi e^{-i\lambda})\phi e^{i\lambda}}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})}\right]$$

$$= \frac{1}{2\pi}\left(\frac{\sigma^2}{1-\phi^2}\right)\left[1 + \frac{(\phi e^{-i\lambda} - \phi^2) + (\phi e^{i\lambda} - \phi^2)}{1 - \phi e^{-i\lambda} - \phi e^{i\lambda} + \phi^2}\right]$$

$$= \frac{1}{2\pi}\left(\frac{\sigma^2}{1-\phi^2}\right)\left[1 + \frac{1 - 1 + \phi(e^{-i\lambda} + e^{i\lambda}) - \phi^2 - \phi^2}{1 - \phi(e^{-i\lambda} + e^{i\lambda}) + \phi^2}\right]$$

$$= \frac{1}{2\pi}\left(\frac{\sigma^2}{1-\phi^2}\right)\left[1 + (-1) + \frac{1-\phi^2}{1-\phi\underbrace{(e^{-i\lambda}+e^{i\lambda})}_{\to\, 2\cos(\lambda)} + \phi^2}\right]$$

$$= \frac{\sigma^2}{2\pi}\left[1 - 2\phi\cos(\lambda) + \phi^2\right]^{-1}$$

AR(1)

$$\boxed{f(\lambda) = \frac{\sigma^2}{2\pi}\left[1 - 2\phi\cos(\lambda) + \phi^2\right]^{-1}}$$

$$\gamma(h) = \begin{cases} \sigma^2/(1-\phi^2) & h = 0 \\ \phi^{|h|}\cdot\gamma(0) & h > 0 \end{cases}$$

Periodgram and $\hat{f}(\lambda)$

Periodgram

$\to$ sample version of $2\pi f(\lambda)$.

$$|x|^2 = x^* \cdot x$$
$$x \in \mathbb{C}$$

$$I_n(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} X_t e^{-it\lambda}\right|^2$$

$$= \sum_{h=-n}^{n} \hat{\gamma}(h)\, e^{-ih\lambda} \qquad \text{if } \lambda = \omega_k$$

$$\omega_k = \frac{2\pi k}{n}, \qquad k = -\left\lfloor \frac{n-1}{2} \right\rfloor, \dots \left\lfloor \frac{n}{2} \right\rfloor \qquad \lfloor \cdot \rfloor = \text{floor}(\cdot).$$

$\uparrow$ fourier frequencies.

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

$$2\pi f(\lambda) = \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h).$$

$$I_n(\lambda) = \sum_{h=-n}^{n} e^{-ih\lambda} \hat{\gamma}(h) \qquad \lambda = \omega_k.$$

→ I_n itself is <u>not</u> a consistent estimator of $f(\cdot)$,

but can be weighted to be consistent.

$$I_n(\omega_k) = \sum_{h=-n}^{n} e^{-ih\omega_k}\,\hat{\gamma}(h)$$

$$\underline{e}_i = \begin{bmatrix} e^{i\omega_k} \\ e^{2i\omega_k} \\ \vdots \\ e^{ni\omega_k} \end{bmatrix} \frac{1}{\sqrt{n}}$$

$$(\underline{e}_i^{*})^{T}\,\underline{e}_j = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$\underline{e}_j$ orthonormal basis for $\mathbb{C}^n$

$j = 1, \dots, n$

Discrete Fourier Transformation

If $\underline{X}$ is in $\mathbb{C}^n$, then it can be written as

$$\underline{X} = \sum_k a_k \underline{e}_k \qquad k = -\lfloor \tfrac{n-1}{2} \rfloor, \dots, \lfloor \tfrac{n}{2} \rfloor .$$

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Then a_k can be calculated by

$$a_k = (\underline{e}_k^*)^T \cdot \underline{X} = \frac{1}{\sqrt{n}} \left[e^{-i\omega_k}, \dots, e^{-ni\omega_k} \right] \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t e^{-it\omega_k} .$$

By the formula

$$a_k = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t e^{-it\omega_k},$$

$[X_1, \dots, X_n]$ can be transformed into $[a_1 \dots, a_n]$.

(Discrete Fourier transformation).

$$I_n(\omega_k) = \frac{1}{n} \left| \sum_{t=1}^{n} X_t e^{-it\omega_k} \right|^2$$

$$= |a_k|^2$$

$$= a_k^* \cdot a_k .$$