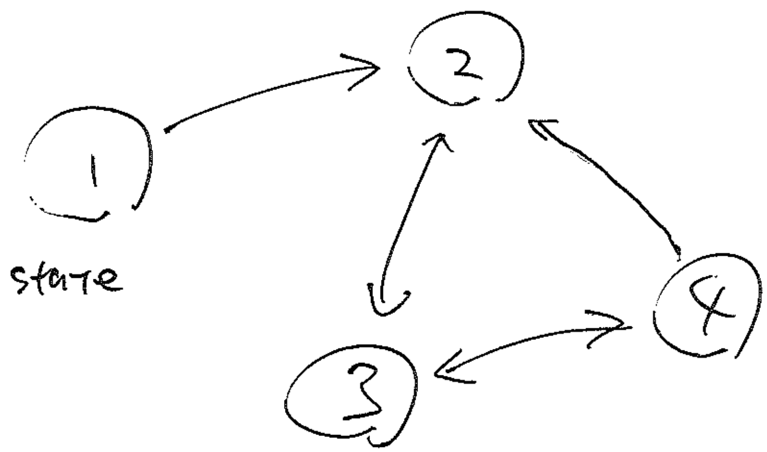
Ch. 6 Continuous-time Markov Chain

$\underset{t \geq 0}{X(t)}$: Continuous time Markov Chain.



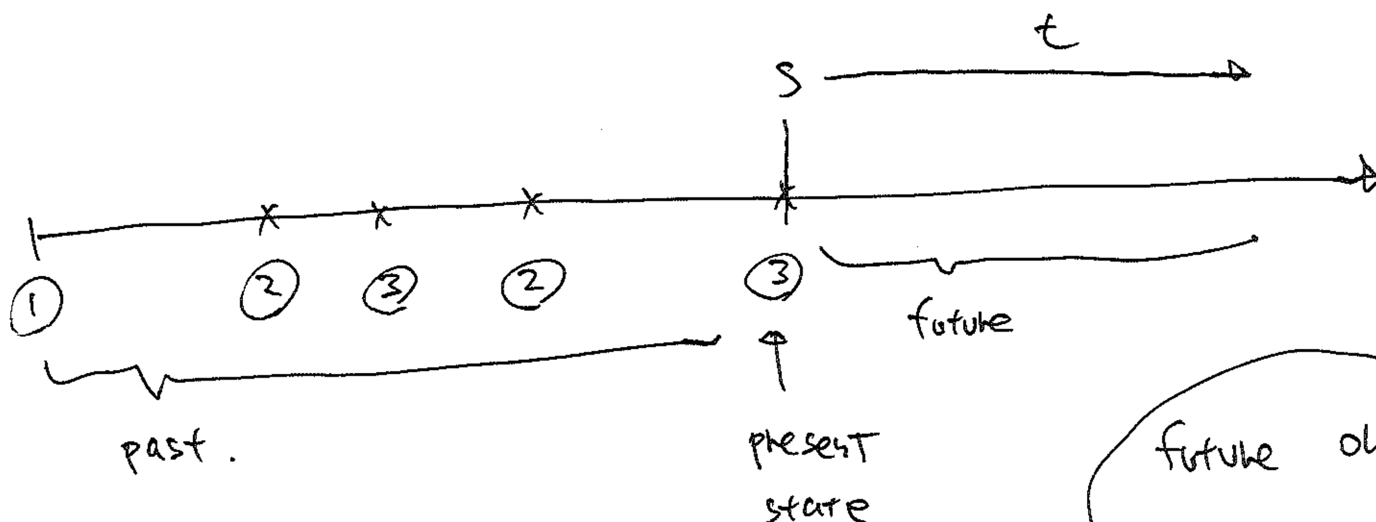
Jump occurs at time $t \in (0, \infty)$.

Markov Property :

$$P\left(X(t+s) = j \mid X(s) = i ,\ X(u) = x(u) \quad 0 \le u < s \right)$$

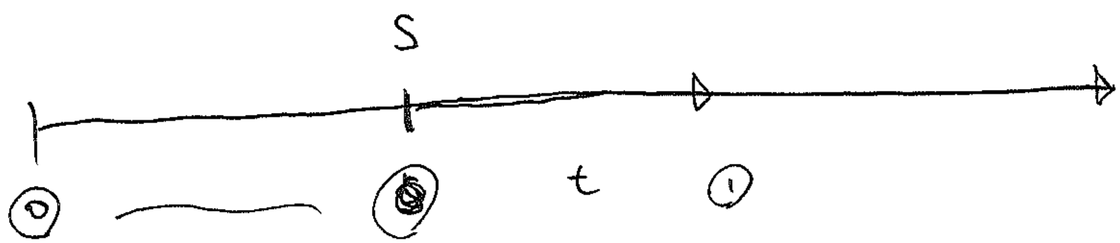
$$= P\left(X(t+s) = j \mid X(s) = i \right) = P_{ij}(t)$$

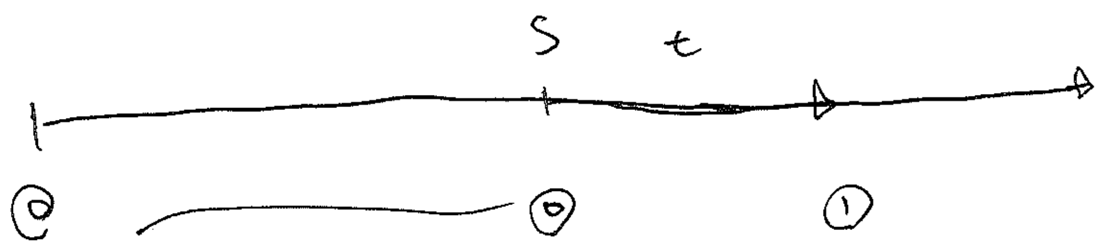
can't depend on s



future only depends on present state

$P_{ij}(t)$ can't depend on s





can't depend on s $\Rightarrow$ $P_{ij}(t)$ has memoryless property

$\Rightarrow$ $P_{ij}(t) \sim \mathrm{Exp}(\lambda)$ in time.

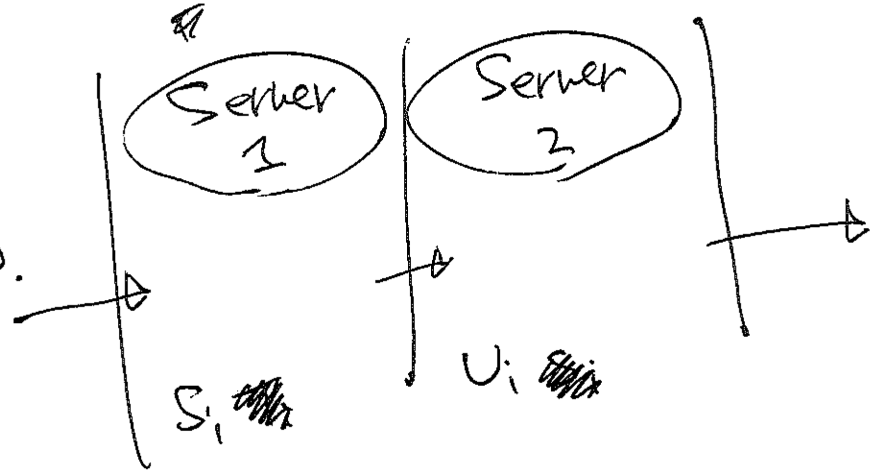
Ross p367

Ex 6.1 Two-Server queue

Poi Proc(λ)

T_i

$\sim Exp(\lambda)$



Service time

$S_i \sim Exp(\mu_1)$

$U_i \sim Exp(\mu_2)$

New customer enters only when both servers are open.

States

- 0 both server open
- 1 customer in S1
- 2 customer in S2

Av. time from 0 to 1 : $\frac{1}{\lambda}$

" 1 to 2 : $\frac{1}{\mu_1}$

" 2 to 0 : $\frac{1}{\mu_2}$

$P_{01} = P_{12} = P_{20} = 1$.

Birth and Death Process

n people in the system.

Birth $\sim \text{Exp}(\lambda_n)$

Death $\sim \text{Exp}(\mu_n)$

$P_{01} = 1$ $\qquad$ $E[\text{time in } 0] = \frac{1}{\lambda_0}$.

$E[\text{time in } 1] = \frac{1}{\lambda_1 + \lambda_2}$

$P_{10} = \frac{\mu_1}{\lambda_1 + \mu_1}$ $\qquad$ $P_{12} = \frac{\lambda_1}{\lambda_1 + \mu_1}$

For state i

$$E[\text{time in state } i] = \frac{1}{\mu_i + \lambda_i}$$

Birth $P_{i\,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}$

Death $P_{i\,i-1} = \dfrac{\mu_i}{\lambda_i + \mu_i}$

Ex. Poisson Process

$\mu_n = 0$

$\lambda_n = \lambda$.

pure birth process with constant birth rate

Ex Birth Process with linear birth rate

$\mu_n = 0$ no death.

$\lambda_n = n\lambda$ if each member takes $\text{Exp}(\lambda)$ to give birth, min $\sim \text{Exp}(n\lambda)$.

Yule process.

Pure Birth Process

$$⓪ \to ① \to ② \to ③ \to ④$$

with birth rate λ.

Differential Eq'n: (with $P_0(0) = 1$)

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t)$$

$$\frac{d}{dt} P_1(t) = \underbrace{-\lambda P_1(t)}_{\text{leaving}} + \underbrace{\lambda P_0(t)}_{\text{coming in}}$$

$$\frac{d}{dt} P_2(t) = -\lambda P_2(t) + \lambda P_1(t)$$

$$\vdots$$

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t)$$

$$\boxed{P_0(t) = e^{-\lambda t}} = P(X(t) = 0 \mid X(0) = 0)$$

$$\frac{d}{dt} P_1(t) = -\lambda P_1(t) + \lambda P_0(t)$$

$$\boxed{P_1(t) = \lambda t e^{-\lambda t}} = P(X(t) = 1 \mid X(0) = 0)$$

$$P_1'(t) = -\lambda \left(\lambda t e^{-\lambda t}\right) + \lambda e^{-\lambda t}$$

$$\frac{d}{dt} P_2(t) = -\lambda P_2(t) + \lambda P_1(t)$$

$$\boxed{P_2(t) = \frac{\lambda^2}{2} t^2 e^{-\lambda t}} = P\left(X_{(t)} = 2 \mid X_{(0)} = 0 \right)$$

$$P_2'(t) = -\lambda \left(\frac{\lambda^2}{2} t^2 e^{-\lambda t} \right) + \lambda^2 t e^{-\lambda t}.$$

$$\boxed{P_3(t) = \frac{\lambda^3}{3!} t^3 e^{-\lambda t}}$$

$$P_n(t) = \frac{\lambda^n}{n!} t^n e^{-\lambda t}.$$

Pure Birth Process

View 2

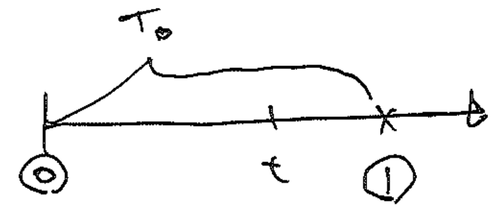
$$0 \xrightarrow{T_0} 1 \xrightarrow{T_1} 2 \xrightarrow{T_2} 3 \xrightarrow{T_3} 4 \xrightarrow{T_4} 5 \longrightarrow t$$

If $T_i \sim Exp(\lambda)$.

$$P_0(t) = P(X(t) = 1 \mid X(0) = 0)$$

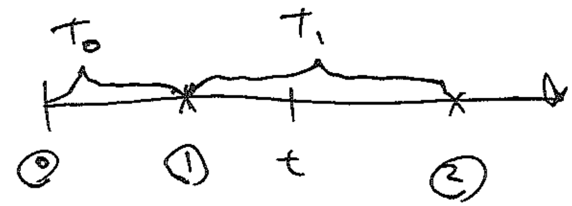
$$= \; ?$$

$$P_0(t) = P(T_0 > t)$$



$$= e^{-\lambda t}.$$

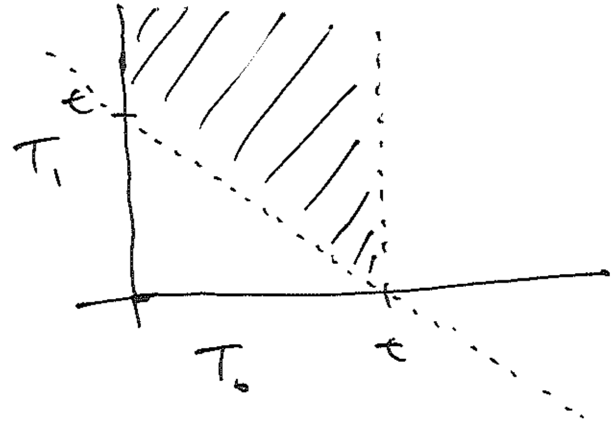
$$P_1(t) = P(T_0 < t \cap T_1 > t - T_0)$$



$$= P(T_0 < t \cap T_0 > t - T_1)$$

$$= P(t - T_1 < T_0 < t)$$

$$P(t - T_1 < T_0 < t)$$



$$= \int_0^t \int_{-t_0+t}^{\infty} \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda t_0} \, dt_1 \, dt_0$$

$t - T_1 < T_0$

$T_1 > -T_0 + t$

$$= \int_0^t \lambda e^{-\lambda t_0} \left(-e^{-\lambda t_1} \Big|_{-t_0+t}^{\infty} \right) dt_0$$

$$= \int_0^t \lambda e^{-\lambda t_0} e^{-\lambda(t-t_0)} \, dt_0 .$$

$$= \int_0^t \lambda e^{-\lambda t} \, dt_0 = \boxed{\lambda t e^{-\lambda t}}$$

$$P_2(t) = P\left(T_0 + T_1 < t \ \cap \ T_2 > t - (T_0 + T_1) \right)$$

$$\left.\begin{array}{l} T_0 + T_1 \sim GAM(2, \tfrac{1}{\lambda}) \\ T_2 \sim Exp(\lambda) \end{array}\right\} \text{indep.}$$

$$= \int_0^t \int_{-x+t}^{\infty} f_{GAM}(x) \, \cancel{f} \cdot \lambda e^{-\lambda t_2} \, dt_2 \, dx$$

$$= \boxed{\frac{\lambda^2}{2} t^2 e^{-\lambda t}}$$

Pure Birth Process

Differential Eqn :

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t)$$

$$\vdots$$

i.e. $\boxed{\text{Poi Proc } (\lambda)}$

$\Updownarrow$

Event time Distribution

$$1 \xrightarrow[\mathrm{Exp}(\lambda)]{} 2 \xrightarrow[\mathrm{Exp}(\lambda)]{} 3 \xrightarrow[\mathrm{Exp}(\lambda)]{} 4 \longrightarrow \cdots$$

Birth Process with λ depending on states

$$(0) \xrightarrow{\lambda_0} (1) \xrightarrow{\lambda_1} (2) \xrightarrow{\lambda_2} (3) \xrightarrow{\lambda_3} (4)$$

$$\frac{d}{dt} P_0(t) = -\lambda_0 P_0(t)$$

$$\frac{d}{dt} P_1(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t)$$

$$\frac{d}{dt} P_2(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t)$$

$$\vdots$$

$$\boxed{P_0(t) = e^{-\lambda_0 t}}$$ same as before.

General Solution for $P_n(t)$:

$$\boxed{P_n(t) = e^{-\lambda_n t} \int_0^t e^{\lambda_n s} \lambda_{n-1} P_{n-1}(s)\, ds}$$

$$\frac{d}{dt} P_n(t) = -\lambda_n P_n(t) + e^{-\lambda_n t} e^{\lambda_n t} \lambda_{n-1} P_{n-1}(t).$$

$$P_0(t) = e^{-\lambda_0 t}$$

$$P_1(t) = e^{-\lambda_1 t} \int_0^t e^{+\lambda_1 s} \lambda_0 e^{-\lambda_0 s} \, ds$$

$\lambda_0 \neq \lambda_1$

$$= e^{-\lambda_1 t} \lambda_0 \int_0^t e^{-(\lambda_0 - \lambda_1) s} \, ds$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{(\lambda_1 - \lambda_0) s} \Big|_0^t$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \cdot \left(e^{(\lambda_1 - \lambda_0) t} - 1 \right)$$

$$= \boxed{\frac{\lambda_0}{\lambda_1 - \lambda_0} \cdot \left(e^{-\lambda_0 t} - e^{-\lambda_1 t} \right)} = P_1(t)$$

Is that same as ....

$$0 \xrightarrow[\text{Exp}(\lambda_0)]{} 1 \xrightarrow[\text{Exp}(\lambda_1)]{} 2 \xrightarrow[\text{Exp}(\lambda_2)]{} 3 \longrightarrow \cdots$$

$$P_1(t) = P(T_0 < t \cap T_1 > t - T_0)$$

$$= P(t - T_1 < T_0 < t)$$

$$= \int_0^t \int_{-t_0+t}^{\infty} \lambda_1 e^{-\lambda_1 t_1} \cdot \lambda_0 e^{-\lambda_0 t_0} \, dt_1 \, dt_0$$

$$= \int_0^t \lambda_0 e^{-\lambda_0 t_0} \cdot \left(-e^{-\lambda_1 t} \Big|_{-t_0+t}^{\infty} \right) dt_0$$

$$= \int_0^t \lambda_0 e^{-\lambda_0 t_0} e^{-\lambda_1 (t-t_0)} dt_0$$

$$= \lambda_0 e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - \lambda_0) t_0} dt_0$$

$$= \lambda_0 e^{-\lambda_1 t} \frac{1}{(\lambda_1 - \lambda_0)} e^{(\lambda_1 - \lambda_0) t_0} \Big|_0^t$$

$$= \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1 \right) e^{-\lambda_1 t}$$

$$= \boxed{\frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{-\lambda_0 t} - e^{-\lambda_1 t} \right)}$$

same as before.