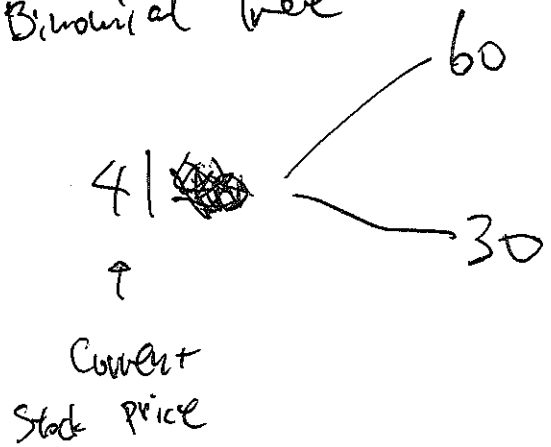
10. Binomial Option Pricing I

Binomial Tree

1yr

Payoff.

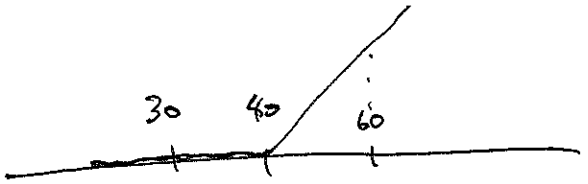
Binomial Tree



$20

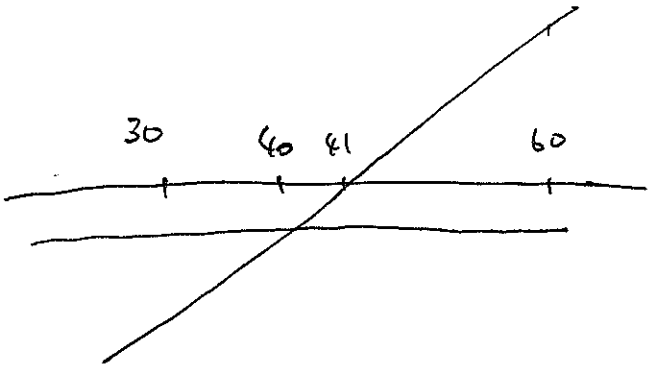
$0

Portfolio A: 40 strike 1 year call.



Portfolio B : long Δ amount of Stock, and borrow

$B at risk-free rate of 8%. (cont. comp.)



Payoff

$$\begin{cases} \Delta 60 e^{\delta T} + B e^{.08T} \\ \Delta 40 e^{\delta T} + B e^{.08T} \end{cases}$$

δ ↑ dividend

Q: for what Δ and B, payoff will be the same as 40-strike call option?

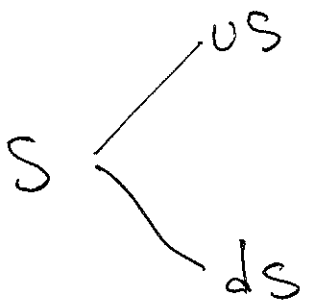
$T = 1$ yr

$\delta = 0$

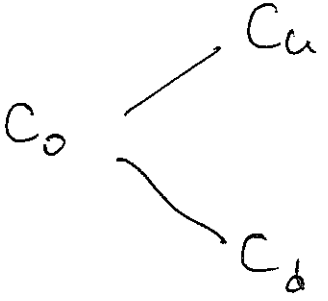
$$\begin{cases} \Delta 60 + B e^{.08} = 20 \\ \Delta 30 + B e^{.08} = 0 \end{cases}$$

$$\begin{cases} \Delta = 2/3 \\ B = -18.462 \end{cases}$$

Stock Price



option value



$$\begin{cases} \Delta u S e^{\delta h} + B e^{rh} = C_u \\ \Delta d S e^{\delta h} + B e^{rh} = C_d \end{cases}$$

$u = 1+i$ $\quad$ i = rate of capital gain $\quad$ h = length of period

$d = 1+i$

Cost of the option $\boxed{\Delta S + B} = C_0$

Portfolio A

40-strike Call option

Premium

$8.871

Portfolio B

Long $\Delta = \frac{2}{3}$ Stock

Borrow $B = -18.462$

Initial Cost

$\Delta S + B$

$= \frac{2}{3}(41) + (-18.462)$

$= \$8.871$

Stock price movements must be

$$u > e^{(r-\delta)h} > d.$$

otherwise there's arbitrage.

IF $\delta = 0$.

$$u > e^{rh} > d.$$

$$\| \quad 1+i$$

risk free rate

If Call option is overpriced

Say $9 premium

		Stock Price in 1yr: 30	60
+9 ←	Sell Call	0	-20
-8.891	2/3 Long Stock	20	40
	Repay $(18.462)e^{.08}$	-20	-20
		0	0

↓

.129

risk-free profit.

Δ = # of stocks

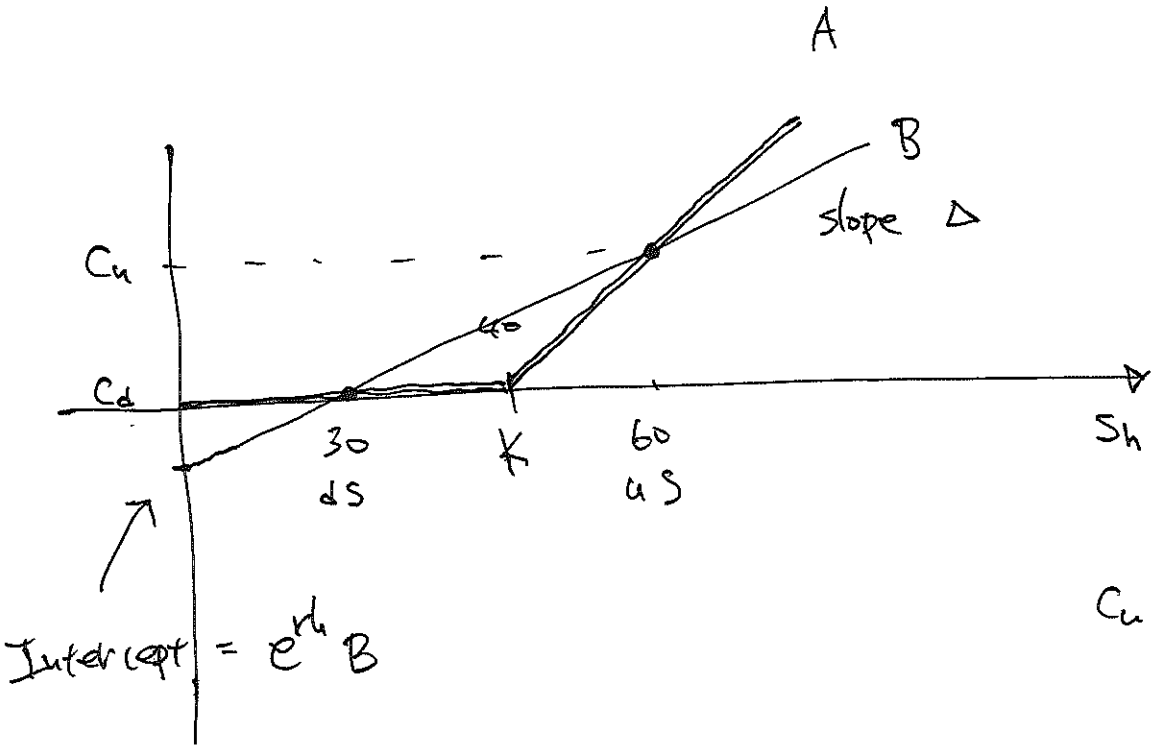
= sensitivity of option to a change in S

$\Delta S + B$ = option price

if S change by \$1,

Option price change by Δ.

Binomial Option Pricing



$C_u = uS - K$

Incoporating Uncertainty

Suppose the stock price is certain.

time = t . S_t

time = $t+h$ $S_{t+h} = S_t e^{(r-\delta)h}$ = Delivery price of forward.

[redacted].

Stock price = risk-free investment at rate r.

Now introduce uncertainty σ

$$uS_t = F_{t,t+h}\, e^{\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h}\, e^{-\sigma\sqrt{h}}$$

σ : std. dev. of risk free rate r.
(volatility)

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

$\sigma = 0 \Rightarrow$ risk free investment.

Example.

Instead of $30 S_0 = $41 $60, use

volatility σ = 30%.

$\begin{cases} r = .08 \\ \delta = 0 \end{cases}$ as before.

$$uS = S_0 e^{.08 + .3} = 59.954$$

40-strike Call

$$dS = S_0 e^{.08 - .3} = 32.903$$

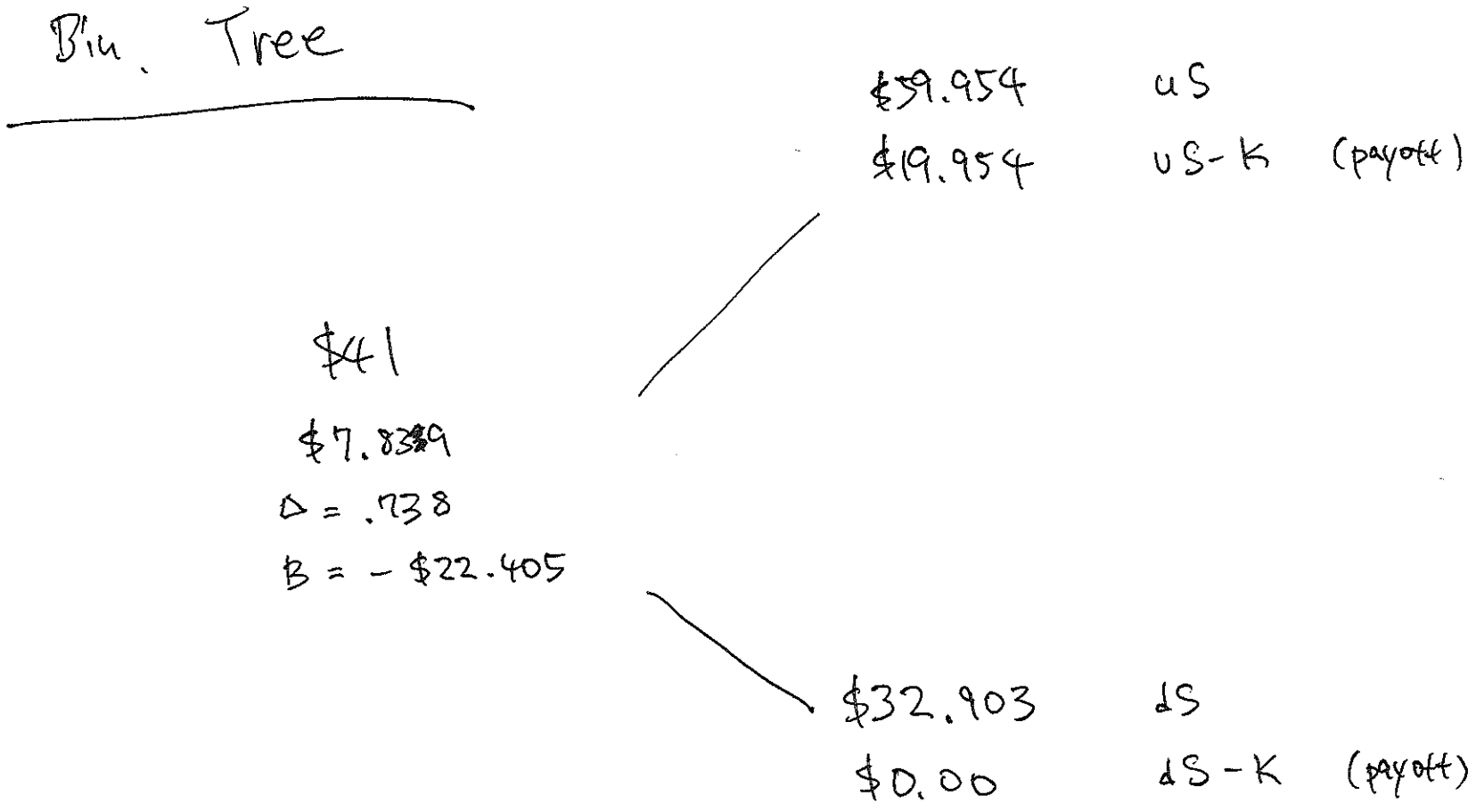
$$\begin{cases} \Delta dS e^{\delta h} + Be^{rh} = 59.954 - 40 \\ \Delta uS e^{\delta h} + Be^{rh} = 32.903 - 40 \end{cases}$$

Calculate

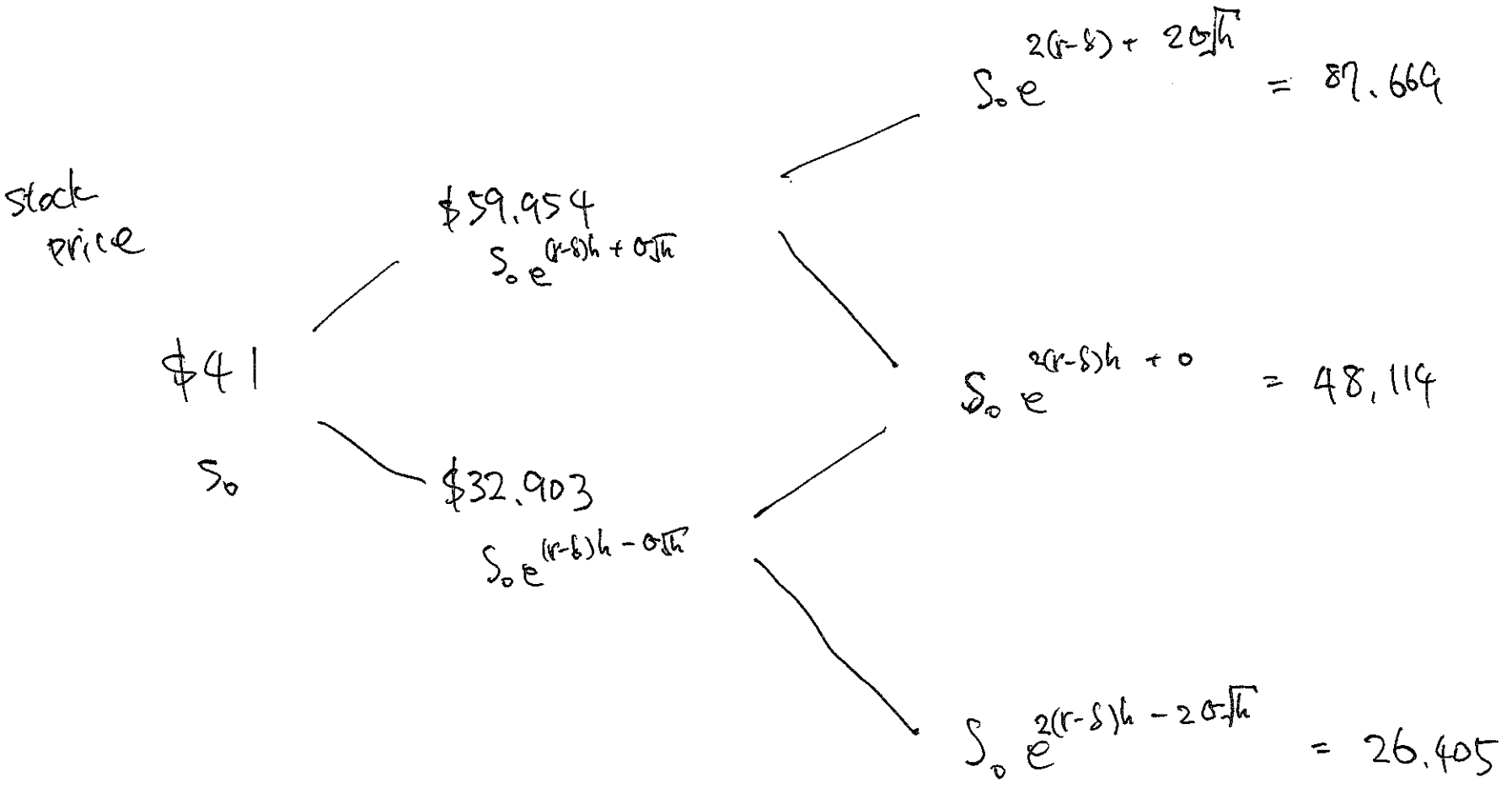
$$\begin{cases} \Delta = .7376 \\ B = -22.405 \end{cases}$$

$$\Delta S_0 + B = (.7376)41 - 22.405 = \boxed{7.839}$$ option price

Bin. Tree


$59.954 uS
$19.954 uS-K (payoff)
$41
$7.839
Δ = .738
B = -$22.405
$32.903 dS
$0.00 dS-K (payoff)

10.2 More than two periods



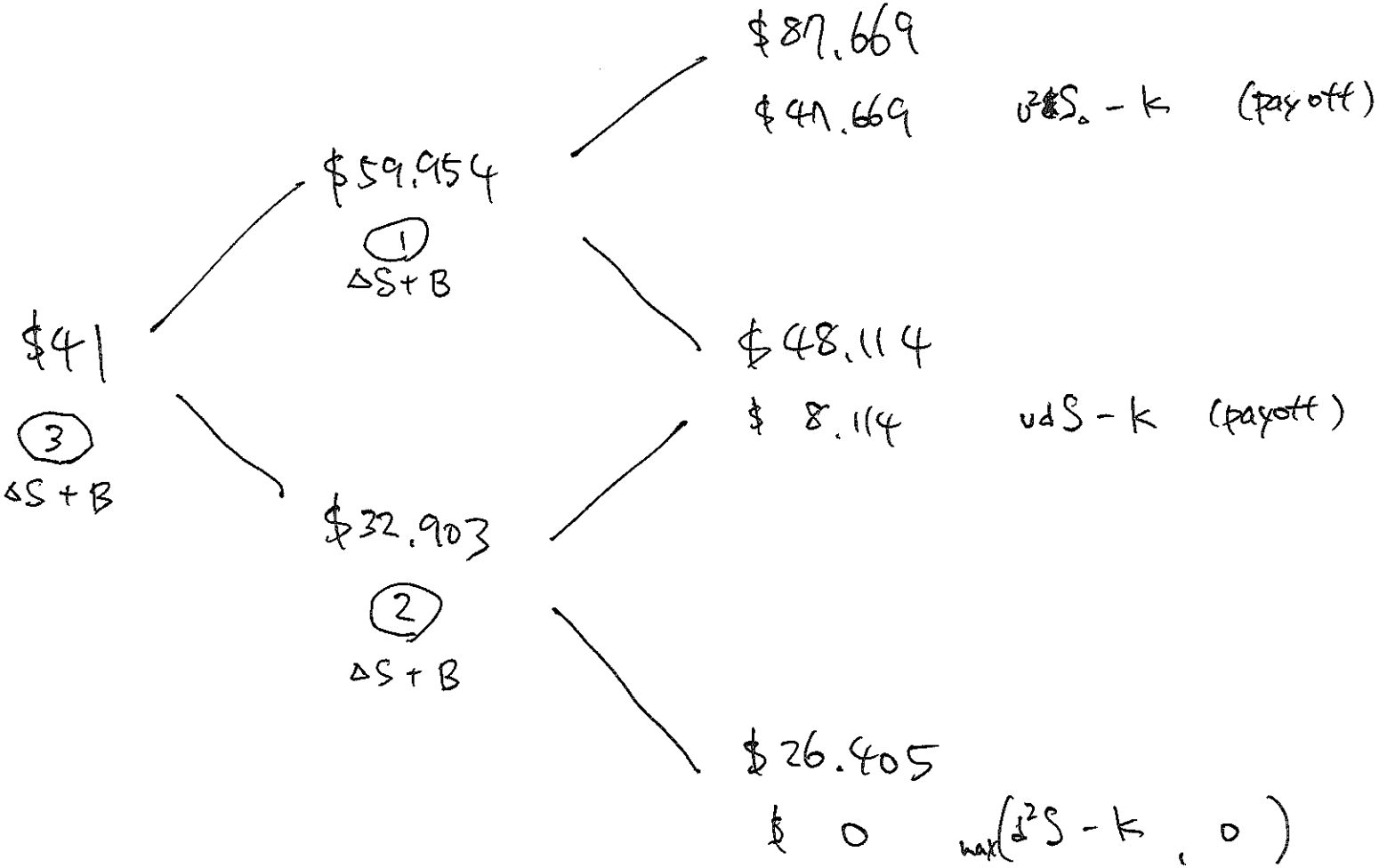
Alternatively.

$$\begin{cases} u = e^{(r_* - \delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

$$u^2 S_0 = 87.669$$

$$ud\, S_0 = 48.114$$

$$d^2 S_0 = 26.405$$



①, ②, ③ Option price

$$(1) = \Delta\, 59{,}954 + B$$

$$\begin{cases} \Delta\,(87.669)\,e^{8h} + B e^{.08h} = 47.669 \\ \Delta\,(48.114)\,e^{8h} + B e^{.08h} = 8.114 \end{cases}$$

$\delta = 0$

$h = 1$

$$\begin{cases} \Delta = 1 \\ B = -36.925 \end{cases}$$

$$(1) \quad \Delta\, 59{,}954 + B = 23.029$$

(2) $= \Delta(32.903) + B$

$$\begin{cases} \Delta(48.114)e^{8h} + Be^{.08h} = 8.114 \\ \Delta(26.405)e^{8h} + Be^{.08h} = 0 \end{cases}$$

$$\begin{cases} \Delta = .394 \\ B = -9.111 \end{cases}$$

(2) $\Delta(32.903) + B = 3.187$

(3) $= \Delta(\quad 41 \quad) + B$

$$\begin{cases} \Delta(\quad 39.954 \quad) + B e^{.08h} = 23.029 \\ \Delta(\quad 32.903 \quad) + B e^{.08h} = 3.187 \end{cases}$$

Option price in 1 period,

Stock price in 1 period

$$\begin{cases} \Delta = .734 \\ B = -19.337 \end{cases}$$

(3) $= 10.737$