Simulating Poisson Process

Monte Carlo Simulation

Kinetic Monte Carlo Simulation.

Ex. Ross p 315 Nth event vs Mth event.

2 Poi. Proc $\begin{cases} N_1(t) & \lambda_1 \\ N_2(t) & \lambda_2 \end{cases}$ indep.

what is $P(S^1_n < S^2_m)$

= P (n th event in Proc. 1
is before m th event
in Proc 2.)

We saw before

$$P(S_1^1 < S_1^2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

$$S_1 \sim Exp(\lambda_1)$$
$$S_2 \sim Exp(\lambda_2).$$

We could compute as.

$$P(S_n^1 < S_m^2) = \frac{\boxed{}}{},$$

$$S_n^1 \sim GAM(n, \tfrac{1}{\lambda_1})$$
$$S_m^2 \sim GAM(m, \tfrac{1}{\lambda_2}).$$

$$P(S_n^1 < S_m^2) = P(S_n^1 - S_m^2 < 0)$$

Monte Carlo Simulation

$$\begin{bmatrix} X_1 \\ \vdots \\ X_{1000} \end{bmatrix} < \begin{bmatrix} Y_1 \\ \vdots \\ Y_{1000} \end{bmatrix} = \begin{bmatrix} \text{Yes}, \\ \text{No} \\ \text{No} \\ \vdots \\ \text{Yes} \end{bmatrix} \leftarrow \text{See How many \% are 'Yes'}$$

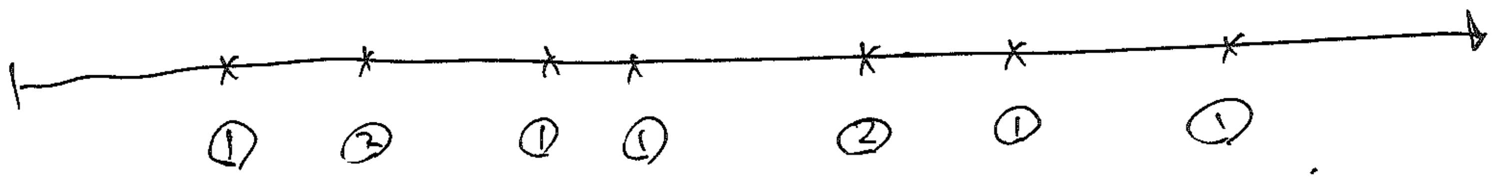
$$\approx P(S_n^1 - S_m^2 < 0)$$

i.i.d. sample from $GAM(\underset{\alpha}{\underset{\parallel}{n}}, \underset{\beta}{\underset{\parallel}{\tfrac{1}{\lambda_1}}})$

i.i.d. sample from $GAM(m, \frac{1}{\lambda_2})$

in R: $\text{rgamma}(1000, \overset{\alpha}{\overset{\parallel}{n}}, \overset{1/\beta}{\overset{\parallel}{\lambda_1}})$

Alternatively, use View 1 and see



type

Poi Proc. w. rate $(\lambda_1 + \lambda_2)$.

events

$$\begin{cases} 1 & \text{w. prob. } \dfrac{\lambda_1}{\lambda_1+\lambda_2} \\ 2 & \text{w. p. } \dfrac{\lambda_2}{\lambda_1+\lambda_2}. \end{cases}$$

$(S^1_n < S^2_m)$ = n_{th} 1 before m_{th} 2.

n th <u>Head</u> before m th <u>Tail</u>

$$X \sim \text{Neg. Binomial}\,(\overset{r}{\overset{\shortparallel}{n}}\,,\ \overset{p}{\overset{\shortparallel}{\tfrac{\lambda_1}{\lambda_1+\lambda_2}}})$$

$\boxed{X = \#\text{ of throw until } n \text{ th Head}}$

$$P(n\text{th H before } m\text{th T})$$

$$= P(X < n+m)$$

$$= P(X \le n+m-1).$$

CDF of neg. bin.

$$P(S^1_n < S^2_m)$$

$$= \text{CDF of } NB\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right) \text{ at } x = n+m-1.$$

$$= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}.$$

$$= \text{pnbinom}\left(m-1, \; n, \; \frac{\lambda_1}{\lambda_1+\lambda_2}\right).$$

<u>in R</u>: X = # of <u>Tails</u> before n th Head.

Σ

Conditional Joint distribution of arrival time

$$S_i = \sum_{k=1}^{i} T_k \sim GAM\left(i, \frac{1}{\lambda}\right).$$

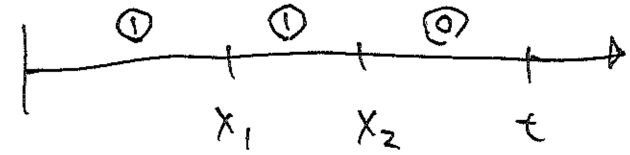
What is joint dist. of S_1, S_2 given $N(t) = 2$?

Let. $x_1 \leq x_2 \leq t$

$$P\left(S_1 \leq x_1,\ S_2 \leq x_2 \mid N(t) = 2\right).$$

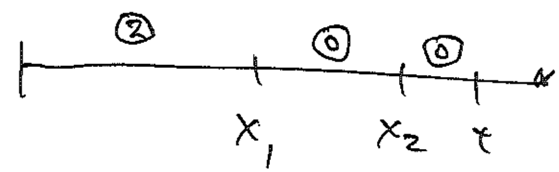
$$= \frac{P\left(S_1 \leq x_1,\ S_2 \leq x_2,\ N(t) = 2\right)}{P(N(t) = 2)}.$$

$$P(S_1 \le x_1,\ S_2 \le x_2,\ N(t) = 2)$$



$$= P\big(N(x_1) = 1,\ N(x_2) - N(x_1) = 1,\ N(t) - N(x_2) = 0\big)$$

$$+ P\big(N(x_1) = 2,\ N(t) - N(x_1) = 0\big)$$



$$
\begin{aligned}
&P(N(x_1) \geq 1, N(x_2) \geq 2, N(t) = 2) \\
&\quad = \quad P(N(x_1) = 1,\ N(x_2) - N(x_1) = 1,\ N(t) - N(x_2) = 0) \\
&\quad \quad\quad +P(N(x_1) = 2,\ N(t) - N(x_1) = 0) \\
&\quad = \quad \frac{e^{-\lambda x_1}\lambda x_1}{1!}\,\frac{e^{-\lambda(x_2 - x_1)}\lambda(x_2 - x_1)}{1!}\,e^{-\lambda(t-x_2)} \\
&\quad \quad\quad +\frac{e^{-\lambda x_1}(\lambda x_1)^2}{2!}e^{-\lambda(t-x_1)} \\
&\quad = \quad \frac{\lambda^2(x_1 x_2 - x_1^2)}{1!}e^{-\lambda t} + \frac{(\lambda x_1)^2}{2!}e^{-\lambda t} \\
&\quad = \quad \frac{\lambda^2(x_1 x_2)}{1!}e^{-\lambda t} - \frac{(\lambda x_1)^2}{2!}e^{-\lambda t}
\end{aligned}
$$

$$P(S_1 \le x_1,\ S_2 \le x_2 \mid N(t) = 2)$$

$$= \frac{\lambda^2 (x_1 x_2) e^{-\lambda t} - (\lambda x_1)^2 e^{-\lambda t}/2!}{e^{-\lambda t} (\lambda t)^2/2!}$$

$$= \frac{1}{t^2}\left(2!\, x_1 x_2 - x_1^2\right).$$

Cond'l Joint CDF

take $\frac{\partial^2}{\partial x_1 \partial x_2}$ for joint pdf.

$x_1, x_2 = t$ gives 1.

$$f_{S_1,S_2}(x_1, x_2 \mid N(t)=2)$$

$$= \frac{d^2}{dx_1 dx_2} \frac{1}{t^2}\left(2!\, x_1 x_2 - x_1^2\right)$$

$$= \frac{2!}{t^2} \qquad 0 \le x_1 \le x_2 \le t,$$

$\therefore$ Joint dist of $S_1, S_2 \mid N(t) = 2$ is

same as $U_{(1)}, U_{(2)}$

ordered iid Unif.

Remark: Thm 5.2 Ross

Joint dist. of

$$(S_1, S_2, \ldots S_k) \mid N(t) = k$$

$\sim$ orderd k iid Unif$(0, t)$.