Black - Scholes Equation

Geometric Brownian Motion

$$\frac{dS}{S} = (\alpha - \delta)\, dt + \sigma\, dz$$

$S(t)$: stock price = Geo. Bro. Mo.

Option Price

$V(S(t), t)$

risk free rate

r

$$\frac{dW}{W} = r\, dt$$

Investment.

$$dW = r\, W\, dt$$

$$\underline{C_0 = \Delta S + B} \qquad C_0 - \Delta S - B = 0$$

$$I = V(S(t), t) + N S(t) + W(t) = 0 .$$

Total investment.

option   short stock   lend ~~[illegible]~~

$$dI = \underbrace{dV(S(t), t)}_{\text{by Ito's lemma}} + N\, dS(t) + dW(t)$$

$$= V_t'\, dt + V_s'\, dS + \frac{1}{2}\sigma^2 S^2 V_s''\, dt + N(dS + \delta S\, dt)$$

$$+ r W(t)\, dt$$

delta-headge

$$\Delta = -V_s' = N \qquad \text{\# of stocks to short.}$$

$$dI = V_t' dt + \underbrace{V_s' dS} + \frac{1}{2}\sigma^2 S^2 V_s'' dt + \underbrace{N(dS} + \delta S dt) + rW_{(t)} dt$$

$$= V_t' dt + \frac{1}{2}\sigma^2 S_{(t)}^2 V_s'' dt - V_s' \delta S_{(t)} dt + rW_{(t)} dt.$$

Lending is to finance the transaction

$$W_{(t)} = V_s' S - V$$

$$dI = V_t' dt + \frac{1}{2}\sigma^2 S_{(t)}^2 V_s'' dt - V_s' \delta S_{(t)} dt$$

$$+ r \underbrace{W_{(t)}} dt$$

$$= V_t' dt + \frac{1}{2}\sigma^2 S_{(t)}^2 V_s'' dt + (r-\delta) V_s' S_{(t)} dt - rV dt$$

$$\left.\begin{array}{l}\text{Zero - investment}\\ \text{Zero - risk}\end{array}\right\} \Rightarrow \text{Zero - return.}$$

$$dI = 0.$$

$$\boxed{V_t' + \frac{1}{2}\sigma^2 S_{(t)}^2 V_S'' dt + (r-\delta) V_s' S_{(t)} - rV = 0.}$$

Black - Scholes Equation

Black - Scholes Equation

- stochastic pde
- Option prices must satisfy. otherwise there's arbitrage.
- Assumptions
  1. Asset price $S(t) \sim GBM$.
  2. divident is constant at δ.
  3. r is constant, and can lend or borrow at same rate.
  4. no transaction costs.

Zero-Coupon Bond

[illegible]

Zero-Coupon Bond of \$1 at time T.

Current time $= t$.

$$\text{PV of Bond} = e^{-r(T-t)}$$

$$= V(T, t)$$

We must have

$$V(T, T) = 1 \quad \text{Boundary condition}$$

B-S Eq'n

$$\boxed{V_t' + \frac{1}{2}\sigma^2 S_{(t)}^2 V_s'' dt + (r-\delta) V_s' S_{(t)} - rV = 0}$$

Bond price does not depend on stock price

$$\Rightarrow V_s' = 0$$

$$V_s'' = 0$$

$$\boxed{V_t' = rV}$$

if $V = e^{-r(T-t)}$,

then

$$V_t' = r\, e^{-r(T-t)}.$$

Prepaid Forward

$$F^{P}_{0,T} = S_0 e^{-\delta T}.$$

$$V(S(t), t) = S(t)\, e^{-\delta(T-t)}$$

Since you get the stock at time T,

$$V(S(T), T) = S(T).$$ Boundary Condition.

$$V(S_{(t)}, t) = S_{(t)} e^{-\delta(T-t)}$$

$$V_s' = e^{-\delta(T-t)}$$

$$V_s'' = 0$$

$$V_t' = \delta S_{(t)} e^{-\delta(T-t)}$$

$$\boxed{V_t' + \frac{1}{2}\sigma^2 S_{(t)}^2 V_s'' dt + (r-\delta) V_s' S_{(t)} - rV = 0}$$

$$\delta S_{(t)} e^{-\delta(T-t)} + 0 + (r-\delta) S_{(t)} e^{-\delta(T-t)} - r S_{(t)} e^{-\delta(T-t)}$$

$$= 0.$$

Call price

B-S formula

$$C = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

$$d_{1,2} = \frac{\ln(S_0/K) + (r - \delta \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

must change $T \Rightarrow (T-t)$.

$\Rightarrow$ each term satisfies B-S Eq'n.

Black ~~and~~ Scholes formula

$$C_0 = e^{-rT} E\left[\max(0, S_T - k)\right]$$

$$= e^{-rT}\left\{E\left[S_T - k \mid S_T > k\right] \cdot P(S_T > k) + E\left[0 \mid S_T < k\right] P(S_T < k)\right\}$$

$$\underbrace{S(t)\, e^{-\delta(T-t)}\, N(d_1)} - \underbrace{K\, e^{-r(T-t)}\, N(d_2)}$$

① Asset-or-nothing option

$\parallel$

$\boxed{\text{pays } S(T) \text{ if } S(T) > K \quad \text{o/w pays } 0.}$

② Cash-or-nothing option.

$\parallel$

$\boxed{\text{pays } \$1 \text{ if } S(T) > K \quad \text{o/w pays } 0.}$

(1) $S(t) e^{-\delta(T-t)} N(d_1)$

$$d_1 = \frac{\ln(S(t)/K) + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

As $t \to T$, if $S(T) > K$,

$d_1 = \infty$ $\quad N(d_1) = 1$

If $S(T) < K$

$d_2 = -\infty$ $\quad N(d_2) = 0.$

① $S(t)\, e^{-\delta(T-t)} N(d_1)$

$$= \begin{cases} S(T) & \text{if } S(T) > K \\ 0 & \text{if } S(T) < K \end{cases}$$

Asset - or - nothing

② $K \cdot e^{-r(T-t)} N(d_2)$

$$d_2 = \begin{cases} \infty \\ -\infty \end{cases} \qquad N(d_2) = \begin{cases} 1 & S(T) > K \\ 0 & S(T) < K \end{cases}$$

$$K e^{-r(T-t)} N(d_2)$$

$$= \begin{cases} K \\ 0 \end{cases}$$

cash - or - nothing.