4.3 Classification of States

Accessible : You can get to state j from state i.

$$i \to j$$

Communicate : $i \leftrightarrow j$

(state i communicates with itself by definition).

Irreducible : all states communicates.

Ex.

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 \\ .5 & .25 & .25 \\ 0 & .33 & .67 \end{bmatrix}$$

Is this MC irreducible ?

Ex.

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ .25 & .25 & .25 & .25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is this MC irreducible ?

Class : states that communicate are in same class.

class

$\{1, 2\}$ → 1, 2 communicates.

$\{3\}$ → Can go 1, 2, 3 or 4.

$\{4\}$ → stuck. → absorbing state.

Recurrent State : P(process will come back to that state ever) $= 1$.

transient State : P(") < 1.

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ .25 & .25 & .25 & .25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which states are recurrent?

which one transient?

$\{1, 2\}$ – recurrent.

$\{3\}$ – transient.

Corollary 4.2

If state i is recurrent, and $i \leftrightarrow j$
then state j is also ~~rec~~ recurrent.

(same goes for transience)

State i is recurrent if the process ~~always~~ keeps coming back to it. (Assuming process starts from i)

$$\Updownarrow$$

State i is recurrent if $E(\#\text{ of visits}) = \infty$

(state i is transient if $E(\#\text{ of visits}) < \infty$.)

Let

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases} \quad \text{(process visits state } i\text{)}.$$

then

$$\left(\begin{array}{c}\text{\# of visits} \\ \text{to} \\ \text{state } i\end{array}\right) = \sum_{n=0}^{\infty} I_n .$$

So we need to look at

$$E(\text{\# of visits}) = E\left(\sum_{n=0}^{\infty} I_n\right).$$

[Assuming $X_0 = i$].

$$E\left(\sum_{n=0}^{\infty} I_n\right) = \sum_{n=0}^{\infty} E(I_n)$$

① because $E(\ )$ can go inside any summation

$$= \sum_{n=0}^{\infty} P(X_n = i \mid X_0 = i)$$

② because for any indicator I_n. $E(I_n) = P(I_n = 1)$.

$$= \sum_{n=0}^{\infty} P_{ii}^{n}$$

(2) Indicator Function

$$I_n = \begin{cases} 1 \\ 0 \end{cases}$$

$$E(I_n) = 0 \cdot P(I_n = 0) + 1 \cdot P(I_n = 1)$$

$$= P(I_n = 1).$$

Formula.

$$\boxed{E\left(\begin{array}{c}\#\text{ of visits to}\\ \text{state } j\end{array}\right) = \sum_{n=1}^{\infty} P_{jj}^{n}}$$

(starting from j)

Remark:

Not all states of a finite MC can be transient.

Remark 2:

You cannot go to recurrent states to transient states.

recurrent states : P(process will come back to state i) = 1.

$E(\#\text{ of visits}) = \infty$.

$$\begin{cases} \text{positive recurrent} : E(\text{time until next visit}) < \infty \\ \text{null recurrent} : E(\text{time until next visit}) = \infty \end{cases}$$

positive recurrency is a <u>class</u> property.

Remark: In <u>finite-state</u> MC, all recurrent states are positive recurrent.

$\rightarrow$ All states communicate.

$\Downarrow$

the MC is irreducible

$\Downarrow$

(all states) must be recurrent.

Ex. 4.14

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ .25 & .25 & 0 & 0 & .5 \end{bmatrix}$$

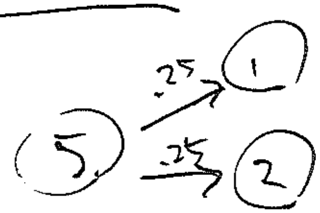
→ 3 classes : $\{1, 2\}$ - recurrent

$\{3, 4\}$ - recurrent.

$\{5\}$

Is $\{5\}$ transient ?

State 5



Can't enter 5 from any other state.

$\rightarrow$ transient.

Alternatively look at

$$E(\#\text{ of visits}) = \sum_{n=1}^{\infty} P_{55}^{n}$$

if $< \infty$ then transient.

$$E\left(\#\text{ of visits to state } 5\right) = \sum_{n=0}^{\infty} P_{55}^{n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

$$= \frac{1}{1-\frac{1}{2}} = 2. < \infty.$$

state 5 is transient.

Ex 4.13

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & .5 & .5 \\ 1 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .5 & 0 & .5 \end{bmatrix}$$