Ch. 2. Valuation of Annuities

annuity : a series of periodic payments.

annuity - certain

vs

life-contingent annuity.

$$1 + x + x^2 + \cdots + x^k = S \quad — \quad ①$$

$$x + x^2 + x^3 + \cdots + x^{k+1} = Sx \quad — \quad ②$$

Subtracting ② from ①,

$$1 - x^{k+1} = S - Sx$$

$$S = \frac{1 - x^{k+1}}{1 - x}$$

$$\boxed{1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}} \quad (2.1)$$

P.72

Example 2.1

\$30 deposit on the last day of each month.

Annual interest rate of 9% compounded monthly.

Interest is paid on the last day of each month.

May 31, 1998 ~ Dec 31, 2009 (140 mo.)

→ Account balance on Dec 31, 2009 including payment made + interest paid on that day?

(nominal) annual rate 9% $= i^{(12)}$

$j = \frac{.09}{12} = .0075$ effective mthly rate.

140 payments of \$30. Accumulated value on Dec 31, 2009

$$30\left[(1+j)^{139} + (1+j)^{138} + \cdots + (1+j) + 1\right]$$

$$= 30\left[\frac{1-(1+j)^{140}}{1-(1+j)}\right] \quad \text{using } (2.1)$$

$$= 7385.91$$

2.1 Level Payment Annuities

n equal payments of 1

interest per payment period $= i$

Accumulated value

$$s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots (1+i) + 1$$

$$= \sum_{t=0}^{n-1} (1+i)^{t} \cancel{t} = \frac{1-(1+i)^{t}}{1-(1+i)}$$

$$= \frac{(1+i)^{n}-1}{i} \qquad (2.3)$$

p.34

- accumulated value is found at the time of and including the final payment.

→ Annuity Immediate

"... payments occur at end of each year ..."

$$\boxed{s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}}$$

Ex 2.2

nominal annual rate of interest 9%

Compounded semi-annually.

$i^{(2)} = .09$

Deposit

May 1, Nov. 1 of each year from 2008 to 2015.

16 deposits

Accumulated value on Nov 1, 2015 = \$7000 (includes last deposit + interest)

→ How much each deposit should be?

Ex 2.2

May 1, 2008 ~ Nov 1, 2015

2 x 8yrs = 16 deposits.

9% Annual Interest. → 4.5% semi-annual.

$$7000 = X \cdot s_{\overline{16|}.045}$$

$$X = \frac{s_{\overline{16|}.045}}{7000} = \frac{(1.045)^{16} - 1}{.045} \cdot \frac{1}{7000}$$

$$= 308.11$$

Ex 2.3

$i^{(12)} = 9\%$

\$30 deposit end of each month.

May 1998 ~ Apr 2014. (16 years), (192 deposits)

Account is untouched until end of Apr 2019.

→ Accumulated Value ?

Ex 2.3 Some time after final payment.

A: End of the month of 16^{th} b'day. $\rightarrow$ 192 deposits.

12×16

$$30 \cdot s_{\overline{192}|.0075} = 12,792.31$$

From his A to End of 21^{st} b'day $\rightarrow$ $5 \times 12 = 60$ mo.

$$\text{Accumulated value} = (12,792.21) \cdot (1.0075)^{60}.$$

$$s_{\overline{n}|i} \cdot (1+i)^{k} = \frac{(1+i)^{n} - 1}{i} \cdot (1+i)^{k}$$

$$= \frac{(1+i)^{n+k} - (1+i)^{k}}{i}$$

$$= \frac{(1+i)^{n+k} - 1}{i} - \frac{(1+i)^{k} - 1}{i}$$

$$= s_{\overline{n+k}|} - s_{\overline{k}|} .$$

Ex 2.4 non-Level interest rates

\$30 deposit end of each mo.

May 1998 ~ Dec 2009 (140 mo.)

$i^{(12)} = 9\%$ until Dec 2003. (68 mo.)

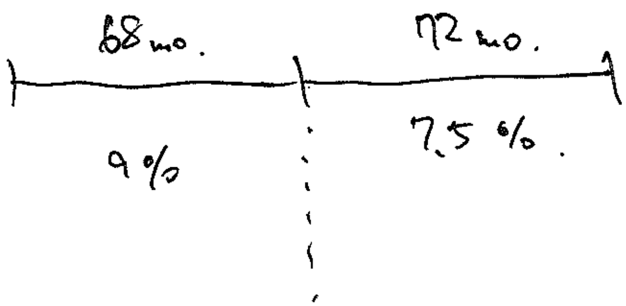
$i^{(12)} = 7.5\%$ Jan 2004 ~ (72 mo.)

→ Acc. Val. on Dec 2009 ?

$\frac{.09}{12} = .0075$ $\qquad$ $\frac{.075}{12} = .00625$

$$30 \cdot s_{\overline{68|}\,.0075}\,(1.00625)^{72}$$

$$+\ 30\, s_{\overline{72|}\,.00625} = \begin{array}{c} 4147.86 \\ + \\ 2717.36 \end{array}$$



Ex 2.5 payment amount change

10 monthly payments of \$50 each is followed by

14 monthly payments of \$75 each.

$j = 1\%$ (eff. monthly rate)

→ Acc. Val. at the time of final payment?

$$50\, s_{\overline{10}|.01}\,(1.01)^{14} + 75 \cdot s_{\overline{14}|.01}$$

$$= 1722.36.$$

Present Value

Ex 2.6

Withdraw \$1000 each year, for 4 years.

starting a year from today.

$i = 6\%$ (eff. ann. rate)

→ How much does she need to deposit today?

Deposit X today

$i = .06$.

choose reference time point : today. $\quad v = \frac{1}{1+i}$

$$X = 1000v + 1000v^2 + 1000v^3 + 1000v^4$$

$$= 1000(v + v^2 + v^3 + v^4)$$

$$= 3,465.11$$

$$1000\left(v + v^2 + v^3 + v^4\right)$$

$$= 1000\,(v)\left[1 + v + v^2 + v^3\right]$$

$$= 1000\,(v)\,\frac{1-v^4}{1-v}$$

$$= 1000\,\frac{1}{(1+i)} \cdot \frac{1-v^4}{1-\frac{1}{1+i}}$$

$$= 1000 \cdot \frac{1-v^4}{1+i-1} \qquad = 1000 \cdot \frac{1-v^4}{i}$$

Series of equally spaced payments 1.

1 payment period before the payments begin

$$a_{\overline{n}|i} = v + v^2 + \dots + v^n$$

$$= \sum_{i=1}^{n} v^t$$

$$\boxed{a_{\overline{n}|i} = \frac{1 - v^n}{i}}$$

Ex 2.7 Loan Repayment.

Loan of 12,000.

a) Monthly payments for 3-years. starting one month after purchase. $i^{(12)} = 12\%$

b) Monthly payments for 4 years. starting one mo. after purchase. $i^{(12)} = 15\%$.

→ Find monthly payments and total amount paid.

Ex 2.7 Loan payments.

Loan of 12,000

a)

$$12{,}000 = P_1 \cdot a_{\overline{36}|.01}$$

$P_1 = 398.57$

$P_2 = 333.97$

b)

$$12{,}000 = P_2 \cdot a_{\overline{48}|.0125}$$

$36 \cdot P_1 = 14348.52$

$48 \cdot P_2 = 16030.56$

Ex 2.8

In the previous example, what if

1st payment is made 9 month after the

purchase ?

Ex 2.8

not $\ddot{a}$.

a) $12{,}000 = P_1' \cdot v^8 a_{\overline{36}|.01}$

b) $12{,}000 = P_2' \cdot v^8 a_{\overline{48}|.0125}$

$P_1' = 431.60$

$P_2' = 368.86$

Deferred Annuity

n-payment annuity of 1 per period

valued $k+1$ payment period before the first payment

$$v^k \cdot a_{\overline{n}|i} = a_{\overline{n+k}|} - a_{\overline{k}|} \qquad (2.8)$$

→ Non-level interest rates

→ Relationship b/w $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$

$$s_{\overline{n}|i} = (1+i)^n * a_{\overline{n}|i}$$

$$v^n s_{\overline{n}|i} = a_{\overline{n}|i}$$

Perpetuity : infinite period annuity.

$$\lim_{n \to \infty} a_{\overline{n}|i} = \lim_{n \to \infty} \frac{1 - v^n}{i} = \frac{1}{i}.$$

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

Perpetuity - immediate

Ex 2.9

$i = 8\%$.

How much do you need to deposit so that you can withdraw \$800 a year indefinitely?

Ex 2.9 Perpetuity.

annual interest rate $i = .08$

deposit 10,000.

Annual payment P.

$$10{,}000 = P \cdot a_{\overline{\infty}|i}$$

$$= P \cdot \frac{1}{i}$$

$$P = 10{,}000 \cdot (.08) = \boxed{\$800}$$

Ex 2.10 Valuation of Perpetuity.

Perpetuity - immediate pays X per year.

Brian : 1st n payments.
Colleen : next n payments
Jeff : rest of payments.

Brian's share of present value = 40 % of pres. val. of entire perpetuity.

→ Find K = Jeff's share of present value

Ex 2.10 Valuation of Perpetuity.

Present value of perpetuity

$$X \cdot a_{\overline{\infty}|i} = \frac{X}{i}.$$

P.V. of Brian's portion

$$X \cdot a_{\overline{n}|i} = X\left(\frac{1-v^n}{i}\right)$$

$$= (40\%) \cdot \frac{X}{i}$$

$$X\left(\frac{1-v^n}{i}\right) = .4\left(\frac{X}{i}\right)$$

$$\Rightarrow \quad 1 - v^n = .4$$

P.V. of Colleen's portion

$$X\,v^{n} a_{\overline{4}|i} = v^{4} \cdot X\, a_{\overline{4}|i}$$

$$= (.6)\left(.4 \frac{X}{i}\right)$$

$$= .24 \frac{X}{i}$$

P.V. of Jeff's portion

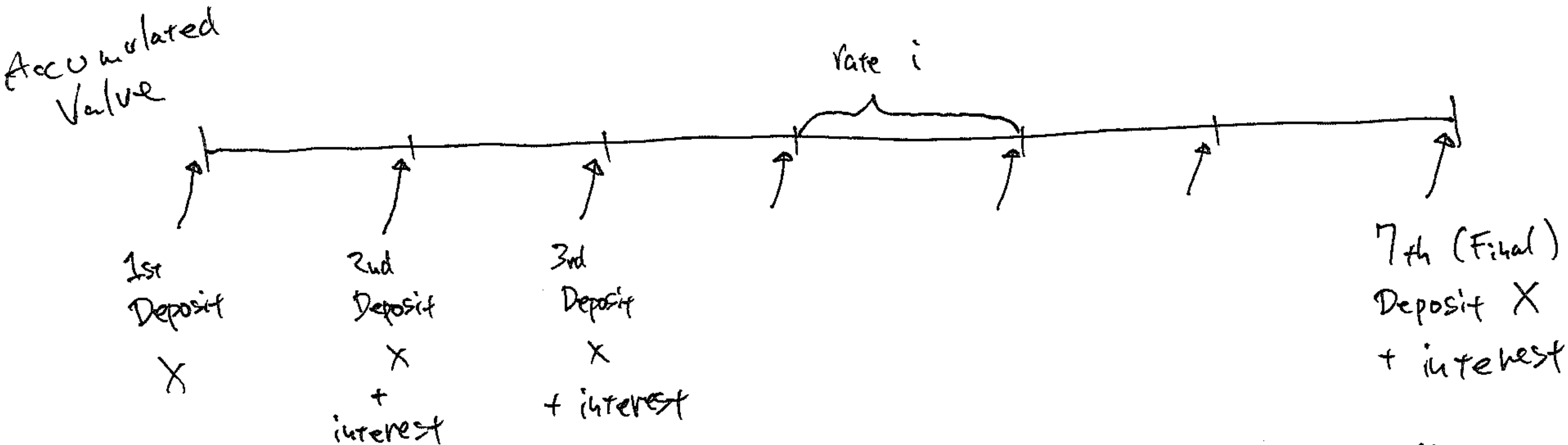
$$1 - .4 - .24 = \boxed{.36}$$

K = 36 %

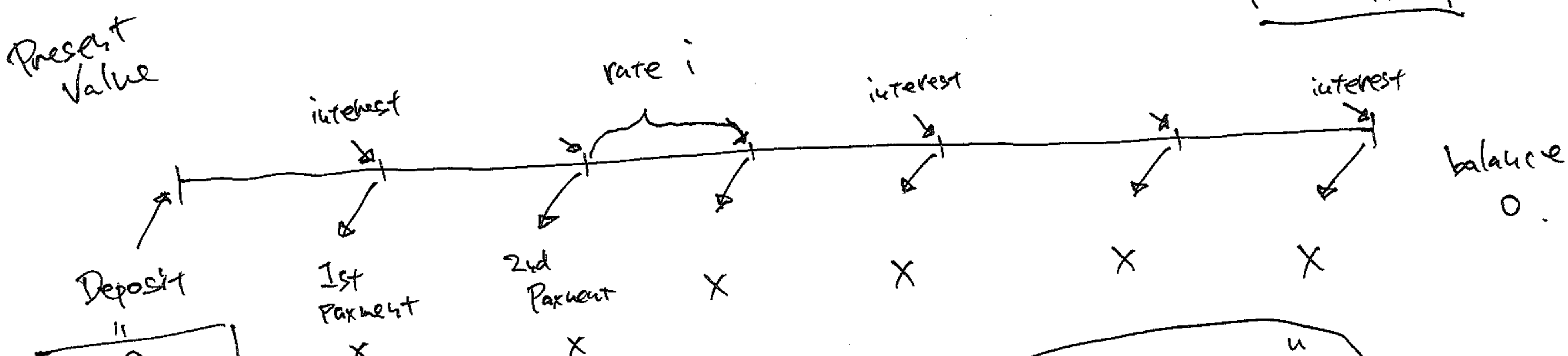
$$\boxed{(.36)\frac{X}{i}}$$

Annuity - Immediate

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$



$$= \boxed{X \cdot s_{\overline{7}|i}}$$



$$= \boxed{X \cdot a_{\overline{6}|i}}$$

$$a_{\overline{n}|i} = \frac{1 - v^n}{i}$$

$$v = \frac{1}{1+i}$$

Annuity - Immediate

: "... payments are made at the end of each period ..."

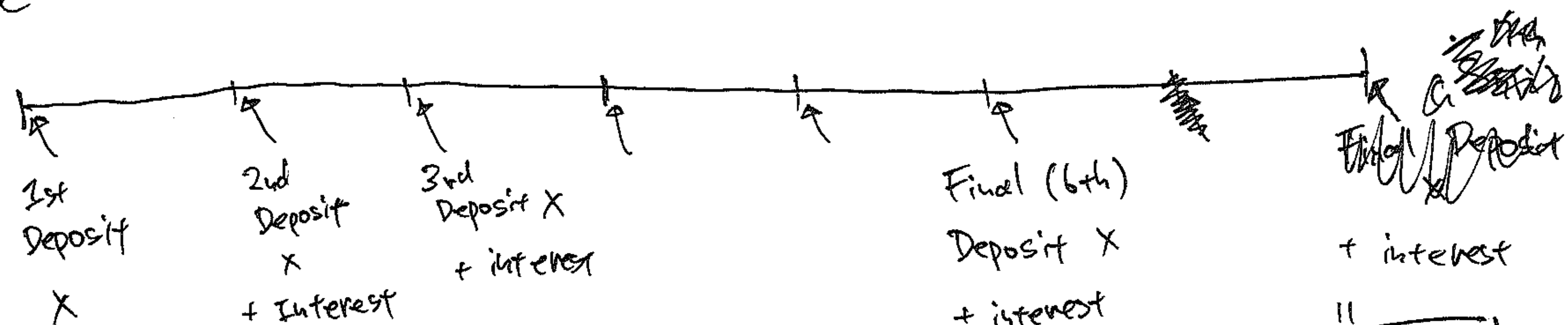
Annuity - Due

"

...

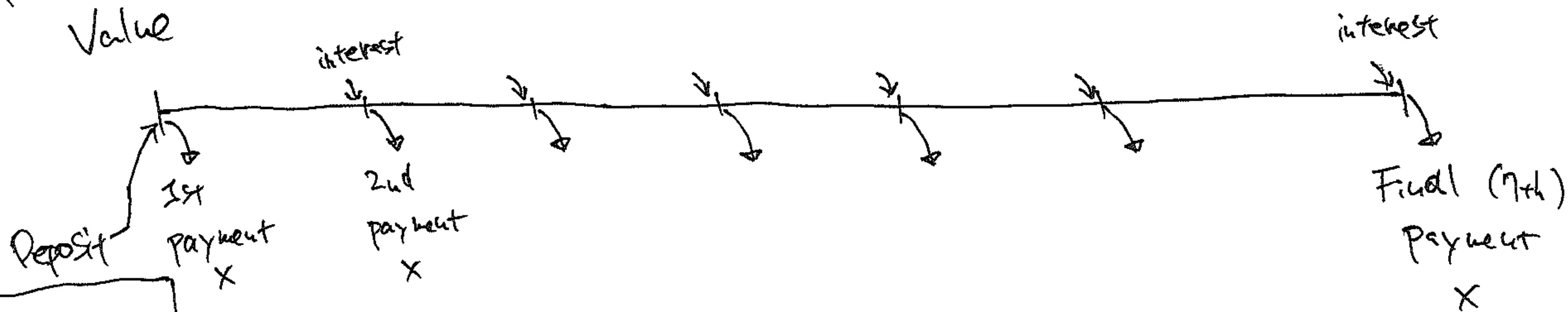
Annuity - Due

Accumulated Value



$$\boxed{X \cdot \ddot{s}_{\overline{7}|i}}$$

Present Value



$$\boxed{X \cdot \ddot{a}_{\overline{7}|i}}$$

Includes 1st payment

Annuity - Due

$$\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 + \cdots + (1+i)^n$$

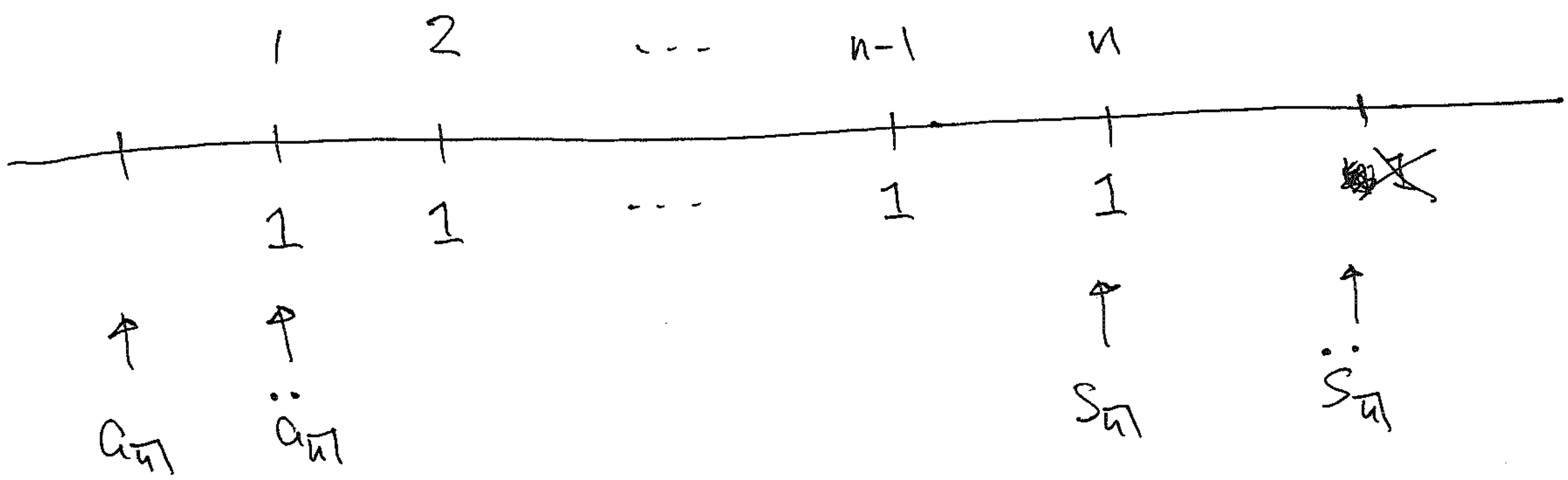
$$= \boxed{\frac{(1+i)^n - 1}{d}}$$

$$d = \frac{i}{1+i}$$

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \cdots + v^{n-1}$$

$$= \boxed{\frac{1 - v^n}{d}}$$

Payment No.



$$\ddot{a}_{\overline{n}|} = (1+i)\, a_{\overline{n}|}$$

$$(1+i)\, s_{\overline{n}|} = \ddot{s}_{\overline{n}|}$$

Perpetuity Due

$$\ddot{a}_{\overline{\infty}|i} = (1+i)\, a_{\overline{\infty}|i}$$

$$= (1+i) \cdot \frac{1}{i}$$

$$= 1 + a_{\overline{\infty}|i}$$

2.2 Level Payment Annuities

Ex 2.12

At exactly each $\frac{1}{4}$-year . $1000 deposited.

a) 9% nominal annual rate compounded monthly.

b) 10% effective annual rate compounded every $\frac{1}{4}$-year.

4 times a year × 16 yrs = 64 deposits.

a)
b) } $1000 \cdot s_{\overline{64|}\,j} = 1000 \cdot \dfrac{(1+j)^{64} - 1}{j}$.

a) Nominal 9 % annual rate → $\frac{9}{12} = .75\%$ monthly.

$(1.0075)^3 = 1.0227$ effective rate for $\frac{1}{4}$ year.

$j = .0227$.

↑

If you round.

b) 10 % effective annual rate compounded every 1/4 year.

$$(1+j)^4 = 1.10$$

$$j = (1.10)^{1/4} - 1 = .0241$$

//.

If you round.

Ex 2.12 b)

10 % effective rate compounded 4-times a year for 16 years.

$$1000 \cdot s_{\overline{64}|j} = \underbrace{4000\, s^{(4)}_{\overline{16}|.1}}_{m\text{-thly payable annuity.}}$$

$(1+j)^4 = 1.10$

$$\boxed{s^{(m)}_{\overline{n}|i} = \frac{(1+i)^n - 1}{i^{(m)}}}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i$$

m - thly annuity

Present Value

$$a^{(m)}_{\overline{n}|i} = \frac{1 - v^n}{i^{(m)}} = a_{\overline{n}|i} \cdot \frac{i}{i^{(m)}}$$

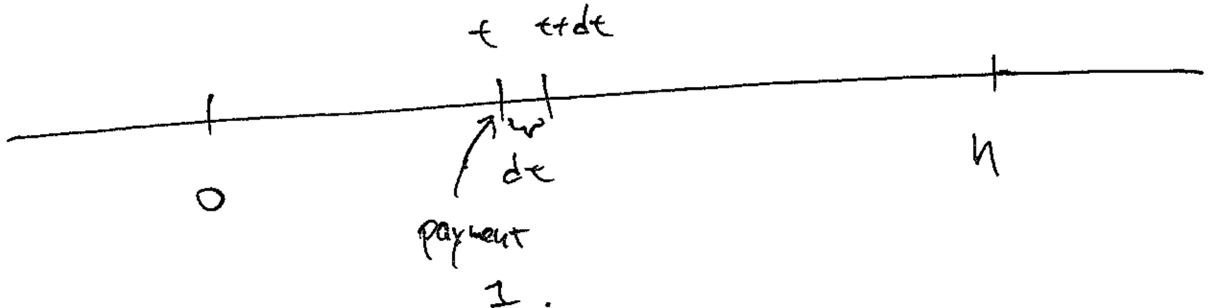
Value $\frac{1}{m}$ - year before the 1st payment.

Continuous Annuities

$s_{\overline{n}|i}^{(m)}$ ← what if $m \to \infty$?

Suppose ~~you~~ payments of 1 is made each dt.

with effective rate i for period dt.



$$\int_0^n (1+i)^{n-t}\,dt = \bar{s}_{\overline{n}|i}$$

$$\bar{s}_{\overline{n}|i} = \int_0^n (1+i)^{n-t}\,dt$$

$$= \int_0^n e^{\ln(1+i)\cdot(n-t)}\,dt$$

$$= -\frac{e^{\ln(1+i)(n-t)}}{\ln(1+i)}\Bigg|_0^n$$

$$= \frac{-(1+i)^{n-t}}{\ln(1+i)}\Bigg|_0^n$$

$$= \frac{(1+i)^n - 1}{\ln(1+i)}.$$

$$\bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\ln(1+i)}$$

$$= \frac{i}{\ln(1+i)} \cdot \frac{(1+i)^n - 1}{i} = \frac{i}{\ln(1+i)}\, s_{\overline{n}|i}$$

$$\lim_{m\to\infty} s^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \frac{(1+i)^n - 1}{i^{(m)}} = \bar{s}_{\overline{n}|i}$$

Example 2.13 continuous annuity.

\$12 deposits every day in 2004 + 2005. 9% eff. A. rate.

\$15 " in 2006 12% eff. A. rate

Find accumulated amount

a) using daily deposits

b) using continuous approximation.

Ex 2.13

2004	2005	2006	
\$12	\$12	\$15	
9%	9%	12%	eff. Ann. rate.

a) $j_1 = i^{(365)}$

$= (1.09)^{1/365} - 1 = .00023631$

$j_2 = (1.12)^{1/365} - 1 = .000311$

$12 \cdot s_{\overline{730}|j_1}(1.12) + 15\, s_{\overline{365}|j_2} = 12\left[\frac{1.09^2 - 1}{j_1}\right](1.12) + 15\left[\frac{1.12 - 1}{j_2}\right]$

$= 16502.59$

b)

$\$12 \times 365 = 4380$

$\$15 \times 365 = 5475$.

$$4380\, \bar{s}_{\overline{2}|.09}\,(1.12) + 5475\, \bar{s}_{\overline{1}|.12}$$

$$= 4380\left[\frac{(1.09)^2 - 1}{\ln(1.09)}\right](1.12) + 5475\left[\frac{(1.12) - 1}{\ln(1.12)}\right]$$

$$= 16504.75$$

///

Continuous Annuity : present Value

$$\bar{a}_{\overline{n}|i} = \int_0^n v^t \, dt = \frac{1 - v^n}{\ln(1+i)} = \frac{i}{\ln(1+i)} \cdot a_{\overline{n}|i}$$

$$\lim_{m \to \infty} a^{(m)}_{\overline{n}|i} = \bar{a}_{\overline{n}|i}$$

Suppose $a(t_1, t_2)$ is accumulated value at t_2 of amount 1 deposited at time t_1.

Continuous Annuity of 1
$$\begin{cases} \int_0^n a(t, n)\, dt & \text{accu. value at } n \\ \int_0^n \frac{1}{a(0,t)}\, dt & \text{present value at } 0 \end{cases}$$

With Force of Interest δ_r,

$$a(t_1, t_2) = \exp\left\{ \int_{t_1}^{t_2} \delta_r \, dr \right\}.$$

$$\bar{a}_{\overline{n}|\delta_r} = \int_0^n \frac{1}{a(0,t)}\, dt = \int_0^n e^{-\int_0^t \delta_r \, dr}$$

$$\bar{s}_{\overline{n}|\delta} = \int_0^n a(t,n)\, dt = \int_0^n e^{\int_t^n \delta_r \, dr}$$

p103

2.3 Nonconstant Payments

what if each payment grows

1. Geometrically
2. Linearly.

① Ex. 1st payment 26,000, but grows by 4% to accomodate the inflation. annual interest rate 9%.

$r = .04 \qquad i = .09$

PV of
Constant payments

$$x\nu + x\nu^2 + x\nu^3 + \cdots + x\nu^n$$

$$= x\nu(1 + \nu + \cdots + \nu^{n-1})$$

$$= x\nu\left(\frac{1-\nu^n}{1-\nu}\right) = x\left(\frac{1-\nu^n}{i}\right) = x\,a_{\overline{n|}\,i}$$

PV of
Geometric progression

$$x\nu + x(1+r)\nu^2 + x(1+r)^2\nu^3 + \cdots + x(1+r)^{n-1}\nu^n$$

$$= x\nu(1 + (1+r)\nu + \cdots + (1+r)^{n-1}\nu^{n-1})$$

$$= x\nu\left(\frac{1-\nu_0^n}{1-\nu_0}\right)$$

$$\nu_0 = (1+r)\nu$$

Geometric Progression

$$PV = X \, v \left(\frac{1 - v_0^n}{1 - v_0} \right)$$

$$v_0 = (1+r)v = \frac{1+r}{1+i} = \frac{1}{1+i_0}$$

$$1 + i_0 = \frac{1}{v_0} = \frac{1+i}{1+r} \quad \text{(real rate of interest)}$$

$$PV = X \frac{v}{v_0} v_0 \left(\frac{1 - v_0^n}{1 - v_0} \right) = X \frac{v}{v_0} a_{\overline{n}|i_0}$$

$$\frac{v}{v_0} = \left(\frac{1}{1+i}\right)\left(\frac{1+i}{1+r}\right) = \frac{1}{1+r}$$

$$= X \boxed{\frac{a_{\overline{n}|i_0}}{1+r}}$$

PV of arithmetic progression

Payments

$$X, \quad 2X, \quad 3X, \quad 4X, \quad \ldots, nX$$

$$PV = Xv + 2Xv^2 + 3Xv^3 + \cdots + nXv^n =: S.$$

then

$$\frac{S}{v} = X + 2Xv + 3Xv^2 + \cdots + nXv^{n-1}$$

$$\frac{S}{v} - S = X + Xv + Xv^2 + \cdots Xv^{n-1} - nXv^n$$

$$\downarrow$$

$$S\left(\frac{1}{v} - 1\right) = S\left((1+i) - 1\right) = iS$$

$$iS = X(1+v+v^2+\cdots+v^{n-1}) - nXv^n$$

$$= X\left\{ (1+i)a_{\overline{n}|i} - nv^n \right\}$$

PV of arithmetic progression

$$= S = X\left(\frac{(1+i)a_{\overline{n}|i} - nv^n}{i}\right)$$

$$= X\,(Ia)_{\overline{n}|i}$$

similarly,

$$(Is)_{\overline{n}|i} = \frac{(1+i)s_{\overline{n}|i} - n}{i}$$

Summary

Geometric Progression $\qquad i_0 = \dfrac{1+i}{1+r}$

$$PV = \frac{a_{\overline{n}|i_0}}{1+r} \qquad\qquad FV = PV\,(1+i)^n$$

Arithmetic progression

$$(Ia)_{\overline{n}|i} = \left(\frac{a_{\overline{n}|i}(1+i) - n v^n}{i}\right) \qquad\qquad (IS)_{\overline{n}|i} = \left(\frac{s_{\overline{n}|i}(1+i) - n}{i}\right)$$

Ex 2.17

20-yr annuity

Payments begin in a year.

eff. ann rate 11 %.

each payment grows by 4 %.

1st PMT = 26,000.

PV = ?

$$PV = 26{,}000 \frac{a_{\overline{20|}\, i_0}}{1+r} = 26000 \left(\frac{1 - v_0^{20}}{i_0} \right) \frac{1}{1+r}$$

$$v_0 = .93694$$

$$i_0 = \frac{1.11}{1.04} = 1.0673$$

$$1+r = 1.04$$

$$PV = 26000 \, (10.82) \left(\frac{1}{1.04} \right)$$

$$= 26000 \, (10.40323)$$

$$= 270{,}484$$

Geometric increase (inflation) } may not coincide.
Payment Period

Ex 2.18

payments increase 12 % each year.

\$25 /mo 2001

\$28 /mo 2002

\$31.36 /mo 2003

⋮

$i^{(12)} = .12$

→ Find Accu. Value in 18 years.

Ex 2.18

Payment amount change once a year.

Payment - monthly.

A	on Dec. 31 Year A.	on Dec 31, Year 2018
2001	$25\, S_{\overline{12}\rceil .01}$	$25\, S_{\overline{12}\rceil .01}\,(1.12)^{17}$
2002	$25\,(1.12)\, S_{\overline{12}\rceil .01}$	$25\,(1.12)\, S_{\overline{12}\rceil .01}\,(1.12)^{16}$
2003	$25\,(1.12)^{2}\, S_{\overline{12}\rceil .01}$	$25\,(1.12)\, S_{\overline{12}\rceil .01}\,(1.12)^{15}$
$\vdots$		$\vdots$
2018	$25\,(1.12)^{17}\, S_{\overline{12}\rceil .01}$	$25\,(1.12)^{17}\, S_{\overline{12}\rceil .01}$

total $25\,(1.12)^{17}\, S_{\overline{12}\rceil .01} \cdot 18$

Divident Discount Model

Value of a share of Stock.

= Present Value of future dividents

→ Assume constant increase in amount of the divident paid.

→ next dividend payable one year from now is amount $\boxed{k}$.

→ Annual ~~growth~~ compound growth rate of dividend is $\boxed{r}$

→ interest rate $\boxed{i}$

PV

$$= K\left[v + (1+r)v^2 + (1+r)v^2 + \cdots \right]$$

$$= Kv\left[\frac{1}{1-(1+r)v}\right] =$$

$$= K\left(\frac{1}{1+i}\right)\left[\frac{1}{1-\frac{1+r}{1+i}}\right] = \frac{K}{i-r}$$

<u>theoretical price of the stock.</u>

Take lim of Geo. Prog.

$$\lim_{n\to\infty} K\left(\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}\right) = \frac{K}{i-r}$$

if $(1+r)v < 1$.

Ex 2.19

- Stock X pays dividend \$50 end of 1st year.
- each subsequent dividend 5% greater.
- Purchase at at price to earn 10% eff. ann. yield.
- Immediately after 10^{th} dividend, stock sold for price P,
- eff. Ann. Yield over 10 years was 8%.

Find P.

Ex 2.19

Dividend : \$50 and grows geometrically with 5%.

If he held onto the Stock:

$r = .05$

$j = .10$

$$\text{Present Value} = \frac{50}{.1 - .05}$$

$$= 1000$$

He sold the stock after 10 yrs.

$r = .05$

$j = .08$

$$\text{Present Value} = 50\left[\frac{1 + \left(\frac{1+r}{1+i}\right)^{10}}{i - r}\right]$$

$$= 409.18$$

$$\text{Present Value} = \left(\frac{1}{1.08}\right)^{10} P.$$

$$1000 = 409.18 + \left(\frac{1}{1.08}\right)^{10} P$$

$$P = 1275.54$$