Ch. 1

Interest Rate

Annual interest rate : i

$\$100 \longrightarrow \$100(1+i)$ future value of \$100.

today in 1 yr

$\$100\left(\frac{1}{1+i}\right) \longrightarrow \$100$

today in 1yr

Present value of \$100 in 1 yr.

Compound interest

$$c(1+i)^t$$

\$ 1000   5 % annual int.

in 1 yr $1000(1.05) = 1050$

3 yr $1000(1.05)^3 = 1157.6$

5 yr $1000(1.05)^5 = 2078.6$

Average Ann. Rate

Yr	1	2	3	4	5
	2.73 %	3.02 %	5.17 %	5.39 %	7.91 %

What is average annual interest rate?

$$C\,(1.0273)(1.0302)(1.0517)(1.0539)(1.0791) = C\,(1.2658)$$

in 5 yrs.

→ Average rate $(1.2658)^{1/5} - 1 = .04827$

4.83 %

Ex 1.3

5 % ann. int. rate

+ 1000	on	1/1/2005
− 200		1/1/2007
100		1/1/2008
− 250		1/1/2010

interest credited on 12/31 each yr.

How much $ left on 12/31/2011 ?

Ans.

$$1000\,(1.05)^7$$
$$-200\,(1.05)^5$$
$$100\,(1.05)^4$$
$$-250\,(1.05)^2$$

$$997.77.$$

Accumulation Factor

$$a(t) = (1+i)^t$$

Accumulation Amount

$$A(t) = A(0)(1+i)^t$$

Getting Effective Interest rate

$$\text{effective rate } i \text{ over time period} = \frac{\text{money you make}}{\text{money you had}}$$

→ Then convert it to annual rate.

Ex.

① Lend \$500. ~~[illegible]~~ paid back \$540 after 6mo.

$$\frac{540-500}{500} = \frac{4}{50} = .08.$$

8% return in 6 mo.

$\rightarrow \quad (1.08)^2 = 1.1664 \qquad$ 16.64% effective ann. interest rate.

(2) Borrowed \$10,000.

paid back \$11500 2 yrs later.

$$\frac{11500 - 10000}{10000} = \frac{1500}{10000} = .15$$

15 % interest in 2 yrs.

$$(1.15)^{1/2} = 1.072381$$

7.24 % Effective ann. int. rate.

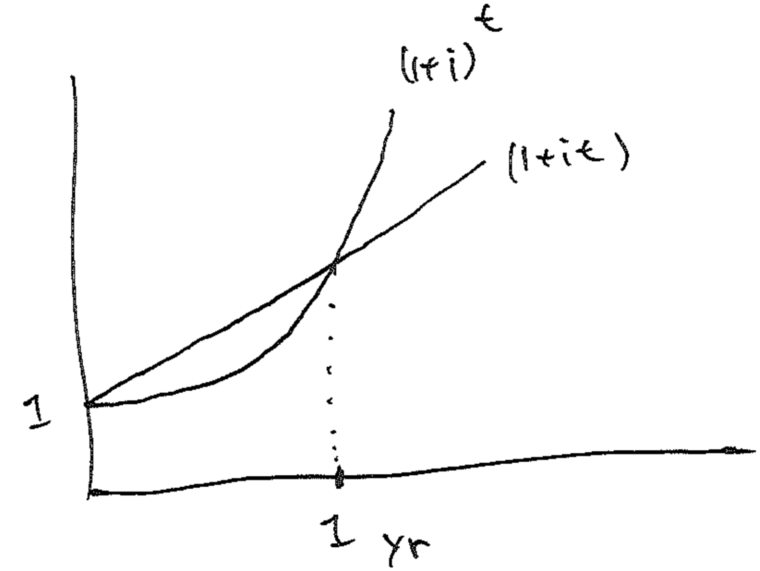
Simple Interest

Compounding Int. $(1+i)^t$

Simple Int $(1+it)$

often used when t is fraction of a year.

$\frac{\text{mo}}{12}$ or $\frac{\text{days}}{365}$



Ex

Borrowed \$1000 at rate of 5% per annum.

for 90 days. How much to pay back?

Comp. int $\quad 1000(1.05)^{90/365} = 1012.103$

Simple int. $\quad 1000\left(1+.05\cdot\frac{90}{365}\right) = 1012.329$

Present Value

$i = 5\%$

$100\left(\frac{1}{1.05}\right) \longrightarrow \$100$

Today.

in 1yr

Present Value.

$$\nu = \frac{1}{(1+i)}$$

Present value factor.

$$C \longrightarrow C(1+i)^n$$

today in n yrs.

$$B\nu^n \longleftarrow B$$

today in n yrs.

<u>Ex.</u>

$i = 5\%$

Need $\$10{,}000$ in 10 yrs.

How much do i need to deposit today?

$\boxed{\text{Review :}}$ PV and FV

$$v^n \longleftarrow \$100 \longrightarrow (1+i)^n$$

today

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

Ex.

6 % effective ann. rate

① Need \$7000 in 5 yrs

How much do I need to deposit? (today)

② Have \$3500. How much will it be in 4 yrs?

$1+i = 1.06 \qquad v = .9434$

① $7000\, v^5 = 5230.807$

② $3500\,(1+i)^4 = 4418.669$

③

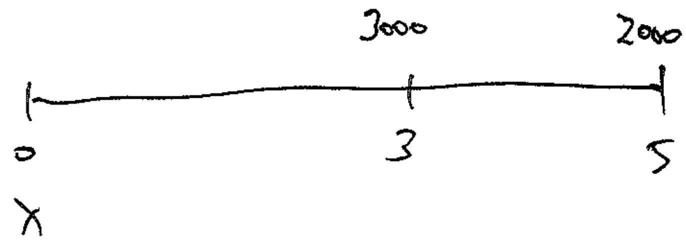
$i = 6\%$

(3) Need $3000 in 3 yrs. $2000 in 5 yrs.

How much do I need today?

(4) Have $6000. Take $3000 in 5yrs. and all remaining in 8 yrs. How much do I get in 8 yrs?

3



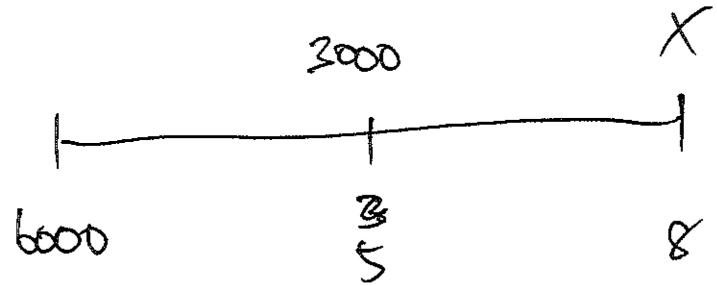
$$X = 3000v^3 + 2000v^5$$ time : 0

$$2000 + 3000(1+i)^2 = X(1+i)^5$$ time : 5

$$3000 + 2000v^2 = X(1+i)^3$$ time : 3

$$X = 4013.374$$

4



$X = 5990.04$

Equation of Value

(balance out)

→ Transactions must have equal value at wherever the time point you pick.

Nominal rate of interest

\$100 $\longrightarrow$ \$~~[illegible]~~ 112
in 1 yr

12 % effective annual interest rate.

Nominal ~~[illegible]~~ ~~[illegible]~~
annual interest rate of 12 %
compounded monthly

$$(1.01)^{12} = 1.126825$$

12.68 % effective rate

Nominal annual interest rate of 12 %

Compounded semi annually

$$(1.06)^2 = 1.1236$$

Compounded quarterly

$$(1.03)^4 = 1.125509$$

Compounded m-thly

$$\left(1 + \frac{.12}{m}\right)^m = 1 + i \qquad i = \text{effective rate}.$$

If $m \to \infty$

$$\left(1+\frac{.12}{m}\right)^m \to e^{.12} = 1.127497$$

effective rate

12.75 %.

(Nominal) 12 % Compounded Continuously $\to e^{.12}$ Accumulation factor.

$$e^{.12} = (1+i)$$

$$.12 = \ln(1+i)$$

$\uparrow$

(Nominal) Continuous interest rate

Ex

Nominal Rate of int conversion.

Force of Interest

$$\delta_t = \frac{A'(t)}{A(t)}$$

FoI

$A(t)$ = accumulation amount function

Recall

$$i^{(m)} = \text{Nominal Int Rate} = m \cdot \left(\begin{array}{c} \text{Int Rate} \\ \text{in } \cancel{\text{the}} \ \frac{1}{m} \text{ year} \end{array} \right)$$

$$= m \left(\frac{\$ \text{ made}}{\$ \text{ had}} \right)$$

$$= m \left(\frac{A(t+\frac{1}{m}) - A(t)}{A(t)} \right)$$

what if $m \to \infty$?

$$i^{(\infty)} = \lim_{m \to \infty} m \left(\frac{A(t+\frac{1}{m}) - A(t)}{A(t)} \right)$$

$$= \frac{1}{A(t)} \cdot \lim_{h \to 0} \frac{A(t+h) - A(t)}{h}$$

$$= \frac{A'(t)}{A(t)} = \text{Force of Interest } \delta_t .$$

+ Instantaneous rate of growth per dollar invested at time t.

FoI - Simple Int.

$$A(t) = A(0)(1 + it)$$

$$A'(t) = A(0)(i)$$

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{i}{1+it}$$

decreasing
function in time.

FoI - Compounding Int.

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$

$$= e^{x \ln a} (\ln a)$$

$$A(t) = A(0)(1+i)^t$$

$$A'(t) = A(0)(1+i)^t \ln(1+i)$$

$$\delta_t = \frac{A'(t)}{A(t)} = \ln(1+i)$$

Constant in t

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt} \ln(A(t))$$

$$\int_0^n \delta_t \, dt = \ln(A(t)) \Big|_0^n$$

$$\int_0^n \delta_t \, dt = \ln(A(n)) - \ln A(0) = \ln\left(\frac{A(t)}{A(0)}\right)$$

$$e^{\int_0^n \delta_t \, dt} = \frac{A(n)}{A(0)}$$

$$\boxed{A(n) = A(0) \, e^{\int_0^n \delta_t \, dt}}$$

Formulas

$\delta_t = \ln(1+i)$ Comp.

$= \frac{i}{1+it}$ Simp.

FoI $\qquad \delta_t = \frac{A'(t)}{A(t)}$

$$A(t) = A(0)\, e^{\int_0^n \delta_t \, dt}$$

$$A(0) = A(t)\, e^{-\int_0^n \delta_t \, dt}$$

Example FoI

Example 1.14: Force of Interest

Given $\delta_t = .08 + .005t$, calculate the accumulated value over five years of an investment of 1000 made at each of the following times:

a) Time 0

$$A(5) = 1000e^{\int_0^5 (.08+.005t)dt} = 1588.04.$$

b) Time 2

$$A(7) = 1000e^{\int_2^7 (.08+.005t)dt} = 1669.46.$$

Constant Force of Interest

If force of interest is constant (as in compund interest rate with constant rate i), we have $\delta_t = \ln(1+i)$ and

$$
\begin{aligned}
A(n) &= A(0)e^{\int_0^n \delta_t dt} \\
&= A(0)e^{\delta_t n}
\end{aligned}
$$

Example 1.15: Overnight Rate

Bank A requires an overnight loan of 10,000,000 and is quoted a nominal annual rate of interest convertible daily of 12% by Bank B.

a) Calculate the amount of interest Banle A must pay for the one-day loan.

With $i^{(365)} = .12$, one-day rate of interest is .12/365.

$$10,000,000(.12/365) = 3,288.21.$$

b) Suppose the loan was quoted at an annual force of interest of 12%. Calculate the interest Bank A must pay in this case.

If $\delta_t = .12$, we have $n = 1/365$ and

$$10,000,000(e^{(.12/365)}) - 10,000,000 = 3,288.21.$$

Real Rate of Interest

1.7: Inflation and 'Real' rate of Interest

Definition: Real Rate of Interest

With annual interest rate i and annual inflation rate r, real rate of interest for the year is

$$i_{real} = \frac{i - r}{1 + r}$$

Commonly, $i - r$ is used as real rate of interest, but it is not theoretically correct because ....

Example 1.16: The Real Rate of Interest

Smith invests 1000 for one year at effective annual rate 15.5%. At the time Smith makes the investment, the cost of a certain consumer item is 1. One year later, when interest is paid and principal returned to Smith, the cost of the item has become 1.10. What is the annual growth rate in Smith's purchasing power with respect to the consumer item?

- At the start of the year, Smith can buy 1000 items.
- At teh end of the year, he receives 1000(1.155) = 1155.
- He can then buy 1155/1.10 = 1050 items. So his purchasing power grew by 5%.

$$i_{real} = \frac{i - r}{1 + r} = \frac{.155 - .1}{1.10} = 0.05$$

$$i - r = .155 - .1 = .055.$$

$$i_{real} = \left(\frac{1+i}{1+r}\right) - 1$$

$$= \frac{i-r}{1+r}$$

Rate of Discount

1.5: Effective and Nominal Rates of Discount

- Interest payable in arrears vs payable in advance.
- Borrow \$1000 for one year at a quoted rate of 10% with interest payable in advance.
- Get \$1000, pay \$100 now, return \$1000 at the end of year.
- Effective annual interest rate

$$\frac{100}{900} = .1111.$$

- **Rate of Discount** of 10% = Effective interest rate of 11.11%.

Effective Annual Rate of Discount

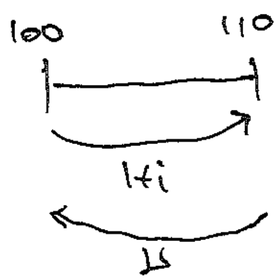
- $$d = \frac{A(1) - A(0)}{A(1)}$$

- Present value of 1 due one period from now is (*t* period from now is)

$$\nu = 1 - d \qquad \nu^t = (1 - d)^t.$$

- Equivalence with *i* (effective rate of interest)

$$d = \frac{i}{1+i} \quad and \quad i = \frac{d}{1-d}$$



$1+i = 1.1$

$\nu = (1.1)^{-1} = .9091$

$110(1-d)$

$\frac{10}{110} = .0909$

Definition: Simple Discount

- With a quoted annual discount rate d, present value of 1 payable t years from now is

$$PV = (1 - dt.)$$

t is usually less than a year.

Definition: Nominal Annual Rate of Discount

- A nominal annual rate of discount compounded or convertible m times per year refers to a discount compounding period of $1/m$ years.

Equivalence

$$\begin{aligned} & \text{(Nominal) annual discount rate of 14\%, compounded monthly} \\ = \; & \text{(Nominal) annual discount rate of14\%, conveltible monthly} \\ = \; & \text{(Nominal) annual discount rate of14\%, convertible 12 times per year} \\ = \; & \text{Effective annual discount rate of 13.14\%: } (1-.14/12)^{12} = 0.8686 \end{aligned}$$

Actuarial Notation

- d = effective annual rate.
- $d^{(m)}$ = nominal annual rate convertible m times per year.

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m$$

i.e.

$$d^{(m)} = m[1 - (1 - d)^{1/m}]$$

Example 1.12: Equivalent effective and nominal rate of discount

Suppose the effective annual rate of interest is 12%. Find the equivalent nominal annual rates for m = 1,2,3,4,6,8,12,52,365, ∞.

m	$1-(1-d)^{1/m}$	$d^{(m)} = m\left[1-(1-d)^{1/m}\right]$
1	.107143	.107143
2	.0551	.1102
3	.0371	.1112
4	.0279	.1117
6	.0187	.1123
8	.0141	.1125
12	.0094	.1128
52	.0022	.1132
365	.0003	.11331
∞	$\lim\limits_{m\to\infty} m\left[1-(1-d)^{1/m}\right] = -\ln(1-d) = .113329$	