7. Cashflow Duration and Immunization

Market Value of Bond is very sensitive to Yield rate.

Ex 7.1

Maturity	1	2	3	4
Spot Rate	$S_0(1) = .05$	$S_0(2) = .1$	$S_0(3) = .15$	$S_0(4) = .2$

- 4-yr zero-coupon bond with $F = 100$. bought today.
- Suppose spot rate stays the same for 4 more years.

Book value vs Market Value of the bond?

Book Value.

$t =$	1	2	3	4
$S_0(t)$	.05	.1	.15	.2

$t=0$. $\quad 100\, v^4 = 100 \left(\frac{1}{1.20}\right)^4 = 48.23$

$i = .2$

$t=1 \quad 100\, v^3 = 57.87$

$t=2 \quad 100\, v^2 = 69.44$

$t=3 \quad 100\, v^1 = 83.33$

$t=4 \quad 100 = 100$

Market Value

t	1	2	3	4
$S_0(t)$	.05	.1	.15	.2

$t=0 \qquad 100\,v^4 = 100\left(\frac{1}{1.20}\right)^4 = 48.23$

$t=1 \qquad 100\,v^3 = 100\left(\frac{1}{1.15}\right)^3 = 65.75$

$t=2 \qquad 100\,v^2 = 100\left(\frac{1}{1.10}\right)^2 = 82.6\cancel{4}$

$t=3 \qquad 100\,v^1 = 100\left(\frac{1}{1.05}\right)^1 = 95.24$

$t=4 \qquad 100$

BV
48.23
57.87
64.44
83.33
100

7.1 Duration of Zero-Coupon Bond

$\boxed{\text{Modified Duration of the bond}}$:

Sensitivity of the bond price to change in yield rate.

(rate of change/dollar invested)

Zero - coupon bond maturing in n - year

$$PV = v^n = \left(\frac{1}{1+i}\right)^n = (1+i)^{-n}$$

modified duration

$$D_M = -\frac{\frac{d}{di} PV}{PV} = -\frac{-n(1+i)^{-n-1}}{(1+i)^{-n}} = n\left(\frac{1}{1+i}\right) = \boxed{nv}$$

convention. per dollar invested.

i goes up, market value of bond goes down with rate nv.

$\boxed{\text{Macaulay Duration}}$ 1938

$$\boxed{D = DM(1+i) \quad = \quad n \qquad \text{for zero-coupon bond.}}$$

$$\begin{pmatrix}\text{Sensitivity of the bond value} \\ \text{to } i\end{pmatrix} = \begin{pmatrix}\text{time until} \\ \text{maturity}\end{pmatrix},$$

Ex 7.2 10% yied rate → 9.99%

Term to Maturity	1-Year	10-Year	30-Year
Bond Price $P(i)$	90.909091	38.55433	5.7309
Modified Duration	0.909091	9.090910	27.2727
Macaulay Duration	1.000000	10.000000	30.0000
Approximate Change in Price Using DM	0.008264	0.035049	0.015630
Approximate Relative Change in Price	0.000091	0.000909	0.002727
Actual Price at Yield Rate 9.99%	90.917356	38.589400	5.746507
Actual Change in Price	0.008265	0.035067	0.015652
Actual Relative Change in Price	0.000091	0.000910	0.002731

Duration for General series of Cashflows

$K_1, K_2, \ldots, K_n$: series of cash flows.

at

time $= 1, 2, 3, \ldots, n$.

$$PV = K_1 v + K_2 v^2 + \cdots + K_n v^n$$

$$= \sum_{t=1}^{n} K_t v^t$$

Modified Duration

$$DM = -\frac{\frac{d}{di}PV}{PV} = \frac{\sum_{t=1}^{n} K_t \, t \, (1+i)^{-t-1}}{PV}$$

$$PV = \sum_{t=1}^{n} K_t (1+i)^{-t}$$

Macaulay Duration

$$D = DM(1+i) = \frac{\sum_{t=1}^{n} K_t \, t \, (1+i)^{-t}}{\sum_{t=1}^{n} K_t (1+i)^{-t}}$$

Duration for a Coupon Bond

Bond with coupon rate r (per period)

maturing in n period.

$$PV = F\nu^n + Fr\, a_{\overline{n}|i}$$

$$= F\nu^n + \sum_{t=1}^{n} Fr\,\nu^t$$

$$D = -\frac{\frac{d}{di}PV}{PV}(1+i) = \boxed{\frac{n \cdot F\nu^n + Fr\sum_{t=1}^{n} t\,\nu^t}{F\nu^n + Fr\sum_{t=1}^{n}\nu^t}}$$

Mcauley Duration

Effective Duration

$$DE = -\frac{PV_{t_0+h} - PV_{t_0-h}}{2h} \cdot \left(\frac{1}{PV_{i_0}} \right)$$

7.2 Asset - Liability Matching and Immunization

looked from $t = 0$.

$$\left.\begin{array}{ll} \text{liability due} & L_t \\ \text{asset income} & A_t \end{array}\right\} \text{ most match.}$$

but both are function of i.

Ex 7.6

$$L_0 = 0 \qquad L_1 = 1 \qquad L_2 = 1.$$

$i = 10\%$

$$PV_L = v + v^2 = 1.735537.$$

There are many ways to prepare for this L_1, L_2.

(i)

$A_0 = 0 \quad A_1 = 1 \quad A_2 = 1$

$\uparrow$ one-yr zero coupon bond

$\uparrow$ 2-yr zero-coupon bond.

$$PV_A = \nu + \nu^2 = 1.735537.$$

(ii)

$A_0 = 1.735537$ $\qquad A_1 = 0$ $\qquad A_2 = 0$.

$\uparrow$

Cash deposits

$PV_A = 1.735537$.

(iii)

$A_0 = 0$ $\qquad A_1 = 0$ $\qquad A_2 = 2.1$

$\uparrow$ line of credit. $\qquad \uparrow$ pay all.

$PV_A = 2.1\,v^2 = 1.735537$.

Redington Immunization

$$PV_A(i_0) = \sum_{t=0}^{n} A_t v^t = \sum_{t=0}^{n} L_t v^t = PV_L(i_0)$$

$$PV(i_0) = \sum_{t=0}^{n} (A_t - L_t) v^t = 0$$

A-L matching

Immunization

Redington
1952

$$\begin{cases} PV_A(i)\big|_{i_0} = PV_L(i)\big|_{i_0} & ① \\ \frac{d}{di} PV_A(i)\big|_{i_0} = \frac{d}{di} PV_L(i)\big|_{i_0} & ② \\ \frac{d^2}{di^2} PV_A(i)\big|_{i_0} > \frac{d^2}{di^2} PV_L(i)\big|_{i_0} & ③ \end{cases}$$

then liability cashflows are immunized.

i.e., if $i \neq i_0$, $PV_A(i) > PV_L(i)$

Immunization

$$\textcircled{1} \quad \sum_{t=0}^{n} A_t v^t = \sum_{t=0}^{n} L_t v^t$$

$$\textcircled{2} \quad \sum t A_t v^t = \sum t L_t v^t$$

$$\textcircled{3} \quad \sum t^2 A_t v^t = \sum t^2 L_t v^t$$

Convexity

$$\frac{\left.\frac{d^2}{di^2} PV_A(i)\right|_{i_0}}{\left.PV_A(i)\right|_{i_0}}$$

Ex 7.7

3 employees, 50 y.o. 53 y.o. 55 y.o.

$$\begin{cases} 10{,}000 \text{ /year until } 65, \\ 100{,}000 \text{ when } 65. \end{cases}$$

(Payments continue even after death.)

Ex 7.7

~~[illegible]~~

To meet the liability, purchase zero coupon bonds due t_1 and t_2.

Assume term structure is flat at 10%

a) $t_1 = 0$ $t_2 = 15$

b) $t_1 = 6$ $t_2 = 12$

c) $t_1 = 2$ $t_2 = 14$

Determine amount of bond needed, and if immunized or not.

liabilities

$$PV = 100{,}000\,v^{10} + 10{,}000\,a_{\overline{10}|.1} = 100{,}000$$

$$PV = 100{,}000\,v^{12} + 10{,}000\,a_{\overline{12}|.1} = 100{,}000$$

$$PV = 100{,}000\,v^{15} + 10{,}000\,a_{\overline{15}|.1} = 100{,}000$$

$$PV_{\text{of total liabilities}} = 300{,}000$$

$$= \sum_{t=1}^{15} L_t\, v^t$$

$$L_1, \ldots, L_{10}, L_{11}, L_{12}, L_{13}, L_{14}, L_{15}$$

$L_1, \ldots$	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}
30,000	$\downarrow$ 130,000	20,000	$\downarrow$ 120,000	10,000		$\downarrow$ 110,000

PV $$X v^{t_1} + Y v^{t_2} = \sum_{t=1}^{15} L_t v^t$$ $i = 10\%$.

$\frac{d}{di}$PV $$t_1 X v^{t_1} + t_2 Y v^{t_2} = \sum_{t=1}^{15} t L_t v^t$$

Two-eq'n, two-unknowns.

$\rightarrow$ solve for X, Y.

$t_1 = 0$ $\qquad$ $t_2 = 15$

a) $X = 149{,}194.85$ $\qquad$ $Y = 629{,}950.53$

$$\underbrace{t_1^2 X v^{t_1} + t_2 Y v^{t_2}}_{33{,}931{,}158} > \underbrace{\sum_{t=1}^{15} t^2 L_t v^t}_{22{,}709{,}898}$$

Yes, Immunized.

$t_1 = 6$ $t_2 = 12$

b) $X = 395{,}035.30$

$Y = 241{,}699.38$

$\underbrace{LHS > RHS}$ NO.

19,117,390 22,709,818

$t_1 = 2$ $t_2 = 14$

c) $X = 195{,}407.21$

$Y = 525{,}977.96$

$LHS > RHS$ Yes.

27,793,236

Full Immunization

$$\sum A_t v^t \geq \sum L_t v^t \quad \text{for any } i > 0.$$