Ch 2: Probability

Contents

June 15, 2017

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Preliminaries

[ToC]

1.1 Sample Space and Events

[ToC]

What would you say...

$$\text{Probability of getting \#3 when you throw a die } = ?$$

- **Experiment** is any action or process whose outcome is subject to uncertainity.
- **Sample Space** of an experiment is **a set of all possible outcomes**. In this case, it's $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
- **Event** is any subset of the sample space $\mathcal{S}$.

Probability of equally likely outcomes

If each outcomes in $\mathcal{S}$ is equally likely, then probablity of an event A is

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } \mathcal{S}}$$

Example: Thorw a die.

$$\begin{aligned} \mathcal{S} &= \{1, 2, 3, 4, 5, 6\} \\ P(\text{ 1 }) &= 1/6 \\ P(\text{ even }) &= 3/6 \end{aligned}$$

Example: Throw a fair coin.

$$\mathcal{S} = \{H, T\}$$

$$P(\text{ Head}) = 1/2$$

Two ways to write $\mathcal{S}$

Example: Throw a fair coin twice, record number of heads.

$$\mathcal{S} = \{0, 1, 2\}$$

$$P(\text{ two heads}) = ?$$

Example: Throw a fair coin twice, record number of heads.

$$\mathcal{S} = \{(T, H), (H, T), (H, H), (T, T)\}$$

$$P(\text{ two heads}) = ?$$

Example: Throw a fair coin twice, record number of heads.

$$\mathcal{S} = \{0, 1, 2\}$$

$$P(\text{ two heads}) = 1/3$$

(Wrong, because 0,1,2 are not equally likely)

Example: Throw a fair coin twice, record number of heads.

$$\mathcal{S} = \{(T, H), (H, T), (H, H), (T, T)\}$$

$$P(\text{ two heads}) = \frac{\{(H, H)\}}{\{(T, H), (H, T), (H, H), (T, T)\}} = \frac{1}{4}$$

(Correct, because each element in $\mathcal{S}$ is equally likely)

Example: Throw a die twice.

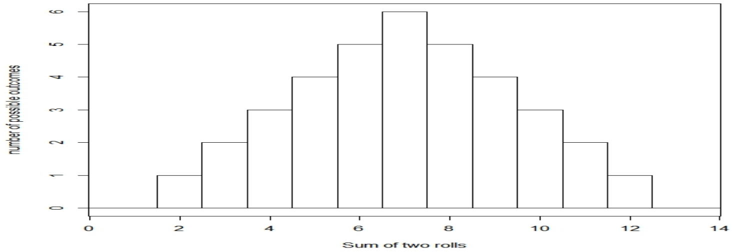
What is the probability the sum of the two numbers will equal 7?
What is the probability the sum of the two numbers will equal 4?

What is the probability the (min of two numbers is greater than 4)?

Write $\mathcal{S}$ in a form (First Throw, Second Throw):

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	**(1,6)**
(2,1)	(2,2)	(2,3)	(2,4)	**(2,5)**	(2,6)
(3,1)	(3,2)	(3,3)	**(3,4)**	(3,5)	(3,6)
(4,1)	(4,2)	**(4,3)**	(4,4)	(4,5)	(4,6)
(5,1)	**(5,2)**	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)



What is the probability the (min of two numbers is greater than 4)?

(1,1)	(1,2)	(1,3)	(1,4)	**(1,5)**	**(1,6)**
(2,1)	(2,2)	(2,3)	(2,4)	**(2,5)**	**(2,6)**
(3,1)	(3,2)	(3,3)	(3,4)	**(3,5)**	**(3,6)**
(4,1)	(4,2)	(4,3)	(4,4)	**(4,5)**	**(4,6)**
(5,1)	**(5,2)**	**(5,3)**	**(5,4)**	**(5,5)**	**(5,6)**
(6,1)	**(6,2)**	**(6,3)**	**(6,4)**	**(6,5)**	**(6,6)**

So what does it really mean...

Throw a die,

$$P(\text{get\#3}) = \frac{1}{6}$$

1.2 Interpretation of Probability

[ToC]

- Relative frequeny gets closer and closer to probability as number of trial increases.

$$[\text{Relative Frequency}] \quad \Rightarrow \quad [\text{Probability}] \qquad \text{as } n \to \infty \ .$$

$$\frac{[\text{num of times the die shows 3}]}{[\text{number of rolls}]} \quad \Rightarrow \quad [\text{Probability}]$$

2

Counting Techniques

[ToC]

Counting Formulas

[ToC]

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make?

Counting Formulas

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6! sequences

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make with only using 3 cards?

Counting Formulas

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make?

$$6! \text{ sequences}$$

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make with only using 3 cards?

$$6 \cdot 5 \cdot 4 = \frac{6!}{3!} = 120 \text{ sequences}$$

Counting Formulas 1

- When you have n subjects, there are $n!$ ways to order.
- When you have k subjects out of n subjects, there are $n!/(n-k)!$ ways to order.

Counting Formulas

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different groups can you make with 3 cards?

Counting Formulas

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$$ABC$$
$$ACB$$
$$BAC$$
$$BCA$$
$$CAB$$
$$CBA$$

Counting Formulas

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different groups can you make with 3 cards?

$$ABC$$
$$ACB$$
$$BAC$$
$$BCA$$
$$CAB$$
$$CBA$$

- If order does matter, then there are $6!/3! = 120$ sequences.
- However, it's counting same group of 3 cards $3! = 6$ times.
- So number of groups should be $120/6 = 20$ groups.

-
$$\frac{120}{6} = \frac{6!}{3!}\frac{1}{3!}$$

- When you choose k subjects out of n, without regard to order, there are
$$\binom{n}{k} = \frac{n!}{(n-k)! \;\; k!}$$
possible combinations.

- This is read as "n choose k".

- Some calculater write this as ${}_nC_r$

2.1 Counting Formulas

[ToC]

1. n subjects

$$n! \quad \text{sequences}$$

2. Use k out of n subjects,

$$n!/(n-k)! \quad \text{sequences}$$

3. Choose k subjects out of n, without regard to order,

$$\binom{n}{k} = \frac{n!}{(n-k)! \;\; k!} \quad \text{groups}$$

2.2 Example: Batting Orders

[ToC]

1. There are 9 players in a basebal team. How many different batting orders are possible?

2. What if you have 15 players ? (only 9 can play)

Example: Batting Orders

1. There are 9 players in a basebal team. How many different batting orders are possible?

$$9! = 362,880 \text{ orders}$$

2. What if you have 15 players ? (only 9 can play)

$$15!/6! = 1,816,214,400 \text{ orders}$$

Example: Boys and Girls

1. There are 5 boys and 5 girls. If they have to sit in a line, how many ways are there?

2. If no two boys and no two girls can sit together, how many ways are there?

Example: Boys and Girls

1. There are 5 boys and 5 girls. If they have to sit in a line, how many ways are there?

   $10!$ ways

2. If no two boys and no two girls can sit together, how many ways are there?

   $2(5!)(5!)$ ways

Exercise: 20 people in a party

If everybody shakes hand with everybody, how many handshakes occur?

Exercise: Three Kinds of Light Bulbs in a Box

A box contains four 40w bulbs, five 60w bulbs, and six 75w bulbs. Three bulbs are selected at once in random.

-

$$
\begin{aligned}
P(\text{ exactly two 75w}) &= \\
P(\text{ same rating}) &= \\
P(\text{ one from each rating}) &=
\end{aligned}
$$

- We must calculate this as

$$P(A) = \frac{\text{number of ways in event A}}{\text{number of total ways}}$$

Exercise: Three Kinds of Light Bulbs in a Box

A box contains four 40w bulbs, five 60w bulbs, and six 75w bulbs. Three bulbs are selected at once in random. What is $P($ exactly two 75w$)$?

1. We are calculating WITHOUT order:

$$
\begin{aligned}
T &= \text{Total number of outcomes} = \binom{15}{3} \\
P(\text{ exactly two 75w}) &= \binom{6}{2}\binom{9}{1} / T. \\
P(\text{ same rating}) &= \binom{4}{3} + \binom{5}{3} + \binom{6}{3} / T. \\
P(\text{ one from each rating}) &= \binom{4}{1}\binom{5}{1}\binom{6}{1} / T.
\end{aligned}
$$

Example: Two kids moved into

a house next to yours. You found out of of them is a girl. What's the probability that the other one is also a girl?

3

Set Operations

[ToC]

3.1 Set Operations and Definitions

[ToC]

Let

$$\begin{aligned} S &= \{1,2,3,4,5,6,7,8\}, \\ A &= \{1,3,5,7\}, \qquad B = \{2,4,5\} \qquad C = \{8,9\} \end{aligned}$$

then

$$\begin{aligned} \text{Union: } A \cup B &= \{1,2,3,4,5,7\} \\ \text{Intersection: } A \cap B &= \{5\} \\ \text{Complement: } A' &= \{2,4,6,8\} \\ \text{Null Set: } &= \{\varnothing\} \\ \text{Disjoint if } A \cap B &= \{\varnothing\} \\ \text{Exhaustive if } A \cup B \cup C &= S \end{aligned}$$

3.2 Axioms of Probability

Ax1 For any event A, $P(A) \geq 0$.

Ax2 $P(\mathcal{S}) = 1$.

Ax3 If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

These three axioms imply that $P(\varnothing) = 0$.

3.3 Probability Formulas

[ToC]

For any three event A, B and C,

1. $P(A) \leq 1$ (from Axiom 2)
2. $P(A) + P(A') = 1$. Therefore, $P(A) = 1 - P(A')$ (from Axiom 3)
3. $P(B) = P(B \cap A) + P(B \cap A')$ (from Axiom 3)
4. Inclusion-Exclusion $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. Inclusion-Exclusion for three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \quad + P(A \cap B \cap C)$$

6. DeMorgan's Law: For any events A, B and C,

$$\begin{aligned}
A' \cap B' &= (A \cup B)' \\
A' \cup B' &= (A \cap B)' \\
A' \cap B' \cap C' &= (A \cup B \cup C)' \\
A' \cup B' \cup C' &= (A \cap B \cap C)'
\end{aligned}$$

3.4 Example: Project Funding

[ToC]

There are 3 projects that has applied for the grant. Let A_i represent an event that project i gets funded. There are 3 projects. Given

$$P(A_1) = .22, \quad P(A_2) = .25, \quad P(A_3) = .28$$

and

$$\begin{aligned} P(A_1 \cap A_2) &= .11, & P(A_1 \cap A_3) = .05, \\ P(A_2 \cap A_3) &= .07, & P(A_1 \cap A_2 \cap A_3) = 0.01, \end{aligned}$$

Calculate the probability of :

1. $P($ At least one of project 1 and 2 get award $)$

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Calculate the probability of :

1. $P($ At least one of project 1 and 2 get award $)$

$$= P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = .36$$

2 P(Neither project 1 nor 2 get award)

3 P(At least one of 3 project gets award)

2 P(Neither project 1 nor 2 get award)

$$= P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2) = .64$$

3 P(At least one of 3 project gets award)

$$\begin{aligned} = \; & P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) \\ & -P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = .53 \end{aligned}$$

4 $P($ None of the project get award $)$

5 $P($ Only project 3 is awarded $)$

4 $P($ None of the project get award $)$

$$= P(A_1' \cap A_2' \cap A_3') = P\Big((A_1 \cup A_2 \cup A_3)'\Big) = 1 - P\Big((A_1 \cup A_2 \cup A_3)\Big) = .47$$

5 $P($ Only project 3 is awarded $)$

$$= P(A_3) - P(A_3 \cap A_1) - P(A_3 \cap A_2) + P(A_1 \cap A_2 \cap A_3)$$

$$P\Big((A_1' \cap A_2') \cup A_3\Big)$$

$$= P(A_3) + P(A_1' \cap A_2' \cap A_3') = .75$$

4

Conditional Probability

[ToC]

4.1 Conditional Probability and Tree Diagram

[ToC]

- Conditional Probability of event A given that the event B has occurred, is denoted as $P(A|B)$, and defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- That is to same thing as

$$P(A \cap B) = P(A|B) \cdot P(B).$$

Example: Made in Plant A

Electrical product made in two plants. A sales man picks up a product randomly.

	Defective	Non-defective	Total
Plant A	6	14	20
Plant B	4	26	30
Total	10	40	50

$$P(\text{ defective }) = \frac{10}{50} \qquad P(\text{ made in plant A}) = \frac{20}{50}$$

$$P(\text{ defective} \mid \text{Made in plant A}) \quad = \qquad = \quad \frac{6}{20}$$

$$\frac{P(\text{ defective} \cap \text{Made in plant A})}{P(\text{Made in A })} \quad = \quad \frac{6/50}{20/50} \quad = \quad \frac{6}{20}$$

Independence

[ToC]

5.1 Independence

[ToC]

- Two events A and B are independent if

$$P(A|B) = P(A), \qquad \text{or} \qquad P(A \cap B) = P(A) \cdot P(B).$$

Events are said to be dependent otherwise.

- This turns Inclusion-Exclusion formula to:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \end{aligned}$$

- Mutually exclusive events cannot be independent.

Exercise: Aircraft Seam

An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are independent of each other.

If only 1% of all rivets needs to be reworked, what is the probability that a seam needs to be reworked.

Exercise: Aircraft Seam

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If only 1% of all rivets needs to be reworked, what is the probability that a seam needs to be reworked.

$$
\begin{aligned}
P(\text{ a seam needs rework}) &= P(\text{ at least one of 25 revet are defect}) \\
&= 1 - P(\text{ all of 25 revet are good}) \\
&= 1 - P(R_1 \text{ is good }) \cdot P(R_2 \text{ is good }) \cdots P(R_{25} \text{ is good }) \\
&= 1 - P(\text{ a rivet is good})^{25} \\
&= 1 - (.99)^{25} = .222
\end{aligned}
$$

If we have

$$1 - (.999)^{25} = 0.025$$

6

Law of Total Probability

[ToC]

6.1 The Law of Total Probability

[ToC]

- Recall formula:

$$P(B \cap A) = P(B|A)P(A).$$

Then for event B, can be written using formula #2,

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B|A)P(A) + P(B|A')P(A') \end{aligned}$$

- Instead of A, A', if A_1, A_2, A_3 are mutually exclusive and exhaustive events, we can write

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$

6.2 Bayes' theorem

[ToC]

- Bayes theorem says

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Example: Red and White balls in Two Urns

- There are two urns, urn A and urn B.
- Urn A contains 5 red balls, 2 white.
- Urn B contains 3 red balls, 4 white.
- Fair coin flip decides which urn to be used.
- Somebody flip a coin, and drew one ball from an urn. You don't know which urn was used. Ball dran was red.
- What is the probability that urn A was used?

Example: Testing for Disease

- 1 in 1000 adults is afflicted with this disease.
- Test for this disease is 99% accurate on infected patients.
- Test is 98% accurate on non-infected patients.
- If test comes back positive, what is the chance that you are actually infected?

$$P(\text{Infected}) =$$
$$P(\text{Pos}|\text{Infected}) =$$
$$P(\text{Pos}|\text{Not Infected}) =$$

Example: Testing for Disease

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- If test comes back positive, what is the chance that you are actually infected?

$$P(\text{Infected} \mid \text{Pos}) = ?$$

$$P(\text{Infected}) = 0.001, \quad P(\text{Pos}|\text{Infected}) = .99, \quad P(\text{Pos}|\text{Not Infected}) = .02$$

- Using the Baye's theorem,

$$P(I|Pos) = \frac{P(I \cap Pos)}{P(Pos)} = \frac{P(Pos|I)P(I)}{P(Pos|I)P(I) + P(Pos|I')P(I')}$$

$$= \frac{(.99)(.001)}{(.99)(.001) + (.02)(.999)} = 0.0472$$

So if your test comes back positive, you have only 5% chance of having the disease.

- On the other hand, If the test comes back negative,

$$P(I'|B') = \frac{P(I' \cap P')}{P(P')} = \frac{P(P'|I')P(I')}{P(P'|I')P(I') + P(P'|I)P(I)}$$

$$= \frac{(.98)(.999)}{(.98)(.999) + (.01)(.001)} = 0.999989$$

If your test comes back negative, you probably don't have the disease.