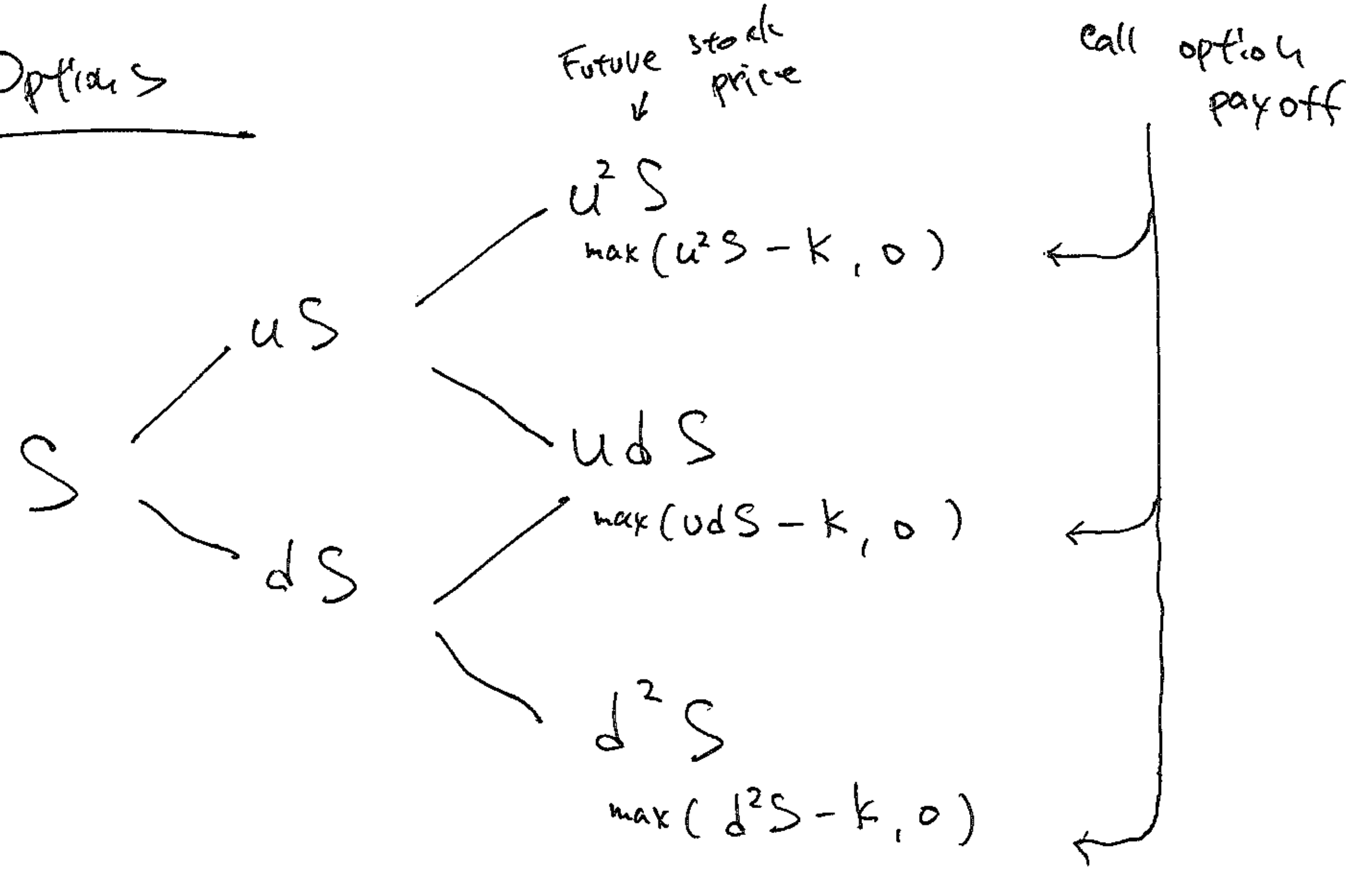
Binomial Options



$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

# of stocks

How much you borrow

$$\begin{cases} \Delta u S e^{\delta h} + B e^{rh} = C_u \\ \Delta d S e^{\delta h} + B e^{rh} = C_p \end{cases}$$

$$\boxed{\Delta = e^{-\delta h}\,\frac{C_u - C_d}{S(u-d)}}$$

$$\boxed{B = e^{-rh}\,\frac{uC_d - dC_u}{u-d}}$$

Option price

$$\boxed{C_0 = \Delta S + B}$$

$$\cancel{C_0 = \frac{e^{-\delta h}}{u-d}\left[C_u - C_d + uC_d - dC_u\right]}$$

$$\cancel{= \frac{e^{-\delta h}}{u-d}\left[(1-d)C_u + (1-u)C_d\right]}$$

$$C_0 = \Delta S + B$$

$$= e^{-rh}\left[\left(\frac{e^{(r-\delta)h} - d}{u-d}\right) C_u + \left(\frac{u - e^{(r-\delta)h}}{u-d}\right) C_d\right]$$

$$\boxed{C_0 = e^{-rh}\left[p^* C_u - (1-p^*)\, C_d\right]}$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

Ex

$$\begin{cases} r = .08 \\ \delta = 0 \\ \sigma = .2 \\ h = 1 \end{cases}$$

$u = e^{(r-\delta)h - \sigma\sqrt{h}} = 1.3231$

$d = \cancel{3}\, e^{(r-\delta)h - \sigma\sqrt{h}} = .8869$

strike price

$K = 30$

$S = 30$

③

$uS = 39.694$

①

$dS = 26.608$

②

$u^2S = 52.520$

22.520

$udS = 35.205$

5.205

$d^2S = \cancel{25}$

23.599

0

(1)

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = .4502$$

$$C_0 = e^{-rh}\left[p^* \, 22.520 + (1-p^*)\, 5.205\right]$$

$$= \cancel{6.116}$$

$$12$$

(2)

$$C_0 = e^{-rh}\left[p^* \, 5.205 + (1-p^*)\, 0\right]$$

$$=$$

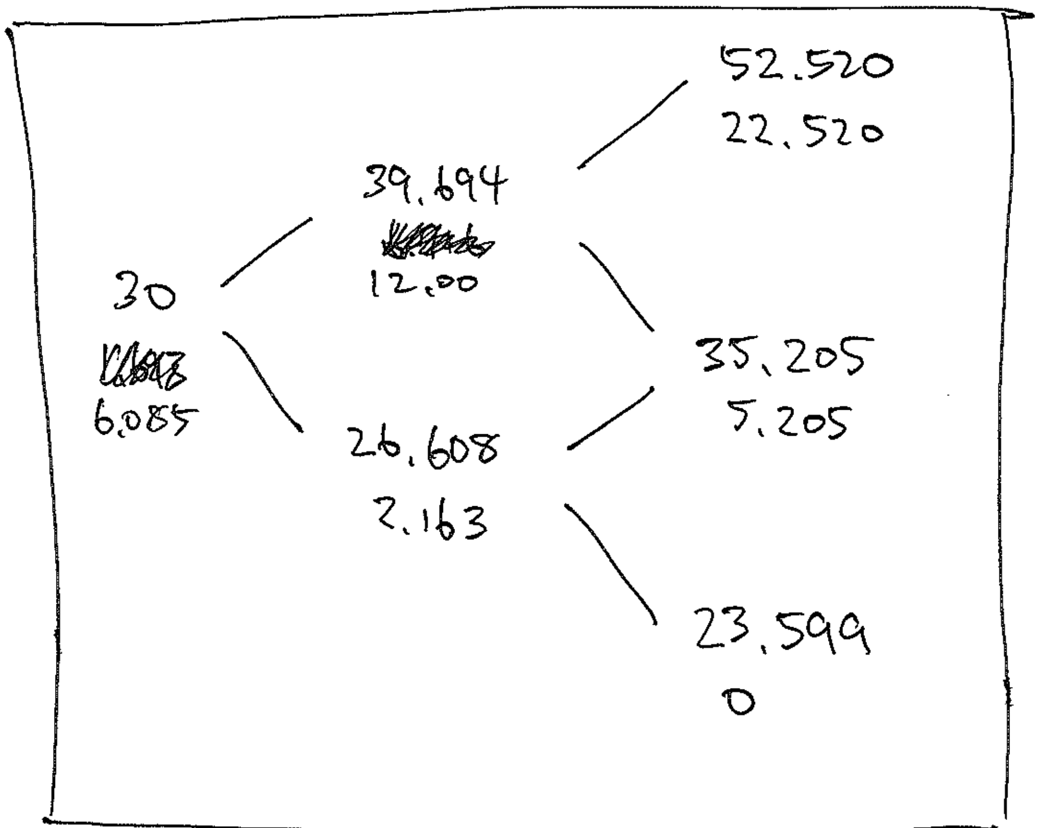
$$2.163$$

(3)

$$C_0 = e^{-rh}\left[p^* \, 6.716 + (1-p^*) \, 2.163\right]$$

$= \cancel{11.6932}$

6.085



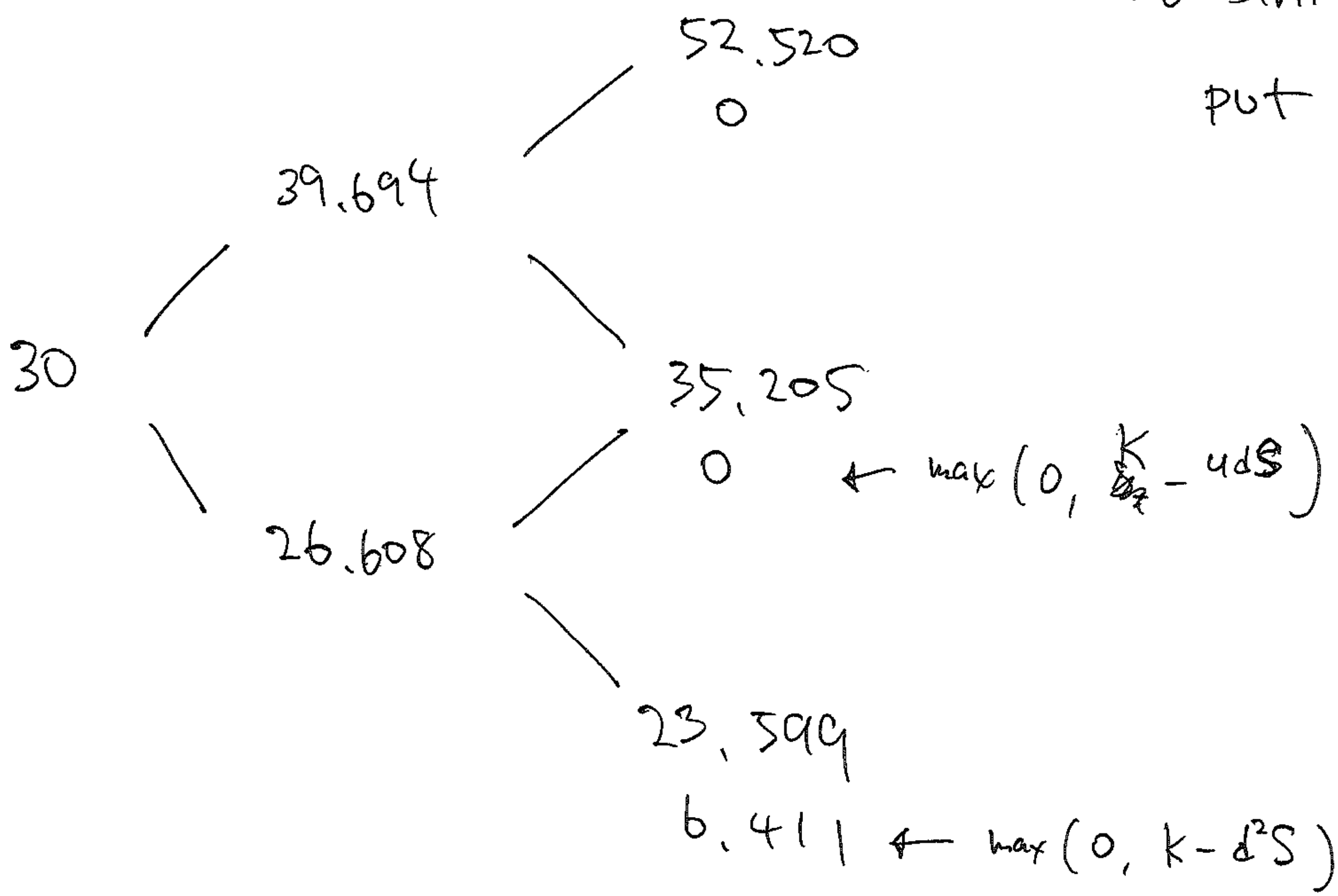
Price of 30-Strike Call in 2yrs

European

$n=2$

Put Option

30-strike put.



rest of the calculation is the same.

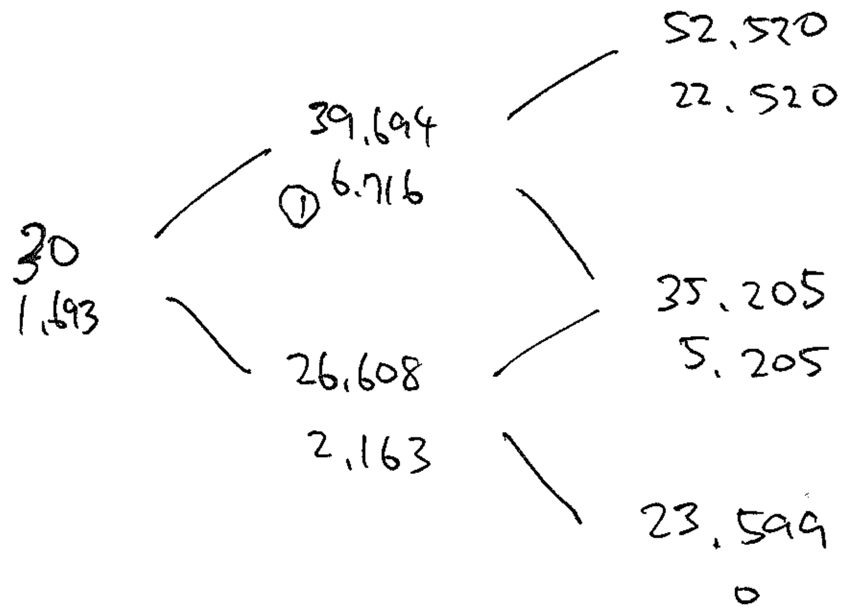
American Options

Value of $\overset{\text{call}}{\text{option}}$ if exercised $= \max(0, S-K)$

American call:

$$p(S, K, t)$$

$$= \max\left(S-K,\ e^{-rh}\left[p^* \, p(uS, K, t+h) + (1-p^*)\, p(dS, K, t+h) \right] \right).$$



European Call

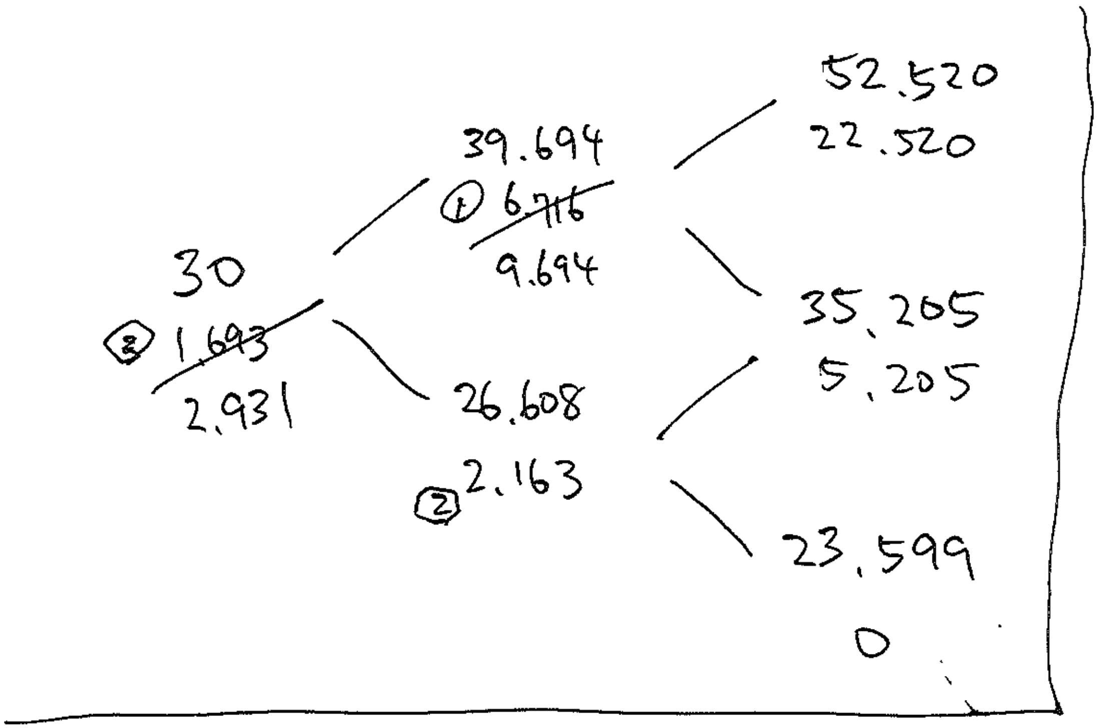
30 - strike

① 6.716 = unexercised.

39.694 - 30 = 9.694. exercised.

→ value is higher when exercised.

① = max (6.716 , 9.694) = 9.694.



American Call

30 - strike

② is unchanged.

③ is now

$$C_0 = e^{-rh}\left[p^* 9.694 + (1-p^*) 2.163\right]$$

$$= 2.931$$

Risk - Neutral Pricing

Binomial Option pricing Formula :

$$C_0 = \Delta S + B$$

$$= e^{-rh} \underbrace{\left[p^* C_u + (1-p^*) C_d \right]}_{E(\text{Op. Value})}$$

where

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

If Option Value $= C_0$ today and

$$\text{Value} = \begin{cases} C_u & \text{with probability } p^* \\ C_d & \text{with probability } 1-p^* \end{cases}$$

$$E(\text{Option Value in 1 period}) = p^* C_u + (1-p^*) C_d$$

$$PV\left[E(\text{Opt. Val. in 1 period})\right] = e^{-rh}\left[p^* C_u + (1-p^*) C_d\right]$$

$$\parallel$$

$$C_0 .$$

So p^* is a probability ?

but we assigned no probability.

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}.$$

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h + \sigma\sqrt{h}} \end{cases}$$

$$E(\text{Opt. Value in 1 period}) = p^* C_u + (1-p^*) C_d.$$

$$= p^* u S + (1-p^*) d S$$

$$= e^{(r-\delta)h} S$$

$$= \text{Forward Price} : F_{0,T}$$

p^* is risk-neutral probability.

i.e. p^* is a probability such that if

$$\begin{cases} C_u & \text{w. prob. } p^* \\ C_d & \text{w. prob. } (1-p^*) \end{cases},$$

then

$$C_0 = PV(\text{Future Contract of the stock}).$$

why is it risk-neutral?

$$p^* u S + (1-p^*) d S = e^{(r-\delta)h} S$$

$$p^* u S e^{\delta h} + (1-p^*) d S e^{\delta h} = \underbrace{e^{rh} S}_{\text{risk-free growth}}.$$

E(stock values) = Risk-free investment.

For Risk-neutral investers, they're the same.

Early Exercise

If somebody exercise call option early, he

- → receives the stock + dividends
- → must pay strike price now
- → lose Insurance.

100-Strike Call

$r = 5\%$

$\delta = 5\%$.

$S = 120$.

$$\left.\begin{array}{lll} \text{dividend} & 120\,(.05) = \$6 \\ \text{Interest Cost} & 100\,(.05) = \$5. \end{array}\right\} \rightarrow \text{better to exercise.}$$

→ Insurance is lost. (S may go down).

If $\sigma = 0$

volatility

$rK \leq \delta S$ then you should exercise.

you don't need insurance

b/c $\sigma = 0$.

If $\sigma \neq 0$, then

Fig 11.1 p345.

Pricing an Option using real Probabilities

In reality, it should be more like

true growth rate of stock.

$$p u S + (1-p) d S = e^{\alpha h} S$$

true probability of uS happening.

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

solve for p,

$$p = \frac{e^{\alpha h} - d}{u - d}.$$

must have $u > e^{\alpha h} > d$.

true yield rate of option.

$$C_0 = e^{-\gamma h}\left[p u S + (1-p) d S \right]$$

$$\Delta S e^{\alpha h} + B e^{r h} = \underbrace{(\Delta S + B)}_{\text{Initial investment}} e^{\gamma h}$$

So

$$e^{\gamma h} = \frac{\Delta S}{\Delta S + B} e^{\alpha h} + \frac{B}{\Delta S + B} e^{r h}$$

It turns out, that

$$C_0 = e^{-\gamma h}\left[p u S + (1-p) d S \right]$$ (using true probability)

is algebraically equivalent to

$$C_0 = e^{-rh}\left[p^* u S + (1-p^*) d S \right]$$ (using risk-neutral prob.)

Risk - neutral

[Buy Call Option] = [Buy Δ Stock + Borrow B].

premium = Initial Cost

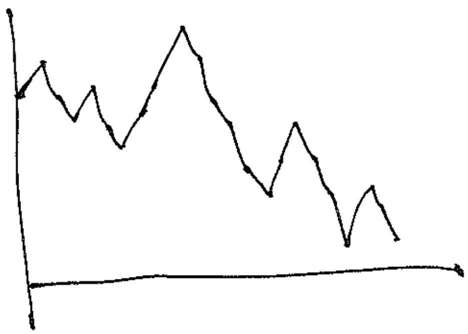
C_0 = $\Delta S + B$

Pay off = Pay off

$\max(0, S-k)$ = S_T -

Binomial Tree and Lognormality

Random Walk.



$$Y_i = \begin{cases} +1 & \text{w.p. } .5 \\ -1 & \text{w.p. } .5 \end{cases}$$

each step is independent.

$$Z_n = \sum_{i=1}^{n} Y_i$$

(random walk).

Can we use this to model stock prices?

$$E(Z_n) = E\left(\sum_{i=1}^{n} Y_i\right)$$

$$= \sum_{i=1}^{n} E(Y_i) = 0 ,$$

$$V(Z_n) = V\left(\sum_{i=1}^{n} Y_i\right)$$

by independence

$$= \sum_{i=1}^{n} V(Y_i)$$

$$= n$$

$$\begin{cases} +1 & .5 \\ -1 & .5 \end{cases}$$

$$E(Y^2) = 1 .5 + 1 .5$$

$$= 1 .$$

$$E(Y) = 0 .$$

$$V(Y) = E(Y^2)$$

Problems with RW

1. RW can go negative. Stock can't

2. $\pm \$1$ step may not be approprate.

3. Stock grows on average. RW has 0 mean.

Solution

Let

RW = Continuously Compounded returns.

$S_u = S e^{(r-\delta)h + \sigma\sqrt{h}}$ one step up

$S_d = S e^{(r-\delta)h - \sigma\sqrt{h}}$ one step down.

$r_{t,t+h} = (r-\delta)h + \sigma\sqrt{h} = \ln\left(\frac{S_u}{S}\right)$

Ex.

	day 1	2	3	4
Stock price	100	103	97	98

$$\begin{tikzcd}
100 \arrow[r, bend right, ".02956"'] \arrow[rrr, bend right=60, "-0.0202"'] & 103 \arrow[r, bend right, "-.06002"'] & 97 \arrow[r, bend right, ".01026"'] & 98
\end{tikzcd}$$

$$\ln\left(\frac{103}{100}\right) = .02956$$

$$\ln\left(\frac{97}{103}\right) = -.06002$$

$$\ln\left(\frac{98}{97}\right) = .01026$$

$$-0.0202 = (.02956) + (-.06002) + (.01206)$$

$$\ln\left(\frac{98}{100}\right) = -0.0202$$

Ex

Stock price $\$100$ $\$10$

percentage return $\dfrac{10-100}{100} = -90\%$.

Cont. Comp. return $\ln\left(\dfrac{10}{100}\right) = -2.30$. $= \ln(1+i)$, $i = -.9$

Ex

Stock price $100

Cont. Comp. return -500%.

$$100\,e^{-5} = .6738$$

$$\$100 \xrightarrow[1\text{ year.}]{} .6738$$

SD of return

Cont. comp. rate

$$r_{annual} = \sum_{i=1}^{12} r_{monthly}$$

$$V(r_{annual}) = \sum_{i=1}^{12} V(r_{mo.})$$ assuming independence.

$$\sigma^2 = 12\, \sigma_{mo}^2.$$

$$\sigma = \sqrt{12}\, \sigma_{mo}.$$

$$\sigma^* = \sqrt{1}\,\sigma^*_{annual}$$

$$= \sqrt{12}\,\sigma^*_{monthly}$$

$$= \sqrt{4}\,\sigma_{quarterly}$$

$$= \sqrt{365}\,\sigma_{daily}$$

$$\sigma_h = \sigma\sqrt{h}$$

SD for 1 period.

# of periods.

Binomial Model

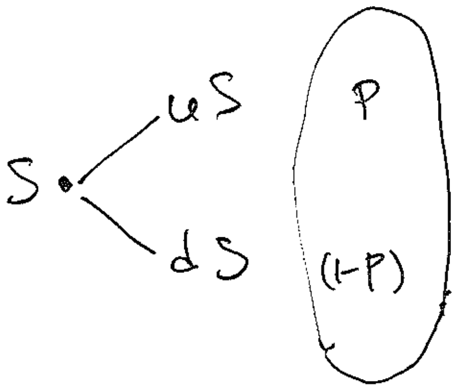
$$S_{t+h} = S e^{(r-\delta)h \pm \sigma\sqrt{h}} \leftarrow \text{RW}.$$

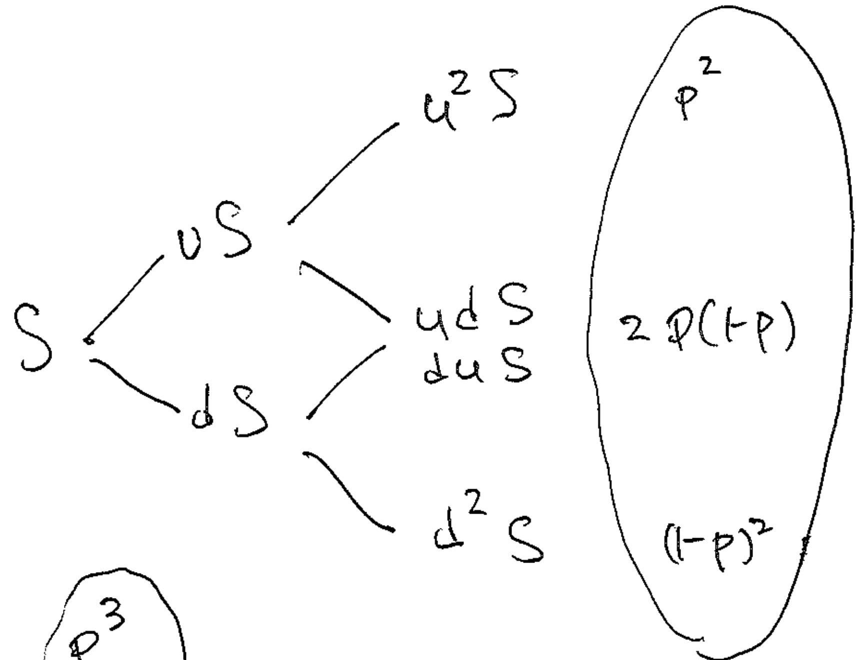
1. Stock <u>Cannot</u> be negative.

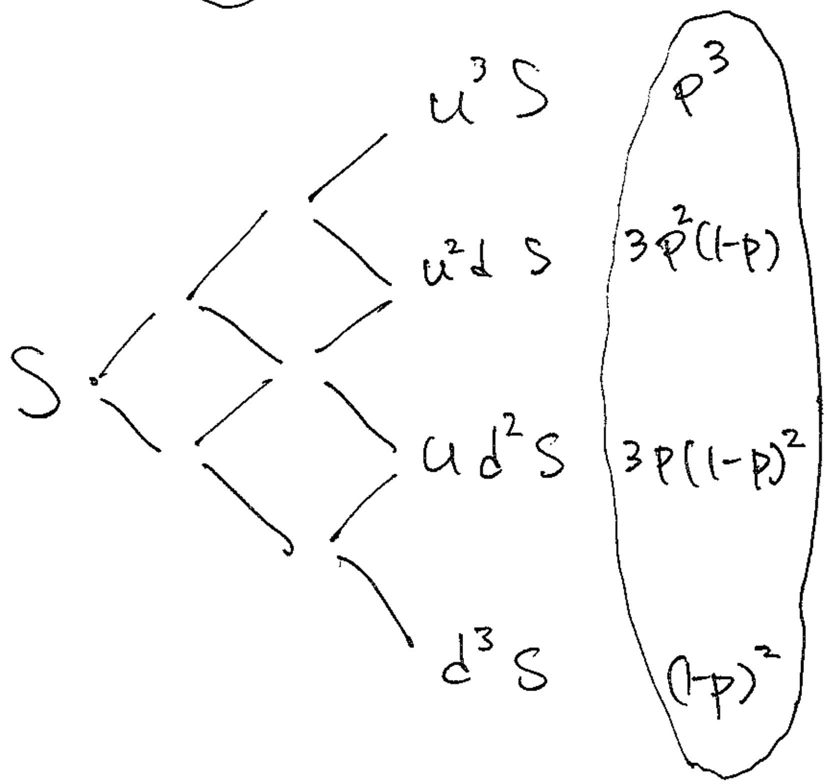
2. we can adjust σ and h. (step size, step period)

3. Stock always grow by $(r-\delta)h$. so return ~~被~~ will be positive on average.

Normality and RW

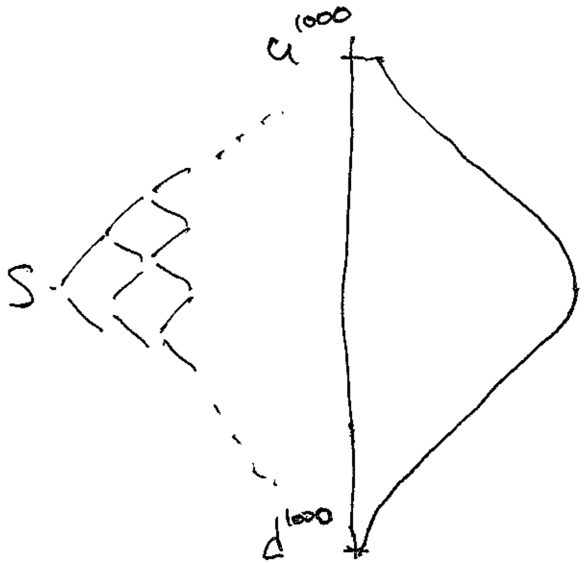






Binomial Distribution

After 1000 steps



Binomial (n, p)

Binomial $\longrightarrow$ Normal.

e^{Normal} = lognormal

Let $X \sim N(\mu, \sigma^2)$

$Y = e^X \sim LN(\mu, \sigma^2)$.

Lognormal Distribution

$$Y \sim LN(\mu, \sigma^2)$$

$$\ln(Y) \sim N(\mu, \sigma^2)$$

$$E(Y) = E(e^{X}) \qquad X \sim N(\mu, \sigma^2)$$

$$E(Y^2) = E(e^{2X}) \qquad E(e^{X}) \neq e^{E(X)}$$

Moment Generating Function for Normal

$$M(t) = E(e^{Xt}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M_{(1)} = E\left(e^{X(1)}\right) = e^{\mu + \frac{\sigma^2}{2}} = E(Y)$$

$$M_{(2)} = E\left(e^{X(2)}\right) = e^{2\mu + 2\sigma^2} = E(Y^2)$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

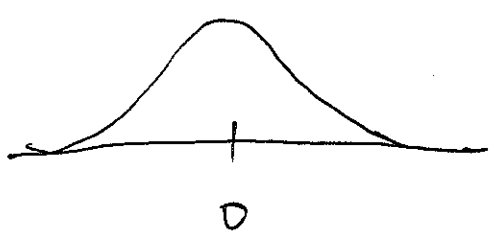
$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$= \left[e^{\sigma^2} - 1\right] e^{2\mu + \sigma^2}$$

Lognormal

$$Y = e^X$$

$X \sim N(0, \sigma^2)$ $\qquad$ $Y \sim LN(0, \sigma^2)$

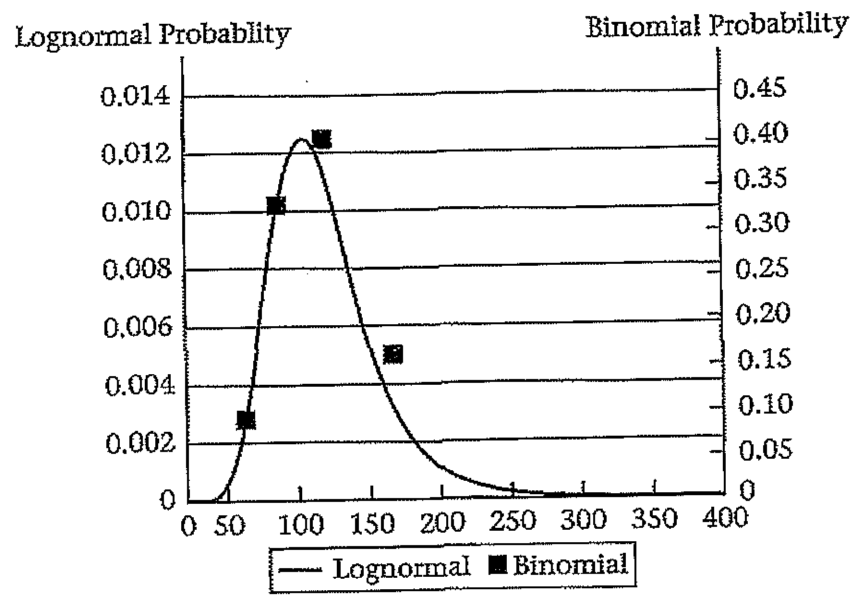


$$E(Y) = e^{\sigma^2/2}$$

$$V(Y) = \left(e^{\sigma^2} - 1\right) e^{\sigma^2}$$

FIGURE 11.8

Comparison of lognormal distribution with three-period binomial approximation.



[3]The expression $\binom{n}{i}$ can be computed in Excel using the combinatorial function, *Combin*(n, i).

FIGURE 11.9

Comparison of lognormal distribution with 25-period binomial approximation.