Ex 5.17 $^{\text{ROSS}}$ Coupon Collecting Problem

m types of coupons.

Coupon = type j with prob. p_j. $\sum_{j=1}^{m} p_j = 1$

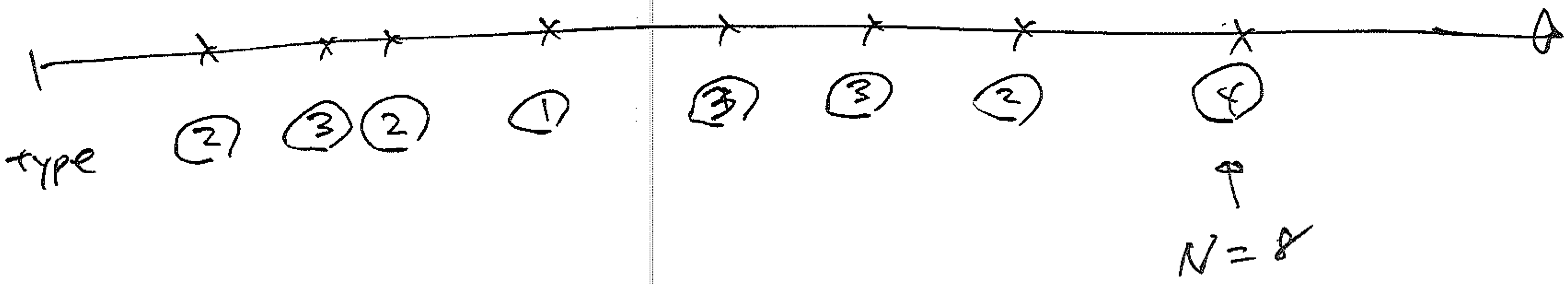
$\boxed{N : \text{\# of coupons needed to collect all } \underline{\underline{m}} \text{ types.}}$

$E(N) = \, ?$

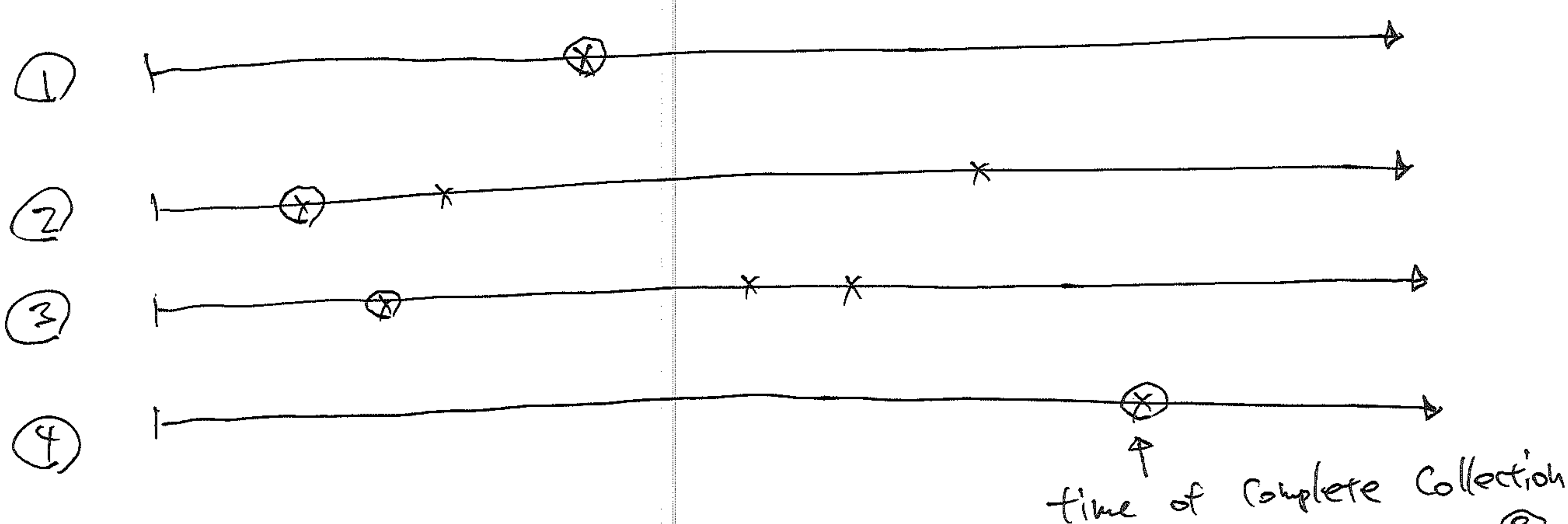
View 1

m = 4

Poi Proc



View 2



View 1

1 Poi. Proc. (λ) $\left\{ \begin{array}{l} 1 \\ \vdots \\ m \end{array} \right.$ types.

Choose $\lambda = 1$.

View 2

indep. $\left\{ \begin{array}{l} 1 \text{ Poi Proc with rate } (\lambda P_1) \\ \vdots \\ m \text{ Poi. Proc with rate } (\lambda P_m) \end{array} \right.$

X_1 = time of 1st event

$\vdots$

X_m = time of 1st event.

$$\boxed{\begin{aligned} X &= \text{time of complete Collection} \\ &= \max(X_1, \ldots, X_m). \end{aligned}}$$

View 2

X_i = time of 1st event

$\sim \mathrm{Exp}(\lambda p_i)$

$$P(X < t) = P(\max(X_1, \ldots, X_m) < t)$$

$$= \prod_{i=1}^{m} P(X_i < t)$$

$$= \prod_{i=1}^{m} \left(1 - e^{-\lambda p_i t}\right) \qquad = F_X(t).$$

$$E(X) = \int_0^\infty \left(1 - F_X(t)\right) dt$$

since X is non-negative.

$$F_X(t) = \prod_{i=1}^{m} \left(1 - e^{-\lambda P_i t}\right).$$

Now translate $E(X)$ back to $E(N)$.

↳ go back to <u>View 1</u>.

X = time of Nth event.

$$= \sum_{i=1}^{N} T_i$$

where T_i is time between events in view 1.

$$E(X) = E\left(\sum_{i=1}^{N} T_i\right)$$

$$= E\left[E\left(\sum_{i=1}^{N} T_i \;\middle|\; N=n\right)\right]$$

$$= E\left[\sum_{i=1}^{N} \underbrace{E(T_i)}_{\substack{\parallel \\ \lambda\, =\, 1.}}\right]$$

$$= E[N].$$

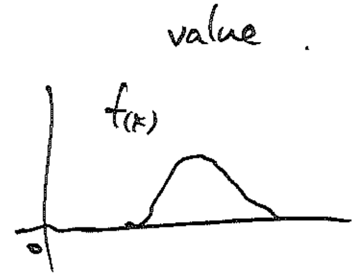
///.

Ex 5.15 Ross

Gift Problem

Gift arrives as Poi Proc. (λ).

Each gift has value $\underset{iid}{\sim} f(x)$



Can only accept 1 gift.

There's cost of waiting c per unit time.

maximize E(Return).

$$\boxed{\text{Decision Rule: pick threshold } y. \quad \text{Accept 1st gift above } y.}$$

Gift $\sim PP(\lambda)$

each arrival $\begin{cases} \text{Accept} & \text{w.p.} \quad 1 - F(y) = q \\ \text{Not Accept} & \text{w.p.} \quad \cancel{1-}\, F(y) = p \end{cases}$

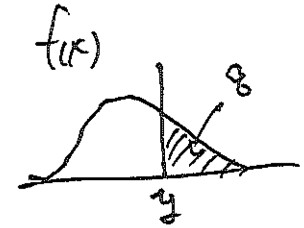
$\rightarrow$ Accepted Gift $\sim PP(\lambda \cdot q)$

Time until 1st accepted gift $\sim Exp(\lambda \cdot q)$

$$E(\text{Return}) = E\left(\begin{array}{c}\text{Value of}\\ \text{Accepted Gift}\end{array}\right) - c\cdot E\left(\begin{array}{c}\text{Time until}\\ \text{Acc. Gift}\end{array}\right)$$

$$= E\left(X \mid X > y\right) - c\left(\frac{1}{\lambda q}\right)$$

$$E\left(X \mid X > y\right) = \int_{\text{all } x > y} X \cdot f_{X|X>y}\, dx$$



$$= \int_y^\infty X \cdot \frac{f(x)}{q}\, dx$$

$$E(\text{Return}) = \frac{1}{q}\left[\int_y^\infty x \cdot f(x)\, dx - \frac{c}{\lambda}\right]$$

Find y that maximize $E(\text{Ret})$.

(note $q = 1 - F(y)$). let $\overline{F}(y) = 1 - F(y)$.

$$\frac{d}{dy} E(Ret(y))$$

by product Rule

$$= \frac{d}{dy}\left(\frac{1}{\bar{F}(y)}\right)\left[\int_y^\infty x \cdot f(x)\,dx - \frac{c}{\lambda}\right]$$

$$+ \left(\frac{1}{\bar{F}(y)}\right) \cdot \frac{d}{dy}\left[\int_y^\infty x \cdot f(x)\,dx - \frac{c}{\lambda}\right].$$

$$\begin{cases} \frac{d}{dy}\left(\frac{1}{\bar{F}(y)}\right) = \frac{d}{dy}\frac{1}{(1-F(y))} = \frac{-1}{(1-F(y))^2}\left(-f(y)\right). \\ \frac{d}{dy}\int_y^\infty x \cdot f(x)\,dx = -y \cdot f(y) \end{cases}$$

Fundamental Thm of Calc.

$$\frac{d}{dy} E\left(Ret(y)\right)$$

$$= \frac{f(y)}{(1-F(y))^2}\left[\int_y^\infty x\cdot f(x)\,dx - \frac{c}{\lambda}\right] - \frac{y\cdot f(y)}{1-F(y)} \overset{set}{\equiv} 0.$$

multiply $\bar{F}^2_{(y)}$ and get.

$$f(y)\left[\int_y^\infty x\cdot f(x)\,dx - \frac{c}{\lambda}\right] - \bar{F}(y)\, y\cdot f(y) = 0.$$

$$\int_y^\infty x \cdot f(x)\, dx - \frac{c}{\lambda} = \bar{F}(y) \cdot y$$

$$\bar{F}(y) = P(X > y) = \int_y^\infty f(x)\, dx \, .$$

$$\int_y^\infty x \cdot f(x)\, dx - \frac{c}{\lambda} = y \cdot \int_y^\infty f(x)\, dx$$

$$\int_y^\infty (X - y)\, f(x)\, dx = \frac{c}{\lambda}$$

$$(X-y)^+ = \begin{cases} X - y & \text{if it's} \geq 0. \\ 0 & \text{if it's} < 0. \end{cases}$$

$$\boxed{E\left[(X-y)^+\right] = \frac{c}{\lambda}}$$

eq'n for optimal ~~[illegible]~~ y.

$$E\left[(X-y)^+\right]$$

$\uparrow$

non-increasing function of y.

X is always positive

i.e. $y \uparrow$. $(X-y)^+$ always $\downarrow$.

If, ∴

$$E(X) < \frac{c}{\lambda}.$$

Then since

$$E\left((X-y)^+\right) \le E(X) \quad \text{for any } y \ge 0.$$

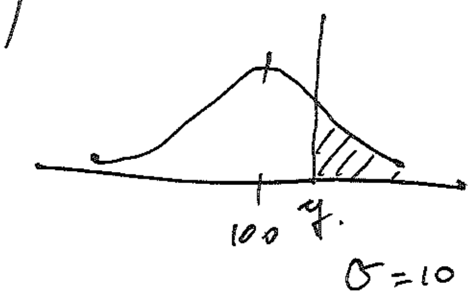
There's no solution for $E\left((X-y)^+\right) = \frac{c}{\lambda}$. //.

If $E(X) \geq \frac{c}{\lambda}$, then solve

$$E[(X-y)^+] = \frac{c}{\lambda}$$ to find y.

Example.

Gift value $\sim N(100, 10^2)$



$c = 5$

$\lambda = 1$

$y = ?$

Solve

$$E[(X-y)^+] = \frac{c}{\lambda}.$$