Exponential Distribution

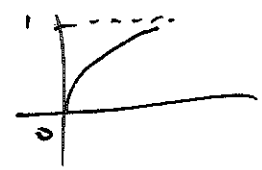
$$X \sim \text{Exp}(\lambda)$$

pdf

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0$$

CDF

$$F(x) = 1 - e^{-\lambda x} \qquad x > 0$$



$$E(X) = 1/\lambda$$

$$V(X) = 1/\lambda^2$$

MGF

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

Memoryless Property

$X \sim Exp(\lambda)$

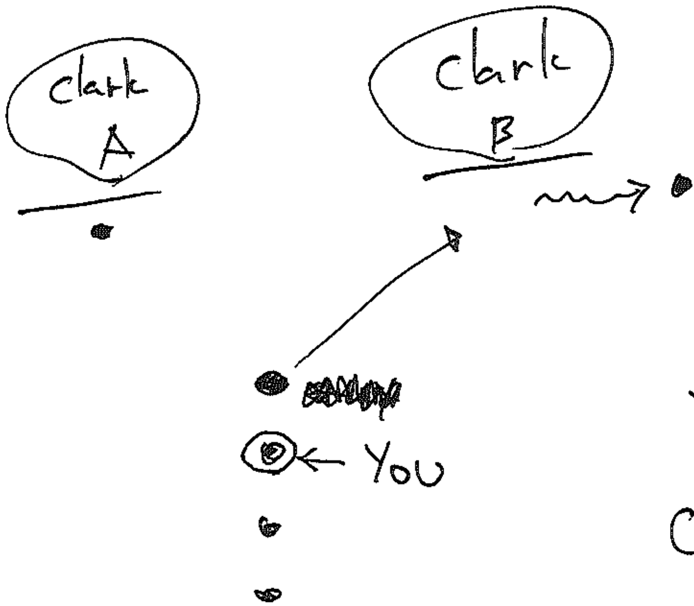
$$P(X \geq s) = e^{-\lambda s}$$

$$P(X \geq s+t \mid X \geq t) = P(X \geq s)$$

i.e. $X - t \mid X \geq t \sim Exp(\lambda)$

__Ex__. X = wait time for clark at Bank.

$X_1 \sim Exp(\lambda)$
$X_2 \sim Exp(\lambda)$.   __indep__.



You have been in line for 5 min.

Customer at Clark B just left.

What is $P(\text{clark}_A \text{ will be done sooner})$?

Memoryless Property in Expectation

$X \sim Exp(\lambda)$

$$E(X-a \mid X \geq a) = E(X)$$

Ex 5.4 p285 $\boxed{\text{Insurance with Deductible}}$

$X :=$ \$ of damage in Auto Accidents.

$X \sim E(\lambda)$ $\quad E(X) = 1000.$

$(\lambda = \frac{1}{1000})$

Deductible $= 400$.

$\left.\begin{array}{l} E(\text{pay}) = ? \\ \\ SD(\text{pay}) = ? \end{array}\right\}$ by Insurance Co.

$$\text{Pay}(x) = \begin{cases} 0 & x < 400 \\ x - 400 & x \geq 400 \end{cases}$$

damage

$$E(\text{Pay} \mid x < 400) = 0.$$

$$E(\text{Pay} \mid x \geq 400) = E(X) = 1000$$

by memoryless.

$E(\text{pay})$

$$= \underbrace{E(\text{pay} \mid X < 400)}_{0} \cdot P(X < 400) + \underbrace{E(X - 400 \mid X \geq 400)}_{\substack{\text{memory less} \\ \shortparallel \\ 1000}} \cdot P(X \geq 400)$$

$$= 0 \cdot + E(X) \cdot P(X \geq 400)$$

$$= 1000 \cdot e^{-\lambda \cdot 400}$$

//.

③

Convolution of $Exp(\lambda)$

$$\left.\begin{array}{l} X_1 \sim Exp(\lambda) \\ X_2 \sim Exp(\lambda) \\ \vdots \\ X_k \sim Exp(\lambda) \end{array}\right\} \text{ independent.}$$

$$T = X_1 + \cdots + X_k \sim GAM(k, \beta)$$

$$\beta = \frac{1}{\lambda}.$$

$$E(T) = k\beta$$

$$V(T) = k\beta^2$$

Minimum of Exponential RVs

$$\left.\begin{array}{l} X_1 \sim \mathrm{Exp}(\lambda) \\ X_2 \sim \text{"} \\ \vdots \\ X_n \sim \text{"} \end{array}\right\} \text{ independent.}$$

$$T = \min(X_1, \dots, X_n) \sim \quad ?$$

$$S = \max(X_1, \dots, X_n) \sim \quad ?$$

$$F_{min}(x) = 1 - \left(1 - F_X(x)\right)^n$$

$$= 1 - e^{-n\lambda x}$$

$$\boxed{\min(X_1, \dots, X_n) \sim Exp(n\lambda)}$$

$$F_{max}(x) = F_X(x)^n$$

$$= \left(1 - e^{-\lambda x}\right)^n$$

Min of Two Exp

$$X_1 \sim \mathrm{Exp}(\lambda_1)$$

$$X_2 \sim \mathrm{Exp}(\lambda_2)$$

$$\min(X_1, X_2) \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$$

Who finishes first ?

Server 1 $\quad X_1 \sim Exp(\lambda_1)$ $\quad$ independent.

Server 2 $\quad X_2 \sim Exp(\lambda_2)$

$$P(X_1 < X_2) = ?$$

$$= \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

Ex.

$$X_1 \sim \mathrm{Exp}(5) \qquad E(X_1) = \frac{1}{5}$$

$$X_2 \sim \mathrm{Exp}(8) \qquad E(X_2) = \frac{1}{8}$$

$$P(X_2 < X_1) = \frac{8}{5+8} = \frac{8}{13}.$$

$$P(X_1 \leq X_2) = \frac{5}{5+8} = \frac{5}{13}.$$

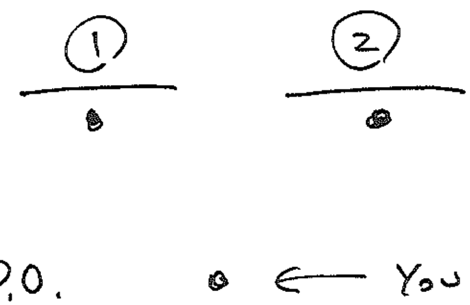
Ross 5.8

Ex. Post Office

$$\left.\begin{array}{l}\text{Clerk 1} \sim \mathrm{Exp}(\lambda_1) \\ \text{Clerk 2} \sim \mathrm{Exp}(\lambda_2)\end{array}\right\} \text{ind.}$$

You just went in.

~~Both~~ clerk are busy.



$T :=$ ~~waiting time~~ Time you spend in P.O.

$E(T) = ?$

X_1 = time until clerk 1 finish

X_2 = time until clerk 2 finish.

$$T = \min(X_1, X_2) + \underbrace{\text{time for your job}}_{S}$$

$$E(T) = E\begin{pmatrix}\text{time to} \\ \text{go up to A or B}\end{pmatrix} + E\begin{pmatrix}\text{wait time} \\ \text{for me} \\ \text{to be served}\end{pmatrix}$$

$$\uparrow$$

$$E(\min(X_1, X_2)) = \frac{1}{\lambda_1+\lambda_2}.$$

$$E\left(T \,\middle|\, \begin{matrix}\text{go to} \\ A\end{matrix}\right) = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}$$

$$E\left(T \,\middle|\, \begin{matrix}\text{go to} \\ B\end{matrix}\right) = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}$$

$$E(T) = E\left(T \,\middle|\, \text{go to } A\right) \cdot P\left(\text{go to } A\right) + E\left(T \,\middle|\, \text{go to } B\right) \cdot P\left(\text{go to } B\right)$$

$$= \left[\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}\right]\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) + \left[\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}\right]\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

$$= \boxed{\frac{3}{\lambda_1+\lambda_2}}$$