Brownian Motion

and

Itô's Lemma

Black - Scholes Assumption

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma\, dZ(t)$$

$dS(t)$ = instantaneous change in Stock Price.

α = continuously - compounded return

σ = " SD (volatility)

$dZ(t)$ = instantaneous change in $Z(t)$. (Continuous normal process.)

$S(t)$: Geometric Brownian Motion.

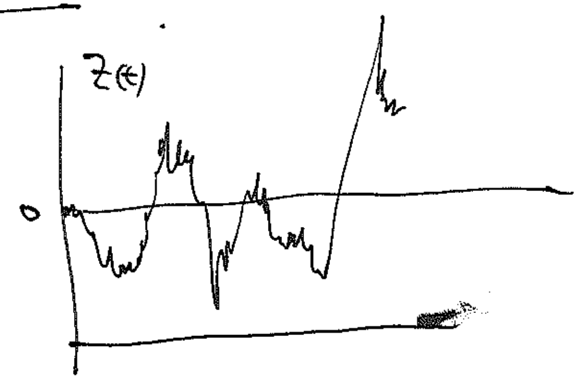
$$\frac{dS(t)}{S(t)} = \alpha \, dt + \sigma \, dz(t)$$

Stochastic Differential Eqn.

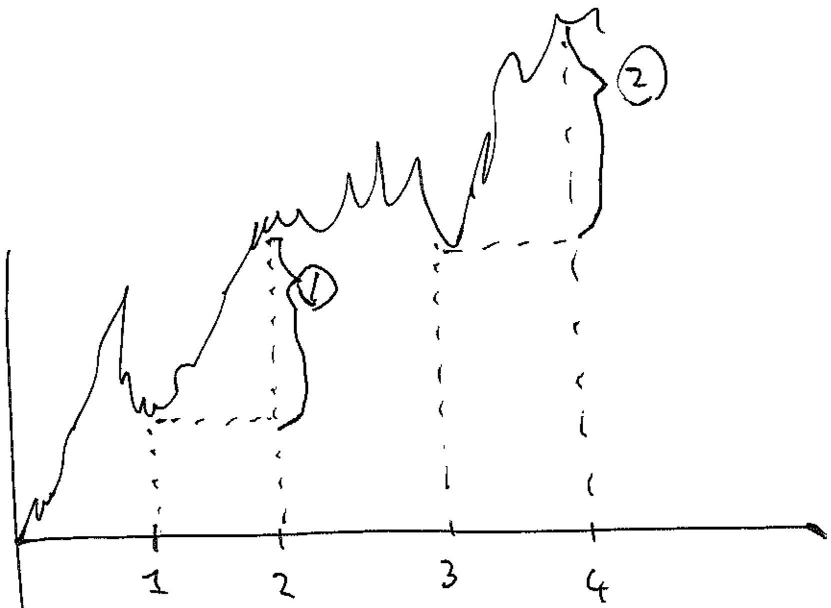
→ $S(t)$ has lognormal distribution at any fixed point in time.

→ We are interested in path of $S(t)$ rather than $S(T)$ at fixed T.

Definition of Brownian Motion



- $Z(0) = 0$
- $Z(t+s) - Z(t) \sim N(0, s)$
- Nonoverlapping increments are independent.
- $Z(t)$ is continuous.



$$1 = Z(2) - Z(1) \sim N(0,1)$$

$$2 = Z(3) - Z(2) \sim N(0,1)$$

Random Walk.

$Z_i \sim \text{iid } N(0,1)$

$$Y_n = \sum_{i=1}^{n} Z_i$$

$$E(Y_n) = \sum_{i=1}^{n} E(Z_i) = 0.$$

$$Y_1 = Z_1$$

$$Y_2 = Z_1 + Z_2$$

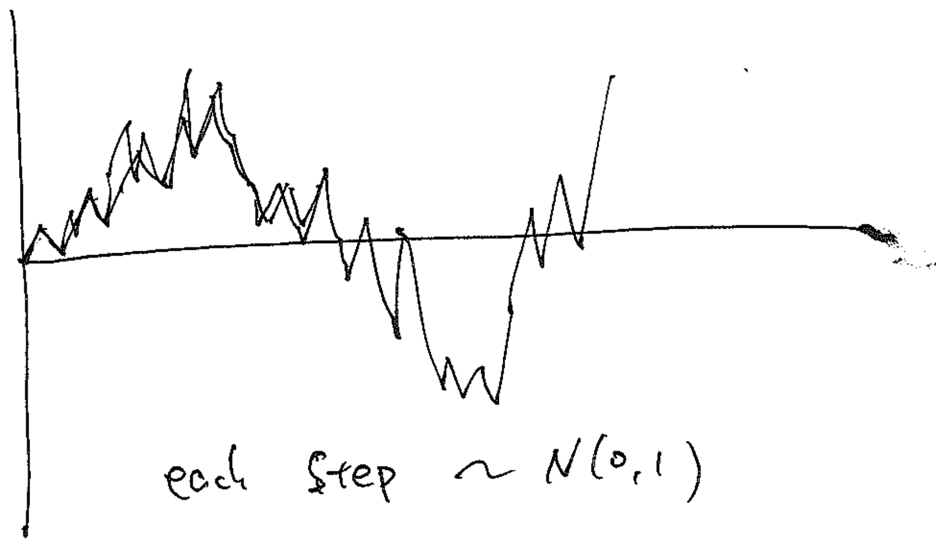
$$Y_3 = Z_1 + Z_2 + Z_3$$

$$Y_4 = Z_1 + \cdots + Z_4$$

$$\vdots$$

$$V(Y_n)$$

$$= \sum_{i=1}^{n} V(Z_i) = \sum_{i=1}^{n} 1 = n.$$



each step $\sim N(0,1)$

time is discrete, $t = 1, 2, 3, 4 \ldots$

Brownian Motion (Weiner Process) = $\lim_{n\to\infty}$ of random walk with $N\left(0, \frac{t}{n}\right)$ step with n steps between $(0, t)$.

$$Z(t) = \sum_{i=1}^{n} Z_i \qquad Z_i \sim N\left(0, \frac{1}{n}\right)$$

$$Var(Z(t)) = \sum_{i=1}^{n} Var(Z_i)$$

$$= n \cdot \frac{t}{n} = t.$$

$z_i \sim N(0, \frac{t}{n})$ is same thing as

$\sqrt{\frac{t}{n}}\, z_i \qquad z_i \sim N(0,1)$.

Let $\frac{t}{n} = h$: increments in time.

$$\boxed{Z(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{h}\, z_i} \qquad z_i \sim N(0,1).$$

Bro. Mo.

Quadratic Variation of Brownian Motion

$$\sum_{i=1}^{n} \left[\underbrace{Z(ih) - Z((i-1)h)} \right]^2$$

$h =$ time increment

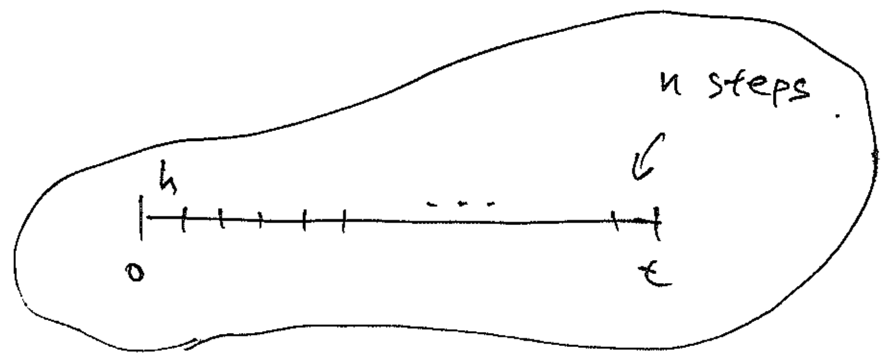
$= \dfrac{t}{n}$.

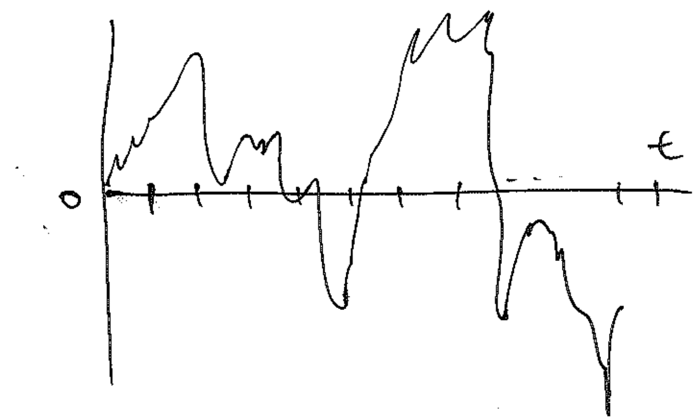
$$= \sum_{i=1}^{n} \left[\sqrt{h}\, z_i \right]^2$$

$z_i \sim N(0,1)$

$$= h \sum_{i=1}^{n} z_i^2$$

$$= t \left[\frac{1}{n} \sum_{i=1}^{n} z_i^2 \right]$$





$$\lim_{n\to\infty} QV = \lim_{n\to\infty} t \underbrace{\left[\frac{1}{n} \sum_{i=1}^{n} z_i^2 \right]}_{\text{mean of } z_i^2}$$

$$\to t \quad E[z_i^2] = t$$

Not random

Total Variation of Brownian Motion

$$\sum_{i=1}^{n} \left| Z(ih) - Z((i-1)h) \right|$$

$$= \sum_{i=1}^{n} \left| \sqrt{h}\, Z_i \right| \qquad Z_i \sim N(0,1)$$

$$= \sqrt{h} \sum_{i=1}^{n} |Z_i|$$

$$= \sqrt{t} \cdot \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |Z_i| \right]$$

$$\lim_{n\to\infty} TV = \sqrt{t} \cdot \left[\frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n} |z_i| \right]$$

$$= \underbrace{\sqrt{t}}_{\text{const}} \cdot \underbrace{\sqrt{n}}_{\to\,\infty} \underbrace{\left[\frac{1}{n} \cdot \sum_{i=1}^{n} |z_i| \right]}_{\to\, E(|z_i|)}$$

$$= \infty$$

Arithmetic Brownian Processes

$Z(t)$ - Bro. Mo.

$$E\left(Z(t+\delta) - Z(t)\right) = 0.$$

$$V\left(Z(t+\delta) - Z(t)\right) = \delta.$$

Generalize to allow different values.

$X(t)$ - Arithmetic Bro. Mo.

$$X(t+h) - X(t) = \alpha h + \sigma [Z(t+h) - Z(t)]$$

$$E(X(t+h) - X(t)) = \alpha h$$

$$V(X(t+h) - X(t)) = \sigma^2 h$$

$$X(T) - X(0) \sim N(\alpha T, \sigma^2 T)$$

$$X(T) - X(0) = \alpha T + \sigma Z(T)$$

$$dX(t) = \alpha\, dt + \sigma\, dZ(t).$$

$$X(T) = X(0) + \int_0^T \alpha\, dt + \int_0^T \sigma\, dZ(t).$$

→ X can become negative

→ mean and variance does not depend on X.

Ornstein - Uhlenbeck Process

- modification of arithmetic Bro. Mo.
- "mean - reversion"

$$dX(t) = \underbrace{\lambda[\alpha - X(t)]\,dt}_{} + \sigma\, dZ(t)$$

$X(t)$ above $\alpha \Rightarrow$ neg. mean

$X(t)$ below $\alpha \Rightarrow$ pos. mean.

Itô process

$$dX(t) = \alpha\, dt + \sigma\, dZ(t).$$

In general, $\alpha = \alpha(X(t))$ and $\sigma = \sigma(X(t))$.

mean is function of $X(t)$

volatility is function of $X(t)$.

Suppose $\alpha(X(t)) = \alpha \cdot X(t)$

$\sigma(X(t)) = \sigma \cdot X(t)$.

Geometric Brownian Motion

$$dX_{(t)} = \alpha \cdot X_{(t)}\, dt + \sigma X_{(t)} \cdot dZ_{(t)} .$$

$$\underbrace{\frac{dX_{(t)}}{X_{(t)}}}_{\substack{\text{percentage} \\ \text{change in } X_{(t)}.}} = \underbrace{\alpha}_{\text{constant}}\, dt + \underbrace{\sigma}_{\text{constant}}\, dZ_{(t)}$$

$$X(T) - X(0) = \int_0^T \alpha X_{(t)}\, dt + \int_0^T \sigma X_{(t)}\, dt$$

$X(t) \sim$ Geometric Bro. Mo.

$Z(t)$ = Bro. Mo.

$$\frac{X(t+h) - X(t)}{X(t)} = \alpha h + \sigma [\underbrace{Z(t+h) - Z(t)}_{\sqrt{h}\, Z_i}]$$

finite step Approximation

$Z_i \sim N(0,1)$

$$X(t) \sim LN\left(\ln(X(0)) + (\alpha - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t \right)$$

Drift vs Noise

$$X(t+h) - X(t) = \underbrace{\alpha X(t)\, h}_{\substack{\text{mean} \\ \text{'Drift'} \\ \text{Deterministic}}} + \underbrace{\sigma X(t) \sqrt{h}\, Z_i}_{\substack{\cancel{\text{variance}}\ \text{volatility.} \\ \text{random.}}}$$

For small h, volatility dominates over drift, b/c

$$\frac{\sigma X(t) \sqrt{h}}{\alpha X(t)\, h} = \frac{\sigma}{\alpha \sqrt{h}} \longrightarrow \infty, \quad \text{as } h \to 0.$$

For large h,

$$\frac{\sigma}{\alpha \sqrt{h}} \longrightarrow 0 \quad \text{as } h \to \infty.$$

Multiplication Rules

$$[X_{(t+h)} - X_{(t)}]^2 = [\alpha X_{(t)} h + \underbrace{\sigma X_{(t)} \sqrt{h}\, z_i}_{\text{noise term.}}]^2$$

$$= \alpha^2 X_{(t)}^2 h^2 + 2\alpha\sigma X_{(t)}^2 h^{1.5} z_i$$

$$+ \underbrace{\sigma^2 X_{(t)}^2 h z_i^2}_{\text{Dominant term.}}$$

$$[X_{(t+h)} - X_{(t)}]^2 \approx \sigma^2 X_{(t)}^2 h \qquad E(z_i^2) = 1.$$

$$(dX_{(t)})^2 \approx \sigma^2 X_{(t)}^2 dt$$

$$dt \propto h$$

$$dZ(t) \propto \sqrt{h} \quad \Rightarrow \quad (dZ(t))^2 \propto h.$$

"proportional to"

"of order of"

$$(dX(t))^2 \propto dt \propto h$$

$$dX(t) \propto \sqrt{h}$$

Back to Itô process

$$dX(t) = \left(\alpha(X(t), t) - \delta(X(t), t)\right) dt$$

$$+ \sigma(X(t), t) \, dZ(t) .$$

α, δ, σ depends on $X(t)$ and t,

Consider functions of $X(t)$, and t

$V(X(t), t)$ = e.g. option price.

By taylor expantion

$$V(X+dX,\ t+dt) = V(X,t) + V_X'\,dX + V_t'\,dt$$

$$+ \frac{1}{2}V_X''(dX)^2 + \frac{1}{2}V_t''\underbrace{(dt)^2}_{h^2} + V_{st}''\underbrace{(dX)(dt)}_{h^{1.5}}$$

$$+ \text{higher order terms}$$

$$dt \approx h$$

$$(dX)^2 \approx h$$

$$(dt)^2 \approx h^2$$

$$(dX)(dt) \approx h^{1.5}$$

$$V_X' = \frac{d}{dX}V$$

$$V_t' = \frac{d}{dt}V$$

Itô's Lemma

$$\underbrace{V(x+dx,\ t+dt) - V(x,t)}$$

$$\boxed{dV(x,t) = V_x'\,dx + \frac{1}{2}V_x''(dx)^2 + V_t'\,dt}$$

if $V(\cdot)$ is twice differentiable function.

Delta - Gamma approximation

$$C_t - C_0 = \Delta\,(\text{change in } S(t)) + \frac{1}{2}\,\Gamma\cdot(\text{change in } S(t))^2$$

$$+\ \theta\,(\text{change in } T)$$

$$dV(X(t), t) = \underbrace{V'_x}_{\Delta}\, dX + \frac{1}{2}\,\underbrace{V''_x}_{\Gamma}\,(dX)^2 + \underbrace{V'_t}_{\theta}\, dt$$

$$X(t) = S(t)$$

Using

$$dX = [\alpha_{(\cdot)} - \delta_{(\cdot)}]\, dt + \sigma_{(\cdot)}\, dZ ,$$

$$(dX)^2 = \sigma^2_{(\cdot)}\, dt ,$$

Itô's Lemma can be written as

$$dV = V_x'\, dx + \frac{1}{2} V_x''\,(dx^2) + V_t'\, dt$$

$$= V_x' \{(\alpha_{(\cdot)} - \delta_{(\cdot)})\, dt + \sigma_{(\cdot)}\, dZ\}$$

$$+ \frac{1}{2} V_x'' \{\sigma^2_{(\cdot)}\, dt\}$$

$$+ V_t'\, dt$$

$$\boxed{dV = \left\{\left[\alpha_{(\cdot)} - \delta_{(\cdot)}\right] V'_x + \frac{1}{2} V''_x \sigma^2_{(\cdot)} + V'_t\right\} dt + \sigma_{(\cdot)} V'_x \, dZ}$$

Itô's Lemma

$$\left.\begin{aligned} \alpha_{(\cdot)} &= \alpha X_{(t)} \\ \delta_{(\cdot)} &= \delta X_{(t)} \\ \sigma_{(\cdot)} &= \sigma X_{(t)} \end{aligned}\right\} \longrightarrow X_{(t)} = \text{Geometric Brownian Motion.}$$