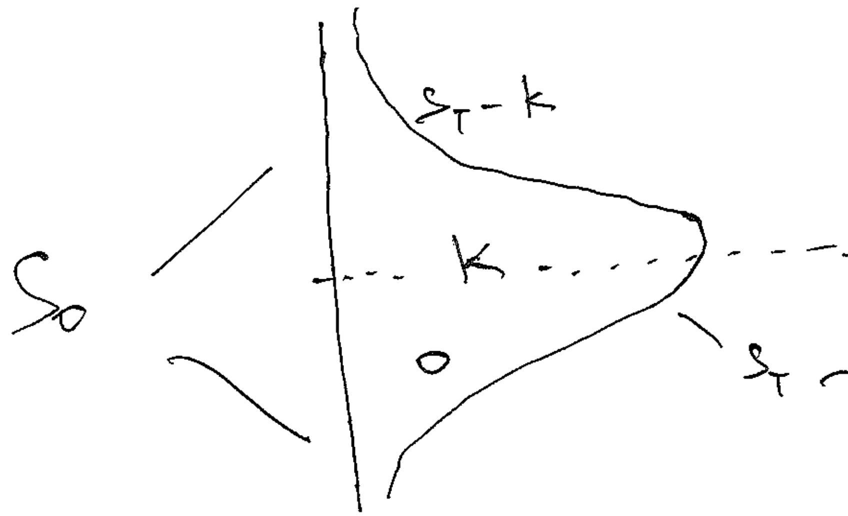
Monte Carlo Valuation

Black - Scholes formula



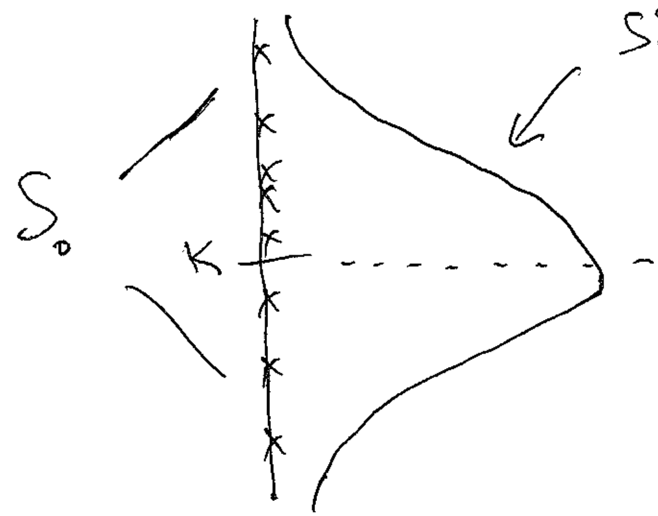
$S_T \sim LN\left(\left(r - \delta - \frac{1}{2}\sigma^2\right)T, \ \sigma^2 T\right)$

$C_T = \max(S_T - K, 0)$

$C_0 = e^{-rT} E(C_T)$

Monte Carlo valuation = do the E( ) by simulation.

Monte Carlo Valuation



Simulate $S_T \sim LN$

for each realization of S_T,
Calculate C_T.

$\downarrow$

Average C_T after many iteration.

$\downarrow$

$$\overline{C_T} \approx E(C_T)$$

$$C_0 = e^{-rT}(\overline{C_T})$$

Risk - neutral view

$$S_T \sim LN\left((r-\delta-\tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right) \quad \star$$

$$\Rightarrow E(S_T) = e^{(r-\delta)T} \qquad \text{risk - neutral.}$$

- Monte Carlo valutation using $\star$ holds risk-neutral view of the market.

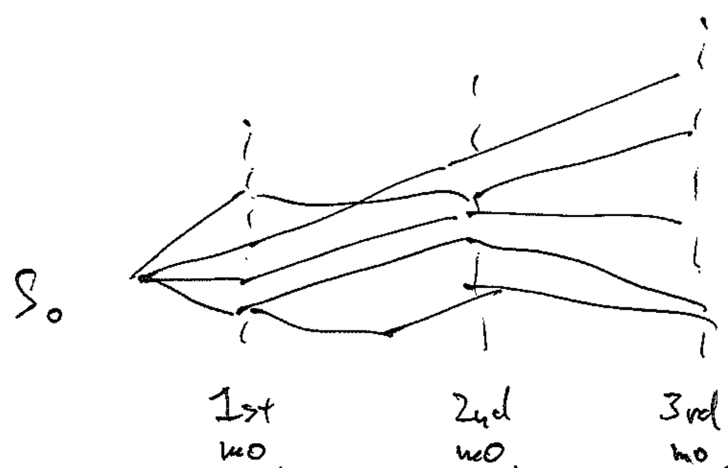
- It is possible to complete MC with true probability, but much more demanding computationary.

in P.

in r.

Pros of MC valuation

- B-S only computes C_T at time T.
- Some options are <u>path</u> dependent. (Asian option)
- Use MC to simulate each <u>path</u>.



Simulating Lognormal Random Variable

$$S_T \sim LN(\mu, \sigma_0^2)$$

$$S_T = e^X \qquad X \sim N(\mu, \sigma_0^2)$$

$$X = \mu + \sigma_0 Z \qquad Z \sim N(0,1)$$

$$S_T = e^{\ln S_0 + (r - \delta - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\cdot Z}$$

simulate

$$= S_0 \cdot e^{(r-\delta-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\cdot Z}$$

Monte Carlo Valuation

$$C_0 = e^{-rT}\left[\frac{1}{n}\sum_{i=1}^{n} C_T^i\right]$$

i = index for iteration

$$C_T^i = \max(0, S_T^i - K)$$

$$S_T^i = S_0\, e^{(r-\delta-\sigma^2/2)T + \sigma\sqrt{T}\cdot z^i}$$

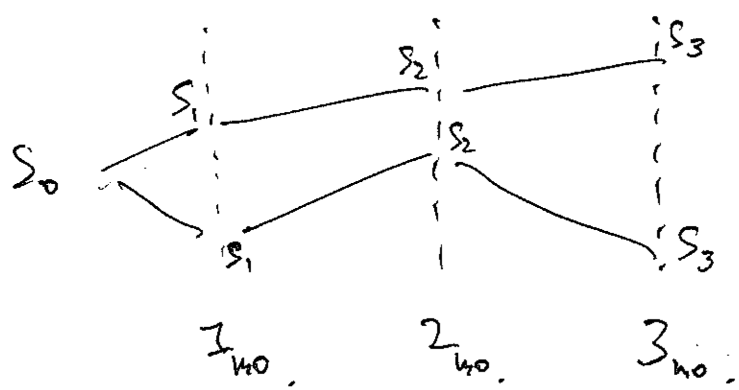
ith simulated $N(0,1)$

Arithmetic Asian Option with MC

$T = 3$ mo.

Average value of end of 1st, 2nd and 3rd mo.

$$C_{Asian} = e^{-rT} E\left[\max\left(\frac{S_1 + S_2 + S_3}{3} - K, 0\right)\right]$$



Accuracy of Monte Carlo.

How close $\frac{1}{n}\sum_{i=1}^{n} C_T^i$ is to $E(C_T)$?

By CLT

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n} C_T^i}_{\overline{C_T}} \sim N\left(E(C_T),\ \frac{V(C_T)}{n} \right)$$

use sample SD of C_T^i.

$\overline{C_T}$ is within $E(C_T) \pm 1.96\sqrt{\frac{V(C_T)}{n}}$ 95% of times.

If $\widehat{SD}(C_T) = 4.05$ and $C_0 \approx \$3$. and you want 1% accuracy $= (.03)$. then you need

$$1.96 \frac{4.03}{\sqrt{n}} = .03 \quad \Rightarrow \quad n \doteq 70{,}000 \text{ iterations}$$

Efficient Monte Carlo Valuation

$$\bar{C}_T \to E(C_T) \quad \text{as}$$

$$\bar{C}_T \sim N\left(E(C_T), \frac{V(C_T)}{n} \right)$$

Can I do better?

(faster)

Antithetic variates

Calculate $\dot{C}_T^i$, $\tilde{C}_T^i$ where

$\dot{C}_T^i$ and $\tilde{C}_T^i$ are negatively correlated,

Then let

$$C_T^i = \frac{\dot{C}_T^i + \tilde{C}_T^i}{2} \quad \text{and}$$

$$E(C_T^i) \approx \overline{C_T^i}.$$

$$\mathrm{Var}\left(\frac{\dot{C}_T^i + \tilde{C}_T^i}{2}\right)$$

$$= \frac{1}{4}\Big[\underbrace{\mathrm{Var}(\dot{C}_T^i)}_{V(C_T^i)} + \underbrace{\mathrm{Var}(\tilde{C}_T^i)}_{V(C_T^i)} + 2\underbrace{\mathrm{Cov}(\dot{C}_T^i, \tilde{C}_T^i)}_{\text{neg.}}\Big]$$

$$< \frac{V(C_T^i)}{2}$$

Antithetic with $n/2$ each

$$< \frac{V(C_T^i)}{2 \; n/2}$$

regular MC with n.

$$\frac{V(C_T^i)}{n}$$

Control Variates

C_T,

$E(T) = \mu_T$.

const

$$C_T^* = C_T + c\,(T - \mu_T)$$

random

$$\mathrm{Var}(C_T^*) = \mathrm{Var}(C_T) + c^2\,\mathrm{Var}(T - \mu_T) + 2c\,\mathrm{Cov}(C_T, T - \mu_T)$$

$$= \mathrm{Var}(C_T) + c^2\,\mathrm{Var}(T) + 2c\,\mathrm{Cov}(C_T, T)$$

what c minimize $\mathrm{Var}(C_T^*)$?

Take $\frac{d}{dc}$ and set to 0.

$$2C\,\mathrm{Var}(T) + 2\,\mathrm{Cov}(C_T, T) \equiv 0.$$

$$\boxed{C = -\frac{\mathrm{Cov}(C_T, T)}{\mathrm{Var}(T)}}$$

Put it back in, we get,

$$\mathrm{Var}(C_T^*) = \mathrm{Var}(C_T) + \left(\frac{\mathrm{Cov}(C_T, T)}{\mathrm{Var}(T)}\right)^2 \cdot \mathrm{Var}(T) + 2\,\frac{-\mathrm{Cov}(C_T, T)}{\mathrm{Var}(T)} \cdot \mathrm{Cov}(C_T, T)$$

$$\mathrm{Var}(C_T^*) = \mathrm{Var}(C_T) - \frac{\mathrm{Cov}^2(C_T, T)}{\mathrm{Var}(T)}$$

$$\frac{\mathrm{Cov}^{\cancel{2}}(C_T, T)}{\sqrt{\mathrm{Var}(C_T)\,\mathrm{Var}(T)}} = \mathrm{Corr}(C_T, T) = \rho$$

$$\mathrm{Var}(C_T^*) = \mathrm{Var}(C_T)\left[1 - \rho^2\right]$$

Pick T so that ρ^2 is close to 1

$$C_T^{*i} = C_T^i + -\left(\frac{\mathrm{Cov}(C_T, T)}{\mathrm{Var}(T)}\right)\left[T^i - \mu_T\right]$$

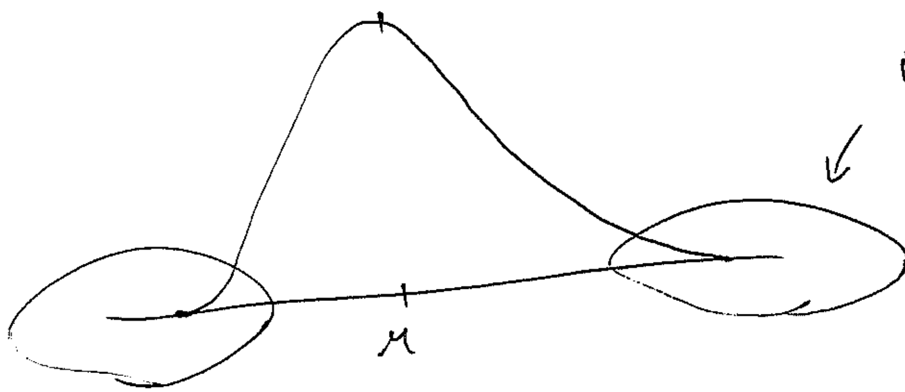
Compute or estimate.

$$E(C_T) \approx \overline{C_T^*}$$

Lognormal Stock Model with Poisson Jumps

$$S_T \sim LN\left(\ln(S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)T \;,\; \sigma^2 T \right)$$

Tail distribution is too little



Often you see some large values here.

Modify Lognormal Model as ...

original model:

$N(0,1)$ $\downarrow$

$$S_T = S_0 e^{(r-\delta-\frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\cdot Z}$$

$m \sim Poi(\lambda k)$

$W_{(i)} \sim N(0,1)$

modified model

$$S_T = \left[S_0 e^{(r-\delta-\frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\, Z}\right]\underbrace{\left[e^{m(a_j - \frac{1}{2}\sigma_j^2) + \sigma_j \sum_{i=1}^{m} W_{(i)}}\right]}_{\text{Jump component.}}$$

S_T is still lognormally distributed.

→ Since the jump component is Lognormally distributed, new S_T is still Lognormal, (prod of LN)

→ m = # of jumps,

$m \sim \text{Poi}(\lambda t)$, where λ = av. # of jumps per year.

→ Because of Poisson, m is often 0. (no jumps).

FIGURE 19.7

Simulated stock price paths over 10 years (3650 days). One stock cannot jump; the other is the same except that jumps can occur. The simulation assumes that $\alpha = 8\%$, $\delta = 0$, $\sigma = 30\%$, $\lambda = 3$, $\alpha_J = -2\%$, and $\sigma_J = 5\%$.

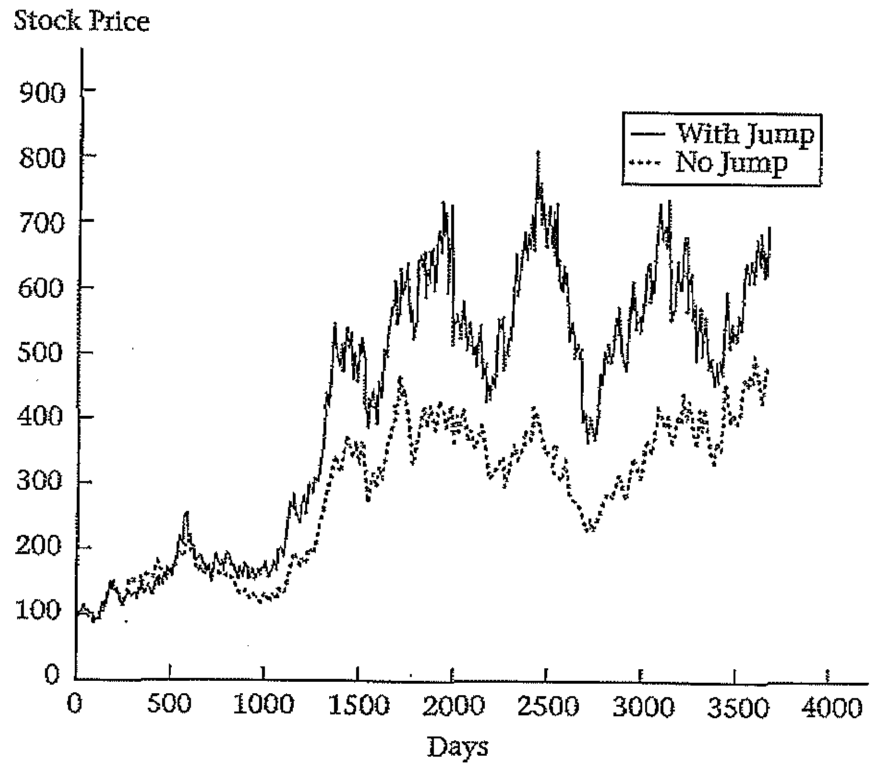
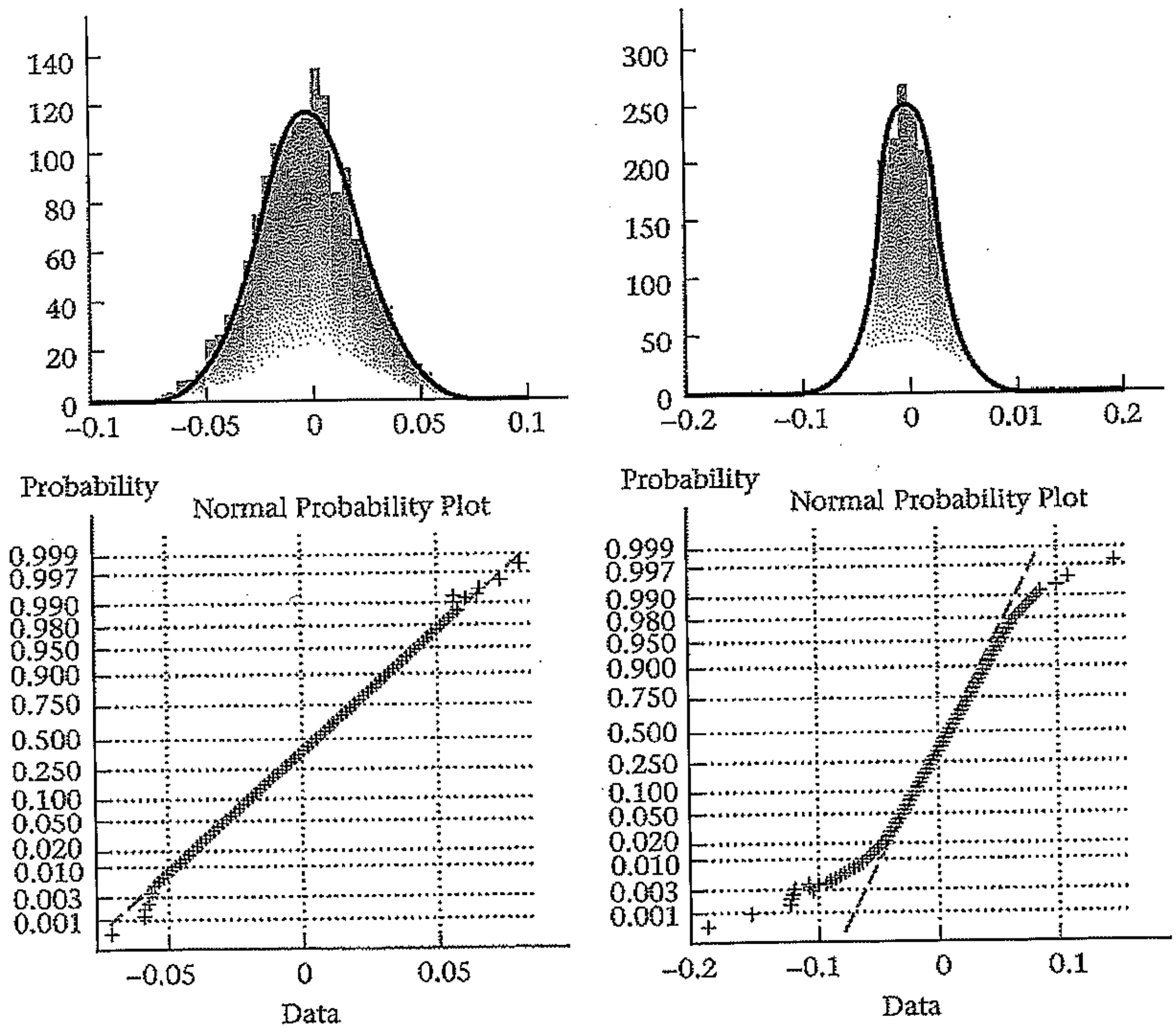


FIGURE 19.8



Histograms and normal probability plots for the daily returns generated from the two series in Figure 19.7. Graphs on the left are for the no-jump series.