Fall 2013 [3450: 489/689] Math Topics: Time Series Analysis and Forecasting

Midterm 75min

Name: Solution

This exam is closed notes, closed book. You are allowed to use handheld calculator only.

1. Suppose process Y_t is a random walk,

$$Y_t = \sum_{i=1}^{t} e_i$$

where $e_t \sim IID(0, \sigma^2)$. Is Y_t stationary? Show your work.

$$V(Y_t) = \sum_{i=1}^{t} V(e_i) = t\sigma^2$$

Y_t is not stationary b/c $V(Y_t)$ depends on t.

2. Given MA(2) model

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

where $e_t \sim IID(0, \sigma^2)$, Derive formula for ACVF functions for lag up to 5.

$$\gamma(0) = Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})$$

$$= \boxed{\sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2}$$

$$\gamma(1) = Cov(e_{t+1} - \theta_1 e_t - \theta_2 e_{t-2},\ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})$$

$$= \boxed{-\theta_1 \sigma^2 + \theta_2^2 \sigma^2}$$

$$\gamma(2) = Cov(e_{t+2} - \theta_1 e_{t+1} - \theta_2 e_t,\ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})$$

$$= \boxed{-\theta_2 e_t}$$

$$\gamma(3) = \boxed{0}$$

$$\gamma(4) = \boxed{0}$$

$$\gamma(5) = \boxed{0}$$

3. Let Y_t be ARMA(1,1) model,

$$Y_t - .6Y_{t-1} = e_t - .3e_{t-1},$$

where $e_t \sim IID(0, \sigma^2 = 2)$. Calculate numerical value of $Cov(Y_t, e_{t-1})$.

$$Cov(Y_t, e_{t-1})$$

$$= Cov(.6Y_{t-1} + e_t - .3e_{t-1},\ e_{t-1})$$

$$= .6\,Cov(Y_{t-1}, e_{t-1}) - .3\sigma^2$$

$$Cov(Y_{t-1}, e_{t-1}) = Cov(.6Y_{t-2} + e_{t-1} - .3e_{t-2},\ e_{t-1})$$

$$= \sigma^2$$

$$Cov(Y_{t-1}, e_{t-1}) = .6\sigma^2 - .3\sigma^2 = \boxed{.3\sigma^2}$$

$$= \boxed{.6}$$

4. Suppose ARMA(1,1) model

$$Y_t - .6Y_{t-1} = e_t - .3e_{t-1}$$

is causal. That means this process can be expressed as

$$Y_t = \sum_{i=0}^{\infty} \psi_i e_{t-i}.$$

Obtain numerical value of ψ_2.

$$(1-.6x)(\psi_0 + \psi_1 x + \psi_2 x + \cdots) = (1-.3x)$$

$$\psi_0 = 1$$

$$\psi_1 - .6\psi_0 = -.3 \Rightarrow \psi_1 = -.3 + .6\psi_0 = .3$$

$$\psi_2 - .6\psi_1 = 0 \Rightarrow \psi_2 = 0 + .6\psi_1 = \boxed{.18}$$

5. For time series data $Y_1, \ldots, Y_{10}$, you are considering MA(1) model with mean,

$$Y_t = \mu + e_t + .7e_{t-1},$$

where $e_t \sim IID(0, \sigma^2 = 2)$. Your sample mean was 1.153. Should we use model with μ or can we set $\mu = 0$? (i.e. Determine if $\bar{Y}$ is significantly different from 0.)

$$\bar{Y} \sim N\Big(\mu,\ \frac{1}{n} \underbrace{\sum_{h=-n}^{n} \Big(1 - \frac{|h|}{n}\Big) \gamma(h)}_{\nu}\Big)$$

For MA(1)

$$\begin{cases} \gamma(0) = (1+\theta^2)\sigma^2 = 2.98 \\ \gamma(1) = -\theta\sigma^2 = 1.4 \\ \gamma(h) = 0 \quad h > 1 \end{cases}$$

$$\nu = \gamma(0) + 2 \cdot \Big(1 - \frac{1}{10}\Big)\gamma(1) = 5.5$$

$$\bar{Y} \sim N\Big(\mu,\ \frac{5.5}{10}\Big)$$

95% CI for μ

$$\bar{Y} \pm 1.96\sqrt{\frac{5.5}{10}} = \underbrace{1.153 \pm 1.453}_{\text{Includes } 0.}$$

$\Rightarrow \bar{Y}$ not significantly different from 0.

6. Suppose you are given time series data $X_1, \ldots, X_{200}$. In software R, function `acf( )` gives you plot of sample acf with blue dotted line drawn horizontally, so that you can determine if the sample ACF looks like that of IID sequence or not. At what value would this blue line be?

If $X_1, \ldots, X_n$ are iid,

$$\underset{\text{sample ACF}}{\hat{\rho}} \sim N\left(0, \frac{1}{n}\right).$$

Therefore, the blue dotted lines are drawn at

$$\pm 1.96 \frac{1}{\sqrt{n}}.$$

So if $n = 200$ they are at

$$\boxed{\pm .139}.$$

7. Consider AR(3) model

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \phi_3 Y_{t-3} = e_t$$

where $e_t \sim IID(0, \sigma^2)$. Yule-Walker equation is written as

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(2) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{bmatrix}$$

Derive the second line of the equation

$$\gamma(1)\phi_1 + \gamma(0)\phi_2 + \gamma(1)\phi_3 = \gamma(2).$$

Multiply both side of 1st eqn by Y_{t-2}, and take expectation,

$$E\left[Y_{t-2}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \phi_3 Y_{t-3})\right] = E\left[Y_{t-2} \cdot e_t\right]$$

$$\gamma(2) - \phi_1 \gamma(1) - \phi_2 \gamma(0) - \phi_3 \gamma(1) = 0$$

We obtain

$$\gamma(1)\phi_1 + \gamma(0)\phi_2 + \gamma(1)\phi_3 = \gamma(2).$$

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8. Consider the AR(p) model,

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = e_t$$

where $e_t \sim IID(0, \sigma^2)$. Given data $Y_1, \ldots, Y_n$, what is the formula for one-day-ahead linear predictor, $\hat{Y}_{n+1}$, that has minimum MSE, $E(Y_{n+1} - \hat{Y}_{n+1})^2$? BRIEFLY explain how the formula is derived.

$$\hat{Y}_{n+1} = \phi_1 Y_n + \phi_2 Y_{n-1} + \cdots + \phi_p Y_{n-p-1}$$

This is because if we try to find linear predictor $\hat{Y}_{n+1} = a_0 + a_1 Y_n + a_2 Y_{n-1} + \cdots + a_n Y_1$, such that minimize MSE, we end up with Yule-walker equ,

$$\Gamma_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{bmatrix}.$$

So

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

9. Consider a model with linear deterministic model

$$Y_t = a + bt + e_t$$

where $e_t \sim IID(0, \sigma^2)$ and a, b are constant. Show that ∇Y_t does not have the linear trend.

$$\nabla Y_t = Y_t - Y_{t-1}$$

$$= a + bt + e_t - (a + b(t-1) + e_{t-1})$$

$$= b + e_t - e_{t-1}$$

$\uparrow$

trend is constant now.