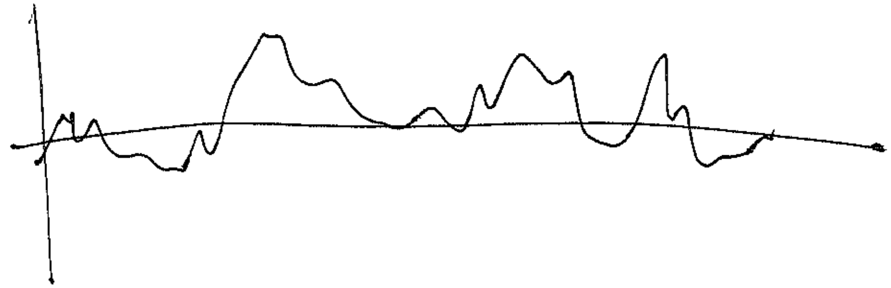
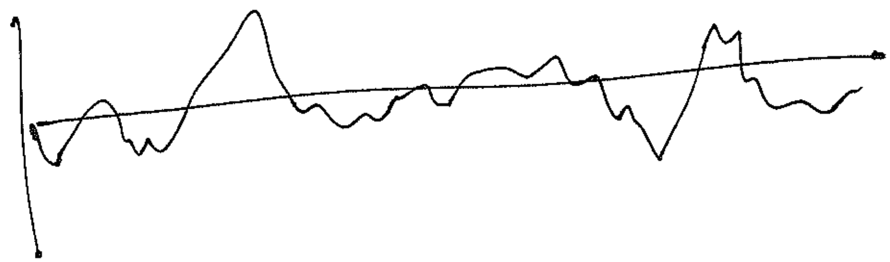
Multivariate Time Series

X_{t1}



X_{t2}



are they independent?,

correlated?, If so, How?

Mean Vector

$$\underline{\mu}_t = \begin{bmatrix} E(X_{t1}) \\ E(X_{t2}) \end{bmatrix}$$

Covariance Matrix

$$\underline{\Gamma}(h) = \begin{bmatrix} Cov(X_{t+h,1}, X_{t1}) & Cov(X_{t+h,1}, X_{t2}) \\ Cov(X_{t+h,2}, X_{t1}) & Cov(X_{t+h,2}, X_{t2}) \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{bmatrix}$$

$$\gamma_{11}(h) = \gamma_{11}(-h) = \mathrm{Cov}(X_{t+h,1}, X_{t1})$$

$$\gamma_{12}(-h) = \mathrm{Cov}(X_{t-h,1}, X_{t2})$$

$$= \mathrm{Cov}(X_{t1}, X_{t+h\,2})$$

$$= \mathrm{Cov}(X_{t+h\,2}, X_{t1})$$

$$= \gamma_{21}(h).$$

not equal to

$$\gamma_{12}(h) = \mathrm{Cov}(X_{t+h\,1}, X_{t})$$

$$\Gamma(h) = \begin{bmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{bmatrix}$$

$$\Gamma(-h) = \begin{bmatrix} \gamma_{11}(-h) & \gamma_{12}(-h) \\ \gamma_{21}(-h) & \gamma_{22}(-h) \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_{11}(h) & \gamma_{21}(h) \\ \gamma_{12}(h & \gamma_{22}(h) \end{bmatrix}$$

$$= \Gamma^{T}(-1)$$

Example

$$X_{t1} = e_t \qquad e_t \sim \mathrm{IID}(0,1)$$

$$X_{t2} = e_t + .75\, e_{t-2}$$

$\mathrm{Cov}(X_{t1+h}, X_{t2})$

$\gamma_{11}(0) = 1$

$\gamma_{11}(1) = 0$

$\vdots$

$\gamma_{22}(0) = 1 + .75^2$

$\gamma_{22}(1) = 0$

$\gamma_{22}(2) = .75$

$\gamma_{22}(3) = 0$

$\vdots$

$\gamma_{12}(0) = 1$

$\gamma_{12}(1) = 0$

$\vdots$

Ex

$$X_{t1} = e_t$$

$$X_{t2} = e_t + .75\, e_{t-2}$$

		$Cov(X_{t+h,1}, X_{t2})$	$Cov(X_{t+h,2}, X_{t1})$
$\gamma_{11}(0) = 1$	$\gamma_{22}(0) = 1 + .75^2$	$\gamma_{12}(0) = 1$	$\gamma_{21}(0) = 1$
$(1) = 0$	$(1) = 0$	$(1) = 0$	$(1) = 0$
$\vdots$	$(2) = .75$	$\vdots$	$(2) = .75$
	$(3) = 0$		$(3) = 0$
	$\vdots$		$\vdots$

Testing if two Stationary series are independent

B-D p.236

$\boxed{\text{Thm } 1}$ Let

X_{t1} = Linear Process with IID $(0, \sigma_1^2)$ error Z_{t1}

X_{t2} = " IID$(0, \sigma_2^2)$ error Z_{t2}

then,

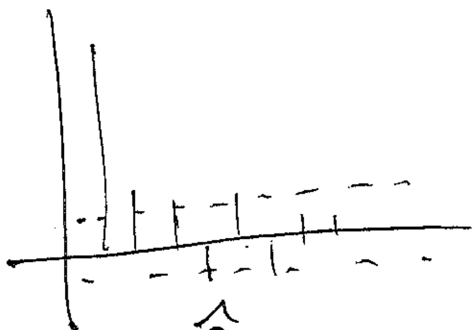
$$\sqrt{n}\,\hat{\rho}_{12}(h) \sim N\Big(0,\ \sum_{j=-\infty}^{\infty} \rho_{11}(j)\,\rho_{22}(j)\Big)$$

cov $\displaystyle\sum_{j=-\infty}^{\infty} \rho_{11}(j)\,\rho_{22}(j+k-h)$.

$$\sqrt{n}\,\hat{\rho}_{21}(h) \sim N\big(0,\ \text{"}\ \big)$$

X_{t1}

X_{t2}

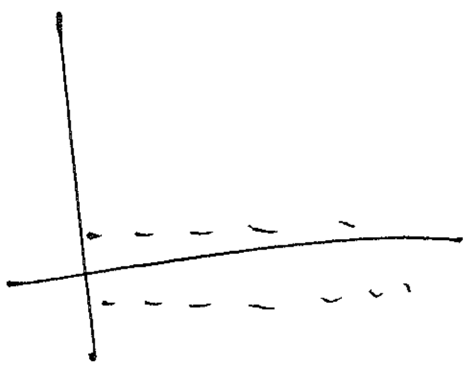


$\hat{\rho}_{11}$

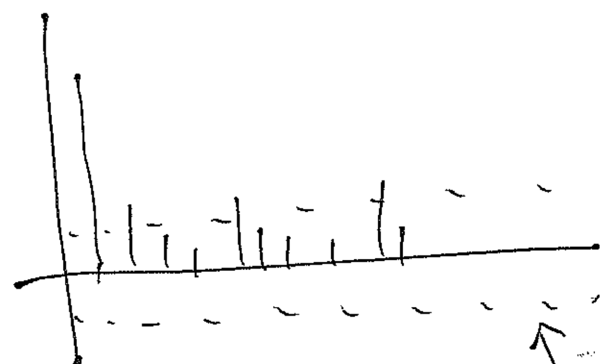
$\hat{\rho}_{12}$

dotted line cannot be used.

unless



$\hat{\rho}_{21}$



$\hat{\rho}_{22}$

dotted line can be used.

Pre-whitening

$\hat{\rho}_{12}(h)$ can't be used to see if X_{t1} and X_{t2} are independent, because it's variance depends on ρ_{11} and ρ_{22}.

$\longrightarrow$ Instead of checking X_{t1}, X_{t2}.

See if $\hat{e}_{t1}$, $\hat{e}_{t2}$ are independent.

Pre-whitening 2

$$X_{t1} = e_t + .5\, e_{t-1} \qquad e_t \sim IID(0, \sigma_1^2)$$

$$X_{t2} = a_t + .3\, a_{t-1} + .4\, a_{t-2} \qquad a_t \sim IID(0, \sigma_2^2)$$

$X_{t1} \sim MA(1)$

$X_{t2} \sim MA(2)$

} independent.

Fit X_{t1} with $MA(1)$, get residuals $\hat{e}_t$

Fit X_{t2} with $MA(2)$, get $\hat{a}_t$

$\rightarrow$ See if $\hat{e}_t$ and $\hat{a}_t$ look independent..

Thm ① when $X_{t1} = e_t$, $e_t \sim IID(0, \sigma_1^2)$.

$$\begin{cases} \rho_{11}(0) = 1 \\ \rho_{11}(h) = 0 \quad h > 0 \end{cases}$$

$$\sqrt{n}\ \hat{\rho}_{12}(h) \sim N\left(0, \sum_{j=-\infty}^{\infty} \rho_{11}(j)\, \rho_{22}(j)\right) \quad h > 0$$

$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\, \rho_{22}(j) = \rho_{11}(0) \cdot \rho_{22}(0)$$

$\hat{\rho}_{12}(h) \sim N\left(0, \frac{1}{n}\right)$ just like $\hat{\rho}_{11}(h)$ and $\hat{\rho}_{22}(h)$.

$h > 0$.

$\rightarrow$ use dotted line.

Multivariate ARMA

VMA

Example

$$X_{t1} = e_t + .5a_{t-1} \qquad e_t \sim IID(0, \sigma_1^2)$$

$$X_{t2} = a_t + .3e_{t-1} + .4a_{t-2} \qquad a_t \sim IID(0, \sigma_2^2)$$

$$\underline{X} = \begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_t \\ a_t \end{bmatrix} + \begin{bmatrix} 0 & .5 \\ .3 & 0 \end{bmatrix}\begin{bmatrix} e_{t-1} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & .4 \end{bmatrix}\begin{bmatrix} e_{t-2} \\ a_{t-2} \end{bmatrix}$$

$$\begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} = \left(\mathbb{I} + \Theta_1 B + \Theta_2 B^2\right)\begin{bmatrix} e_t \\ a_t \end{bmatrix}$$

VAR

$$X_{t1} - .3\, X_{t-1,1} = e_t \qquad e_t \sim IID(0, \sigma_1^2)$$

$$X_{t2} - .5\, X_{t-1,1} + .3\, X_{t-2,2} = a_t \qquad a_t \sim IID(0, \sigma_2^2)$$

$$\begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} - \begin{bmatrix} .3 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} X_{t-1,1} \\ X_{t-1,2} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -.3 \end{bmatrix} \begin{bmatrix} X_{t-2,1} \\ X_{t-2,2} \end{bmatrix} = \begin{bmatrix} e_t \\ a_t \end{bmatrix}$$

$$\left(\mathbb{I} - \Phi_1 B - \Phi_2 B^2 \right) \begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} = \begin{bmatrix} e_t \\ a_t \end{bmatrix}$$

VARMA(p,q)

$$\underline{X}_t = \begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} \qquad \underline{e}_t = \begin{bmatrix} e_{t1} \\ e_{t2} \end{bmatrix}$$

$$\underline{X}_t - \Phi_1 \underline{X}_{t-1} - \Phi_2 \underline{X}_{t-2} - \cdots - \Phi_p \underline{X}_{t-p}$$

$$= \underline{e}_t + \Theta_1 \underline{e}_{t-1} + \Theta_2 \underline{e}_{t-2} + \cdots + \Theta_q \underline{e}_{t-q}$$

$$\underline{e}_t \sim IID(0, \Sigma)$$

Causality

If $\det\left(\mathbb{I} - \Phi_1 z - \cdots - \Phi_p z^p\right)$ has all roots outside of unit circle, then VARMA(p,q) is causal, i.e.

matrix — scalor in complex plane

$$\underline{X}_t = \sum_{j=0}^{\infty} \Psi_j \, \underline{e}_{t-j}$$

matrix

Matrix Ψ_j can be found recursively by eqn.

$$\Psi_j = \Theta_j + \sum_{k=1}^{\infty} \Phi_k \Psi_{j-k} \qquad j = 0, 1, \ldots$$

$$(\Theta_0 = \mathbb{I}).$$