451/551 - HW on Ch. 4 Continuous RV

Due Wed Oct 12th

The final answer must be clealy indicated.

Name: Solution.

1. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \frac{2}{y^3} & \text{for } y > 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Obtain the CDF of the loss.

$$F(x) = \int_1^x \frac{2}{y^3}\,dy = \frac{2}{-2} y^{-2} \Big|_1^x = \boxed{\begin{cases} 0 & x \le 1 \\ 1 - \frac{1}{x^2} & x > 1 \end{cases}}$$

   (b) What is the expected value of the benefit paid under the insurance policy?

$$\text{Benefit} = \begin{cases} Y & 1 < y \le 10 \\ 10 & 10 < y \end{cases}$$

$$E[\text{Benefit}]$$

$$= \int_{\text{all}} (\text{Benefit}) \cdot f(y)\,dy = \int_1^{10} y \cdot f(y)\,dy + \int_{10}^{\infty} 10 \cdot \frac{2}{y^3}\,dy$$

$$= -2y^{-1} \Big|_1^{10} + 10\left[1 - F(10)\right]$$

$$= 2\left[-\frac{1}{10} + 1\right] + 10\left[\frac{1}{100}\right] = \boxed{\frac{19}{10}}$$

2. The number of days between the beginning of a calendar year and the moment a high risk driver is involved in an accident is exponentially distributed. An insurance company estimates probability that high-risk drivers will be involved in an accident during the first 50 days of a calendar year is 40%. What is the probability that high-risk driver is involved in an accident during the first 80 days of a calendar year?

$$P(X \le 50) = .4$$

$$1 - e^{-\frac{50}{\lambda}} = .4$$

$$-50/\lambda = \ln(.6)$$

$$\lambda = \frac{-50}{\ln(.6)} = 97.88$$

$$P(X \le 80) = 1 - e^{-\frac{80}{97.88}}$$

$$= \boxed{.5584}$$

X

3. The lifetime of a printer costing 200 is exponentially distributed with mean of 4 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first 2 years following its purchase, and a 60% refund if it fails during the third year. For a single printer sold, what is the expected refund payment?

$$X \sim \text{Exp}(4)$$

$$E[\text{refund}] \qquad \text{refund} = \begin{cases} 200 & 0 < x \le 2 \\ 120 & 2 < x \le 3 \\ 0 & 3 < x \end{cases}$$

$$= \int_{\text{all } x} (\text{refund}) \cdot f(x)\, dx$$

$$= \int_0^2 200 \left(\frac{1}{4} e^{-x/4}\, dx\right) + \int_2^3 120 \left(\frac{1}{4} e^{-x/4}\right) dx$$

$$= 200\, F_X(2) + 120\,(F_X(3) - F_X(2))$$

$$= 200\,(.393) + 120\,(.1341)$$

$$= \boxed{94.79}$$

$$F_X(x) = 1 - e^{-x/\lambda}$$

for $\text{Exp}(\lambda)$

4 ~~(a)~~ Suppose a study conducted 8 years ago shows that the size of claims followed an exponential distribution, and the probability that a claim would be less than 800 was 0.25. The actuary feels that the result of the study are still valid, but as a result of inflation, size of every claim made today would be 20% more than a similar claim made 8 years ago. Calculate the probability that the size of a claim made today is less than 800.

8 yrs ago. $X \sim Exp(\lambda)$

$$P(X < 800) = .25 = 1 - e^{-\frac{800}{\lambda}}$$

$\Rightarrow$ solve for λ : $\lambda = 2780.85$

Today. $Y = 1.2X \sim Exp(?)$

$$P(Y < 800) = P(1.2X < 800)$$

$$= P\left(X < \frac{800}{1.2}\right) = P(X < 667)$$

$$= 1 - e^{-\frac{667}{\lambda}}$$

$\lambda = 2780.85$

$$= \boxed{.213}$$

5 ~~16~~ The value, v, of an appliance is based on the number of years since purchase, t, as follows: $v(t) = 10 - .1t^2$ for $0 < t < 10$. If the appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance. After seven years, the warranty pays nothing. The time until failure of the appliance has an exponential distribution with mean 8. Calculate the expected payment from the warranty.

X : time until failure $\sim Exp(8)$

$$Y : \overset{\text{warranty}}{\text{payment}} = \begin{cases} 10 - .1X^2 & 0 < X \le 7 \\ 0 & 7 < X \end{cases}$$

$$E[Y] = E[g(X)]$$

$$= \int_0^7 (10 - .1X^2) \, \frac{1}{8} \cdot e^{-X/8} \, dx$$

$$= \int_0^7 \frac{10}{8} \cdot e^{-\frac{x}{8}} \, dx - \frac{.1}{8} \int_0^7 x^2 e^{-X/8} \, dx$$

$$= 10\left(-e^{-X/8}\right)\Big|_0^7 - \frac{1}{80}\left[+60.215\right]$$

$$= 5.\overset{83}{\cancel{88}}14 - .7527 = \boxed{5.079}$$

$$\int_0^7 x^2 e^{-x/8}\,dx = x^2\left(-8e^{-x/8}\right)\Big|_0^7 + \int_0^7 2x\,8e^{-x/8}\,dx$$

$dv = 2x \quad u = -8e^{-x/8}$

Int. by parts

$$= -163.41 + 16\int_0^7 x e^{-x/8}\,dx$$

Ibyp again $\quad dv = 1 \quad u = -8e^{-x/8}$

$$= -163.41 + 16\left[x\left(-8e^{-x/8}\right)\Big|_0^7 + 8\int_0^7 e^{-x/8}\,dx\right]$$

$$= (-163.41) + (-373.508) + \underbrace{16\cdot 8^2\left(-e^{-7/8} + 1\right)}$$

$$= -60,215$$

Back to last page.

6 4. Under a group insurance policy, an insurer agrees to pay 90% of the medical bills incurred during the year by employees of a small company, up to a maximum total of 1 million dollars. If total bill is between 1 and 2 million dollars, insurers will pay 90% of the first million, and 80% of amount over a million dollars. If total bill exceeds 2 million, the policy will pay 1.7 million dollars. The total amount of bills incurred, X, measured in millions has pdf

$$f(x) = \frac{x(4-x)}{9} \quad \text{for} 0 < x < 3$$

and 0 otherwise. Calculate the total amount, in millions of dollars, the insurer would expect to pay under this policy.

$$T = \begin{cases} .9X & \text{if } 0<X<1 \\ .9 + .8(X-1) & \text{if } 1<X<2 \\ 1.7 & \text{if } 2<X<3 \end{cases}$$

$$E[T] = \int_0^1 .9x\, f(x)\, dx + \int_1^2 [.9 + .8(x-1)]\, f(x)\, dx + \int_2^3 1.7\, f(x)\, dx$$

$$= .9\int_0^1 x f(x)\, dx + \int_1^2 (.9-.8) f(x)\, dx + .8\int_1^2 x f(x)\, dx + \int_2^3 1.7 f(x)\, dx$$

① ② ③ ④

$$\int_a^b x f(x)\,dx = \frac{1}{9}\int_a^b x^2(4-x)\,dx$$

$$= \frac{1}{9}\left(\frac{4}{3}x^3 - \frac{x^4}{4}\right)\Big|_a^b$$

$$\int_a^b f(x)\,dx = \frac{1}{9}\int_a^b x(4-x)\,dx$$

$$= \frac{1}{9}\left(2x^2 - \frac{1}{3}x^3\right)\Big|_a^b$$

$$E[T] = .9\left[\frac{1}{9}\left(\frac{4}{3}x^3 - \frac{x^4}{4}\right)\Big|_0^1\right] \quad 1$$

$$+ .8\left[\frac{1}{9}\left(\frac{4}{3}x^3 - \frac{1}{4}x^4\right)\Big|_1^2\right] \quad 3$$

$$+ .1\left[\frac{1}{9}\left(2x^2 - \frac{1}{3}x^3\right)\Big|_1^2\right] \quad 2$$

$$+ 1.7\left[\frac{1}{9}\left(2x^2 - \frac{1}{3}x^3\right)\Big|_2^3\right] \quad 4$$

$$= \boxed{1.339}$$

7 8. Let X_1, X_2, X_3 be independent and identically distributed random variables each with density function

$$f(x) = \sqrt{2} - x \qquad \text{for } 0 < x < \sqrt{2},$$

and 0 otherwise. What is the probability that exactly 1 of X_1, X_2, X_3 does not exceed .5?

P (exactly one does not exceed .5)

$$= P(X_1 < .5 \cap X_2 > .5 \cap X_3 > .5)$$
$$+ P(X_1 > .5 \cap X_2 < .5 \cap X_3 > .5)$$
$$+ P(X_1 > .5 \cap X_2 > .5 \cap X_3 < .5)$$

$$= 3 \cdot P(X_1 < .5) \cdot P(X_1 > .5)^2$$

$$= 3 \cdot p \cdot (1-p)^2 = \boxed{.305}$$

$$p = \int_0^{.5} (\sqrt{2} - x)\, dx = \sqrt{2}\,x - \frac{x^2}{2} \Big|_0^{.5} = .582$$

8. Suppose the time until failure of the appliance failing has an exponential distribution with mean 8.

(a) What is the probability that appliance will fail within 1 year?

$$P(X<1) = 1-e^{-\frac{1}{8}} = \boxed{.1175}$$

(b) If 10 appliances are sold, what is the probability that the first failure occurs later than 1 year?

$$P(\min(X_1, \dots, X_{10}) > 1) = P(X_1 > 1)^{10}$$

$$= \left(e^{-\frac{1}{8}}\right)^{10}$$

$$= e^{-\frac{10}{8}}$$

$$= \boxed{.2865}$$