University of Akron, Dept. of Statistics

3470:451/551 Theoretical Statistics I

Ch 4: Continuous Random Variables

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Textbook: Wackerly, Mendenhall, and Scheaffer 7e (2008)

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Contents

2.1 Continuous RV

Continuous Random Variable is a r.v. whose range is a interval on a real line or a disjoint union of such intervals. It also must satisfy that for any constant c, $P(X = c) = 0$.

Probability density function (pdf) of continuous r.v. X is a functon $f(x)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Pdf must satisfy:

1. $f(x) \geq 0$ for all x.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

Cumulative Distribution Function (CDF) of r.v. X is a functon $F(x)$ defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

If X is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$ then at every x at which the derivative $F'(x)$ exists,

$$F'(x) = f(x).$$

Cdf must satisfy:

1. $F(-\infty) = 0$ and $F(\infty) = 1$.
2. non-decreasing.
3. right continuous.

For any number a and b with $a < b$,

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

Percentiles

Let p a number between 0 and 1. The $(100 \times p)$th percentile of the distribution of a continuous r.v. X, denoted η_p, is a number such that

$$F(\eta_p) = p$$

Example

Let rv X have pdf

$$f(x) = \begin{cases} Kx^2 & \text{if } 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

1. What is K?
2. What is $P(1.5 \leq X \leq 2)$
3. What is $F(x)$
4. What is 70th percentile of X?

Example: K 1-24

An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function $f(x) = 3x^{-4}$ for $1 < x$, and 0 otherwise. Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2?

Example: K 1-13

The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively. What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

Expected Values

Expected or mean value of a continuous r.v. X with pdf f(x) is

$$E\big[X\big] = \int_{-\infty}^{\infty} x f(x) dx.$$

To get expectation of a function of X, $g(X)$,

$$E\big[g(X)\big] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

If $g(\cdot)$ is a linear function,

$$E(aX + b) = aE(X) + b.$$

Variance

Variance of a continuous r.v. X with pdf f(x) is

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E\Big[(x-\mu)^2\Big]$$

and standard deviation (SD) of X is

$$\sigma = \sqrt{\sigma^2}.$$

Or we can still use alternative formula

$$V(X) = E(X^2) - [E(X)]^2.$$

Example Let rv X have pdf

$$f(x) = \begin{cases} Kx^2 & \text{if } 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

1. What is $E(X)$?

2. What is $V(X)$?

Example: K 1-10 modified

An auto insurance company insures an automobile for one year under a policy with a 1000 deductible and upper limit of 15,000. If there is a damage, the amount can be modeled by (in thousands) pdf $f(x) = 3e^{-x/3}$ for $0 < x$. What is the expected claim payment?

Example: K 1-12

A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E(X)$.

2.2 Popular Distributions

2.2.1 Uniform

[top]

$X \sim Unif(a,b)$

$$\text{pmf}: \quad f(x) = \frac{1}{b-a} \qquad \text{for } x \in [a,b]$$

$$\text{CDF}: \quad F(x) = \frac{x-a}{b-a} \qquad \text{for } x \in [a,b]$$

$$\text{mean and var}: \quad E(X) = \frac{b+a}{2} \qquad V(X) = \frac{(b-a)^2}{12}$$

$$\text{MGF}: \quad M(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$$

```
dunif(2, a,  b)       #pmf at x=2
punif(2, a,  b)       #CDF at x=2
qunif(.5, a,  b)      #Inv CDF at q=.5
runif(1000, a, b)     # random sample of size 1000
x=seq(-1,4,.01);  plot(x,dunif(x,1,3), type="l", ylim=c(0,1))  #plot pdf
x=seq(-1,4,.01);  plot(x,punif(x,1,3), type="l", ylim=c(0,1))  #plot CDF
```

2.2.2 Normal

[top]

$X \sim \mathrm{N}(\mu, \sigma^2)$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R} \\
\text{CDF}: \quad F(x) &= \int_{-\infty}^{x} f(t)dt \\
\text{mean}: \quad E(X) &= \mu \\
\text{var}: \quad V(X) &= \sigma^2 \\
\text{MGF}: \quad M(t) &= e^{\mu t + \frac{\sigma^2}{2} t^2}
\end{aligned}
$$

μ is location parameter, and σ is scale parameter.

```
dnorm(2, mu, sigma)        #pmf at x=2
pnorm(2, mu, sigma)        #CDF at x=2
qnorm(.5, mu, sigma)       #Inv CDF at q=.5
rnorm(1000, mu, sigma)     # random sample of size 1000.
x=seq(-4,4,.01);  plot(x,dnorm(x,0,1), type="l", ylim=c(0,1))  #plot pdf
x=seq(-4,4,.01);  plot(x,pnorm(x,0,1), type="l", ylim=c(0,1))  #plot CDF
```

2.2.3 Log-normal

[top]

$X \sim$Log-normal(μ, σ)

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\sqrt{2\pi}\sigma x} e^{-[\ln(x)-\mu]^2/(2\sigma^2)} \\
\text{CDF}: \quad F(x) &= \Phi\Big(\frac{\ln(x)-\mu}{\sigma}\Big) \\
\text{mean}: \quad E(X) &= e^{\mu+\sigma^2/2} \\
\text{var}: \quad V(X) &= e^{2\mu+\sigma^2} \times (e^{\sigma^2} - 1) \\
\text{MGF}: \quad M(t) &=
\end{aligned}
$$

$\ln(X) \quad \sim \; \mathrm{N}(\mu, \sigma^2)$

```
dlnorm(2, u, si)        #pmf at x=2
plnorm(2, u, si)        #CDF at x=2
plnorm(.5,u, si)        #Inv CDF at q=.5
rlnorm(1000, u, si)     # random sample of size 1000.  mean should be a*b
x=seq(-1,5,.01);  plot(x,dlnorm(x, 0, 1), type="l", ylim=c(0,1))  #plot pdf
x=seq(-1,5,.01);  plot(x,plnorm(x, 0, 1), type="l", ylim=c(0,1))  #plot CDF
```

2.2.4 Exponential

[top]

$X \sim$Exp(β)

$$
\begin{aligned}
\text{pdf}: \quad p(x) &= \frac{1}{\beta}\, e^{-\frac{x}{\beta}} \quad \text{for } x > 0 \\
\text{CDF}: \quad F(x) &= 1 - e^{-x/\beta} \quad \text{for } x > 0 \\
\text{mean}: \quad E(X) &= \beta \\
\text{var}: \quad V(X) &= \beta^2 \\
\text{MGF}: \quad M(t) &= \left[\frac{1}{1 - t\beta}\right]
\end{aligned}
$$

β is a scale parameter

```
dexp(2, 1/b)        #pmf at x=2
pexp(2, 1/b)        #CDF at x=2
pexp(.5, 1/b)       #Inv CDF at q=.5
rexp(1000, 1/b)     # random sample of size 1000.  mean should be b
x=seq(-1,5,.01);  plot(x,dexp(x,1/2), type="l", ylim=c(0,1))  #plot pdf
x=seq(-1,5,.01);  plot(x,pexp(x,1/2), type="l", ylim=c(0,1))  #plot CDF
```

2.2.5 Gamma

$X \sim \text{Gam}(\alpha, \beta)$ [top]

$$\text{pdf}: \quad f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad x > 0$$

$$\text{CDF}: \quad F(x) = \frac{\Gamma(x/\beta;\alpha)}{\Gamma(\alpha)} \qquad x > 0$$

$$\text{mean and var}: \quad E(X) = \alpha\beta \qquad V(X) = \alpha\beta^2$$

$$\text{MGF}: \quad M(t) = \left[\frac{1}{1-t\beta}\right]^\alpha$$

α is a shape parameter, β is a scale parameter

```
dgamma(2, a, scale=b)        #pmf at x=2
pgamma(2, a, scale=b)        #CDF at x=2
pgamma(.5, a, scale=b)       #Inv CDF at q=.5
rgamma(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,10,.01);  plot(x,dgamma(x,2,scale=2), type="l", ylim=c(0,.5))  #plot pdf
x=seq(-1,10,.01);  plot(x,pgamma(x,2,scale=2), type="l", ylim=c(0,.5))  #plot CDF
```

Gamma Func: $\Gamma(\alpha) = \int_0^1 x^{\alpha-1} e^{-x} dx,$ Incomplete Gamma Func: $\Gamma(x,\alpha) = \int_0^x t^{\alpha-1} e^{-t} dt$

2.2.6 Chi-square

[top]

$X \sim \chi^2(\nu)$

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\Gamma(\nu/2)\,2^{\nu/2}}\; x^{\nu/2-1}\, e^{-x/2} \\
\text{CDF}: \quad F(x) &= \Gamma(x/2;\nu/2)/\Gamma(\nu/2) \\
\text{mean}: \quad E(X) &= \nu \\
\text{var}: \quad V(X) &= 2\nu \\
\text{MGF}: \quad M(t) &= \left[\frac{1}{1-2t}\right]^{\nu/2} \qquad t < 1/2
\end{aligned}$$

same as Gam$(\nu/2, 2)$

```
dchisq(2, a, scale=b)        #pmf at x=2
pchisq(2, a, scale=b)        #CDF at x=2
pchisq(.5, a, scale=b)       #Inv CDF at q=.5
rchisq(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,4,.01);  plot(x,dchisq(x,3), type="l", ylim=c(0,.5))  #plot pdf
x=seq(-1,4,.01);  plot(x,pchisq(x,3), type="l", ylim=c(0,.5))  #plot CDF
```

2.2.7 Beta

$X \sim$Beta(α, β) $\qquad \alpha > 0, \beta > 0$ [top]

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1 \\
\text{CDF}: \quad F(x) &= \frac{\beta(x;\alpha,\beta)}{\beta(\alpha,\beta)} \\
\text{mean and var}: \quad E(X) &= \frac{\alpha}{\alpha+\beta}, \qquad V(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
\text{MGF}: \quad M(t) &= 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}
\end{aligned}
$$

```
dbeta(2, a, scale=b)        #pmf at x=2
pbeta(2, a, scale=b)        #CDF at x=2
pbeta(.5, a, scale=b)       #Inv CDF at q=.5
rbeta(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,2,.01);  plot(x,dbeta(x,2,2), type="l", ylim=c(0,2))  #plot pdf
x=seq(-1,2,.01);  plot(x,pbeta(x,2,2), type="l", ylim=c(0,2))  #plot CDF
```

$$
\begin{aligned}
\text{Beta function} \quad \beta(\alpha,\beta) &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
\text{Incomplete Beta func} \quad \beta(x;\alpha,\beta) &= \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt
\end{aligned}
$$

2.2.8 Cauchy

[top]

$X \sim \text{Cau}(\theta, \sigma) \qquad \theta \in \mathbb{R}, \sigma > 0$

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\pi\sigma} \frac{1}{1 + (\frac{x-\theta}{\sigma})^2} \qquad x \in \mathbb{R} \\
\text{CDF}: \quad F(x) &= \int_0^x f(t) dt \quad \text{for } t > 0 \\
\text{mean and var}: \quad E(X) &= \text{Does not exist} \qquad V(X) = \text{Does not exist} \\
\text{MGF}: \quad M(t) &= \text{ Does not exsit}
\end{aligned}$$

Special case of student's t, when df=1.

2.2.9 Weibull

[top]

$X \sim \text{Wei}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{\alpha}{\beta}\, x^{\alpha-1}\, e^{-x^{\alpha}/\beta} \qquad 0 \leq x \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } 0 \leq x \\
\text{mean}: \quad E(X) &= \beta^{1/\alpha}\Gamma(1+1/\alpha) \\
\text{var}: \quad V(X) &= \beta^{2/\alpha}[\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)] \\
\text{moments}: \quad E(X^n) &= \beta^{n/\alpha}[\Gamma(1+n/\alpha)] \\
\text{MGF}: \quad M(t) &= \text{Exists only for } \alpha \geq 1.
\end{aligned}
$$

If $\alpha = 1$, it is $\text{Exp}(\beta)$

2.2.10 Student-t

[top]

$X \sim \mathrm{t}(\nu) \qquad \nu = 1, 2, 3, \ldots$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } t > 0 \\
\text{mean and var}: \quad E(X) &= 0, \quad \nu > 1 \qquad V(X) = \frac{\nu}{\nu-2}, \quad \nu > 2 \\
\text{moments}: \quad E(X^2) &= \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{\nu-n}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\nu^{n/2} \qquad \text{if } n < \nu \text{ and even. } 0 \text{ if odd.} \\
\text{MGF}: \quad M(t) &= \text{Does not exsit}
\end{aligned}
$$

```
dt( 2, v)        #pmf at x=2
pt( 2, v)        #CDF at x=2
pt(.5, v)       #Inv CDF at q=.5
rt(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-4,4,.01);  plot(x,dt(x,5), type="l", ylim=c(0,1))  #plot pdf
x=seq(-4,4,.01);  plot(x,pt(x,5), type="l", ylim=c(0,1))  #plot CDF
```

2.2.11 F

[top]

$X \sim \mathrm{F}(\nu_1, \nu_2) \qquad \nu_1, \nu_2 = 1, 2, 3, \ldots$

$$\begin{aligned}
\text{pdf :} \quad f(x) &= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}\frac{x^{(\nu_1-2)/2}}{\left(1+(\frac{\nu_1}{\nu_2})x\right)^{(\nu_1+\nu_2)/2}} && 0 \leq x \\
\text{CDF :} \quad F(x) &= \int_0^x f(t)dt \quad \text{for } x > 0 \\
\text{mean :} \quad E(X) &= \frac{\nu}{\nu_2 - 2} \quad \nu_2 > 2 \\
\text{var :} \quad V(X) &= 2(\frac{\nu_2}{\nu_2-2})^2\frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)} \quad \nu_2 > 4 \\
\text{moments} \quad E(X^n) &= \frac{\Gamma(\frac{\nu_1+2n}{2})\Gamma(\frac{\nu_2+2n}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_2}{\nu_1}\right)^n, && n < \frac{\nu_2}{2} \\
\text{MGF :} \quad M(t) &= \text{Does Not Exist}
\end{aligned}$$

2.2.12 Overlay plots in R

[top]

```
#-- Overlay with N(0,1) pdf --
plot(x, dt(x,5), type='l')
lines(x, dnorm(x,0,1), col='red')
```

```
#-- Overlay with N(0,1) pdf (method 2 - have to specify plot range) --
plot(x, dt(x,5), type='l', xlim=c(-5,5), ylim=c(0,.4))
par(new=T)
plot(x, dnorm(x,0,1), type='l', xlim=c(-5,5), ylim=c(0,.4), col='red')
```

2.2.13 Distributional Relations

[top]

- When X and Y are independent $\text{Exp}(\lambda)$, $X + Y$ is $\text{Gam}(2, \lambda)$.
- When you have n iid Exponential r.v. with mean of λ, $\min(X_1, \ldots, X_n)$ is Exponenital with mean λ/n.
- Beta(1,1) is same as Unif(0,1).
- Cauchy is same as t(1).
- When X, and Y are independent U(0,1), X/Y is Cauchy.
- When X is $\text{U}(-\pi/2, \pi/2)$, $\tan(X)$ is Cauchy.
- If we have two independent r.v. $X_1 \sim\text{Gam}(\alpha_1, \beta)$ and $X_2 \sim\text{Gam}(\alpha_2, \beta)$ and

$$Y = \frac{X_1}{X_1 + X_2} \quad \sim \quad \text{Beta}(\alpha_1, \alpha_2)$$

- That is same as to say, if we have two independent r.v. $X_1 \sim \chi^2(\alpha_1)$ and $X_2 \sim \chi^2(\alpha_2)$

$$Y = \frac{X_1}{X_1 + X_2} \sim \text{Beta}\Big(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}\Big)$$

- When U is $\chi^2(r_1)$ and V is $\chi^2(r_2)$, $\frac{U/r_1}{V/r_2}$ is $F(r_1, r_2)$,

- $F_{1,\nu}$ is same as $t^2(\nu)$

- Gamma function: $\frac{1}{2}! = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

2.3 Scale Parameter

[top]

- If you transform r.v. X to $Y = \theta X$,

$$
\begin{aligned}
F_Y(y) &= P(\theta X \leq y) = P(X \leq y/\theta) = F_X(x/\theta) \\
f_Y(y) &= \frac{1}{\theta} f_X\left(\frac{x}{\theta}\right)
\end{aligned}
$$

  θ is called the scale parameter.

- if Y has scale parameter θ, then

$$\frac{X}{\theta} \text{ has same distribution with } \theta = 1.$$

Example

If $X \sim$Exp(λ), then λ is a scale parameter. Thus

$$\frac{X}{\lambda} \sim \text{Exp}(1).$$

Example

If $X \sim$Gam(α, β), then β is a scale parameter. Thus

$$\frac{X}{\lambda} \sim \text{Gam}(\alpha, 1).$$

Example

If $X \sim$Gam(α, β), then $X/\beta \sim$Gam($\alpha, 1$). We can write cdf of Gam($\alpha, 1$) as

$$F_X(x) = \frac{1}{\Gamma(\alpha)} \underbrace{\int_0^x y^{a-1} e^{-y} dy}_{\text{(lower) incomplete gamma func}} = \frac{\Gamma(x, \alpha)}{\Gamma(\alpha)}, \qquad 0 < x < \infty$$