Forwards and Futures

Types of Forwards

	Pay at	Receive Asset at	Price
Outright Purchase	$t=0$	$t=0$	S_0
Fully leveraged Purchase	$t=T$	$t=0$	$S_0 e^{rT}$
Prepaid forward	$t=0$	$t=T$	?
Forward	$t=T$	$t=T$	$(?)\cdot e^{rT}$

Prepaid Forwards

Pay at $t=0$. receive at $t=T$.

→ could miss dividends.

If there are no dividends to be paid,

$$F^P_{0,T} = S_0 \qquad (\text{paid at } t=0).$$

r = Force of interest.

$$A(t) = A(0)\, e^{\int_0^T r\, dt} = A(0)\, e^{rT}.$$

Present Value

$$A(0) = A(t)\, e^{\int_T^0 r\, dt} = A(t)\, e^{-rT}.$$

If $F^P_{0,T} \neq S_0$, then there's an arbitrage

Suppose $F^P_{0,T} \neq S_0$, say A_0.. then you can

		$t=0$		$t=T$
$A_0 > S_0$	Sell prepaid forward at	A_0.		deliver asset $(-S_T)$
	Buy same stock at	$-S_0$.	$\longrightarrow$	S_T
$A_0 < S_0$	Buy prepaid forward	$-A_0$.	$\longrightarrow$	S_T
	Short sale stock	S_0		$-S_T$

Prepaid Forwards with Dividends

Discrete dividends

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} PV(D_{t_i})$$

D_{t_i} = dividend paid at $t = t_i$

Ex 5.1

$S_0 = 100$.

\$1.25 dividend quarterly. 1st one in 3-mo.

annual risk free rate 10%. Continuously compounded.

1-year prepaid forward price?

Review.

nominal annual rate $i^{(m)} = 8\%$.

effective annual rate $1+i = \left(1+\frac{i^{(m)}}{m}\right)^m$.

$m = 2 \qquad 1+i = 1.0816$

$m = 3 \qquad 1+i = 1.0824$

$m = \infty \qquad 1+i = \lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m = e^{i^{(m)}} = e^{.08} = 1.0833$.

Force of Interest $r = \ln(1+i) = \ln\left(e^{i^{(m)}}\right) = i^{(\infty)} =$

$m = \infty$

Ex 5.1

Continuously compounded (nominal) annual rate $i^{(\infty)} = 10\%$

Continuously compounded nominal quarterly rate $j^{(\infty)} = ?$

annual

$$e^{\int_0^1 .10\, dt} = e^{.10(1)} = 1+i$$

quarterly

$$e^{\int_0^1 r\, dt} = e^{r(1)} = \underset{\text{effective quarterly}}{1+j} = (1+i)^{1/4}$$

Continuously compounded

annual rate

$$e^{I} = 1+i$$

quarterly rate

$$e^{r} = (1+i)^{1/4}$$

annual

10%.

$$r = \frac{10\%}{4} = 2.5\%.$$

annual rate

$$F_{0,T}^{P} = 1000 - \sum_{i=1}^{4} 1.25\, e^{-(.1)(i/4)}$$

quarterly rate

$$F_{0,T}^{P} = 1000 - \sum_{i=1}^{4} 1.25\, e^{-(.025)(i)}$$

$$F_{0,T}^{P} = 95.30$$

Price of Prepaid Fwd with continuous dividends

→ approximation

→ ~~~~ Dividend yield is fixed.

Daily dividend $\frac{\delta}{365} S_0$. δ : annual dividend yield.

$\quad \parallel$

$\quad i^{(365)}$

If we reinvest all the dividends,

$$A(t) = A(0)\left(1 + \frac{\delta}{365}\right)^{365t} \longrightarrow A(0)\, e^{\delta t}$$

$$\delta = i^{(\infty)}.$$

$$\boxed{F^{P}_{0,T} = S_0 e^{-\delta T}}$$

w/ cont. dividends

Forward Contract

pay at $t=T$, receive at $t=T$.

$F_{0,T}$

when there's no dividends.

$$F_{0,T} = FV(S_0) = S_0 e^{rT}$$

r = risk-free rate
continuously compounded.

If Forward price is not $S_0 e^{rT}$,

there's arbitrage

Say $F_{0,T} = A_0 \neq S_0 e^{rT}$

$A_0 > S_0 e^{rT}$

	$t=0$	$t=T$
sell forward	0	A_0
borrow at rate r	S_0	$\longrightarrow -S_0 e^{rT}$
buy asset.	$-S_0$	$\longrightarrow S_T$ → must give away by forward contract.
	0	$A_0 - S_0 e^{rT}$

$A_0 < S_0 e^{rT}$

	$t=0$	$t=T$
buy forward	0	$-A_0$
lend at rate r	$-S_0$	$\longrightarrow$ $S_0 e^{rT}$
short asset	S_0	$-S_T$ asset will come from forward.

Forward

Discrete Dividends

$$F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$$

Continuous Dividends

$$F_{0,T} = S_0 e^{(r-\delta)T} = e^{rT} \underbrace{S_0 e^{-\delta T}}_{\text{PV of stock, w/o dividends.}}$$

Futures Contracts

Futures

→ almost like forwards.

Agreement to buy/sell asset at given price in future dates.

Differences from forwards

→ standardized - dates, locations, procedures, quantity.

→ Settled daily

→ There's daily price limits in future markets.

e.g. if S&P500 ↓ more than 5%, then trading stops for some period.

Example $\boxed{\text{S\&P500 futures}}$ at Chicago Mercantile Exchange

Size $\$250 \times$ SP500 index ~~\$~~

Cash settled based on ~~[illegible]~~ SP500 price on 3-rd Friday.

+ Suppose you want to long SP500 Futures for \$2.2 million.

SP500 today $= 1100$

1 contract $= \$250 \cdot 1100 = \$.275$ mil.

2.2 mil $=$ 8 contracts.

Broker finds somebody to sell (short) this to YOU.

Broker will require 'Margin' from both buyer

and sellar.

If next day price = 1099,

$$8 \text{ contracts} = \underbrace{8 \times 250 \times 1100}_{\text{must pay for SP500}} = 2{,}200{,}000$$

but SP500 only worth

$$8 \times 250 \times 1099 = 2{,}198{,}000$$

$$-2000$$

Margin will acquire interest.