6.5 Limiting Probabilities

Birth and Death Process

$$0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow$$

Birth rate λ_i

Death rate μ_i

starting from state 0,

Forward Eq'n

leaving   incoming

$$\frac{d}{dt} P_0(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

$$\frac{d}{dt} P_1(t) = -(\lambda_1 + \mu_1) P_1(t) + \lambda_0 P_0(t) + \mu_2 P_2(t)$$

$$\frac{d}{dt} P_2(t) = -(\lambda_2 + \mu_2) P_2(t) + \lambda_1 P_1(t) + \mu_3 P_3(t)$$

$$\vdots$$

What happens when $t \to \infty$?

Assume $\lim_{t \to \infty} P_j(t) = P_j$ exists. ☆ Then ....

P_j exists $\Rightarrow$ All $\frac{d}{dt} P_j(t) \to 0$ as $t \to \infty$.

then we have

leaving		incoming
$\lambda_0 P_0$	$=$	$\mu_1 P_1$
$(\lambda_1 + \mu_1) P_1$	$=$	$\lambda_0 P_0 + \mu_2 P_2$
$(\lambda_2 + \mu_2) P_2$	$=$	$\lambda_1 P_1 + \mu_3 P_3$
	$\vdots$	

$$\sum_j P_j = 1$$

P_j shouldn't depend on initial state.

limiting prob. exists when ...

Ⓐ All states in MC communicate.

$P(i \text{ to } j) > 0$ for all i, j.

Ⓑ MC is positive recurrent.

$E(\text{time for } i \text{ to } i) < \infty$.

(Sufficient condition)

Ergodic chain = P_j exists.

B-D process

Going out   Coming in

$$\lambda_0 P_0 = \mu_1 P_1$$

$$(\lambda_1 + \mu_1) P_1 = \mu_2 P_2 + \lambda_0 P_0$$

$$(\lambda_2 + \mu_2) P_2 = \mu_3 P_3 + \lambda_1 P_1$$

$$\vdots$$

Add each eq'n to one before,

and get

$$\lambda_0 P_0 = \mu_1 P_1 \quad \rightarrow \quad P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$\lambda_1 P_1 = \mu_2 P_2 \quad \rightarrow \quad P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

$$\lambda_2 P_2 = \mu_3 P_3 \quad \rightarrow \quad P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$$

$$\vdots \qquad\qquad \vdots$$

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \quad n \geq 0 .$$

$$\rightarrow \quad P_n = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} P_0 .$$

[illegible]

[illegible]

Since $\sum_{n=0}^{\infty} P_n = 1$.

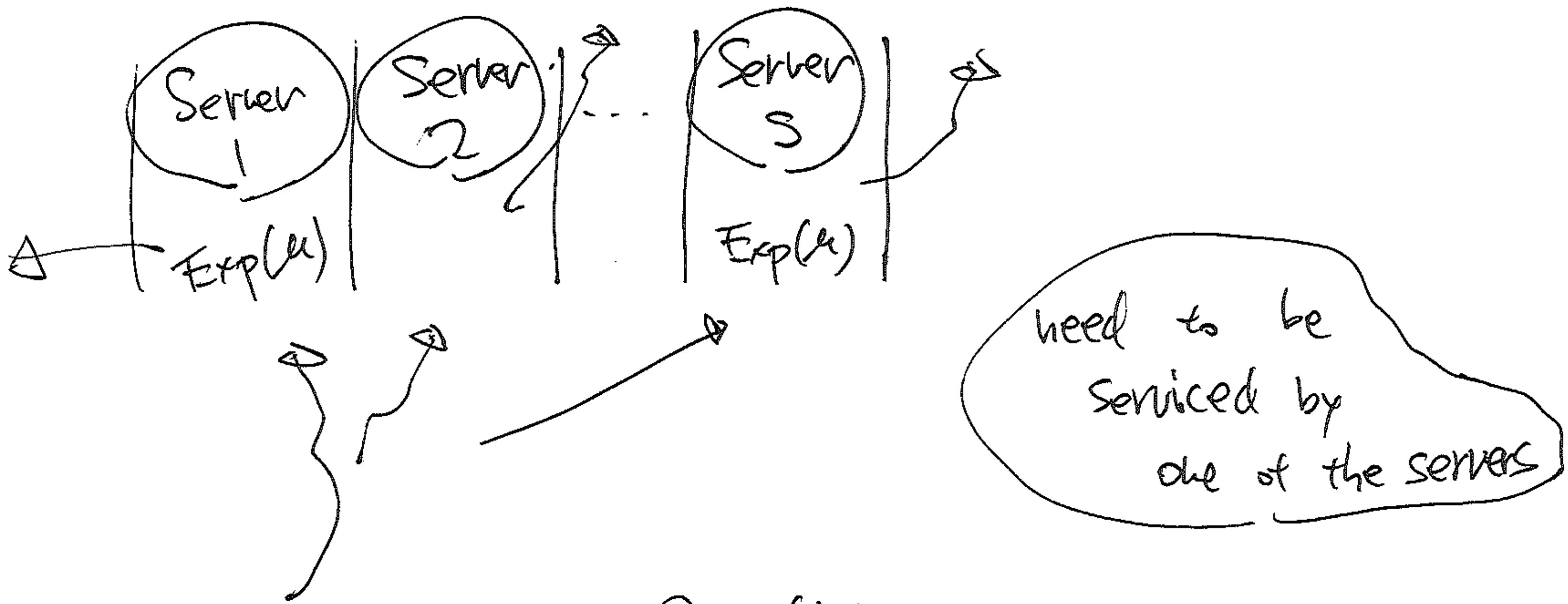
$$1 = P_0 + \underbrace{\left(\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \right)}_{A} P_0$$

A must be finite ← necessary + sufficient cond.

$$\boxed{P_0 = \frac{1}{1+A} \qquad P_n = \left(\frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \right) \frac{1}{1+A}}$$

$n \geq 1$

Ex 6.6 M/M/s queue



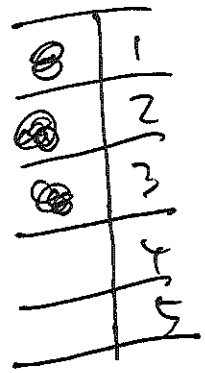
Customer $\sim$ PoiProc (λ)

$\boxed{\text{\# of Customers in queue}}$

$\sim$ B-D proc with $\begin{cases} \mu_n = \\ \lambda_n = \end{cases}$

Say $S = 5$

$n = 3$



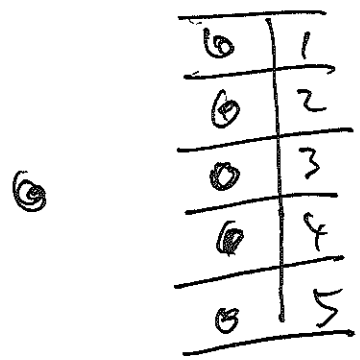
μ_n min of 3 indep $\mathrm{Exp}(\mu)$.

$= \mathrm{Exp}(3\mu)$.

$\lambda_n = \lambda$.

$\mu_n = 3\mu$

$n = 6$



$\mu_n = 5\mu$

$\lambda_n = \lambda$.

$$\begin{cases} \mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases} \\[2em] \lambda_n = \lambda \qquad\qquad n \ge 0. \end{cases}$$

M/M/s queue.

$$A = \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} < \infty \quad \text{means}$$

$$= \sum_{n=s+1}^{\infty} \frac{\lambda^n}{(s\mu)^n} + \sum_{n=1}^{s} \frac{\lambda^n}{(n\mu)^n}$$

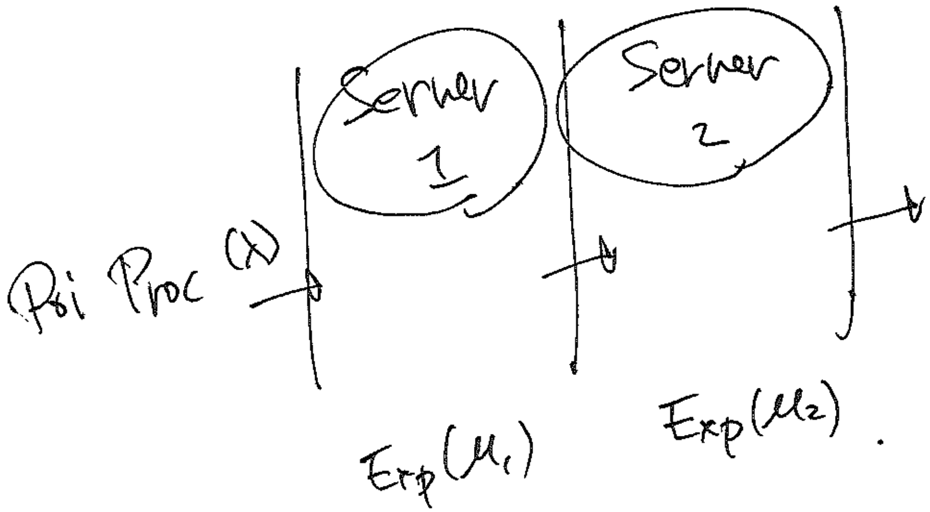
$$\sum_{n=s+1}^{\infty} \frac{\lambda^n}{(s\mu)^n}$$

$$= \left(\frac{\lambda}{s\mu}\right)^{s+1} \sum_{n=0}^{\infty} \left(\frac{\lambda}{s\mu}\right)^n \quad < \infty$$

$$\underset{\text{iff.}}{\iff} \quad \frac{\lambda}{s\mu} < 1$$

Then P_j exists.

Ex 6.1 Two servers in row queue.



new customer enters only when both servers are open.

states

$$\begin{cases} 0 & \text{both server open} \\ 1 & \text{customer in S1} \\ 2 & \text{" in S2} \end{cases}$$

Is this B-D process?

Yes.

Balance Eq'n.

Coming in = Going out

$$\begin{cases} \lambda P_0 = \mu_2 P_2 \\ \mu_1 P_1 = \lambda P_0 \\ \mu_2 P_2 = \mu_1 P_1 \end{cases}$$

Solve in terms of P_0.

$$\begin{cases} P_1 = \dfrac{\lambda}{\mu_1} P_0. \\ P_2 = \dfrac{\lambda}{\mu_2} P_0 \end{cases}$$

$P_0 + P_1 + P_2 = 1$.

$$P_0 + P_1 + P_2 = 1$$

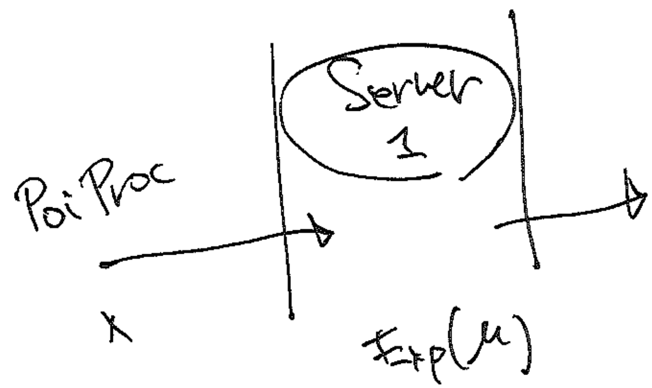
$$P_0 + \frac{\lambda}{\mu_1} P_0 + \frac{\lambda}{\mu_2} P_0 = 1$$

$$P_0 = \frac{1}{\left(1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}\right)}$$

$$P_1 = \frac{\lambda}{\mu_1} P_1$$

$$P_2 = \frac{\lambda}{\mu_2} P_2$$

Ex 6.5   M/M/1   Queue.



$\boxed{\text{\# of customers in queue}}$

= B-D process with $\mu_n = \mu$ constant rate.

$\lambda_n = \lambda$.

Example 6.14

M/M/1 Queue.

$$\begin{cases} \lambda_n = \lambda \\ \mu_n = \mu \end{cases}$$

$$\begin{cases} P_n = \dfrac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n (1+A)} & n \geq 1. \\[2ex] P_0 = \dfrac{1}{1+A} & \text{where } A = \displaystyle\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_0 \cdots \mu_n}. \end{cases}$$

$$A = \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_0} = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\lambda}{\mu} \left(\frac{1}{1-\frac{\lambda}{\mu}} \right)$$

$$P_0 = \frac{1}{1+A} = \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_n = \left(\frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_0} \right) \frac{1}{1+A} = \left(\frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Ex 6.13

Machine Repair Model

M machines. 1 repair man.

lifetime $\sim Exp(\lambda)$. fix time $\sim Exp(\mu)$.

$E($ # of machines broken $)$

$E($ % of time each machine is in use $)$.

n machines broken, $n \geq 0$.

$\mu_n = \mu \qquad n \geq 1 \qquad$ repair rate

$$\lambda_n = \begin{cases} (M-n)\lambda & n \leq M \\ 0 & n > M \end{cases} \qquad \text{break rate.}$$

MC irriducible
+
pos. recurrent.

$$P_0 = \frac{1}{1 + A}$$

$$= \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1}}$$

$$= \frac{1}{1 + \sum_{n=1}^{M} \frac{(M-(n-1))\lambda \cdot \cdots \cdot M\lambda}{\mu^n}}$$

$$= \frac{1}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}$$

$$P_n = \frac{\lambda_{n-1} \cdots \lambda_0}{(1 + A)\,\mu_n \cdots \mu_1}$$

$$= \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!} \left[\frac{1}{1+A} \right] \qquad n = 0, 1, \ldots, M.$$

E(# of Machines broken)

$$= \sum_{n=0}^{M} n P_n$$

$E($ % of time each machine is in use $)$

$$= 1 - \underbrace{\frac{\sum_{n=0}^{M} n P_n}{M}}_{\text{\% of machines broken}}$$

$= E($ % of #s of machines in use $)$