4.6 Mean Visit Times

Mean Return Time

$$m_{jj} = E(\text{time until } j \text{ to } j)$$

$$= E(\min(n \geq 1 ; X_n = j \mid X_0 = j))$$

$$= \frac{1}{\pi_j}$$

$\underline{\pi}$ = limiting distribution

Mean Visit Time

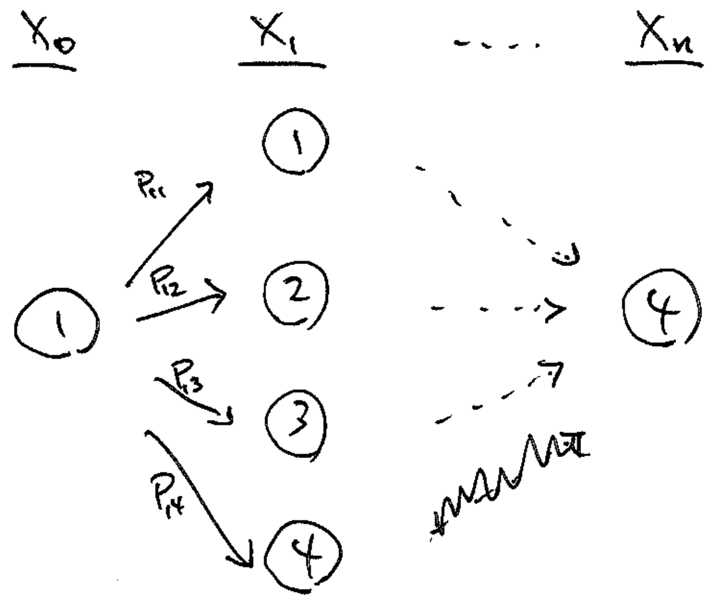
$$m_{ij} = E(\text{time until } i \text{ to } j)$$

$$= E\left(\min(n \geq 1,\ X_n = j) \mid X_0 = i \right)$$

E.g.

~~what about~~

$$m_{14} = E(\text{time to } 4 \mid X_0 = 1)$$



$$
\begin{aligned}
m_{14} = \; & 1 + E(\text{time to } 4 \mid X_1 = 1) \cdot P_{11} \\
& + 1 + E(\text{time to } 4 \mid X_1 = 2) \cdot P_{12} \\
& + 1 + E(\text{time to } 4 \mid X_1 = 3) \cdot P_{13} \\
& + 1 \cdot P_{14}
\end{aligned}
$$

$$m_{14} = 1 + \sum_{i=1}^{3} m_{i4} \cdot P_{1i} = 1 + \begin{bmatrix} P_{11} & P_{12} & P_{13} \end{bmatrix} \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix}$$

$$m_{24} = 1 + \sum_{i=1}^{3} m_{i4} \cdot P_{2i} = 1 + \begin{bmatrix} P_{21} & P_{22} & P_{23} \end{bmatrix} \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix}$$

$$m_{34} = 1 + \sum_{i=1}^{3} m_{i4} \cdot P_{3i} = 1 + \begin{bmatrix} P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix}$$

$$\begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix}$$

$$\underline{m} = \underline{e} + Q\,\underline{m}$$

solve for $\underline{m}$

$$\underline{m} = \underline{e} + Q\underline{m}$$

$$\underline{m} - Q\underline{m} = \underline{e}$$

$$\underline{m} = (\mathbb{I} - Q)^{-1}\underline{e}$$

$$\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ identity matrix.

__Thm:__ unique solution exists when state space is finite

Ex

$$\mathbb{P} = \begin{bmatrix} .5 & .5 & 0 \\ .5 & .5 & 0 \\ .33 & .33 & .34 \end{bmatrix}$$

what are m_{i1} ?

what about m_{i3} ?

Ex.

$$\mathbb{P} = \begin{bmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix}$$

$$\begin{cases} SS = 1 \\ SR = 2 \\ RS = 3 \\ RR = 4 \end{cases}$$

What is m_{i1} ?

Linear Equations by Conditioning

Example.

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & .5 & .5 \\ 1 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .5 & 0 & .5 \end{bmatrix}$$

irreducible.

$$\underline{\pi} = [.25 \quad .25 \quad .125 \quad .375]$$

$$\frac{1}{\underline{\pi}} = [4 \quad 4 \quad 8 \quad 2.67] \quad \text{Mean Rec. Times}$$

Ex. coin toss surprize (Tijms p91)

Flip a coin record ~~last~~ first 3 flips

HHH

THH

1	begin	
2	H	first toss
3	T	
4	HH	last 2 toss
5	HT	
6	TT	
7	TH	
8	TTH	← stop win
9	THH	← stop loss.

TTH before HHT

$$
\mathbb{P} = \begin{array}{ll} & \\ \text{beg.} & 1 \\ H & 2 \\ T & 3 \\ HH & 4 \\ HT & 5 \\ TT & 6 \\ TH & 7 \\ TTH & 8 \\ THH & 9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 \\
0 & & & .5 & .5 & 0 & 0 & 0 & 0 \\
0 & & & & & .5 & .5 & 0 & 0 \\
0 & & & & & .5 & & .5 & 0 \\
0 & & & & .5 & 0 & 0 & 0 & .5 \\
0 & & & & 0 & 0 & 0 & 1 & 0 \\
0 & & & & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

You can take 1st 7 linear eqn and solve

$\pi_i = P(\text{goes to state } 8 \text{ from state } i)$

$$\pi_1 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3$$

$$\pi_2 = \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5$$

$$\pi_3 = \frac{1}{2}\pi_6 + \frac{1}{2}\pi_7$$

$$\pi_4 = \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5$$

$$\pi_5 = \frac{1}{2}\pi_6 + \frac{1}{2}\pi_7$$

$$\pi_6 = \frac{1}{2}\pi_6 + \frac{1}{2}\cdot 1$$

$$\pi_7 = \frac{1}{2}\pi_5 + \frac{1}{2}\,0$$

$\pi_0 \ldots \pi_7$

$$= \left(\frac{2}{3}\ \frac{2}{3}\ \frac{2}{3}\ \frac{2}{3}\ \frac{2}{3},\ 1,\ \frac{1}{3}\right)$$

$\pi_0 = \frac{2}{3}$ (not $\frac{1}{2}$) $\cancel{\$}$.

TTH more likely than THH.

why?