Ch 2: Probability

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Textbook: Wackerly, Mendenhall, and Scheaffer 7e (2008)
(August 30, 2017)

1

Preliminaries

[ToC]

1.1 Interpretation of Probability (2.2)

[ToC]

Roll a die:

$$P\Big(\text{get }\#\ 3\Big) = \frac{1}{6}$$

- What does this mean?
- When do we actually see $1/6$ in our real-life dice-rolling?

1.2 Interpretation of Probability

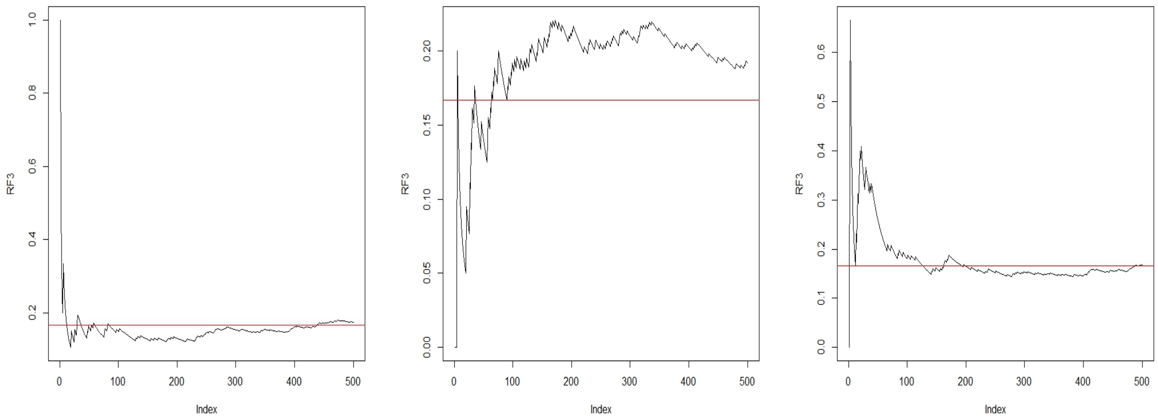
[ToC]

- Relative frequeny gets closer and closer to probability as number of trial increases.

$$[\text{Relative Frequency}] \qquad \Rightarrow \qquad [\text{Probability}] \qquad \text{as } n \to \infty \ .$$

$$\frac{\{\text{num of times the die shows 3}\}}{\{\text{number of rolls}\}} \quad \Rightarrow \quad [\text{Probability}]$$

Frequentist interpretation of Probability



```
n=500

Bin <- numeric(0)
for (i in 1:n){
  A  <- sample(1:6, size=1, replace=TRUE)  # roll a die once
  X  <- (A==3)                           # 1 for yes 0 for no
  Bin[i]  <- X
}

RF3 <- cumsum(Bin)/(1:n)
plot( RF3, type="l")
abline(h=1/6, col="red")
```

How Relative Frequency Converges to Probability

```
RF3at500  <- numeric(0)
RF3at1000 <- numeric(0)
for (j in 1:500) {
  n=1000

  Bin <- numeric(0)
  for (i in 1:n){
    A  <- sample(1:6, size=1, replace=TRUE)  # roll a die once
    X  <- (A==3)                           # 1 for yes 0 for no
    Bin[i]  <- X
  }

  #plot(Bin, type="l")
  #plot(cumsum(Bin))

  RF3 <- cumsum(Bin)/(1:n)
  #plot( RF3, type="l")
  #abline(h=1/6, col="red")

  RF3at500[j]  <- RF3[500]
  RF3at1000[j] <- RF3[1000]
}
hist(RF3at500, xlim=c(0,1)) ; abline(v=1/6, col="red")
hist(RF3at10000, xlim=c(0,1)) ; abline(v=1/6, col="red")
```

Sample Space and Events

- **Experiment** is any action or process whose outcome is subject to uncertainity.
  (e.g. roll a die)

- **Sample Space** of an experiment is **a set of all possible outcomes**.
  (e.g. $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$)

- **Event** is any subset of the sample space $\mathcal{S}$.
  (e.g. {1,2,3z})

Example: Throw a die

- sample space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.
- an event $A =$(number less than 4) $=$
- an event $B =$(number is odd) $=$
- What is

$$P(A) = ?$$

1.3 When each outcome is equally likely

[ToC]

If each element in $\mathcal{S}$ is equally likely, then for any event A,

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \mathcal{S}}$$

1.4 Examples: Flip a coin twice

[ToC]

Let X =(number times you get head).

- How do you write $\mathcal{S}$?
- What is $P(X = 2)$?

Example: Sum of two dice

Roll two dice, then sum the two number. Let X be the sum.

- How do you write $\mathcal{S}$?
- What is the most likely outcome ?

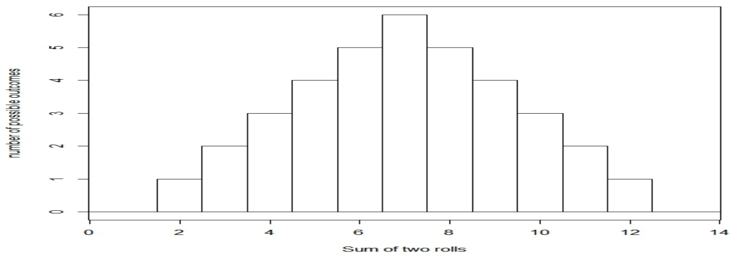
Write $\mathcal{S}$ in a form (First Throw, Second Throw):

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$$P(X = 7) = ?$$

$$P(X < 4) = ?$$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	**(1,6)**
(2,1)	(2,2)	(2,3)	(2,4)	**(2,5)**	(2,6)
(3,1)	(3,2)	(3,3)	**(3,4)**	(3,5)	(3,6)
(4,1)	(4,2)	**(4,3)**	(4,4)	(4,5)	(4,6)
(5,1)	**(5,2)**	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)



What if X =minimum of two numbers

$$P(X > 4) = ?$$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example: Kids next door

- Family just moved in next door. You're told that they have two kids.
- Next morning, you saw one of them are a girl.
- Now what is P(both of the kids are girls)?

Example: Three Cards

- There are three cards in a bag. Front and back of the cards are identical.
- One card has two side in red. One card has two sides in green. One card has one side red and one side green.
- Randomly, one card is chosen. (Bag is large compared to the cards)
- Chosen card is green on the side that is facing you. What is the probability that the other side is also green?

2

Calculating Probability through Set Operatons

[ToC]

2.1 Set Operations (2.3)

[ToC]

$$
\begin{aligned}
\mathcal{S} &= \{1,2,3,4,5,6,7,8\}, \\
C_1 &= \{1,3,5,7\}, \qquad C_2 = \{2,4,5\} \qquad C_3 = \{6,8\}
\end{aligned}
$$

then

$$
\begin{aligned}
\text{Union: } C_1 \cup C_2 &= \{1,2,3,4,5,7\} \\
\text{Intersection: } C_1 \cap C_2 &= \{5\} \\
\text{Complement: } C_2^c &= \{1,3,6,7,8\} \\
\text{Disjoint if } C_1 \cap C_3 &= \{\varnothing\} \\
\text{Exhaustive if } C_1 \cup C_2 \cup C_3 &= \mathcal{S}
\end{aligned}
$$

Distributive law of unions and intersections

Unions and intersections can be distributed:

$$\begin{aligned} C_1 \cap (C_2 \cup C_3) &= (C_1 \cap C_2) \cup (C_1 \cap C_3) \\ C_1 \cup (C_2 \cap C_3) &= (C_1 \cup C_2) \cap (C_1 \cup C_3) \end{aligned}$$

DeMorgan's law

Distributive law of complement over union or intersection

$$(C_1 \cap C_2)^c = C_1^c \cup C_2^c$$
$$(C_1 \cup C_2)^c = C_1^c \cap C_2^c$$

$$(C_1 \cap C_2 \cap C_3)^c = C_1^c \cup C_2^c \cup C_3^c$$
$$(C_1 \cup C_2 \cup C_3)^c = C_1^c \cap C_2^c \cap C_3^c$$

$$\left(\cap_{k=1}^{\infty} C_k \right)^c = \cup_{k=1}^{\infty} C_k^c$$
$$\left(\cup_{k=1}^{\infty} C_k \right)^c = \cap_{k=1}^{\infty} C_k^c$$

2.2 Calculating Probability through Set Operatons (2.4)

[ToC]

Axioms of probability (p.30)

P is a probability set function if

1. $P(C_1) \geq 0$, for all event $C_1 \in \mathcal{B}$.
2. $P(\mathcal{S}) = 1$.
3. If C_n is a sequence of disjoint events in $\mathcal{B}$, then

$$P\Big(\cup_{n=1}^{\infty} C_n\Big) = \sum_{n=1}^{\infty} P(C_n)$$

2.3 Probability Formulas

[ToC]

1. $P(A^c) = 1 - P(A)$
2. $P(A) = P(A \cap B) + P(A \cap B^c)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Inclusion-Exclusion
4. $(C_1 \cap C_2)^c = C_1^c \cup C_2^c$ DeMorgan's law
5. $(C_1 \cup C_2)^c = C_1^c \cap C_2^c$

Formula 1

$$P(A^c) = 1 - P(A)$$

Formula 2

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

Formula 3 (Inclusion-Exclusion)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Inclusion-Exclusion Formula Extended

- two events

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

- three events

$$\begin{aligned} P(C_1 \cup C_2 \cup C_3) &= P(C_1) + P(C_2) + P(C_3) \\ &\quad -P(C_1 \cap C_2) - P(C_2 \cap C_3) - P(C_1 \cap C_3) \\ &\quad +P(C_1 \cap C_2 \cap C_3) \end{aligned}$$

- four events

$$\ldots.$$

2.4 Examples: Project Funding

[ToC]

There are 3 projects that has applied for the grant. Let A_i represent an event that project i gets funded. There are 3 projects. Given

$$P(A_1) = .22, \quad P(A_2) = .25, \quad P(A_3) = .28$$

and

$$\begin{aligned} P(A_1 \cap A_2) &= .11, \qquad P(A_1 \cap A_3) = .05, \\ P(A_2 \cap A_3) &= .07, \qquad P(A_1 \cap A_2 \cap A_3) = 0.01, \end{aligned}$$

Calculate the probability of :

1. $P($ At least one of project 1 and 2 get award $)$

Example: Project Funding

1. $P($ At least one of project 1 and 2 get award $)$

2 P(Neither project 1 nor 2 get award)

3 P(At least one of 3 project gets award)

4 $P($ None of the project get award $)$

5 $P($ Only project 3 is awarded $)$

1 $P($ At least one of project 1 and 2 get award $)$

$$= \quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = .36$$

2 $P($Neither project 1 nor 2 get award$)$

$$= \quad P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2) = .64$$

3 $P($ At least one of 3 project gets award $)$

$$\begin{aligned} = \quad & P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) \\ & -P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = .53 \end{aligned}$$

4 $P($ None of the project get award $)$

$$= \quad P(A_1' \cap A_2' \cap A_3') = P\Big((A_1 \cup A_2 \cup A_3)'\Big) = 1 - P\Big((A_1 \cup A_2 \cup A_3)\Big) = .47$$

5 $P($ Only project 3 is awarded $)$

$$= \quad P(A_3) - P(A_3 \cap A_1) - P(A_3 \cap A_2) + P(A_1 \cap A_2 \cap A_3)$$

$$P\Big((A_1' \cap A_2') \cup A_3\Big) = P(A_3) + P(A_1' \cap A_2' \cap A_3') = .75$$

Example: Lab work vs Referral

- The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%.
- Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.
- Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

Example

- An insurer offers a health plan to the employees of a large company.
- As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage.
- The proportions of the company's employees that choose coverages A, B, and C are $1/4$, $1/3$, $5/12$, respectively.
- Determine the probability that a randomly chosen employee will choose no supplementary coverage.

Example: Insurance Classification

An auto insurance company has 10,000 policyholders. Each policyholder is classified as

1. young or old
2. male or female
3. married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

Example: TV watching

A survey of a group's viewing habits over the last year revealed the following information:

- 28% watched gymnastics
- 29% watched baseball
- 19% watched soccer
- 14% watched gymnastics and baseball
- 12% watched baseball and soccer
- 10% watched gymnastics and soccer
- 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

3

Counting Techniques (2.6)

[ToC]

3.1 Counting Formulas

[ToC]

select k out of n Binomial coefficient

	without replacement	with replacement
ordered	$\frac{n!}{(n-k)!}$	n^k
not ordered	$\binom{n}{k}$	$\binom{n+k-1}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Counting Formulas (Orderd, without Replacement)

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make?

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different sequences can you make with only using 4 cards?

Counting Formulas 1

- When you have n subjects, there are $n!$ ways to order.
- When you have k subjects out of n subjects, there are $n!/(n-k)!$ ways to order.

Counting Formulas

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different groups can you make with 3 cards?

Counting Formulas (Not ordered, without Replacement)

Example: If you have 6 cards labeled A, B, C, D, E, F, how many different groups can you make with 3 cards?

- When you choose k subjects out of n, without regard to order, there are

$$\binom{n}{k} = \frac{n!}{(n-k)!\ \ k!}$$

possible combinations.

- This is read as "n choose k".

- Some calculater write this as ${}_nC_k$

Binomial Coefficient

- Binomial Coefficient:

$$\binom{n}{k} = \frac{n!}{(n-k)! \ \ k!}$$

- Binomial Expansion:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- Binomial Tree

- Can you expand $(x+y)^7$?

3.2 Counting Formulas (Without Replacement)

[ToC]

1. n subjects

$$n! \quad \text{sequences}$$

2. Use k out of n subjects,

$$\frac{n!}{(n-k)!} = {}_nP_k \quad \text{permutations}$$

3. Choose k subjects out of n, without regard to order,

$$\binom{n}{k} = \frac{n!}{(n-k)!\ \ k!} = {}_nC_k \quad \text{groups}$$

3.3 Example: Batting Orders

[ToC]

---

1. There are 9 players in a baseball team. How many different batting orders are possible?

2. What if you have 15 players ? (only 9 can play)

3.4 Example: Batting Orders

3 What if there are 3 pichers(have to bat 9th) 5 sluggers (have to bat clean up (3rd, 4th, 5th) and 7 players?

4 9 players including John, Jake and Emily. How many ways that John, Jake and Emily will hit in that order?

5 What about three of them bat in a row?

$$9! = 362,880 \text{ orders}$$

$$15!/6! = 1,816,214,400 \text{ orders}$$

$(7!(5!/(5-3)!(3) = 907,200$ orders.)

Example: Boys and Girls

1. There are 5 boys and 5 girls. If they have to sit in a line, how many ways are there?

2. If no two boys and no two girls can sit together, how many ways are there?

3. If all boys have to sit together, how many ways are there?

Exercise: 20 people in a party

If everybody shakes hand with everybody, how many handshakes occur?

Exercise: Cards

How many different sequence can you make with cards $AAABBCCCC$?

Exercise: Binomial Expansion

When you expand $(2x^2 + y)^5$, what is the coefficient for term with x^6y^2?

Exercise: Kids and Gifts

Seven different gifts are distributed among 10 kids. One kid can't get more than 1 gift. How many different ways?

Counting Formulas (Ordered, with Replacement)

Example: PIN number is made of 4 digit number of 0-9. How may possible PINs are there?

Example: Password for a website must be 6 characters long, and for each character, you can use any of alphabet and number. How many different password can you make?

Counting Formulas (Not Ordered, with Replacement)

There are 7 spices A,B,C,D,E,F,G. Suppose you randomly pick 5 pinches of spice, out of 7 with with replacement. How may different combinations can you make?

5 stars, 7-1 dividers

$* \quad * \quad * \quad * \quad *$

$* \quad * \quad * \quad * \quad *$

$* \quad * \quad * \quad * \quad *$

$* \quad * \quad * \quad * \quad *$

$* \quad * \quad * \quad * \quad *$

Counting Formulas

select k out of n Binomial coefficient

	without replacement	with replacement
ordered	$\frac{n!}{(n-k)!}$	n^k
not ordered	$\binom{n}{k}$	$\binom{n+k-1}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

4

Calculating Probability through Sample Points (2.5)

[ToC]

4.1 Example: Fake or Real

[ToC]

- A box contains 35 gems, of which 10 are real diamonds and 25 are fake diamonds.
- Gems are randomly taken out of the box, one at a time without replacement for two times.
- What is the probability that exactly 2 fakes are selected?

Exercise: Three Kinds in a Box

A box contains four 40w bulbs, five 60w bulbs, and six 75w bulbs. Out of 15 total bulbs, 3 are selected randomly at once.

- $$\begin{aligned} P(\text{ exactly two 75w}) &= \\ P(\text{ same rating}) &= \\ P(\text{ one from each rating}) &= \end{aligned}$$

- We must calculate this as

$$P(A) = \frac{\text{number of ways in event A}}{\text{number of total ways}}$$

Exercise: Three Kinds of Light Bulbs in a Box

A box contains four 40w bulbs, five 60w bulbs, and six 75w bulbs. Three bulbs are selected at once in random. What is $P($ exactly two 75w$)$?

1. We are calculating WITHOUT order:

$$\begin{aligned}
T &= \text{Total number of outcomes} = \binom{15}{3} \\
P(\text{ exactly two 75w}) &= \binom{6}{2}\binom{9}{1} / T. \\
P(\text{ same rating}) &= \binom{4}{3} + \binom{5}{3} + \binom{6}{3} / T. \\
P(\text{ one from each rating}) &= \binom{4}{1}\binom{5}{1}\binom{6}{1} / T.
\end{aligned}$$

Example: Birthday Probability

Say there are 30 people in the class. Assume distribution of Birthdays are even, and independent.

$P($ At least one guy has same Bday as you $) =$

$P($ At least one pair of same Bday $) =$

$$
\begin{aligned}
P(\text{ at least one guy has same Bday as you }) &= 1 - P(\text{ nobody has same Bday as you }) \\
&= 1 - (364/365)^{30} = 0.079
\end{aligned}
$$

$$
\begin{aligned}
P(\text{ at least one pair of same Bday }) &= 1 - P(\text{ nobody has same Bday as anybody }) \\
&= 1 - \frac{364}{365}\frac{363}{365}\cdots\frac{335}{365} = 1 - (\frac{1}{365})^{30}\frac{365!}{(365-30)!} = .7
\end{aligned}
$$

For 23 people, it is around 50%

Example: Throw a fair coin n times

What is the number of ways you can get x heads?

$$
\begin{aligned}
P(\text{ filp coin 5 times get 2 heads }) &= \frac{\#\text{ of outcome for getting 2 heads}}{\#\text{ of outcome in } S} \\
&= \frac{(HHTTT), (HTHTT), (HTTHT), (HTTTH), \ldots)}{2^5} \\
&= \binom{5}{2}\frac{1}{2^5}
\end{aligned}
$$

In general,

$$P(\text{ filp fair coin } n \text{ times get } x \text{ heads }) = \binom{n}{x}\frac{1}{2^n}$$

Example: Poker Hands

In a five-card poker, Total possible hands are $T = \binom{52}{5} = 2598960$.

$$P(\text{ 4 of a kind}) =$$

$$P(\text{ 3 of a kind}) =$$

$$P(\text{ full house}) =$$

$$P(\text{ one pair}) =$$

$$T = \binom{52}{5} = 2598960.$$

$$P(\text{ 4 of a kind}) = AAAAX \times 13 = \frac{(4\cdot 3\cdot 2\cdot 1/4!)(48)}{T} \times 13 = \frac{624}{T} = .000240$$

$$\begin{aligned} P(\text{ 3 of a kind}) &= AAAXY \times 13 = \frac{(4\cdot 3\cdot 2/3!)(48\cdot 44/2!)}{T} \times 13 \\ &= \binom{13}{3}\binom{3}{1}\binom{4}{3}\binom{4}{1}\binom{4}{1}/T = \frac{54912}{T} = .0211 \end{aligned}$$

Watch out for the 2 for permutation of XY.

$$P(\text{ full house}) = AAABB \times 13 = \binom{4}{3}\frac{(48)(3)}{2} \times 13 = \binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2}/T = 3744/T = .0014$$

$$P(\text{ one pair}) = AAXYZ \times 13 = \binom{4}{2}\frac{(48)(44)(40)}{3!} \times 13 = \binom{13}{4}\binom{4}{1}\binom{4}{2}\binom{4}{1}^3/T = 1098240/T = .423$$

$$P(\text{ No hand}) = \binom{13}{5}\binom{4}{1}^5 = \frac{(52)(48)(44)(40)(36)}{5!} - \text{ flush and straight}$$

a straight consists of five cards with consecutive numbers of any suit. Now, for straight flash, we have 10 choices for a number of the high card $(A, 13, 12, 11, \ldots, 6, 5)$, and 4 choices for the suit. Once you pick the number and suit for the high card, we don't have any more choices:

5

Conditional Probability and Independence

[ToC]

5.1 Conditional Probability(2.7)

[ToC]

- Probability of A given B.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Only defined when $P(B) > 0$.
- Can also be written in **multiplication form**

$$P(A \cap B) = P(A|B)P(B)$$

Example:

	Defective	non defective	total
factory A	13	100	113
factory B	4	15	19

You select a product randomly

- How should the table look like if they are independent?
- $P(Defective)$
- $P(Defective|A)$

Property

- $P(A|B) \geq 0$.
- If $A_1, A_2, \ldots$ are disjoint,

$$P\Big(\cup_{j=1}^{\infty} A_j | B \Big) = \sum_{i=2}^{\infty} P(A_j|B) \qquad \text{(countable additivity)}$$

- $P(A|A) = 1$.
- $P(A|B)$ is just another probability within the space of event B. All formulas will apply.

  1. $P(A^c|B) + P(A|B) = 1$
  2. $P(A|B) = P(A \cap C|B) + P(A \cap C^C|B)$
  3. $P(A \cap C|B) = P(A|B) + P(C|B) - P(A \cap C|B)$

Example: Red and White

- A box contains 4 red balls and 6 white balls.
- A sample of size 3 is drawn without replacement from the box.
- What is the probability of obtaining 1 red ball and 2 white balls, given that at least 2 of the balls in the sample are white?

Example:

- Let A, B, C and D be events such that $B = A^C$, $C \cap D = \varnothing$.
- $P[A] = 1/4, \quad P[B] = 3/4$
- $P[C \big| A] = 1/2, \quad P[C \big| B] = 1/4,$
- $P[D \big| A] = 1/4, \quad P[D \big| B] = 1/8.$
- Calculate $P[C \cup D]$.

Example: Assembly Lines

- Two assembly lines I and II have the same rate of defectives in their production of voltage regulators.
- Five regulators are sampled from each line and tested.
- Among the total of ten tested regulators, four are defective. Find the probability that exactly two of the defective regulators came from line I.

5.2 Independence

[ToC]

- Two events A and B are independent if

$$P(A|B) = P(A), \qquad \text{or} \qquad P(A \cap B) = P(A) \cdot P(B).$$

  Events are said to be dependent otherwise.

- This turns Inclusion-Exclusion formula to:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \end{aligned}$$

- Mutually exclusive events cannot be independent.

If events are independent

- $P(A \cap B) = P(A)P(B)$
- $P(A \cup B)$
- Events A^c and B are also independent.

Exercise: Aircraft Seam

An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are independent of each other.

If only 1% of all rivets needs to be reworked, what is the probability that a seam needs to be reworked.

Exercise: Aircraft Seam

- An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are independent of each other.
- If only 1% of all rivets needs to be reworked, what is the probability that a seam needs to be reworked.

$$P(\text{ a seam needs rework}) =$$

$$1 - (.99)^{25} = .222$$
$$1 - (.999)^{25} = 0.025$$

Example:

- Let A, B and C be mutually independent events.
- $P(A) = .5, \quad P(B) = .6 \quad \text{and} \quad P(C) = .1.$
- Calculate $P(A^C \cup B^C \cup C)$.

Example: Auto Insurance

An actuary studying the insurance preferences of automobile owners makes the following conclusions:

- An automobile owner is twice as likely to purchase collision coverage as disability coverage.
- The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
- The probability that an automobile owner purchases both collision and disability coverages is 0.15.

What is the probability that an automobile owner purchases neither collision nor disability coverage?

6

Law of Total Probability

[ToC]

6.1 The Law of Total Probability

[ToC]

- Recall formula:

$$P(B \cap A) = P(B|A)P(A).$$

Then for event B, can be written using formula #2,

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B|A)P(A) + P(B|A')P(A') \end{aligned}$$

- Instead of A, A', if A_1, A_2, A_3 are mutually exclusive and exhaustive events, we can write

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$

6.2 Bayes' theorem

[ToC]

- Bayes' formula says:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')} \end{aligned}$$

Example: Red and White balls in Two Urns

- There are two urns, urn A and urn B.
- Urn A contains 5 red balls, 2 white.
- Urn B contains 3 red balls, 4 white.
- Fair coin flip decides which urn to be used.
- Somebody flip a coin, and drew one ball from an urn. You don't know which urn was used. Ball dran was red.
- What is the probability that urn A was used?

Example: Testing for Disease

- 1 in 1000 adults is afflicted with this disease.
- Test for this disease is 99% accurate on infected patients.
- Test is 98% accurate on non-infected patients.
- If test comes back positive, what is the chance that you are actually infected?

$$P(\text{Infected}) =$$
$$P(\text{Pos}|\text{Infected}) =$$
$$P(\text{Pos}|\text{Not Infected}) =$$

Example: Testing for Disease

- 1 in 1000 adults is afflicted with this disease.
- Test for this disease is 99% accurate on infected patients.
- Test is 98% accurate on non-infected patients.
- If test comes back positive, what is the chance that you are actually infected?

$$P(\text{Infected} \mid \text{Pos}) = ?$$

$$P(\text{Infected}) = 0.001, \quad P(\text{Pos}|\text{Infected}) = .99, \quad P(\text{Pos}|\text{Not Infected}) = .02$$

- Using the Baye's theorem,

$$P(I|Pos) = \frac{P(I \cap Pos)}{P(Pos)} = \frac{P(Pos|I)P(I)}{P(Pos|I)P(I) + P(Pos|I')P(I')}$$

$$= \frac{(.99)(.001)}{(.99)(.001) + (.02)(.999)} = 0.0472$$

So if your test comes back positive, you have only 5% chance of having the disease.

- On the other hand, If the test comes back negative,

$$P(I'|B') = \frac{P(I' \cap P')}{P(P')} = \frac{P(P'|I')P(I')}{P(P'|I')P(I') + P(P'|I)P(I)}$$

$$= \frac{(.98)(.999)}{(.98)(.999) + (.01)(.001)} = 0.999989$$

If your test comes back negative, you probably don't have the disease.

6.3 Law of total probability (2.8)

[ToC]

- Partition the event C_1.

$$P(C_1) = P(C_1 \cap A) + P(C_1 \cap A^C).$$

(from which axiom does this follow?)

- Use the multiplication form to get

$$P(C_1) = P(C_1|A)P(A) + P(C_2|A^c)P(A^C)$$

- If B_1, B_2, B_3, B_4 are disjoint and exhaustive,

$$P(C_1) = \sum_{i=1}^{4} P(C_1|B_i)P(B_i)$$

Example: Republicans vs Democrats

Suppose in a certain state, 40% of population are Republicans, and 60% are Democrats. If 70% of Rep and 80% of Democrats are in favor of issuing Bonds, what is the total proportion of votors that are in favor of issuing Bonds?

Example:

- Two bowls each contain 5 black and 5 white balls.
- A ball is chosen at random from bowl 1 and put into bowl 2.
- A ball is then chosen at random from bowl 2 and put into bowl 1.
- Find the probability that bowl 1 still has 5 black and 5 white balls.

Ex: Testing Positive

Suppose we are concerned with a certain type of disease, who is believed to be infecting 1 in every 1000 people. Initial saliva testing method is 99% accurate in testing infected patients, and 98% accurate in testing non-infected patients.

- What is the probability that given a positive result from the test, patient is actually infected?

6.4 Bayes' Formula (2.9)

[ToC]

- with exhaustive events $B_1, B_2, \ldots, B_k$,

$$P(B_1|C_1) = \frac{P(C_1 \cap B_1)}{P(C_1)}$$

$$= \frac{P(C_1|B_1)P(B_1)}{P(C_1)}$$

$$= \frac{P(C_1|B_1)P(B_1)}{\sum_{i=1}^{k} P(C_1|B_i)P(B_i)}$$

6.5 Example: Testing Positive

[ToC]

Suppose more expensive blood test is 99.9% accurate in testing infected patients, and 99.8% accurate in testing non-infected patients. However, due to its cost, only patients who has tested positive in the first saliva test will be tested with blood test.

- What is the probability that given a positive result from both the saliva test, and blood test, the patient is actually infected?

Ex: Driver Class

Suppose all drivers are divided into three categories A,B, and C. It is known that among all drivers, 50% of them are class A driver, 30% are class B driver, and 20% are class C driver. Probabilities for numbers of accidents each driver have in a year are given below:

	0	1	2	3+
A	.7	.15	.1	.05
B	.5	.25	.15	.1
C	.3	.3	.25	.15

A driver had one accident last year. Given this information, what is the probability he is an class A driver? How about probability that he is an class B driver? Class C driver?

M = (.15*.5) + (.25*.3) + (.3*.2)

(.15*.5) / M = .36 (.25*.3) / M = .36 (.3*.2) / M = .29

Example:

A study of automobile accidents produced the following data:

Model Year	Propmtion of all vehhicles	Prob. of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

Example:

An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	.15
Young Adult	16%	.08
Midlife	45%	.04
Senior	31%	.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

Philosophical differences

- Probability that a driver is from class A, when you randomly choose a driver from a pool of drivers who have two one-accident years in a row.
- Randomly choose a driver from entire pool of drivers, then find out that he had two one-accident years in a row. Probability that he is from class A.