Monte Carlo Integration

$$E[X] = \int_{\text{all } x} x\, f(x)\, dx$$

If $X_i \sim f(x)$ is easy to generate, then

we can use

$\bar{X}$ as estimate for $E(X)$.

If $X_1, \ldots, X_n$ are R.S. from $f(x)$.

95% CI for $E(X)$

$$\bar{X} \pm 1.96 \sqrt{\frac{Var(X)}{n}}$$

we can use sample variance from $\{X_1, \ldots, X_n\}$ as estimate for $Var(X)$.

Ex 1.

$$\int_0^1 e^{-x}\,dx$$

See this as

$$E(e^{-U}) = \int_0^1 e^{-x}\cdot 1\,dx$$

when $U \sim \text{Unif}(0,1)$.

Then we can use Monte Carlo simulation to estimate

$$E(e^{-U}),\quad U \sim U(0,1)$$

MC Integration

① Generate $U_1, \ldots, U_n \sim U(0,1)$

② let $X_i = e^{-U_i}$

③ 95 % CI for $E(X)$ is

$$\bar{X} \pm 1.96 \sqrt{\frac{\widehat{Var}(X)}{n}}$$

$\widehat{Var}(X)$ = sample variance of $\{X_1, \ldots, X_n\}$

Ex 2

$$\int_0^5 e^{-x} \, dx$$

See it as

$$5 \int_0^5 e^{-x} \cdot \frac{1}{5} \cdot dx = 5 E\left[e^{-U}\right]$$

$U \sim \text{Unif}(0, 5)$

Method to accelerate

MC simulation of $E(\quad)$

Antithetic Variates

Let X, Y have same distribution.

$$E(X) = E(Y) = E\left(\frac{X+Y}{2}\right).$$

If we use $\bar{X}$ to estimate $E(X)$,

$$\boxed{\begin{array}{l}\text{95 \% CI for } E(X), \\ \qquad \bar{X} \pm 1.96\sqrt{\dfrac{V(X)}{n}}\end{array}}$$

What if we use mean$\left(\frac{X_i+Y_i}{2}\right)$ to

estimate $E\left(\frac{X+Y}{2}\right)$?

If each X_i+Y_i are independent for different i,

$\boxed{95\%\ \text{CI for } E\left(\frac{X+Y}{2}\right)}$

$$\boxed{\text{mean}\left(\frac{X_i+Y_i}{2}\right) \pm 1.96\sqrt{\frac{Var\left(\frac{X_i+Y_i}{2}\right)}{n}}}$$

same as before, but ...

$$\mathrm{Var}\left(\frac{X_i+Y_i}{2}\right) \quad \text{is estimating} \quad \mathrm{Var}\left(\frac{X+Y}{2}\right).$$

$$V\left(\frac{X+Y}{2}\right) = \frac{1}{4} V(X+Y)$$

$$= \frac{1}{4}\left\{ V(X) + V(Y) + \underline{2\,\mathrm{Cov}(X,Y)} \right\},$$

same.

$$= \frac{V(X)}{2} + \frac{1}{2}\mathrm{Cov}(X,Y).$$

If, X_i, Y_i are independent (for same i),

$Cov(X, Y) = 0$,

$$V\left(\frac{X+Y}{2}\right) = \frac{V(X)}{2}.$$

So

$n = 10{,}000$ X only.

$n = 5000$ each. X and Y indep.

$$\pm 1.96\sqrt{\frac{V(X)}{10{,}000}} = \pm 1.96\sqrt{\frac{\frac{V(X)}{2}}{5000}}$$

Same Margin of Error.

If X_i and Y_i are negatively correlated,

$$V\left(\frac{X+Y}{2}\right) = \frac{V(X)}{2} + \frac{1}{2}\underbrace{Cov(X,Y)}_{\text{negative!}}$$

$n = 10{,}000$ X only

$n = 5000$ each X, Y neg. corr.

$$1.96\sqrt{\frac{V(X)}{10{,}000}} > 1.96\sqrt{\frac{\frac{V(X)+Cov(X,Y)}{2}}{5000}}$$

Transformation of ~~Negatively~~ Correlated r.v.

Thm

If X, Y are neg. correlated, then

$g(X), g(Y)$ are also neg. correlated

if $g(\cdot)$ is monotone function.

what is neg. correlated with $U \sim UNIF(0,1)$

and still have $UNIF(0,1)$ distribution?

let $V = 1-U$. $V \sim U(0,1)$.

$$
\begin{aligned}
Corr.(U,V) &= Corr(U, 1-U) \\
&= Corr(U, 1) + Corr(U, -U) \\
&= 0 - Corr(U,U) \\
&= -1.
\end{aligned}
$$

Ex.

$$\int_0^1 e^{-x}\,dx = E\left[e^{-U}\right]$$ where $U \sim UNIF(0,1)$.

① generate $U_1, \dots, U_n \sim U(0,1)$.

$\boxed{V_i = 1 - U_i} \xrightarrow{\text{Compute}} V_1, \dots, V_n \sim U(0,1)$

negatively correlated.

② then let $X_i = e^{-U_i}$

$Y_i = e^{-V_i}$

negatively correlated.

③ CI for $E(X) = E\left(\frac{X+Y}{2}\right)$

$$\boxed{\text{mean}\left(\frac{X_i+Y_i}{2}\right) \pm 1.96\sqrt{\frac{Var\left(\frac{X_i+Y_i}{2}\right)}{n}}}$$