Branching Process

(Ross 4.7)

Population Modeling :

X_0	X_1	X_2	......
initial pop.	Gen. 1	Gen. 2	

Each individual has

$$P(\text{reproduce } j \text{ offspring}) = P_j$$

If $X_k = 0$, then no one left. ~~then~~

so $X_{k+1} = 0$.

$\Rightarrow P_{00} = 1$ state 0 is recurrent.

If $P_0 > 0$, then

$$P_{i0} = P(\text{all of } i \text{ people have } 0 \text{ offspring})$$

$$= P_0^i > 0.$$

Since this is same as

$$P(\text{go from } i \text{ to } 0 \text{ and never come back}) > 0.$$

All state i are transient.

$\Rightarrow$ This process either die out or diverge.

$$
\begin{array}{ccccccc}
\underline{X_0} & & & \underline{X_1} & & & \underline{X_2} \\
5 & \left.\begin{array}{l} Z_1 = 3 \\ Z_2 = 2 \\ \vdots \\ Z_5 = 1 \end{array}\right\} & \xrightarrow[\text{add}]{} & 11 & \left.\begin{array}{l} Z_1 = 0 \\ Z_2 = 1 \\ \vdots \\ Z_{11} = 4 \end{array}\right\} & \longrightarrow & 17 \quad \cdots\cdots
\end{array}
$$

$P(Z_i = j) = P_j$. i.i.d.

(same for all individual.)
(independent ")
(same for each generation).

$P_0 > 0$. it is possible to have no offspring.

distribution

$$Z_k \sim \text{?} \Rightarrow P_0, P_1, P_2, \ldots$$

$$X_n = \sum_{k=1}^{X_{n-1}} Z_k$$

$$\mu = E(Z_k)$$

$$\sigma^2 = V(Z_k)$$

What are $E(X_n)$ and $V(X_n)$?

$$E(X_n) = E\left(\sum_{k=1}^{X_{n-1}} Z_k \right)$$

$$= E\left(E\left(\sum_{k=1}^{X_{n-1}} Z_k \,\middle|\, X_{n-1} = i \right) \right)$$

$$= E\left(\sum_{k=1}^{i} E(Z_k) \right)$$

$$= E(i\mu)$$

$$= \mu \, E(i) = \mu E(X_{n-1}).$$

If $X_0 = a$,

$$E(X_0) = a$$

$$E(X_1) = a\mu$$

$$E(X_2) = a\mu^2$$

$$\vdots$$

$$\boxed{E(X_n) = a\mu^n}.$$

$$\boxed{E(X_n) = \mu E(X_{n-1})}$$

$$\mathrm{Var}(X_n) = E\left[\mathrm{Var}(X_n \mid X_{n-1})\right] + \mathrm{Var}\left[E(X_n \mid X_{n-1})\right]$$

$$\begin{cases} E(X_n \mid X_{n-1}) = E\left(\sum_{k=1}^{\overset{\text{not random}}{X_{n-1}}} Z_k\right) = X_{n-1}\,\mu \\[2em] \mathrm{Var}(X_n \mid X_{n-1}) = \mathrm{Var}\left(\sum_{k=1}^{\overset{\text{not random}}{X_{n-1}}} Z_k\right) = X_{n-1}\,\mathrm{Var}(Z_k) = X_{n-1}\,\sigma^2 . \end{cases}$$

$$\begin{cases} E(X_{n-1}\,\sigma^2) = \sigma^2 E(X_{n-1}) \\[2em] V(X_{n-1}\,\mu) = \mu^2 V(X_{n-1}) . \end{cases}$$

$$\boxed{\mathrm{Var}(X_n) = \sigma^2 a\,\mu^{n-1} + \mu^2 V(X_{n-1}) .}$$

$$X_0 = a$$

$$\mathrm{Var}(X_0) = 0 ,$$

$$V(X_n) = a\sigma^2 \mu^{n-1} + \mu^2 V(X_{n-1})$$

$$\mathrm{Var}(X_1) = a\sigma^2 \mu^0$$

$$\mathrm{Var}(X_2) = a\sigma^2 \mu + a\sigma^2 \mu^2 = a\sigma^2(\mu + \mu^2)$$

$$\mathrm{Var}(X_3) = a\sigma^2 \mu^2 + a\sigma^2(\mu + \mu^2)\mu^2$$

$$= a\sigma^2 \mu^2 (1 + \mu + \mu^2)$$

$$\boxed{\mathrm{Var}(X_n) = \begin{cases} a\sigma^2 \mu^{n-1}\left(\dfrac{1-\mu^n}{1-\mu}\right) & \mu \neq 1 \\ n\sigma^2 a & \mu = 1 \end{cases}}$$

Prob. of Die Out.

$E(X_n) = a\mu^n$ if $\mu < 1$, then

$$\lim_{n\to\infty} E(X_n) = 0.$$

Then observe that

$$E(X_n) = \sum_{j=0}^{\infty} j\, P(X_n = j)$$

$$\geq \sum_{j=1}^{\infty} 1 \cdot P(X_n = j) = P(X_n \geq 1).$$

Therefore, if $\mu < 1$,

$$\lim_{n \to \infty} P(X_n \geq 1) = 0,$$

i.e. population <u>will</u> die out.

(if $\mu = 1$, this is still true).

If $\mu > 1$. (assume $X_0 = 1$).

$P(\text{Die out})$

$$= \sum_{j=0}^{\infty} \underbrace{P(\text{Die out} \mid X_1 = j)}_{\downarrow \; P(\text{Die out})^j} P_j$$

$$\boxed{P(\text{D.O.}) = \sum_{j=0}^{\infty} P(\text{D.O.})^j P_j}$$

$P(\text{D.O.})$ = smallest positive number satisfying above eq'n.