NHPP

intensity func.

$$N(t_2) - N(t_1) \sim NHPP(\lambda(t))$$

Then

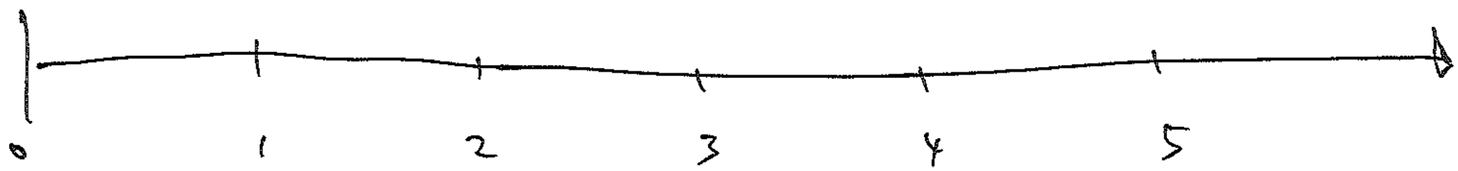
$$P(N(t_2) - N(t_1) = x) = \frac{g(t_1, t_2)^x \cdot e^{-g(t_1, t_2)}}{x!}$$

Poisson pmf

$$\frac{\lambda^x e^{-\lambda}}{x!}$$

Monte Carlo for NHPP

(Approximation)



$g(0,1)$ $g(1,2)$ $g(2,3)$ $g(3,4)$

$PP(g(0,1))$ for 1 day.

To perform kinetic MC, we need distribution of inter-arrival times.

let S_m = time of m th event.

then $S_m = \sum_{i=1}^{m} T_i$ $\qquad$ T_i = ith inter-arrival times.

$P(S_m < t)$ = P(m th event was before t)

= P(In time $0 \sim t$, there was more than m events)

$= P(N(t) \geq m)$

This is CDF of S_m.

$$F_{S_m}(t) = P(N(t) \geq m)$$

$$= \sum_{x=m}^{\infty} \frac{g(0,t)^{x} e^{-g(0,t)}}{x!}$$

(For HPP, $S_m \sim GAM(m, \frac{1}{\lambda})$ and $T_i \sim Exp(\lambda)$.)

Take $\frac{d}{dt}$ to obtain pdf of S_m.

$$f_{S_m}(t) = \sum_{x=m}^{\infty} \frac{x\, g(0,t)^{x-1} e^{-g(0,t)}}{x!} g'(0,t)$$

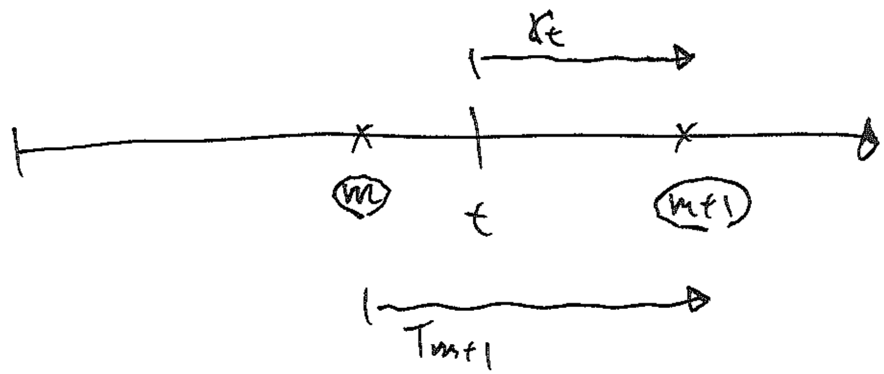
$$+ \sum_{x=m}^{\infty} \frac{g(0,t)^{x} e^{-g(0,t)} (-g'(0,t))}{x!}$$

All term cancell's except the 1st term on the left.

$$f_{S_m}(t) = \frac{g(0,t)^{m-1} e^{-g(0,t)}}{(m-1)!} g'(0,t).$$

Now we turn to

$$\gamma_t = \text{waiting time from } t \text{ to next event}.$$

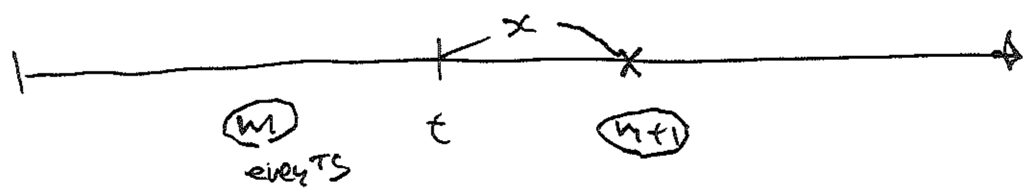


$m = 0, 1, 2, 3, \ldots$

For HPP, $T_{m+1} \sim \mathrm{Exp}(\lambda)$ and $\gamma_t \sim \mathrm{Exp}(\lambda)$ by memoryless.

$$P(\gamma_t > x) = P(T_1 > t + x)$$

$$+ \sum_{n=1}^{\infty} P(S_n \le t,\ S_{n+1} > t + x)$$



First term.

$$N(t+x) = 0$$

$$P(T_1 > t+x) = P(\quad)$$

$$= e^{-g(0,\,t+x)}$$

Second term.

$$\sum_{m=1}^{\infty} P\left(S_m \le t, \quad S_{m+1} > t+x \right)$$

$$= \sum_{m=1}^{\infty} \int_0^t P\left(S_{m+1} > t+x \mid S_m = y\right) \cdot f_{S_m}(y) \, dy$$

$$= \sum_{m=1}^{\infty} \int_0^t P\left(N(t+x) - N(y) = 0 \right) \cdot f_{S_m}(y) \, dy$$

$$= \sum_{m=1}^{\infty} \int_0^t e^{-g(y,\, t-x)} \left(\frac{g(0,y)^{m-1} \, e^{-g(0,y)} \, g'(0,y)}{(m-1)!} \right) dy$$

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a$$

~~since~~

$$= \int_0^t e^{-g(y,t-x)} \cdot e^{g(0,y)} \cdot e^{-g(0,y)} \cdot g'(0,y)\, dy$$

$$= \int_0^t g'(0,y)\, e^{-g(y,t-x)}\, dy$$

Combining, we get

$$P(X_t > x) = e^{-g(0,t+x)} + \int_0^t g'(0,y)\, e^{-g(y,t-x)}\, dy$$

Then we have CDF for γ_t,

$$F_{\gamma_t}(x) = P(\gamma_t \le x) = 1 - P(\gamma_t > x)$$

$$= 1 - e^{-g(0,t+x)} - \int_0^t g'(0,y)\, e^{-g(y,t+x)}\, dy .$$

Take $\frac{d}{dx}$ to get pdf,

$$f_{\gamma_t}(x) = g'(0,t+x)\, e^{-g(0,t+x)}$$

$$+ \int_0^t g'(0,y) \cdot g'(y,t+x) \cdot e^{-g(y,t+x)}\, dy$$

$$g'(a,b) = \frac{d}{dt} g(a,t)\Big|_{t=b}$$

Use Inverse method with $F_{\delta t}(t)$ or

Accept/Reject method with $f_{\delta t}(t)$ to

perform Kinetic Monte Carlo.

Ex.

$$\lambda(t) = e^{-\{bt\}} + a$$

intensity func.

$$g(t_1, t_2) = \int_{t_1}^{t_2} \lambda(u)\, du$$

$$= \frac{1}{b}\left[e^{-bt_1} - e^{-bt_2}\right] + a(t_2 - t_1)$$

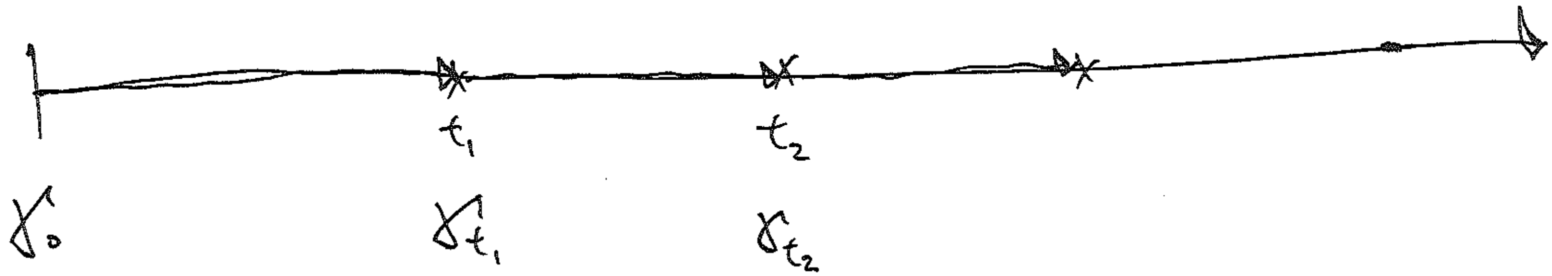
$$g'(0, t+x) = \frac{d}{dx} g(0, t+x)$$

$$= e^{-b(t+x)} + a$$

$$g'(0, y) = \frac{d}{dy} g(0, y) = e^{-b(y)} + a$$

$$g'(y, t+x) = \frac{d}{dx} g(y, t+x) = e^{-b(t+x)} + a$$

Kinetic Monte Carlo for NHPP



Each time γ_t must be generated with new value of t.

Simulating
Nonhomogeneous PP

- Monte carlo
- Kinetic Monte Carlo.

Inverse Method

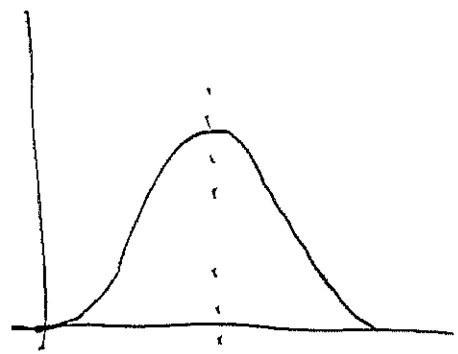
$X \sim F_X(x)$

CDF

Normal

pdf

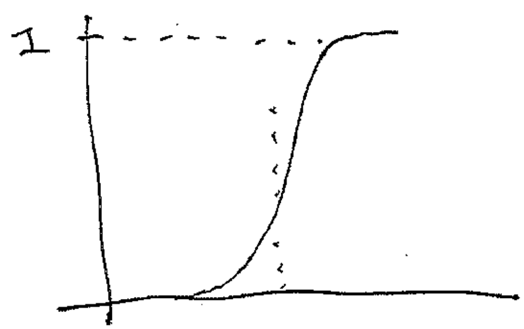
$f_X(x)$



$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma}}$$

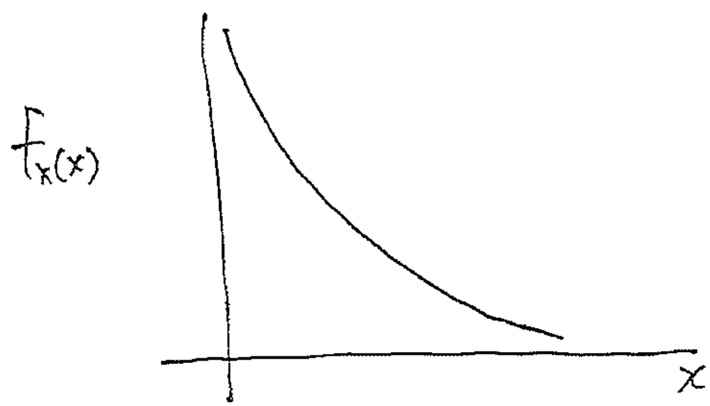
CDF

$F_X(x)$

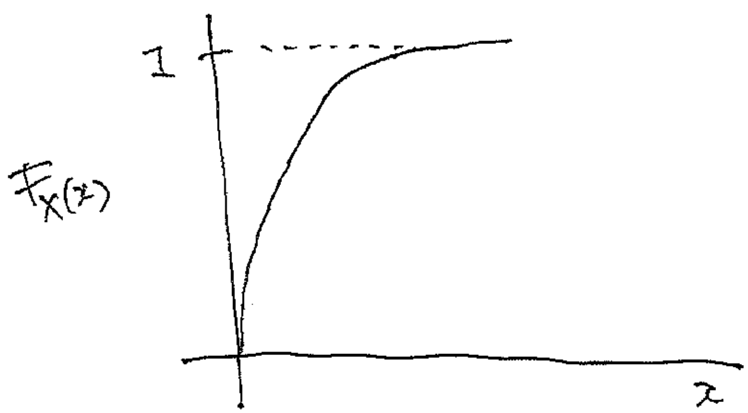


$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$\boxed{\text{Exp}}$



$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad x \geq 0$$



$$F_X(x) = \int_0^x \frac{1}{\lambda} e^{-t/\lambda}\, dt \qquad x \geq 0.$$

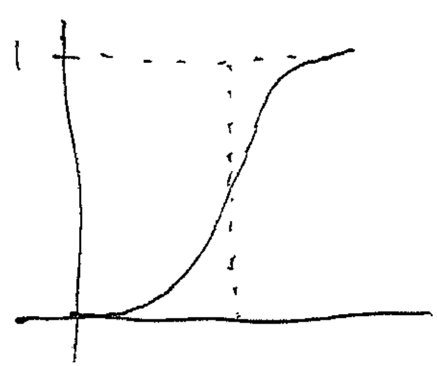
$$= 1 - e^{-x/\lambda}$$

Thm : what ever the distribution is ;

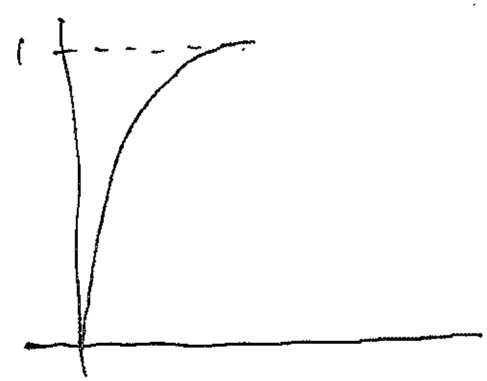
If $X \sim F_X$, then

$$F_X(X) \sim UNIF(0,1).$$

Normal



Exp



Inverse Method for Generating R.V.

Want to generate $X \sim F_X(x)$.

~~$F_X(x)$ is known and [illegible]~~

We know that $$F_X(X) \underset{\text{in distribution}}{=} U.$$

That means, $$X \underset{\text{in dist.}}{=} F_X^{-1}(U).$$

where $U \sim UNIF(0,1)$.

We know how to generate U.

in R: runif(n)

Ex.

$X \sim Exp(5)$,

$$F_X(x) = 1 - e^{-x/5}.$$

let

$$U = 1 - e^{-x/5}.$$

sloving for x,

$$U - 1 = -e^{-x/5}$$

$$-\frac{x}{5} = \ln(1-u)$$

$$x = -5 \cdot \ln(1-u),$$

but, since

$1-u \sim Unif(0,1)$,

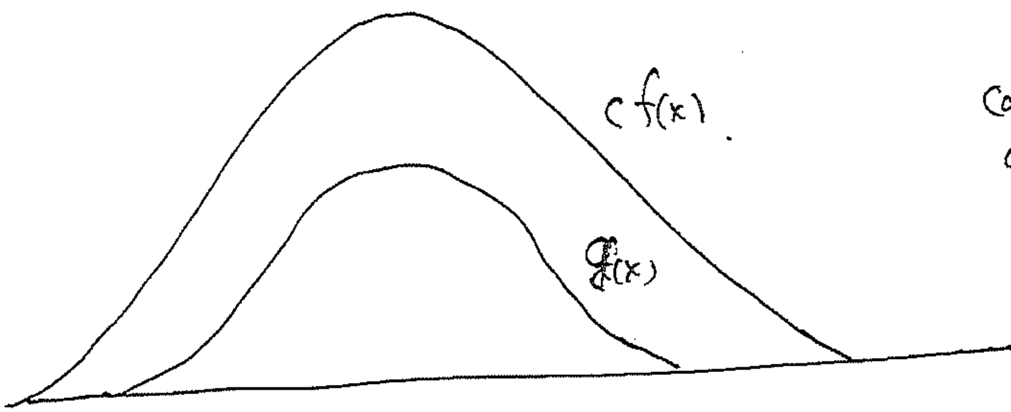
$$\boxed{X = -5\ln(u)}$$

① generate n #s from Unif(0,1). call it u.

② let $X = -5\ln(u)$,

③ X is exponentially distributed with mean 5.

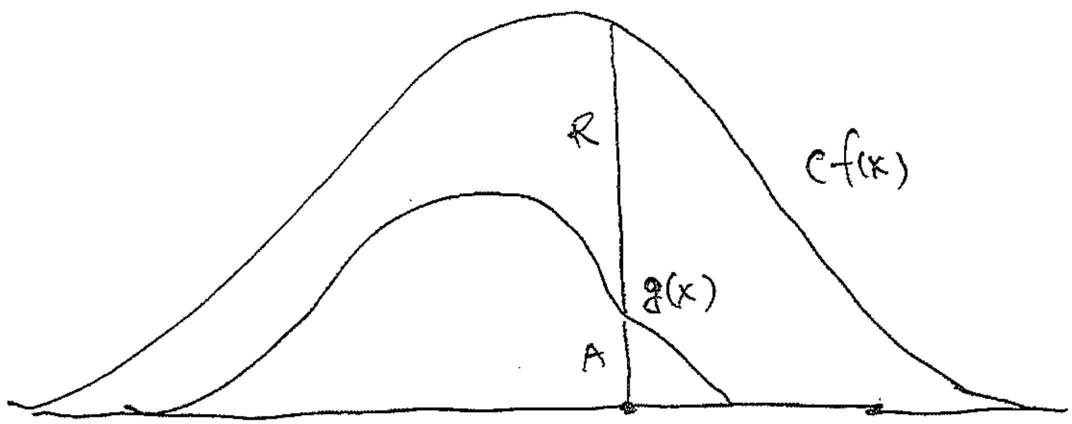
Acceptance - Rejection Method



$$g(x) \leq c\,f(x)$$

can't generate

can generate



① generate X_1 from $f(x)$.

② ~~generate~~ generate $U \sim UNIF(0, cf(X_1))$

③ if $U < g(X_1)$, then accept X_1 as generation from $g(x)$.

if not, disregard X_1.

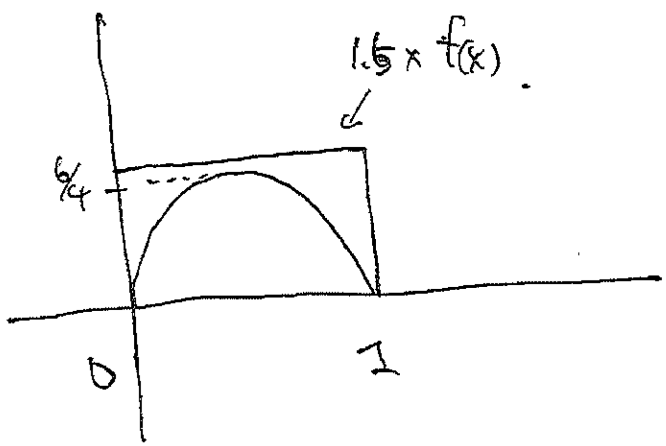
④ repeat.

Ex. A-R for Beta(2,2)

Beta(2,2)

$$g(x) = 6x(1-x).$$

pdf



① generate $X_1 \sim UNIF(0,1)$

② generate $U \sim UNIF(0, 1.5)$

③ if $U < 6X_1(1-X_1)$ accept.

④ repeat.

$f(x)$ = pdf of $UNIF(0,1)$.