Time Reversible Markov Chains

$\boxed{\text{Discrete time MC.}}$

Trace MC Backward:

$$Q_{ij} = P\{X_m = j \mid X_{m+1} = i\}$$

$$= \frac{P\{X_{m+1} = i \mid X_m = j\}\, P(X_m = j)}{P\{X_{m+1} = i\}}$$

$$= P_{ji} \cdot \frac{\pi_j}{\pi_i} \qquad \text{if } \underline{\pi} \text{ exists.}$$

(limit dist.)

$$Q_{ij} = P_{ij} \quad \text{if}$$

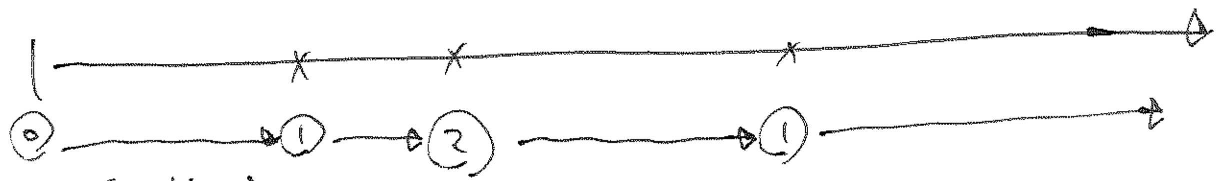
$$Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i} = P_{ij}$$

i.e. $\pi_j P_{ji} = \pi_i P_{ij}$

(rate j to i) = (rate i to j)

then MC is called time reversible.

Continuous time MC



	$\text{Exp}(\lambda_0)$	$\text{Exp}(\lambda_1+\mu_1)$	$\text{Exp}(\lambda_2+\mu_2)$	$\text{Exp}(\lambda_1+\mu_1)$
mean trans time	$\frac{1}{\lambda_0}$	$\frac{1}{\lambda_1+\mu_1}$	$\frac{1}{\lambda_2+\mu_2}$	$\frac{1}{\lambda_1+\mu_1}$
	$\frac{1}{\nu_0}$	$\frac{1}{\nu_1}$	$\frac{1}{\nu_2}$	$\frac{1}{\nu_1}$

If we ignore trans. time, we get Discrete time MC

with $\mathbb{P} = \{P_{ij}\}$.

Suppose this Disc. time MC is ergodic,

and limiting dist $\underline{\pi} = \{ \pi_j \}$ exists.

What is relationship b/w

P_j and π_j, $\frac{1}{\nu_j}$?

$\boxed{\text{Cont. time MC}}$ $\boxed{\text{Disc time MC}}$

limiting dist. over time

limit. dist ignoring time

Av. time spent in state j.

$$P_j = \frac{\pi_j \left(\frac{1}{\nu_j} \right)}{\sum_l \pi_l \frac{1}{\nu_l}}$$

proportion of time spent in state j.

Working back on C.t. MC.

$$P(\text{state } i \text{ in all of } [t, t+s] \mid X(t+s) = i)$$

$$= \frac{P(\text{state } i \text{ in all of } [t, t+s])}{P(X(t+s) = i)}$$

$$= \frac{P(X(t) = i \text{ and transition time} > s)}{P(X(t+s) = i)}$$

$$= \frac{P(X(t) = i)}{P(X(t+s) = i)} e^{-\nu_i s} = \boxed{e^{-\nu_i s}}$$

both P_i

→ reversed c.t. MC has

transition time $\sim \text{Exp}(\nu_i)$

(same as going forward)

Disc. Time MC has reverse trans prob.

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

$Q_{ij} = P_{ij}$ if

$$\pi_i P_{ij} = \pi_j P_{ji}$$

Combine with

$$P_i = \frac{\pi_i \left(\frac{1}{\nu_i}\right)}{\sum_l \pi_l \left(\frac{1}{\nu_l}\right)}, \qquad A P_i \nu_i = \pi_i$$

$$\pi_i P_{ij} = \pi_j P_{ji}$$

becomes

$$\boxed{\cancel{A} P_i \nu_i P_{ij} = \cancel{A} P_j \nu_j P_{ji}}$$

If this is true, c.t. MC is time-reversible

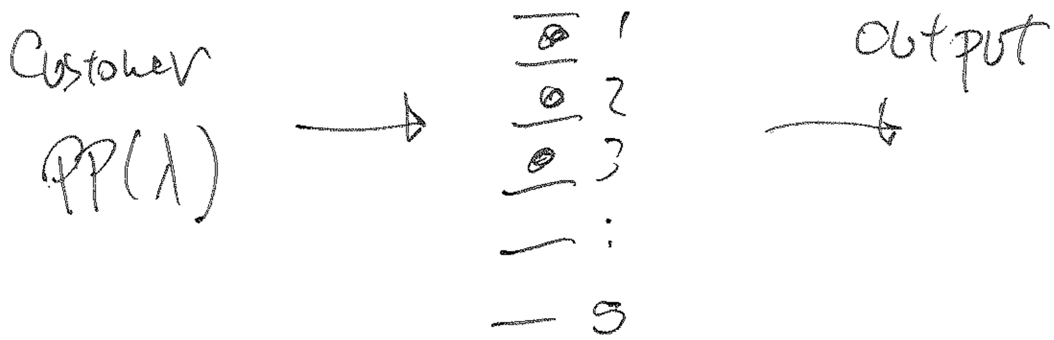
(i.e. look back in time, and process is same MC.)

Ross

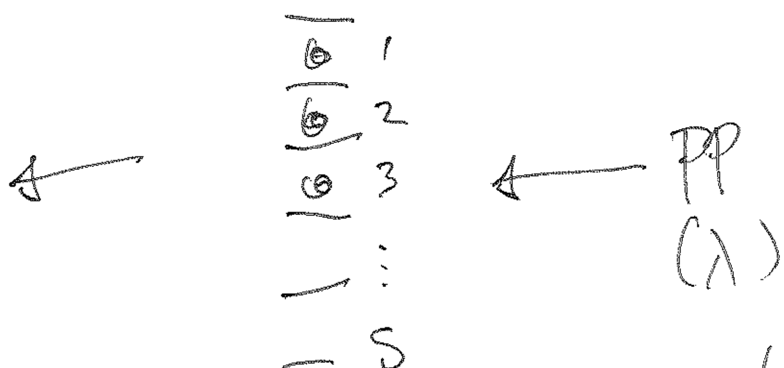
Prop. 6.5

Ergodic B+D process is time reversible

Corollary 66 M/M/s queue.



time-reversed view



Once system is stable, outgoing customer is PP(λ).

Ex 6.17

1st come 1st serve

M/M/1

$\text{PoiProc}(\lambda) \longrightarrow |\,B\,| \longrightarrow$

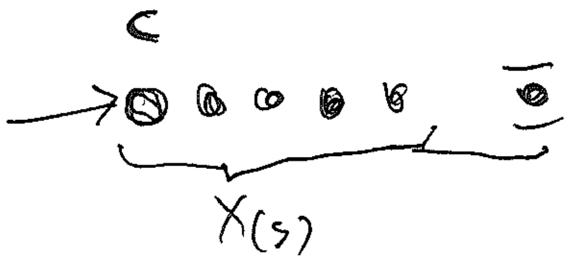
$\text{Exp}(\mu)$

$\frac{\lambda}{\mu} < 1$.

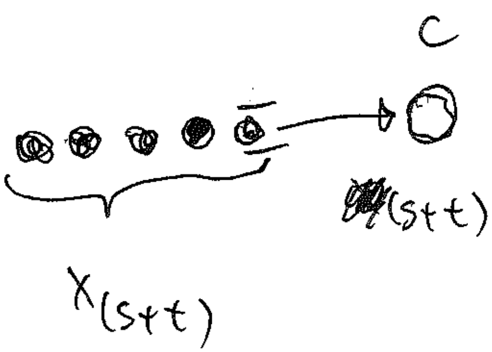
$X(t)$ = Length of line at t.

$\left(X_{(S)} \;\middle|\; c \text{ spend } t \text{ in system} \right)$

when c arrived.



$X(s)$ must depart by time $s+t$.



$X(s) \mid c$ left at time $s+t \sim \text{Poi}(\lambda t)$

Prop. 6.8

If time reversible MC with P_j

truncated ~~at~~ to set $A \subset S$, and remain irreducible, then new limiting prob.

$$P_j^A = \frac{P_j}{\sum_{l \in A} P_l}$$

Ex 6.19 M/M/1. truncate at N.

(customer will not get in line, if line is N).

M/M/1 time reversible,

$$P_j = \left(\frac{\lambda}{\mu}\right)^j\left(1-\frac{\lambda}{\mu}\right)$$

$$P_j^A = \frac{P_j}{\sum_{l=1}^{N} P_l}$$