Birth and Death Process

$\xrightarrow[\text{Birth}]{}$ 6 $\sim Exp(\lambda_5)$

independent

$\xrightarrow[\text{Death}]{}$ 4 $\sim Exp(\mu_5)$

$$P(\text{next event is a Birth}) = \frac{\lambda_5}{\lambda_5 + \mu_5}.$$

$\mu_0 = 0.$

Time until next event $\sim Exp(\lambda_5 + \mu_5)$.

Birth and Death Process

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n people in the system.

Birth $\sim \text{Exp}(\lambda_n)$

Death $\sim \text{Exp}(\mu_n)$

$$P_{01} = 1 \qquad E[\text{time in } 0] = \frac{1}{\lambda_0}.$$

$$E[\text{time in } 1] = \frac{1}{\lambda_1 + \lambda_2}$$

$$P_{10} = \frac{\mu_1}{\lambda_1 + \mu_1} \qquad P_{12} = \frac{\lambda_1}{\lambda_1 + \mu_1}$$

For state i

$$E[\text{time in state } i] = \frac{1}{\mu_i + \lambda_i}$$

Birth $\quad P_{i\,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}$

Death $\quad P_{i\,i-1} = \dfrac{\mu_i}{\lambda_i + \mu_i}$

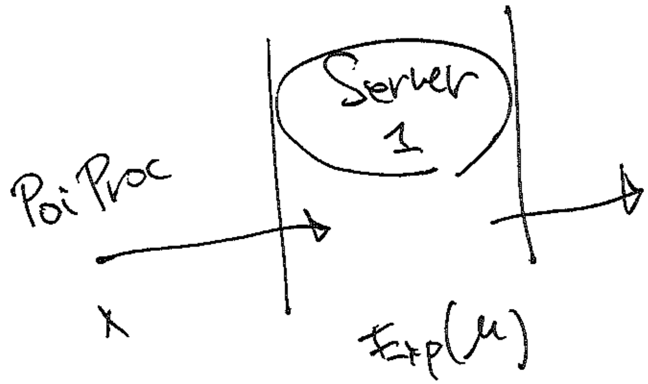
Ex. Linear Growth Process with Immigration

$$\mu_n = n\mu \qquad n \geq 1$$

$$\lambda_n = \underbrace{n\lambda}_{\text{natural growth}} + \underbrace{\theta}_{\text{immigration}} \qquad n \geq 0$$

$$X(t) = [\text{Population at time } t]$$

Ex 6.5 M/M/1 Queue.

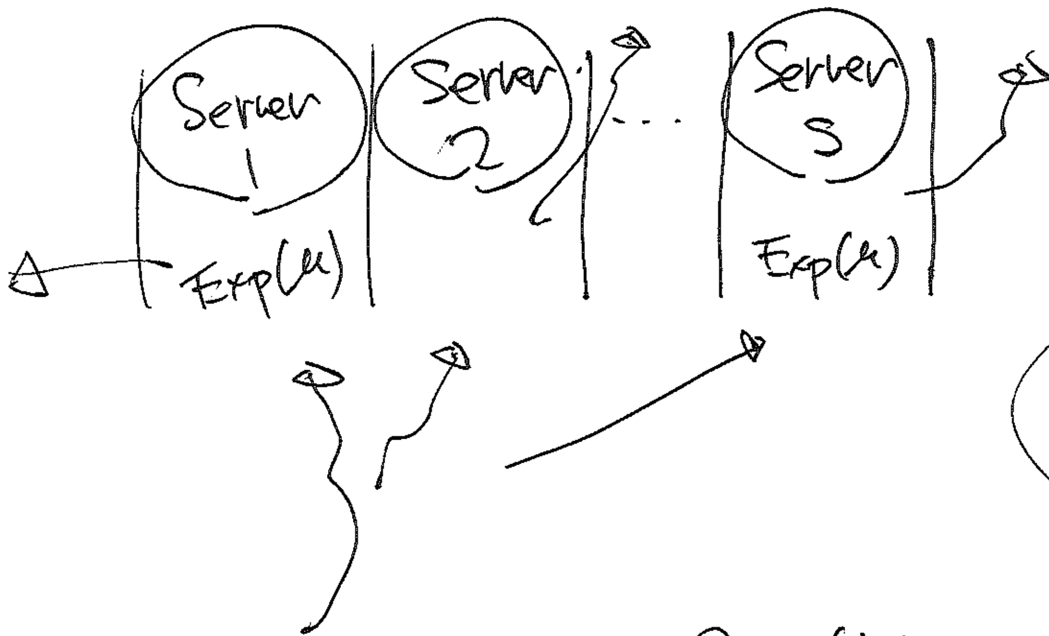


# of customers in queue

= B-D process with $\mu_n = \mu$ constant rate.

$\lambda_n = \lambda$.

Ex 6.6 $M/M/s$ queue



need to be serviced by one of the servers

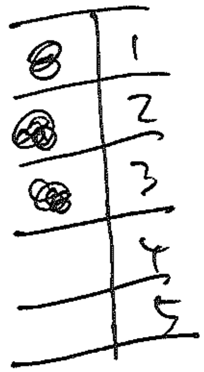
Customer $\sim$ PoiProc(λ)

$\boxed{\text{\# of Customers in queue}}$

$$\sim \text{B-D proc} \quad \text{with} \begin{cases} \mu_n = \\ \lambda_n = \end{cases}$$

Say $S = 5$

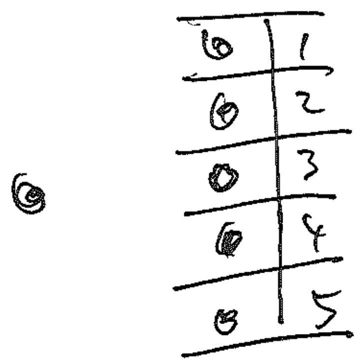
$n = 3$



$$\begin{cases} \mu_n \ \text{min of 3 indep } \mathrm{Exp}(\mu) = \mathrm{Exp}(3\mu). \\ \lambda_n = \lambda. \end{cases}$$

$\mu_n = 3\mu$

$n = 6$



$$\begin{cases} \mu_n = 5\mu \\ \lambda_n = \lambda. \end{cases}$$

$$\begin{cases} \mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases} \\ \\ \lambda_n = \lambda & n \ge 0. \end{cases}$$

M/M/s queue.

B-D process simulation

(0) $X = n$

(1) $B \sim Exp(\lambda_n)$

$D \sim Exp(\mu_n)$

(2) If $B \le D$, then $X = n+1$,

If $B > D$, then $X = n-1$.

Record X and time of events, T.

Use X[max(which(T < t))] to get

state at time t.

Birth-Death Process

$\mu_n, \quad \lambda_n.$ $\quad \mu_0 = 0$.

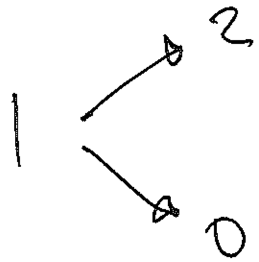
Let

$$T_i = \left\{ \begin{array}{l} \text{time it takes for process to} \\ \text{go from state } i \text{ to } i+1. \end{array} \right\}$$

$$E[T_0] = \frac{1}{\lambda_0}.$$

$$E[T_1] = ?$$

T_1 = time to go 1 to 2.



let $I_i = \begin{cases} 1 & \text{if First transition was to } i+1 \\ 0 & \text{'' } i-1 \end{cases}$.

$$E[T_1 \mid I_1 = 1] = \frac{1}{\lambda_1 + \mu_1}$$

$$E[T_1 \mid I_1 = 0] = \frac{1}{\lambda_1 + \mu_1} + E[T_0] + E[T_1]$$

$$E[T_1] = E\left[E[T_1 \mid I_1] \right]$$

$$= E[T_1 \mid I_1 = 1] \cdot P(I_1 = 1)$$

$$+ E[T_1 \mid I_1 = 0] \cdot P(I_1 = 0)$$

$$= \frac{1}{\lambda_1 + \mu_1} \cdot \frac{\lambda_1}{\lambda_1 + \mu_1}$$

$$+ \left[\frac{1}{\lambda_1 + \mu_1} + \underbrace{E[T_0]}_{\frac{1}{\lambda_0}} + E[T_1] \right] \cdot \frac{\mu_1}{\lambda_1 + \mu_1}$$

$$E[T_1] = \frac{1}{\lambda+\mu_1} + \frac{\mu_1}{\lambda_1+\mu_1}\left(E[T_0] + E[T_1]\right)$$

Solve

$$E[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} E[T_0].$$

Same for $E[T_2]$.

$$\boxed{E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]}.$$

what is

$$V[T_1] = ?$$

$$= \underbrace{V\left[E(T_1 | I_1)\right]}_{①} + \underbrace{E\left[V(T_1 | I_1)\right]}_{②}$$

① Recall.

$$E[T_1 | I_1 = 1] = \frac{1}{\lambda_1 + \mu_1}$$

$$E[T_1 | I_1 = 0] = \frac{1}{\lambda_1 + \mu_1} + E[T_0] + E[T_1]$$

write this as,

$$E[T_1 | I_1] = \frac{1}{\lambda_1 + \mu_1} + (1 - I_1)\Big[E[T_0] + E[T_1]\Big]$$

Then,

$$V\left(E[T_1 \mid I_1]\right) = V\left(\frac{1}{\lambda_1+\mu_1} + (1-I_1)\left[E[T_0]+E[T_1]\right]\right)$$

$$= \left[E[T_0]+E[T_1]\right]^2 V(1-I_1)$$

$$= [\qquad]^2 V(I_1) \longleftarrow \text{1a}$$

$$I_1 = \text{Bernoulli}\left(\frac{\lambda_1}{\lambda_1+\mu_1}\right).$$

$$V(I_1) = p(1-p) = \frac{\lambda_1}{\lambda_1+\mu_1}\cdot\left(\frac{\mu_1}{\lambda_1+\mu_1}\right) = \frac{\lambda_1\mu_1}{(\lambda_1+\mu_1)^2}.$$

(2)

time until any event.

$$V(T_1 \mid I_1 = 1) = V(S_1) = \frac{1}{(\lambda_1 + \mu_1)^2}$$

$$V(T_1 \mid I_1 = 0) = V(S_1 + T_0 + T_1)$$

$$= V(S_1) + V(T_0) + V(T_1) \quad \text{by independence.}$$

$$V(T_1 \mid I_1) = \frac{1}{(\lambda_1 + \mu_1)^2} + (1 - I_1)\left[V[T_0] + V[T_1]\right]$$

(2)

$$E\left[V(T_1 | I_1)\right] = \frac{1}{(\lambda_1+\mu_1)^2} + E(1-I_1)\left[V[T_0] + V[T_1]\right]$$

$$\downarrow$$

$$1 - \frac{\lambda_1}{\lambda_1+\mu_1} = \frac{\mu_1}{\lambda_1-\mu_1}.$$

(1)

$$V\left[E(T_1 | I_1)\right] = \left[E[T_0] + E[T_1]\right]^2 \frac{\lambda_1 \mu_1}{(\lambda_1+\mu_1)^2}$$

$$V[T_1] = (1) + (2).$$

solve for $V[T_1]$.

$$V[T_1] = \frac{1}{\lambda_1(\lambda_1+\mu_1)} + \frac{\mu_1}{\lambda_1} V(T_0) + \frac{\mu_1}{\lambda_1+\mu_1}\left(E[T_0] + E[T_1]\right)^2 .$$

you can use this formula by replacing

$$\begin{cases} 1 \to i \\ 0 \to i-1 . \end{cases}$$

for $i > 1$.

$$\begin{cases} E[T_1] & \checkmark \\ V[T_1] & \checkmark \\ \text{distribution of } T_1 \text{ ?} & \to \text{ simulation.} \end{cases}$$