Seasonal Models

Seasonal ARIMA model

Consider

$$Y_t = e_t - \Theta e_{t-12}$$

MA(12)

$$Cov(Y_{t+1}, Y_t) = Cov(e_{t+1} - \Theta e_{t-11}, e_t - \Theta e_{t-12})$$

$$= 0.$$

$$\mathrm{Cov}(Y_{t+12}, Y_t) = \mathrm{Cov}(e_{t+12} - \Theta e_t, \; e_t - \Theta e_{t-12})$$

$$= -\Theta \sigma^2$$

Correlation only at lag 12.

Seasonal MA(Q) with period s.

$$Y_t = e_t - \Theta_1 e_{t-s} \qquad \text{sMA}(Q=1)$$

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} \qquad \text{sMA}(Q=2)$$

$\Gamma(h)$ will be zero except at lag $s, 2s, 3s, \ldots, Qs$.
$h>0$

Representation with B.

sMA(1) $s=12$.

$$Y_t = e_t - \Theta_1 e_{t-12}.$$

$$= \underbrace{(1 - \Theta_1 B^{12})}_{\text{seasonal characteristic polynomial}} e_t$$

seasonal
characteristic polynomial.

sMA(1) $\quad s = 12$

$$Y_t = e_t - \Theta_1 e_{t-12} - \Theta_2 e_{t-24}$$

$$= \underbrace{\left(1 - \Theta_1 B^{12} - \Theta B^{24}\right)}\, e_t$$

Seasonal AR(P) with period s

$$Y_t = \Phi_1 Y_{t-s} + e_t \qquad sAR(P=1)$$

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + e_t \qquad sAR(P=2)$$

SAR(1) period 12

$$Y_t = \Phi_1 Y_{t-12} + e_t$$

$$E(Y_t \cdot Y_t) = E\left[Y_t \cdot (\Phi_1 Y_{t-12} + e_t)\right]$$

$$\gamma(0) = \Phi_1 \gamma(12) + \underbrace{E(Y_t, e_t)}_{\sigma^2}$$

$k \geq 1$

$$E\left(Y_{t-k} \cdot Y_t\right) = E\left[Y_{t-k} \cdot \left(\Phi_1 Y_{t-12} + e_t\right)\right]$$

$$\gamma(k) = \Phi_1 \gamma(k-12) + 0 .$$

$$\begin{cases} \gamma(0) = \Phi_1 \gamma(12) + \sigma^2 & \text{—— } (2) \\ \gamma(k) = \Phi_1 \gamma(k-12) & \text{—— } (1) \end{cases}$$

$k \geq 1 .$

Take ①. $k = 1$,

$$\gamma(1) = \Phi_1 \gamma(-11)$$

Take ① $k = 11$,

$$\gamma(11) = \Phi_1 \gamma(-1)$$

Since $\gamma(h) = \gamma(-h)$, we have

$$\gamma(1) = \Phi_1 \gamma(11) = \Phi_1^2 \gamma(1).$$

$\Rightarrow \quad \gamma'(1) = 0$, and $\gamma'(11) = 0$.

① says $\quad \gamma'(k) = \Phi_1 \gamma'(k-12)$

$\gamma'(2) = \Phi_1 \gamma'(10) = \Phi_1^2 \gamma'(2) \quad \Rightarrow \quad \gamma'(2) = 0, \ \gamma'(10) = 0.$

$\gamma'(3) = \Phi_1 \gamma'(9) = \Phi_1^2 \gamma'(3) \quad \Rightarrow \quad \gamma'(3) = 0, \ \gamma'(9) = 0.$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$\gamma'(5) = \Phi_1 \gamma'(7) = \Phi_1^2 \gamma'(5) \qquad \gamma'(5) = 0, \ \gamma'(7) = 0$

$\gamma'(6) = \Phi_1 \gamma'(6) = \Phi^2 \gamma'(6) \qquad \gamma'(6) = 0 \quad , \gamma'(6) = 0.$

$$\gamma(0) = \Phi_1 \gamma(12) + \sigma^2 \quad - (2)$$

use (1) with $k = 12$,

$$\gamma(12) = \Phi_1 \gamma(0).$$

$$\gamma(0) = \Phi_1 (\Phi_1 \gamma(0)) + \sigma^2$$

$$= \Phi_1^2 \gamma(0) + \sigma^2$$

$$\gamma(0) = \boxed{\frac{\sigma^2}{1 - \Phi_1^2}}.$$

$$\gamma(12) = \Phi_1 \gamma(0).$$

$$\gamma(13) = \Phi_1 \gamma(1) = 0 .$$

$$\vdots$$

$$\gamma(23) = \Phi_1 \gamma(11) = 0 .$$

$$\gamma(24) = \Phi_1 \gamma(12) = \Phi_1^2 \gamma(0) .$$

$$\vdots$$

$$\gamma(0) = \sigma^2 / (1 - \Phi_1^2) .$$

$$\gamma(1) = 0$$

$$\vdots$$

$$\gamma(11) = 0$$

$$\gamma(12) = \Phi_1 \gamma(0)$$

$$\vdots$$

sAR(1) $\quad s = 12$

$$
\begin{cases}
\gamma(0) = \dfrac{\sigma^2}{1 - \Phi_1^2} \\[2ex]
\gamma(ks) = \Phi_1^{k}\, \gamma(0) \\[2ex]
\text{otherwise} \quad \gamma(\cdot) = 0 .
\end{cases}
$$

Multiplicative sARMA

Consider sMA(1) $s=12$

$$Y_t = \varepsilon_t - \Theta_1 \varepsilon_{t-12}$$

but $\varepsilon_t \sim MA(1)$ model. i.e.

$$\varepsilon_t = e_t - \theta_1 e_{t-1} .$$

$$Y_t = (1 - \Theta_1 B^{12})\,\varepsilon_t$$

$$= \underbrace{(1 - \Theta_1 B^{12})(1 - \theta_1 B)}_{\substack{\text{characteristic} \\ \text{polynomial of}}}\, e_t$$

ARMA $(0, 1) \times (0, 1)_{12}$

ARMA $(p, q) \times (P, Q)_s$

$ARMA(p,q) \times (P,Q)_s$ model

$P = 1 \qquad Q = 1$

$$(1 - \Phi_1 B^s)\tilde{Y}_t = (1 - \Theta_1 B^s)\tilde{e}_t$$

$$(1 - \Phi_1 B^s)\,\Phi(B)\,Y_t = (1 - \Theta_1 B^s)\,\Theta(B)\,e_t$$

$\Phi(B)$: AR(p) poly

$\Theta(B)$: MA(q) poly.

Causality and Invertibility

Causality.

$(1-\Phi_1 x^s)(1-\phi_1 x - \cdots \phi_p x^p)$ must have roots outside of unit circle.

$$(1-\Phi_1 x^s) = 0$$

$$(1-\phi_1 x - \cdots - \phi_p x^p) = 0.$$

they have to have roots outside of unit circle.

similar for Invertibility.

Non-stationary Seasonal ARIMA models

Suppose your time-series are generated by model

$$Y_t = m_t + S_t + X_t .$$

m_t: linear trend

S_t: Seasonality

X_t: stationary t.s.

Seasonality "

$$S_t = S_{t-s}$$

$$\sum_{j=0}^{s-1} S_{t-j} = 0 .$$

Let $\nabla_s = (1 - B^s)$

$$\nabla_{12} Y_t = (1 - B^{12}) Y_t = Y_t - Y_{t-12}.$$

$$= m_t + S_t + X_t - m_{t-12} - S_{t-12} - X_{t-12}.$$

$$= \underbrace{m_t - m_{t-12}}_{\text{new trend}} + \underbrace{X_t - X_{t-12}}_{\text{new stationary T.S.}}$$

Take $\nabla = (1-B)$ to obtain,

$$\nabla \nabla_{12} Y_t = \nabla (m_t - m_{t-12} + X_t - X_{t-12})$$

$$= \nabla m_t - \nabla m_{t-12} + \nabla (X_t - X_{t-12})$$

If $m_t =$ linear

$$= c - c + \nabla X_t - \nabla X_{t-12}$$

$$= \nabla X_t - \nabla X_{t-12} .$$

If $X_t \sim ARIMA(p, 1, q)$ i.e.

$\nabla X_t \sim ARMA(p, q)$

$$\nabla \nabla_{12} Y_t = \nabla X_t - \nabla X_{t-12}$$

$$\nabla X_t \sim \text{ARMA}(p, q)$$

$$\Rightarrow \quad \nabla \nabla_{12} Y_t \sim \text{ARMA}(p, q) \times (P, Q)_{12}$$

$$Y_t \sim \text{ARIMA}(p, 1, q) \times (P, 1, Q)_{12}.$$

Seasonal ARIMA model

$$\text{ARIMA}(p, d, q) \times (P, D, Q)_s$$

$$\nabla^d \nabla_s^D Y_t \sim \text{ARMA}(p, q) \times (P, Q)_s .$$