Lognormal Distribution

Normal and Lognormal

$X \sim N(\mu, \sigma^2)$

$Y \sim LN(\mu, \sigma^2)$

$X = \ln(Y)$

$Y = e^X$

$E(X) = \mu$

$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$

$V(X) = \sigma^2$

$$V(Y) = \left[e^{\sigma^2} - 1\right] e^{2\mu + \sigma^2}$$

Sum of Normals

$$X_1 \sim N(\mu_1, \sigma_1^2)$$

$$X_2 \sim N(\mu_2, \sigma_2^2)$$

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

~~Product of Norm~~

Product of Lognormals

$$Y_1 \sim LN(\mu_1, \sigma_1^2)$$

$$Y_2 \sim LN(\mu_2, \sigma_2^2)$$

$$Y_1 \cdot Y_2 = e^{X_1} \cdot e^{X_2} = e^{X_1 + X_2}$$

$$Y_1 \cdot Y_2 \sim LN(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Looking up probabilities

$X \sim N(\mu, \sigma^2)$

$P(X \leq x)$

$$= P\Big(\underbrace{\frac{X-\mu}{\sigma}}_{\substack{\text{std.} \\ \text{Normal}}} \leq \frac{x-\mu}{\sigma}\Big) = N\Big(\frac{x-\mu}{\sigma}\Big)$$

ie, $\Phi\left(\frac{x-\mu}{\sigma}\right)$

Normal table.

$$Y \sim LN(\mu, \sigma^2)$$

$$P(Y \le y)$$

$$= P(\underbrace{\ln(Y)}_{\uparrow \text{ Normal}} \le \ln(y))$$

$$= N\left(\frac{\ln(y) - \mu}{\sigma}\right)$$

Lognormal model of Stock Prices

$$S_t = S_0 e^{(r-\delta)t \pm \sigma\sqrt{t}}$$ Binomial pricing.

$$\ln\left(\frac{S_t}{S_0}\right) = (r-\delta)t \pm \sigma\sqrt{t}$$

$$\sim N\left((r-\delta)t,\ \sigma^2 t\right)$$

That means,

$$\frac{S_t}{S_0} \sim LN\left((r-\delta)t \,,\ \sigma^2 t \right)$$

but then we have

$$E\left(\frac{S_t}{S_0}\right) = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{(formula)}$$

$$= e^{(r-\delta)t + \frac{1}{2}\sigma^2 t}.$$

We want this to be

$$E\left(\frac{S_t}{S_0}\right) = e^{(r-\delta)t}.$$

Adjustment

Let

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left((r-\delta)t - \frac{1}{2}\sigma^2 t \,,\; \sigma^2 t\right)$$

then

$$\boxed{\frac{S_t}{S_0} \sim LN\left((r-\delta)t - \frac{1}{2}\sigma^2 t \,,\; \sigma^2 t\right)}$$

$$E\left(\frac{S_t}{S_0}\right) = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{(formula)}$$

$$= e^{(r-\delta)t - \frac{1}{2}\sigma^2 t + \frac{1}{2}\sigma^2 t}$$

$$= e^{(r-\delta)t}$$

If

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left((r-\delta)t - \frac{1}{2}\sigma^2 t,\ \sigma^2 t\right)$$

$$= \ln(S_t) - \ln(S_0)$$

Then

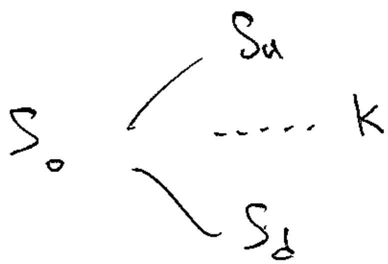
$$\ln(S_t) \sim N\left(\ln(S_0) + \left(r-\delta-\frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right)$$

$$\boxed{S_t \sim LN\left(\ln(S_0) + \left(r-\delta-\frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right)}$$

LN model of stock prices.

$$E(S_t) = e^{\ln(S_0)+(r-\delta)t} = S_0 e^{(r-\delta)t}$$

Probability that $S_t < K$



Call price $= 0$ if $S_t < K$,

$$S_t \sim LN\left(\ln(S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\right)$$

$$P(S_t < K)$$

$$= N\left(\frac{\ln(K) - \mu}{\sigma}\right) \quad \text{(formula)}$$

$$= N\left(\frac{\ln(K) - \ln(S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)$$

$$= N\left(- \frac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right)$$

$$= N(-d_2).$$

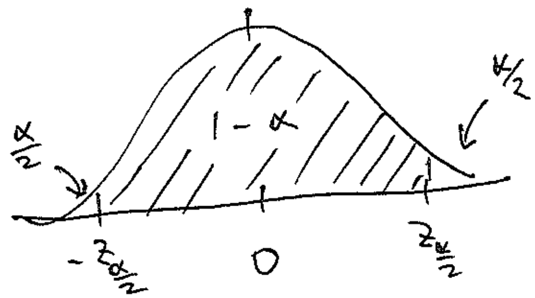
$$P(S_t \geq K)$$

$$= 1 - P(S_t < K) = 1 - N(-d_2)$$

$$= N(d_2).$$

Confidence Intervals for S_t

Standard Normal



$$\ln(S_t) \sim N\Big(\overbrace{\ln(S_0)+(r-\delta-\tfrac{1}{2}\sigma^2)t}^{\mu},\ \overbrace{\sigma^2 t}^{\sigma_1^2}\Big)$$

$$P\left(-z_{\frac{\alpha}{2}} \le \frac{\ln(S_t)-\mu}{\sigma_1} \le z_{\frac{\alpha}{2}}\right)$$

$$= 1-\alpha.$$

$$P\left(\mu - z_{\frac{\alpha}{2}}\sigma_1 \le \ln(S_t) \le \mu + z_{\frac{\alpha}{2}}\sigma_1\right) = 1-\alpha.$$

$$\ln(S_t) \in \underbrace{\ln(S_0)+(r-\delta)t-\tfrac{1}{2}\sigma^2 t}_{\mu} \pm z_{\frac{\alpha}{2}}\underbrace{\sigma\sqrt{t}}_{\sigma_1}$$ with $(1-\alpha)100\%$ probability.

$$\boxed{S_t \text{ is within } S_0\, e^{(r-\delta-\frac{1}{2}\sigma^2)t \pm z_{\frac{\alpha}{2}}\sigma\sqrt{t}} \quad \text{with } (1-\alpha)100\ \%.}$$

$z_{\frac{\alpha}{2}} = 1.96$ if $(1-\alpha)100 = 95\%$.

Conditional Expectation

Value of Call is 0 if $S_T < K$.

Value of Put is 0 if $S_T > K$.

What is

$$E(S_T \mid S_T < K) = ? \qquad S_0$$

$$S_T \sim LN$$

$$= \frac{\int_0^K x \, f_{S_T}(x)\, dx}{\int_0^K f_{S_T}(x)\, dx}$$

$$P(S_T < K) = N(-d_2)$$

pdf of Lognormal

$$Y \sim LN(\mu, \sigma^2)$$

$$\ln Y = X \sim N(\mu, \sigma^2)$$ or

$$\underbrace{P(Y \le y)}_{\text{CDF of } LN} = P(\underbrace{\ln Y}_{X} \le \ln(y))$$

CDF of Normal.

$$F_Y(y) = F_X(\ln y)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{(x-\mu)^2}{2\sigma_0^2}}$$

$$f_Y(y) = f_X(\ln y) \cdot \frac{1}{y}$$

$$= \boxed{\frac{1}{y\sqrt{2\pi}\,\sigma_0} e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}}}$$

pdf of Lognormal.

$$\int_0^K x\, f_{S_T}(x)\, dx$$

$$= \int_0^K x \frac{1}{x\sqrt{2\pi}\,\sigma}\, e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}\, dx$$

$$= \int_0^K \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}\, dx \qquad \leftarrow f_N(\ln x)$$

let $u = \ln x$, $\quad du = \frac{1}{x}\, dx \quad dx = x\, du = e^u\, du$

$$= \int_0^{\ln(K)} \frac{1}{\sqrt{2\pi}\,\sigma}\, \underbrace{e^{-\frac{(u-\mu)^2}{2\sigma^2}}\, e^u}_{*} \cdot du \qquad (1)$$

$$\star = e^{-\frac{1}{2\sigma_0^2}(u-\mu)^2 + u}$$

Take the exponent.

Complete the square

$$-\frac{(u-\mu)^2 - 2\sigma_0^2 u}{2\sigma_0^2}$$

$$= -\frac{1}{2\sigma_0^2}\left[u^2 - 2u\mu + \mu^2 - 2\sigma_0^2 u\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[u^2 - 2u(\mu+\sigma_0^2) + \mu^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[u^2 - 2u(\mu+\sigma_0^2) + \underbrace{(\mu+\sigma_0^2)^2 - (\mu+\sigma_0^2)^2}_{\text{new}} + \mu^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[u - 2u(\mu+\sigma_0^2) + \underbrace{(\mu+\sigma_0^2)^2 + (\mu+\sigma_0^2)^2}_{\text{new}} + \mu^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\{u-(\mu+\sigma_0^2)\}^2 - (\mu+\sigma_0^2)^2 + \mu^2\right]$$

Go back to

$$\star = e^{-\frac{\{u-(\mu+\sigma_0^2)\}^2}{2\sigma_0^2} + \underbrace{\frac{(\mu+\sigma_0^2)^2 - \mu^2}{2\sigma_0^2}}}$$

$$\frac{\mu^2 + 2\mu\sigma_0^2 + \sigma_0^4 - \mu^2}{2\sigma_0^2} = +\mu + \frac{\sigma_0^2}{2}$$

Back to the integral. (1)

$$\int_0^K \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{(u-\mu)^2}{2\sigma_0^2}} e^{u}\, du$$

$$= \int_0^K \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{\{u-(\mu-\sigma_0^2)\}^2}{2\sigma_0^2}} \cdot \underbrace{e^{+\mu+\frac{\sigma_0^2}{2}}}\, du$$

$$= \underbrace{e^{+\mu+\frac{\sigma_0^2}{2}}}_{(3)} \cdot \underbrace{\int_0^K \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{\{u-(\mu-\sigma_0^2)\}^2}{2\sigma_0^2}}\, du}_{(4)} \qquad (2)$$

(4) Normal with mean $\mu-\sigma^2$.

$P(N < K)$

$$S_T \sim LN\Big(\underbrace{\ln S_0 + (r - \delta - \tfrac{1}{2}\sigma^2)T}_{\mu} \, , \, \underbrace{\sigma^2 T}_{\sigma_0^2} \Big)$$

$$(3) = e^{+\mu + \frac{\sigma_0^2}{2}} = e^{+\ln S_0 + (r - \delta - \frac{1}{2}\sigma^2)T + \frac{\sigma^2 T}{2}}$$

$$= S_0 \cdot e^{(r-\delta)T}$$

$$(4) = P(N < \ln(K))$$

$$N \sim N(\mu - \sigma_0^2, \ \sigma_0^2)$$

$$\mu = \ln(S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)T$$

$$\sigma_0^2 = \sigma^2 T .$$

$$= N\left(\frac{\ln(K) - (\mu - \sigma_0^2)}{\sigma_0} \right)$$

$$= N\left(\frac{\ln(K) - \ln(S_0) - (r - \delta - \frac{1}{2}\sigma^2)T + \sigma^2 T}{\sigma\sqrt{T}} \right)$$

$$= N(-d_1)$$

Finally,

$$E(S_T \mid S_T < K)$$

$$= \frac{\int_0^K x\, f_{S_T}(x)\, dx}{\int_0^K f_{S_T}(x)\, dx}$$

$$= \boxed{\frac{N(-d_1)}{N(-d_2)} S_0 e^{(r-\delta)T}}$$

How about $E(S_T \mid S_T > K)$?

$$E(S_T) = E\left[E(S_T \mid A) \right]$$

$$= E(S_T \mid S_T > K) \cdot P(S_T > K)$$

$$+ E(S_T \mid S_T < K) \cdot P(S_T < K)$$

$$\underbrace{e^{\mu+\frac{1}{2}\sigma^2}}_{S_0 e^{(r-\delta)T}} = E(S_T \mid S_T > k) \cdot N(d_2)$$

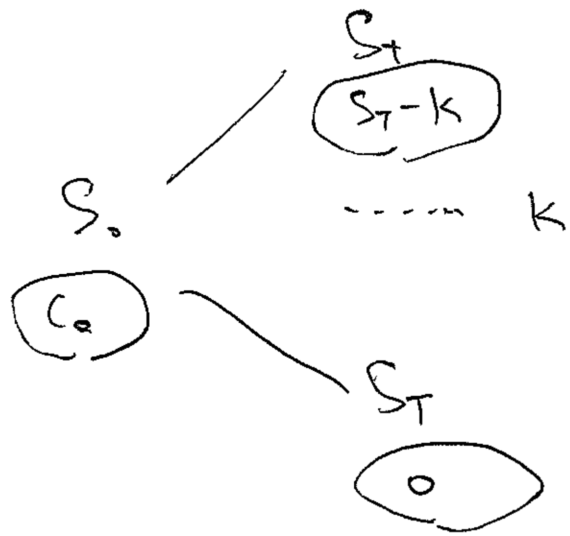
$$S_0 e^{(r-\delta)T} \frac{N(-d_1)}{N(-d_2)} \cdot N(-d_2)$$

$$E(S_T \mid S_T > k) = \frac{S_0 e^{(r-\delta)T}\left(1 - N(-d_1)\right)}{N(d_2)}$$

$$= \boxed{S_0 e^{(r-\delta)T} \frac{N(d_1)}{N(d_2)}}$$

Black - Scholes Formula

Call option



$$C_0 = PV\left(E[\text{Call Values}]\right)$$

$$= e^{-rT} \, E[\text{Call Values}]$$

$$C_0 = e^{-rT} \; E\left[\text{Call Values} \right]$$

$$= e^{-rT} \; E\left\{ E\left[\text{Call Value} \;\middle|\; \text{if } S_T > k \text{ or not} \right] \right\}$$

$$= e^{-rT} \left\{ \underline{E}\left[S_T - k \;\middle|\; S_T > k \right] \cdot P(S_T > k) + \underline{E}\left[0 \;\middle|\; S_T < k \right] \cdot P(S_T \leq k) \right\}$$

$$\cancel{e^{-rT} \left\{ S_0 e^{(r-\delta)T} N(d_1) \cdot N(d_2) + 0 \right\}}$$

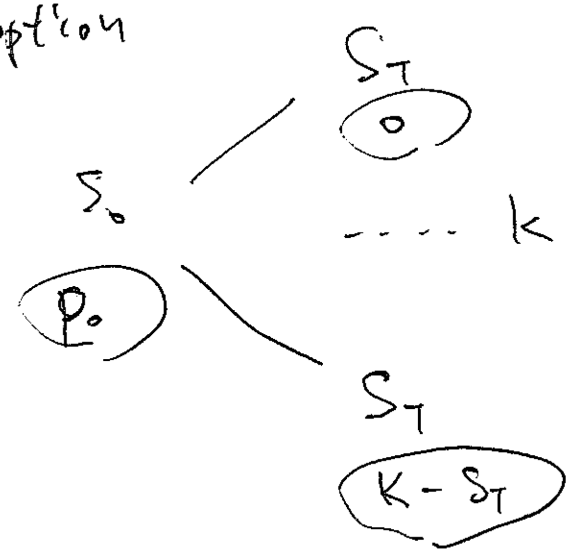
$$= e^{-rT}\left\{ E[S_T \mid S_T > k] - E[k \mid S_T > k] \right\} P(S_T > k)$$

$$= e^{-rT}\left\{ S_0 e^{(r-\delta)T} \frac{N(d_1)}{N(d_2)} - k \right\} N(d_2)$$

$$= \boxed{S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)}$$

B-S call.

Put option



$$p_0 = e^{-rT}\left\{ 0 + E\left(K - S_T \mid S_T < k\right) \cdot P(S_T < k) \right\}$$

$$= e^{-rT}\left\{ K - E\left(S_T \mid S_T < k\right) \right\} P\left(S_T < k\right)$$

$$P_0 = e^{-rT}\left\{ K - S_0 e^{(r-\delta)T} \frac{N(-d_1)}{N(-d_2)} \right\} N(-d_2)$$

$$= \boxed{K e^{-rT} N(-d_2) - S_0 e^{(r-\delta)T} N(-d_1)}$$

Estimating Parameters of Lognormal

$$S_T \sim LN\left(\ln(S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)T \,,\ \sigma^2 T \right)$$

$$E\left[\ln\left(\frac{S_h}{S_0}\right) \right] = (r - \delta - \tfrac{1}{2}\sigma^2) h$$

$$V\left[\ln\left(\frac{S_h}{S_0}\right) \right] = \sigma^2 h$$

Use daily log. return $\ln\left(\frac{S_{t+h}}{S_t}\right)$ and look at sample mean
sample var.