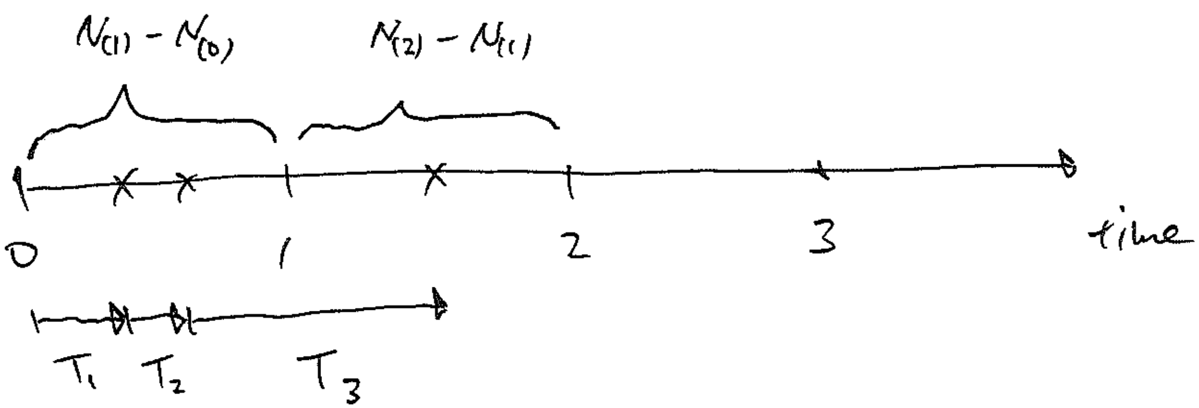
Poisson Process



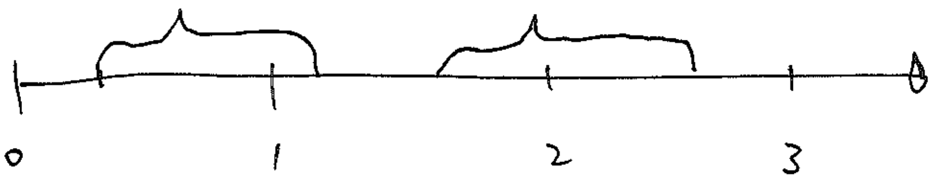
$N_{(1)} - N_{(0)}$ = [# of events b/w time $1 \sim 2$] $\sim \text{Poi}(\lambda)$

mean $= \lambda$

T_1 = [~~time between event~~ inter-arrival time for 1st event] $\sim \text{Exp}(\lambda)$

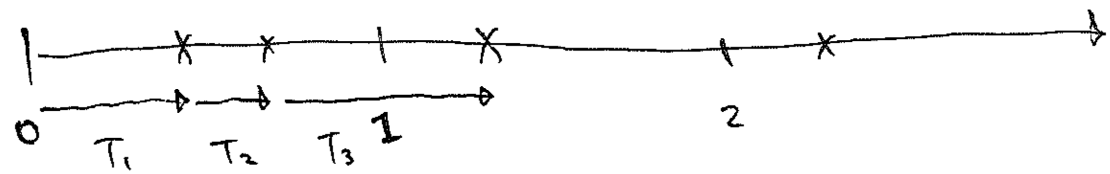
mean $= \frac{1}{\lambda}$

→ # of events between non-overlapping time intervals are independent.



→ each inter-arrival times are independent.

Waiting Time Distribution



T_i = Waiting time until i th event (from $i-1$ th event).

$$T_i \sim \boxed{?}$$

$$P(T_1 > t) = P(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = \boxed{e^{-\lambda t}}$$

$$\to T_1 \sim Exp(\lambda) \qquad E(T_1) = \frac{1}{\lambda}.$$

$$P(T_2 > t \mid T_1 = s) = P(N(s+t) = 1 \mid N(s) = 1)$$

$$= P(N(s+t) - N(s) = 0)$$

$$= \frac{e^{-\lambda(s+t-s)}[\lambda(s+t-s)]^0}{0!}$$

$$= \boxed{e^{-\lambda t}}.$$

Poisson Process

- \# of events in time interval (t_1, t_2).

$N(t_2) - N(t_1) \sim \text{Poi}(\lambda(t_2 - t_1))$, $\quad E(\ ) = \lambda(t_2 - t_1)$.

- time until next event

$T_i \sim \text{Exp}(\lambda) \qquad E(T_i) = \frac{1}{\lambda}$.

- time until ith event

$S_i = \sum_{k=1}^{i} T_k \sim \text{GAM}(i, \frac{1}{\lambda}) \qquad E(S_i) = \frac{i}{\lambda}$.

Ex. 5.13

Immigrate into a territory

rate $\lambda = 1$ per day.

(A) E(time until 10 th immigrant arrives)

(B) P(time b/w 10 th and 11 th immigrant ≥ 2 days)

(C) P(more than 15 immigrant in 10 days).

(A) $E(S_{10}) = \frac{10}{\lambda}$.

(B) $P(T_{11} \geq 2) = e^{-\lambda 2}$

(C) $X \sim POI(\lambda 10)$

$P(X > 15) = 1 - P(X \leq 15)$

$= 1 - \text{ppois}(15, \lambda 10)$.

or
$Y \sim GAM(16, 1)$

$P(Y < 10) = \text{pgamma}(10, \overset{\text{shape}}{16}, \overset{\text{scale}}{1})$.

5.31 two Dr's appointments. 1 pm and 1:30 pm.

Visit time $X \sim Exp(2)$

$$E(X) = \frac{1}{2} \text{ hrs.}$$

Find $E(\text{Waiting time for 2nd patient})$.

5.31

$$P(X > .5) = e^{-2(.5)} = e^{-1}.$$

If $X < .5 \rightarrow 0$ wait for 2nd patient.

If $X > .5 \rightarrow (X - .5 \mid X > .5) =$ wait for 2nd.

$\sim Exp(2)$

mean $= .5$

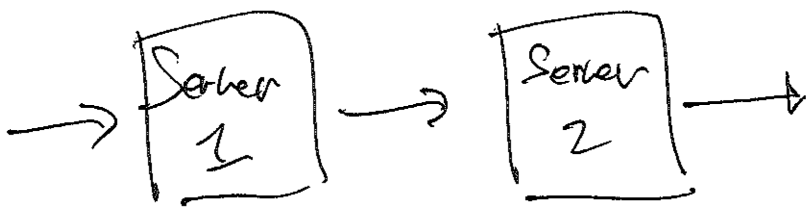
$$E(\text{Wait}) = 0 \cdot (e^{-1})$$

$$+ .5 (1 - e^{-1}) = .316 = \boxed{19 \text{ min}}$$

5.43 Two-server station

Customer ~ $PP(\lambda)$

When new customer arrives, old one goes.



$Exp(1/\mu_1)$ $\quad$ $Exp(1/\mu_2)$

(% of entering customer who completes Server 2?)

Customer will complete the service if

$$P(S_1 + S_2 < T_i)$$

$S_1 \sim Exp(1/\mu_1)$

$S_2 \sim Exp(1/\mu_2)$

$T_2 \sim Exp(\lambda)$

$$= P(T_i > S_1 + S_2 \mid T_i > S_1)\, P(T_i > S_1)$$

$$= \left(\frac{1/\mu_2}{1/\mu_2 + \lambda}\right)\left(\frac{1/\mu_1}{1/\mu_1 + \lambda}\right)$$

by memoryless property,

5-48

n-server $Exp(\mu)$

Customers $\sim PP(\lambda)$

If customer find all servers busy, they leave.

If arrival find all S's busy, then

a) $E(\#$ of busy server found by next arrival$)$

$= E(Bin(n, p))$

$\hookrightarrow P(X < T_i) = \frac{\mu}{\lambda+\mu}$.

$= n\left(\frac{\mu}{\lambda+\mu}\right)$

b) P(Next find all free)

$$= \left(\frac{\mu}{\lambda+\mu}\right)^n$$

c) P(Next find i out of n free)

$$= \binom{n}{i} \left(\frac{\mu}{\mu+\lambda}\right)^i \left(\frac{\lambda}{\lambda+\mu}\right)^{n-i}$$

5-50

Time b/w train $\sim U(0,1)$

Customer $\sim PP(7)$

E[# to get on Next train]

V["]

5.50 $\quad T \sim U(0,1)$

$X = N(T) = \left(\#\text{ of arrival before } \overset{\text{r.v.}}{T}\right) \sim Poi(\eta T)$

$$E[X] = E\big[E[X \mid T]\big] = E[\eta T] = \frac{\eta}{2}$$

$$V[X] = V\big[E(X \mid T)\big] + E\big[V(X \mid T)\big]$$

$$= V(\eta T) + E(\eta T)$$

$$= \eta\left(\frac{1}{12}\right) + \eta\left(\frac{1}{2}\right)$$

5-55 Single Server

PP(λ) Exp(μ)

Customer comes in, and find $n-1$ in line.

when this customer leaves, let

$$X = (\# \text{ in line})$$

$$X \sim ?$$

Alternatively,

length n

arrival

Depart.

$$P = \left(\frac{\mu}{\lambda+\mu}\right)$$

Every Departure/Arrival, is like a flip of a coin.

$X \sim NB(n, p)$ $\qquad$ X = # of tails before n th head.

$$p = \left(\frac{\mu}{\lambda+\mu}\right)$$

$$E(X) = n\left(\frac{1-p}{p}\right) = n\frac{\lambda}{\mu}.$$

$$\text{Time until } A \text{ leaves} = \overset{S}{\overset{\shortparallel}{X_1 + \cdots X_n}} \cancel{T_1 + \cdots + T_n} \sim GAM\left(n, \frac{1}{\mu}\right)$$

$$(\#\text{ in line}) = \#\text{ of arrival before } (X_1 + \cdots + X_n)$$

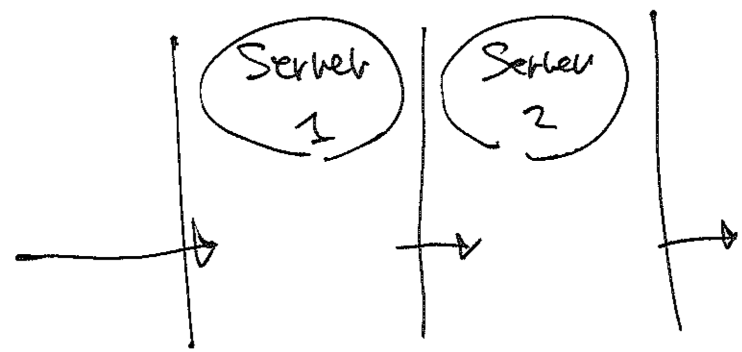
$$X = N(S)$$

$$E(X) = \left(\lambda n \frac{1}{\mu}\right).$$

Can't get pmf of X.

Ross p.349

Prob. 21



$Exp(\lambda_1)$ $Exp(\lambda_2)$.

→ If Server 2 is not open, customer remains at Server 1 even if he's done. ~~there~~.

→ when you enter, there's ~~no~~ 1 customer at S1.

E(you spend in the sys.) = ?

Break down

$$E(\text{Your time in System})$$

$$= E\Big[\underset{①}{(\text{time until S1})} + \underset{②}{(\text{time in S1})}$$

$$+ \underset{③}{\underbrace{(\text{time until S2})}_{\text{if any}}} + \underset{④}{(\text{time in S2})}\Big]$$

① $E\left(\overset{\text{Wait}}{\text{Time}} \text{ until } S1\right) = \dfrac{1}{\mu_1}$

② $E(\text{Time in } S1) = \dfrac{1}{\mu_1}$

④ $E(\text{Time in } S2) = \dfrac{1}{\mu_2}$

③ $E(\overset{\text{wait}}{\text{time}} \text{ until } S2 \text{ if any})$

③a

$$= \begin{cases} 0 & \text{if you finish } S1 \text{ after } A \text{ finish } S2. \\ \text{time until previous Customer } A \text{ is done in } S2 \text{ since you are done in } S1. & \text{otherwise.} \end{cases}$$

③b

③a

X_1 = your time in S1 $\sim Exp(\mu_1)$

A_2 = A's time in S2 $\sim Exp(\mu_2)$

P(you finish S1 <u>before</u> A finish S2)

$$= P(X_1 < A_2)$$

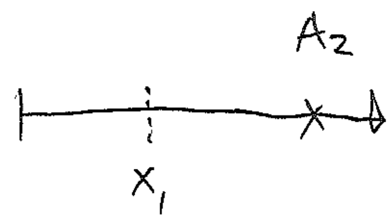
$$= \frac{\mu_1}{\mu_1 + \mu_2}$$

③b

time until A is done in S1

since you are done in S1

$$E\left(\underbrace{A_2 - X_1 \mid X_1 < A_2}_{\sim \mathrm{Exp}(\mu_2)}\right)$$



$\sim \mathrm{Exp}(\mu_2)$ by memoryless. prop.

$$= \frac{1}{\mu_2}$$

$$③ = \frac{1}{\mu_2}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)$$

$E(\text{time in Sys})$

$$= ① + ② + ③ + ④$$

$$= \frac{2}{\mu_1} + \frac{1}{\mu_2}\left(1 + \frac{\mu_1}{\mu_1+\mu_2}\right).$$

P353 Ross.

Prob. 47

Two-Server Parallel Queue.

customer ~ Poi Proc. (λ).

Service times ~ Exp(μ).

If customers find both servers are busy, they leave the system.

Customer leave the system after being served by one server.

a). If both S's are busy, what is

E(time until next customer who enters the system).

= E(time until one server opens)

+ E(time until 1st customer <u>after</u> one server opens).

a) time until one server open

$$= \min(\cancel{\text{Exp}}\, X_1 \,,\, X_2)$$

$X_1 \sim \text{Exp}(\mu)$

$X_2 \sim \text{Exp}(\mu)$

$\min(X_1, X_2) \sim \text{Exp}(2\mu)$.

$$E(\min(X_1, X_2)) = \frac{1}{2\mu}.$$

$X_3 \sim \text{Exp}(\lambda)$

$$E\left(\text{Time until 1st customer } \underline{\text{after}} \text{ one server opens}\right) = E(X_3) = \frac{1}{\lambda}$$

__Ans.__ $\frac{1}{2\mu} + \frac{1}{\lambda}$.

b) Starting both server empty,

what is

E [both server busy].

Let T_i = Time until both server busy starting with i server busy.

want $E[T_0]$.

Note,

$$E[T_0] = \underbrace{E[\text{1st Customer}]}_{\frac{1}{\lambda}} + E[T_1]$$

Now ~~Next~~ Consider T_1, let.

$$X = \text{time until} \left\{ \begin{array}{l} \text{1st Customer leaves} \\ \quad \text{or} \\ \text{2nd Customer arrives} \end{array} \right\}$$

starting with 1 busy server.

what is distribution of X ?

① 1st Customer leaves $\sim \text{Exp}(\mu)$

② 2nd Customer arrives $\sim \text{Exp}(\lambda)$.

$$\underbrace{\left[\text{time until } ① \text{ or } ②\right]}_{X} = \min(①, ②) \sim \text{Exp}(\mu+\lambda).$$

$$E[X] = \frac{1}{\mu+\lambda}.$$

Then

$$E[T_1] = E[X] + E[Y]$$

where

Y = time ~~time~~ ~~until~~ since event in X until both server busy.

$$Y = \begin{cases} 0 & \text{if } X \text{ was arrival time} \\ T_0 & \text{if } X \text{ was departure time.} \end{cases}$$

$$= E\left(E[Y \mid X] \right)$$

$$E[Y] = E[Y \mid X \text{ was arrival}] \cdot P[X \text{ was arrival}]$$
$$+ E[Y \mid X \text{ was departure}] \cdot P[X \text{ was departure}]$$

$$= 0 \cdot \left(\frac{\lambda}{\lambda+\mu} \right)$$
$$+ E[T_0] \cdot \left(\frac{\mu}{\lambda+\mu} \right)$$

$$E[T_0] = \frac{1}{\lambda} + E[T_1]$$

$$= \frac{1}{\lambda} + E[X] + E[Y]$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + E[T_0] \cdot \frac{\mu}{\lambda+\mu}$$

Solving for $E[T_0]$,

$$\boxed{E[T_0] = \frac{2\lambda+\mu}{\lambda^2}}$$