Actuarial Science II

: Rotar

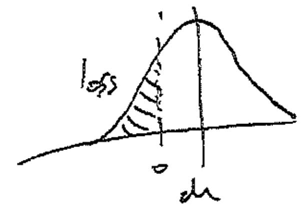
Chapter 1.

Ch. 0

Intro:

X_1 : investment 1. μ, σ^2

X_2 : investment 2. μ σ^2.



$E(X_1) = \mu$ $\quad V(X_1) = \sigma^2$.

$E\left(\frac{X_1+X_2}{2}\right) = \underbrace{\mu}_{\text{same}}$. $\quad V\left(\frac{X_1+X_2}{2}\right) = \underbrace{\frac{\sigma^2}{2}}_{\text{smaller!}}$.

Redistribution of Risk.

Diversification

$X_1, \ldots, X_n$

$$E(\bar{X}) = \mu$$

$$V(\bar{X}) = \frac{\sigma^2}{n}.$$

d. D

Reading CDF:

Discrete

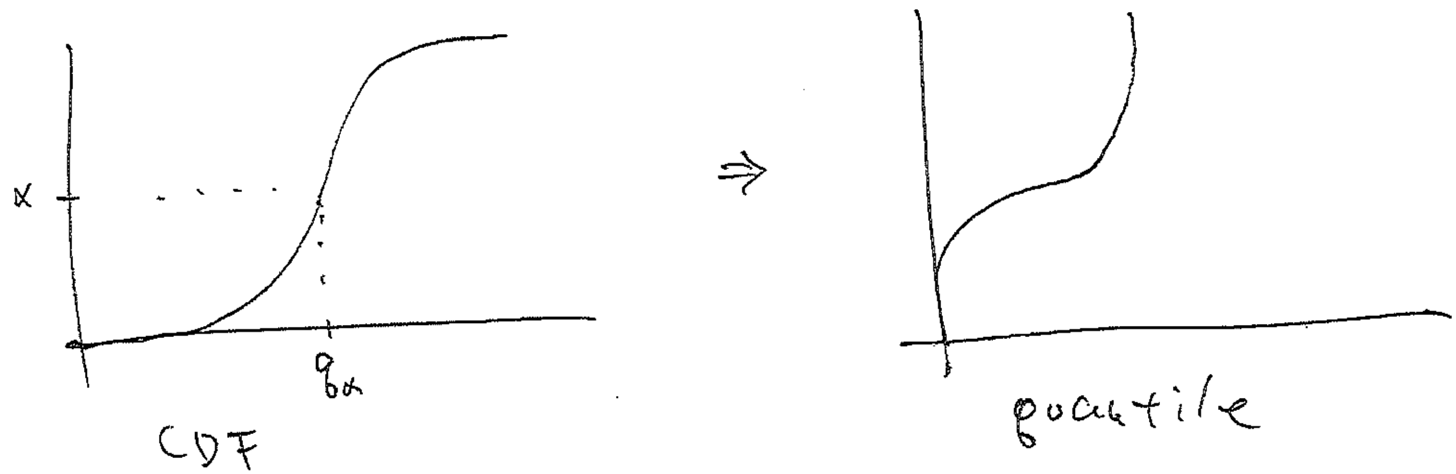
Continuous.

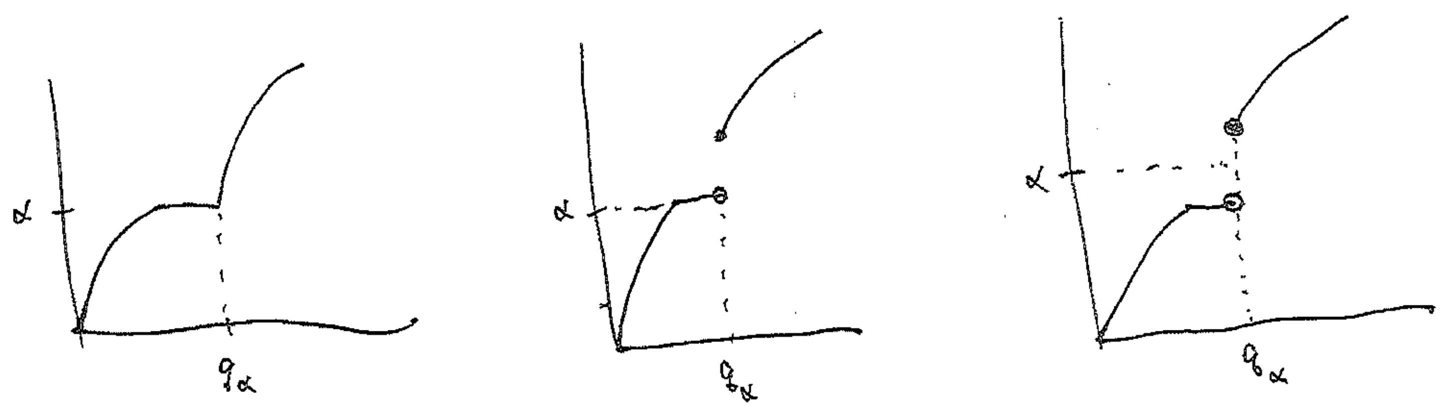
Mixed.

Ch. 0

Quantiles

(inverse of CDF)





Ch. 1

Comparison of RVs

$$X = \begin{cases} 50 & \text{w.p. } .4 \\ -30 & \text{w.p. } .6 \end{cases}$$

$E[X] = 2$

$V[X] = 1536$

$SD[X] = 39.2$

$Y = 1 \quad \text{w.p. } 1$

Which one do you prefer?

$$X = \begin{cases} 1000{,}000 - 1 & \text{w.p. } 1/2{,}000{,}000 \\ -1 & \text{w.p. } 1 - \frac{1}{2M} \end{cases}$$

$$Y = 0$$

which one do you prefer?

$$X = \begin{cases} 0 & \text{w.p. } .999 \\ -20{,}000 & \text{w.p. } .001 \end{cases} \qquad E[X] = -20$$

$$Y = -30 \quad \text{w.p.}$$

which one?

Y is like insurance.

$$X = \begin{cases} 100 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{cases}$$

$$Y = 50.$$

If, $X = Y$. then you are risk-neutral person.

Risk

1. We need some kind of "Measure"

so that we can say $g(X)$

$g(Y)$

$X \gtrsim Y$

X is preferable to Y. or equivalent to Y.

X is not worse than Y.

2. ~~Need to~~ Use Utility Theory

to explain the people's preferance.

Preference Order on the class $\mathcal{X}$

Risk Measure

1. Completeness $X \succsim Y$ $X \precsim Y$ or both.
2. Transitivity $X \succsim Y$ and $Y \succsim Z$ means $X \succsim Z$
3. Motonicity (strict) $P(X \geq Y) = 1$ $X \succsim Y$ $\rho(X) \geq \rho(Y)$ "larger the better"

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4. Translation invariant. $\rho(X+c) = \rho(X) + c$
5. Positive Homogenic $\rho(cX) = c\,\rho(X)$
6. Sub additive $\rho(X+Y) \leq \rho(X) + \rho(Y)$

Mean-value criterion

$$X \succsim Y \iff E[X] \geq E[Y]$$

strictly monotone :

Does not explain preference like

$$X = \begin{cases} 100 & .5 \\ 0 & .5 \end{cases} \qquad Y = \{ 50$$

(Risk - neutral).

Value at Risk Criterion

$q_\gamma(X)$: 100γ th percentile of X.

$$X \succsim Y \iff q_\gamma(X) \geq q_\gamma(Y)$$

γ must be chosen.

Monotone but not

strictly monotone.

Ex 1:

$$X = \begin{cases} 10 & .9 \\ 0 & .1 \end{cases} \qquad Y = \begin{cases} 10 & .93 \\ 0 & .07 \end{cases}$$

$r = .05$ $\quad q_r(X) = 0 = q_r(Y)$ $\quad$ X, Y equivalent.

$r = .08$ $\quad q_r(X) = 0 \quad q_r(Y) = 10$

Y is preferable

Ex 2

$$Y \sim U(0,2)$$

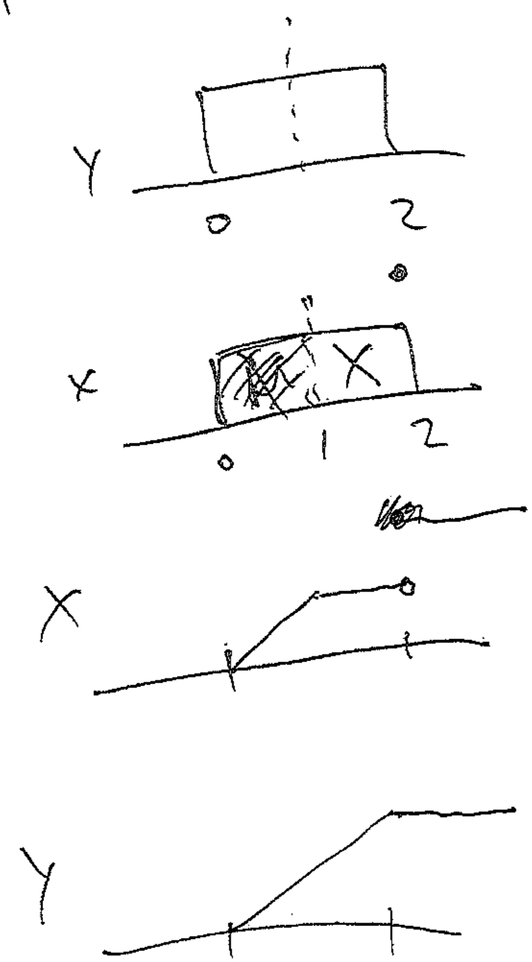
$$X = \begin{cases} Y & Y \le 1 \\ 2 & o/w \end{cases}$$

$Var(X) \sim Var(Y),$

$VaR(.05) =$

$q_{.05}(X) =$

$q_{.4}(X) = q_{.4}(Y)$.



Ex 3.

VaR of $N(m, \sigma^2)$

$$q_r(X) = m + q_r \sigma$$

$$= m + z_{1-r} \sigma$$

Ex 4

$X_1 \; \cdots \; X_{10}$

\$10 in 1 stock $\quad 10X_1 \sim N(10\mu, 100\sigma^2)$ $\quad$ 10 Million.

$$q_{.1}(X_1) = 10\mu - 16.4\,\sigma$$

\$1 each in 10 stocks $\quad X_1 + \cdots + X_{10} \sim N(10\mu, 10\sigma^2)$

better

$$q_{.1}(Y_1) = 10\mu - \underbrace{1.64(\sqrt{10})}_{5.2}\,\sigma$$

Ex 5 (Compare to Ex 4)

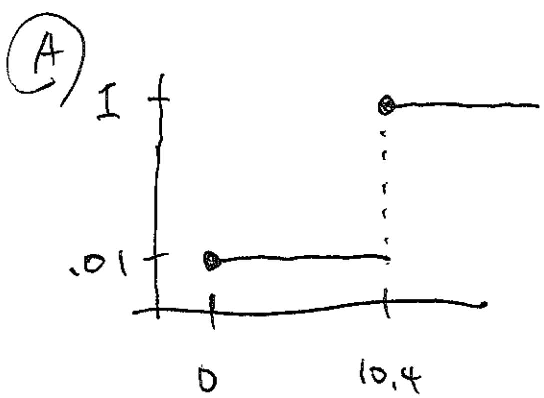
10 indep. assets.

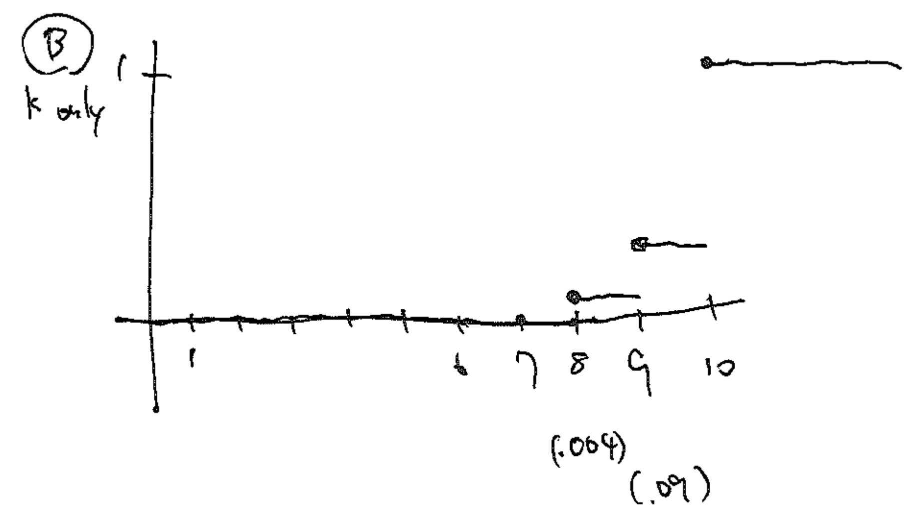
Each Return $\begin{cases} 4\% & .99 \\ -100\% & .01 \end{cases}$

(A) \$10 in 1 asset : $E(\text{Profit}) = 10.4(.99)$

(B) \$1 each in 10 assets : $E(\text{Profit}) = k(1.04)$

$k \sim BIN(10, .99)$





$\gamma = .05$

$q_{.05}(X) = 10.4$

$q_{.05}(Y) = 9(1.04) = 9.36$

X is better!

$\gamma = .009$

$q_{.009}(X) = 0$

$q_{.009}(Y) = 9.36$

Y is better

1.2.5 Tolerance to Risk

$$v(X) = \tau m_X - \sigma_X$$

want more   want less.

Just like $q_r = m_X + k\sigma_X$ from Normal, but above is general.

very popular,

$X \succsim Y \iff v(X) > v(Y)$

Ex 1 p80

$X = 0$
$a \geq 1$

$$Y = \begin{cases} a & \text{w.p. } \frac{1}{a} \\ 0 & \text{"} \quad 1 - \frac{1}{a} \end{cases}$$

$E(Y) = 1$
$Var(Y) = [a - 1]$

$$\rho(Y) = \tau \cdot 1 - \sqrt{[a-1]}$$

$$\rho(X) = \tau \cdot 0 - 0 = 0.$$

as $\rho(Y)$ is neg.

X is better than Y.

Ex 2.

Propos. 1

partial deriv. $g_1(x, y)$ $g_2(x, y)$ are cont. func.

$\frac{\partial}{\partial x} g$ $\frac{\partial}{\partial y} g$

Then for any r.v. X with a finite var

$\exists$ r.v. Y $\in$

$$P(Y \geq X) = 1 \quad \text{while} \quad g(m_X, \sigma_X) > g(m_Y, \sigma_Y).$$

not good idea to use $\underbrace{m_X, \sigma_X}$ as

2 ~~RV~~ Comparison of R.V. and Limit Theorems

2.1 simple model of Ins. with many clients.

$X_1 \cdots X_n$ I.I.D.

$\bar{X} \sim N($

homogeneous group.

$m = E[X_i]$

$c = m + \varepsilon$ premium

$$S_n = X_1 + \cdots + X_n$$

$$P(nc - S_n \geq 0) = P(S_n - mn \leq n\varepsilon)$$

Profit

$$= P(\bar{X} - m \leq \varepsilon)$$

$$P(|\bar{X} - m| \leq \varepsilon) \leq \frac{Var(\bar{X})}{\varepsilon^2}$$ Chebyshev.

$$P(|\bar{X} - m| \leq k\sigma) \leq \frac{1}{k^2}$$

$$P(n\underset{m+\varepsilon}{c} - S_n > 0) = P\left(\frac{\cancel{n}(\bar{X} - m)}{\sigma/\sqrt{n}} \le \cancel{\infty}\right)$$

$$\frac{\varepsilon\sqrt{n}}{\sigma}$$

$$= P(Z \le a)$$

if $\varepsilon = a\frac{\sigma}{\sqrt{n}}$

$$C = m + q_\beta \frac{\sigma}{\sqrt{n}}$$

$$P(\text{profit} \le q_\beta) = \beta .$$

Ex 1

Airline pays $b = 150$ if delay.

10,000 customers

$P(\text{Delay}) = 0.1$

$$X_i = \begin{cases} b & \text{w.p. } .1 \\ 0 & .9 \end{cases}$$

$\mu = 15$

$\sigma = b\sqrt{p(1-p)} = 45$

Choose $\beta = .95$

$$C = 15 + 1.65 \frac{45}{\sqrt{10{,}000}} = 15.74$$

Premium 15.74 Company makes profit 95 % of times.

2.2 St. Petersburg' Paradox.

1738 Bernoulli

Flip coin until head.

	1st flip	2nd	3rd	....	nth
if (H)	\$2	4	8		2^n

$$E(x) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 8\left(\frac{1}{8}\right) + \cdots$$

$$= \infty .$$

How much should dealer charge to play this game? $\$\infty$?

$$\bar{X} \to \infty.$$

$$\frac{X_1 + \cdots + X_n}{n} \to \infty.$$

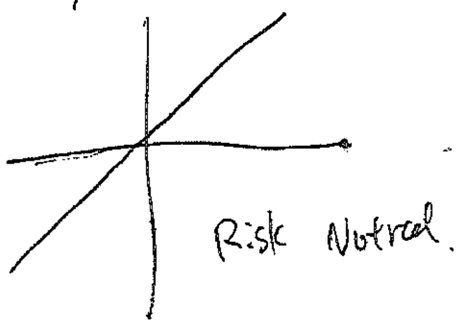
$$\frac{X_1 + \cdots + X_n}{n} \to \log_2 n$$

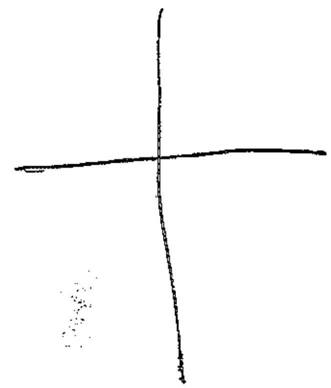
$$\frac{\bar{X}}{\log_2(n)} \to 1.$$

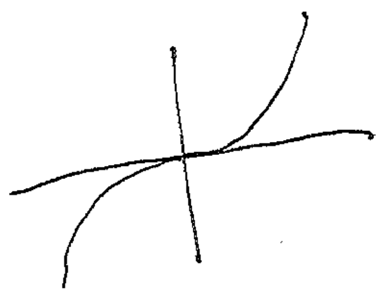
<u>Solution</u> charge $\log_2 n$ to ~~play~~ flip n times.

3. Expected Utility Maximization

Utility Func.







5M

$X = 50 \quad .5 \quad 0$

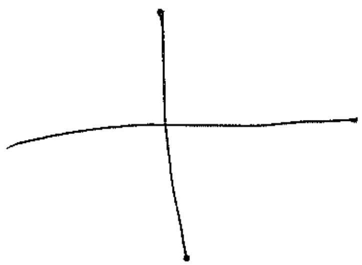
300

$$Y = \begin{cases} 100 & .5 \quad 1 \quad 50 \\ 0 & .5 \quad 0 \quad -50 \end{cases}$$

0

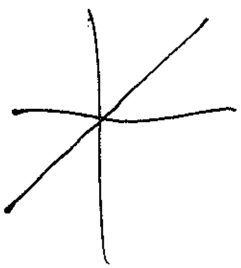
500

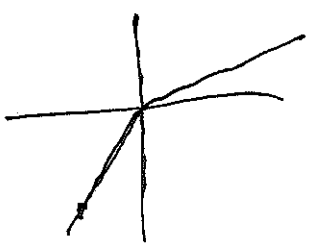
100



$X = 0$ w.p. 1

$$Y = \begin{cases} 500 & .5 \\ -500 & .5 \end{cases}$$



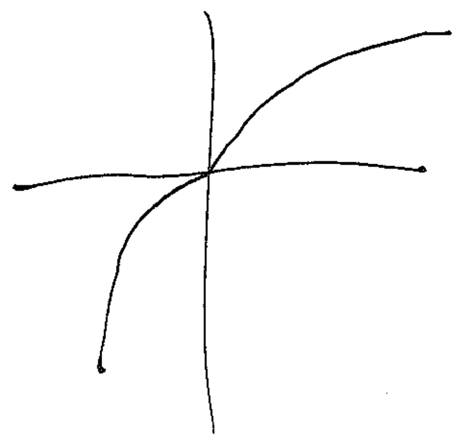


$X = 1$ w.p. 1

$$Y = \begin{cases} 100 & \text{w.p. } .01 \\ -1 & .99 \end{cases}$$

$E[Y] = 1 \quad .99$

$=$



Expected Utility Criterion

$$E(U(X)) \quad \text{vs} \quad E(U(Y)).$$

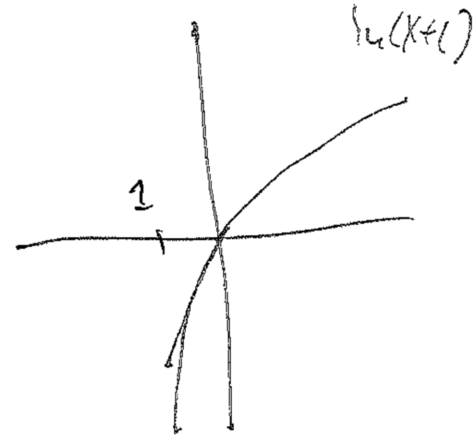
linear transf'n of $U(\cdot)$ does not change the relationship.

$$U(X) > U(Y)$$

then $aU(X) + b > aU(Y) + b.$ $\quad a > 0.$

Ex 3.

$U(x) = \ln(x + 1)$



0

$$\begin{cases} 100 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{cases}$$

$\{ 50$

Utility and Insurance

wealth w

random loss ξ

utility $U(x)$.

premium G ?

$$\begin{cases} X = W - G & \text{issured.} \\ Y = W - \xi & \text{not Insured} \end{cases}$$

$$E[U(W-G)] \overset{?}{\geq} E(U(W-\xi))$$

Otherwise, they don't buy Ins.

Ex 1 Find G_{max}.

$U(x) = 2x - x^2$ (concave?)

$W = 1$

$\xi \sim U(0,1)$

Insured $E(U(w - G)) = U(w - G)$ non-random

Non-Ins $E(U(w - \xi)) = E[2(1-\xi) - (1-\xi)^2]$

$= 2E(1-\xi) - E((1-\xi)^2)$

$= 2(.5) - \left(\frac{1}{12} + .5^2\right)$ $\frac{1}{4} = \frac{3}{12}$

$= 1 - \frac{\quad}{12}$

$$2(1 - G) - (1-G)^2 \geq \frac{8}{12} = \frac{2}{3}.$$

$$2 - 2G - (1 - 2G + G^2) \geq \frac{2}{3}$$

$$-G^2 + 1 - \frac{2}{3} \geq 0$$

$$\frac{1}{3} \geq G^2$$

$$G_{max} = \frac{1}{\sqrt{3}} = \boxed{.577}$$

max premium he/she will pay.

$$E[\xi] = \frac{1}{2} = .5$$

Ex 2 Insurer find H_{min}.

$U(x) = x^{\alpha}$ $\alpha < 1$

$w_1 = 1$

$y = w$ not sell policy

$x = w + H - \xi$

$$W = E(1 + H - \xi)^{\alpha} = E[(H + \eta)^{\alpha}] = \int_0^1$$

$$= \frac{1}{\alpha+1}\left[(H+1)^{\alpha+1} - H^{\alpha+1}\right]$$

if $\alpha = \frac{1}{2}$, $H_{min} = .52$

If $H_{min} > G_{max}$ then insurance is impossible.

- Competition

Drives premium $\begin{cases} \text{Gray} \\ \text{Hain} \end{cases}$

- p depends on w.

- If $u(x) = -e^{-\beta x}$. p doesn't depend on w.

3.4

Risk Aversion.

Cond. Z $\quad X \succsim X + Z_\varepsilon$

indep.

Prop. 3.

cond. Z happens iff $u(x)$ is concave.

Risk averse.

Ins vs No Ins

$$U(W - G_{max}) = E\left(U(W - \xi)\right)$$

$$\leq U\left(E(W - \xi)\right)$$

$$\leq U\left(W - E(\xi)\right)$$

$$G_{max} \geq E(\xi).$$

Jensen's Ineq.

$U(\cdot)$

convex

$$E(U(x)) \geq U(E(x))$$

concave

$$\leq$$

5. Optimal Pmt from point of Insured. p129

X loss r.v.

$$g = (1+\theta)\lambda$$ mean pmt $= E(X)$ for comp. coverage.

g: prem.

θ: rel. sec. loading fac.

Insured may want incomplete coverage.

$$\lambda \neq E(X).$$

Payment func. $r(x)$.

$$E(r(x)) = \lambda.$$

If $r(x)$ is non-decreasing.

$$E(r(x)) = \int_0^\infty (1 - F(x))\, dr(x)$$

$$\frac{dr(x)}{dx} \cdot dx .$$

Ex 1

$r(x) = kx \qquad k < 1$

$$\lambda = E(r(X)) = kE(X)$$

$$\therefore k = \lambda / m$$

Ex 2

Excess of loss ins.

Stop-loss.

$$r(x) = r_d(x) = \begin{cases} 0 & \\ x-d & x > d \end{cases} \quad \text{(deductible)}$$

$$\lambda = E(r(x)) = \int_d^\infty (1 - F_0(x))\, dx$$

Ex 3 . lim. coverage

$$r = \begin{cases} x & x \le S \\ S & x > S \end{cases}$$

$$\lambda = \int_0^S (1 - F(x)) \cdot dx$$

$Y \sim F(y)$.

$$U(F) = E[U(X)] = \int_0^\infty u(x) f(x)\, dx.$$

$$= \int_0^\infty U(x)\, dF(x)$$

Let

$$Y = w - g - X + r(X)$$

g: Pay premium; X: Random Loss; $r(X)$: Insurance Payment.

Can we find $r(\cdot)$ s.t. $U(F)$ is maximum?

Under certain conditions.

Optimal $r(\cdot)$ doesn't depend on shape of $U(\cdot)$ and g,

Optimal $r(\ )$ has shape

$$\begin{cases} 0 & x \le d \\ x-d & x > d \end{cases}$$

Arrow's theorem

$U(x)$ be concave and $r^*(x)$ = dedc.

then for any $r(\cdot)$ from RA.

$E(r(\cdot)) = \lambda$

$$Q(r) \le Q(r^*)$$

Ex

A $\quad U(\cdot) = \sqrt{x}$

B $\quad U(\cdot) = \ln(x)$

$$r_{opt}(\cdot) = \begin{cases} 0 \\ x - d \end{cases}$$

same d. for

same λ.

Ch. 1

2.

3. a ~ e.

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