University of Akron, Dept. of Statistics

3470:651 Probability and Statistics

Common Continuous Distributions

Textbook: Casella and Berger 2ed. (2013)

(October 5, 2016)

Contents

3.1 Uniform

[top]

$X \sim Unif(a,b)$

$$\text{pmf}: \quad f(x) = \frac{1}{b-a} \quad \text{for } x \in [a,b]$$

$$\text{CDF}: \quad F(x) = \frac{x-a}{b-a} \quad \text{for } x \in [a,b]$$

$$\text{mean and var}: \quad E(X) = \frac{b+a}{2} \qquad V(X) = \frac{(b-a)^2}{12}$$

$$\text{MGF}: \quad M(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$$

```
dunif(2, a,  b)       #pmf at x=2
punif(2, a,  b)       #CDF at x=2
qunif(.5, a,  b)      #Inv CDF at q=.5
runif(1000, a, b)     # random sample of size 1000
x=seq(-1,4,.01);  plot(x,dunif(x,1,3), type="l", ylim=c(0,1))  #plot pdf
x=seq(-1,4,.01);  plot(x,punif(x,1,3), type="l", ylim=c(0,1))  #plot CDF
```

3.2 Normal

[top]

$X \sim \mathrm{N}(\mu, \sigma^2)$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R} \\
\text{CDF}: \quad F(x) &= \int_{-\infty}^{x} f(t)dt \\
\text{mean}: \quad E(X) &= \mu \\
\text{var}: \quad V(X) &= \sigma^2 \\
\text{MGF}: \quad M(t) &= e^{\mu t + \frac{\sigma^2}{2} t^2}
\end{aligned}
$$

μ is location parameter, and σ is scale parameter.

```
dnorm(2, mu, sigma)        #pmf at x=2
pnorm(2, mu, sigma)        #CDF at x=2
qnorm(.5, mu, sigma)       #Inv CDF at q=.5
rnorm(1000, mu, sigma)     # random sample of size 1000.
x=seq(-4,4,.01);  plot(x,dnorm(x,0,1), type="l", ylim=c(0,1))  #plot pdf
x=seq(-4,4,.01);  plot(x,pnorm(x,0,1), type="l", ylim=c(0,1))  #plot CDF
```

3.3 Exponential

[top]

$X \sim \text{Exp}(\beta)$

$$
\begin{aligned}
\text{pdf}: \quad p(x) &= \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad \text{for } x > 0 \\
\text{CDF}: \quad F(x) &= 1 - e^{-x/\beta} \quad \text{for } x > 0 \\
\text{mean}: \quad E(X) &= \beta \\
\text{var}: \quad V(X) &= \beta^2 \\
\text{MGF}: \quad M(t) &= \left[\frac{1}{1 - t\beta}\right]
\end{aligned}
$$

β is a scale parameter

```
dexp(2, 1/b)        #pmf at x=2
pexp(2, 1/b)        #CDF at x=2
pexp(.5, 1/b)       #Inv CDF at q=.5
rexp(1000, 1/b)     # random sample of size 1000.  mean should be b
x=seq(-1,5,.01);  plot(x,dexp(x,1/2), type="l", ylim=c(0,1))  #plot pdf
x=seq(-1,5,.01);  plot(x,pexp(x,1/2), type="l", ylim=c(0,1))  #plot CDF
```

3.4 Gamma

$X \sim \text{Gam}(\alpha, \beta)$ [top]

$$\text{pdf}: \quad f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \qquad x > 0$$

$$\text{CDF}: \quad F(x) = \frac{\Gamma(x/\beta; \alpha)}{\Gamma(\alpha)} \qquad x > 0$$

$$\text{mean and var}: \quad E(X) = \alpha\beta \qquad V(X) = \alpha\beta^2$$

$$\text{MGF}: \quad M(t) = \left[\frac{1}{1 - t\beta}\right]^{\alpha}$$

α is a shape parameter, β is a scale parameter

```
dgamma(2, a, scale=b)        #pmf at x=2
pgamma(2, a, scale=b)        #CDF at x=2
pgamma(.5, a, scale=b)       #Inv CDF at q=.5
rgamma(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,10,.01);  plot(x,dgamma(x,2,scale=2), type="l", ylim=c(0,.5))  #plot pdf
x=seq(-1,10,.01);  plot(x,pgamma(x,2,scale=2), type="l", ylim=c(0,.5))  #plot CDF
```

Gamma Func: $\Gamma(\alpha) = \int_0^1 x^{\alpha-1} e^{-x} dx,$ Incomplete Gamma Func: $\Gamma(x, \alpha) = \int_0^x t^{\alpha-1} e^{-t} dt$

3.5 Chi-square

[top]

$X \sim \chi^2(\nu)$

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\Gamma(\nu/2)\, 2^{\nu/2}}\; x^{\nu/2-1}\, e^{-x/2} \\
\text{CDF}: \quad F(x) &= \Gamma(x/2; \nu/2)/\Gamma(\nu/2) \\
\text{mean}: \quad E(X) &= \nu \\
\text{var}: \quad V(X) &= 2\nu \\
\text{MGF}: \quad M(t) &= \left[\frac{1}{1-2t}\right]^{\nu/2} \qquad t < 1/2
\end{aligned}$$

same as $\text{Gam}(\nu/2, 2)$

```
dchisq(2, a, scale=b)        #pmf at x=2
pchisq(2, a, scale=b)        #CDF at x=2
pchisq(.5, a, scale=b)       #Inv CDF at q=.5
rchisq(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,4,.01);  plot(x,dchisq(x,3), type="l", ylim=c(0,.5))  #plot pdf
x=seq(-1,4,.01);  plot(x,pchisq(x,3), type="l", ylim=c(0,.5))  #plot CDF
```

3.6 Beta

$X \sim$Beta$(\alpha, \beta) \qquad \alpha > 0, \beta > 0$ [top]

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1 \\
\text{CDF}: \quad F(x) &= \frac{\beta(x;\alpha,\beta)}{\beta(\alpha,\beta)} \\
\text{mean and var}: \quad E(X) &= \frac{\alpha}{\alpha+\beta}, \qquad V(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
\text{MGF}: \quad M(t) &= 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}
\end{aligned}$$

```
dbeta(2, a, scale=b)        #pmf at x=2
pbeta(2, a, scale=b)        #CDF at x=2
pbeta(.5, a, scale=b)       #Inv CDF at q=.5
rbeta(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-1,2,.01);  plot(x,dbeta(x,2,2), type="l", ylim=c(0,2))  #plot pdf
x=seq(-1,2,.01);  plot(x,pbeta(x,2,2), type="l", ylim=c(0,2))  #plot CDF
```

$$\begin{aligned}
\text{Beta function} \qquad \beta(\alpha,\beta) &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
\text{Incomplete Beta func} \qquad \beta(x;\alpha,\beta) &= \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt
\end{aligned}$$

3.7 Cauchy

[top]

$X \sim \text{Cau}(\theta, \sigma) \qquad \theta \in \mathbb{R}, \sigma > 0$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{1}{\pi\sigma} \frac{1}{1 + (\frac{x-\theta}{\sigma})^2} \qquad x \in \mathbb{R} \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } t > 0 \\
\text{mean and var}: \quad E(X) &= \text{Does not exist} \qquad V(X) = \text{Does not exist} \\
\text{MGF}: \quad M(t) &= \text{ Does not exsit}
\end{aligned}
$$

Special case of student's t, when df=1.

3.8 Weibull

[top]

$X \sim \text{Wei}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$

$$
\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{\alpha}{\beta}\, x^{\alpha-1}\, e^{-x^{\alpha}/\beta} \qquad 0 \le x \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } 0 \le x \\
\text{mean}: \quad E(X) &= \beta^{1/\alpha}\Gamma(1+1/\alpha) \\
\text{var}: \quad V(X) &= \beta^{2/\alpha}[\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)] \\
\text{moments}: \quad E(X^n) &= \beta^{n/\alpha}[\Gamma(1+n/\alpha)] \\
\text{MGF}: \quad M(t) &= \text{Exists only for } \alpha \ge 1.
\end{aligned}
$$

If $\alpha = 1$, it is $\text{Exp}(\beta)$

3.9 Student-t

[top]

$X \sim \mathrm{t}(\nu) \qquad \nu = 1, 2, 3, \ldots$

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } t > 0 \\
\text{mean and var}: \quad E(X) &= 0, \quad \nu > 1 \qquad V(X) = \frac{\nu}{\nu-2}, \quad \nu > 2 \\
\text{moments}: \quad E(X^2) &= \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{\nu-n}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\nu^{n/2} \qquad \text{if } n < \nu \text{ and even. 0 if odd.} \\
\text{MGF}: \quad M(t) &= \text{Does not exsit}
\end{aligned}$$

```
dt( 2, v)        #pmf at x=2
pt( 2, v)        #CDF at x=2
pt(.5, v)       #Inv CDF at q=.5
rt(1000, a, scale=b)     # random sample of size 1000.  mean should be a*b
x=seq(-4,4,.01);  plot(x,dt(x,5), type="l", ylim=c(0,1))  #plot pdf
x=seq(-4,4,.01);  plot(x,pt(x,5), type="l", ylim=c(0,1))  #plot CDF
```

3.10 F

[top]

$X \sim \mathrm{F}(\nu_1, \nu_2) \qquad \nu_1, \nu_2 = 1, 2, 3, \ldots$

$$\begin{aligned}
\text{pdf}: \quad f(x) &= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}\frac{x^{(\nu_1-2)/2}}{\left(1+(\frac{\nu_1}{\nu_2})x\right)^{(\nu_1+\nu_2)/2}} \qquad 0 \leq x \\
\text{CDF}: \quad F(x) &= \int_0^x f(t)dt \quad \text{for } x > 0 \\
\text{mean}: \quad E(X) &= \frac{\nu}{\nu_2 - 2} \quad \nu_2 > 2 \\
\text{var}: \quad V(X) &= 2(\frac{\nu_2}{\nu_2-2})^2\frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)} \quad \nu_2 > 4 \\
\text{moments} \quad E(X^n) &= \frac{\Gamma(\frac{\nu_1+2n}{2})\Gamma(\frac{\nu_2+2n}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_2}{\nu_1}\right)^n, \qquad n < \frac{\nu_2}{2} \\
\text{MGF}: \quad M(t) &= \text{Does Not Exist}
\end{aligned}$$

3.11 Overlay plots in R

[top]

```
#-- Overlay with N(0,1) pdf --
plot(x, dt(x,5), type='l')
lines(x, dnorm(x,0,1), col='red')
```

```
#-- Overlay with N(0,1) pdf (method 2 - have to specify plot range) --
plot(x, dt(x,5), type='l', xlim=c(-5,5), ylim=c(0,.4))
par(new=T)
plot(x, dnorm(x,0,1), type='l', xlim=c(-5,5), ylim=c(0,.4), col='red')
```

3.12 Distributional Relations

[top]

- When X and Y are independent Exp(λ), $X + Y$ is Gam($2, \lambda$).
- When you have n iid Exponential r.v. with mean of λ, $\min(X_1, \ldots, X_n)$ is Exponenital with mean λ/n.
- Beta(1,1) is same as Unif(0,1).
- Cauchy is same as t(1).
- When X, and Y are independent U(0,1), X/Y is Cauchy.
- When X is U($-\pi/2, \pi/2$), $\tan(X)$ is Cauchy.
- If we have two independent r.v. $X_1 \sim$Gam(α_1, β) and $X_2 \sim$Gam(α_2, β) and

$$Y = \frac{X_1}{X_1 + X_2} \;\sim\; \text{Beta}(\alpha_1, \alpha_2)$$

- That is same as to say, if we have two independent r.v. $X_1 \sim \chi^2(\alpha_1)$ and $X_2 \sim \chi^2(\alpha_2)$

$$Y = \frac{X_1}{X_1 + X_2} \;\sim\; \mathrm{Beta}\Big(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}\Big)$$

- When U is $\chi^2(r_1)$ and V is $\chi^2(r_2)$, $\frac{U/r_1}{V/r_2}$ is $F(r_1, r_2)$,
- $F_{1,\nu}$ is same as $t^2(\nu)$
- Gamma function: $\frac{1}{2}! = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

3.13 Scale Parameter

[top]

- If you transform r.v. X to $Y = \theta X$,

$$\begin{aligned} F_Y(y) &= P(\theta X \leq y) = P(X \leq y/\theta) = F_X(x/\theta) \\ f_Y(y) &= \frac{1}{\theta} f_X\left(\frac{x}{\theta}\right) \end{aligned}$$

  θ is called the scale parameter.

- if Y has scale parameter θ, then

$$\frac{X}{\theta} \text{ has same distribution with } \theta = 1.$$

Example

If $X \sim$Exp(λ), then λ is a scale parameter. Thus

$$\frac{X}{\lambda} \sim \text{Exp}(1).$$

Example

If $X \sim$Gam(α, β), then β is a scale parameter. Thus

$$\frac{X}{\lambda} \sim \text{Gam}(\alpha, 1).$$

Example

If $X \sim$Gam(α, β), then $X/\beta \sim$Gam$(\alpha, 1)$. We can write cdf of Gam$(\alpha, 1)$ as

$$F_X(x) = \frac{1}{\Gamma(\alpha)} \underbrace{\int_0^x y^{a-1} e^{-y} dy}_{\text{(lower) incomplete gamma func}} = \frac{\Gamma(x, \alpha)}{\Gamma(\alpha)}, \qquad 0 < x < \infty$$

3.14 Detail Calculations

[top]

3.14.1 Gamma

- Gamma function:

$$\Gamma(\alpha) = \int_0^\infty y^{a-1} e^{-y} dy$$

- $\Gamma(1) = 1$.

- For $\alpha > 1$, integration by parts will show that

$$\Gamma(\alpha) = \int_0^\infty y^{a-1} e^{-y} dy = (a-1) \int_0^\infty y^{a-2} e^{-y} dy = (a-1)\Gamma(\alpha).$$

- When $\alpha \geq 1$, we have

$$\Gamma(\alpha) = (\alpha - 1)! \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- Change parameter of integration to $y = x/\beta$, with $\beta > 0$

$$
\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty y^{a-1} e^{-y} dy \\
&= \int_0^\infty \left(\frac{x}{\beta}\right)^{a-1} e^{-\frac{x}{\beta}} \left(\frac{1}{\beta}\right) dx
\end{aligned}
$$

- That means

$$
1 = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{a-1} e^{-\frac{x}{\beta}} dx
$$

- We let the integrand be pdf

- cdf

$$
F_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^x y^{a-1} e^{-\frac{y}{\beta}} dy \qquad 0 < x < \infty
$$

- α is shape parameter, β is scale parameter.

MGF of Gamma

- $$\begin{aligned}
M(t) = E[e^{tX}] &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty e^{tx} x^{a-1} e^{-\frac{x}{\beta}} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{a-1} e^{-(\frac{1}{\beta}-t)x} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{a-1} e^{-x/(\frac{\beta}{1-t\beta})} dx \\
&= \frac{(\frac{\beta}{1-t\beta})^\alpha}{\Gamma(\alpha)\beta^\alpha(\frac{\beta}{1-t\beta})^\alpha} \int_0^\infty x^{a-1} e^{-x/(\frac{\beta}{1-t\beta})} dx \\
&= \frac{(\frac{\beta}{1-t\beta})^\alpha}{\beta^\alpha} \underbrace{\frac{1}{\Gamma(\alpha)(\frac{\beta}{1-t\beta})^\alpha} \int_0^\infty x^{a-1} e^{-x/(\frac{\beta}{1-t\beta})} dx}_{\text{pdf of gamma integrated from 0 to } \infty} \\
&= \frac{(\frac{\beta}{1-t\beta})^\alpha}{\beta^\alpha} = (1-t\beta)^{-\alpha}
\end{aligned}$$

Sum of independent Exponentials

- Let $X_i \sim_{iid} \text{Exp}(\lambda)$
- Then

$$\sum_{i=1}^{n} X_i \sim \text{Gam}(n, \beta)$$

- proof by mgf
- shape of $\text{Gam}(n, \beta)$ when n is large?

Sum of independent Gammas

- Let $X_i \sim_{iid} \text{Gam}(\alpha_i, \beta)$
- Then

$$\sum_{i=1}^{n} X_i \sim \text{Gam}\Big(\sum_{i=1}^{n} \alpha_i, \beta\Big)$$

- proof by mgf

Posson Process

- Number of events in any time interval $(t_1, t_2)$is $\mathrm{Poi}(\lambda(t_2 - t_1))$.
- Waiting time until next event is $\mathrm{Exp}(\lambda)$ with mean λ.
- Waiting time for 3rd event from now?

Min and Max of Exponential

- Let $X_i \sim_{iid} \text{Exp}(\lambda)$. $i = 1, \ldots, n$.
- cdf for minimum of them?
- cdf for maximum of them?

Memoryless Property

- Probability that a phone lasts more than s year from now, given that it already lasted t year.

$$
\begin{aligned}
P\big(X > t+s | X > t\big) &= \frac{P(X > t+s \cap X > t)}{P(X > t)} \\
&= \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-(t+s)/\lambda}}{e^{-t/\lambda}} = e^{-s/\lambda}
\end{aligned}
$$

Back to Poisson Process

- Waiting time from NOW to the next event.
- Waiting time from NOW to the 3rd event.

Moments of Exponential

- mgf

$$M(t) = (1 - t\beta)^{-1}$$

-

$$\begin{aligned} E(X^k) = \frac{1}{\lambda}\int_0^\infty x^k e^{-x/\lambda} dx &= \frac{\Gamma(k+1)\lambda^{k-1}}{\Gamma(k+1)\lambda^k}\int_0^\infty x^k e^{-x/\lambda} dx \\ &= \Gamma(k+1) = k!\lambda^{k-1} \end{aligned}$$

3.14.2 Chi-square Distribution

- $\chi^2(\nu)$ =Gam($\alpha = \nu/2$, $\beta = 2$)
- $$E(X) = \nu, \qquad V(X) = 2\nu$$
- Sum of ν squared independent $N(0,1)$s.

$$X_i \sim_{iid} N(0,1) \qquad \sum_{i=1}^{\nu} X_i^2 \sim \chi^2(\nu)$$

- $\chi^2(4)$ is

  sum of 4 squared iid N(0,1) sum of 2 iid Exp(2)

- Is Exp(2) same distribution as sum of 2 squared N(0,1)?

- Sum of independent $\chi^2(\nu)$ still χ^2?

3.14.3 Normal

Check if pdf integrates to 1

$$\int_{-\infty}^{\infty} f(x)dx \leq \left(\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2 = \frac{1}{2\pi} \int \int e^{-\frac{x^2+y^2}{2}} dxdy$$

By using polar coordinates, $r = \sqrt{(}x^2 + y^2)$, and $dxdy = dr\, rd\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, drd\theta = 1.$$

MGF of Normal

If $X \sim N(0,1)$,

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+tx} dx \\ &= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx \;=\; e^{\frac{t^2}{2}} \end{aligned}$$

Let $Y = aX + b$. then

$$E(e^{t(aX+b)}) = E(e^{taX}) + e^{bt} = M_X(at) + e^{bt} = e^{\frac{(at)^2}{2}+bt}$$

However, by inspection of pdf, we know that $Y \sim N(b, a^2)$. Therefore, for $X \sim N(\mu, \sigma^2)$, mgf is

$$M(t) = e^{\frac{\sigma^2}{2}t^2+\mu t}$$

Moments of Z

$$\begin{aligned}
M(t) &= e^{t^2/2} \\
M'(t) &= te^{t^2/2} \\
M''(t) &= e^{t^2/2} + t^2 e^{t^2/2} \\
M'''(t) &= te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} \\
M''''(t) &= 3M''(t) + 3t^2 e^{t^2/2} + t^4 e^{t^2/2}
\end{aligned}$$

So $E(Z^2) = 1$, and $E(Z^4) = 3$.

Higher moments of Normal

- We can calculate higher moments of $N(\mu, \sigma^2)$ from moments of $N(0,1)$. Since $X = \sigma Z + \mu$,

$$E[X^k] = E\Big[(\sigma Z + \mu)^k\Big] = E\Big[\sum_{x=1}^{n} \binom{n}{x} (\sigma Z)^x \mu^{n-x}\Big]$$

$$= \sum_{x=1}^{k} \binom{k}{x} \sigma^x E[Z^x] \, \mu^{n-x}$$

- What is the 3rd moment of $N(2,3)$? What is the 4rd moment of $N(0,4)$?

Skewness

- For $X \sim N(\mu, \sigma^2)$,

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3} = 0$$

- Since $Y = X - \mu$ has zero mean,

$$E\left[Y^3\right] = \int y^3 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy = 0$$

Kurtosis

- For $X \sim N(\mu, \sigma^2)$,

$$\frac{E\left[(X-\mu)^4\right]}{\sigma^4} = 3$$

- Measures 'haviness' of the tail.

- Excess Kurtosis: (Kurtosis − 3).

- Larger Kurtosis → Heavier tail

Abs mean of N(0,1)

Let $X \sim$ N(0,1)

$$
\begin{aligned}
E(|x|) &= \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_0^\infty x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} (-e^{-x^2/2}) \Big|_0^\infty \;=\; \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

Chi-square from Normal

- Let X be $N(0,1)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Show that $X^2 \sim \chi^2(1)$.

- Let $Y = X^2$. CDF of Y is,

$$\begin{aligned} P(Y \le y) = P(X^2 \le y) &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-\frac{x^2}{2}} dx \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\sqrt{y}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

Take d/dy to get pdf of Y,

$$\begin{aligned} f_Y(y) &= \frac{2}{\sqrt{2\pi}} e^{-\frac{\sqrt{y}^2}{2}} \left(\frac{1}{2\sqrt{y}} \right) \\ &= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{y^2}{2}} \\ &= \frac{1}{\sqrt{\pi}\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y^2}{\beta}} \end{aligned}$$

note $\Gamma(1/2) = \sqrt{\pi}$. So $Y = X^2 \sim$Gam $\alpha = \frac{1}{2}, \beta = 2$, which is $\chi^2(1)$.

- How do you show that $X_1^2 + X_2^2$ is $\chi^2(2)$ if X_i are iid std normal?

Contaminated Normal

$$X = \begin{cases} Z & \text{w.p. } (1-p) \\ \sigma Z & \text{w.p. } p \end{cases}$$

Truncated Normal

Truncation vs Censoring

- Truncation

$$X | X > d$$

- Censoring

$$X = \begin{cases} X & \text{if } X \le d \\ d & \text{if } X > d \end{cases}$$