451/551 - HW on Ch. 5 Joint Distributions

Due Tue Oct. 20th

The final answer must be clealy indicated.

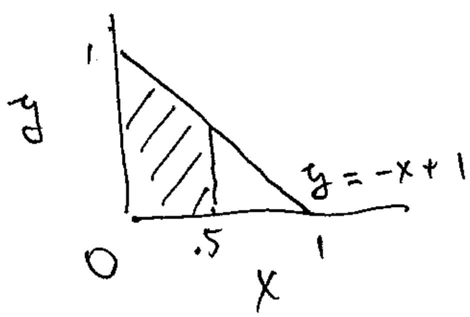
Name: ______________________________

1. A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x,y) = 6(1-(x+y)) \qquad \text{for } x>0, y>0, x+y<1$$

and 0 otherwise. Determine the probability that the portion of a claim representing damage to the house is less than 0.5.

$$P(X<.5) = \iint_{(1)} f(x,y)\,dy\,dx$$



$$= \int_0^{.5}\int_0^{1-x} 6(1-(x+y))\,dy\,dx$$

$$= \int_0^{.5} \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} dx$$

$$= 6\int_0^{\frac{1}{2}} \frac{1}{2}(1-x)^2\,dx = -(1-x)^3\Big|_0^{1/2} = \boxed{.875}$$

2. A joint density function is given by

$$f(x,y) = kxy^2 \qquad \text{for } 0 < x < 1, 0 < y < 2$$

and 0 otherwise, where k is a constant. What is Cov(X,y)? (hint: Are X, Y independent?)

X, Y must be independent, because

$$f(x,y) = k\, h(x) \cdot g(y)$$

for some function $h(\cdot)$ and $g(\cdot)$, and support is cartesian.

$\therefore \ \mathrm{Cov}(X,Y) = 0.$

If $f(x,y) = k\, h(x) \cdot g(y)$, then it has to be

$$f_Y(y) = \int k\, h(x)\, g(y)\, dx = k_2\, g(y)$$

$$f_X(x) = \int k\, h(x)\, g(y)\, dy = k_3\, h(x)$$

So that $f(x,y) = f(x) \cdot f(y)$

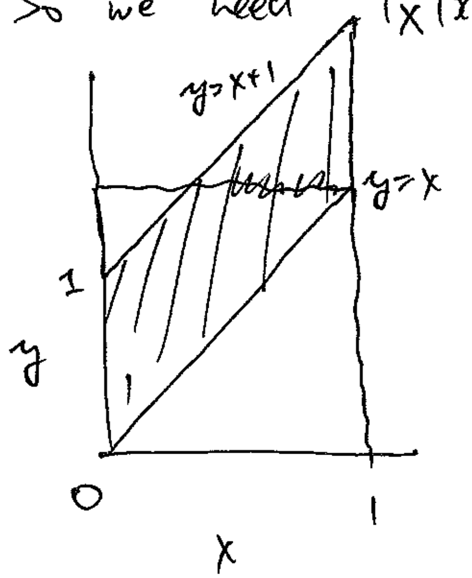
3. The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = 2x \quad for 0 < x < 1, x < y < x+1,$$

and 0 otherwise. What is the conditional expectation Y given that X = x? (i.e. $E[Y|X]$)

Need $f_{Y|X}(y;x) = \dfrac{f(x,y)}{f_X(x)}$.

So we need $f_X(x) = \displaystyle\int_{\text{all } y} f(x,y)\,dy$



$$= \int_x^{x+1} 2x\,dy = 2x$$

$$f_{Y|X}(y;x) = \frac{2x}{2x} = 1 \qquad x<y<x+1$$

$$E[Y|X] = \int_{\text{all } y} y \cdot f_{Y|X}(y)\,dy = \int_x^{x+1} y \cdot 1\,dy$$

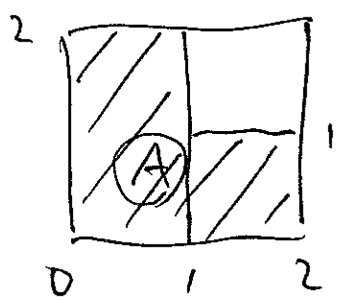
$$= \left.\frac{y^2}{2}\right|_x^{x+1} = \boxed{\frac{2x+1}{2}}$$

4. Suppose a system is consisted by two components. The system needs both of the componens to operate properly. The joint density function of the lifetimes of the components, measured in hours, is $f(x, y) = k(x+y^2)$, where $0 < x < 2$ and $0 < y < 2$. What is the probability that the system fails within the first hour? Set up the integral with proper bounds, but do not evaluate the integral.

$$\text{System life} = \min(X, Y)$$

$$P(\text{Sys. life} < 1) = P(\min(X, Y) < 1)$$

$$= \iint_{A} f(x, y)\, dx\, dy$$



$$= \int_0^2 \int_0^1 k(x+y^2)\, dx\, dy + \int_0^1 \int_1^2 k(x+y^2)\, dx\, dy$$

5. Claim amounts for wind damage to insured homes are modeled by random variable with CDF

$$F(x) = 1 - \frac{1}{x^2} \qquad \text{for } 1 \le x,$$

and 0 for $x < 1$. Claim amounts are assumed to be independent from home to home. Suppose 3 such claims will be made. What is the expected value of the largest of the three claims? Write down the integral with proper bounds, but do not evaluate the integral.

$$Y = \max(X_1, X_2, X_3)$$

$$F_Y(y) = \left(F_X(y)\right)^3 = \left(1 - \frac{1}{y^2}\right)^3 \qquad 1 \le y$$

$$f_Y(y) = F_Y'(y) = 3\left(1 - \frac{1}{y^2}\right)^2 \cdot \left(2\frac{1}{y^3}\right)$$

$$E[Y] = \int_1^\infty y \cdot 6\left(1 - \frac{1}{y^2}\right)^2 \left(\frac{1}{y^3}\right) dy$$

or

$$E[Y] = \int_0^\infty (1 - F_Y(y))\, dy = 1 + \int_1^\infty 1 - \left(1 - \frac{1}{y^2}\right)^3 dy$$

0 up to $y = 1$