Stat 516, Homework 4

Due date: Tuesday, October 31. **Note**: Do *individual* work.

1. (a) Prove that recurrence is a communication class property: $i \leftrightarrow j$ and i is recurrent $\Rightarrow j$ is recurrent.

$$
\begin{aligned}
P_i(X_k = j) &> 0 \\
P_j(X_m = i) &> 0 \\
P_{ij}^{(k)} > 0,\ P_{ii}^{(n)} &> 0,\ P_{ji}^{(m)} > 0 \\
\sum_{n=1}^{\infty} P_{ii}^{(n)} &= \infty = \sum_{n=1}^{\infty} P_i(X_n = i) \\
P_{jj}^{(k+m+n)} &= P_{ij}^{(k)} P_{ii}^{(n)} P_{ji}^{(m)} \\
&= P_{ij}^{(k)} P_{ji}^{(m)} \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty
\end{aligned}
$$

(b) Is the Markov chain with the transition graph below irreducible? Explain. Find its period and its cyclic classes. (Note: the graph has no self-loops, so the transitions from a state i to itself have probability zero.)

The markov chain is irreducible because each state can be reached by every other state given enough time. The cyclic classes of the markov chain are $\{3, 5, 6\}$ $\{1, 2\}$ $\{4, 7\}$ which makes d and the periodicity of the graph 3.

2. Consider a fair simple random walk on $\mathbb{Z}^2$, that is, each state (i, j) has four neighbors (in-/decrease i or j by 1) and the four transitions to these neighbors are equally probable. Show that the state (0,0) is recurrent. Is it positive recurrent?

Let $\rho_k^{(n)}$ be the probability of returning to the k-dimensional origin after n steps.

Given the 1D random walk

$$
\rho_1^{(n)} = \binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}
$$

then for the 2D random walk

$$
\begin{aligned}
\rho_2^{(n)} &= 4^{-2n} \sum_{m=0}^{n} \frac{2n!}{m!m!(n-m)!(n-m)!} \\
&= 4^{-2n} \binom{2n}{n} \sum_{m=0}^{n} \binom{n}{m} \binom{n}{n-m} \\
&= \left(2^{-2n} \binom{2n}{n}\right)^2 \\
&= \left(\rho_1^{(n)}\right)^2 \\
&\sim \frac{1}{\pi n}
\end{aligned}
$$

which means the series converges as $\rho_2^{(n)}$ is less than 1 as n increases.

3. For $\alpha, \beta, \gamma \in (0, 1)$, consider the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\beta & \beta \\ \gamma & 0 & 1-\gamma \end{pmatrix},$$

Show that the associated Markov chain is irreducible and compute its stationary distribution. Do it first by solving the linear equation system from the definition of a stationary distribution. Then derive the stationary distribution again using the "regenerative form of invariant measures."

By linear equations...

$$\begin{aligned}
\pi_1 &= \frac{\gamma\pi_3}{\alpha} \\
\pi_2 &= \frac{\alpha\pi_1}{\beta} \\
\pi_3 &= \frac{\beta\pi_2}{\alpha} \\
1 &= \pi_1 + \pi_2 + \pi_3 \\
1 &= \frac{\gamma\pi_3}{\alpha} + \frac{\gamma\pi_3}{\beta} + \pi_3 \\
1 &= \pi_3(\frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + 1) \\
\pi_3 &= (\frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + 1)^{-1} \\
\pi_1 &= (\frac{\alpha}{\gamma} + \frac{\alpha}{\beta} + 1)^{-1} \\
\pi_2 &= (\frac{\beta}{\alpha} + \frac{\beta}{\gamma} + 1)^{-1}
\end{aligned}$$

By regenerative form of interval measures...

$$\begin{aligned}
E[X] &= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + ... \\
E_1[X_1] &= 1 + 0 + 0 + ... \\
E_1[X_1] &= 1 \\
E_1[X_2] &= \alpha + \alpha(1-\beta) + \alpha(1-\beta)^2 + ... \\
E_1[X_2] &= \frac{\alpha}{\beta} \\
E_1[X_3] &= \alpha + \alpha(1-\gamma) + \alpha(1-\gamma)^2 + ... \\
E_1[X_3] &= \frac{\alpha}{\gamma} \\
\pi_1 = E_1[X_1] * \pi_1 &= (\frac{\alpha}{\gamma} + \frac{\alpha}{\beta} + 1)^{-1} \\
\pi_2 = E_1[X_2] * \pi_1 &= (\frac{\beta}{\alpha} + \frac{\beta}{\gamma} + 1)^{-1} \\
\pi_3 = E_1[X_3] * \pi_1 &= (\frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + 1)^{-1}
\end{aligned}$$

4. (Guttorp 2.2.14) Consider a Markov chain on a finite state space S, with a t.p.m. that has entries $p_{ij} > 0$ for all $i, j \in S$. Let π be the stationary distribution. Show that the n-step transition probabilities $p_{ij}^{(n)}$ satisfy

$$\left|p_{ij}^{(n)} - \pi_j\right| \;\leq\; (1 - d\delta)^n,$$

where $d = |S|$ and $\delta = \min p_{ij}$.

Hint: Divide the terms in the equation $\sum_j (p_{ij} - p_{kj}) = 0$ into those with positive and those with negative values. The sum of the positive terms is bounded by $1 - d\delta$. Now bound $\max_i p_{ij}^{(n+1)} - \min_i p_{ij}^{(n+1)}$ using the Chapman-Kolmogorov equations (i.e., one-step analysis).

$$\pi_j = \sum_{k \in S} \pi_k p_{kj}$$
$$p_{ij}^{(n)} - \pi_j = -\sum_{k \in S} \pi_k \left(p_{kj}^{(n)} - p_{ki}^{(n)}\right)$$