

Stat 516, Homework 1

Due date: Thursday, October 5, at the beginning of class.

1. *Gamma-Poisson mixture*

Recall that if $Z \sim \text{Poisson}(\lambda)$, then $E(Z) = \text{var}(Z) = \lambda$, where $\lambda > 0$ is the intensity parameter. This property of the Poisson distribution is restrictive since often in practice *excess-Poisson variation* is encountered, i.e. the variance of observed count data exceeds the mean. One possible solution to this discrepancy is to use a Gamma-Poisson mixture, which is generated by first randomly drawing $X \sim \text{Gamma}(\alpha, \beta)$, where $\alpha, \beta > 0$ (the parameterization is such that that $E(X) = \alpha/\beta$ and $\text{var}(X) = \alpha/\beta^2$) and then generating $Y \mid X \sim \text{Poisson}(X)$.

(a) Derive $E(Y)$ and $\text{var}(Y)$ and show that $E(Y) < \text{var}(Y)$.

$$\begin{aligned} E[Y] &= E[E(Y|X)] \\ &= E[X] \\ &= \frac{\alpha}{\beta} \end{aligned}$$

$$\begin{aligned} \text{Var}[Y] &= E[\text{Var}(Y|X)] + \text{Var}[E(N|X)] \\ &= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} \\ &= \frac{\beta\alpha + \alpha}{\beta^2} \\ &= \frac{\alpha(\beta + 1)}{\beta^2} \end{aligned}$$

$$\begin{aligned} E[Y] &< \text{Var}[Y] \\ \frac{\alpha}{\beta} &< \frac{\alpha(\beta + 1)}{\beta^2} \\ \alpha\beta &< \alpha(\beta + 1) \\ \beta &< \beta + 1 \\ 0 &< 1 \end{aligned}$$

(b) Derive the marginal distribution of Y .

$$\begin{aligned}
P(Y = k | \alpha, \beta) &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \frac{\lambda^k e^{-\lambda}}{k!} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)k!} \int_0^\infty \lambda^{\alpha+k-1} e^{-\lambda(\beta+1)} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)k!} \frac{\Gamma(\alpha+k)}{(\beta+1)^{\alpha+k}} \int_0^\infty \frac{(\beta+1)^{\alpha+k}}{\Gamma(\alpha+k)} \lambda^{\alpha+k-1} e^{-\lambda(\beta+1)} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)k!} \frac{\Gamma(\alpha+k)}{(\beta+1)^{\alpha+k}} \\
&= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)} \left(\frac{\beta}{\beta+1} \right)^\alpha \left(1 - \frac{\beta}{\beta+1} \right)^k
\end{aligned}$$

2. *Brémaud, 1.7.1.*

Let X_1 and X_2 be two independent random variables taking their values in $\{1, 2, \dots, N\}$, and uniformly distributed, that is, $\Pr(X_i = k) = \Pr(X_2 = k) = \frac{1}{N}$, $1 \leq k \leq N$. Compute $E[X_1 \mid \max\{X_1, X_2\}]$.

$$Z = \max\{X_1, X_2\}$$

$$P(X_1 > Z|Z) = 0$$

$$P(X_1 = Z|Z) = \frac{Z}{2Z-1}$$

$$V = \{1, 2, \dots, Z-1\}$$

$$v \in V$$

$$P(X_1 = v|Z) = \frac{1 - P(X_1 = Z|Z)}{Z-1}$$

$$\begin{aligned} P(X_1 = v|Z) &= \frac{1 - \frac{Z}{2Z-1}}{Z-1} \\ &= \frac{\frac{2Z-1}{2Z-1} - \frac{Z}{2Z-1}}{Z-1} \\ &= \frac{\frac{Z-1}{2Z-1}}{Z-1} \\ &= \frac{Z-1}{(Z-1)(2Z-1)} \\ &= \frac{1}{2Z-1} \end{aligned}$$

$$\begin{aligned} E[X_1|Z] &= \sum_{x=1}^Z p(x)x \\ &= \frac{Z^2}{2Z-1} \sum_{x=1}^{Z-1} \frac{x}{2Z-1} \\ &= \frac{Z^2 + \sum_{x=1}^{Z-1} x}{2Z-1} \end{aligned}$$

3. Brémaud, 1.4.1: Exponential races

Let X_i , $i = 1, \dots, n$, be independent exponential random variables with intensities/rates $\lambda_i > 0$, $i = 1, \dots, n$ respectively. Let $Z = \min(X_1, \dots, X_n)$ and $J = \arg \min_j X_j$. In other words, J is a random index corresponding to $Z = X_J$; J is well defined, because the probability that one or more X_i 's attain the same value is 0. Show that Z and J are independent, and give their respective distributions.

Hint: Start with $\Pr(J = k, Z \geq t)$, then condition on the random variable Z . You may find Brémaud's Exercise 4.2 (*Freezing a Random Variable*) useful, if you keep in mind that for any event A , $\Pr(A) = E(1_A)$. This exercise states that:

Let X_1, \dots, X_n be independent random variables with respective p.d.f.'s f_1, \dots, f_n . Show that

$$E[g(X_1, \dots, X_n)] = \int_{-\infty}^{\infty} E[g(y, X_2, \dots, X_n)] f_1(y) dy$$

and that

$$\Pr(X_1 \leq X_2, \dots, X_1 \leq X_n, X_1 \leq x) = \int_{-\infty}^x \Pr(X_2 \geq y) \times \dots \times \Pr(X_n \geq y) f_1(y) dy.$$

$$\begin{aligned}
F_Z(t) &= P(Z \leq t) = P(\min(X_1, \dots, X_n) \leq t) \\
&= 1 - P(\min(X_1, \dots, X_n) > t) \\
&= 1 - P(X_1 > t, \dots, X_n > t) \\
&= 1 - \prod_{n=1}^N 1 - CDF_{X_i}(t) \\
&= 1 - \prod_{n=1}^N e^{-t\lambda_i} \\
&= 1 - e^{-t \sum_{n=1}^N \lambda_i}
\end{aligned}$$

$$\begin{aligned}
f_Z(t) &= F'_Z(t) = pdf_Z \\
&= \sum_{n=1}^N \lambda_i e^{-t \sum_{n=1}^N \lambda_i} \\
\therefore Z &\sim \exp(\text{rate} = \sum_{n=1}^N \lambda_i)
\end{aligned}$$

if (X_1, \dots, X_N) ind

then $P(X_1 < X_2, \dots, X_1 < X_N) = 1/N$

$\therefore J \sim 1_{[1, 2, \dots, N]}$

$$\begin{aligned}
P(X_1 \leq X_2, \dots, X_1 \leq X_N, X_1 \leq x) &= \int_0^x P(X_2 \geq y) \times \dots \times P(X_N \geq y) f_1(y) dy \\
&= \int_0^x (1 - CDF_{X_2}(y)) \times \dots \times (1 - CDF_{X_N}(y)) \times PDF_{X_1}(y) dy \\
&= \int_0^x \prod_{i=1}^N e^{-\lambda_i y} \\
&= \int_0^x e^{-y \sum_{i=1}^N \lambda_i} \\
&= \frac{1}{N} (1 - e^{-x \sum_{i=1}^N \lambda_i}) \\
&= P(Z \leq x, J = k)
\end{aligned}$$

$$\therefore P(J, Z) = P(J)P(Z)$$

4. Brémaud, 1.8.1: Applying the SLLN

(a) State Kolmogorov's Strong Law of Large Numbers (SLLN).

The sample average converges almost surely to the expected value.

$$\bar{X}_n \xrightarrow{a.s.} \mu \text{ where } n \rightarrow \infty$$

or

$$P(\lim_{n \rightarrow \infty} \bar{X}_n = \mu) = 1$$

(b) Let S_1, S_2, \dots be a sequence of independent and identically distributed (i.i.d.) random variables,

with $\Pr(0 < S_1 < \infty) = 1$ and $\mathbb{E}[S_1] < \infty$. For $t \geq 0$, let $N_t = \sum_{n=1}^{\infty} 1_{(0,t]}(T_n)$, where $T_n = S_1 + \cdots + S_n$. Prove that, almost surely,

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[S_1]}.$$