Stat 516, Homework 3

Due date: Thursday, October 19. **Note**: Do this homework *individually.*

1. (Brémaud 2.1.4) Consider N balls numbered from 1 to N and placed in two urns A and B. Suppose that at stage n, urn A contains X_n balls. One then chooses a ball among the N balls at random (we may suppose that the balls are numbered and that a lottery gives the number of the selected ball, which can be in either of the two urns), and then chooses an urn, A with probability p, B with probability $q = 1-p$. The selected ball is then placed in the selected urn, and the number of balls in urn A is now X_{n+1}. Show that $(X_n)_{n\geq 0}$ is a homogeneous Markov chain, and give its transition probability matrix.

   We can say that this process is a one step Markov process because X_n is independent of any variable X_z where $z \in \{1, 2, \ldots, n-2\}$ given X_{n-1} and is homogeneous because the process is time invariant, that is it is the same process no matter the value of n.

$$p(X_n = x|X_{n-1} = y)) = \begin{cases} \dfrac{x(1-p)}{N} & \text{if x-1=y} \\ \frac{p(N-x)}{N} & \text{if x+1=y} \\ \frac{N-pN-x+2px}{N} & \text{x=y} \\ 0 & \text{otherwise.} \end{cases}$$

   Information about any variable X_z does not change these probabilities.

   The transition matrix can then be shown to be

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & \cdots & \cdots & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ \vdots \\ N \end{array} & \begin{bmatrix} 1-p & p & \ldots & \ldots & 0 \\ \frac{1-p}{N} & \frac{N-pN-1+2p}{N} & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & 0 \\ \vdots & \vdots & \vdots & \frac{pN-2p+1}{N} & \frac{p}{N} \\ 0 & 0 & 0 & 1-p & p \end{bmatrix} \end{array}$$

2. *Simulating gambler's ruin.*
   Write a routine to simulate realizations of the gambler's ruin chain $\{X_n\}$ with probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, $p + q = 1$. The routine should stop simulations as soon as you hit one of the absorbing states. Your input will consist of an initial state i, the maximal state N in the state space $\{0, 1, \ldots, N\}$, and probability of increasing gambler's fortune p. The routine should return a vector of Markov chain states until absorption.

   (a) Provide the source code in any computer language of your choice and output of your routine in the form of 20 random realizations of the Markov chain for input parameters $N = 10$, $i = 3$, and $p = 0.29$.

      Code

```
set.seed(123)
M <- 20
N <- 10
i <- 3
p <- .29

run_gamblers_ruin <- function(M, N, i, p, quietly=TRUE){
  i_start <- as.integer(rep(i, M))
  chain <- lapply(i_start, function(x) x)

  for(j in 1:M){
    end_pos <- chain[[j]]
    if(!quietly){
      cat(paste0("Starting chain number: ", j, "\n"))
    }
    while(end_pos != 0 & end_pos != N){
      if(!quietly){
        cat(paste0(end_pos, "\n"))
      }
      result <- rbinom(1, 1, prob=p)
      action <- 1^result *  (-1)^(1 - result)
      chain[[j]] <- c(chain[[j]], end_pos + action)
      end_pos <- chain[[j]][length(chain[[j]])]
    }
  }

  return(chain)
}

run_gamblers_ruin(M, N, i, p)
```

Output

```
[[1]]
 [1] 3 2 3 2 3 4 3 2 3 2 1 2 1 0

[[2]]
[1] 3 2 1 2 1 0

[[3]]
 [1] 3 2 3 4 3 2 3 2 1 0

[[4]]
```

```
[1] 3 2 1 0

[[5]]
 [1] 3 4 5 4 5 4 3 4 3 2 1 0

[[6]]
[1] 3 2 1 0

[[7]]
[1] 3 2 1 0

[[8]]
[1] 3 2 1 2 1 0

[[9]]
[1] 3 4 3 2 1 0

[[10]]
[1] 3 4 5 4 3 2 1 0

[[11]]
 [1] 3 4 3 4 5 6 5 6 5 6 5 4 3 2 1 0

[[12]]
[1] 3 2 1 0

[[13]]
 [1] 3 2 3 2 1 2 3 4 3 2 1 0

[[14]]
[1] 3 2 1 0

[[15]]
[1] 3 4 3 2 1 0

[[16]]
 [1] 3 2 1 2 1 2 3 2 1 0

[[17]]
 [1] 3 4 3 2 3 4 3 2 3 2 1 0

[[18]]
```

```
[1] 3 2 1 0

[[19]]
[1] 3 2 3 2 1 0

[[20]]
[1] 3 2 1 2 1 2 1 0
```

(b) Use your simulation routine to estimate the probability of reaching the largest state $N = 10$ starting at state 4, denoted $h(4,p)$, for probabilities $p_{i,i+1} = p \in \{0.1, 0.2, \ldots, 0.9\}$. Turn in a graph with estimated $h(4,p)$ plotted against p.

See figure 1 at the end.

3. (Brémaud 2.3.1) Rat and Cat move between two rooms, using different paths. Their motions are independent, governed by their respective transition matrices

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix} \end{array}, \quad \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \end{array}.$$

Cat starts from room 1, Rat from room 2. If they are ever in the same room, Cat eats Rat. How long will Rat survive on the average?

If state a is the scenario where the rat is in room 2 and the cat is in room 1, state b is the scenario where the rat is in room 1 and the cat is in room 2, and state x is the scenario where both animals are in the same room we may rewrite the transition matrix as.

$$\begin{array}{c c} & \begin{array}{ccc} a & b & x \end{array} \\ \begin{array}{c} a \\ b \\ x \end{array} & \begin{bmatrix} .03 & .63 & .34 \\ .54 & .04 & .42 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

then

$$m(x) = 0$$
$$m(a) = 1 + .03m(a) + .63m(b)$$
$$m(b) = 1 + .54m(a) + .04m(b)$$
$$m(b) = \frac{1 + .54m(a)}{.96}$$
$$m(a) = 1 + .03m(a) + .63\Big(\frac{1 + .54m(a)}{.96}\Big)$$
$$m(a) = 2.690355$$

4. (Brémaud 2.3.3) Three characters, A, B, and C, armed with guns, suddenly meet at the corner of a Washington, D.C. street, whereupon they naturally start shooting at one another. Each street-gang

kid shoots every tenth second, as long as he is still alive. The probability of a hit for A, B, and C are α, β, and γ, respectively. A is the most hated, and therefore, as long as he is alive, B and C ignore each other and shoot at A. For historical reasons not developed here, A cannot stand B, and therefore he shoots only at B while the latter is still alive. Lucky C is shot at if and only if he is in the presence of A alone or B alone. What are the survival probabilities of A, B, and C, respectively?

All non listed values are treated as 0.

For A

$$\begin{aligned}
b_A &= 1\\
b_{A,C} &= b_{A,C}(1-\alpha)(1-\gamma)+\alpha(1-\gamma)\\
b_{A,B,C} &= b_{A,B,C}(1-\alpha)(1-\beta)(1-\gamma)+b_{A,C}\alpha(1-\beta)(1-\gamma)\\
b_{A,B,C} &= \frac{\alpha^2(1-\beta)(1-\gamma)^2}{(1-(1-\alpha)(1-\gamma))(1-(1-\alpha)(1-\beta)(1-\gamma))}
\end{aligned}$$

for B

$$\begin{aligned}
b_B &= 1\\
b_{B,C} &= b_{B,C}(1-\beta)(1-\gamma)+\beta(1-\gamma)\\
b_{A,B,C} &= b_{A,B,C}(1-\alpha)(1-\beta)(1-\gamma)+b_{B,C}(1-\alpha)(\gamma+\beta-\beta\gamma)\\
b_{A,B,C} &= \frac{(\beta(1-\gamma)(1-\alpha))(\gamma+\beta-\beta\gamma)}{(1-(1-\beta)(1-\gamma))(1-(1-\alpha)(1-\beta)(1-\gamma))}
\end{aligned}$$

for C

$$\begin{aligned}
b_C &= 1\\
b_{B,C} &= b_{B,C}(1-\beta)(1-\gamma)+\gamma(1-\beta)\\
b_{A,C} &= b_{A,C}(1-\alpha)(1-\gamma)+\gamma(1-\alpha)\\
b_{A,B,C} &= b_{A,B,C}(1-\alpha)(1-\beta)(1-\gamma)+b_{A,C}\alpha(1-\beta)(1-\gamma)+\\
&\quad b_{B,C}(1-\alpha)(\gamma+\beta+\beta\gamma)+\alpha(\gamma+\beta+\beta\gamma)\\
b_{A,B,C} &= \frac{\frac{\alpha\gamma(1-\alpha)(1-\beta)(1-\gamma)}{1-(1-\alpha)(1-\gamma)}+\frac{\gamma(1-\alpha)(1-\beta)(\gamma+\beta-\beta\gamma)}{1-(1-\beta)(1-\gamma)}+\alpha(\gamma+\beta-\beta\gamma)}{1-(1-\alpha)(1-\beta)(1-\gamma)}
\end{aligned}$$

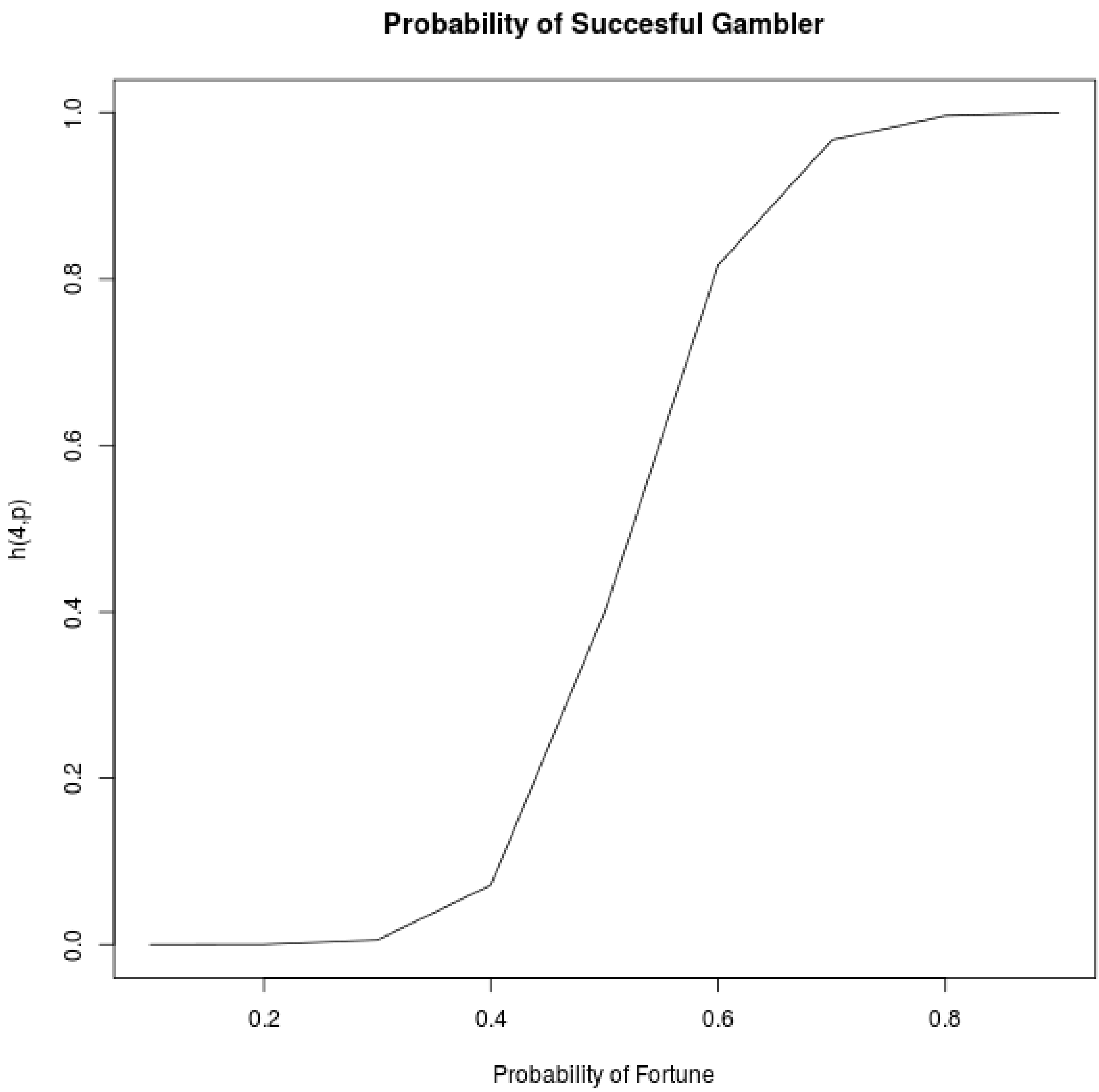

Figure 1: A Gamblers Ruin.