

Matrix Algebra

Geometric Algorithms

Lecture 10

Practice Problem

Write the matrix for the transformation which projects vectors in \mathbb{R}^2 vertically onto the line $y = 2x + 3$ in \mathbb{R}^2 .

Answer

Objectives

1. Connect questions about matrix equations and linear transformations
2. Motivate matrix multiplication
3. Define matrix multiplication
4. Look at the algebra of matrix multiplication

Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

Recap

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector \mathbf{x} to the vector $A\mathbf{x}$

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

What about other questions?

One-to-One and Onto Transformations

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Do the columns of A have full span?

Are the columns of A linearly independent?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Wait, what's going on with this second one?

A New Perspective on Linear Independence

$A\mathbf{x} = \mathbf{0}$ has a unique solution $\equiv A\mathbf{x} = \mathbf{b}$ has at most one solution for any choice of \mathbf{b}

why? :

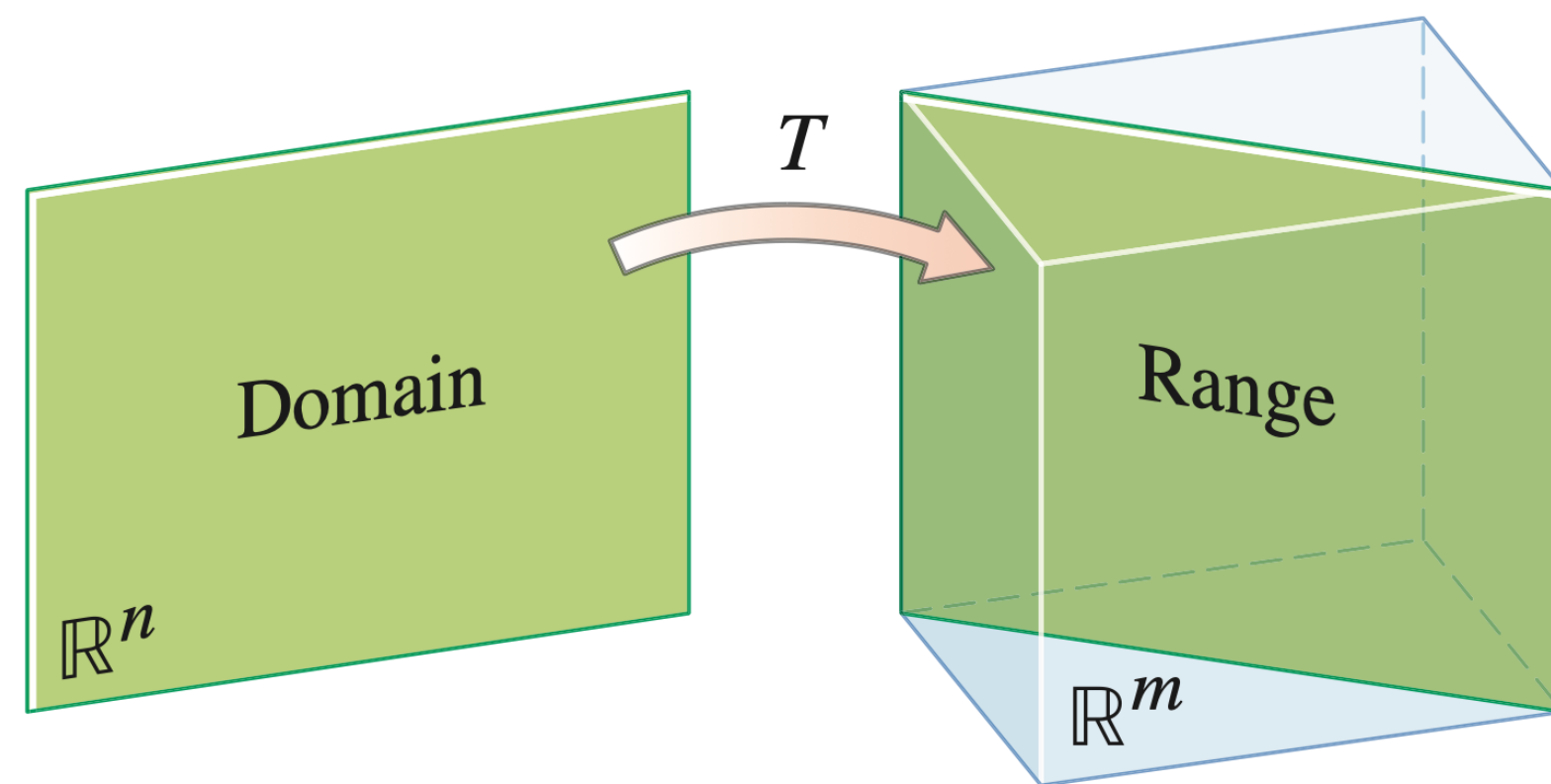
Onto Transformations

Onto Transformations

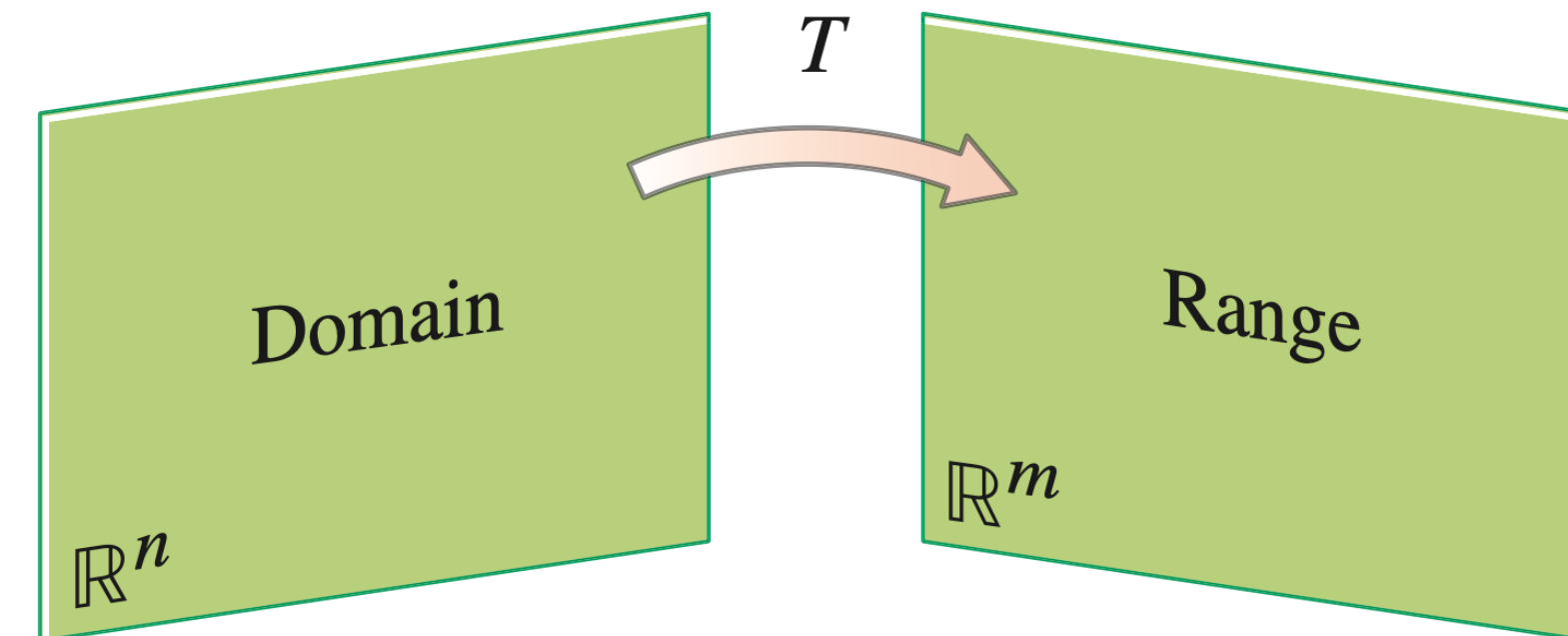
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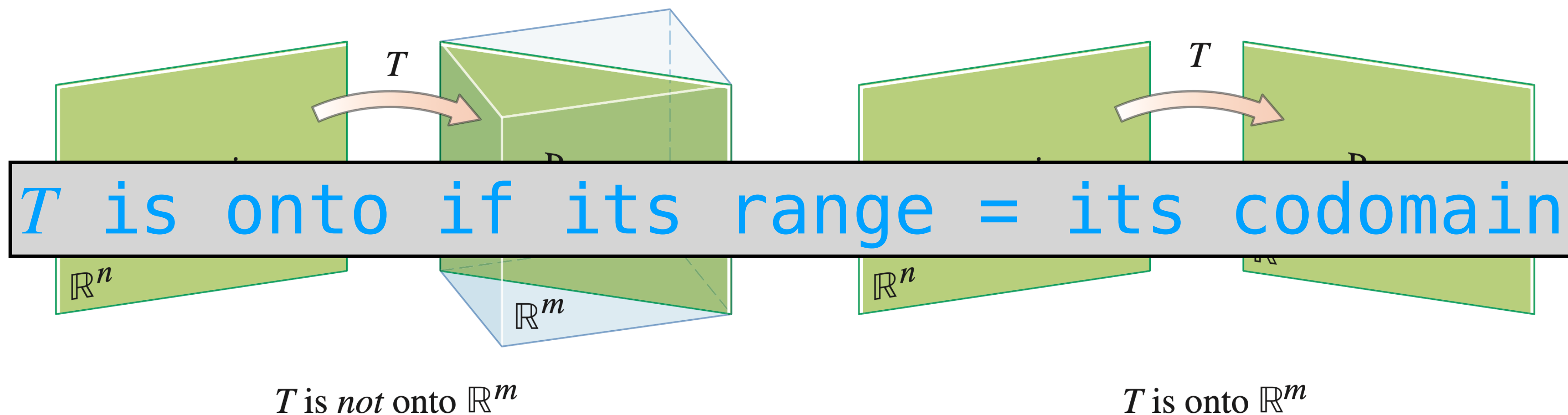
T is not onto \mathbb{R}^m



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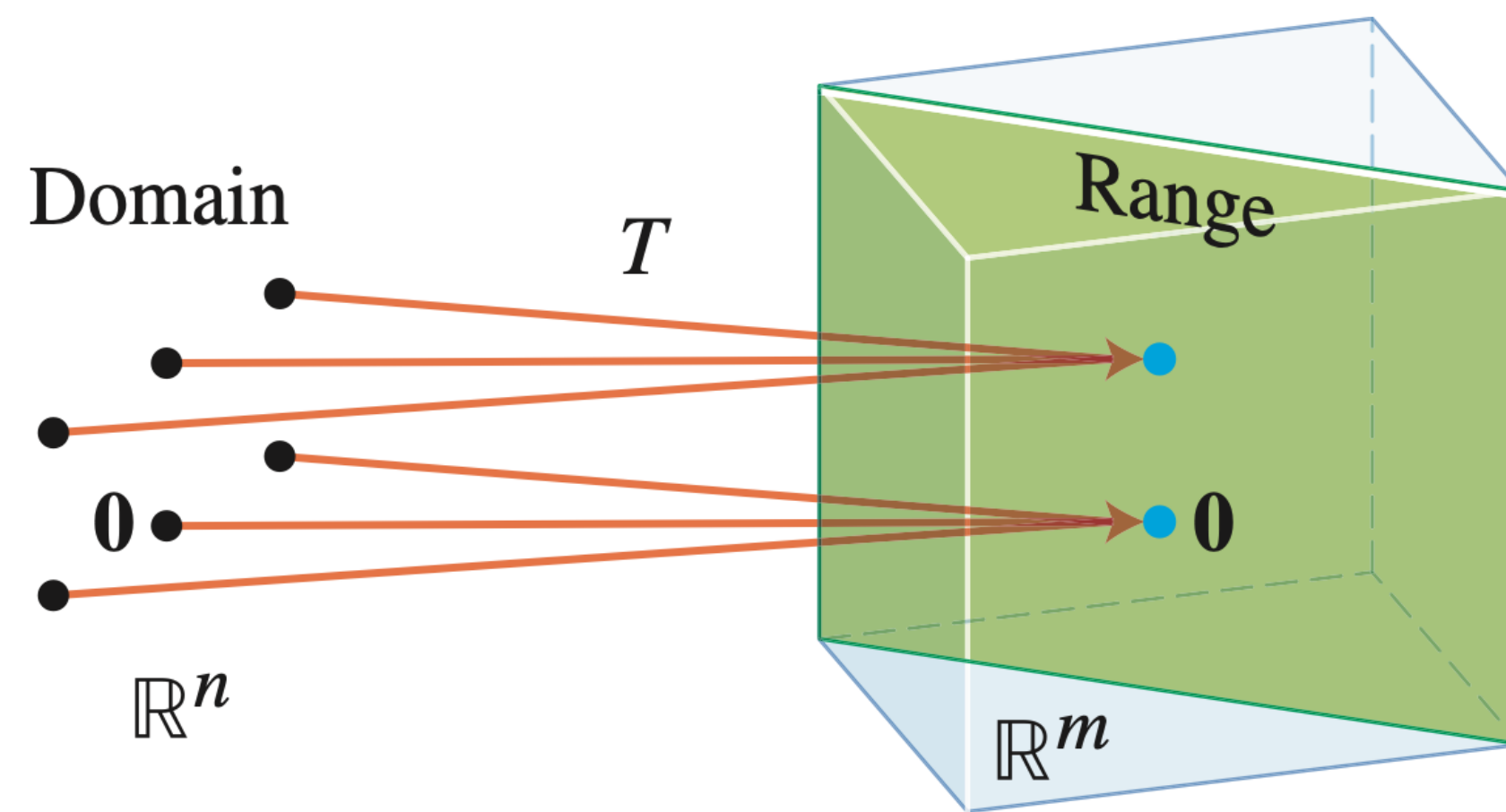
One-to-one Transformations

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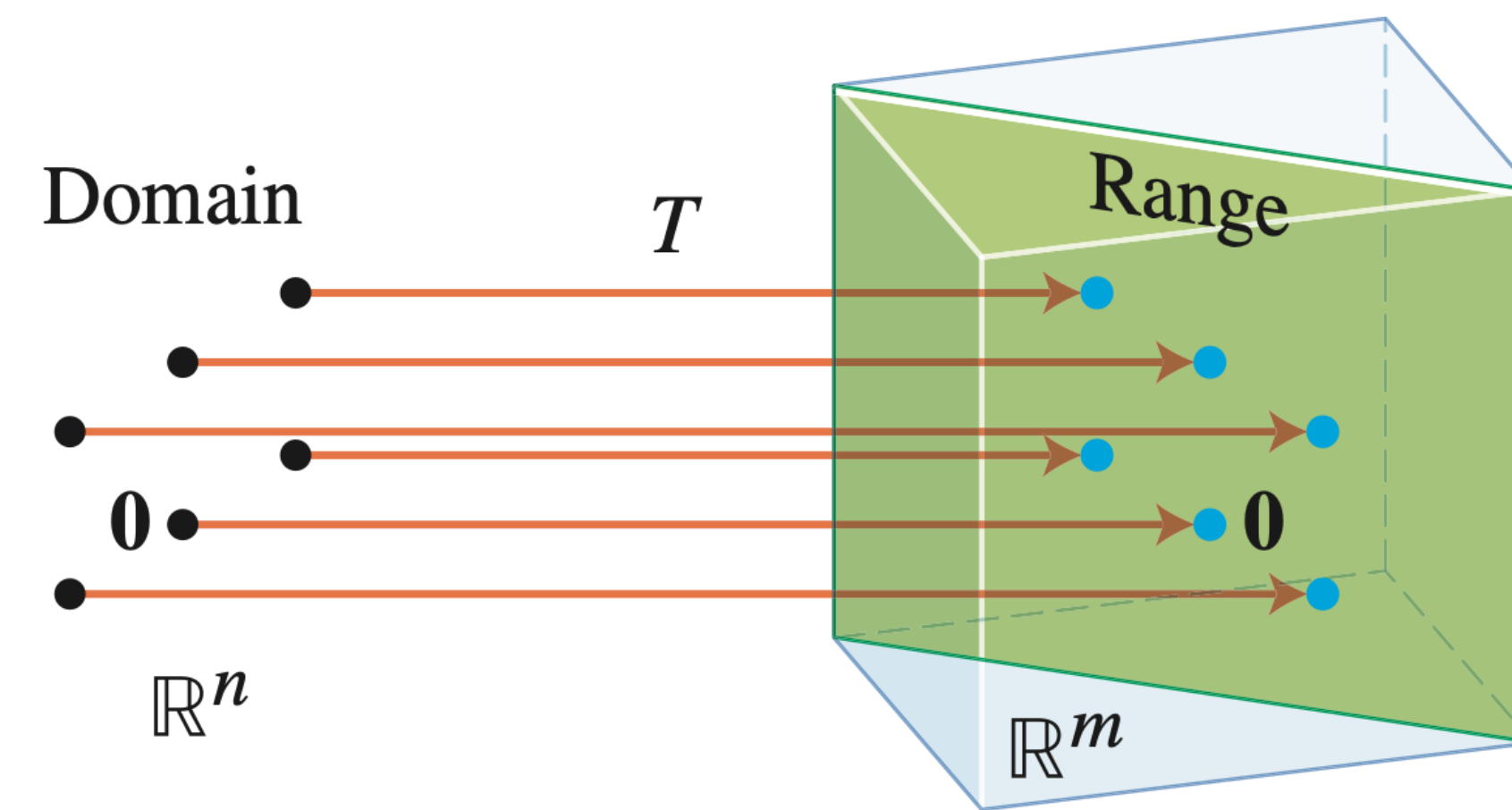
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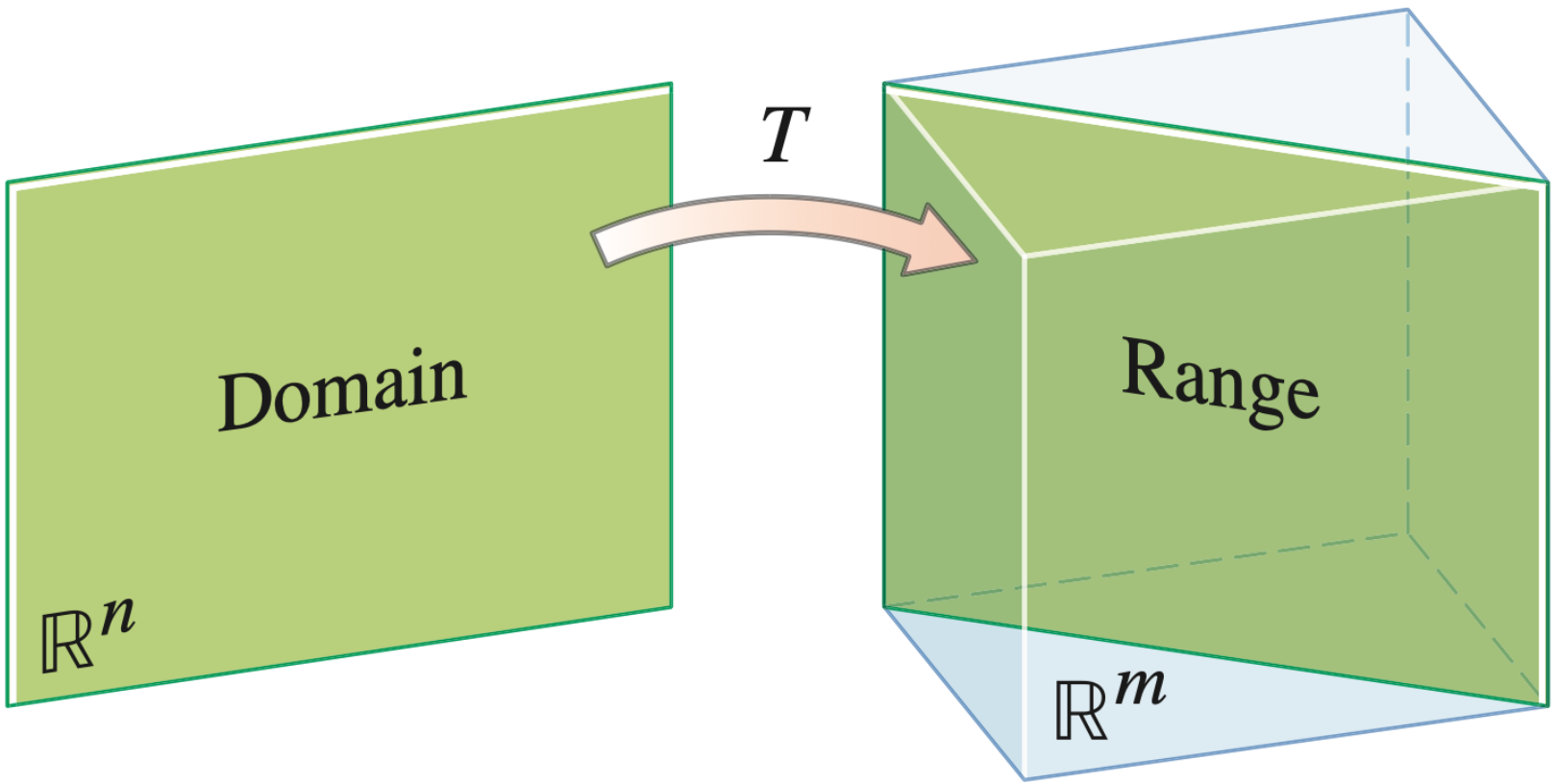


T is *not* one-to-one

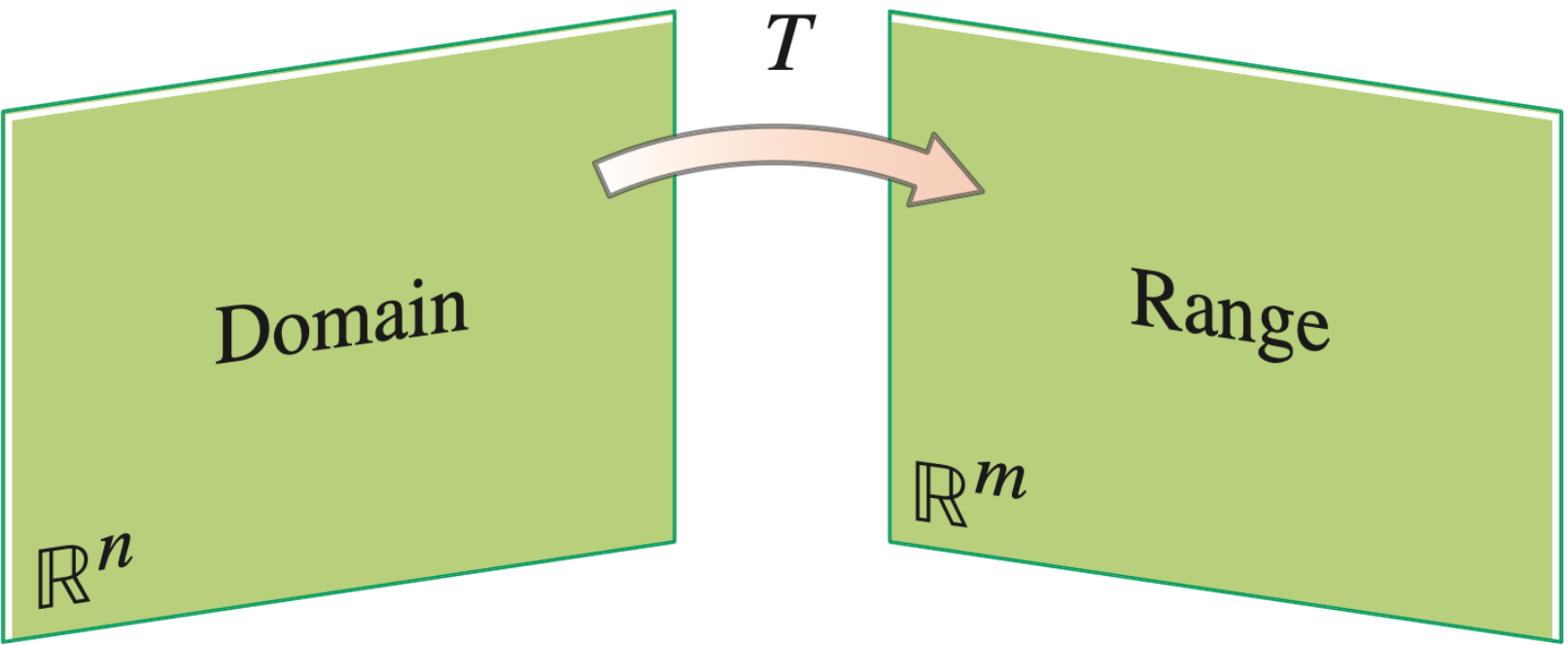


T is one-to-one

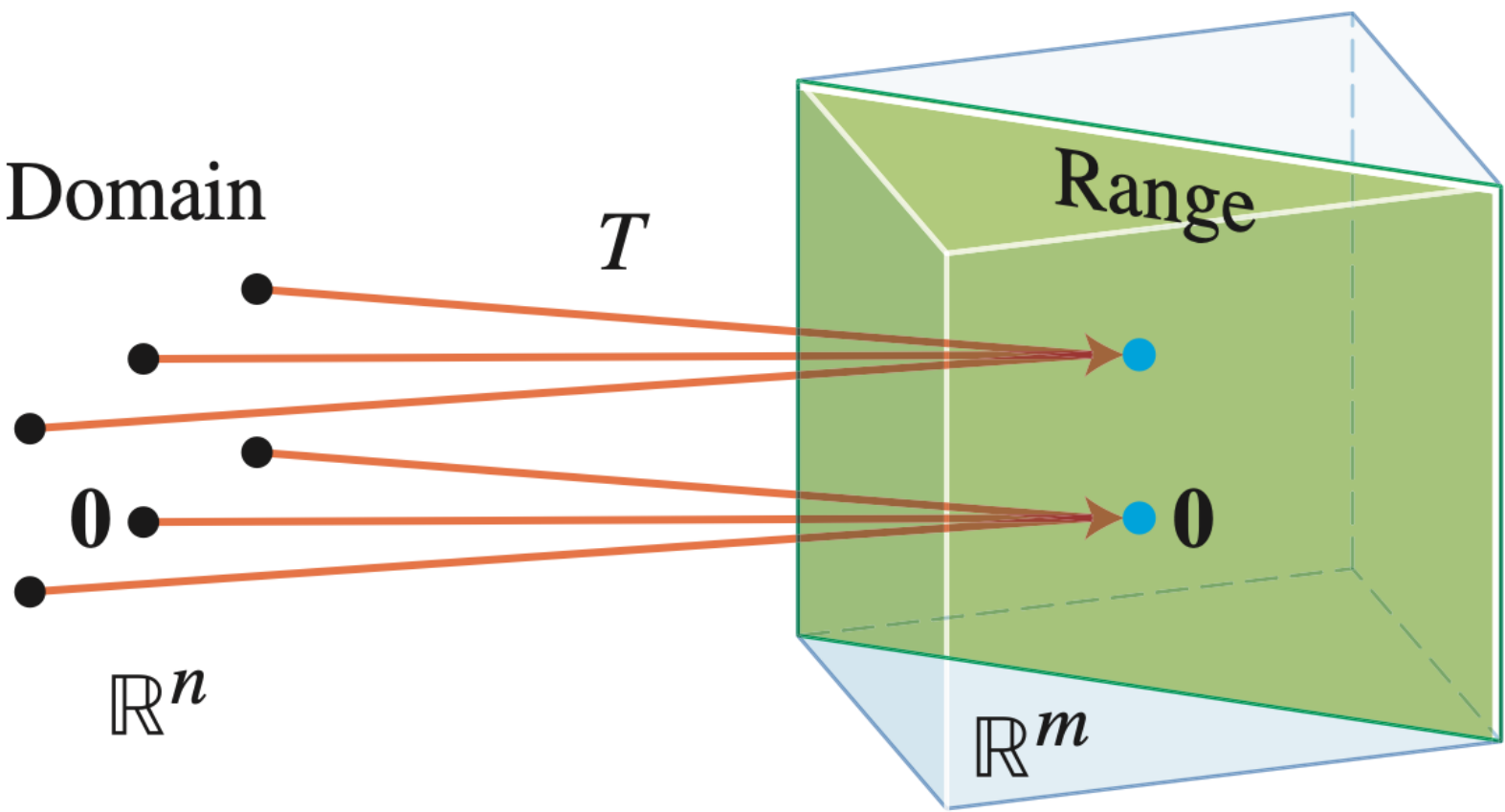
Comparing Pictures



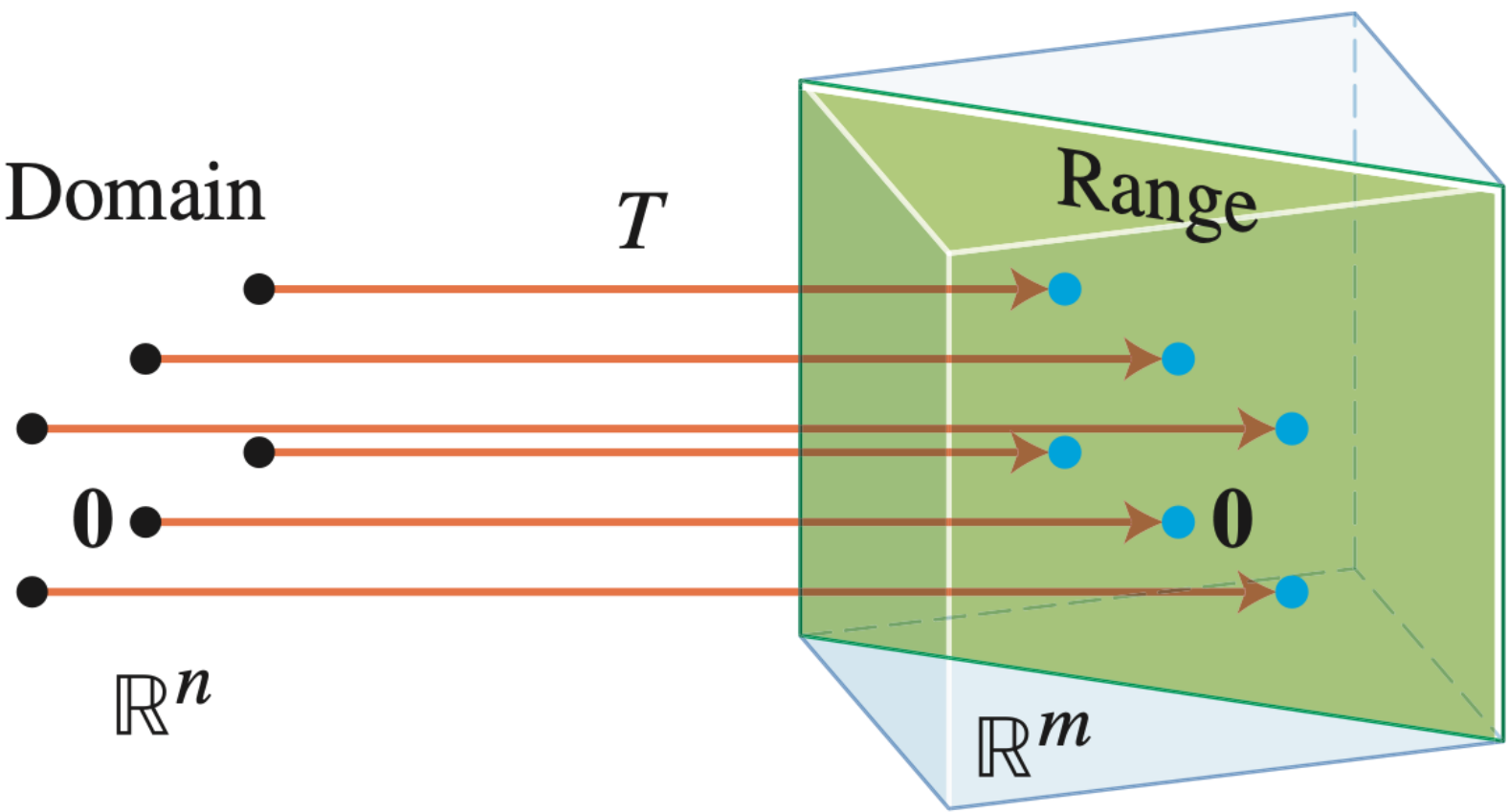
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Taking Stock: Onto

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- » A has a pivot position in every row

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- » The columns of A are linearly independent
- » A has a pivot position in every column

How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto.

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row.

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

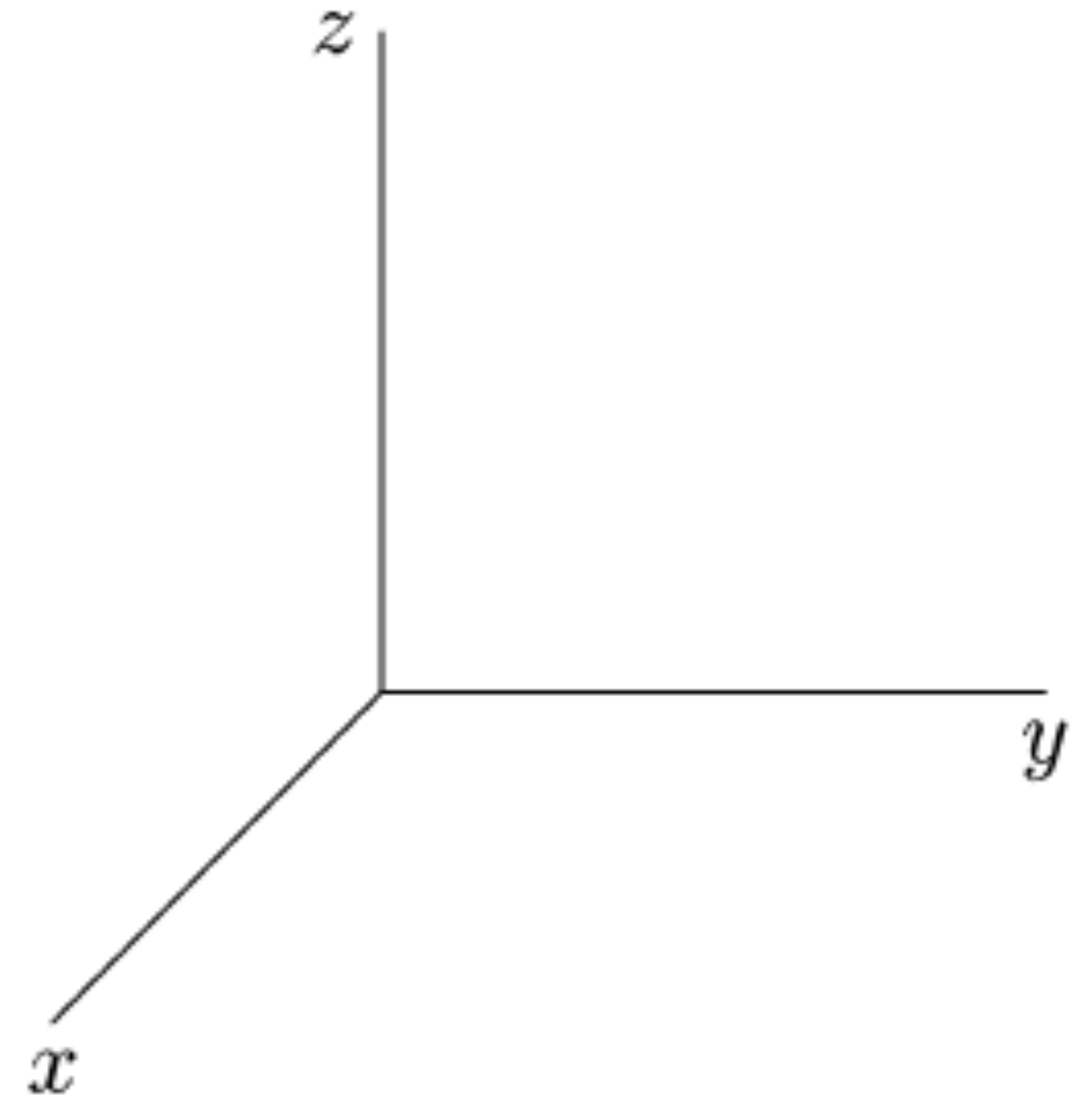
why? :

Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why? :

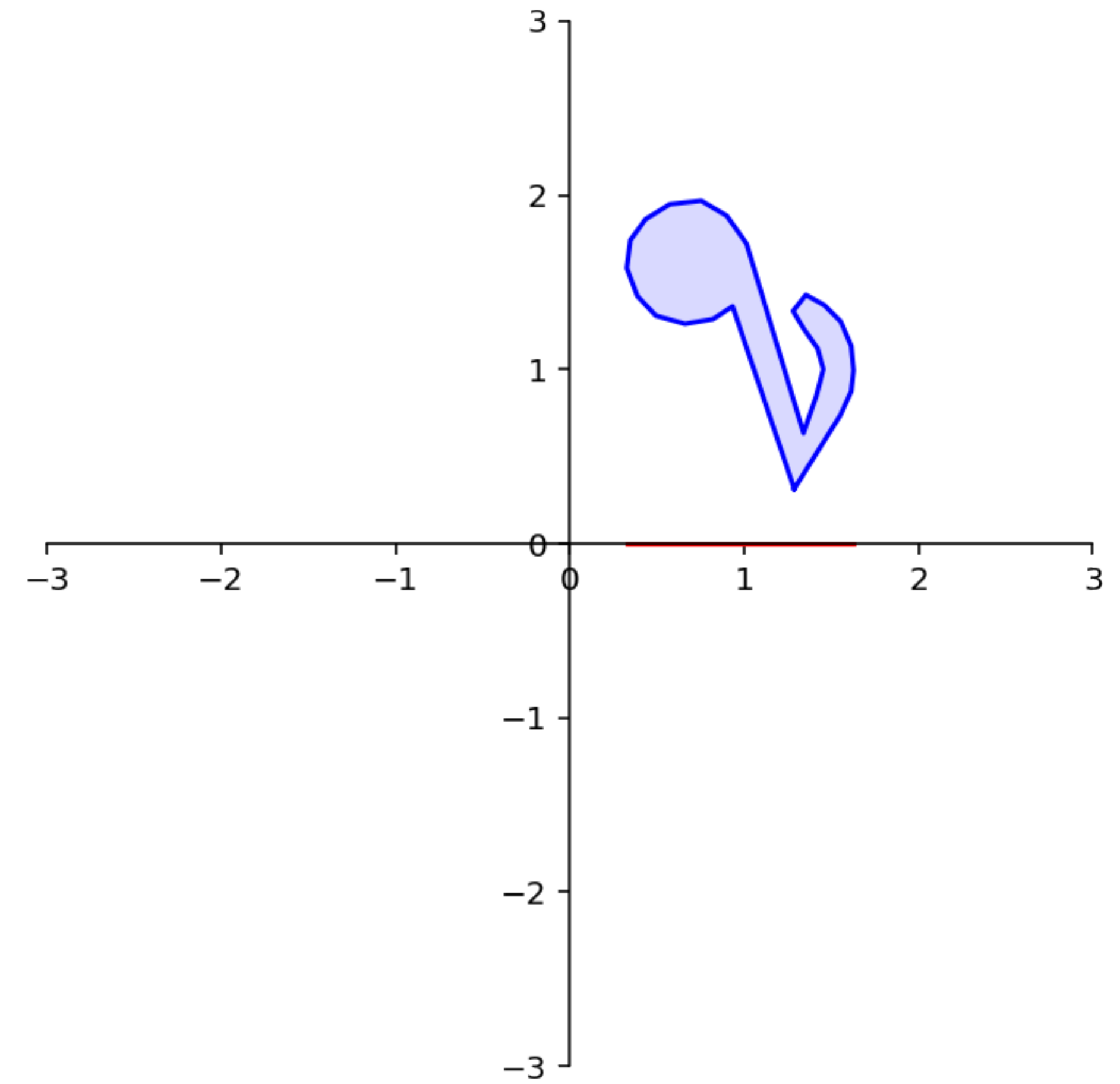


Example: not 1-1, not onto

Projection onto the x_1 -axis:

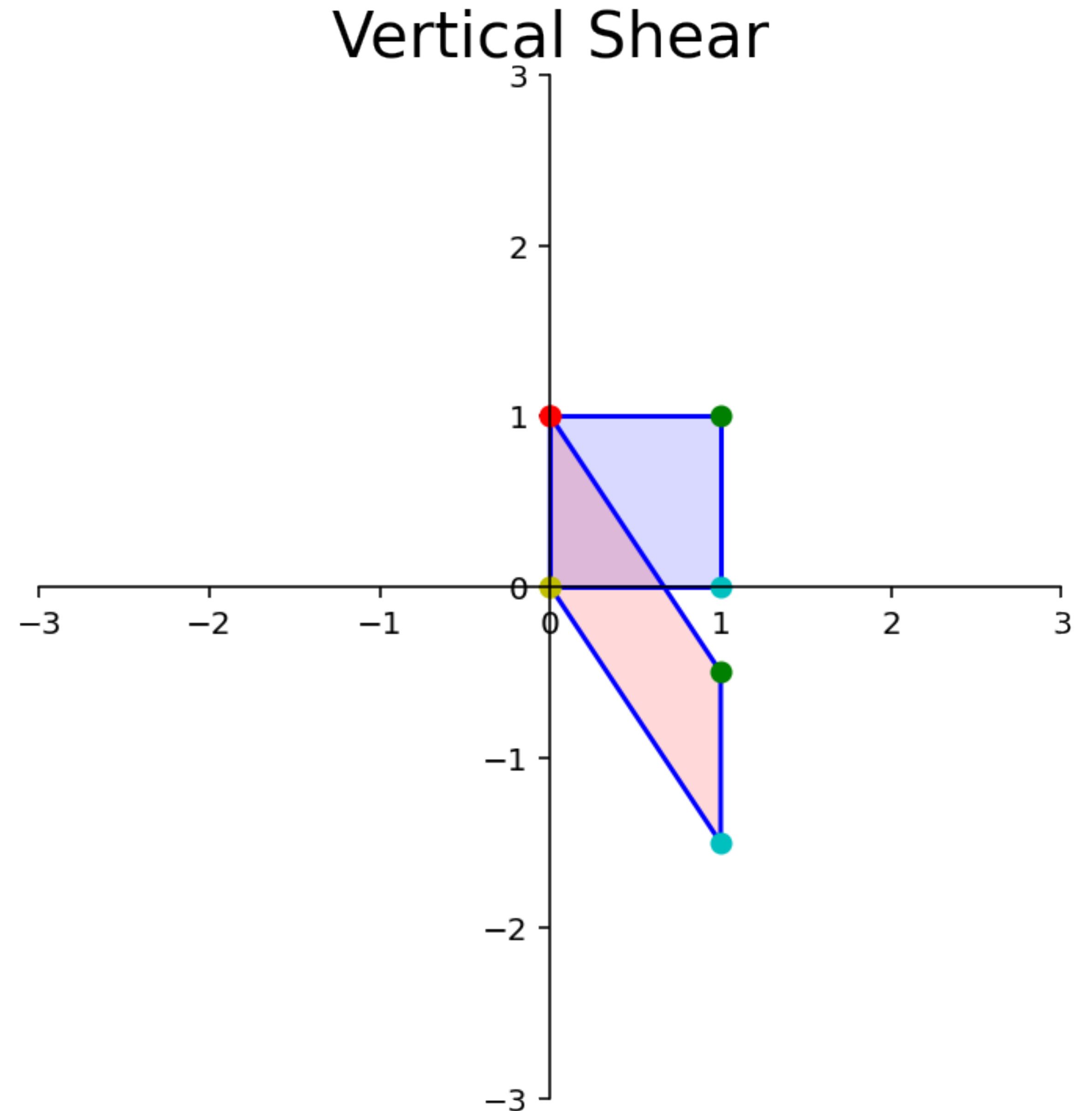
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

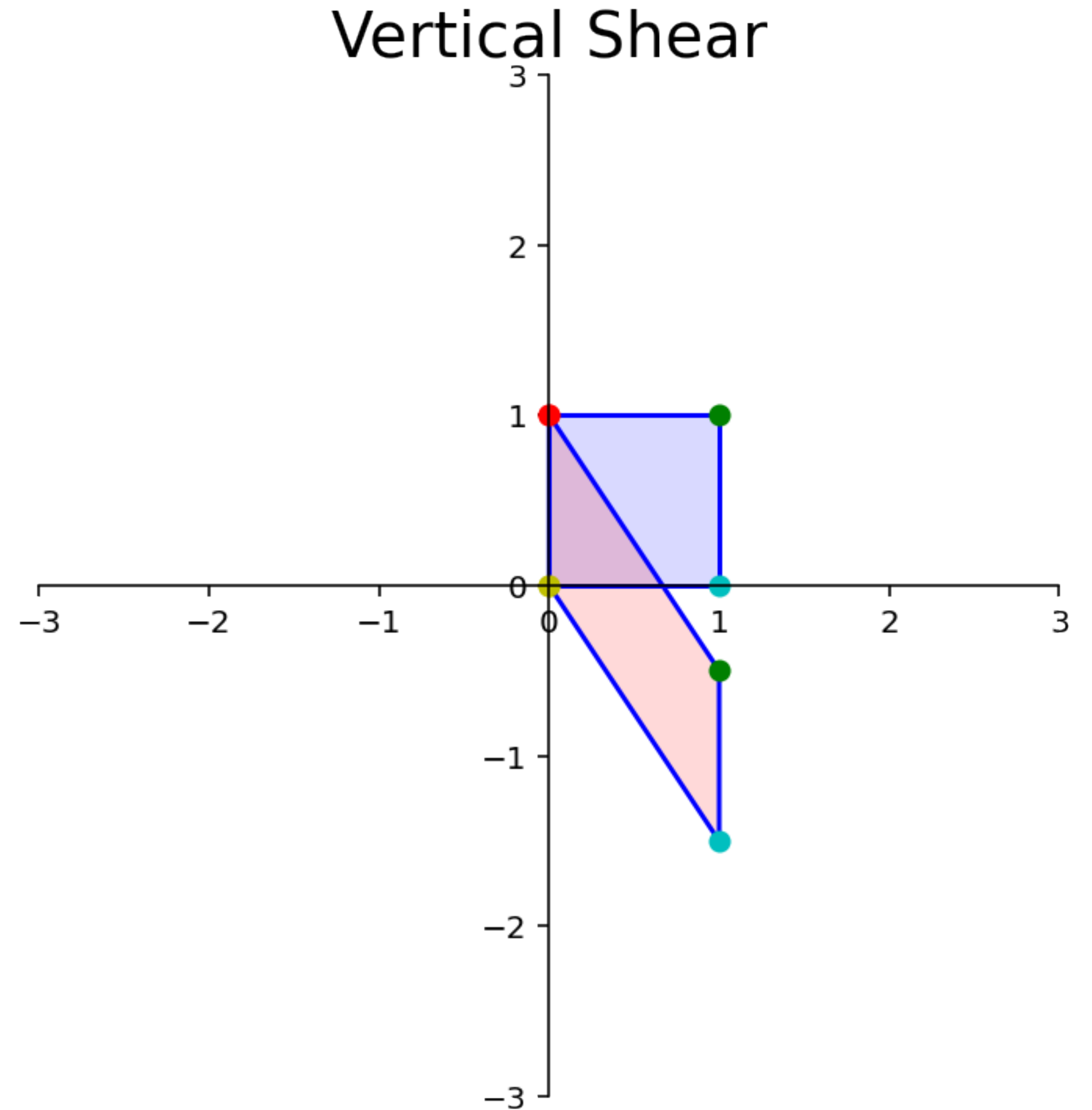


Question

*Is vertical shearing
a 1-1 transformation?
Justify your answer.*



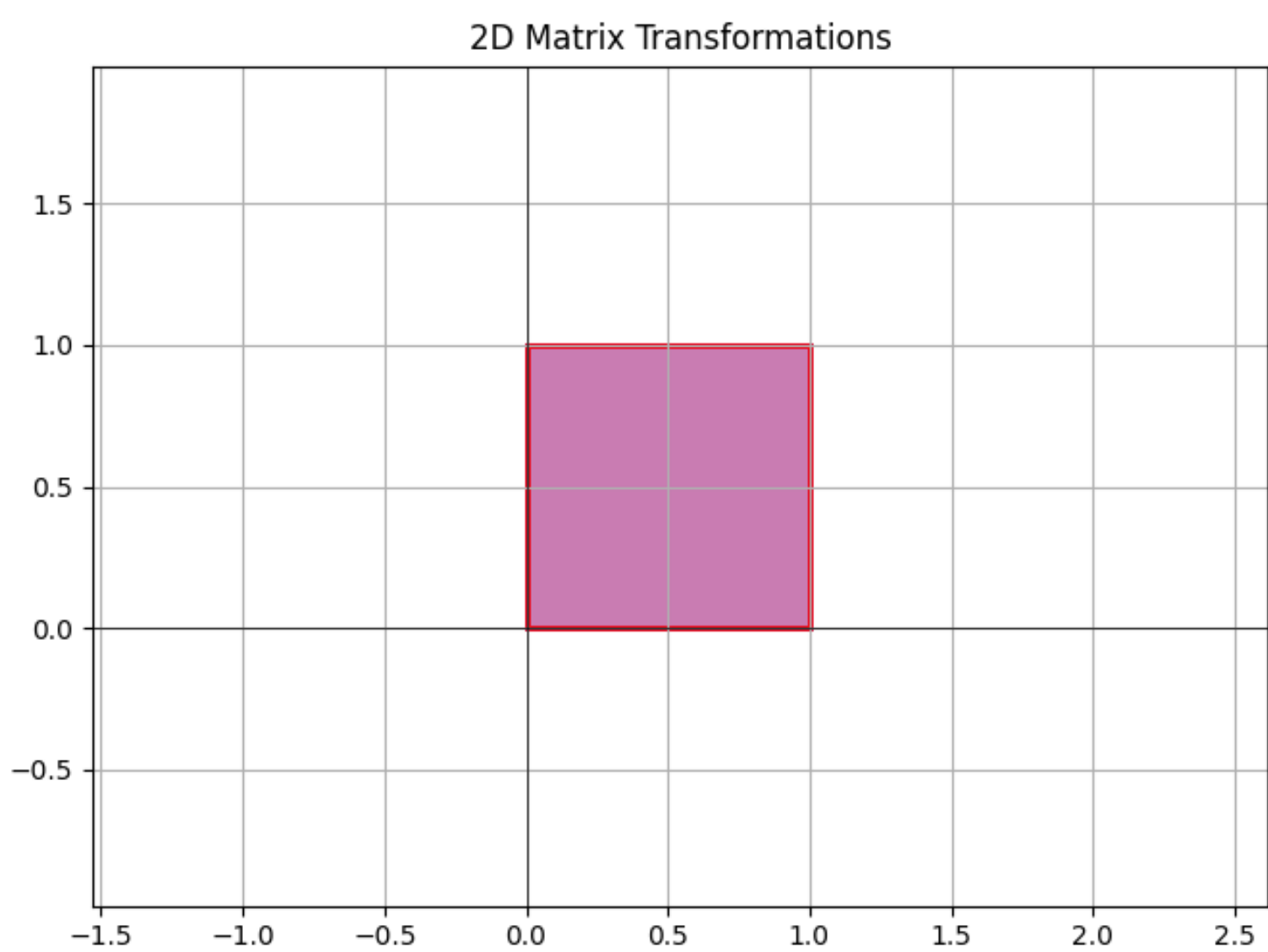
Answer: Yes



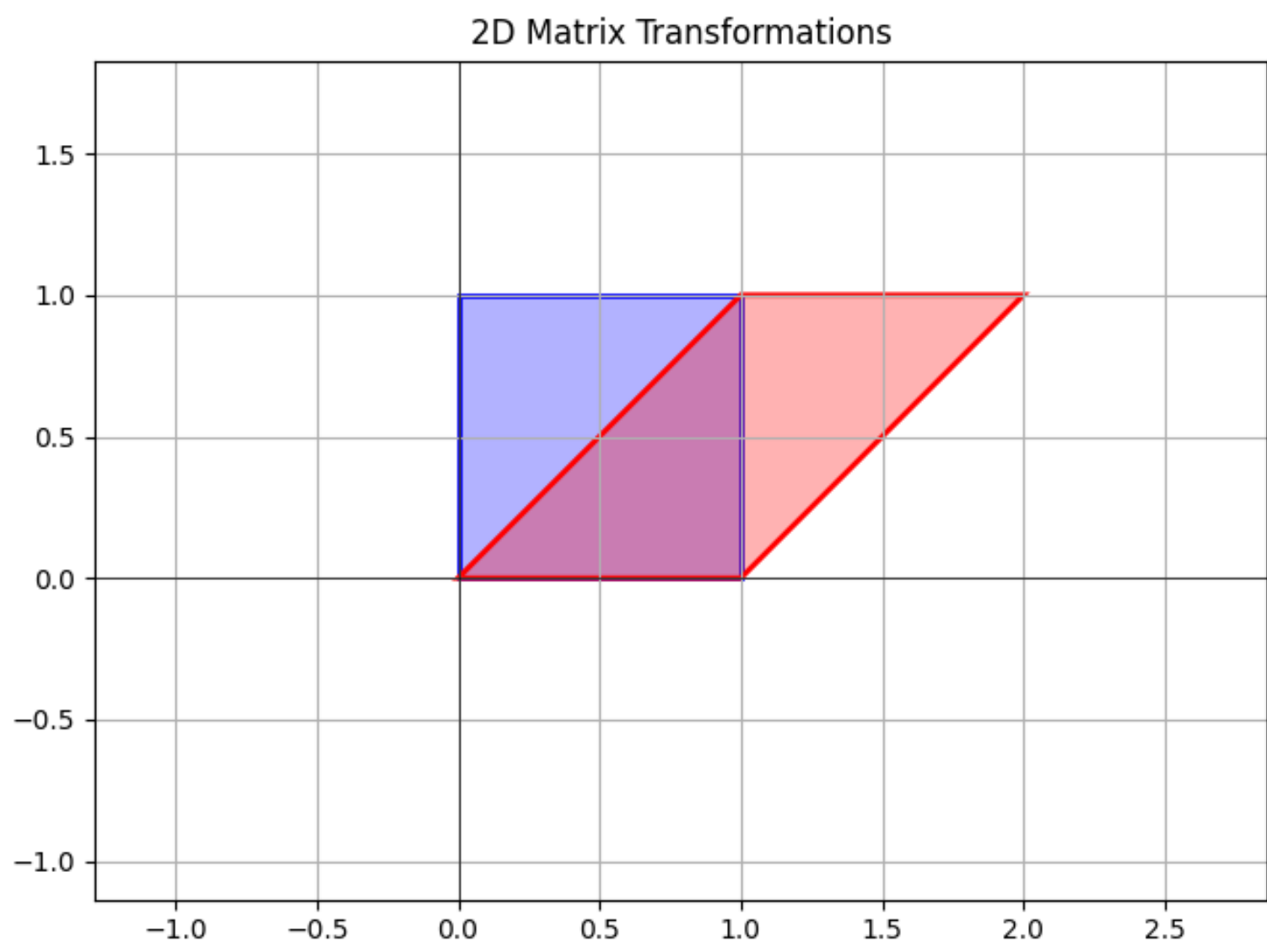
moving on...

Composing Linear Transformations

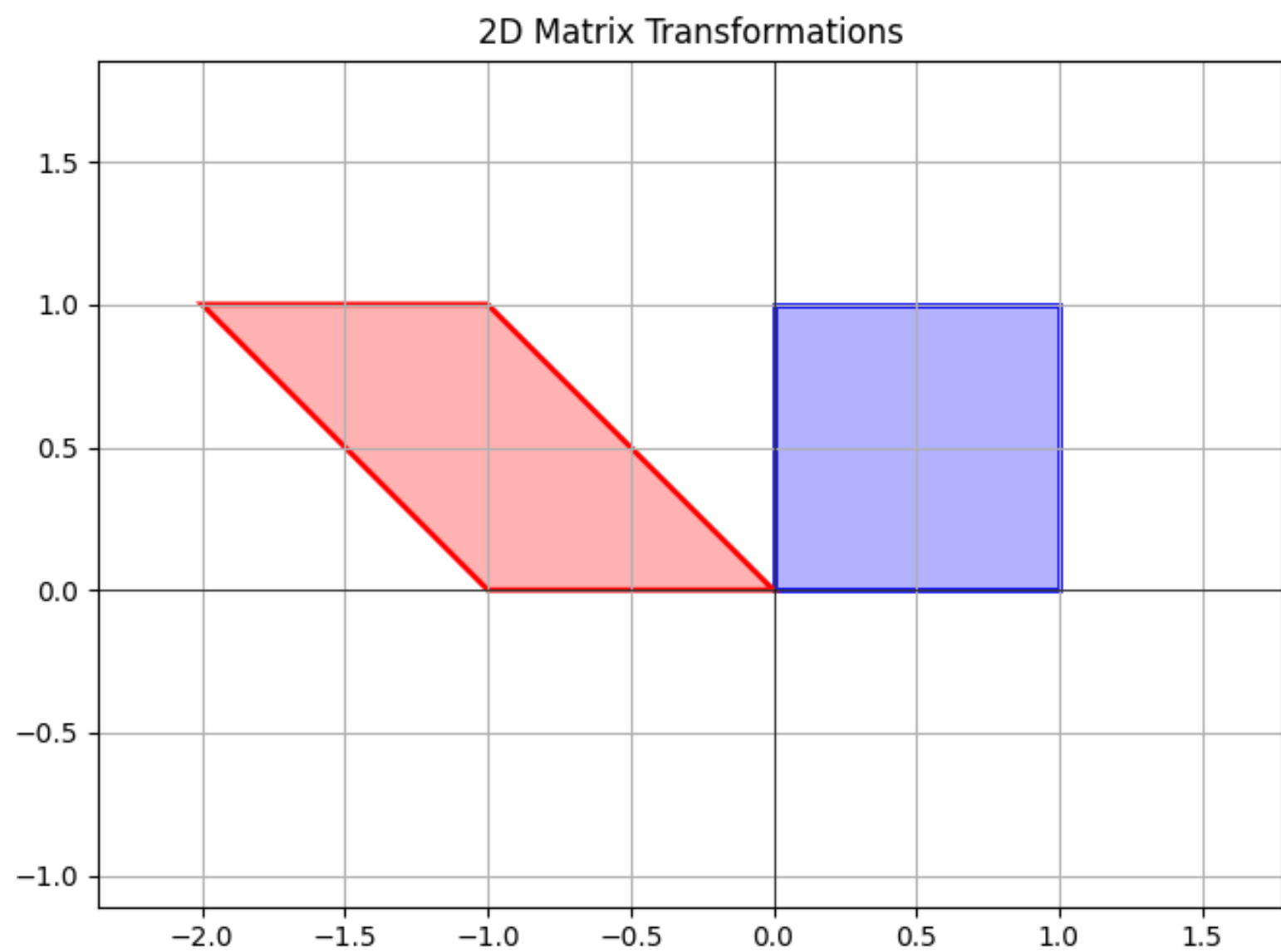
Shearing and Reflecting (Geometrically)



shear



reflect

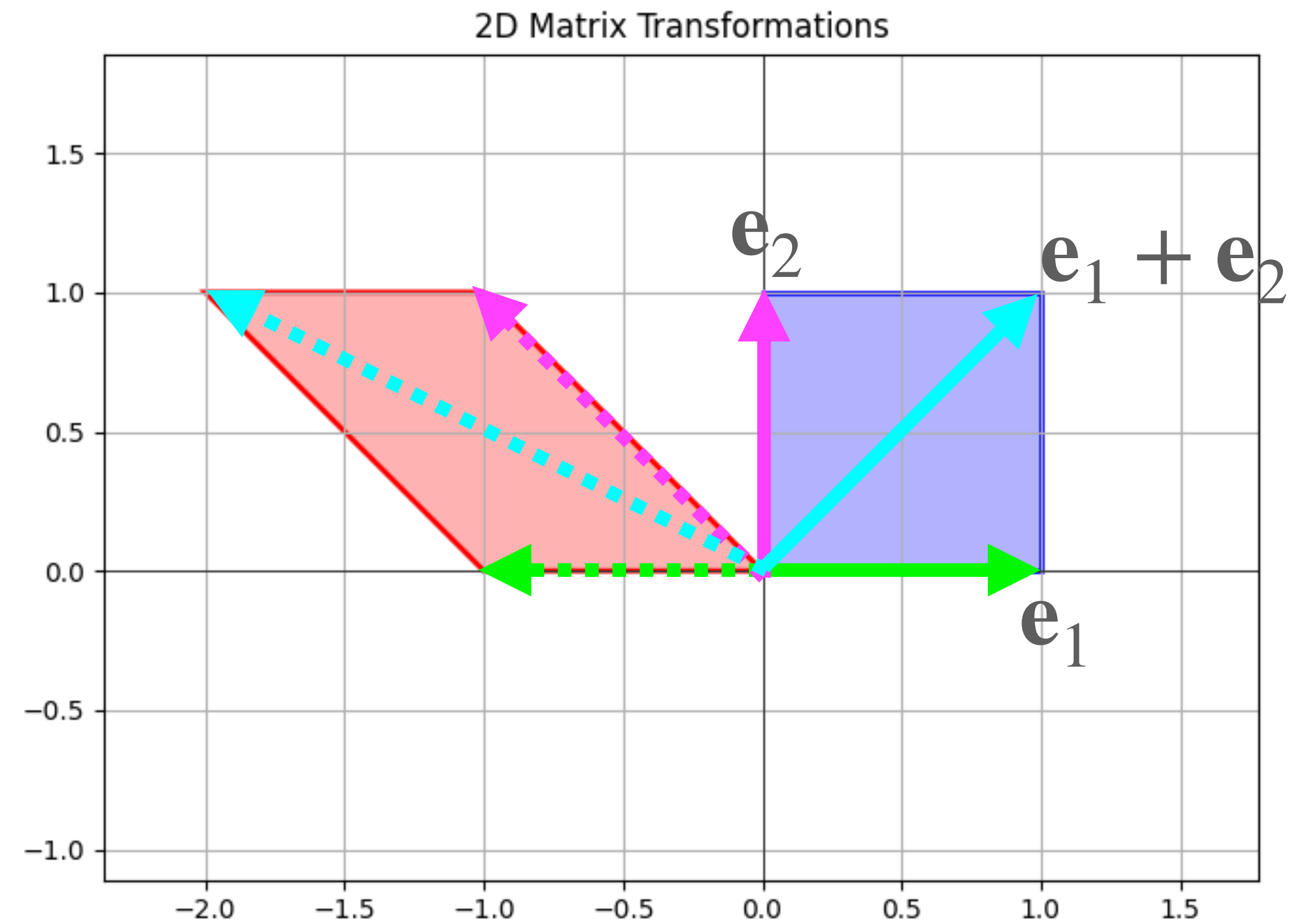


Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect shear

First multiply by shear matrix, then multiply
by reflection matrix

Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply
by reflection matrix

This gives us the same transformation.

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

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Fact. The composition of two linear transformations is a linear transformation.

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Verify:

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Fact. The composition of two linear transformation is a linear transformation.

Verify:

This means the composition of two matrix transformation can be represented as a *single* matrix.

The Key Question

*Given two linear transformations,
how do we compute the matrix which
implements their composition?*

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Matrix Multiplication

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Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices arranged in a sequence separated by an equals sign. The first matrix is a 5x3 matrix, represented by a blue vertical line on the left labeled m and a red horizontal line on top labeled n . The second matrix is a 3x4 matrix, represented by a red vertical line on the left labeled n and a purple horizontal line on top labeled k . The third matrix is a 5x4 matrix, represented by a blue vertical line on the left labeled m and a purple horizontal line on top labeled k . Each matrix contains asterisks representing elements. Below each matrix is its dimension notation: $(m \times n)$ for the first, $(n \times k)$ for the second, and $(m \times k)$ for the third. The m and n in the first notation are in blue and red boxes respectively. The n and k in the second notation are in red and purple boxes respectively. The m and k in the third notation are in blue and purple boxes respectively.

$$\begin{matrix} m \\ \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{array} \right] \end{matrix} \begin{matrix} n \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix} = \begin{matrix} m \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix}$$

$(m \times n)$ $(n \times k)$ $(m \times k)$

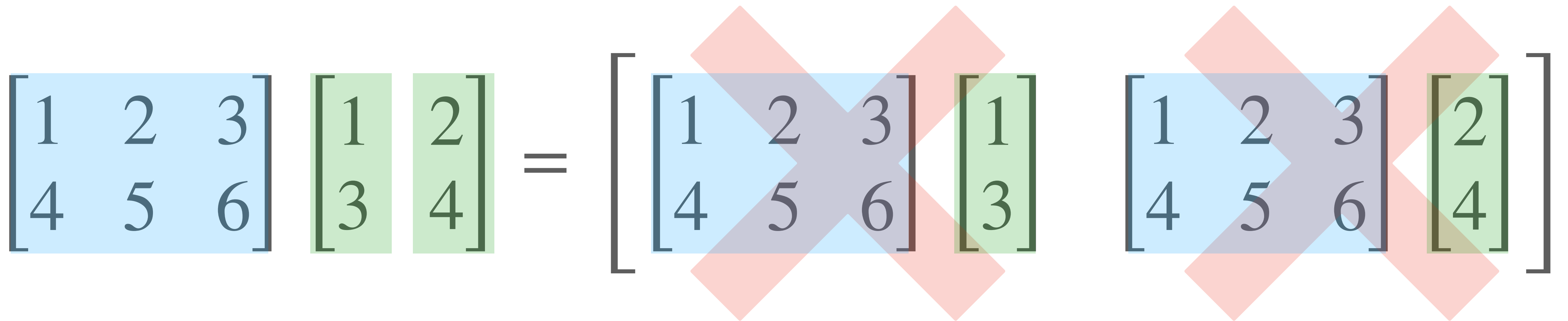
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$


These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its top row highlighted in light blue. The second matrix is a 3x4 matrix with its first column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the top-left element is highlighted in light purple, representing the result of multiplying the first row of the first matrix by the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements marked with asterisks (*). The second matrix is a 3x4 matrix, also with all elements marked with asterisks. The third matrix is a 5x4 matrix, also with all elements marked with asterisks. The second matrix is positioned between the first and third matrices, with an equals sign (=) to its right. The first matrix has its second row highlighted in light blue. The second matrix has its first column highlighted in light red. The third matrix has its first row highlighted in light purple. This visualizes the calculation of the element in the second row of the first matrix and the first column of the third matrix, which is the element in the second row and first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The second row of A is highlighted in light blue, the second column of B is highlighted in light red, and the resulting second row of C is highlighted in light purple. The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

Diagram illustrating the Row-Column Rule for matrix multiplication. It shows a 5x3 matrix A with its second row highlighted in light blue, a 3x4 matrix B with its third column highlighted in light red, and their product matrix AB which is 5x4 with the element at row 2, column 3 highlighted in light purple. The matrices are represented by asterisks inside brackets.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The third row of A is highlighted in light blue, the first column of B is highlighted in light red, and the first row of C is highlighted in light purple. The matrices are represented as grids of asterisks (*).

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

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$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its third row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the third row and fourth column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices arranged in an equation: $A \cdot B = C$.

- Matrix A is a 5x3 matrix. Its third row is highlighted in light blue.
- Matrix B is a 3x4 matrix. Its third column is highlighted in light red.
- Matrix C is a 5x4 matrix. Its third row is highlighted in light purple.

The matrices are represented by asterisks (*) in their respective positions, with the highlighted rows and columns indicating the specific elements being multiplied to form the element in the third row of the resulting matrix C .

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with a light blue horizontal band highlighting its fourth row. The second matrix is a 3x4 matrix with a light red vertical band highlighting its second column. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the fourth row and second column is highlighted with a light purple square, representing the result of the dot product of the highlighted row and column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication using three matrices represented by grids of asterisks (*).
1. The first matrix is a 5x3 grid. Its fourth row is highlighted in light blue.
2. The second matrix is a 3x4 grid. Its third column is highlighted in light red.
3. An equals sign (=) follows.
4. The third matrix is a 5x4 grid. The element at the intersection of the fourth row and third column is highlighted in light purple, representing the result of the dot product of the highlighted row and column from the first two matrices.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C .

Matrix A is represented by a grid of asterisks. The bottom row is highlighted in light blue.

Matrix B is represented by a grid of asterisks. The first column is highlighted in light red.

Matrix C is represented by a grid of asterisks. The bottom-left element is highlighted in light purple.

The equation is shown as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (left): The bottom row is highlighted in light blue.
- A 3x4 matrix B (middle): The second column is highlighted in light red.
- The resulting 5x4 matrix AB (right): The element at the intersection of the bottom row of A and the second column of B is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (left): The bottom row is highlighted in light blue.
- A 3x4 matrix B (middle): The third column is highlighted in light red.
- The resulting 5x4 matrix AB (right): The bottom-right element is highlighted in light purple.

The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in all cells; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in all cells; the fourth column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in all cells; the bottom-right element (row 5, column 4) is highlighted in light purple, representing the dot product of the first matrix's bottom row and the second matrix's fourth column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

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This is just vector multiplication.

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

(also find a pair where they are the same)

Answer: Rotation and Reflection

Computational Aspects of Matrix Multiplication

Matrix Operations in Numpy

Let `a` and `b` be 2D numpy arrays and let `c` be a floating point number.

» `a @ b` (matrix multiplication)

» `a + b` (matrix addition)

» `c * a` (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more.

A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Suppose A and B are $n \times n$ matrices.

This operations takes n multiplications and n divisions ($2n$ FLOPS total)

Repeating for each entry gives $\sim 2n^3$ FLOPS

A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

The main part of this procedure is highly parallelizable.

A Note on Parallelization

```
a = np.array(...)  
b = np.array(...)  
prod = np.zeros([a.shape[0], b.shape[1]])  
for i in range(a.shape[0]):  
    for j in range(b.shape[1]):  
        prod[i, j] = np.dot(a[i], b[:, j])
```

The main part of this procedure is highly parallelizable.

One processor per entry gets you to $\sim 2n$ FLOPS

A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers.

Best leave it to experts (or do research in the area).

LAPACK is the state of the art library for matrix operations.

numpy uses LAPACK

Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations.

Matrix multiplication coincides with composition of linear transformations.

There is an algebra of matrices which is consistent with the algebra of vectors.