Practice Final Solutions

CAS CS 132

Fall 2024

Problem 1

A. $A \sim \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

x_1 is free

$x_2 = 5x_4$

$x_3 = -x_4$

x_4 is free

B. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \\ 1 \end{bmatrix}$

C. $B - \lambda I \overset{\text{swp}}{\sim} \begin{bmatrix} 2-\lambda & -3 & 1 & -4 \\ 0 & -\lambda & 2 & 7 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$

$\det(B - \lambda I) = \lambda^2(\lambda - 2)(3 - \lambda)$

D. No. 0 is an eigenvalue

Problem 2

A.

$\lambda = -1, 8$

$$\begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2+3}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 24 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix}$$

B.

$$B = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -18 \\ 3 & 8 \end{bmatrix}$$

Problem 3

A.

$$A = \begin{bmatrix} 4 & -8 \\ 2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -8 & 4 \\ 6 & 2 \end{bmatrix}$$

B.

$$\det A = 24 + 16 = 40$$

$$\det B = -16 - 24 = -40$$

C. $\text{dist}((-8,6),(4,2)) = \left((-8-4)^2 + (6-2)^2\right)^{1/2}$

$$= 144 + 16 = \sqrt{160} = 4\sqrt{10}$$

$$\left\langle \alpha \begin{bmatrix} 12 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle = 0$$

$$\alpha = \frac{\left\langle \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle} = \frac{40}{160} = \frac{1}{4}$$

$$\frac{1}{4}\begin{bmatrix}12\\-4\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}$$

$$\|\vec{r}\|=\left\|\begin{bmatrix}3\\-1\end{bmatrix}-\begin{bmatrix}4\\2\end{bmatrix}\right\|=\left\|\begin{bmatrix}-1\\-3\end{bmatrix}\right\|=\sqrt{10}$$

$$\text{area}=\frac{1}{2}\left(4(\sqrt{10})\cdot\sqrt{10}\right)=20$$

(half the determinant

Problem 4.

A. False

B. True

C. True

D. False

E. False

F. True

G. True

Problem 5

A.

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

B

$$T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} 674 \text{ rem. } 1 \\ 3\,\overline{)2023} \\ \underline{18} \\ 22 \\ \underline{21} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

$$T^{2023} = T^1 = T = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

C. No. There are only 3 possible matrices for T^n, all have 0 entries

D. Yes

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

one free variable means exactly 1 steady state distribution

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Problem 6

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 3 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 15 \end{bmatrix}$$

$\sigma_1 = \sqrt{15}$ $\sigma_2 = 2$