

Practice Final Solutions

CAS CS 132

Fall 2024

# Problem 1

$$A. \quad A \sim \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_1$  is free

$$x_2 = 5x_4$$

$$x_3 = -x_4$$

$x_4$  is free

$$B. \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \\ 1 \end{bmatrix}$$

$$C. \quad B - \lambda I = \begin{bmatrix} 2-\lambda & -3 & 1 & -4 \\ 0 & -\lambda & 2 & 7 \\ 0 & 0 & -\lambda & -2 \\ 0 & 0 & 1 & 3-\lambda \end{bmatrix}$$

$$\sim \begin{bmatrix} 2-\lambda & -3 & 1 & -4 \\ 0 & -\lambda & 2 & 7 \\ 0 & 0 & -\lambda & -2 \\ 0 & 0 & \lambda & (3-\lambda)\lambda \end{bmatrix}$$

$$\sim \begin{bmatrix} 2-\lambda & -3 & 1 & -4 \\ 0 & -\lambda & 2 & 7 \\ 0 & 0 & -\lambda & -2 \\ 0 & 0 & 0 & (3-\lambda)\lambda - 2 \end{bmatrix}$$

$$\frac{1}{\lambda} (2-\lambda) \lambda^2 ((3-\lambda)\lambda - 2)$$

$$= (2-\lambda)(3\lambda - \lambda^2 - 2) =$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2) = (\lambda-2)^2(\lambda-1)$$

D. No. 0 is an eigenvalue

Problem 2

A.  
 $\lambda = -1, 8$

$$\begin{cases} \begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ \begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix} \end{cases}$$

$$D = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2+3} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$A = P D P^{-1}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8 & 24 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & -54 \\ 9 & 26 \end{bmatrix}$$

B.

$$B = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -18 \\ 3 & 8 \end{bmatrix}$$

### Problem 3

A.

$$A = \begin{bmatrix} 4 & -8 \\ 2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} -8 & 4 \\ 6 & 2 \end{bmatrix}$$

B.

$$\det A = 24 + 16 = 40$$

$$\det B = -16 - 24 = -40$$

$$\begin{aligned} \text{C. } \text{dist}((-8, 6), (4, 2)) &= \left( (-8-4)^2 + (6-2)^2 \right)^{1/2} \\ &= 144 + 16 = \sqrt{160} = 4\sqrt{10} \end{aligned}$$

$$\left\langle \alpha \begin{bmatrix} 12 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle = 0$$

$$\alpha = \frac{\left\langle \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 12 \\ -4 \end{bmatrix} \right\rangle} = \frac{40}{160} = \frac{1}{4}$$

$$\frac{1}{4} \begin{bmatrix} 12 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\|\vec{v}\| = \left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\| = \sqrt{10}$$

$$\text{area} = \frac{1}{2} (4(\sqrt{10}) \cdot \sqrt{10}) = 20$$

(half the determinant)



Problem 4.

A. False

B. True

C. True

D. False

E. False

F. True

G. True

# Problem 5

A.

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

B

$$T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r}
 674 \text{ rem. } 1 \\
 3 \overline{) 2023} \\
 \underline{18} \phantom{00} \\
 22 \phantom{00} \\
 \underline{21} \phantom{00} \\
 13 \phantom{00} \\
 \underline{12} \phantom{00} \\
 1
 \end{array}$$

$$T^{2023} = T^1 = T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

C. No. There are only 3 possible matrices for  $T^n$ , all have 0 entries

D. Yes

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

one free variable means exactly 1 steady  
state distribution

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

## Problem 6

$$A A^T = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 3 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 15 \end{bmatrix}$$

$$\sigma_1 = \sqrt{15}$$

$$\sigma_2 = 2$$