# Dimension and Rank

## Geometric Algorithms
## Lecture 17

CAS CS 132

# Practice Problem

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

*Consider the subspace $H$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that $\mathbf{v}_3$ and $\mathbf{v}_4$ form a basis for $H$.*

# Answer

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix}2\\1\\2\end{bmatrix}$$

*Hint. Show that* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are in the span of* $\mathbf{v}_3$ *and* $\mathbf{v}_4$

$\{\vec{v}_3, \vec{v}_4\}$ is L.I. ✓

$\text{span}\{\vec{v}_3, \vec{v}_4\} = H \Leftarrow$ $v_1 \in \text{span}\{\vec{v}_3, \vec{v}_4\}$

$v_2 \in \text{span}\{\vec{v}_3, \vec{v}_4\}$

$x_1\vec{v}_3 + x_2\vec{v}_4 = \vec{v}_1$

$$\begin{array}{ccc} v_3 & v_4 & v_1 \end{array}$$
$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 2\\0 & 2 & 0\end{bmatrix} \sim \text{RREF}$$

$x_1\vec{v}_3 + x_2\vec{v}_4 = \vec{v}_2$

$$\begin{bmatrix}1 & 2 & -1\\0 & 1 & 0\\0 & 2 & 1\end{bmatrix} \sim \text{RREF}$$

# Objectives

1. Discuss the coordinate systems.

2. Introduce the fundamental notion of <u>dimension</u>, which quantifies how "large" a space is

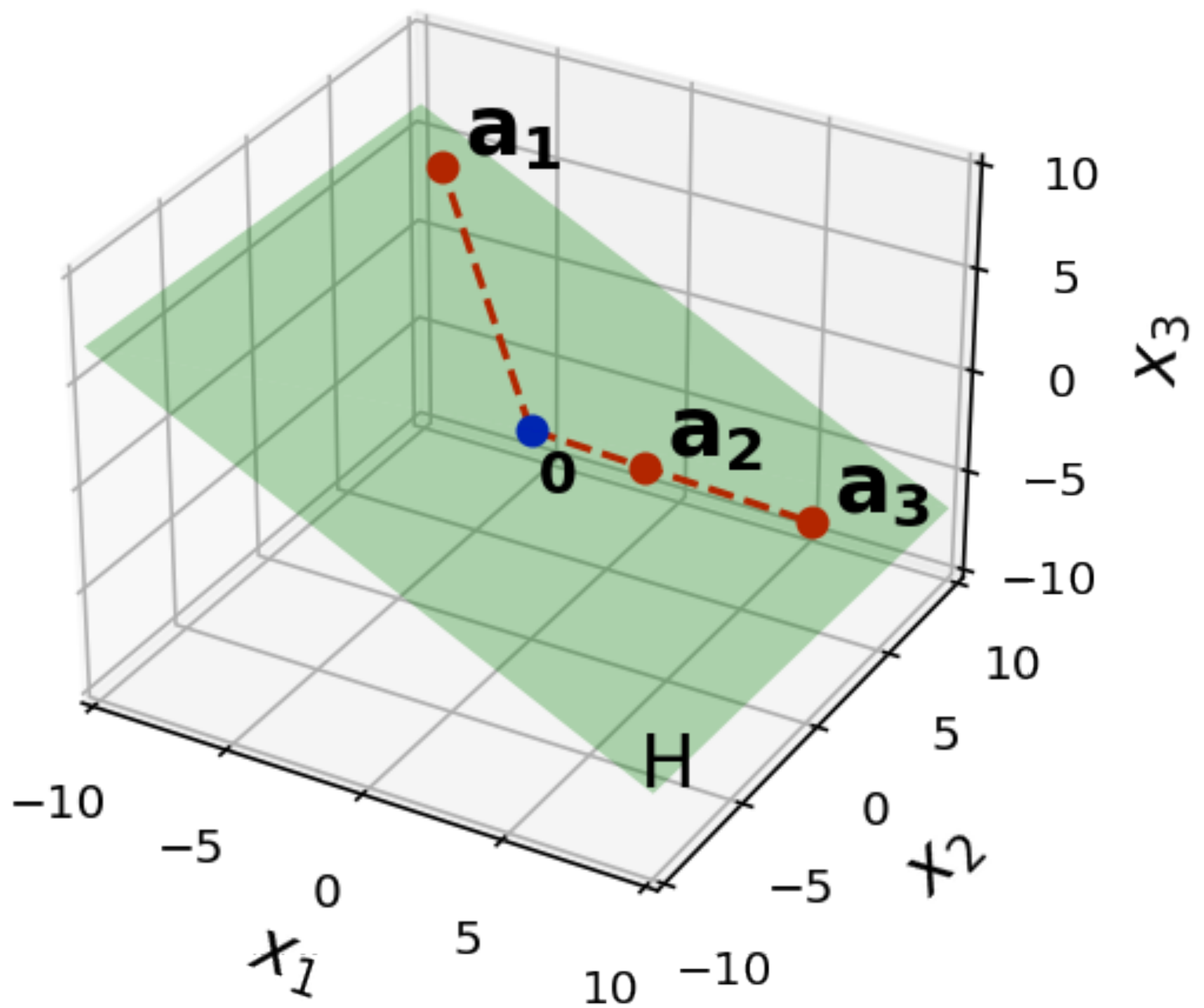
3. Relate the dimension of the column space and the null space of a matrix

# Keywords

```
basis

column space

null space

coordinate system

change of basis

dimension

rank

rank theorem

invertible matrix theorem (extended)
```

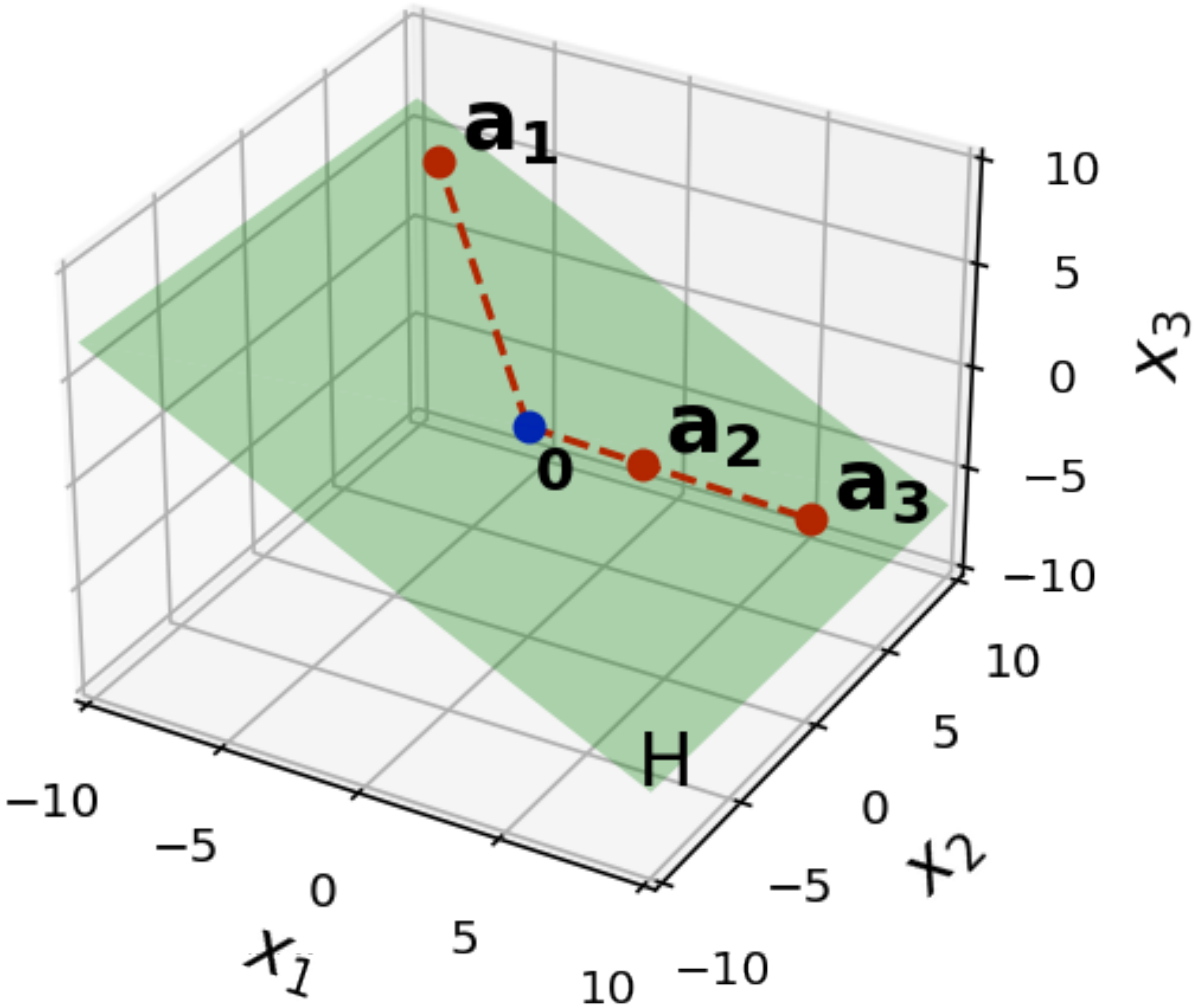
# Recap

# Recall: The Idea Behind Subspaces



# Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"



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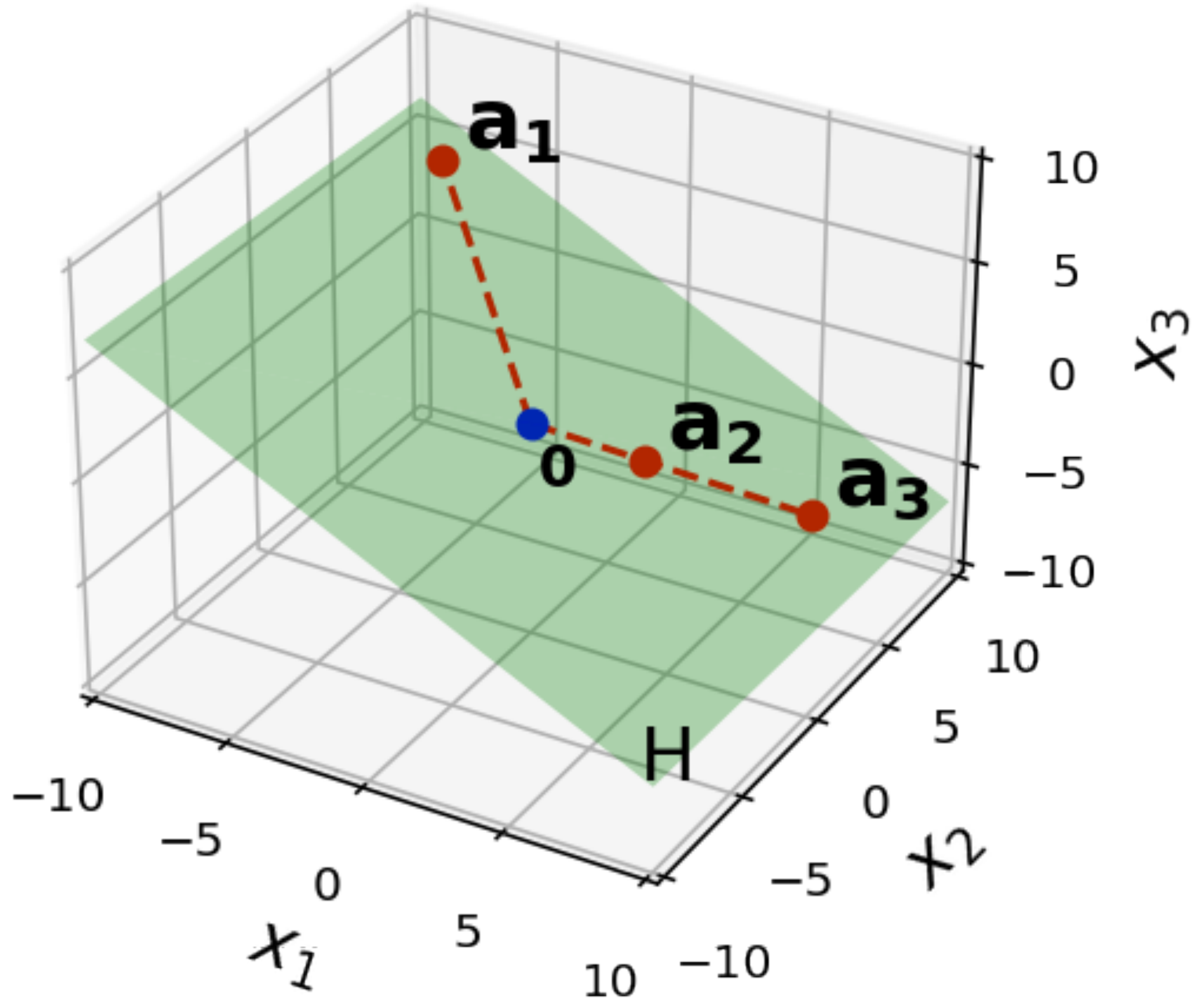
A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$



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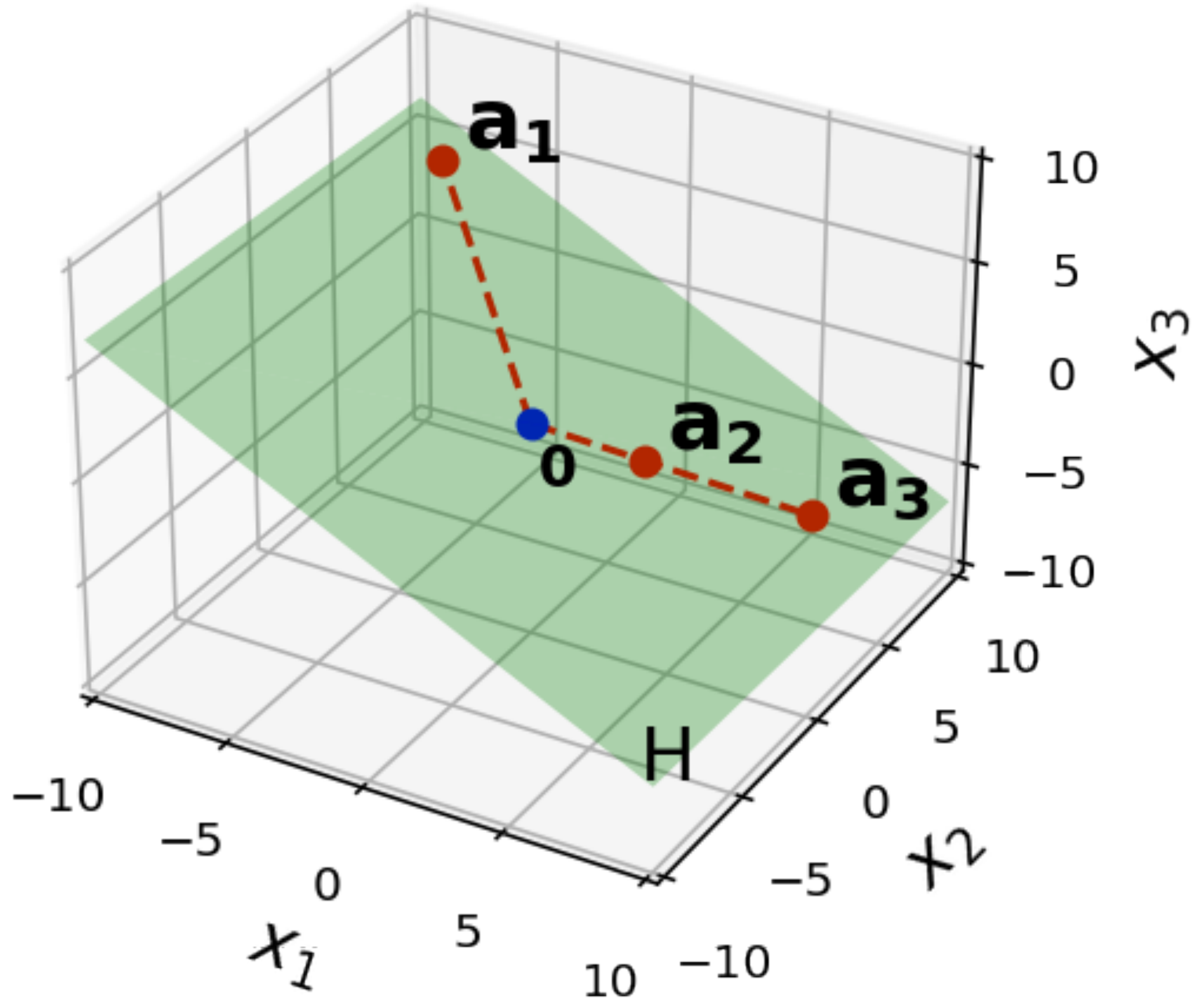
Subspaces *generalize* of this idea.



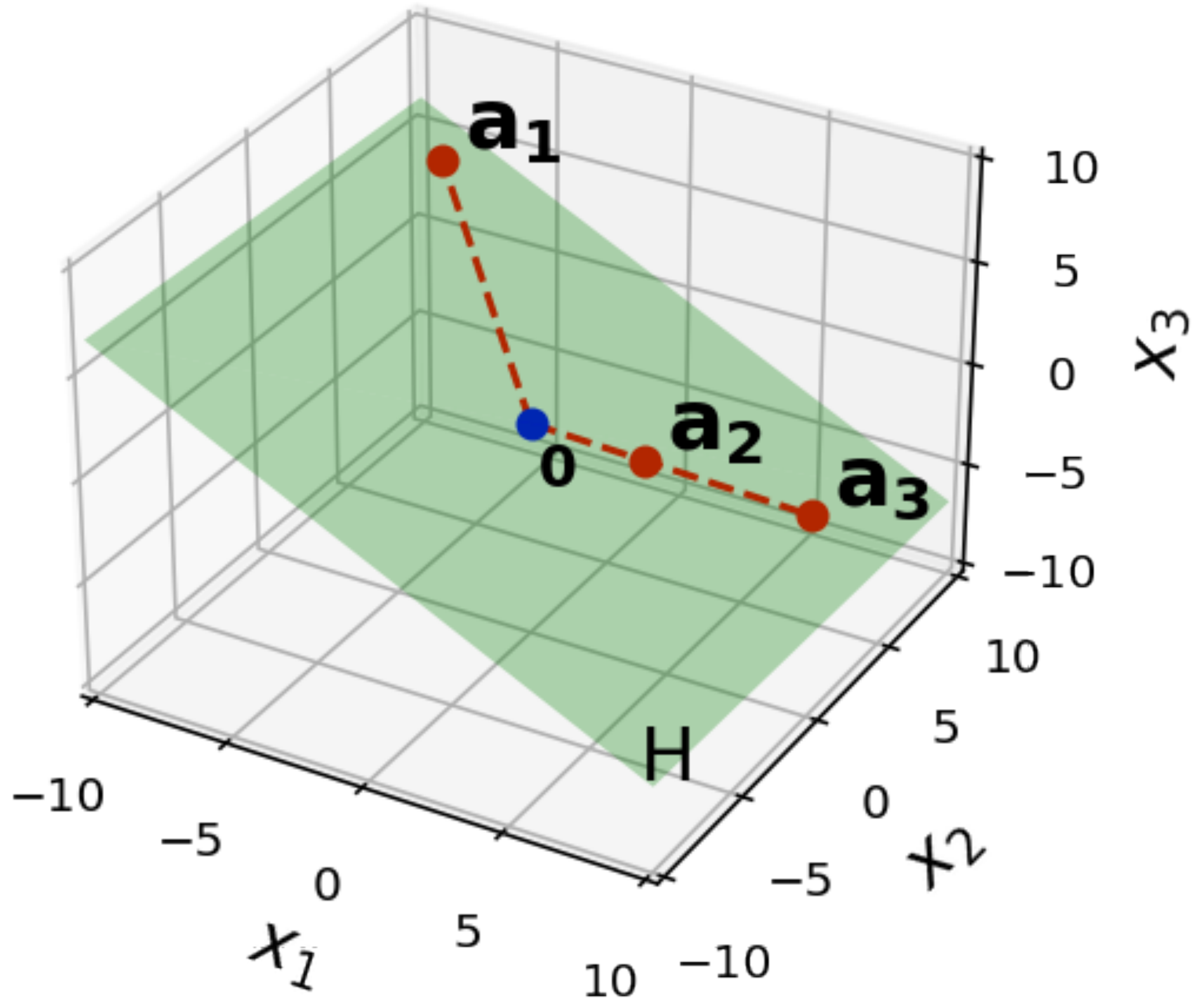
# Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"

A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$



# Recall: Subspace (Algebraic Definition)

**Definition.** A **subspace** of $\mathbb{R}^n$ is a set $H$ of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in $H$, the vector $\mathbf{u} + \mathbf{v}$ is in $H$
2. for every $\mathbf{u}$ in $H$ and scalar $c$, the vector $c\mathbf{u}$ is in $H$

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**!! Subspaces must "live" somewhere !!**

# Column Space

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**The column space of a matrix is the span of its columns.**

# Column Space

$\text{span}\{a_1, \ldots, a_k\} =$

$\text{ran}(T)$ where $T(\vec{x}) = [a_1 \ldots a_k]\vec{x}$

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**The column space of a matrix is the span of its columns.**

**The column space of a matrix is the <u>range</u> of the linear transformation it implements.**

# Subspace of What?

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \cdots & | & | \end{bmatrix} \end{array}$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\text{Col}(A)$

is a subspace of

$\mathbb{R}^m$

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$$A\mathbf{x} = \mathbf{0}$$

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**The null space of a matrix $A$ is the set of all vectors that are mapped to the zero vector by $A$.**

# Subspace of What?

$n$ columns

$m$ rows

$$A\mathbf{v} = \mathbf{0}$$

$m \times n$ $\quad n \times 1$ $\quad m \times 1$

$\mathbf{v}$ **is a vector in $\mathbb{R}^n$**

$\mathrm{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

# Recall: Basis

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**Definition.** A **basis** for a subspace $H$ of $\mathbb{R}^n$ is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors that spans $H$ (in symbols: $H = \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$).

**A basis is a *minimal* set of vectors which spans all of $H$.**

# Recall: Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

$x_1 = 2x_2 + x_4 - 3x_5$
$x_2$ is free
$x_3 = (-2)x_4 + 2x_5$
$x_4$ is free
$x_5$ is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

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**!! this transformation is only linear !!**
**!! in the case of homogeneous equations !!**

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1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.
3. The resulting vectors form a basis for $\mathrm{Nul}(A)$.

# Recall: Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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**The idea.** What if we cover up the non-pivot columns?

# Recall: Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \text{[redacted]} \quad \mathbf{a}_3 \quad \text{[redacted]} \quad \text{[redacted]}] \sim \begin{bmatrix} 1 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]} \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

**The idea.** What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \quad \mathbf{a}_3]$ has 2 pivots.

$$[a_1 \quad a_3] \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

# Recall: Column Space and Reduced Echelon form

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**The idea.** What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \ \ \mathbf{a}_3]$ has 2 pivots.

So the pivot columns are <u>linearly independent</u>.

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# Recall: Column Space and Reduced Echelon form

$$\begin{array}{ccc} 2 & 1 & \\ \mathbf{a}_1 & \mathbf{a}_2 & \text{[redacted]} \end{array} \sim \begin{array}{c} \begin{array}{ccc} 2 & 1 & \end{array} \\ \begin{bmatrix} 1 & -2 & \\ 0 & 0 & \text{[redacted]} \\ 0 & 0 & \end{bmatrix} \end{array}$$

**Observation.** $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

# Recall: Column Space and Reduced Echelon form

$$\begin{array}{ccc} 2 & 1 & \\ \end{array}$$
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So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

row op. don't change relationship between columns

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So $\mathbf{2a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

**In general, every non-pivot column of $A$ can be written as a linear combination pivots in front of it.**

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**In general, every non-pivot column of $A$ can be written as a linear combination pivots in front of it.**

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> $\text{Col}(A)$.

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**The takeaway.** The pivot columns of $A$ form a basis for $\mathsf{Col}(A)$.

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**The takeaway.** The pivot columns of $A$ form a basis for $\text{Col}(A)$.

**!! IMPORTANT !!**

**Choose the columns <u>of $A$</u>.**

*($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\text{Col}(A)$)*

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1. Find the pivot columns in an echelon form of $A$.

2. The associated columns <u>in $A$</u> form a basis for $\mathrm{Col}(A)$.

# Example

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

*Find a bases for the column space and null space of $A$.*

# Answer

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$x_1 = -9 x_3$

$x_2 = 5 x_3 - 2 x_5$

$x_3$ is free

$x_4 = - x_5$

$x_5$ is free

$$x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of Nul$(A)$

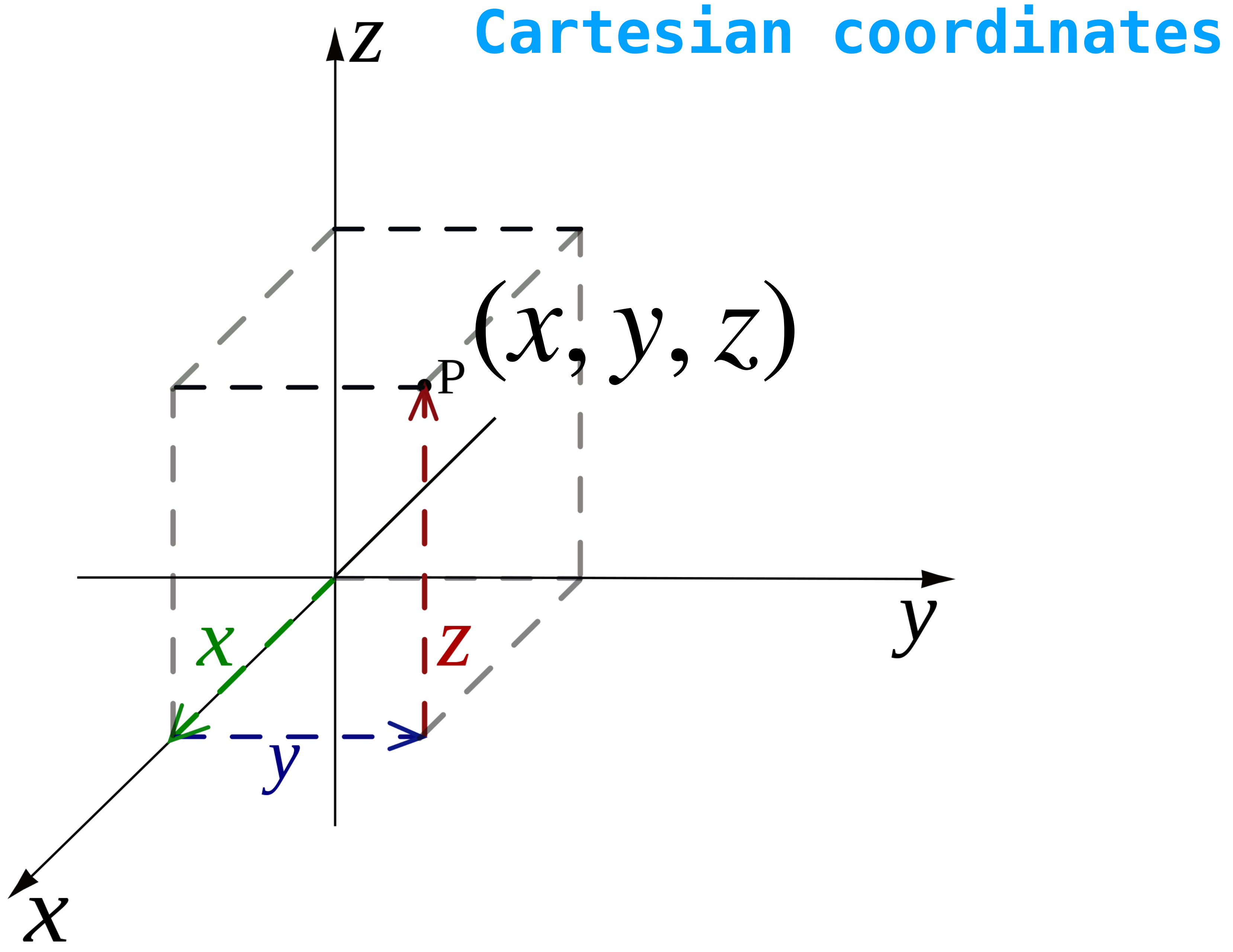
moving on...

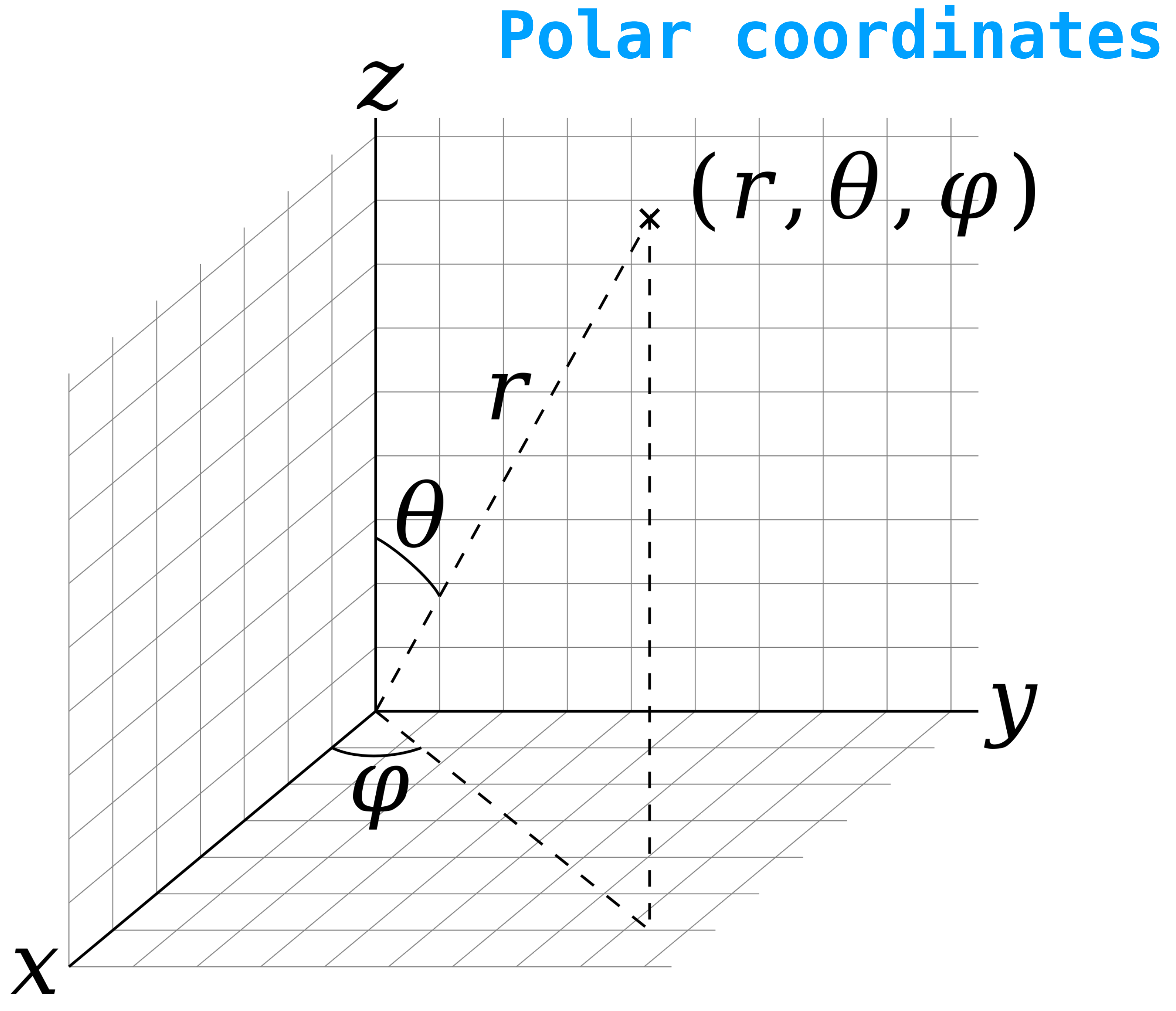
# Coordinate Systems

# At a High Level

A coordinate system is a way of representing positions in terms of a sequence of numbers.

**Examples.**



# Question (Conceptual)*

*And a bit of a trick question

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*Is* $(2.3, 0.01, 5)$ *a polar coordinate or a cartesian coordinate?*

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**This question is non-sensical.**

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# Question (Conceptual)*

*Is* $(2.3, 0.01, 5)$ *a polar coordinate or a cartesian coordinate?*

**This question is non-sensical.**

It's just a sequence of numbers. We need to be *told* if it should be interpreted in the **polar** coordinate system or the **Cartesian** coordinate system.

*And a bit of a trick question

# Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace $H$, there is **exactly one way** to write every vector in $H$ as a linear combination of vectors in $\mathscr{B}$.

Verify: $\mathcal{B} = \{v_1, \ldots, v_k\}$ $\quad u \in H$

Suppose

$$\sum_{i=1}^{k} \alpha_i \vec{v}_i = \sum_{i=1}^{k} \beta_i \vec{v}_i = \vec{u}$$

$$\sum_{i=1}^{k} \alpha_i \vec{v}_i - \sum_{i=1}^{k} \beta_i \vec{v}_i = \boxed{\sum_{i=1}^{k} (\alpha_i - \beta_i) v_i} = \vec{h} - \vec{h} = \boxed{\vec{0}}$$

$\alpha_i - \beta_i = 0$ for all $i$,

$$\boxed{\therefore \alpha_i = \beta_i \text{ for all } i}$$

# Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace $H$, there is **exactly one way** to write every vector in $H$ as a linear combination of vectors in $\mathscr{B}$.

Every basis provides a way to write down *coordinates* of a vector.

And every time we write down a vector, we are **assuming a coordinate system.**

what do we mean by this?

# A Thought Experiment

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Then one day, you get tired of talking about "abstract" vectors, you want to work with *numbers.*

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Because we've learned everything up to now, we know that there is a basis $\mathbf{b}_1$, $\mathbf{b}_2$,..., $\mathbf{b}_n$ for the space $\mathbb{R}^n$.

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

# A Thought Experiment

Because we've learned everything up to now, we know that there is a basis $\mathbf{b}_1$, $\mathbf{b}_2$,..., $\mathbf{b}_n$ for the space $\mathbb{R}^n$.

So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

unique

$$\mathbf{v} = \boxed{2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \\ -0.1 \end{bmatrix}$$

**and then choose those weights as a representation of $v$ as a sequence of numbers**

# But wait...

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`This` *`depends`* `on the choice of basis.`

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2$,..., $\mathbf{c}_n$ then we would get some other representation.

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2$,..., $\mathbf{c}_n$ then we would get some other representation.

$$\mathbf{v} = (-10)\mathbf{c}_1 + (4.3)\mathbf{c}_2 + \ldots + 0\mathbf{c}_n = \begin{bmatrix} -10 \\ 4.3 \\ \vdots \\ 0 \end{bmatrix}$$

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2$,..., $\mathbf{c}_n$ then we would get some other representation.

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**Every basis defined a different coordinate system**

# Standard Basis

$$\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \boxed{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \boxed{5}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \boxed{7}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The standard basis defines the Cartesian coordinate system for $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

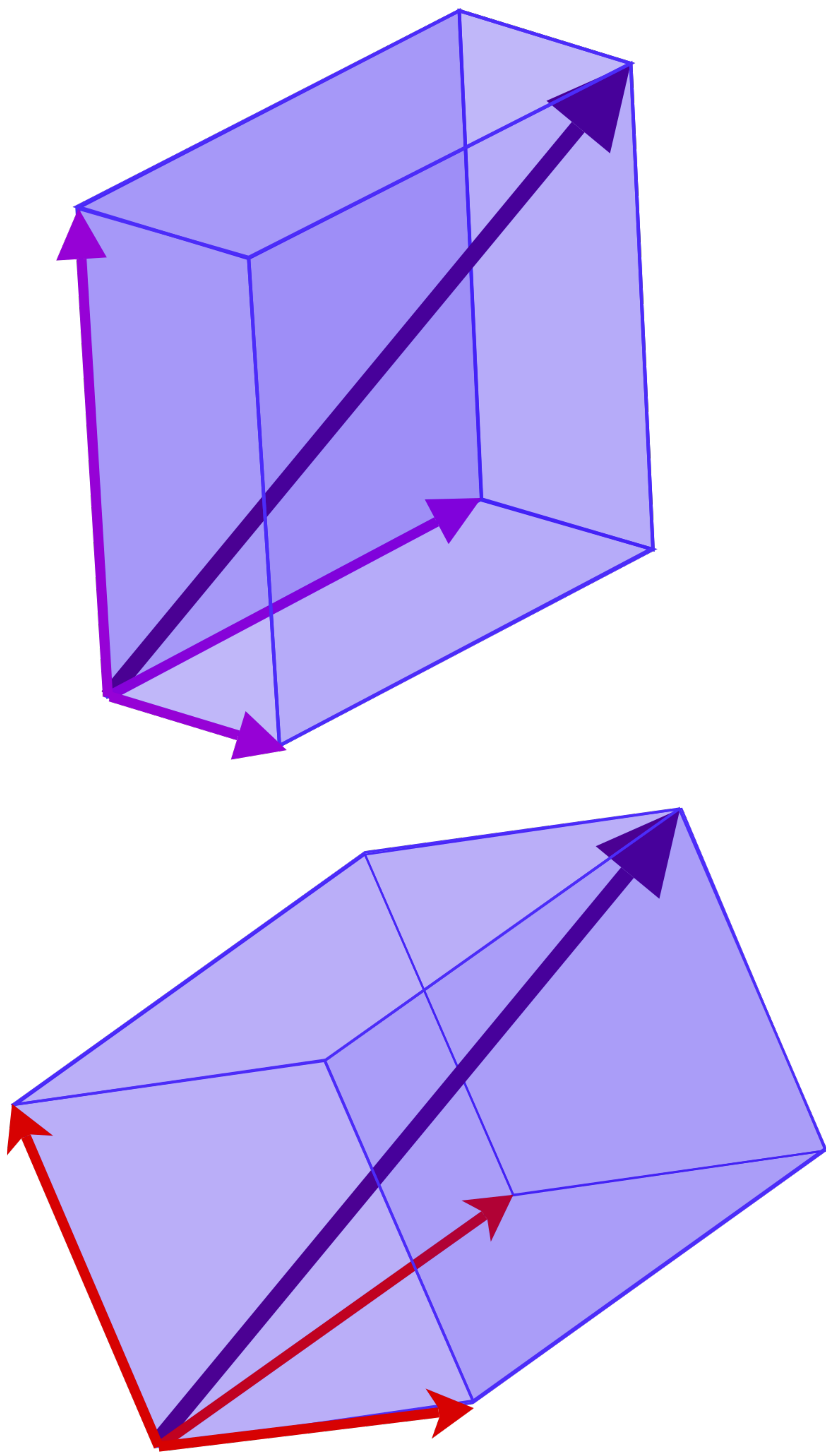
*Column vectors are just weights for a linear combination of the standard basis*

but we can also use
different coordinate systems

# How to think about this

Changing the coordinate system "warps space".

**The question is:** how do we represent a vector $\mathbf{v}$ in the warped space if we wanted it to "be in the same place"?



https://commons.wikimedia.org/wiki/File:3d_two_bases_same_vector.svg

# Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a subspace $H$ of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis of $H$ where

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_k\mathbf{b}_k$$

# Coordinate Vectors

Let $\mathbf{v}$ be a vector in a subspace $H$ of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis of $H$ where

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$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

# Coordinate Vectors and the Standard Basis

When we write down a vector $\mathbf{v}$ in $\mathbb{R}^n$, we're really writing down a coordinate vector **relative to the standard basis** $\mathscr{E}$**.**

$$[\mathbf{v}]_{\mathscr{E}} = \mathbf{v}$$

# How do we find coordinate vectors?

For an arbitrary basis $\mathscr{B}$, to determine $[\mathbf{v}]_{\mathscr{B}}$, we need to find weights $a_1, \ldots, a_k$ such that

$$a_1\mathbf{b}_1 + \ldots + a_k\mathbf{b}_k = \mathbf{v}$$

This is just <u>solving a vector equation</u>.

# Example: 2D Case

Write the coordinate vector for $\begin{bmatrix}1\\6\end{bmatrix}$ relative to the basis $\mathcal{B}$ $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ for $\mathbb{R}^2$

$$-2\begin{bmatrix}1\\0\end{bmatrix}+3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\6\end{bmatrix}$$

$$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\6\end{bmatrix} \qquad \boxed{\begin{bmatrix}1\\6\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}-2\\3\end{bmatrix}}$$

$$\begin{bmatrix}1&1&1\\0&2&6\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&-2\\0&1&3\end{bmatrix}$$

# Example: 2D Case (Geometrically)

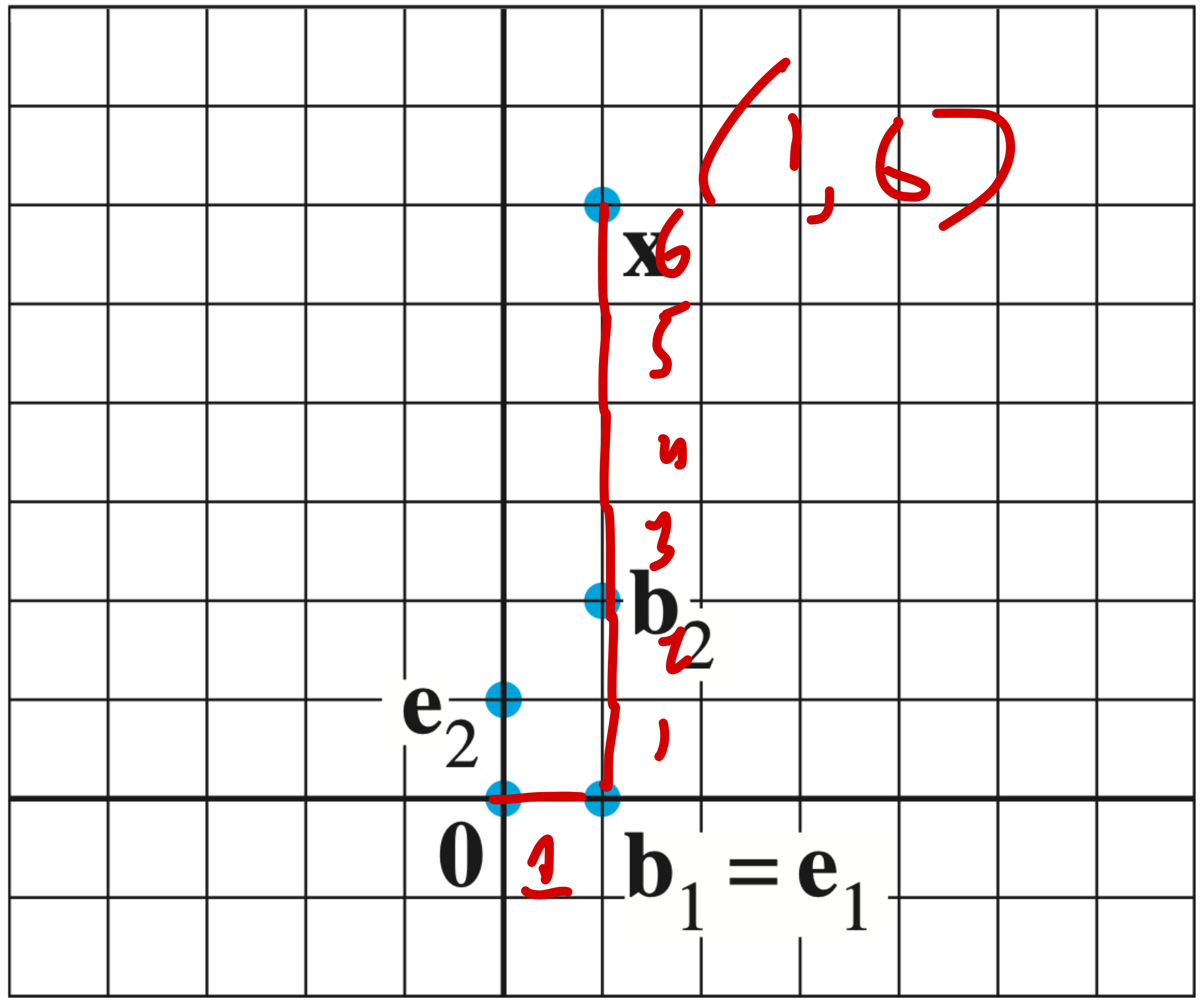



**FIGURE 1** Standard graph paper.

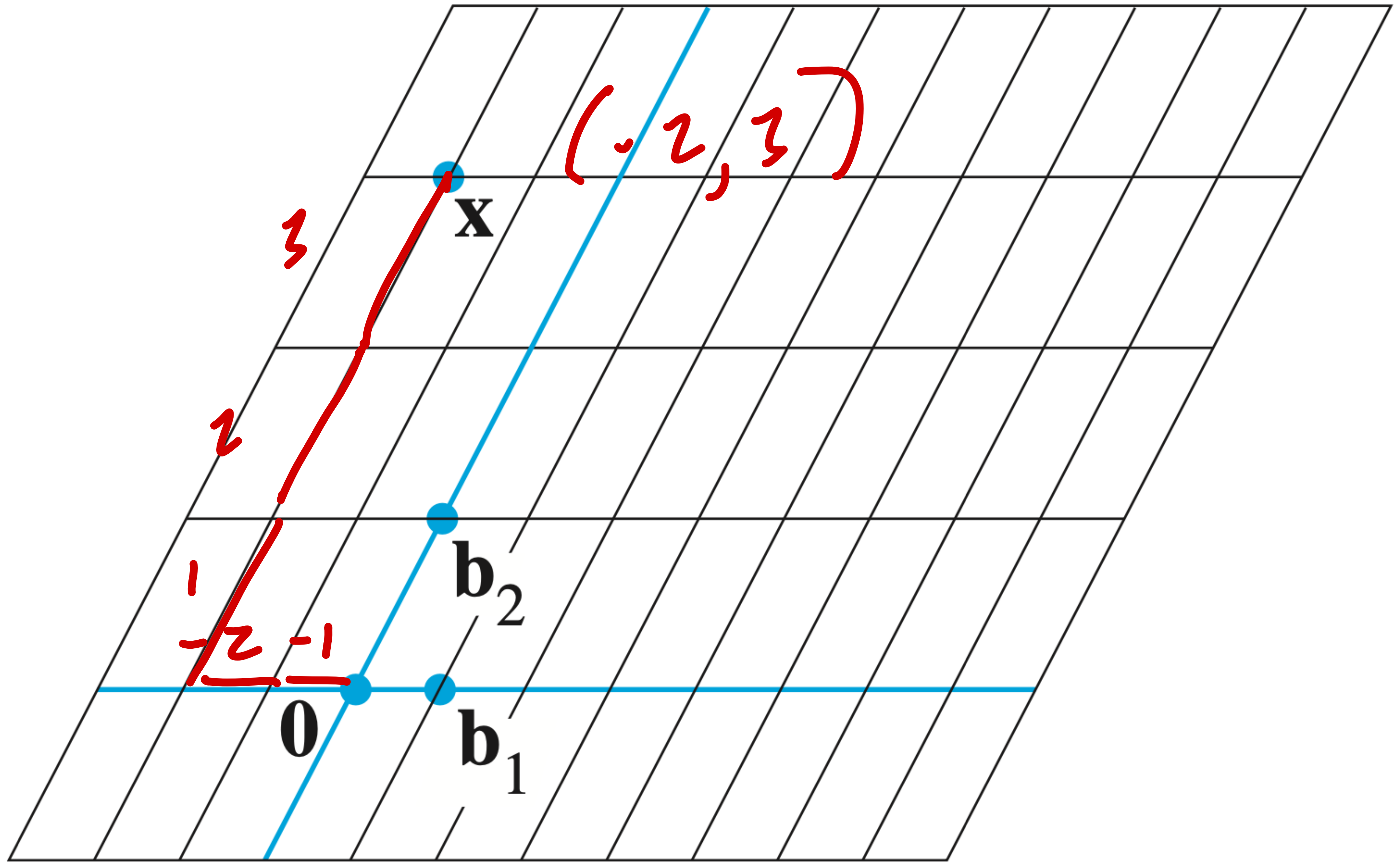



**FIGURE 2** $\mathcal{B}$-graph paper.

$\mathscr{B}$ **defines a "different grid for our graph paper"**

# How To: Coordinate Vectors

**Question.** Find the coordinate vector for $\mathbf{v}$ in the subspace $H$ relative to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$.

**Solution.** Solve the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_k\mathbf{b}_k = \mathbf{v}$$

A solution $(a_1, \ldots, a_k)$ means

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

# Example: 3D Case

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

Find the coordinate vector for $\mathbf{u}$ relative to the basis $\mathcal{B}$ $\{\mathbf{v}_1, \mathbf{v}_2\}$ of a subspace $H$ (of $\mathbb{R}^3$):

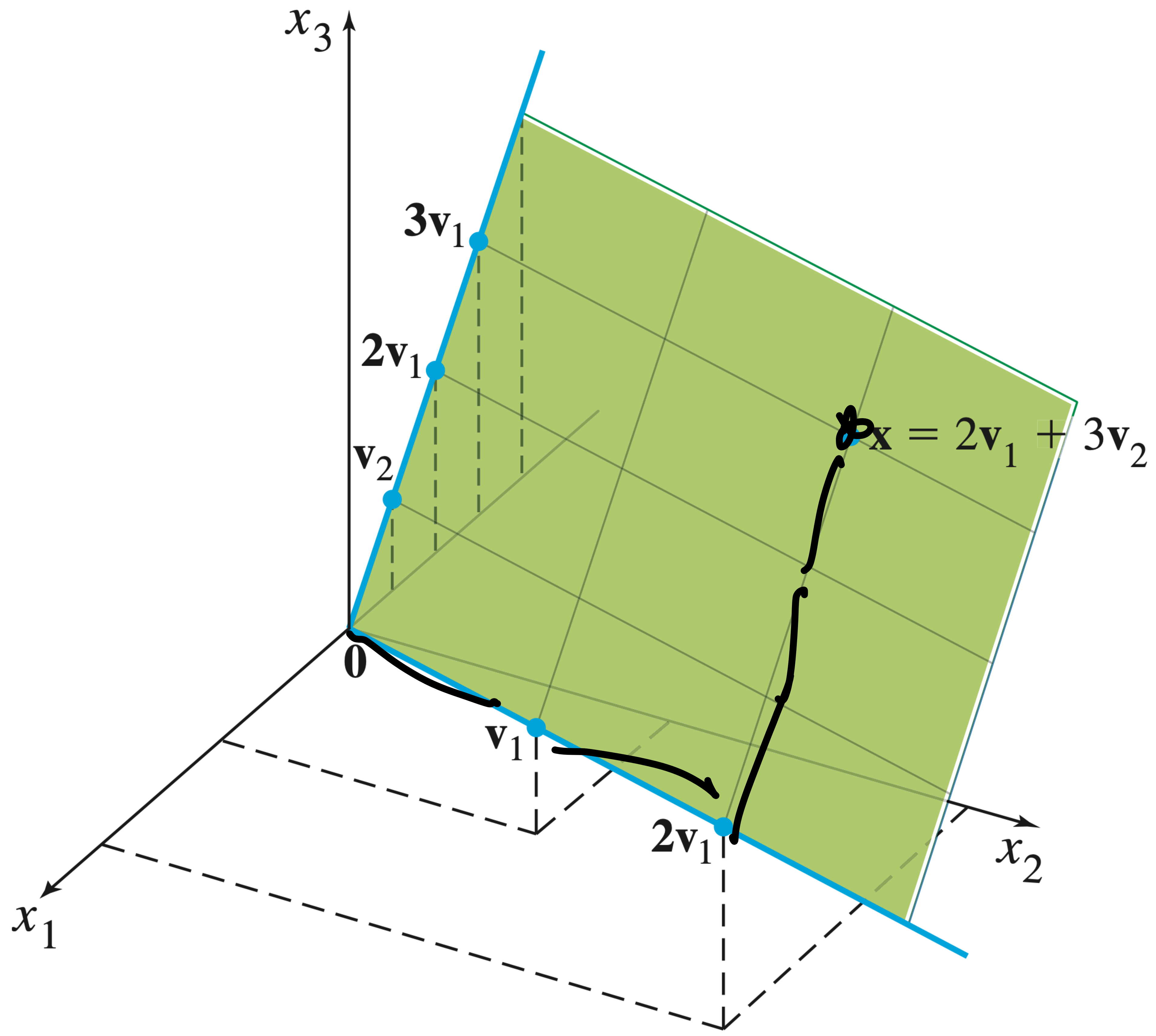
$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{u}$$

$$2\vec{v}_1 + 3 v_2 = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

# An Aside: Coordinates and one-to-one correspondences




Linear Algebra and its Applications (Lay, Lay, McDonald)

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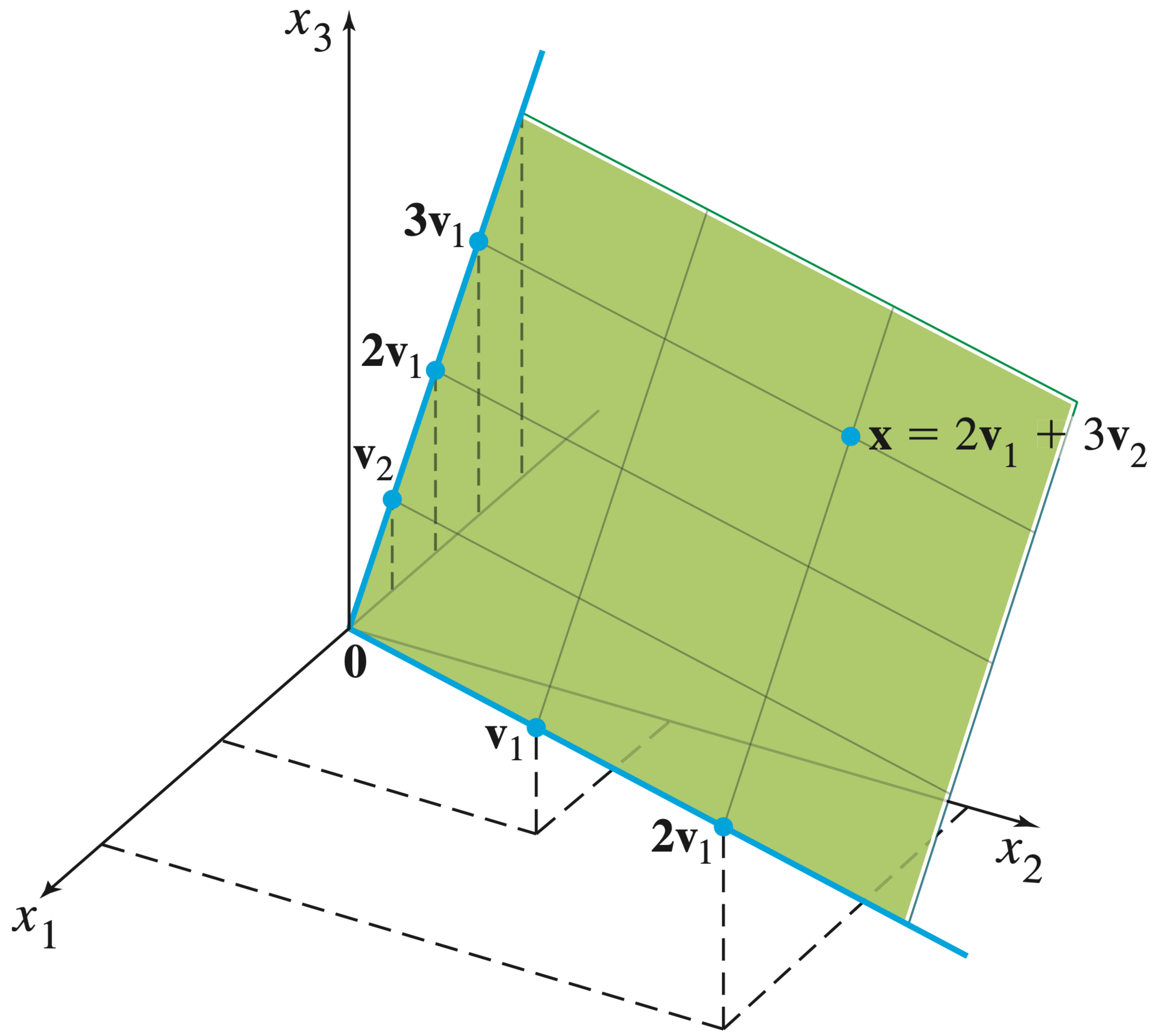
In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a <u>one-to-one correspondence</u> from $H$ to $\mathbb{R}^2$. This is also sometimes called an **isomorphism**.

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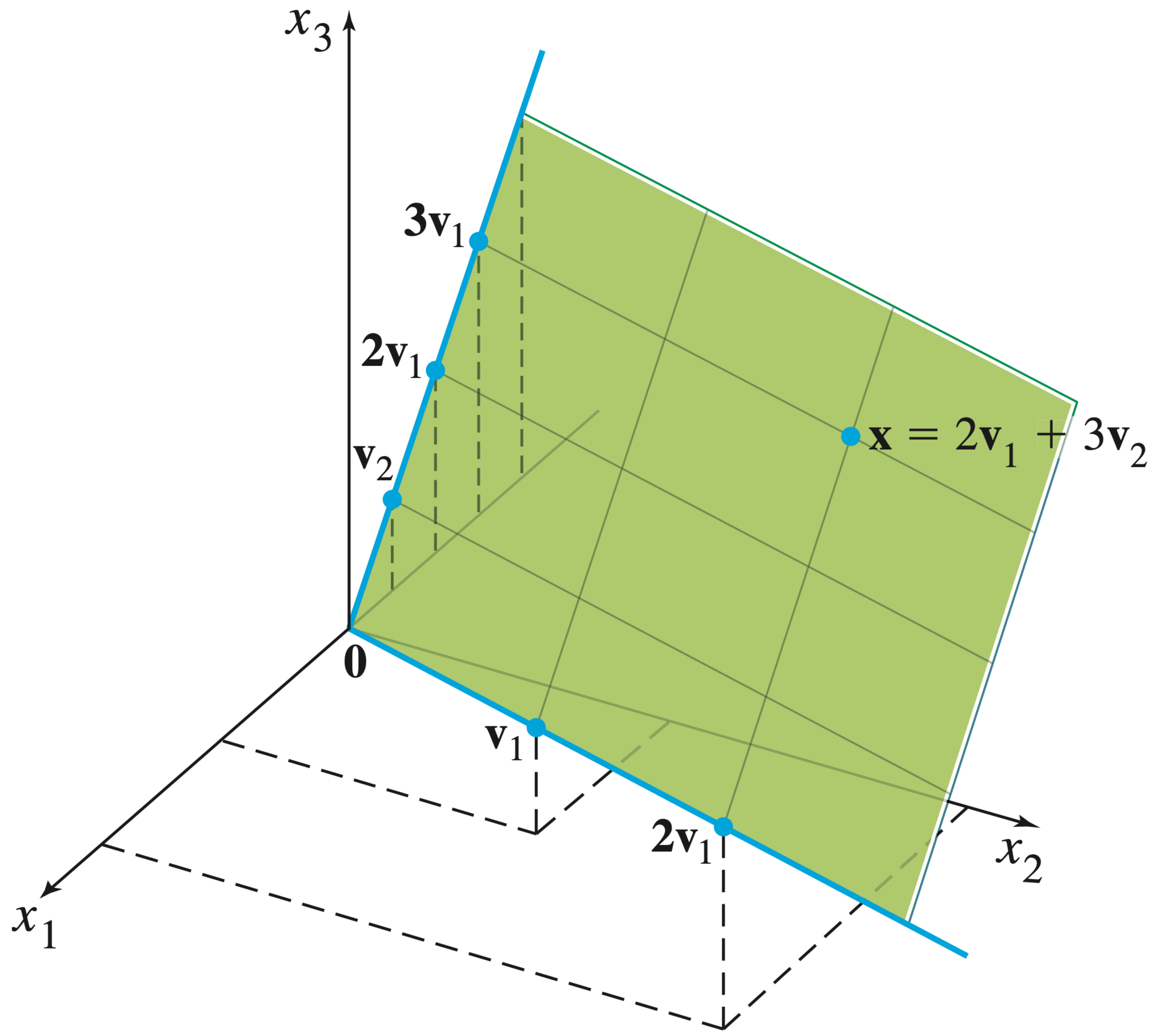


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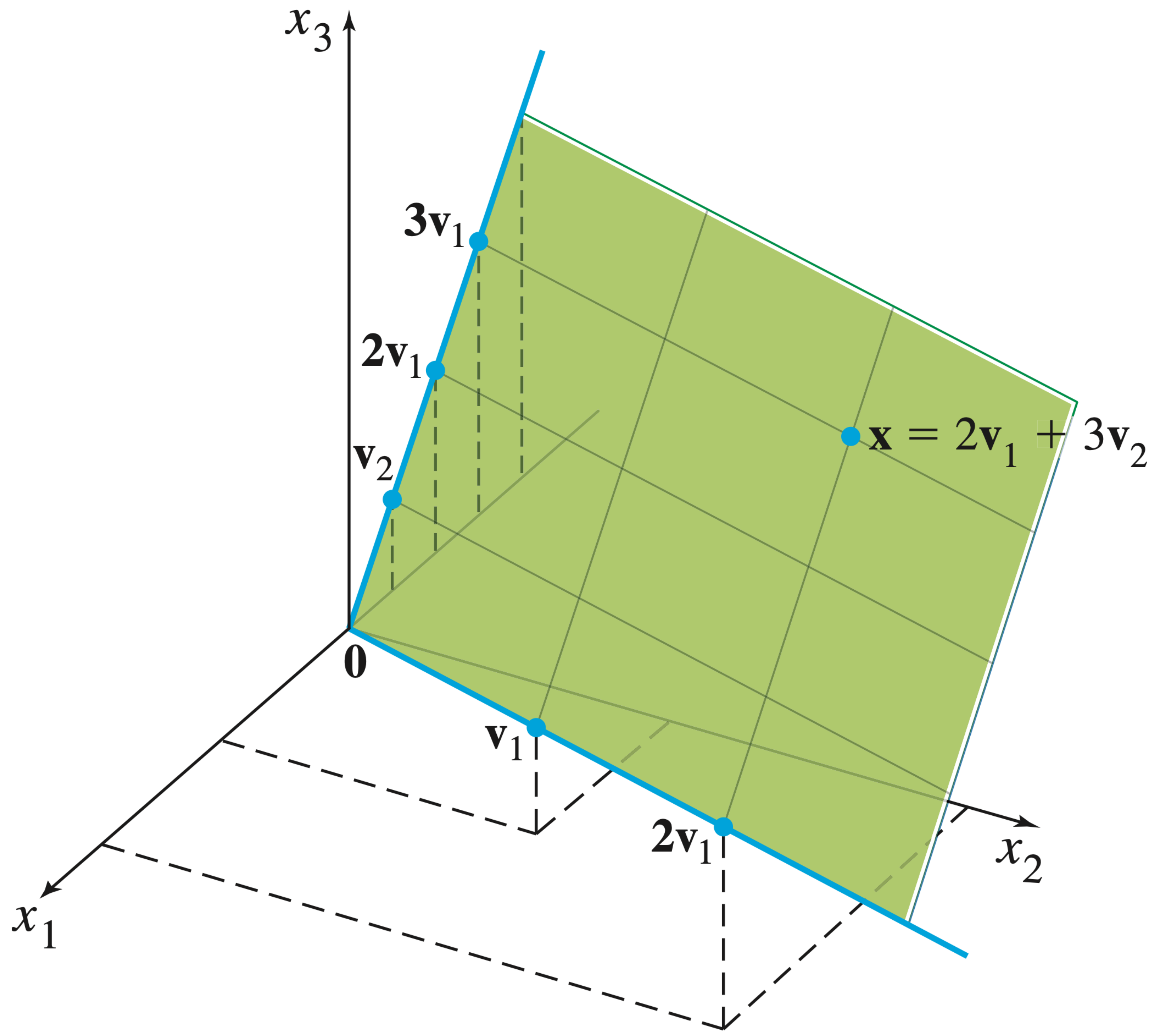


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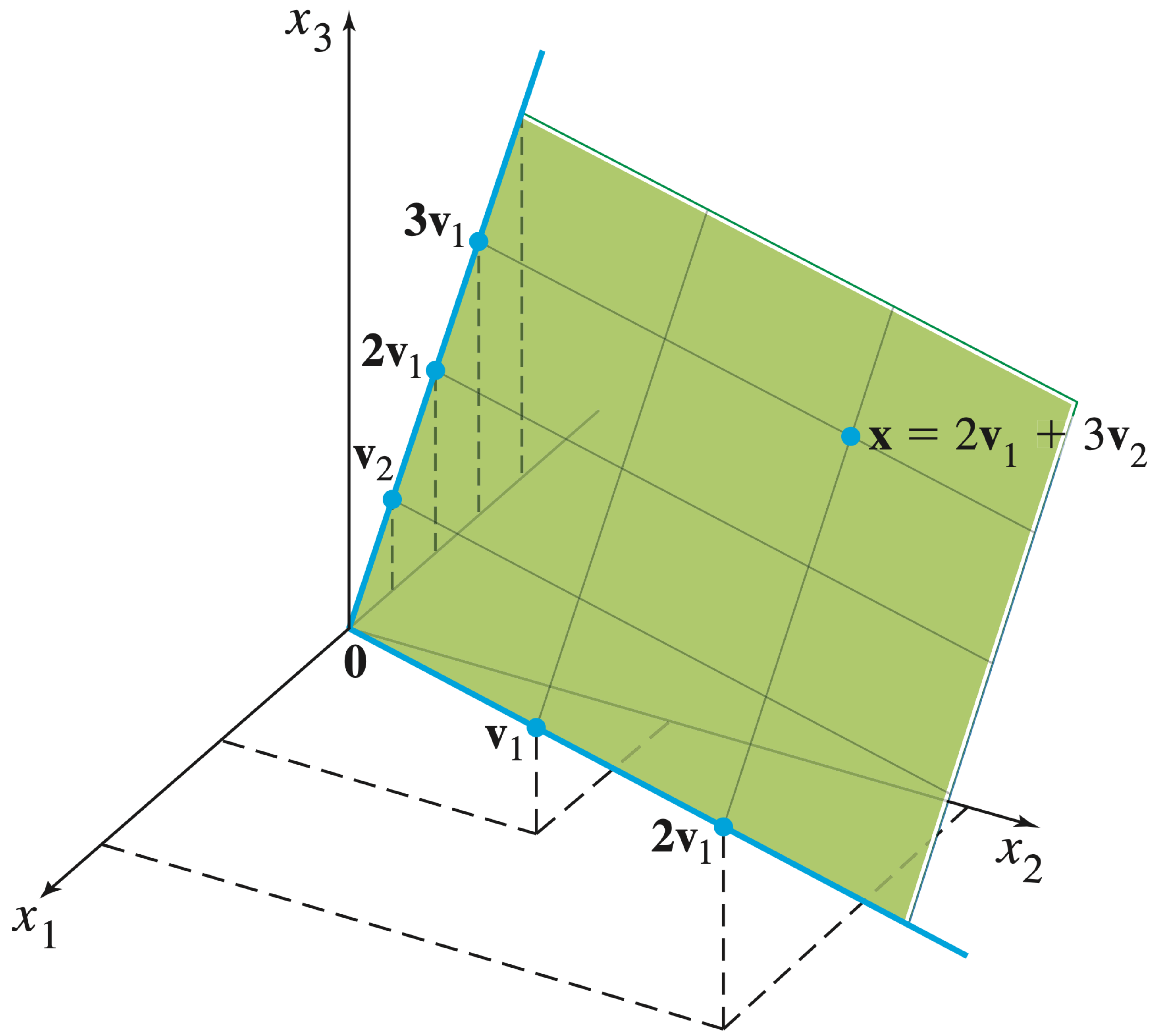
So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to $\mathbb{R}^2$.

This is a formal way of saying that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a "copy of $\mathbb{R}^2$."




Linear Algebra and its Applications (Lay, Lay, McDonald)

# Question

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

*Suppose* $[\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, *where* $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. *Find* $\mathbf{u}$.

# Answer

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad [\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$u = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 2 \end{bmatrix}$$

# Dimension and Rank

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**This number is a measure of how "large" $H$ is.**

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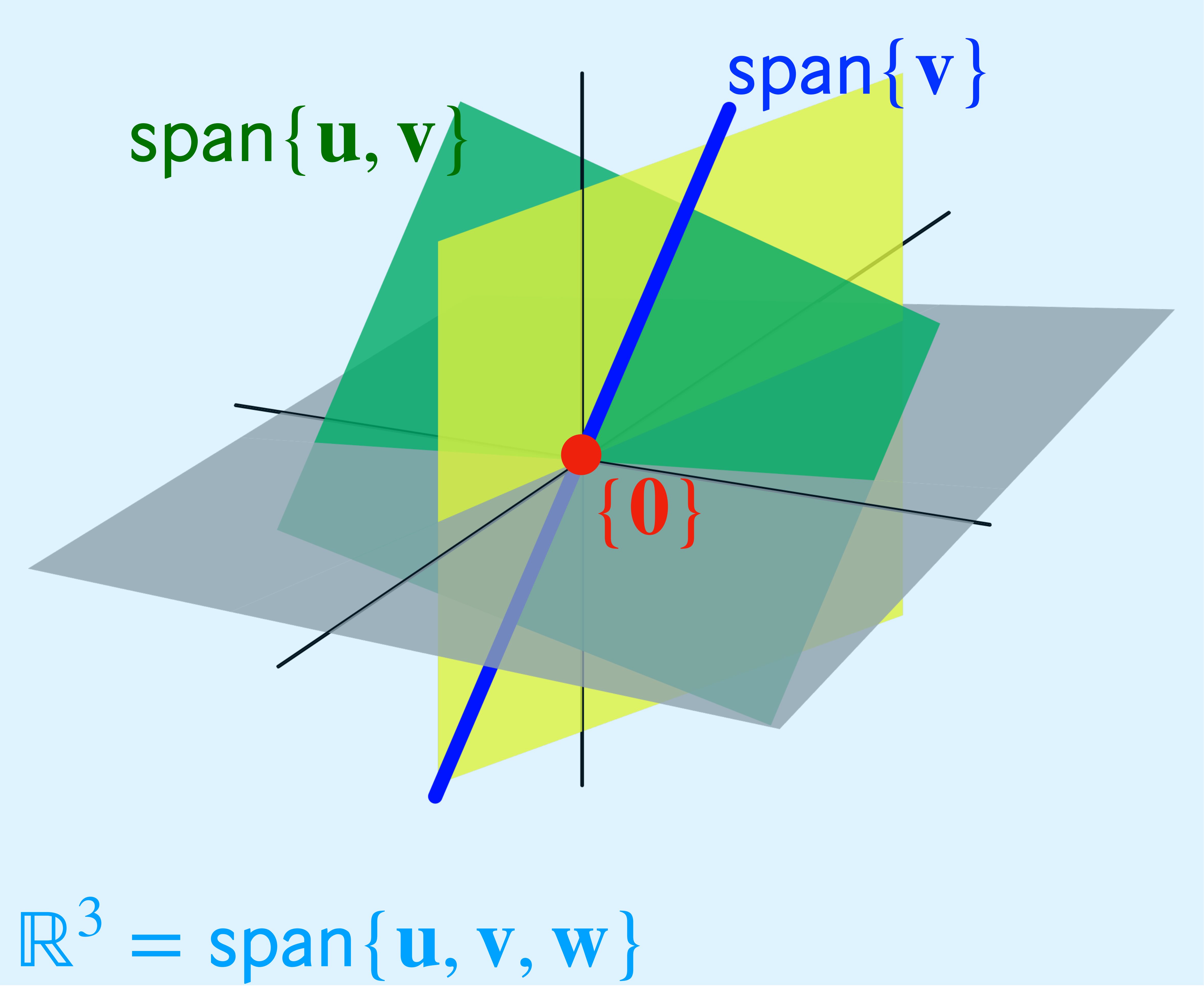
- » a plane (through the origin) is a 2D subspace
- » a line (through the origin) is a 1D subspace

# Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of $\mathbb{R}^3$

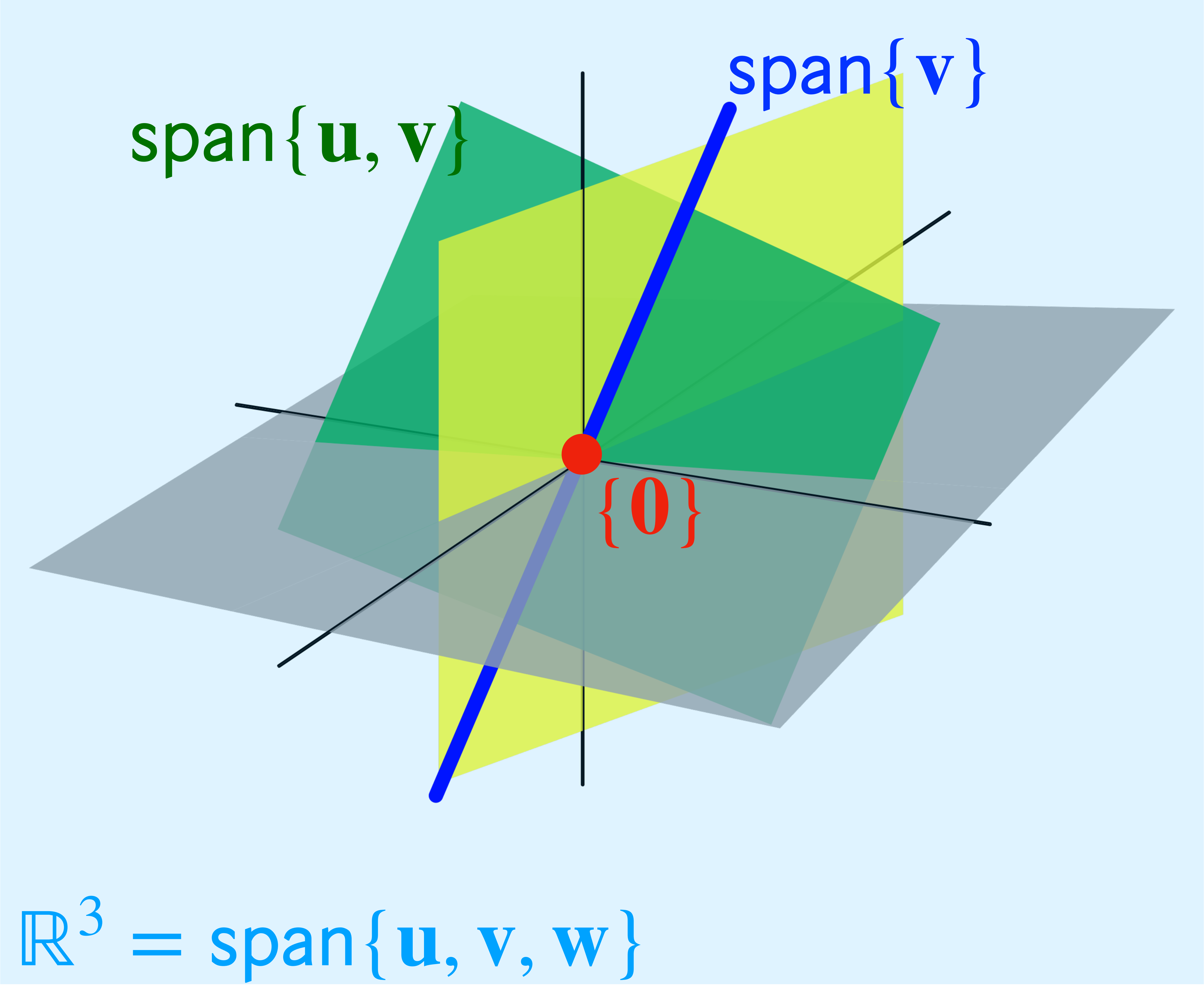



https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

# Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. 0-dimensional subspace
2. 1-dimensional subspaces
3. 2-dimensional subspaces
4. 3-dimensional subspace




https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

How does this connect to null space and column space?

# Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of free variables in a general form solution.

$x_1 = 2x_2 + x_4 - 3x_5$

$\boxed{x_2}$ is free

$x_3 = (-2)x_4 + 2x_5$

$\boxed{x_4}$ is free

$\boxed{x_5}$ is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

# Dimension of the Null Space

The **dimension of** $\mathrm{Nul}(A)$ is the number of <u>free variables</u> in a general form solution to $A\mathbf{x} = \mathbf{0}$.

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The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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# Dimension of the Column Space

The **dimension of** $\mathrm{Col}(A)$ is the number of basic variable in our solution, or equivalently the number of pivot columns of $A$.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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# Rank

**Definition.** The **rank** of a matrix $A$, written $\text{rank}(A)$ or $\text{rank}\, A$, is the dimension of Col(A).

**This is just terminology.**

full rank ≈ full span of colmns

# Rank-Nullity Theorem

**Theorem.** For an $m \times n$ matrix $A$,

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n$$

Verify:

**This is incredibly important.**

# Rank-Nullity Theorem

**Theorem.** For an $m \times n$ matrix $A$,

$$\dim(\mathrm{Col}(A)) + \overset{\text{nullity}}{\boxed{\dim(\mathrm{Nul}(A))}} = n$$

Verify: # of free vars + # of basic vars. = # vars

**This is incredibly important.**

# The Intuition

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For a $m \times n$ matrix $A$, its columns space $\mathrm{Col}(A)$ *could* have $n$ dimensions.

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**Example.** If a "line's worth of stuff" is pulled into the null space (and mapped to $\mathbf{0}$) then

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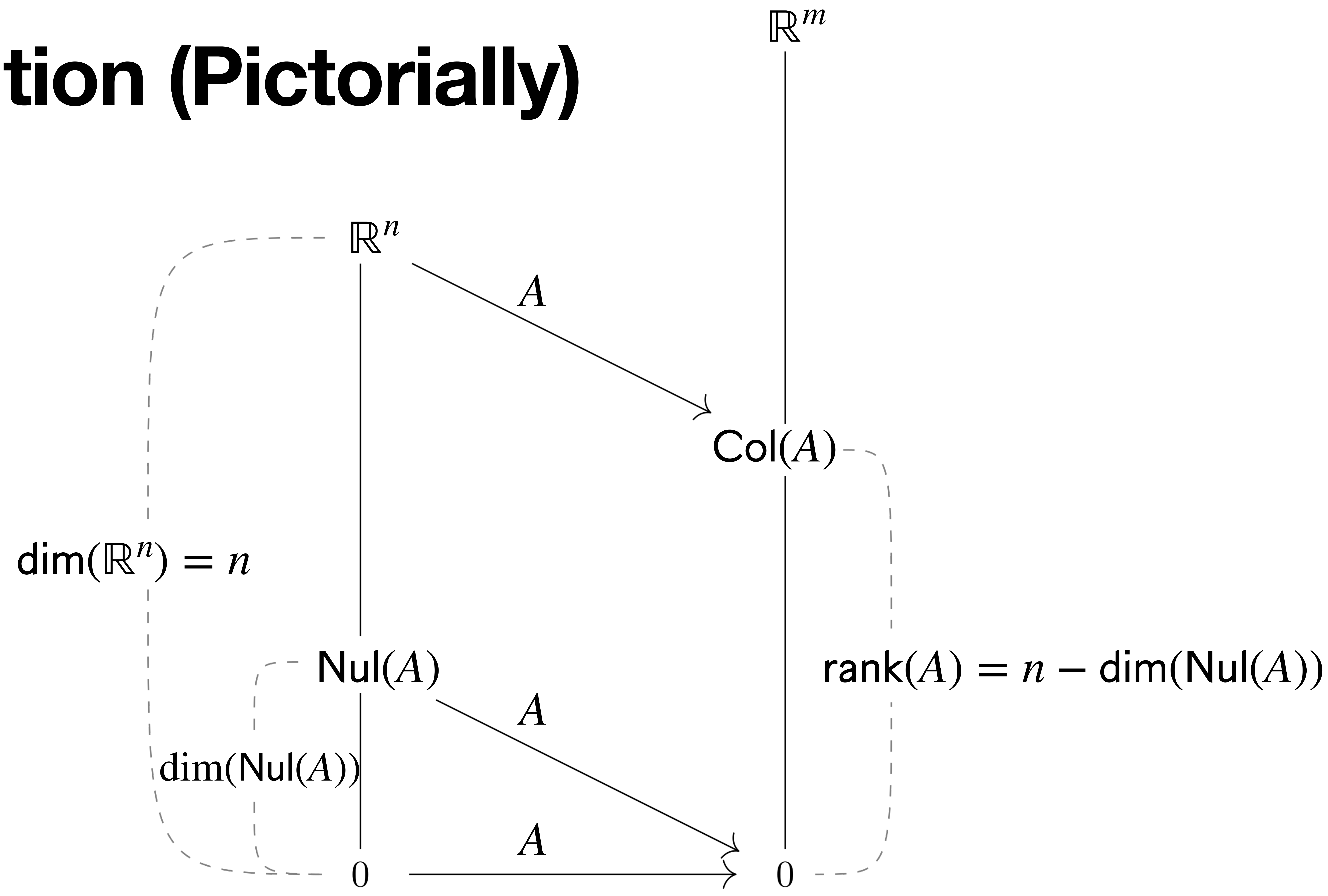
**The null space "takes away" some of the dimensions of the column space.**

# The Intuition (Pictorially)




https://commons.wikimedia.org/wiki/File:Rank-nullity.svg

# Question (Conceptual)

Let $A$ be a $5 \times \boxed{7}$ matrix such that $\dim(\mathrm{Nul}(A)) = 3$.
What is the dimension of $\mathrm{Col}(A)$?

$$7 - 3 = \boxed{4} = \dim(\mathrm{Col}(A))$$

$$\underbrace{\dim(\mathrm{Col}(A))}_{=\,4} + \underbrace{\dim(\mathrm{Nul}(A))}_{=\,3} = 7$$

# Answer: 4

$A$

# Extending the IMT

**Theorem.** For an $n \times n$ invertible matrix $A$, the following are logically equivalent (they must all by true or all by false.

» $\text{Col}(A) = \mathbb{R}^n$
» $\dim(\text{Col}(A)) = n$
» $\text{rank}(A) = n$
» $\text{Nul}(A) = \{\mathbf{0}\}$
» $\dim(\text{Nul}(A)) = 0$

# Summary

We can find bases for the column space and null space by looking at the reduced echelon form of a matrix.

Column vectors are written in terms of a coordinate system, which we can change.

Dimension is a measure of how large a space is.