Matrix Inverses

Geometric Algorithms
Lecture 11

CAS CS 132

Practice Problem(s)

1. *Compute* $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

2. *Find a pair of 2D linear transformations* T_1 *and* T_2 *such that* T_1 *followed by* T_2 *is not the same as* T_2 *followed by* T_1.

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Connect everything(!)

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

eterminant

Invertible Matrix Theorem

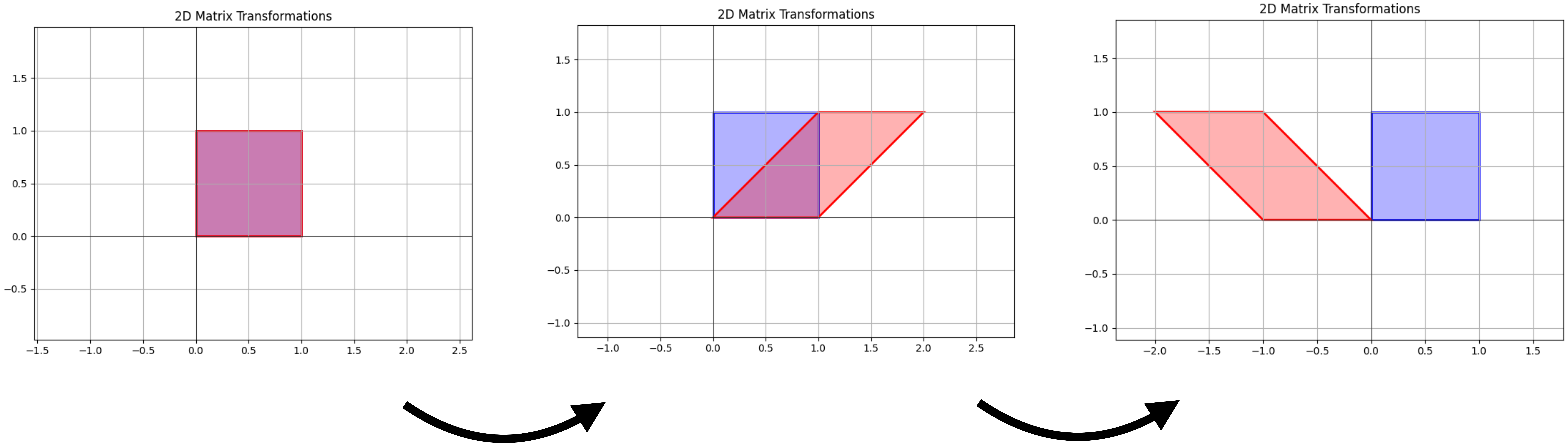
Question (Conceptual)

Find a pair of 2D linear transformations T_1 *and* T_2 *such that* T_1 *followed by* T_2 *is not the same as* T_2 *followed by* T_1*.*

(also find a pair where they <u>are</u> the same)

Recap: Matrix Multiplication

Recall: Composition



shear

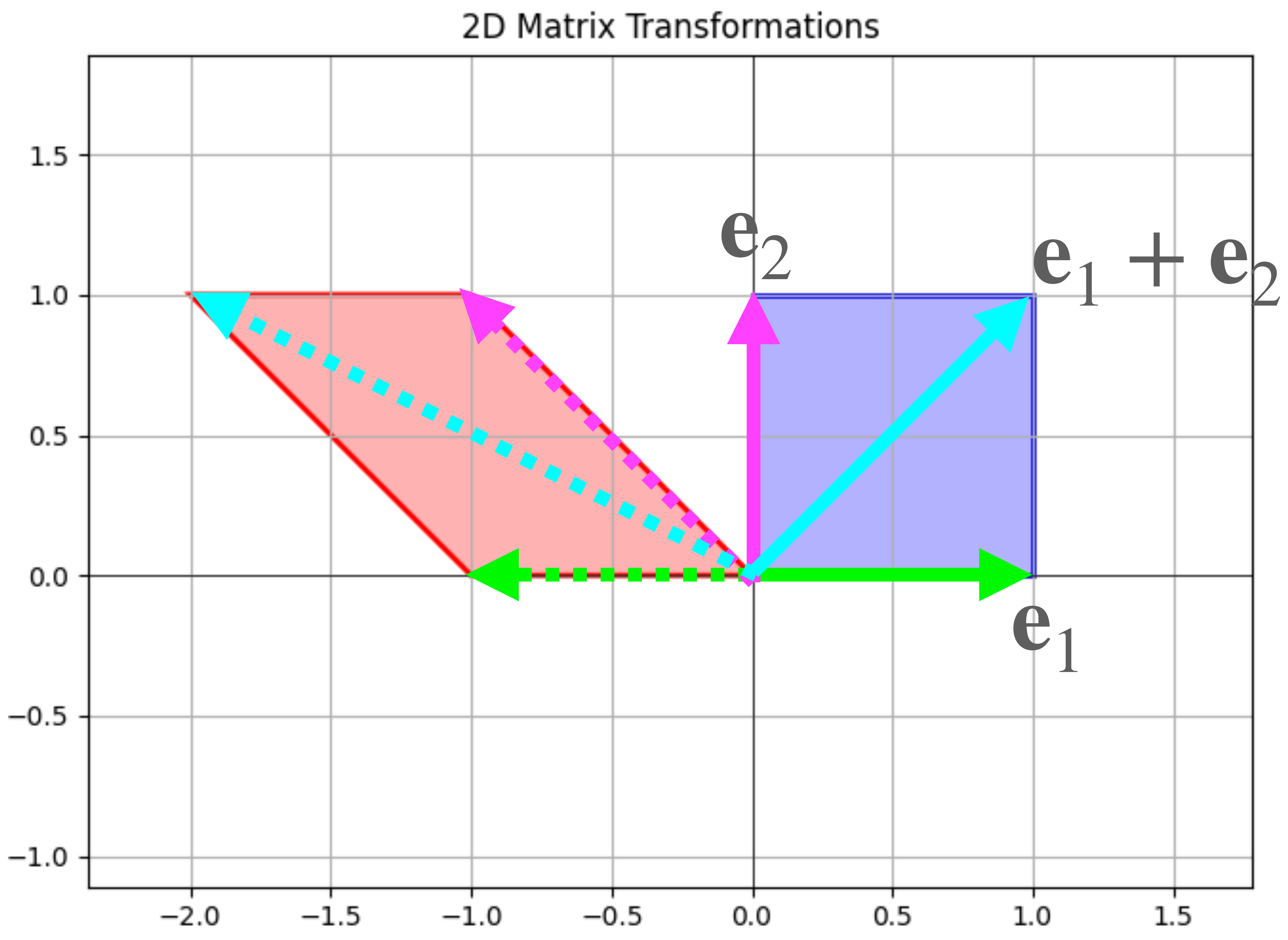
reflect

Recall: Composition

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



General Composition (2D)

$$A\left(\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}\right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \end{array} \begin{array}{cc} & k \\ n & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} = \begin{array}{cc} & k \\ m & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

$(m \times n)$ $(n \times k)$ $(m \times k)$

Important Note

Even if AB is defined, it may be that BA is <u>not</u> defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix}$$

Non-Example

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These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector $\mathbf{v}$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B, the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Matrix Operations

Connection with Matrix-Vector Multiplication

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`What about when the right matrix is a single column?`

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

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We can think of $[A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; \dots \;\; A\mathbf{b}_p]$ as collection of simultaneous matrix–vector multiplications

Matrix "Interface"

multiplication	what does AB mean when A and B are matrices?
addition	what does $A + B$ mean when A and B are matrices?
scaling	what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication — what does AB mean when A and B are matrices?

addition — what does $A + B$ mean when A and B are matrices?

scaling — what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$

Matrix Addition

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

e.g. $2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$

Matrix Addition and Scaling

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This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the same size and r and s are scalars ($\mathbb{R}$)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = AI_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

More Matrix Operations

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)

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Transpose

Definition. For a $m \times n$ matrix A, the **transpose of** A, written A^T, is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$ Important: the order reverses!

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^T A^T$.

Example: $\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^T$

Transposes and Inner Products

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For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

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$$\underset{1 \times n}{\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}} = \underset{1 \times 1}{?}$$

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$\boxed{1} \times \boxed{n} \qquad \boxed{n} \times \boxed{1} \qquad \boxed{1} \times \boxed{1}$

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
is $\mathbf{u}^T\mathbf{v}$ defined?

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

Transposes and Inner Products

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$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Transposes and Inner Products

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Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

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$10^0 = 1$, so it stands to reason that $A^0 = I$.

(we want $A^0 A^k = A^{0+k} = A^k$)

Matrix Powers (Computationally)

We can use **numpy.linalg.matrix_power**

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16}$$

Why?:

Final Warnings about Matrix Multiplication

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Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. *Choose A and B such that they have <u>all</u> nonzero entries.*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

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`transpose` A^T

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`scaling`	cA

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So Far: Matrix Operations

transpose	A^T	
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addition (subtraction)	$A + B$	$A + (-1)B = A - B$
multiplication (powers)	AB	A^k

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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The identity matrix implements the "do nothing" transformation. For any $\mathbf{v}$,

$$I\mathbf{v} = \mathbf{v}$$

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$$IA = AI = A$$

These may be different sizes

Recall: The Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

2×2 $\quad 2 \times 4$ $\quad 2 \times 4$ $\quad 4 \times 4$ $\quad 2 \times 4$

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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How do we solve this equation?

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$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

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Basic Algebra

$$1x = 5$$

How do we solve this equation?

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

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How do we solve this equation?

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Wouldn't it be nice...

$$A\mathbf{x} = \mathbf{b}$$

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A^{-1} is the **multiplicative inverse** of A

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$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

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Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

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Do all matrices have inverses?

Do all matrices have inverses?

No. If they did, then every linear system would have a solution.

When does a matrix have an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

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» that can have inverses.

Matrix Inverses

Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1–axis in $\mathbb{R}^2$ is it's own inverse.

Verify:

Example: No inverse

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a <u>unique</u> solution for any choice of $\mathbf{b}$.

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» exactly one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

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- at most one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

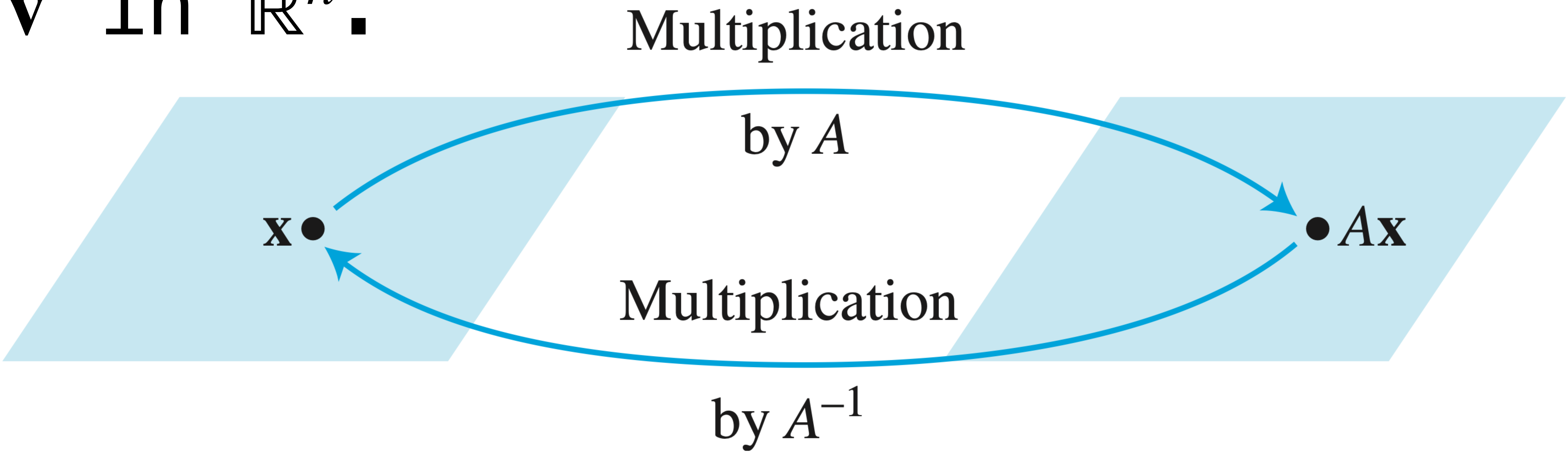
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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Non-Example. Projection onto the x_1-axis.

Connection to Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

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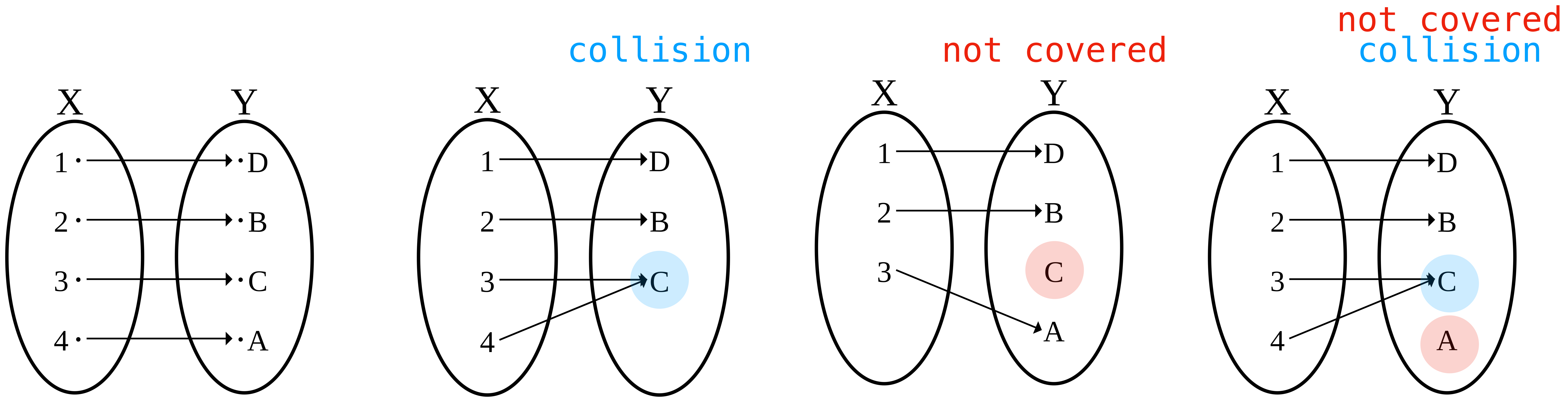
Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



1–1 correspondence

onto, not 1–1

1–1 not onto

not 1–1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

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Answer 1: Try to compute it.

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\mathbf{A}\mathbf{b}_1 = \mathbf{e}_1 \qquad \mathbf{A}\mathbf{b}_2 = \mathbf{e}_2 \qquad \mathbf{A}\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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The inverse is defined only if the determinant is nonzero.

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The inverse is defined only if the determinant is nonzero.

(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

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$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C, demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix.*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen.*

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every $\mathbf{b}$

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Verify:

Recall: Onto Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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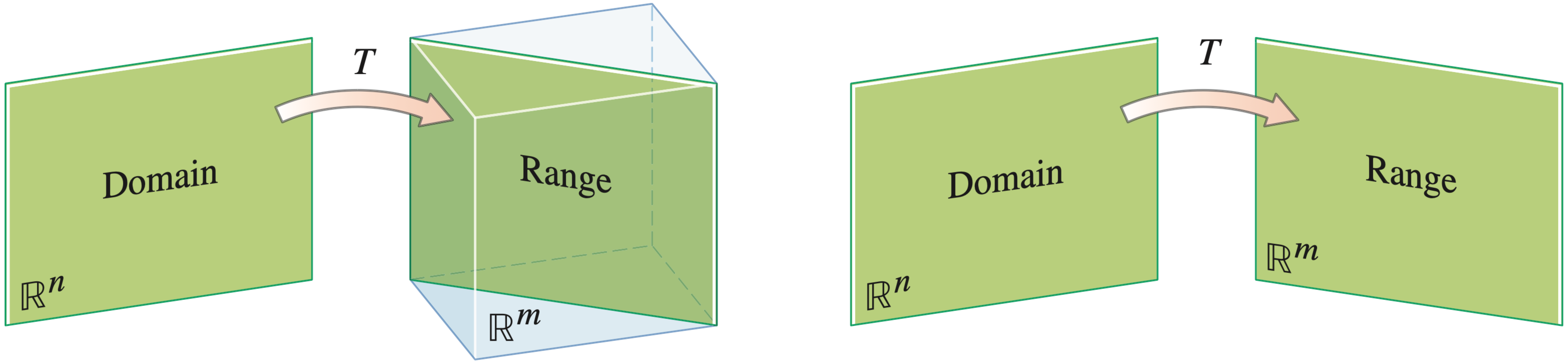

T is *not* onto $\mathbb{R}^m$

T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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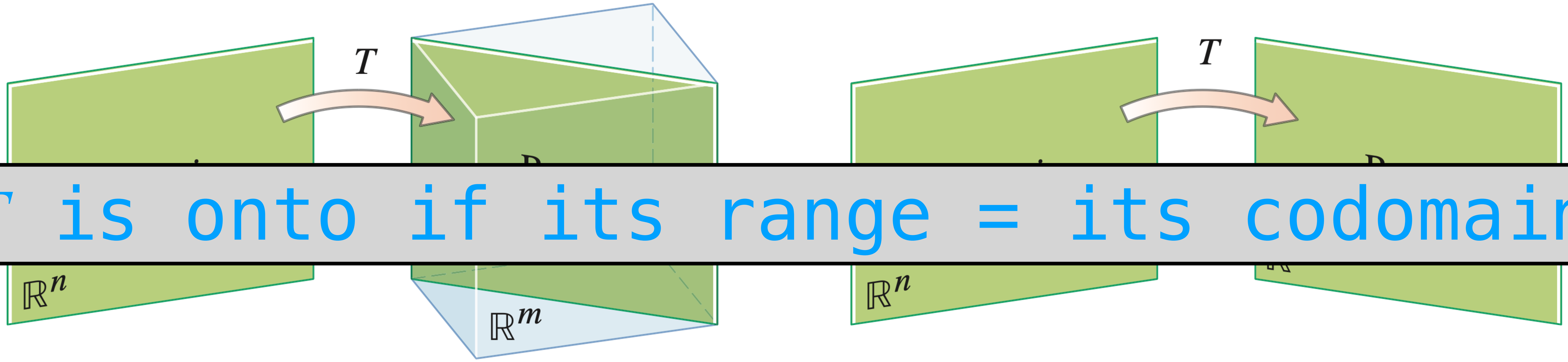


T is *not* onto $\mathbb{R}^m$ $\qquad$ T is onto $\mathbb{R}^m$

T is onto if its range = its codomain

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

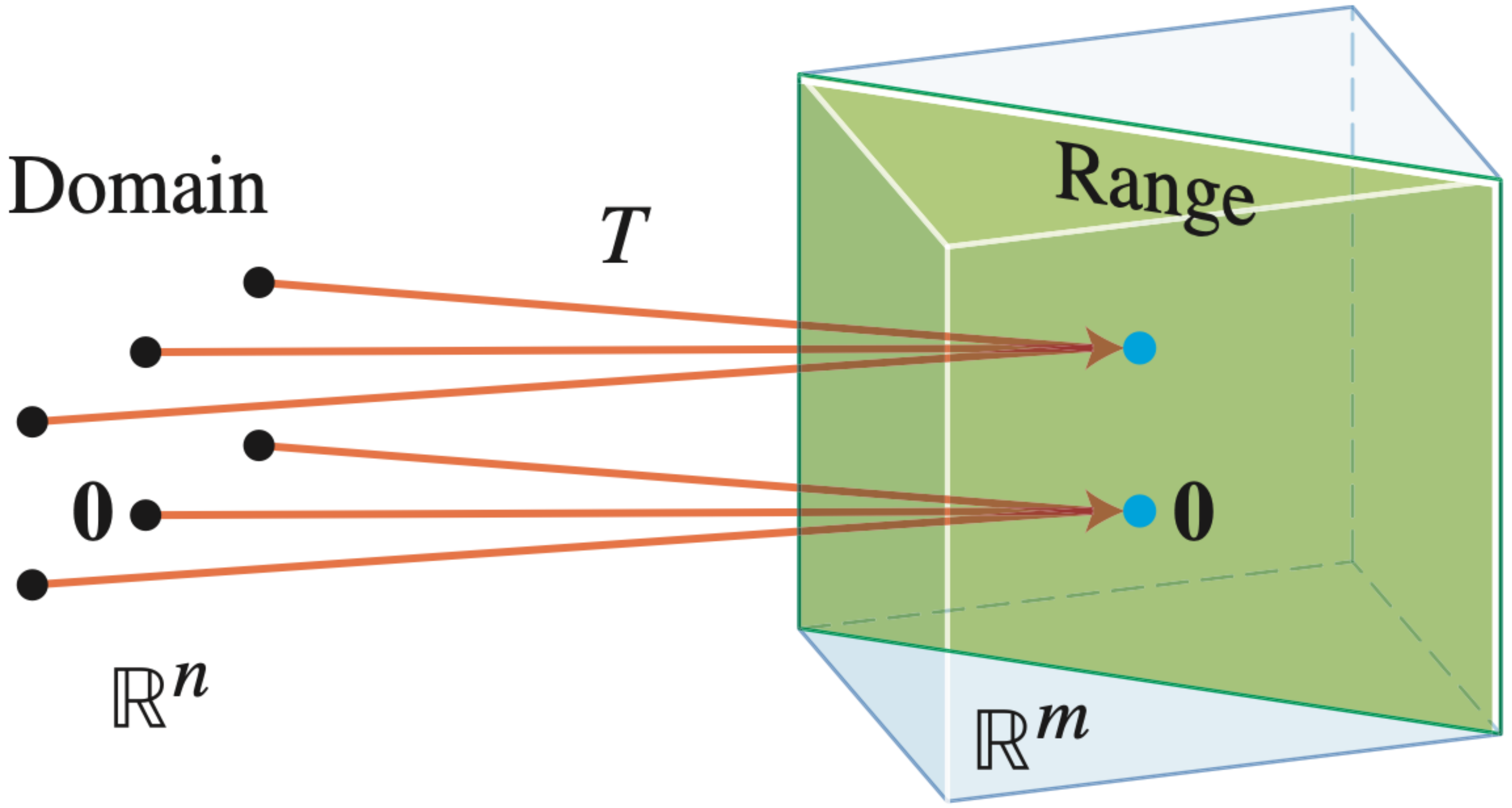
Recall: One-to-one Transformations

Recall: One-to-one Transformations

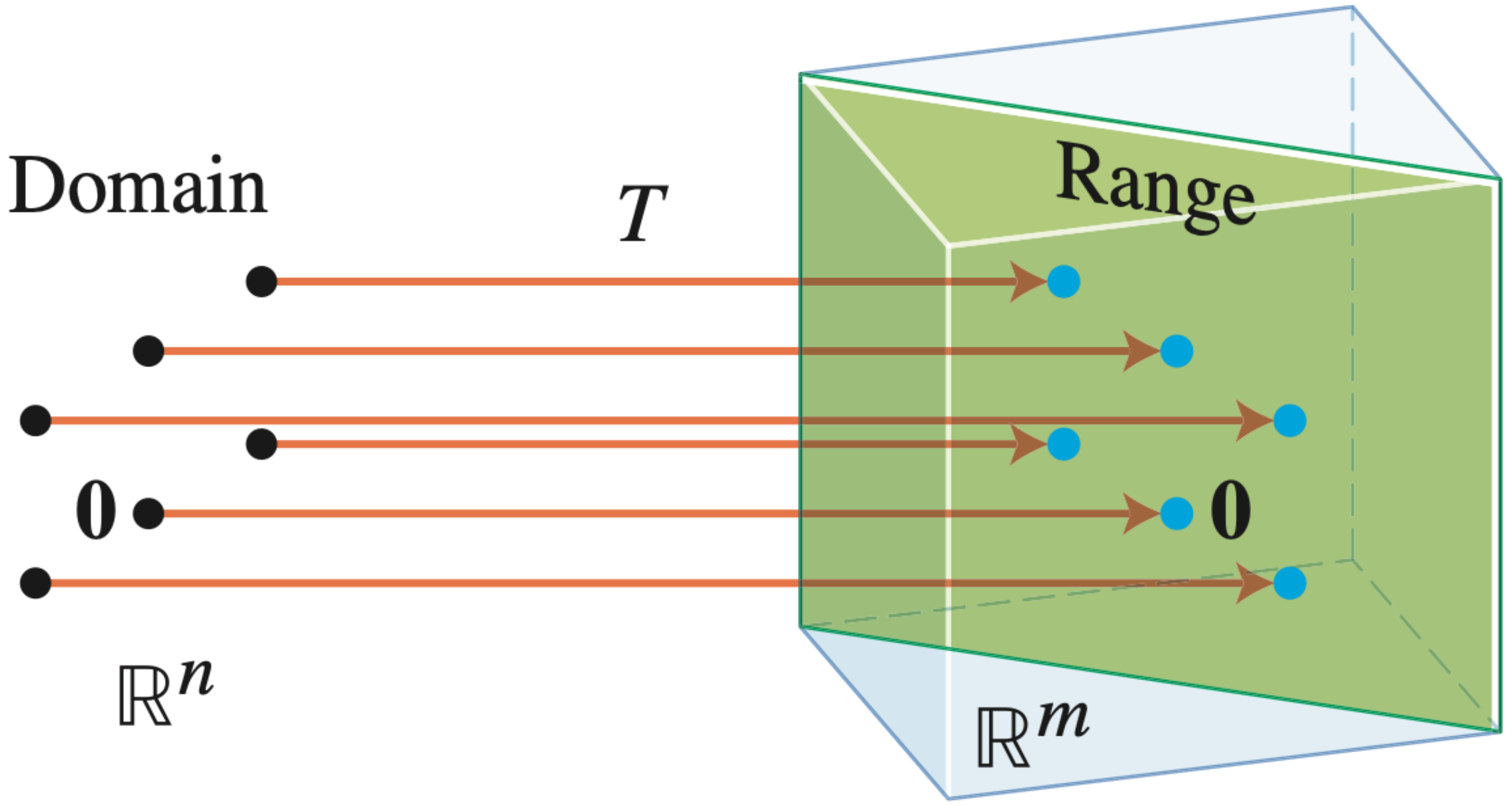
Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

Recall: One-to-one Transformations

Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).



T is *not* one-to-one



T is one-to-one

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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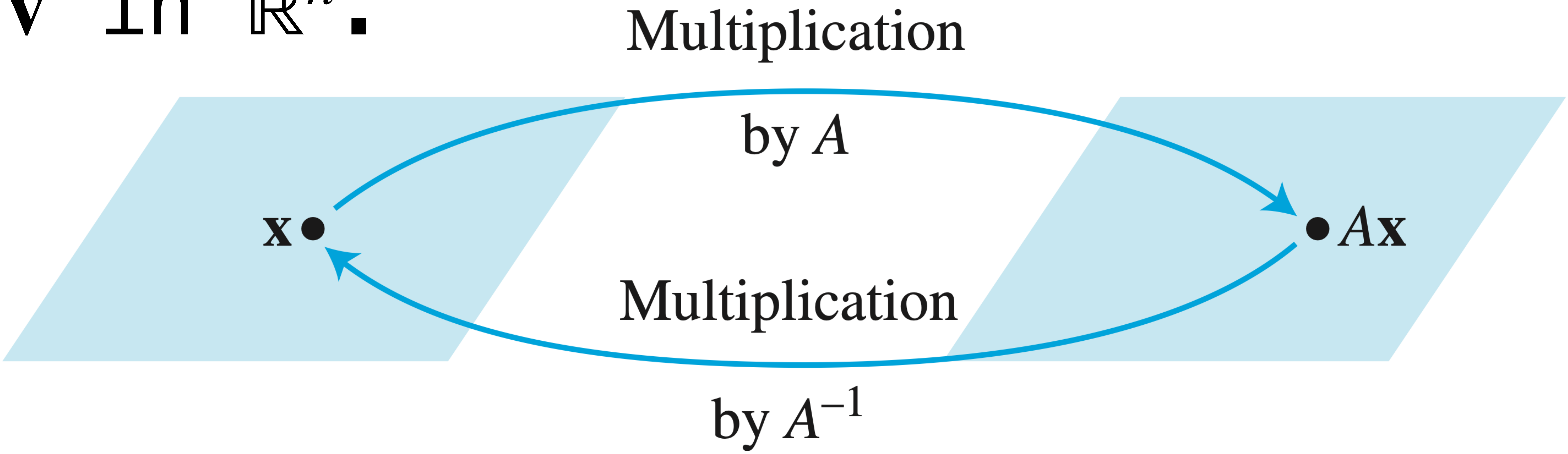
$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



Recall: One-to-One Correspondence

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
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13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

These all express the **same thing**

(this is a stronger statement than we just verified)

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7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

These all express the **same thing**

(this is a stronger statement than we just verified)

!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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Warning. Remember this only applies square matrices.

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True *or* ***False:*** *If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), the B is also invertible.*

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ *is invertible, then is* $\left[(\mathbf{a}_1+\mathbf{a}_2-2\mathbf{a}_3) \quad (\mathbf{a}_2+5\mathbf{a}_3) \quad \mathbf{a}_3\right]$ *also invertible? Justify your answer.*

Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by <u>row operations</u>

Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.