Practice Final Solutions (2023)

CAS CS 132

Fall 2024

Problem 1

A. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

C $y_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 1^{-1} & 0^{+1} & -1^{-1} & 1^{-4} \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 1^{-1} & 1^{+2} & 0^{+3} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1^{-1} & 4^{-1} \\ 0 & 1 & -2^{+2} & -3^{+2} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & \overset{-1}{1} & \overset{-1}{4} \\ 0 & 1 & -2^{+2} & -1^{+3} \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \overset{+1}{-1} & 0 & \overset{-1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$y_1 = 1$
$y_2 = 3$
$y_3 = -1$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3+1 \\ -1+3 \\ 1+(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

D. $$3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+1 \\ 3+0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

Problem 2

A. True

B. False

C. False

D. False

E. True

F. True

G. False

H. False

Problem 3

A.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow 3R_1} \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

B. $E^T = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow 3R_2} \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_3} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C.

$$(AE)^T = E^T A^T$$

$$\begin{bmatrix} 11 & 11 & 11 \\ 22 & 22 & 22 \\ 33 & 33 & 33 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 22 & 22 & 22 \\ 11 & 11 & 11 \\ 33 & 33 & 33 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow 3R_2} \begin{bmatrix} 22 & 22 & 22 \\ 33 & 33 & 33 \\ 33 & 33 & 33 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_3}$$

$$\begin{bmatrix} 88 & 88 & 88 \\ 33 & 33 & 33 \\ 33 & 33 & 33 \end{bmatrix}$$

$$AE = \begin{bmatrix} 88 & 33 & 33 \\ 88 & 33 & 33 \\ 88 & 33 & 33 \end{bmatrix}$$

Problem 4

A. $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{bmatrix} \sim$

$$\begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 1-\lambda & (3-\lambda)(1-\lambda) \end{bmatrix} \sim$$

$$\begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & (3-\lambda)(1-\lambda)+1 \end{bmatrix}$$

$$\det(A - \lambda I) = \frac{1}{\cancel{(1-\lambda)}}(1-\lambda)^{\cancel{2}}((3-\lambda)(1-\lambda)+1)$$

$$= (1-\lambda)(\lambda^2 - 4\lambda + 3 + 1) =$$

$$(1-\lambda)(\lambda-2)^2$$

B.

$$A - I = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$A - 2I = \begin{bmatrix} -1 & 1 & 4 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Nul}(A - 2I) = \text{span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$$

c. A is not diagonalizable.

There is no eigenbasis of $\mathbb{R}^3$ for A.

Problem 5

A. No

B. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

C. $A \sim \begin{bmatrix} 0 & 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

x_1 is free

$x_2 = -2x_3 - 6x_5$

x_3 is free

$x_4 = -2x_5$

x_5 is free

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

D. $B \overset{swap}{\sim} \begin{bmatrix} 3 & -3 & 2 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

$$\overset{swap}{\sim} \begin{bmatrix} 3 & -3 & 2 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & -4 & 1 \end{bmatrix}$$

$$\overset{swap}{\sim} \begin{bmatrix} 3 & -3 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\det(B) = (-1)^3 \, 3 \, 2 \, (-4)(7) =$$

$$24 \cdot 7 = 140 + 28$$

$$= 168$$

E. Yes

Problem G

A.

$$\begin{array}{ccc} X & \vec{\beta} & \vec{y} \\ \begin{bmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \\ \cos x_3 & \sin x_3 \\ \cos x_4 & \sin x_4 \end{bmatrix} & \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = & \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \end{array}$$

B.

$$\cos(x + \hat{\alpha}) =$$

$$\cos x \sin(\hat{\alpha}) + \sin x \cos(\hat{\alpha})$$

By the properties of least squares

$$\left\| X \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} - \vec{y} \right\| \leq \left\| X \begin{bmatrix} \sin(\hat{\alpha}) \\ \cos(\hat{\alpha}) \end{bmatrix} - \vec{y} \right\|$$

C. $\hat{\beta}_1 \cos x + \hat{\beta}_2 \cos x =$

$\cos(x + \theta)$ where $\sin\theta = \hat{\beta}_1$

$\cos\theta = \hat{\beta}_2$

so $\tan\theta = \frac{\hat{\beta}_1}{\hat{\beta}_2}$ and $\hat{\alpha} = \tan^{-1}\left(\frac{\hat{\beta}_1}{\hat{\beta}_2}\right)$