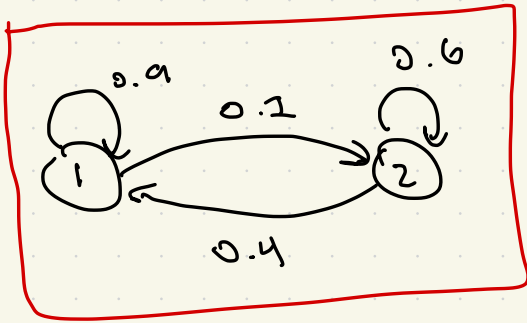


Homework 7 Solutions

CAS CS 132

Fall 2024

Problem 1.1



A is regular because A^1 has strictly positive entries

$$(A - I) \sim \begin{bmatrix} 1 & -6 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 6x_2 \\ x_2 &\text{ is free} \end{aligned}$$

$$x_1 + x_2 = 1$$

$$6x_2 + x_2 = 1$$

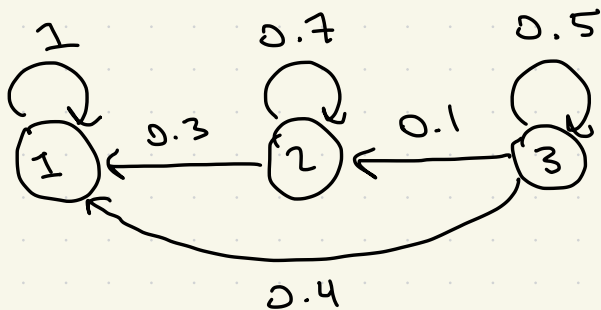
$$7x_2 = 1$$

$$x_2 = 1/7$$

$$\begin{bmatrix} 6/7 \\ 1/7 \end{bmatrix}$$

is the unique steady state vector

Problem 1.2



A is not regular because the first column of A^k is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for any k .

(Also, the product of upper triangular matrices is upper triangular)

$$A - I \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

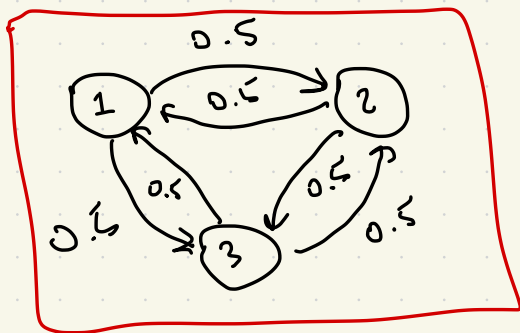
x_1 is free

$$x_2 = 0$$

$$x_3 = 0$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the unique steady state vector.

Problem 1.3



A is regular because A has strictly positive entries.

$$(A - I) \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned}$$

$$x_1 + x_2 + x_3 = 1$$

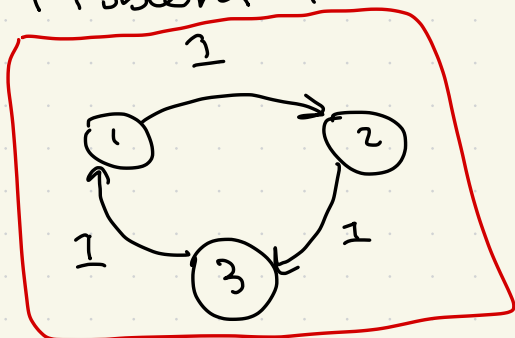
$$3x_3 = 1$$

$$x_3 = 1/3$$

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ is the unique}$$

steady state vector

Problem 1.4



A is not regular because

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So every power of A is the same as A^0 , A^1 or A^2

$$A - I \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

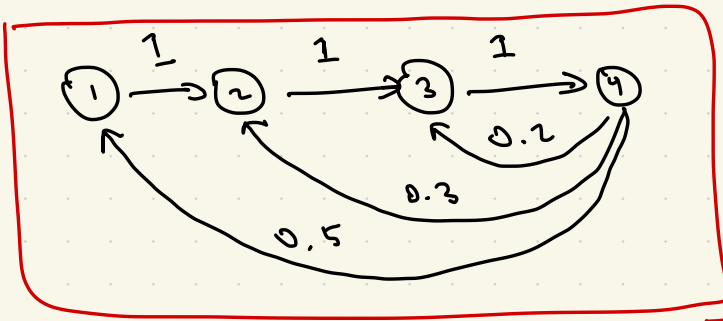
$$x_2 = x_3$$

x_3 is free

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

is the unique steady state vector

Problem 1.5



A is regular because A ⁶ has strictly positive entries

$$A - I \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{2} x_4$$

$$x_2 = \frac{4}{5} x_4$$

$$x_3 = x_4$$

x_4 is free

$$x_1 + x_2 + x_3 + x_4 = 1$$

$$\frac{5}{10} x_4 + \frac{8}{10} x_4 + \frac{10}{10} x_4 + \frac{10}{10} x_4 = 1$$

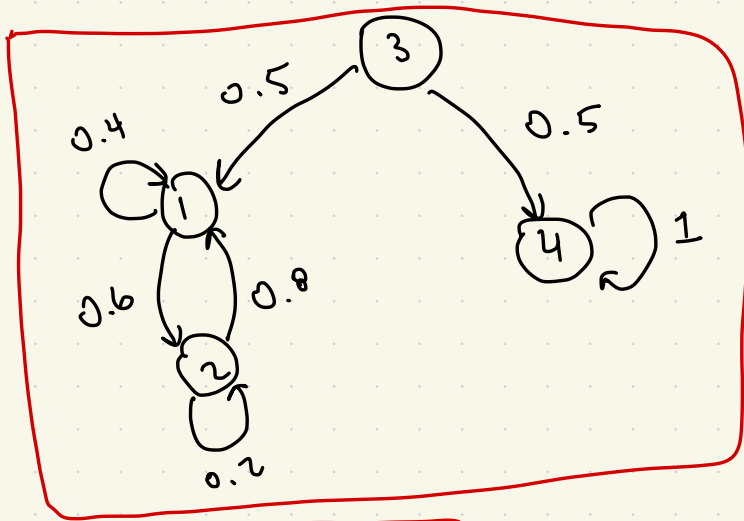
$$\frac{33}{10} x_4 = 1$$

$$x_4 = \frac{10}{33}$$

$$\begin{bmatrix} 5/33 \\ 8/33 \\ 10/33 \\ 10/33 \end{bmatrix}$$

is the unique steady state vector

Problem 1.6



A is **not regular** because the last column of A^k is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for any k .

$$A - I \sim \begin{bmatrix} 1 & -4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= \frac{4}{3} x_2 \\ x_2 &\text{ is free} \\ x_3 &= 0 \\ x_4 &\text{ is free} \end{aligned}$$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a steady state vector, it is **not unique**

Problem 2

$$\begin{matrix} & S & H & L \\ \begin{matrix} S \\ H \\ L \end{matrix} & \begin{bmatrix} 0.7 & 0.4 & 0.3 \\ 0.2 & 0.55 & 0.1 \\ 0.1 & 0.05 & 0.6 \end{bmatrix} \end{matrix}$$

$$A - I \sim \begin{bmatrix} 1 & 0 & -35/11 \\ 0 & 1 & -18/11 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 35/11 x_3$$

$$x_2 = 18/11 x_3$$

x_3 is free

$$x_1 + x_2 + x_3 = 1$$

$$35/11 x_3 + 18/11 x_3 + 1/11 x_3 = 1$$

$$64/11 x_3 = 1$$

$$x_3 = 11/64$$

$$x_1 = 35/64$$

$$x_2 = 18/64$$

$$x_3 = 11/64$$

it is more likely to

stay steady

Problem 3.1

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{11}{2} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{11}{2} \end{bmatrix}$$

Problem 3.2

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 3 & 2 & 10 & 9 \\ -3 & 2 & -2 & -9 \end{bmatrix} \quad \begin{array}{l} R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 - R_1 \end{array}$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 2 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2 \end{array}$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad R_4 \leftarrow R_4 - R_3$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 3.3

$$U = \begin{bmatrix} 2 & 8 & -4 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -7 \end{bmatrix} \text{ is in echelon form}$$

$$\text{and } A \xrightarrow{\quad\quad\quad} U$$

inverse of
row ops.

$$\left\{ \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_3 \leftarrow R_3 + 2R_2 \end{array} \right.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Problem 4.1

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{bmatrix}$$

Problem 4.2

nonzero entries in L : 5997

nonzero entries in U : 5997

nonzero entries in A^{-1} : 3792976

Problem 4.3

Yes it is consistent. It's slower to invert, roughly by 3 times

