Markov Chains

Geometric Algorithms
Lecture 13

CAS CS 132

Practice Problem

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_2 \\ 2x_2 \\ x_2 + bx_3 \end{bmatrix}$$

For what values of b is the above transformation singular? Explain your answer.

(i.e., not invertible)

Find the inverse of the matrix implementing the above transformation, given $b = 1$.

Solution

$b = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & \boxed{b} \end{bmatrix}$$

$b = 0$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \text{RREF}$$

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$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$$

Objectives

1. Motivate linear dynamical systems
2. Analyze Markov chains and their properties
3. Learn to solve for steady-states of Markov chains
4. Connect this to graphs and random walks

Keywords

linear dynamical systems

recurrence relations

linear difference equations

state vector

probability vector

stochastic matrix

Markov chain

steady-state vector

random walk

state diagram

Recap: Algebraic Graph Theory

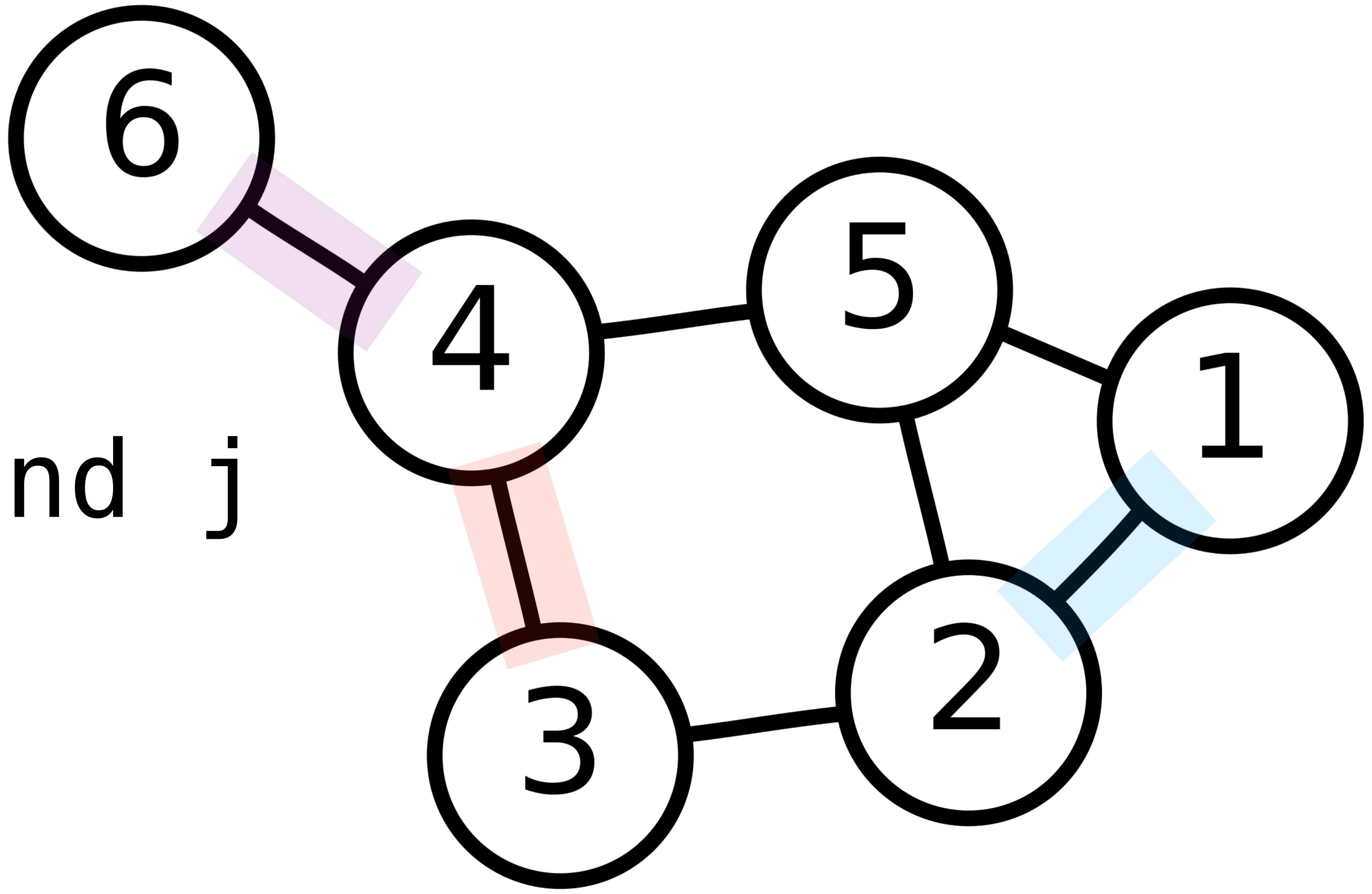
Recall: Adjacency Matrices

$$\begin{array}{c c} & \begin{array}{cccccc} & A_{12} & & A_{34} & & A_{46} \end{array} \\ \begin{array}{c} \\ A_{21} \\ \\ A_{43} \\ \\ A_{64} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Let G be an simple graph with its nodes labeled by numbers 1 through n.

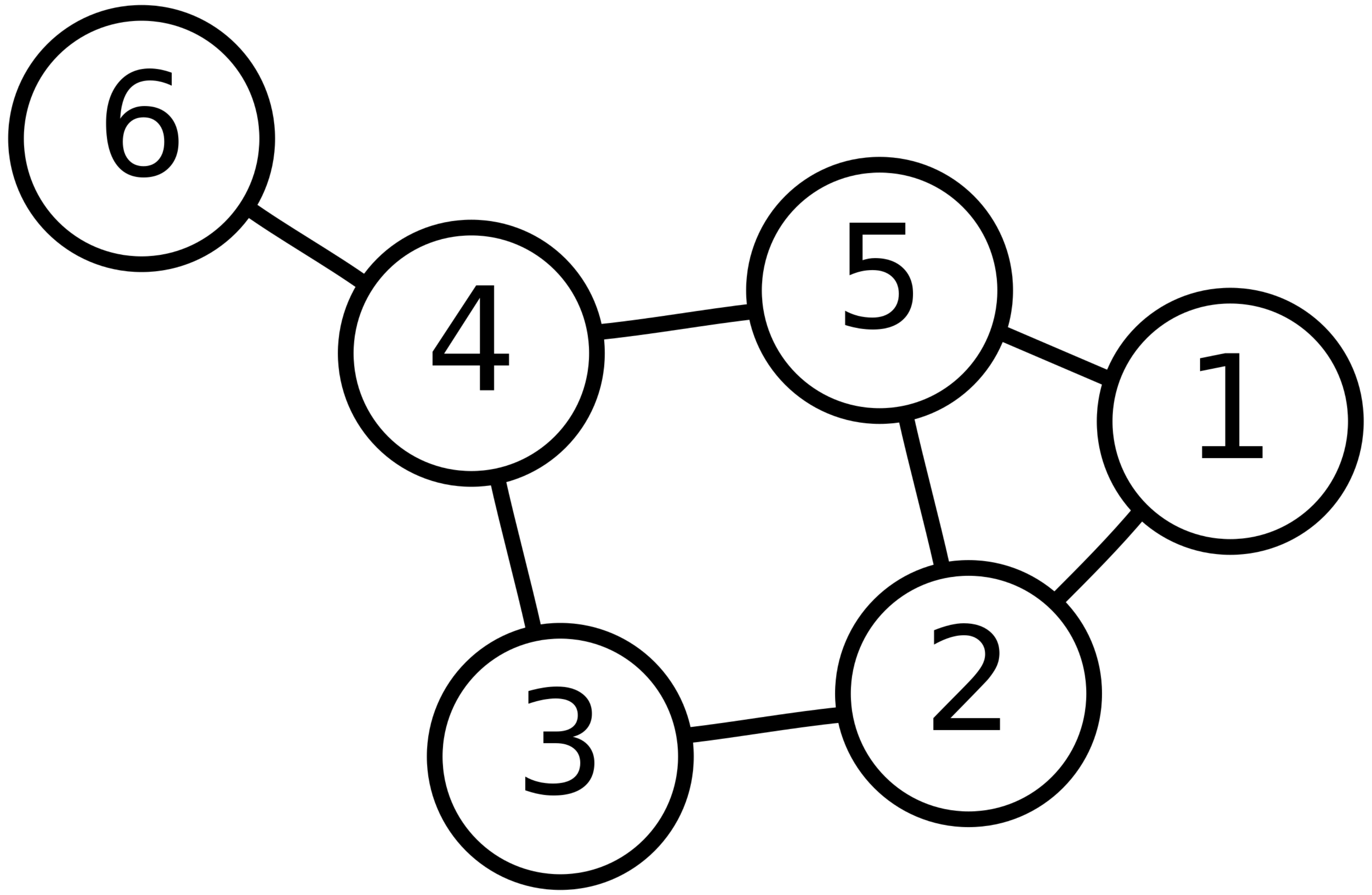
We can create the **adjacency matrix** A for G as follows.

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between i and j} \\ 0 & \text{otherwise} \end{cases}$$



Recall: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$



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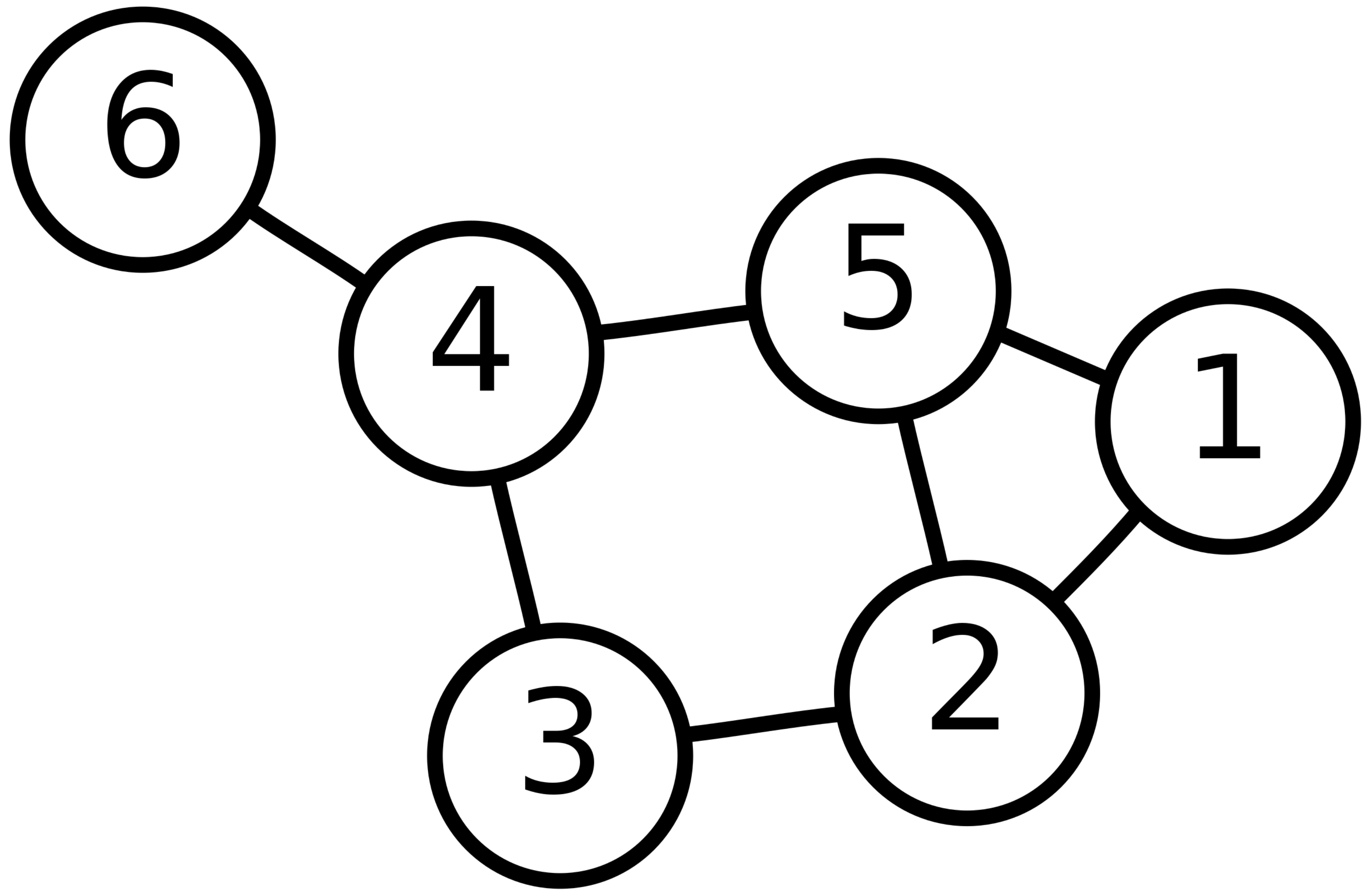

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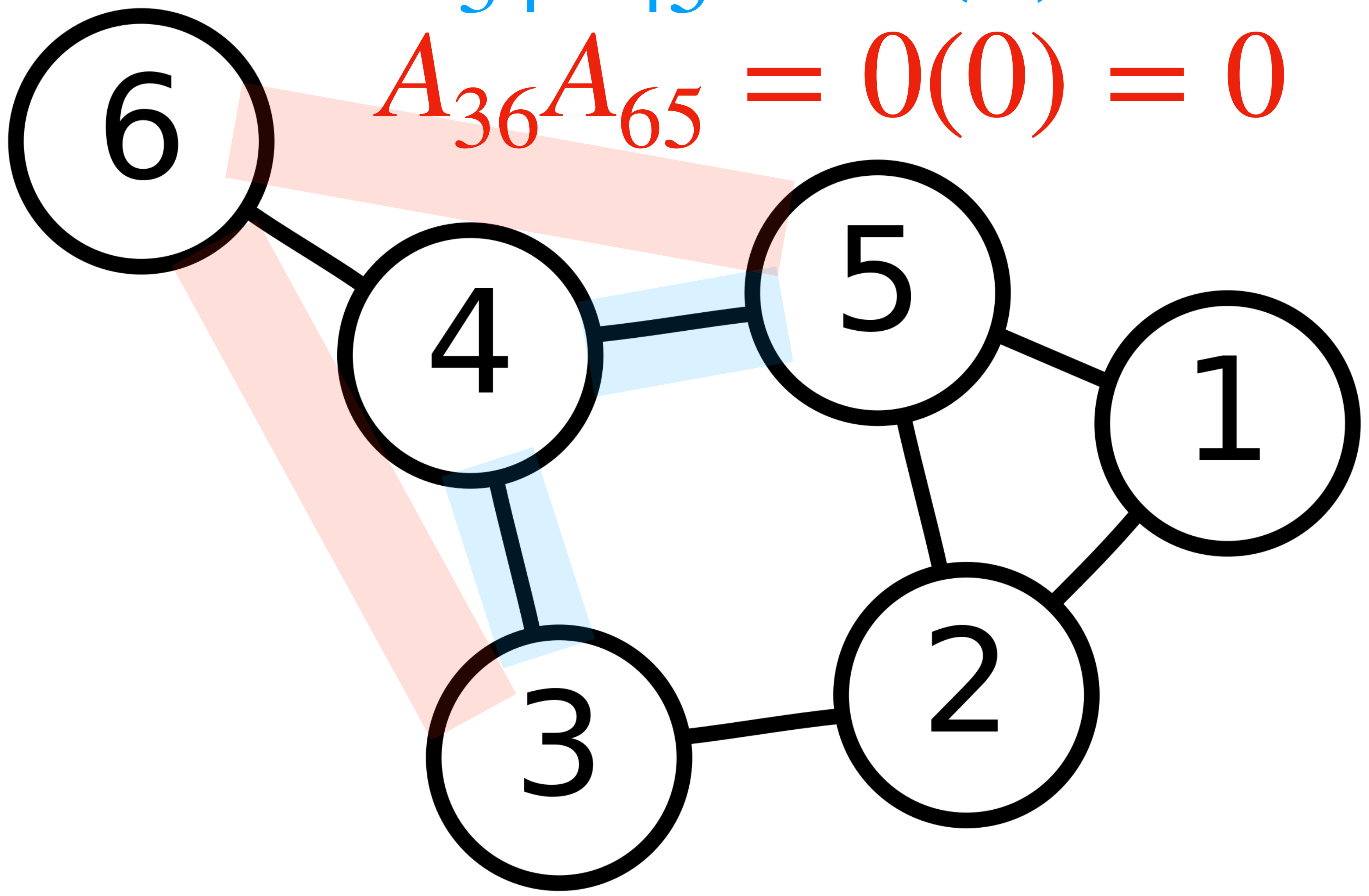
$A_{34}A_{45} = 1(1) = 1$

$A_{36}A_{65} = 0(0) = 0$



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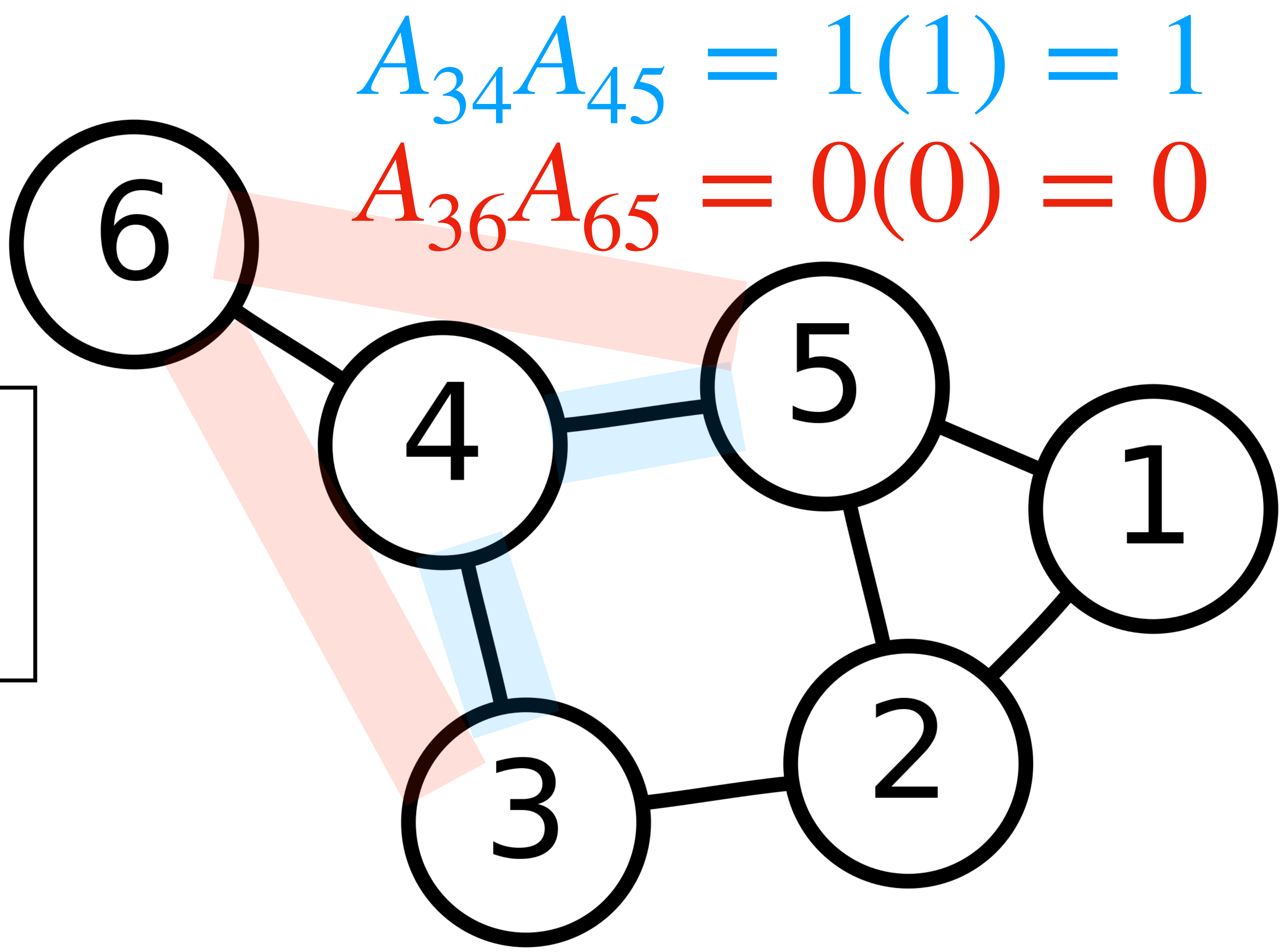
$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$

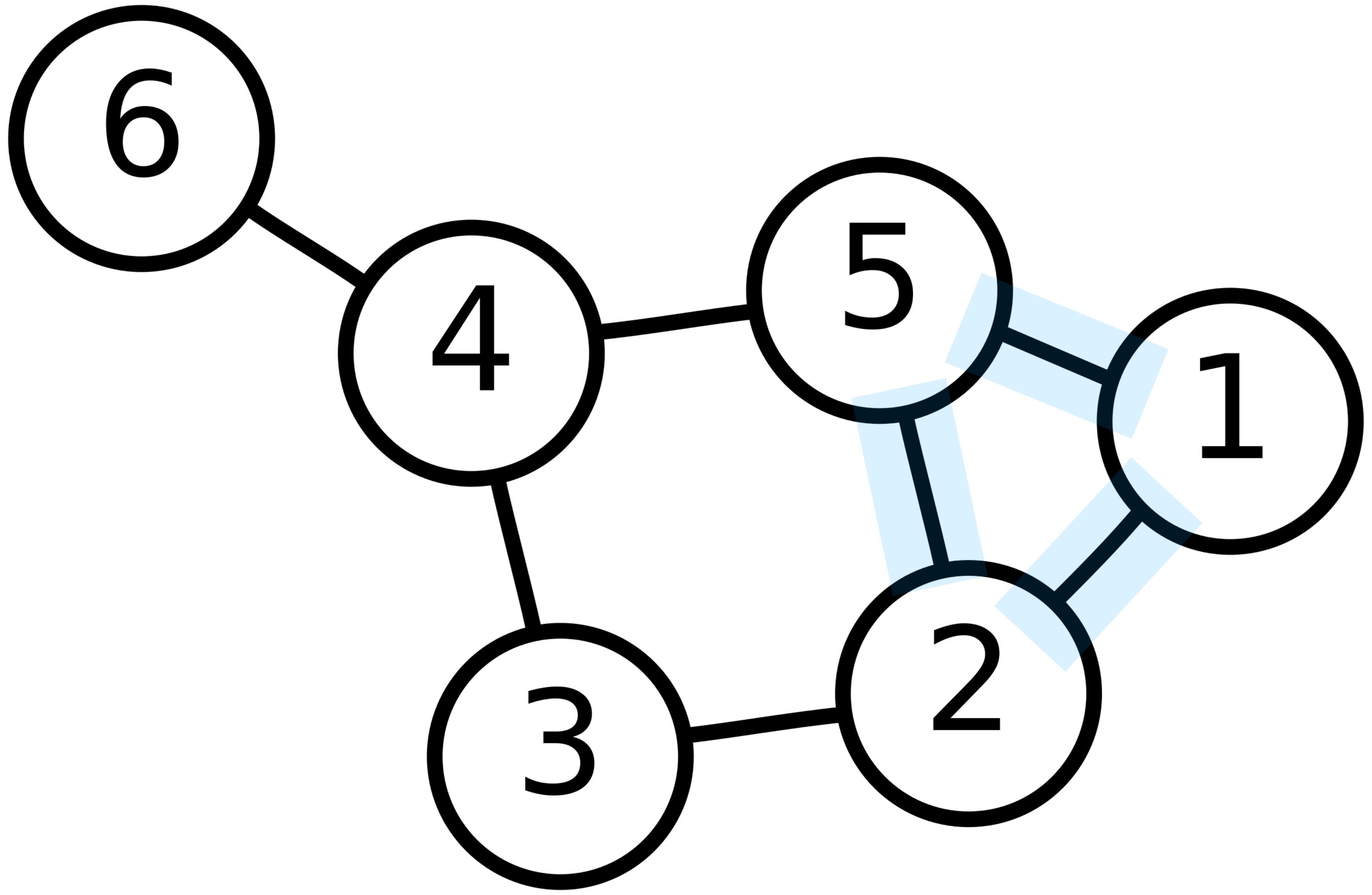
$$(A^2)_{ij} = \boxed{\textbf{number of 2-step paths from i to j}}$$



Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)

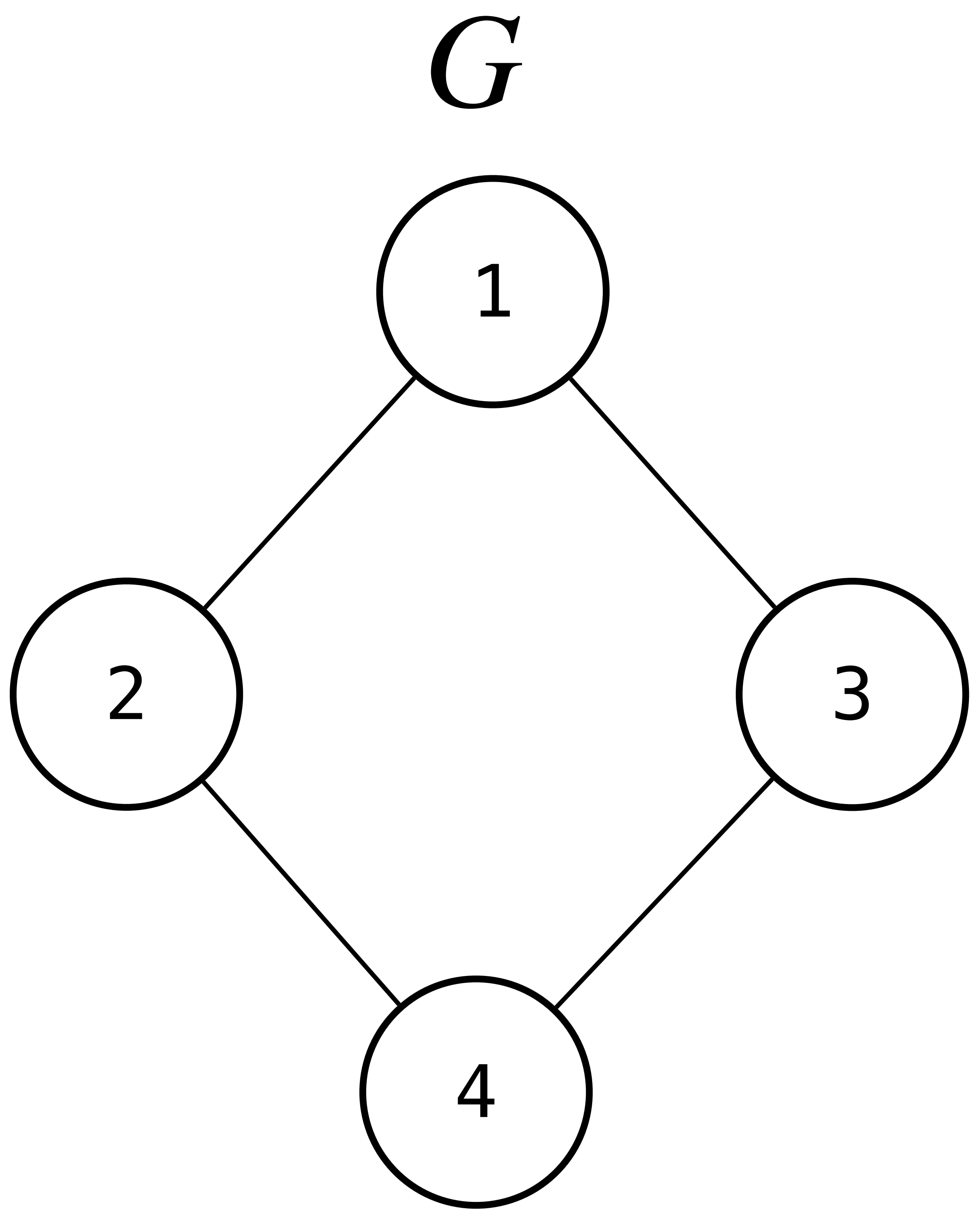


Another Application: Reachability

Question: If A^2 gives us information about length 2 paths, then what about A^k?

A^k **gives us information about** k**–length paths.**

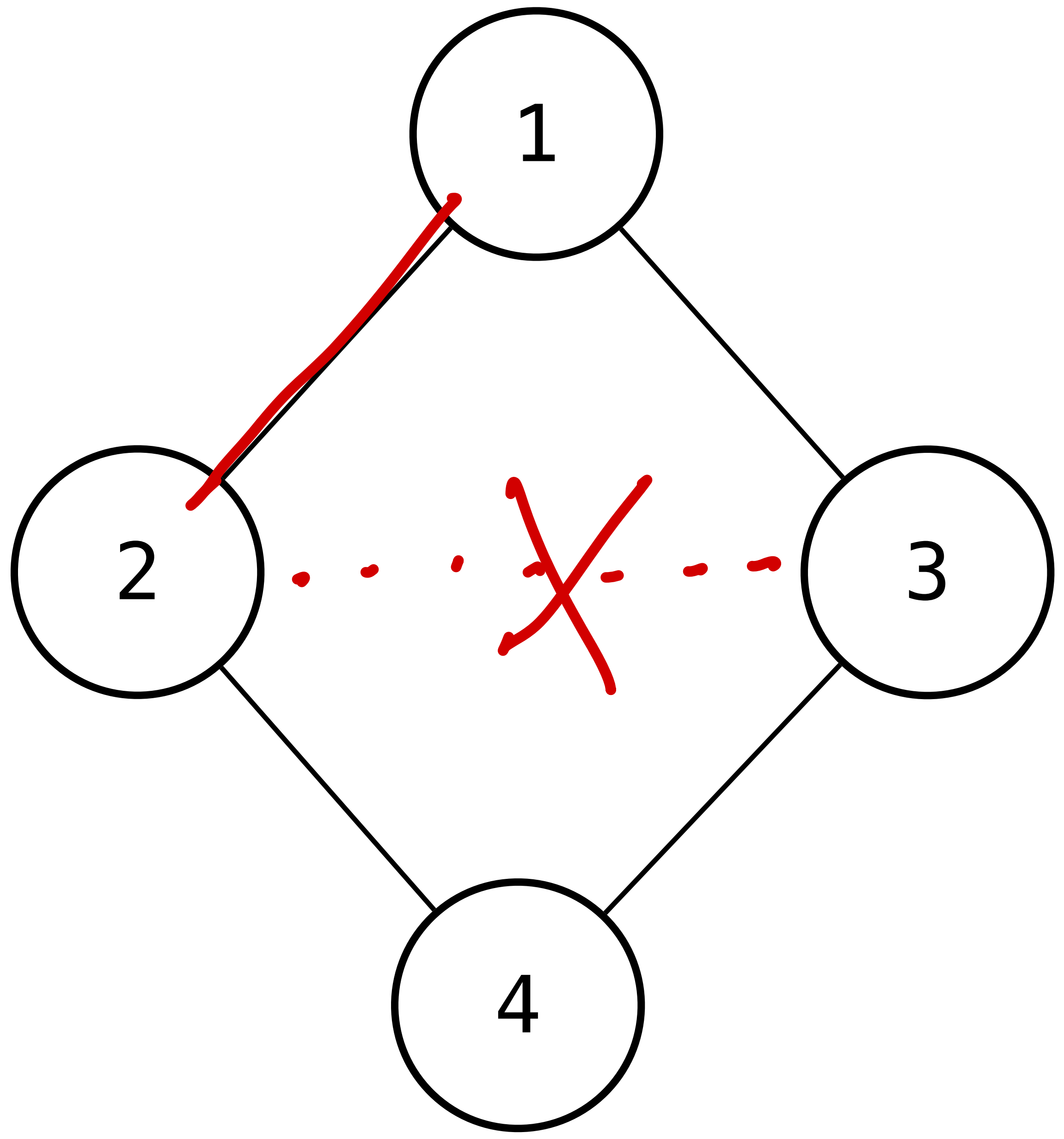
Example



Example

$$\begin{array}{cc} & \begin{array}{cccc} & 12 & & \end{array} \\ 21 & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & \overset{23}{0} & 1 \\ 1 & \overset{32}{0} & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = \texttt{adjacency matrix for } G$$

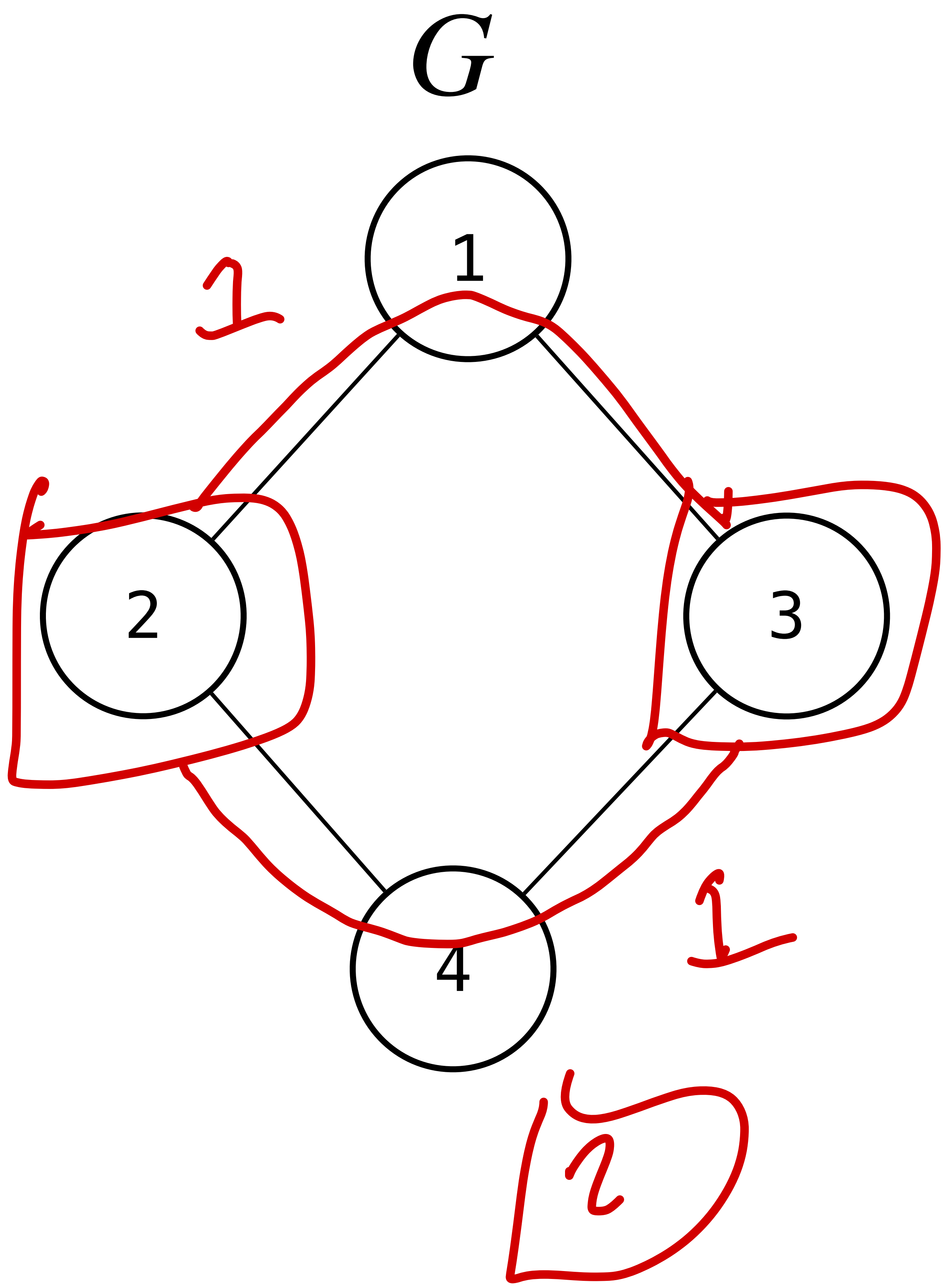
G



Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$



Example

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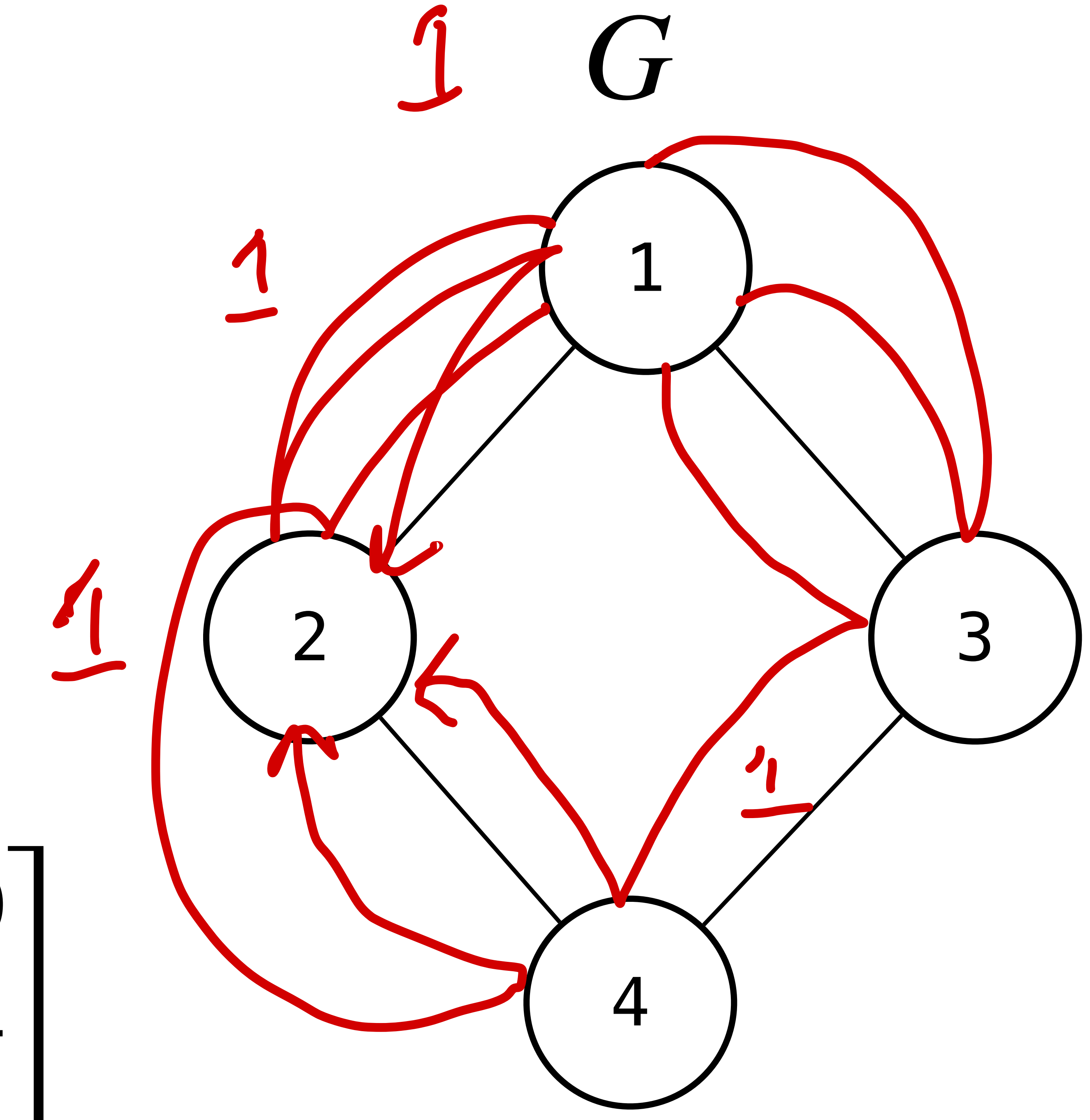
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} & 12 & & \end{array} \\ \begin{array}{c} \\ 21 \\ \\ \\ \end{array} & \begin{bmatrix} 0 & \boxed{4} & 4 & 0 \\ \boxed{4} & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} \end{array}$$



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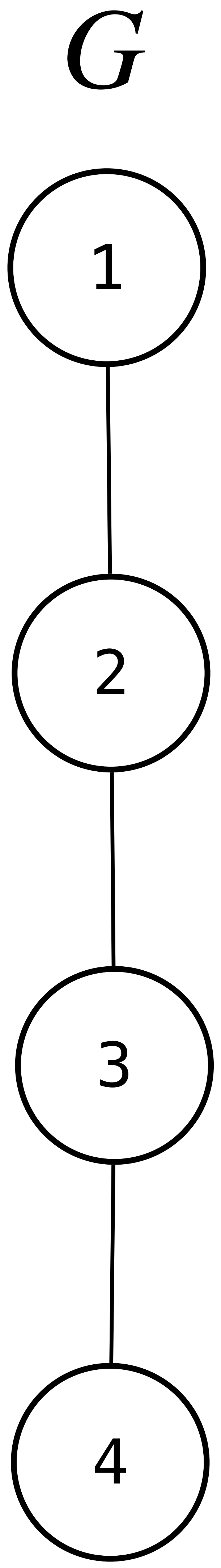

Example

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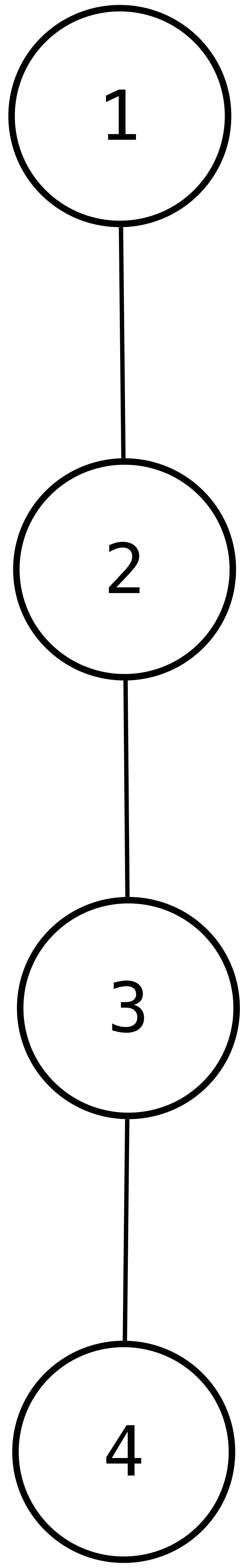
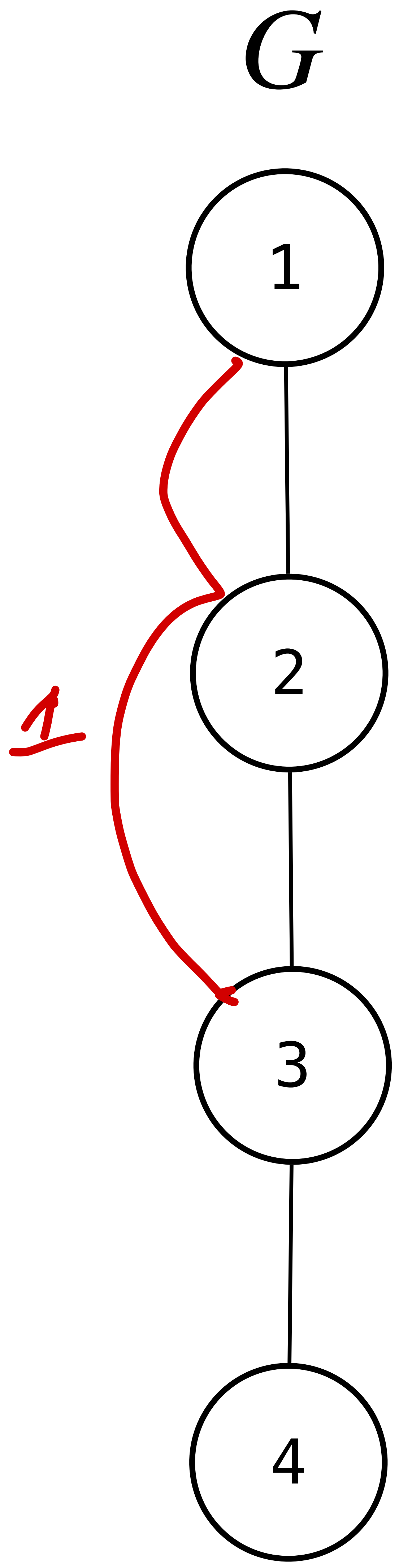


Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{array}{cc} & \begin{matrix} & & 13 & 14 \end{matrix} \\ \begin{matrix} \\ \\ 31 \\ 41 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



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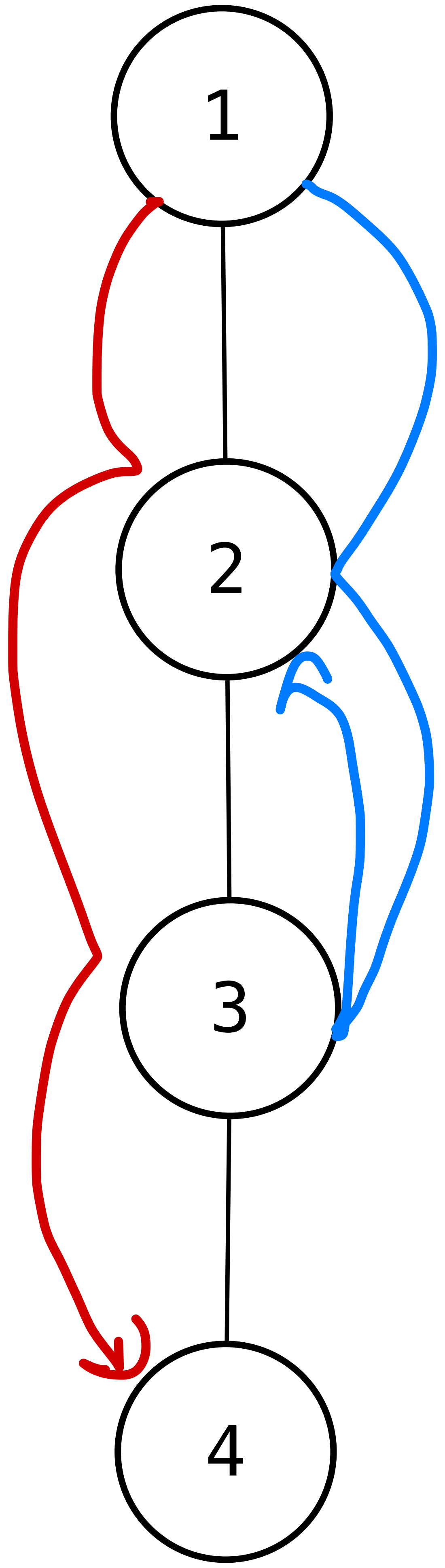
G

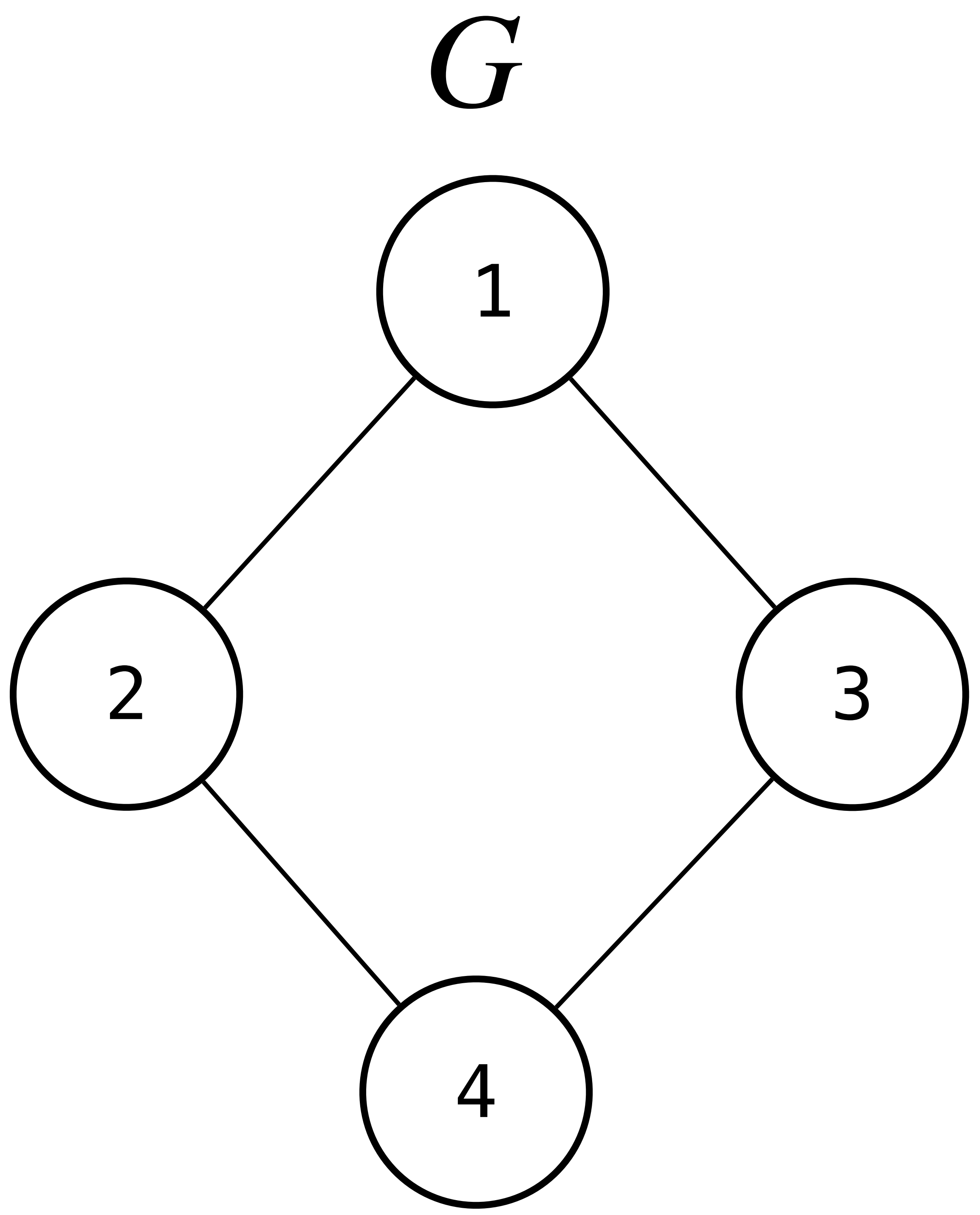


Another Application: Reachability

Theorem: Let G be a simple graph.

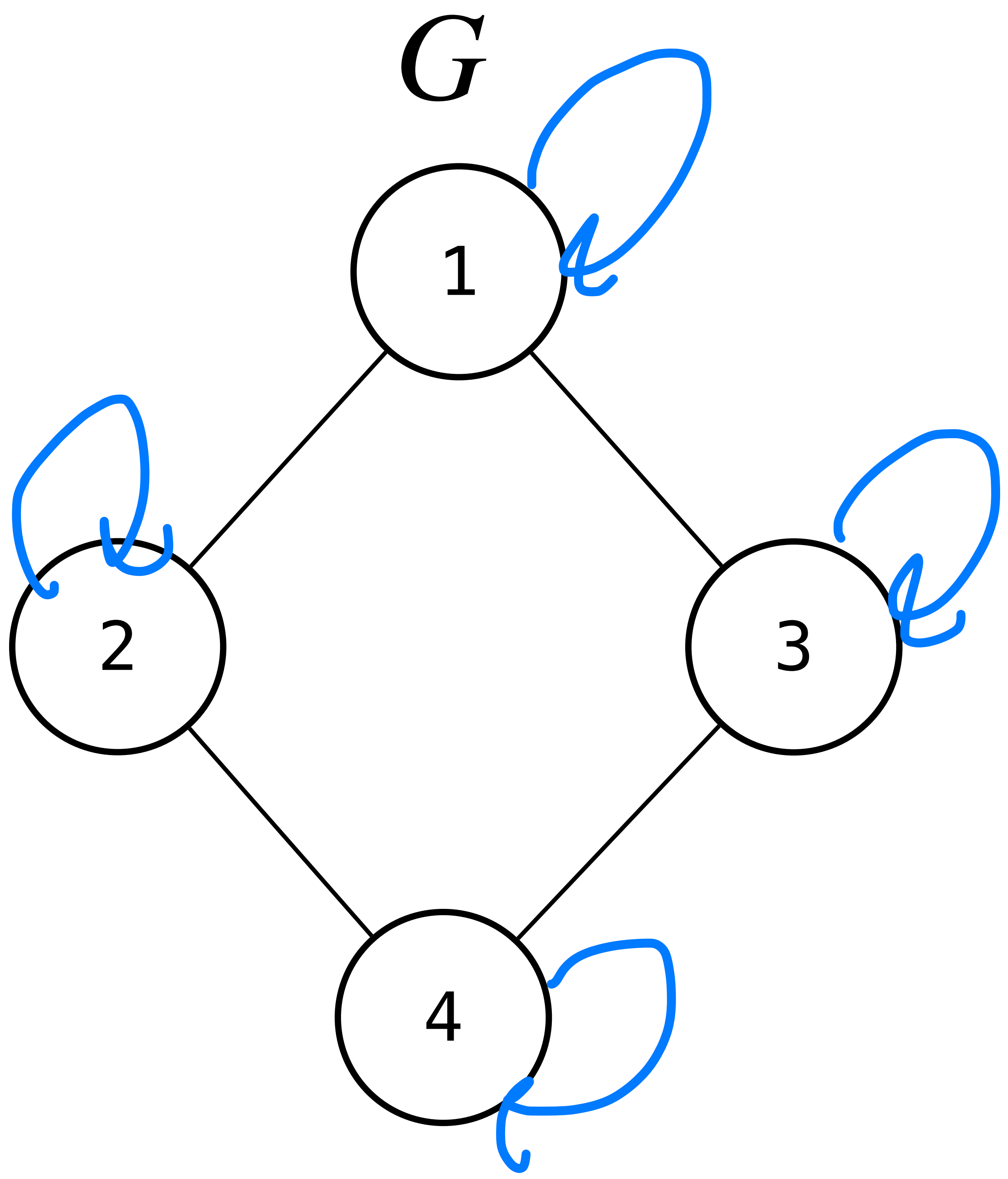
- $(A_G^k)_{ij}$ is the number of paths of length exactly k from v_i to v_j.
- $\left((A_G + I)^k\right)_{ij}$ is nonzero if and only if there is a path of length at most k from v_i to v_j.

Example



Example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = (\texttt{adjacency matrix for } G) + I$$

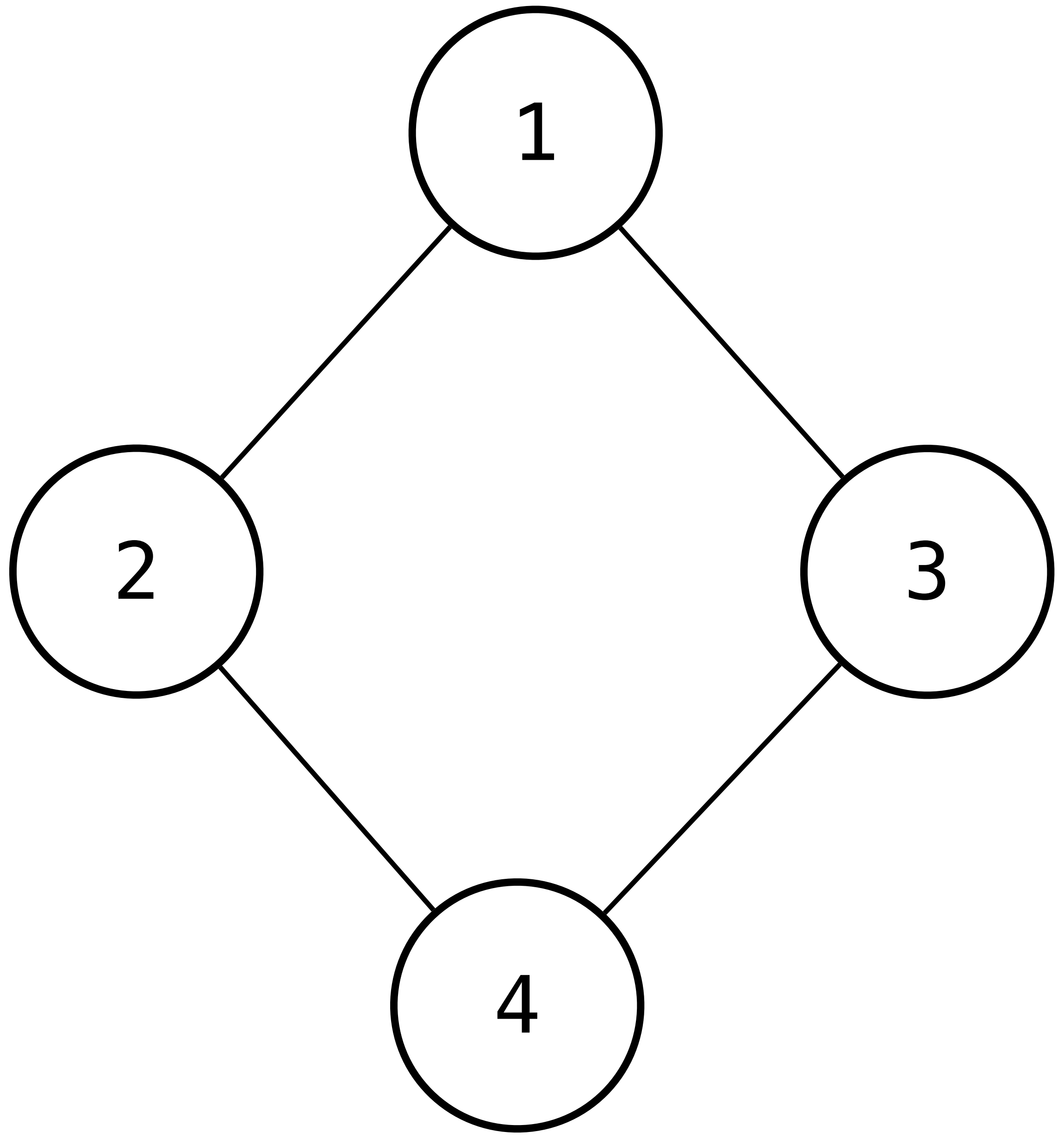


Example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \texttt{(adjacency matrix for } G\texttt{) + } I$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

G



Example

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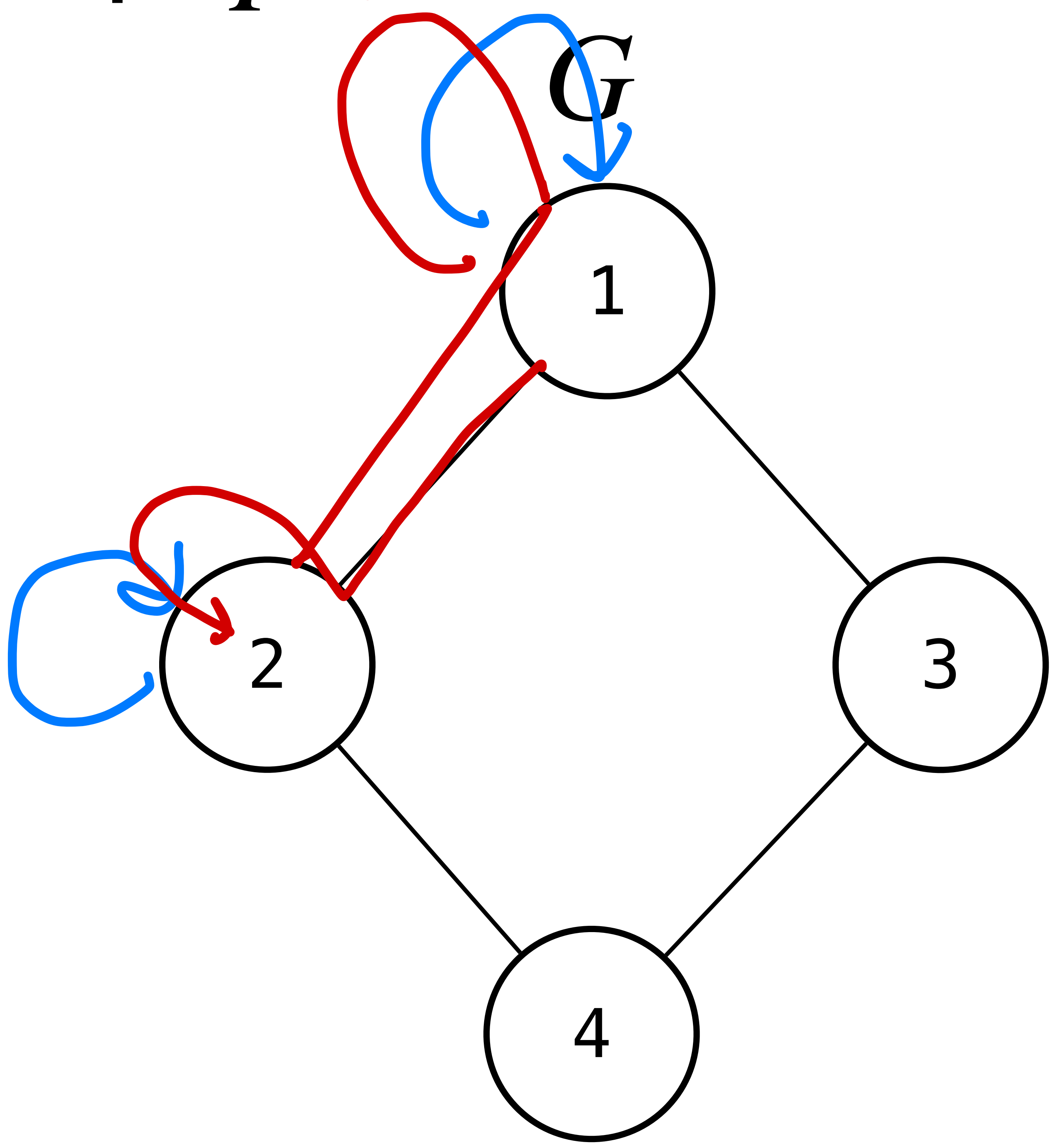
1



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How To: Reachability

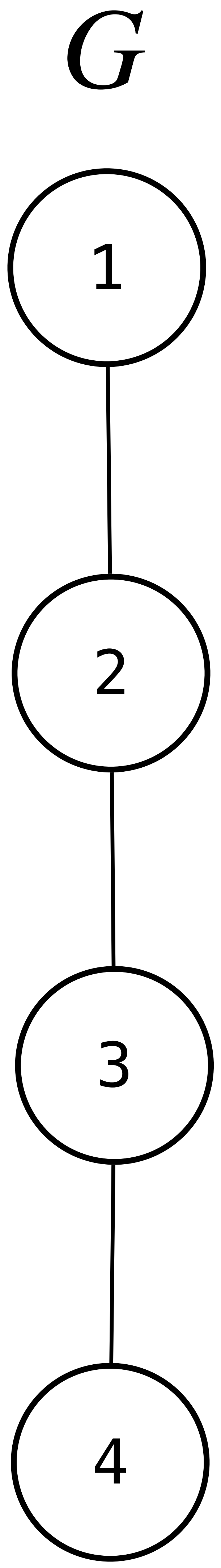
Question: Given a simple graph G determine how many nodes, v_i can reach in at least k steps.

Answer: Find $(A_G + I)^k$ and count the number of nonzero elements in column i.

(This could be useful for homework 6.)

Another Example

Determine the $(A_G + I)^2$ *and* $(A_G + I)^3$ *and interpret the results.*



Markov Chains: Motivation

Change

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```
Things change.
```

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Things change from one state of affairs to another state of affairs.

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Things change often in unpredictable ways.

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If something changes unpredictably, what can we say about it?

Dynamical Systems

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Definition (Informal). A **dynamical system** is a thing (typically with interacting parts) that changes over time.

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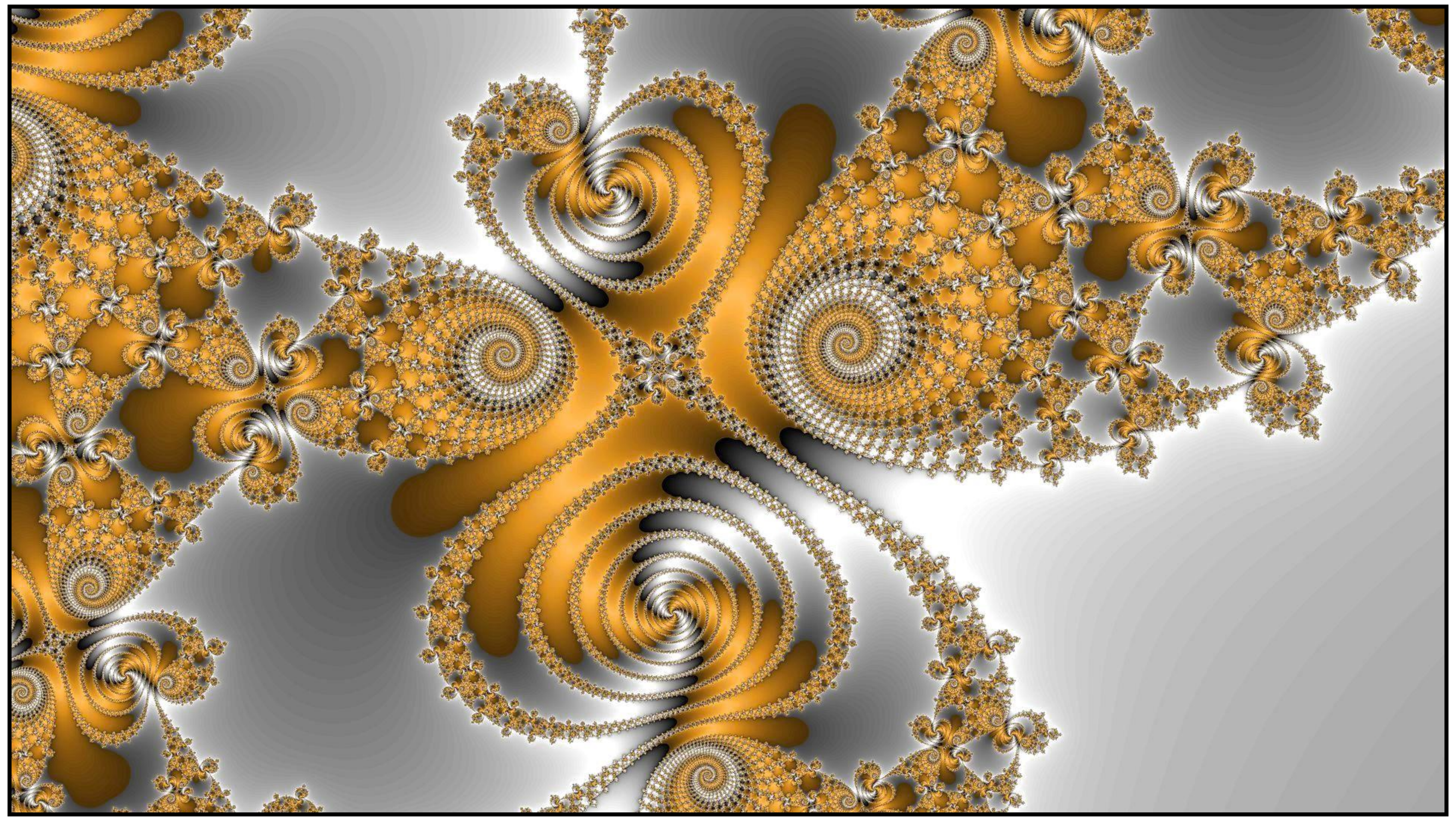
Examples.

» economics (stocks)
» physical/chemical systems
» populations
» weather

An Aside: Chaos Theory



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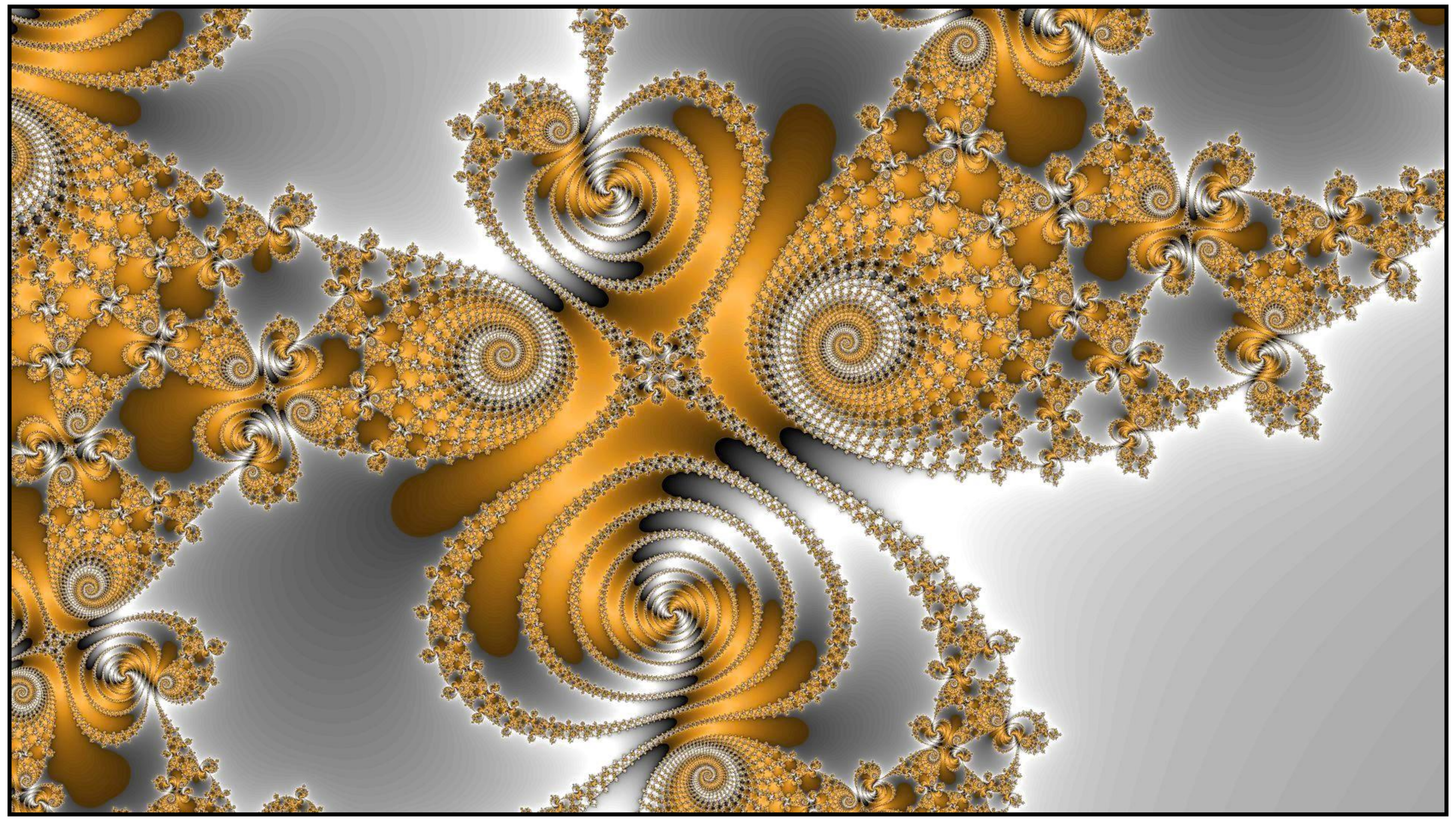
Complex systems like the weather or the economy look nearly random.



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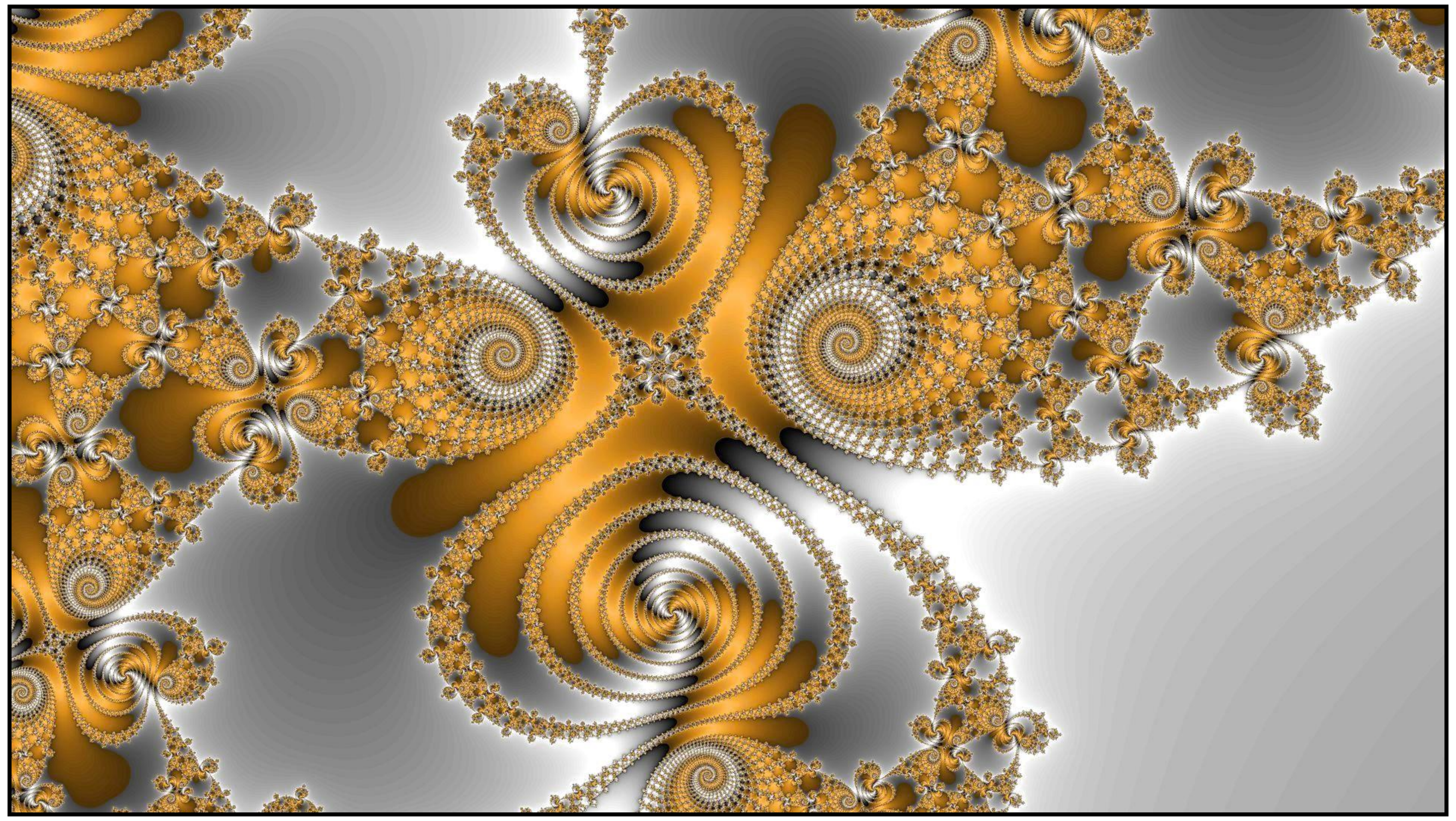

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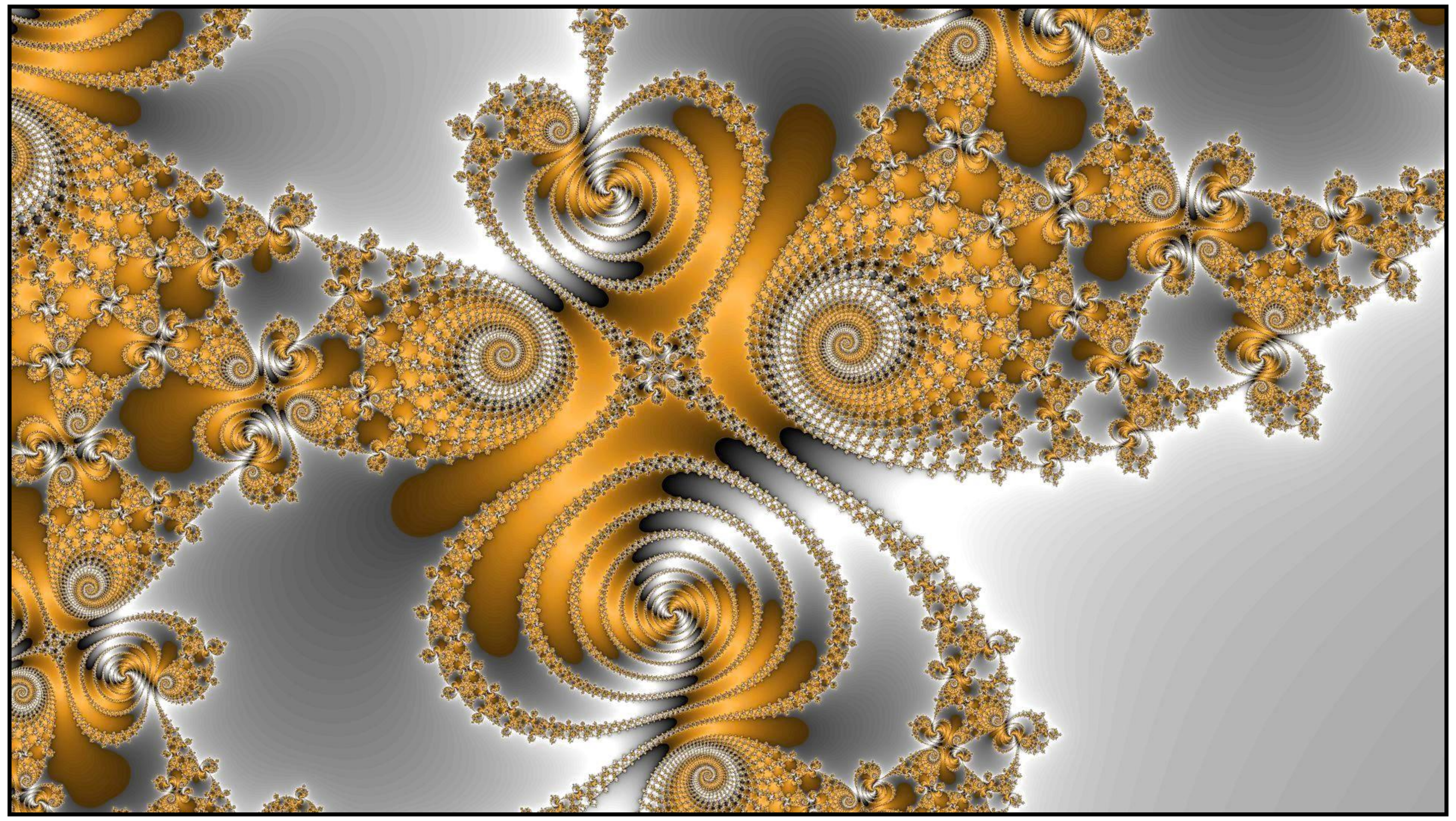
Often it's useful to consider chaotic systems in terms of global properties.



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Motivating Questions

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What does a dynamical system look like "in the
long view?"

Does it reach a kind of equilibrium? (think
heat diffusion)

Or does some part of the system dominate over
time? (think the population of rabbits without
a predator)
```

(Linear) Dynamical Systems

Linear Dynamical Systems

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Definition. A **(discrete time) linear dynamical system** is a described a $n \times n$ matrix A. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after i time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

$A\vec{v}_0 = \vec{v}_1$

$A\vec{v}_1 = \vec{v}_2$

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A tells us how our system evolves over time.

Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after i time steps:

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State Vectors

$$
\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) = A^2 v_0 \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \quad A^3 r_0 \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}
$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number k time steps.

This is also called a *recurrence relation* or a *linear difference function.*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad r_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\vec{v}_i = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^i \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i+1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad r_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\vec{v}_i = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^i \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_{i+3} \\ F_{i+2} \end{bmatrix}$$

How to: Determining State Vectors

Question. Determine the state vector $\mathbf{v}_i$ for the linear dynamical system with matrix A given the initial state vector $\mathbf{v}_0$.

Solution. Compute

$$\mathbf{v}_i = A^i \mathbf{v}_0$$

Matrix Powers in NumPy

```
numpy.linalg.matrix_power(a, k)
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There is a function in NumPy for doing matrix powers.

Use can use this when you need to take a large power of a matrix.

It's much faster than doing each multiplication individually because it uses the "repeated squaring" trick.

Warm up: Population Dynamics

The Setup

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We're working for the census. We have 2024 population measurements for a city and a suburb which are geographically coincident.

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We're working for the census. We have 2024 population measurements for a city and a suburb which are geographically coincident.

We find by analyzing previous data that each year:

- » 5% of the population moves from city → suburb
- » 3% of the population moves from suburb → city

Fundamental Question

Can we make any predictions about the population of the city and suburb in 2044?

Note: No immigration, emigration, birth, death, etc. **The overall population stays fixed.**

Setting up Linear Equations

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If city_0 = 2024 city pop. = 600,000
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If city_0 = 2024 city pop. = 600,000
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then the pop. in 2025 are given by:

$$\text{city}_1 = (0.95)\text{city}_0 + (0.03)\text{suburb}_0$$

$$\text{suburb}_1 = (0.05)\text{city}_0 + (0.97)\text{suburb}_0$$

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then the pop. in 2025 are given by:

$$\text{city}_1 = (0.95)\text{city}_0 + (0.03)\text{suburb}_0$$
$$\text{suburb}_1 = (0.05)\text{city}_0 + (0.97)\text{suburb}_0$$

people who stayed

people who left

Setting up a Matrix

$$\begin{bmatrix} \text{city}_1 \\ \text{suburb}_1 \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} \text{city}_0 \\ \text{suburb}_0 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix}$$

5

In 202~~4~~, we expect the population of the city to decrease.

Setting up a Matrix

$$\begin{bmatrix} \text{city}_2 \\ \text{suburb}_2 \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} \text{city}_1 \\ \text{suburb}_1 \end{bmatrix} = \begin{bmatrix} 565{,}440 \\ 434{,}560 \end{bmatrix}$$

In 2025, we expect the population of the city to *continue* to decrease.

Will it decrease indefinitely?

Setting up a Matrix

$$\begin{bmatrix} \text{city}_k \\ \text{suburb}_k \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} \text{city}_{k-1} \\ \text{suburb}_{k-1} \end{bmatrix}$$

This is a *linear dynamical system.*

So we want to guess what the population will look like in 20 years, we need to compute

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}^{20} \begin{bmatrix} \text{city}_0 \\ \text{suburb}_0 \end{bmatrix}$$

demo

Markov Chains

Stochastic Matrices

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

`What's special about this matrix?`

- `Its entries are nonnegative.`
- `Its columns sum to 1.`

`This should make us think probability.`

Stochastic Matrices

Definition. A $n \times n$ matrix is **stochastic** if its entries are nonnegative and its columns sum to 1.

Example.

$$\begin{bmatrix} \begin{bmatrix}0.7 & 0.1 & 0.3\\ 0.2 & 0.8 & 0.3\\ 0.1 & 0.1 & 0.4\end{bmatrix} & \begin{matrix}0.0\\ 0.0\\ 0.0\end{matrix} \\ \begin{matrix}0.0 & 0.0 & 0.0\end{matrix} & 1.0 \end{bmatrix}$$

Markov Chains

Definition. A **Markov chain** is a linear dynamical system whose evolution function is given by a <u>stochastic</u> matrix.

(We can construct a "chain" of state vectors, where each state vector only depends on the one before it.)

Key Property of Stochastic Matrices

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Stochastic matrices <u>redistribute</u> the "stuff" in a vector.

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Stochastic matrices <u>redistribute</u> the "stuff" in a vector.

Theorem. For a stochastic matrix A and a vector $\mathbf{v}$,

$$\begin{gathered}\text{sum of entries of } \mathbf{v} \\ \| \\ \text{sum of entries of } A\mathbf{v}\end{gathered}$$

Key Property of Stochastic Matrices

The sum of the entries of $\mathbf{v}$ can be computed as

$$\mathbf{1}^T\mathbf{v} = \langle \mathbf{1}, \mathbf{v} \rangle = \sum_{i=1}^{n} 1\, v_i = \sum_{i=1}^{n} v_i$$

So the previous statement can be written

$$\mathbf{1}^T(A\mathbf{v}) = \mathbf{1}^T\mathbf{v}$$

Key Property of Stochastic Matrices

$$\mathbf{1}^T(A\mathbf{v}) = \mathbf{1}^T\mathbf{v}$$

A is stochastic

Let's verify this:

More General Solutions

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In our example, we analyzed the dynamics of a *particular* population.

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What if we're interested more generally in the behavior of the process for *any* population?

More General Solutions

In our example, we analyzed the dynamics of a *particular* population.

What if we're interested more generally in the behavior of the process for *any* population?

We need to shift from a population vector to a population ***distribution*** vector.

Returning to the Example

$$\begin{bmatrix} \text{city}_k \\ \text{suburb}_k \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} \text{city}_{k-1} \\ \text{suburb}_{k-1} \end{bmatrix}$$

Returning to the Example

$$\begin{bmatrix} \text{city}_k \\ \text{suburb}_k \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix}^k \begin{bmatrix} \text{city}_0 \\ \text{suburb}_0 \end{bmatrix}$$

Returning to the Example

$$\begin{bmatrix} \text{city}_k \\ \text{suburb}_k \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix}^k \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$$

Returning to the Example

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But what if we start of with a different population?

Returning to the Example

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But what if we start of with a different population?

Do we have to do all our work over again?

Returning to the Example

$$\begin{bmatrix} \text{city}_k \\ \text{suburb}_k \end{bmatrix} = \begin{bmatrix} 0.95 & 0.3 \\ 0.05 & 0.97 \end{bmatrix}^k \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

60% of pop. in city

40% of pop. in suburb

Not really.

But rather than thinking in terms of populations, we need to think about **how the population is distributed**.

Probability Vectors

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» distributions of collections of things

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These are really the same thing.

Probability Vectors (Example)

The vector $\begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix}$ represents the distribution

where we choose:

1 with probability $1/3$

2 with probability $1/6$

3 with probability $1/2$

$$\begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$$

Probability Vectors (Example)

The vector $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ represented the distribution of the population, but we can also think of this as:

If we choose a random person from the population we'll get someone:

in the city with probability 0.6

in the suburbs with probability 0.4

The point

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We'll be interested in the dynamics of Markov chains on <u>probability vectors</u>.

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Since stochastic matrices preserve $\mathbf{1}^T\mathbf{v}$, they *transform* one distribution into another.

Can we say something about how the distribution changes in the long run?

Steady-State Vectors

Steady-State Vectors

Definition. A **steady-state vector** for a stochastic matrix A is a probability vector $\mathbf{q}$ such that

$$A\mathbf{q} = \mathbf{q}$$

A steady-state vector is *not changed* by the stochastic matrix. They describe equilibrium distributions.

Returning to the Example

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How do we interpret a steady-state vector for our example?

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The populations in the city and the suburb stay the same over time.

Returning to the Example

How do we interpret a steady-state vector for our example?

The populations in the city and the suburb stay the same over time.

The same number of people are moving into and out of the city each year.

Fundamental Questions

Do steady states exist?

Are they unique?

How do we find them?

Finding Steady-State Vectors

$$A\mathbf{q} = \mathbf{q}$$

Let's solve this equation for $\mathbf{q}$.

Finding Steady-State Vectors

$$A\mathbf{q} - \mathbf{q} = \mathbf{0}$$

Let's solve this equation for $\mathbf{q}$.

Finding Steady-State Vectors

$$A\mathbf{q} - I\mathbf{q} = \mathbf{0}$$

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$$(A - I)\mathbf{q} = \mathbf{0}$$

`Let's solve this equation for` $\mathbf{q}$.

Finding Steady-State Vectors

$$(A - I)\mathbf{q} = \mathbf{0}$$

Let's solve this equation for $\mathbf{q}$.

This is a matrix equation.
So we know how to solve it.

How to: Steady-State Vectors

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Question. Determine if the Markov chain with stochastic matrix A has a steady-state vector. If it does, find such a vector.

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How to: Steady-State Vectors

Question. Determine if the Markov chain with stochastic matrix A has a steady-state vector. If it does, find such a vector.

Solution. Solve the equation $(A - I)\mathbf{x} = \mathbf{0}$ and find a solution whose entries sum to 1 (this will be possible given a free variable).

If there is no such solution, the system does not have a steady state.

Example $(A - I)\vec{x} = 0$

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

$x_2 = 5/8$

$x_1 = 3/8$

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}}$$

$$A\begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 \\ 5 & .3 \end{bmatrix} \sim \begin{bmatrix} .5 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{l} x_1 = 3/5\, x_2 \\ x_2 \text{ is free} \end{array}}$$

$x_1 + x_2 = 1$

$3/5\, x_2 + x_2 = 1$

$8/5\, x_2 = 1$

$\boxed{x_2 = 5/8}$

demo

Existence vs Convergence

If $(A - I)\mathbf{x} = \mathbf{0}$ infinitely many solutions, then it has a stable state.

This **does not** mean:

» the stable state is unique
» the system converges to this state

Convergence

Convergence

Definition. For a Markov chain with stochastic matrix A, an initial state $\mathbf{v}_0$ **converges** to the state $\mathbf{v}$ if $\lim_{k\to\infty} A^k \mathbf{v}_0 = \mathbf{v}$.

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Example of Non-Convergence

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Non-Example. I is a stochastic matrix and

$$I\mathbf{v} = \mathbf{v}$$

for any choice of $\mathbf{v}$.

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Example of Non-Convergence

Non-Example. I is a stochastic matrix and

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for any choice of $\mathbf{v}$.

So this system does not have a unique steady state.

And no vectors converge to the same stable state.

Regular Stochastic Matrices

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Definition. A stochastic matrix A is **regular** if A^k has all positive entries for *some nonnegative* k.

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Definition. A stochastic matrix A is **regular** if A^k has all positive entries for *some nonnegative* k.

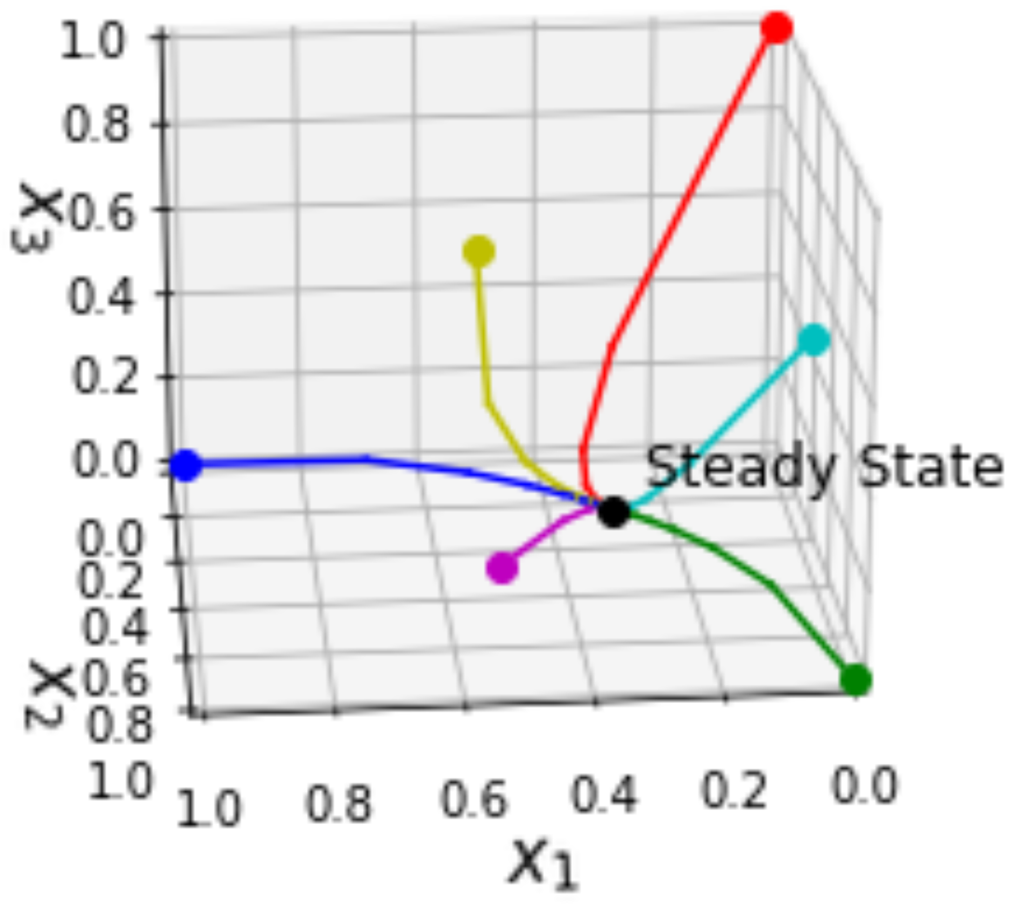
Theorem. A regular stochastic matrix P has a unique steady state, and

every probability vector converges to it

Mixing

This process of converging to a unique steady state is called "mixing."

This theorem says, after some amount of mixing, we'll be close to the stable state, **no matter where we started.**



How to: Regular Stochastic Matrices

Question. Show that A is regular, and then find it's unique steady state.

Solution. Find a power of A which has all positive entries, then solve the equation $(A - I)\mathbf{x} = \mathbf{0}$ as before.

$$\begin{bmatrix} 0.5 & 0.4 & 0 \\ 0.5 & 0.4 & 0.5 \\ 0 & 0.2 & 0.5 \end{bmatrix}$$

Example

$$\begin{bmatrix} 0.5 & 0.4 & 0 \\ 0.5 & 0.4 & 0.5 \\ 0 & 0.2 & 0.5 \end{bmatrix}^2 =$$

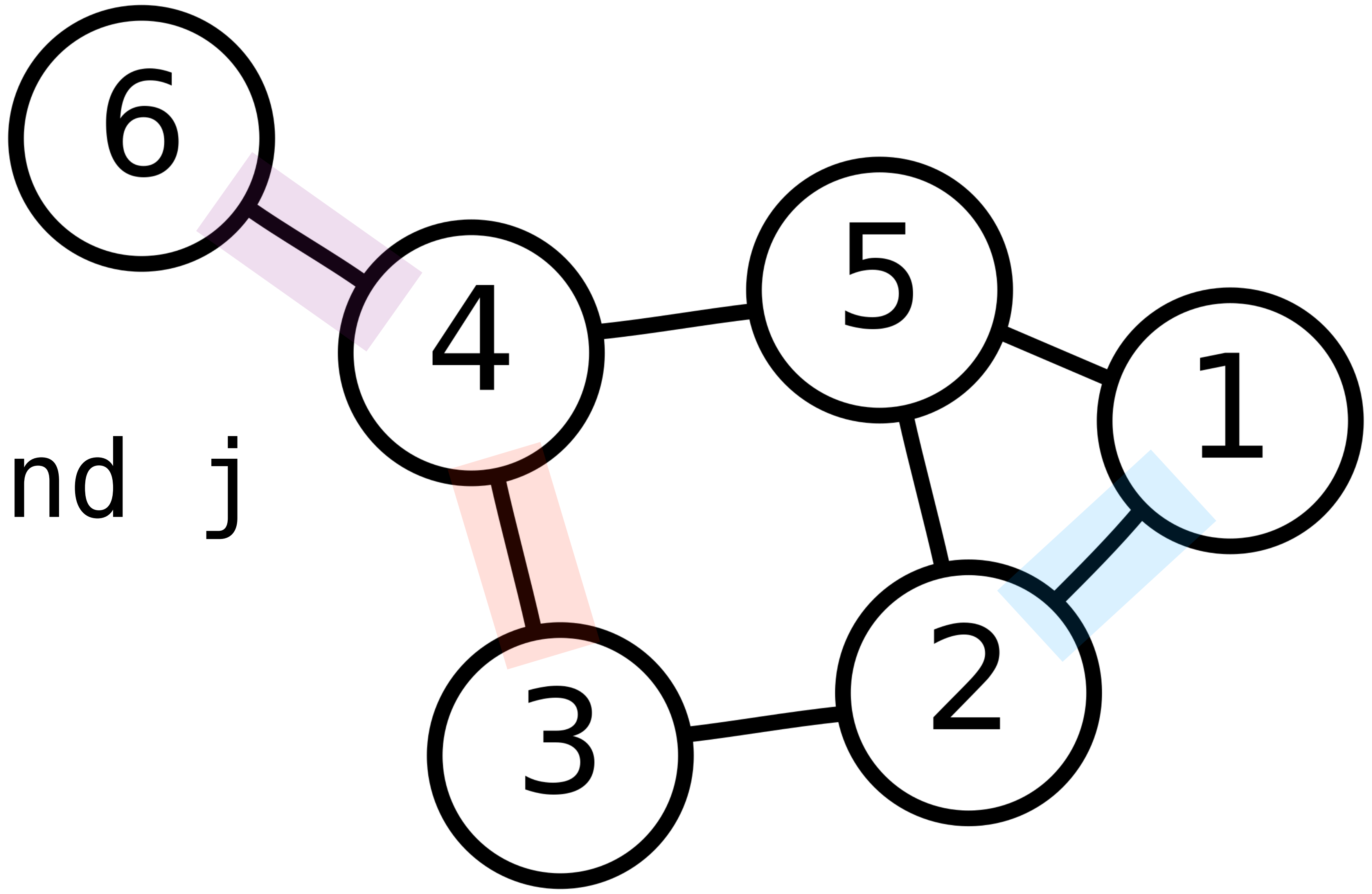
Random Walks

Recall: Adjacency Matrices

$$\begin{array}{cc} & \begin{array}{cccccc} & A_{12} & & A_{34} & & A_{46} \end{array} \\ \begin{array}{c} \\ A_{21} \\ \\ A_{43} \\ \\ A_{64} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Let G be an undirected unweighted graph with its nodes labeled by numbers 1 through n.

We can create the **adjacency matrix** A for G as follows.



$$A_{ij} = \begin{cases} 1 & \text{there is an edge between i and j} \\ 0 & \text{otherwise} \end{cases}$$

Random Walk

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» go to the kth vertex according to some order

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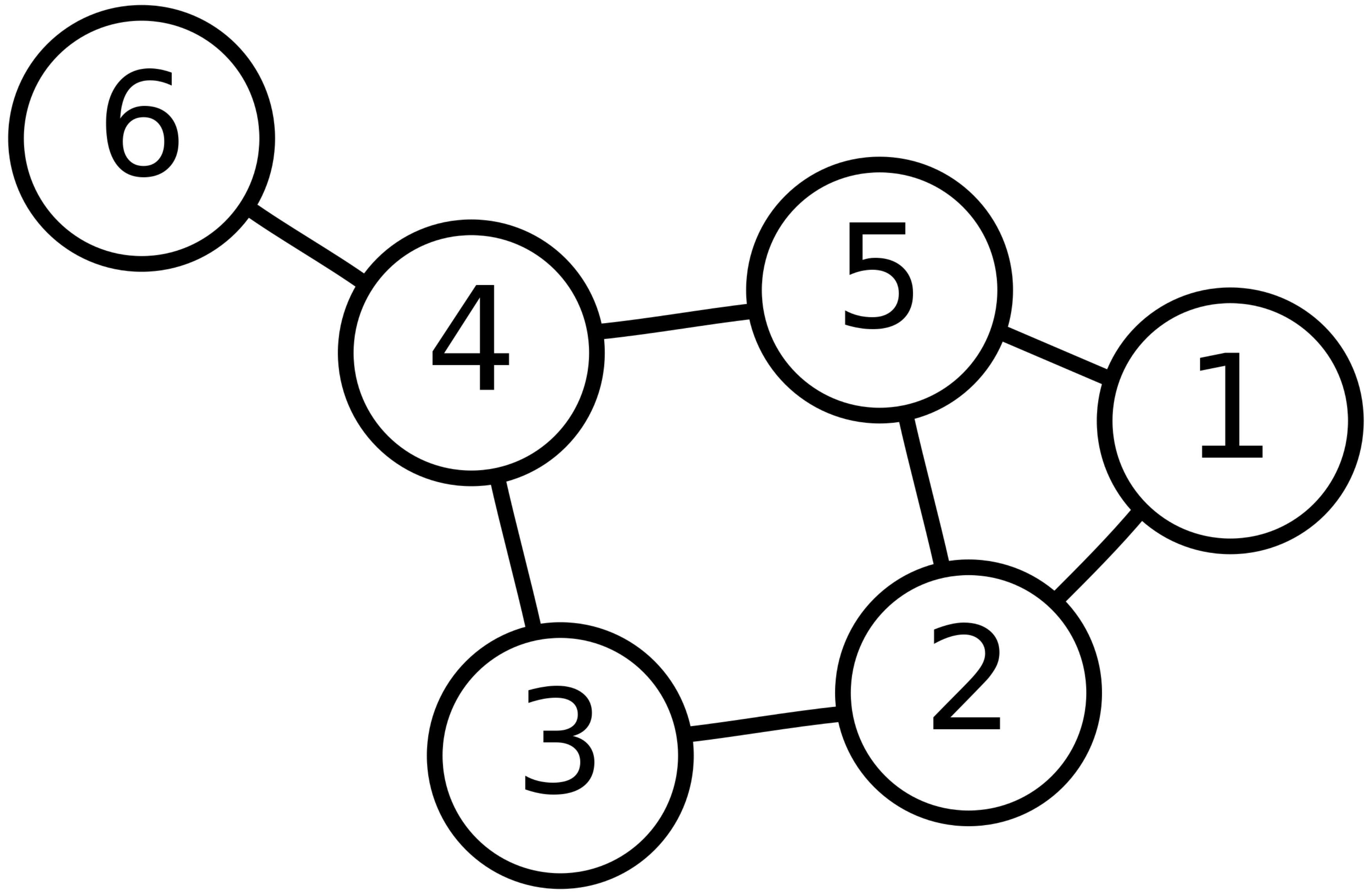
» repeat

Example

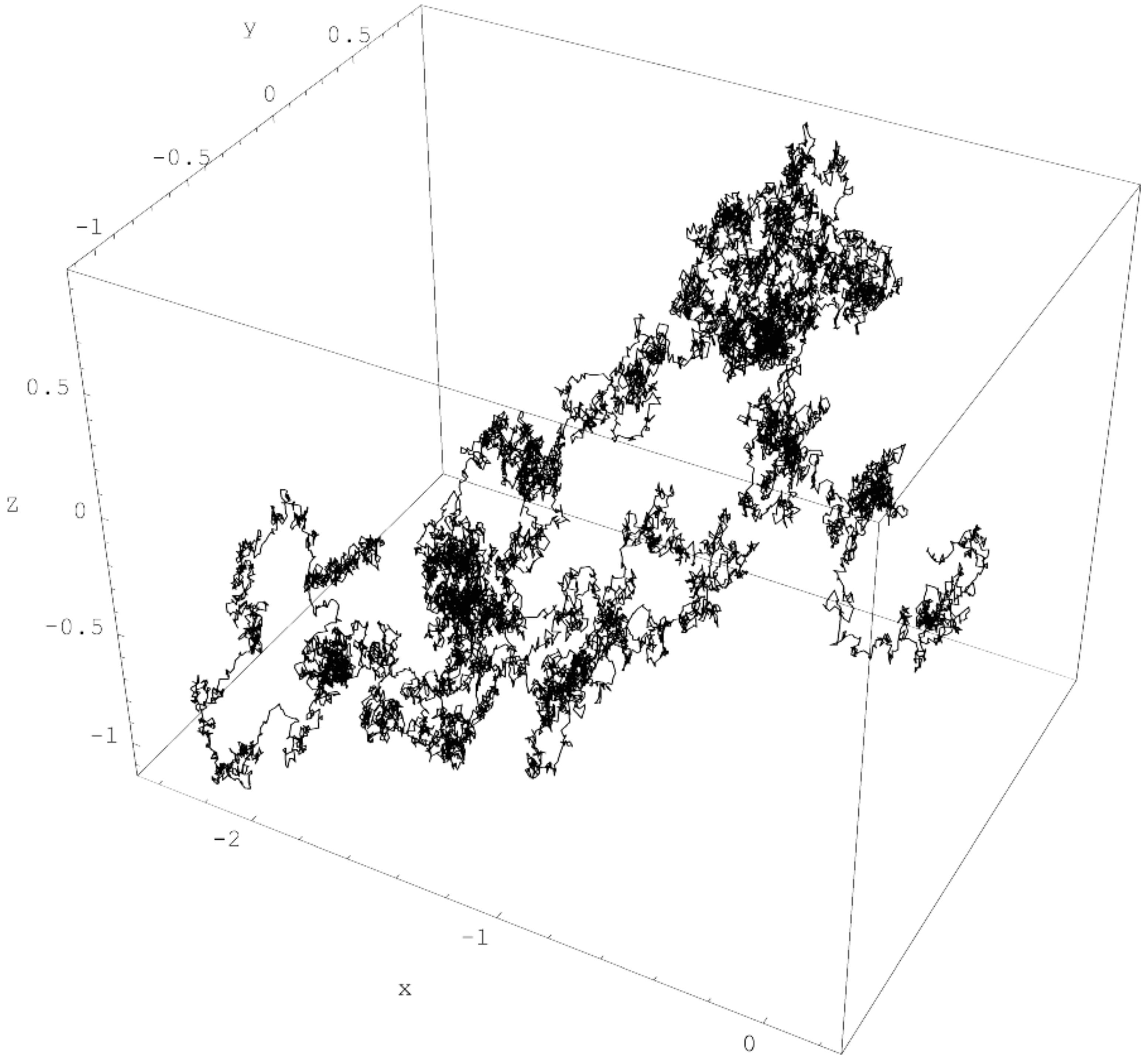


Applications of Random Walks

Brownian Motion is a random walk in 3D space.

Random walks are to simulate complex systems in physics and in economics.

They are also used to design algorithms.



https://commons.wikimedia.org/wiki/File:Wiener_process_3d.png

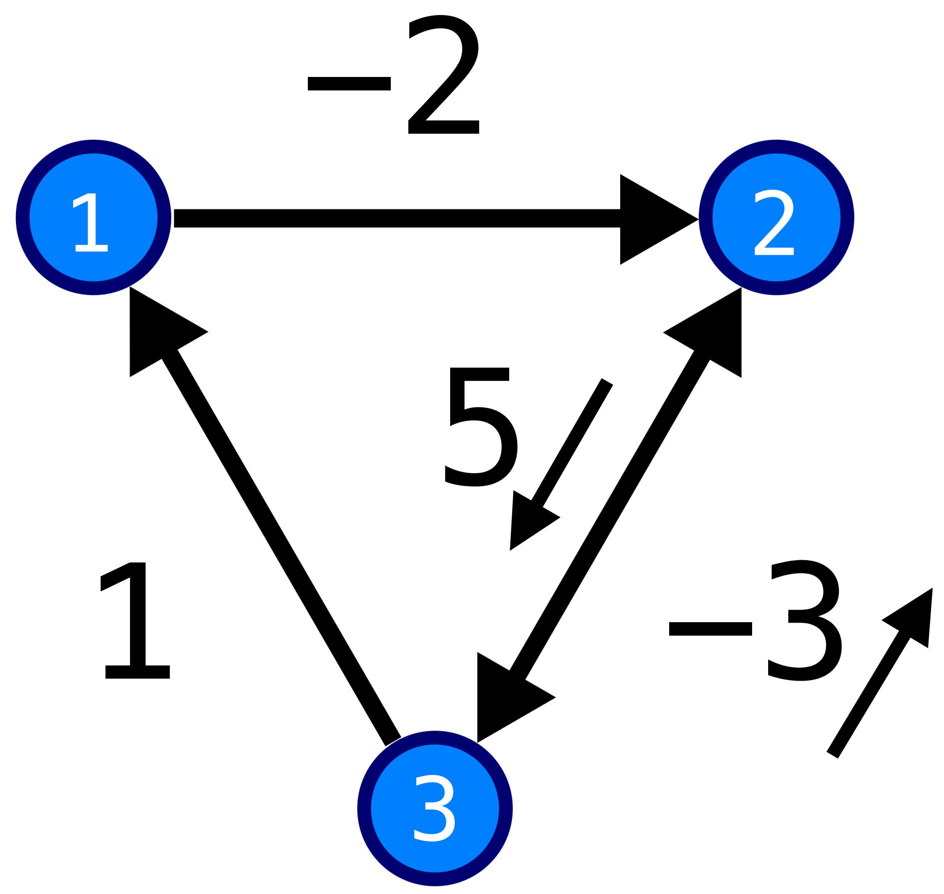
General Adjacency Matrices

We can extend the notion of an adjacency matrix to directed and weighted graphs.

$$A_{ij} = \begin{cases} w_{ji} & \text{there is an edge from } j \text{ to } i \\ 0 & \text{otherwise} \end{cases}$$

Example.

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -3 \\ 0 & 5 & 0 \end{bmatrix}$$



State Diagrams

Definition. A **state diagram** is a directed weighted graph whose adjacency matrix is stochastic.

Example.

Naming Convention Clash

The nodes of a state diagram are often called <u>states</u>.

The vectors which are dynamically updated according to a linear dynamical system are called <u>state vectors</u>.

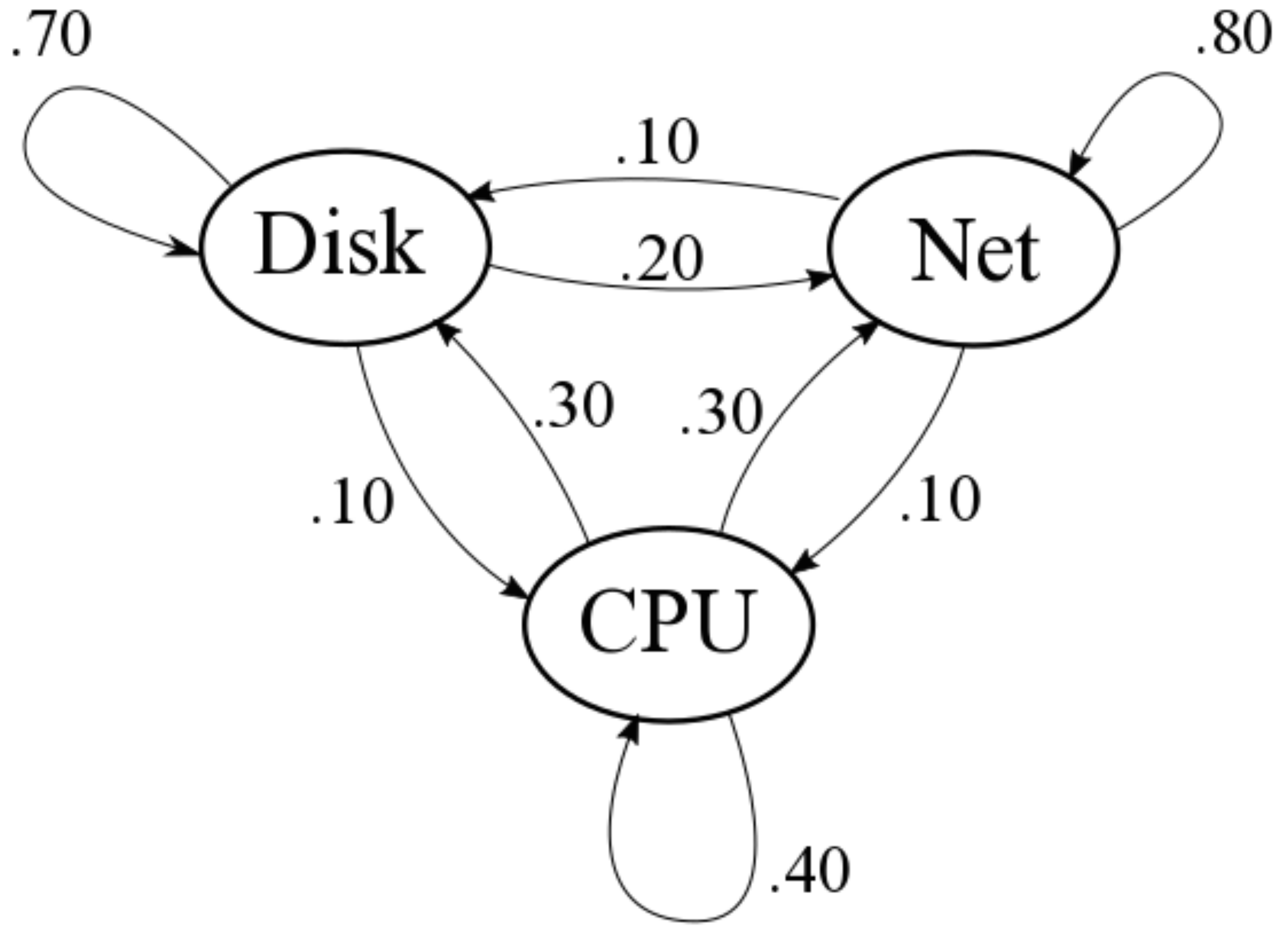
This is an unfortunate naming clash.

Example: Computer System

Imagine a computer system in which tasks request service from disk, network or CPU.

In the long term, which device is busiest?

This is about finding a stable state.

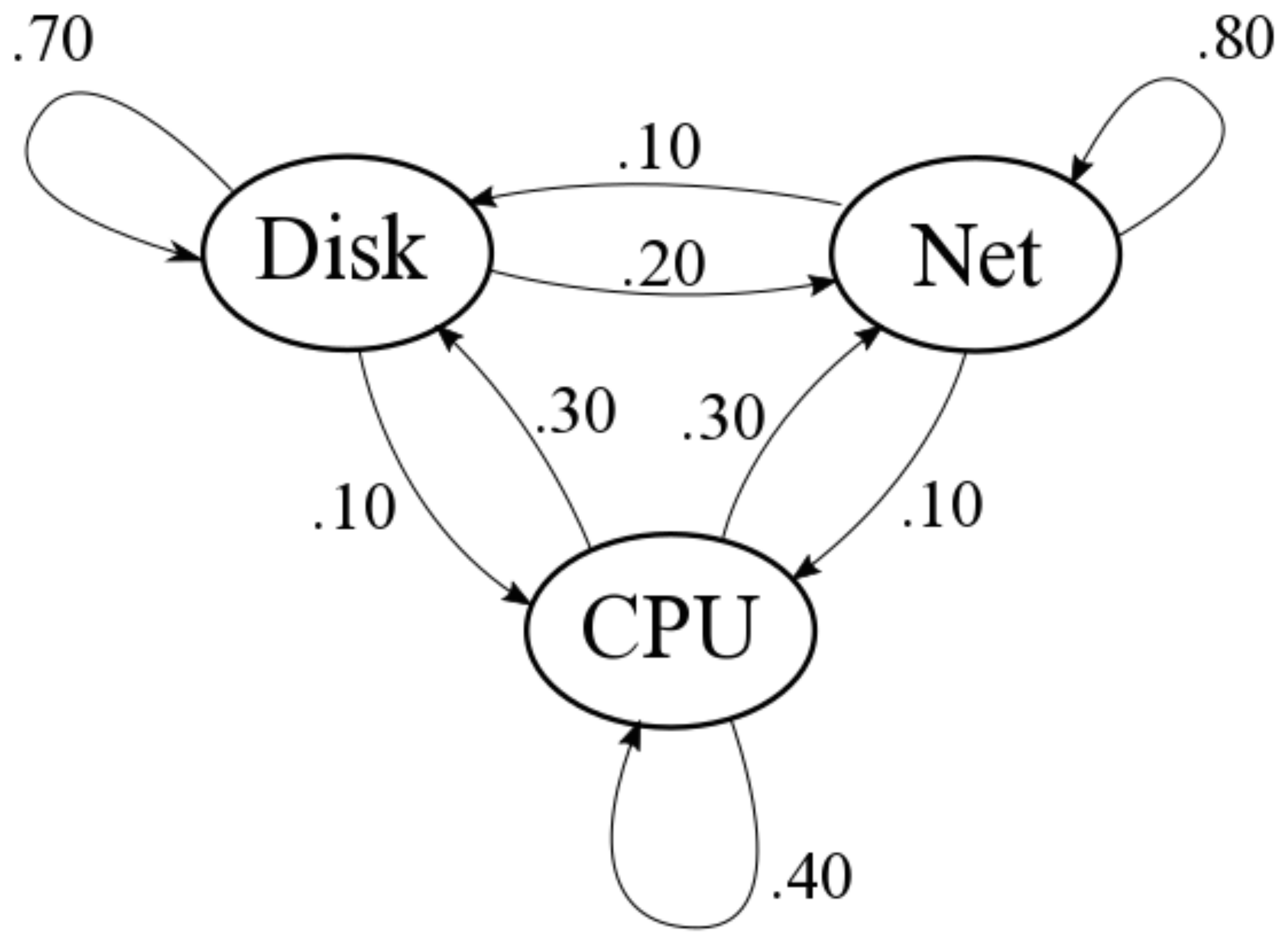


How To: State Diagram

Question. Given a state diagram, find the stable state for the corresponding linear dynamical system.

Solution. Find the adjacency matrix for the state diagram and go from there.

Example



Random Walks as Linear Dynamical Systems

Once we have a stochastic matrix, we can reason about random walks *as linear dynamical systems*.

What are its steady states?

How do we interpret these steady states?

Random Walks on State Diagrams

A **random walk** on a state diagram starting at v is the following process:

» choose a node v is connected to according to the *distribution* given by the edge weights

» go to that node

» repeat

Random Walks on State Diagrams

A **random walk** on a state diagram starting at v is the following process:

» c … o
the

» g

» repeat

Stable states of linear dynamical systems are stable states of random walks on state diagrams.

Steady-States of Random Walks

Theorem. Let A be the stochastic matrix for the graph G. The probability that a random walk starting at i of length k ends on node j is

$$(A^k)_{ji}$$

A transforms a distribution for length k walks to length $k+1$ walks.

Steady States of Random Walks

If a random walk goes on for a sufficiently long time, then the probability that we end up in a particular place becomes fixed.

If you wander for a sufficiently long time, it doesn't matter where you started.

Summary

Markov chains allow us to reason about dynamical systems that are dictated by some amount of randomness.

Stable states represent global equilibrium.

We can think of Markov chains as random walks on state diagrams.