

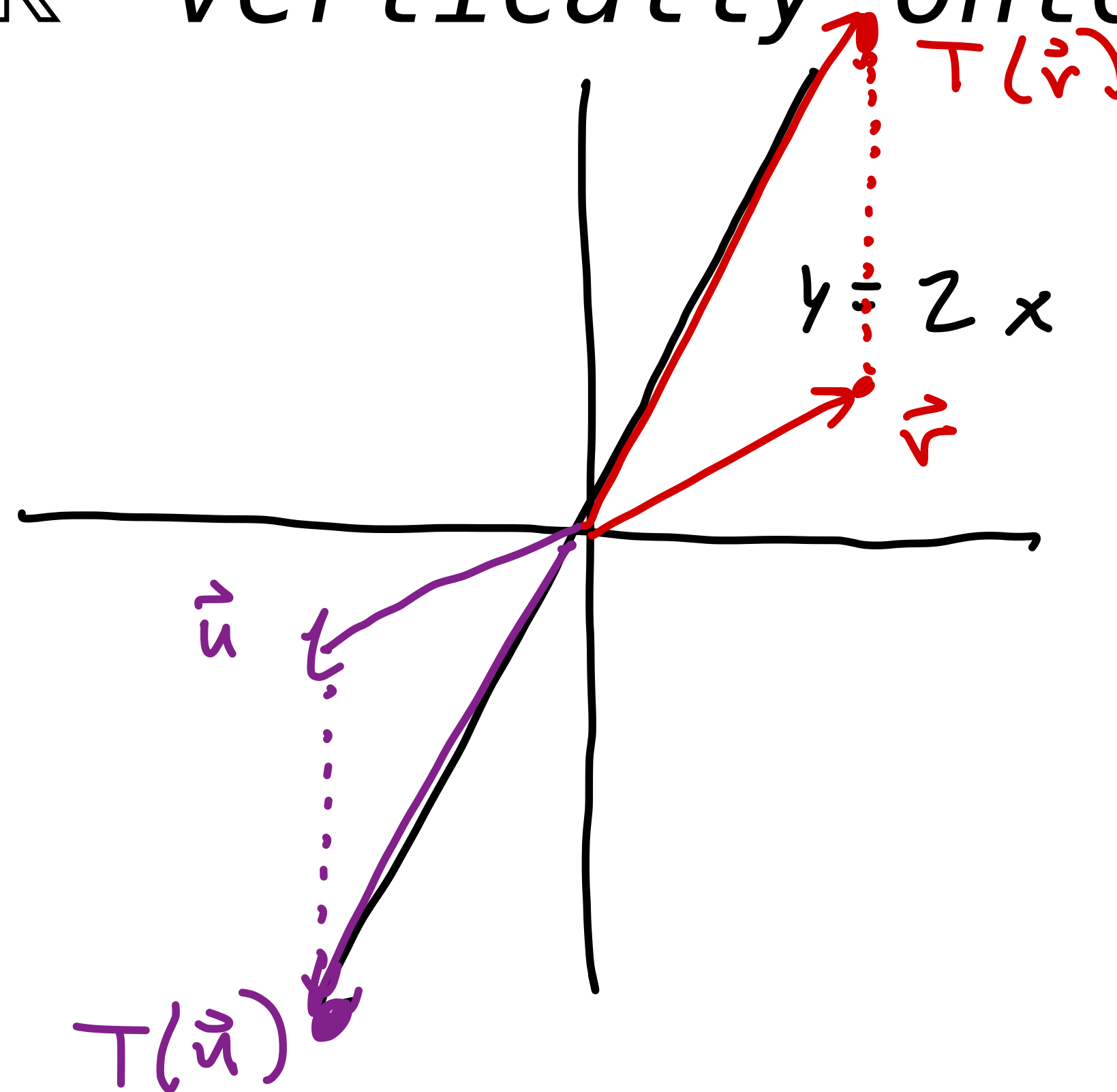
Matrix Algebra

Geometric Algorithms

Lecture 10

Practice Problem

Write the matrix for the transformation which projects vectors in \mathbb{R}^2 vertically onto the line $y = 2x$ ~~in~~ in \mathbb{R}^2 .



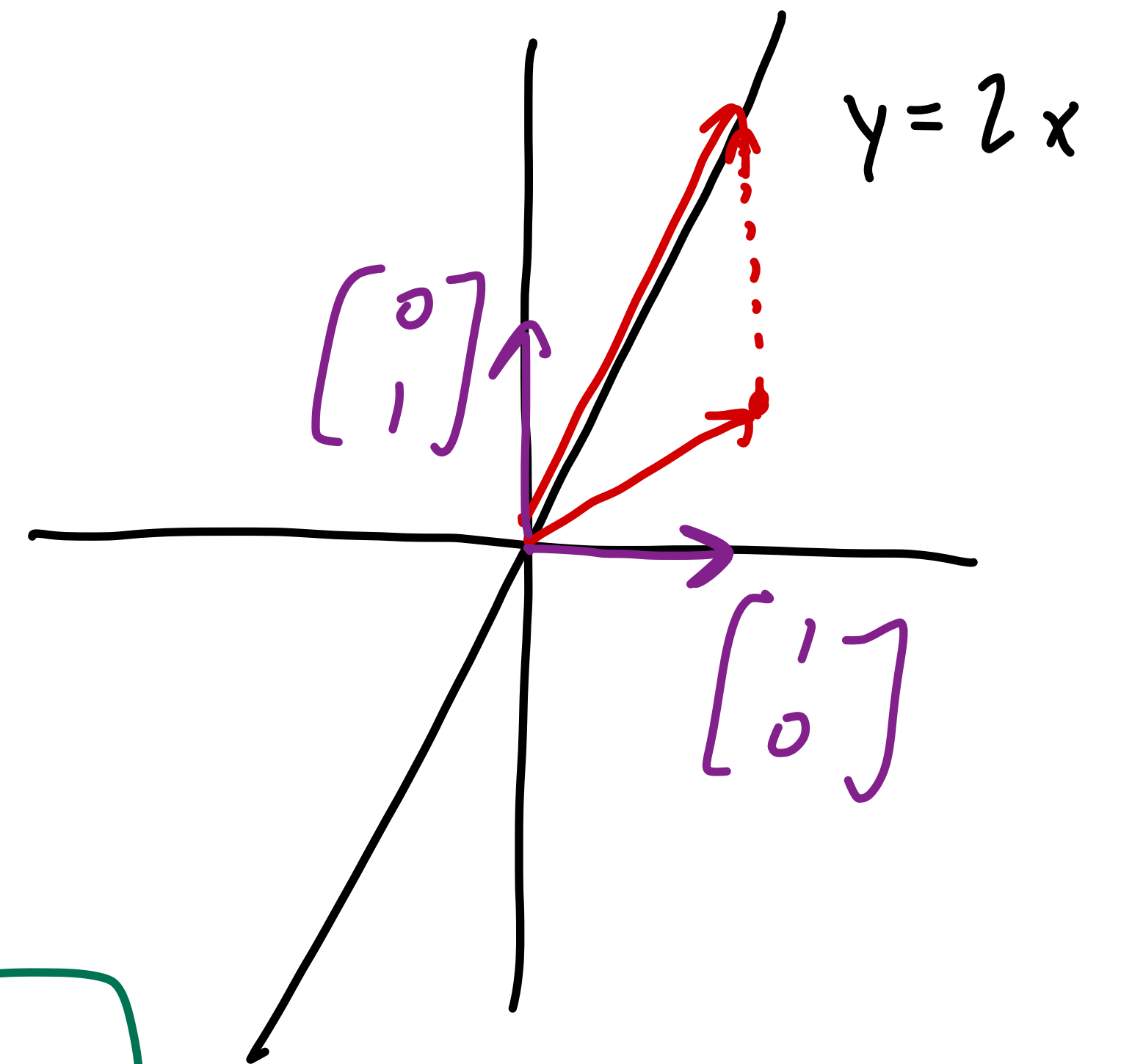
Answer

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \boxed{\begin{bmatrix} x \\ 2x \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}$$



$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x + 0 \\ 2x + 0 \end{bmatrix} = \boxed{\begin{bmatrix} x \\ 2x \end{bmatrix}}$$

Objectives

1. Connect questions about matrix equations and linear transformations
2. Motivate matrix multiplication
3. Define matrix multiplication
4. Look at the algebra of matrix multiplication

Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

Recap: Geometry of Linear Transformations

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector \mathbf{x} to the vector $A\mathbf{x}$

Recall: Motivating Questions

What kind of functions can we define in this way?

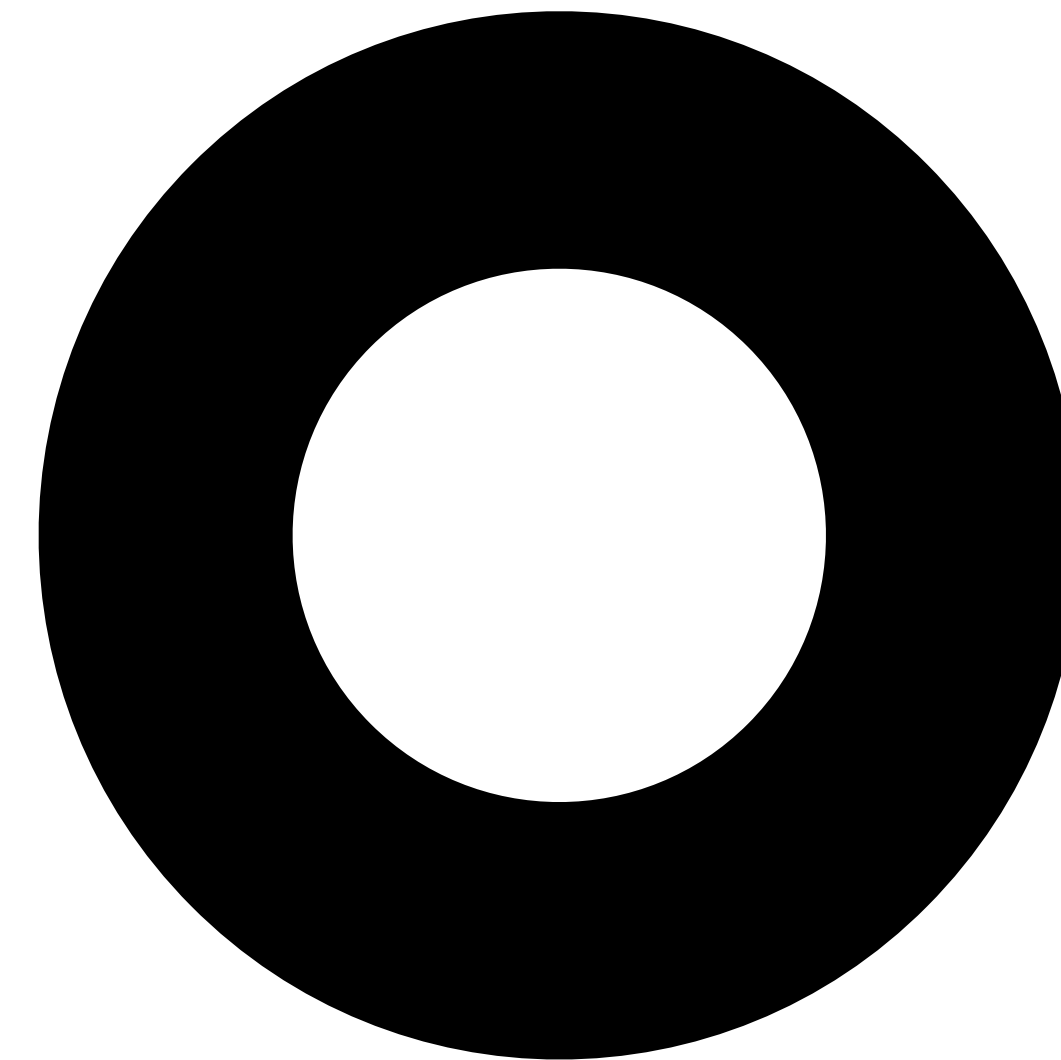
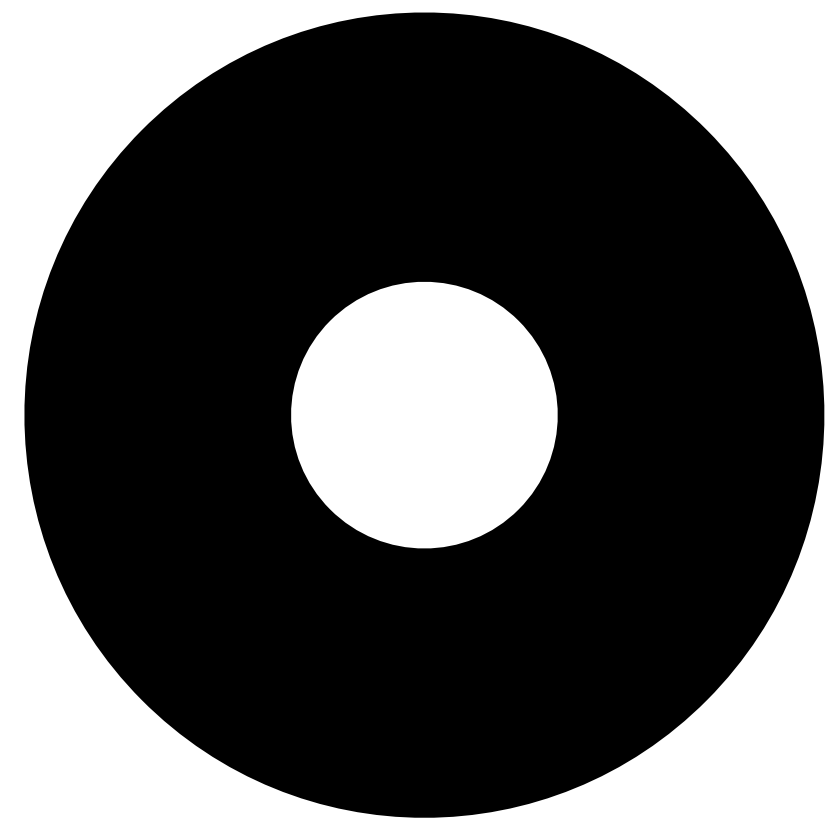
How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Motto

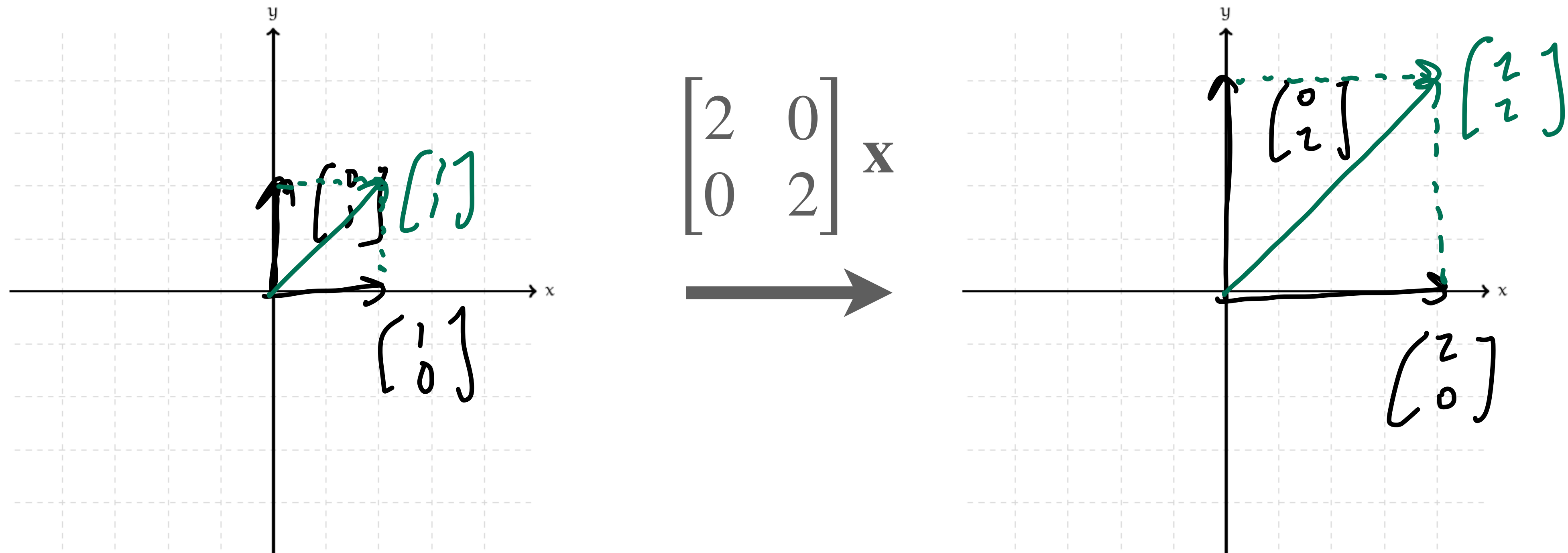
Matrix transformations change
the "shape" of a set of ~~set of~~
vectors (points).

Example: Dilation



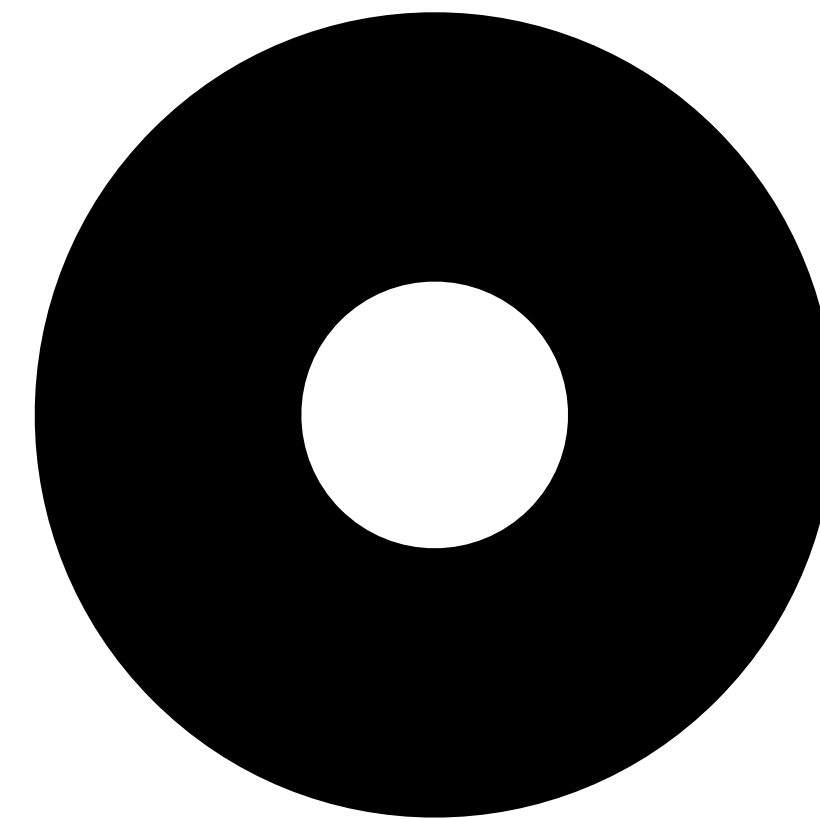
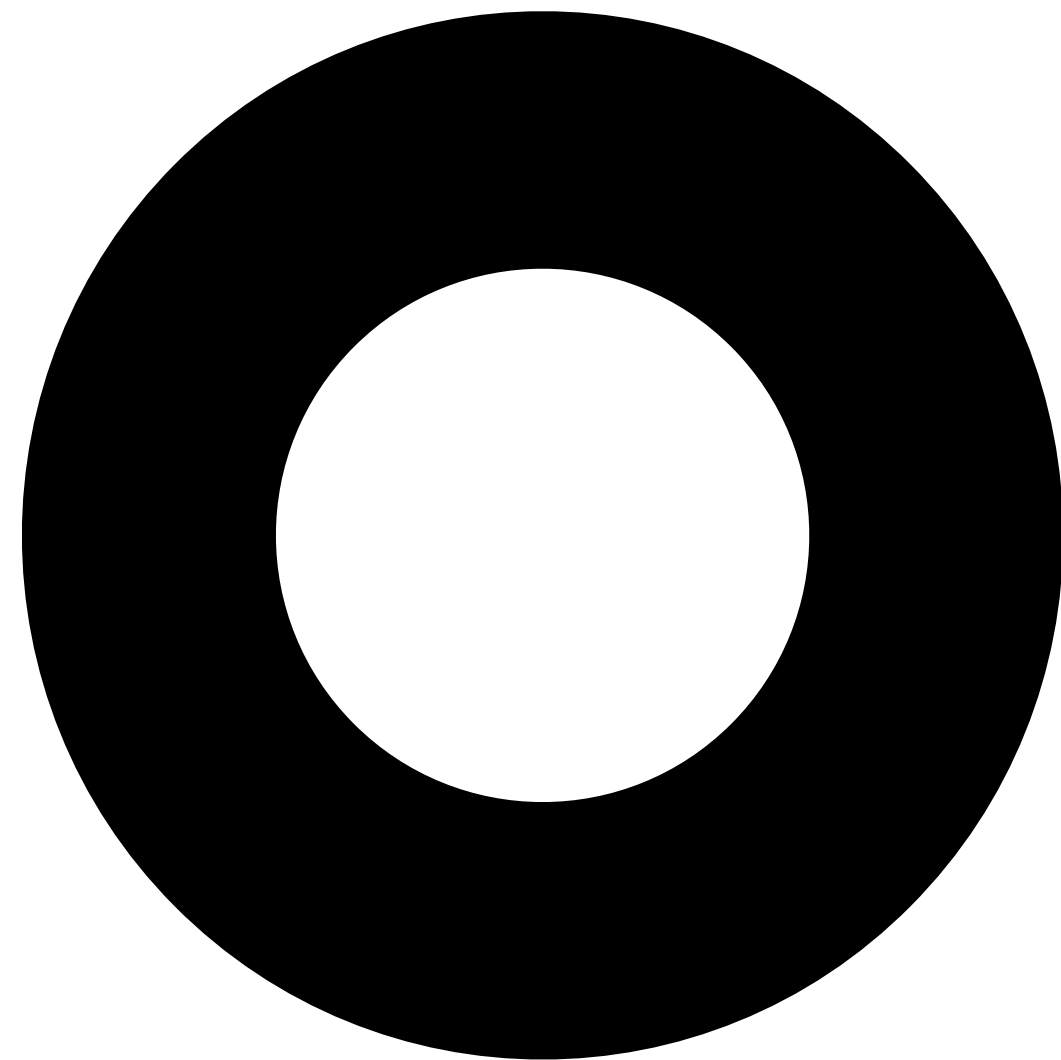
Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



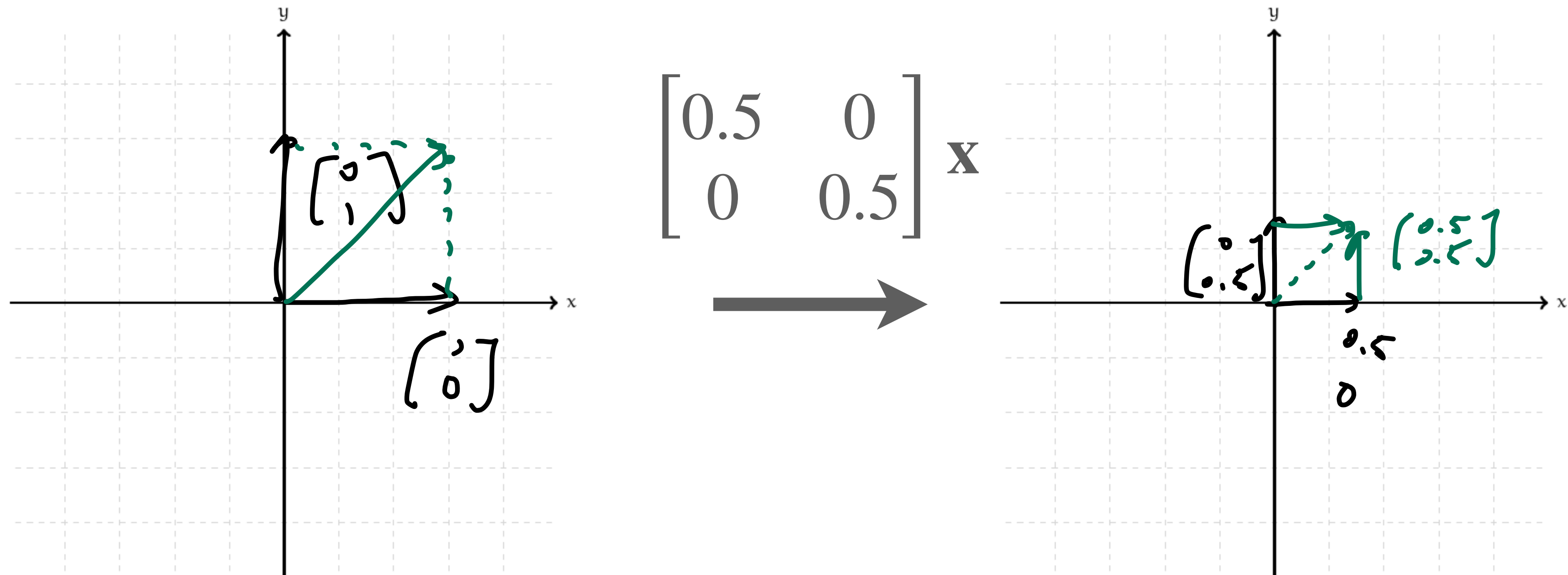
if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



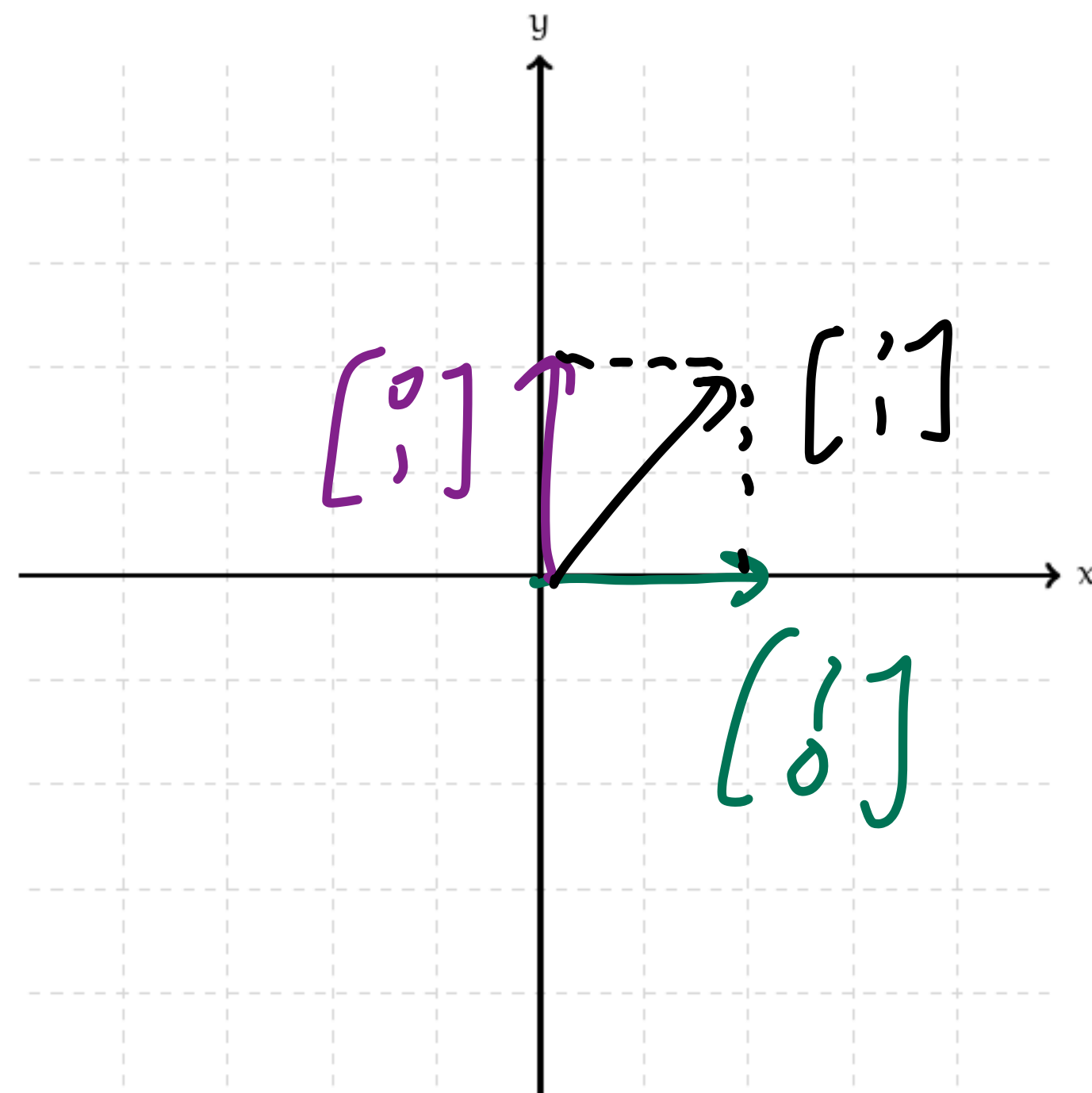
if $0 \leq r \leq 1$, then the transformation
pulls points towards the origin.

Example: Shearing



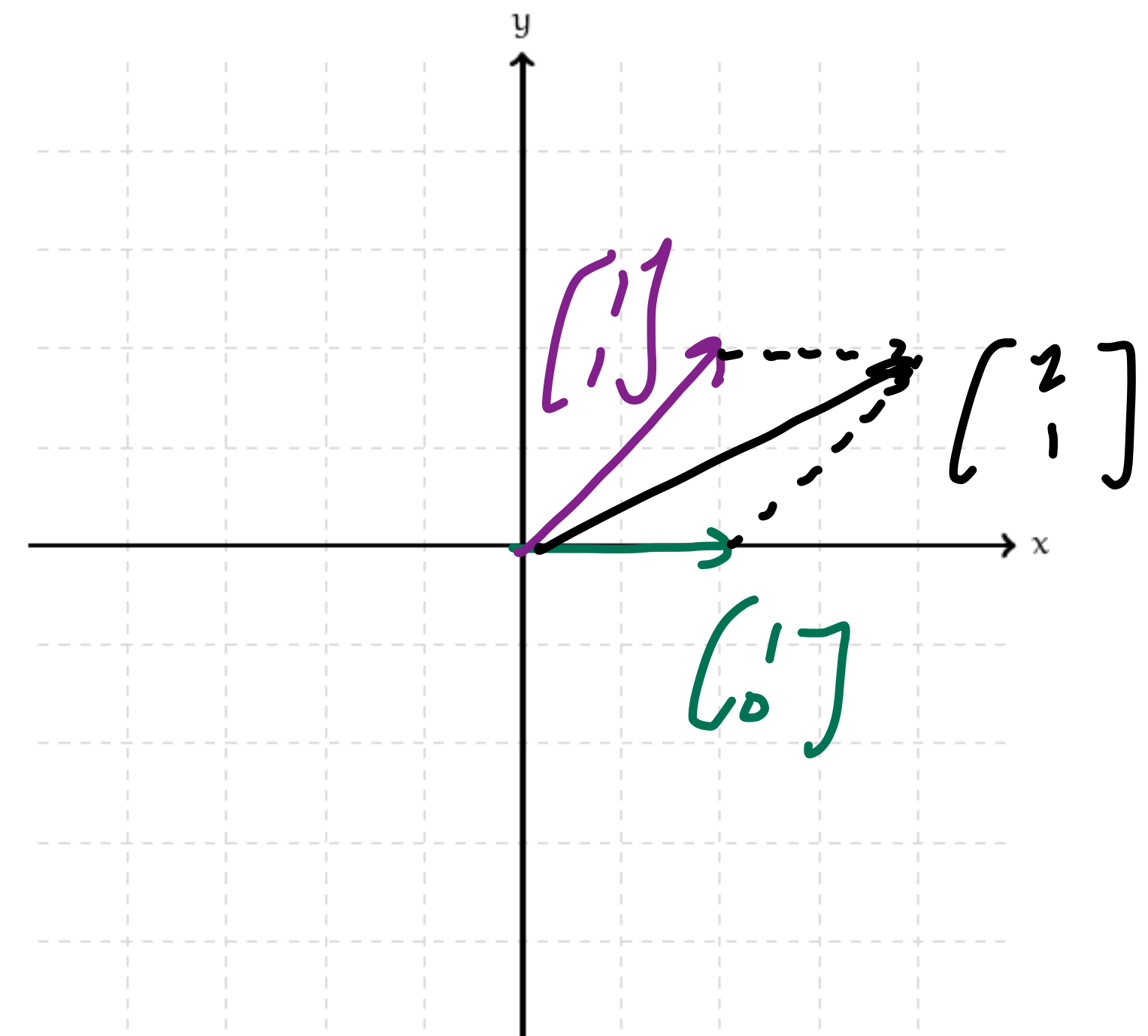
Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

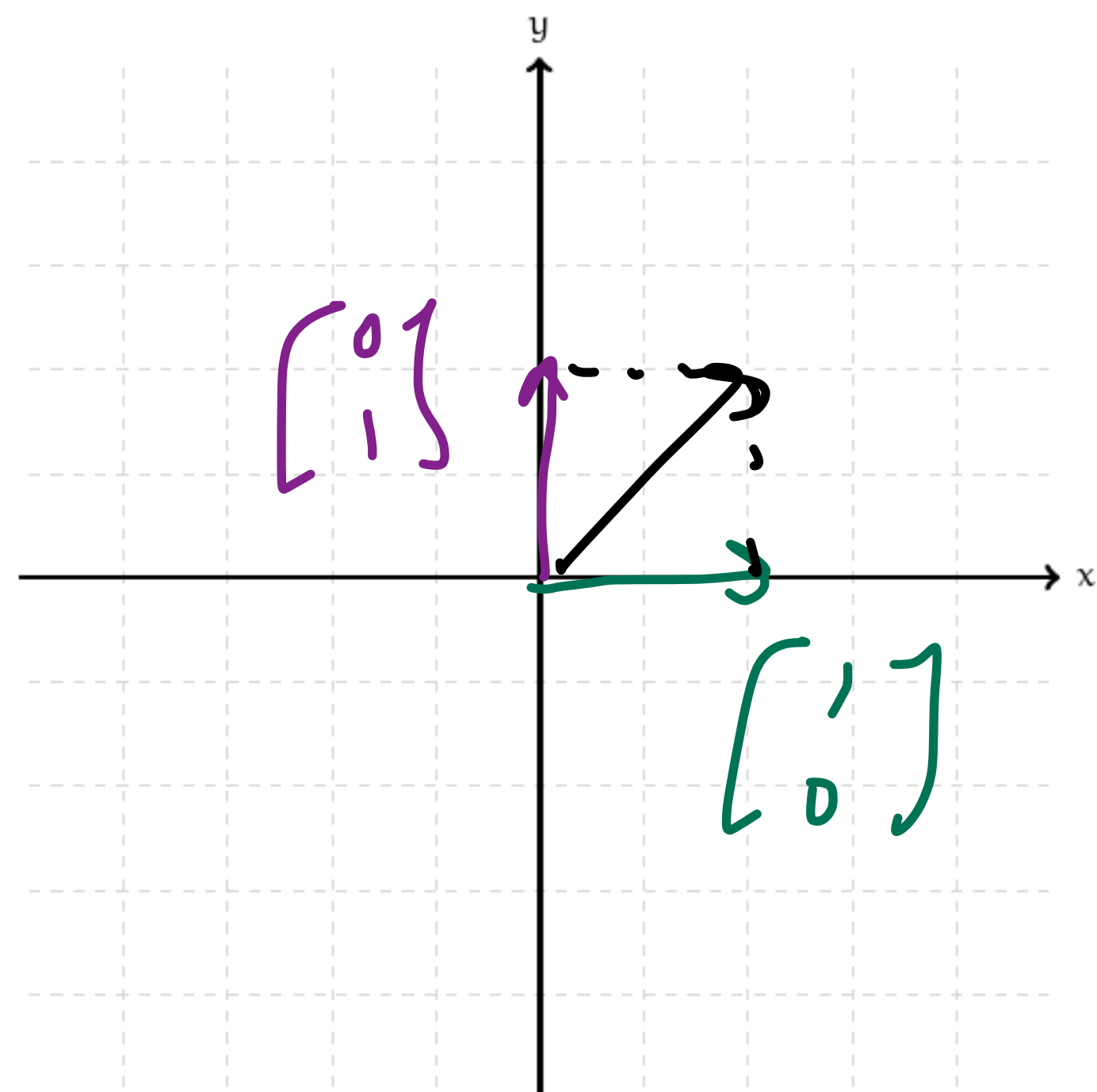
A large black arrow points from the left coordinate system to the right coordinate system, indicating a transformation.



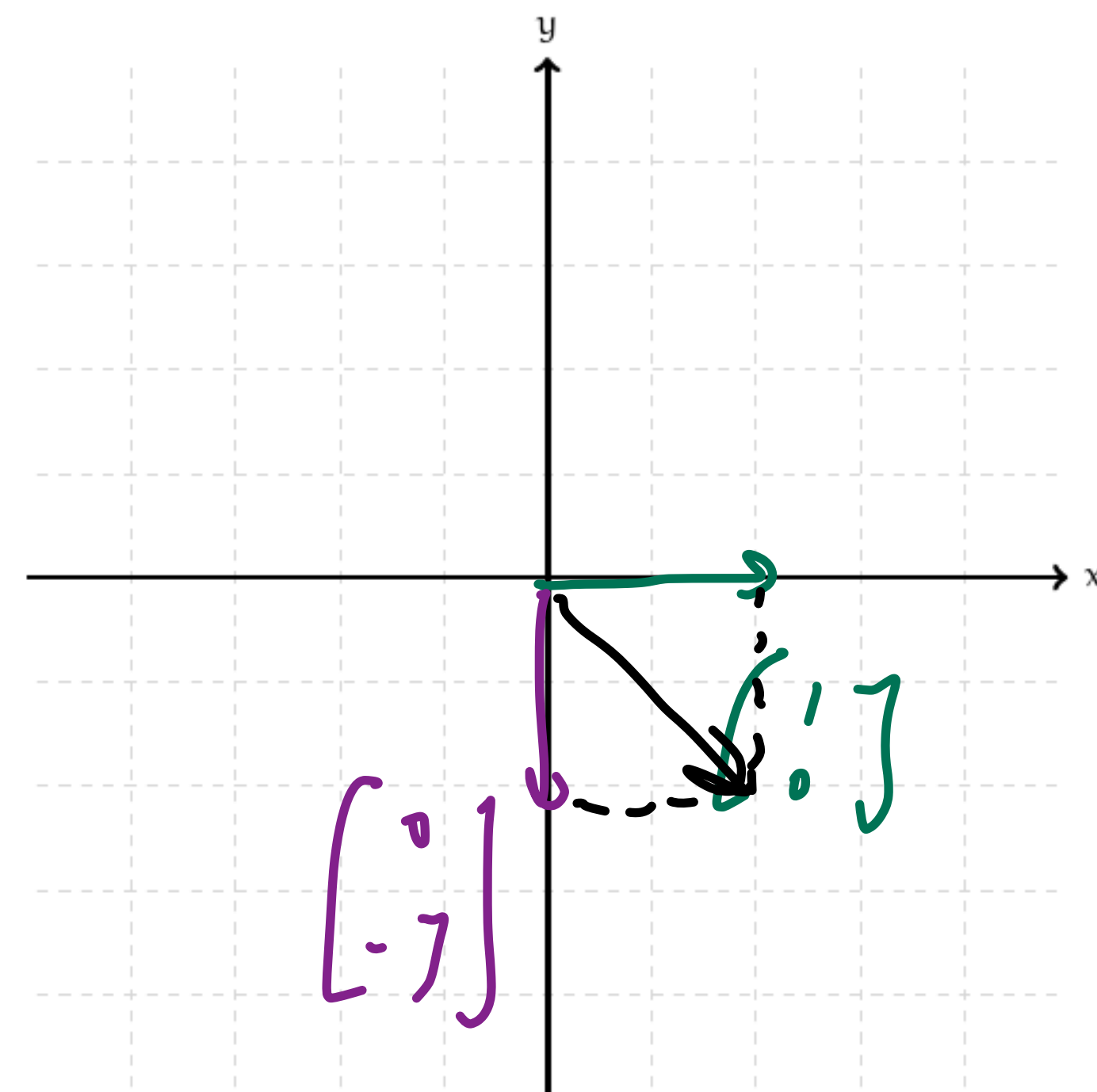
Imagine shearing like with rocks or metal.

Example: Reflection

$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

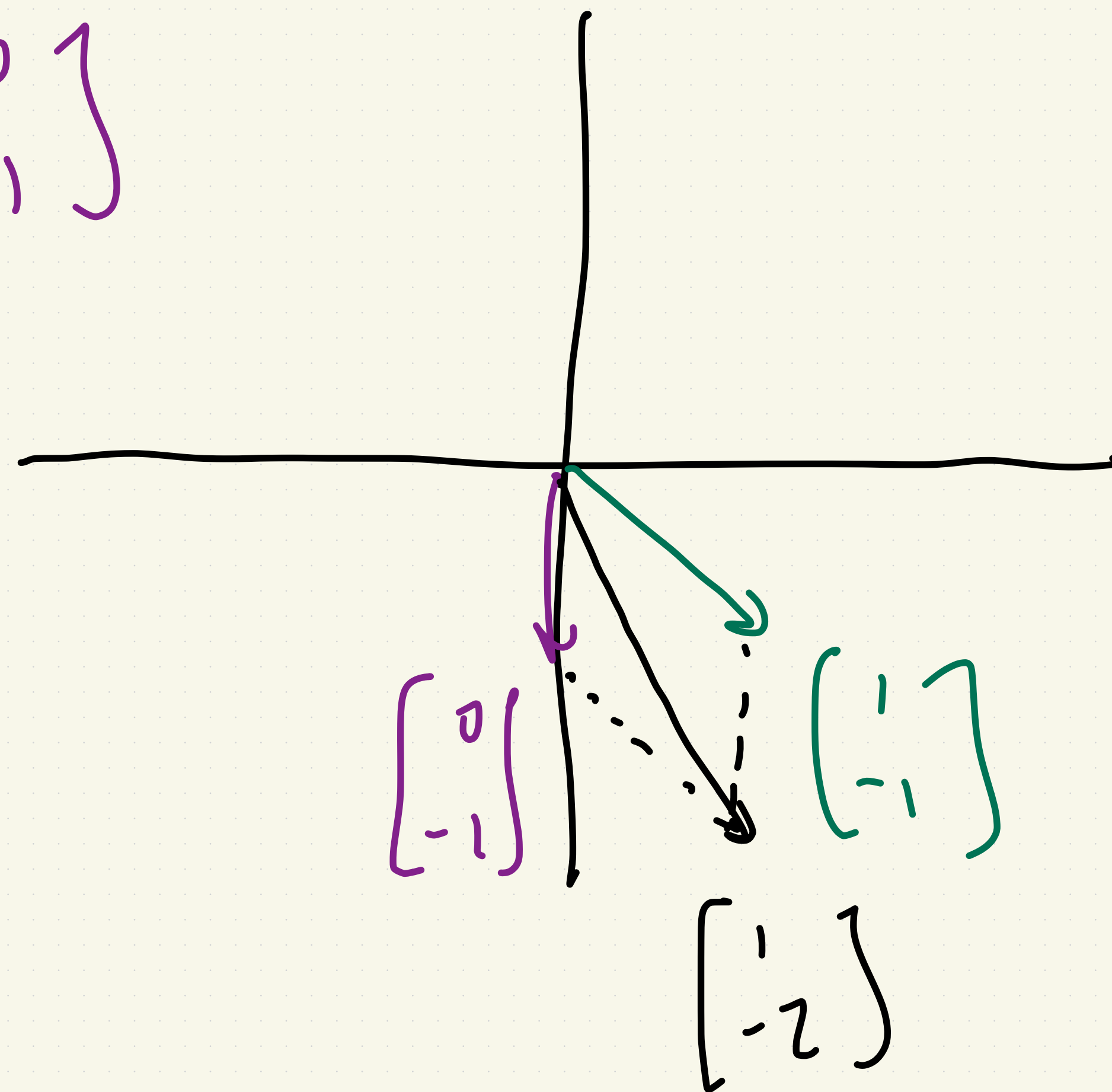
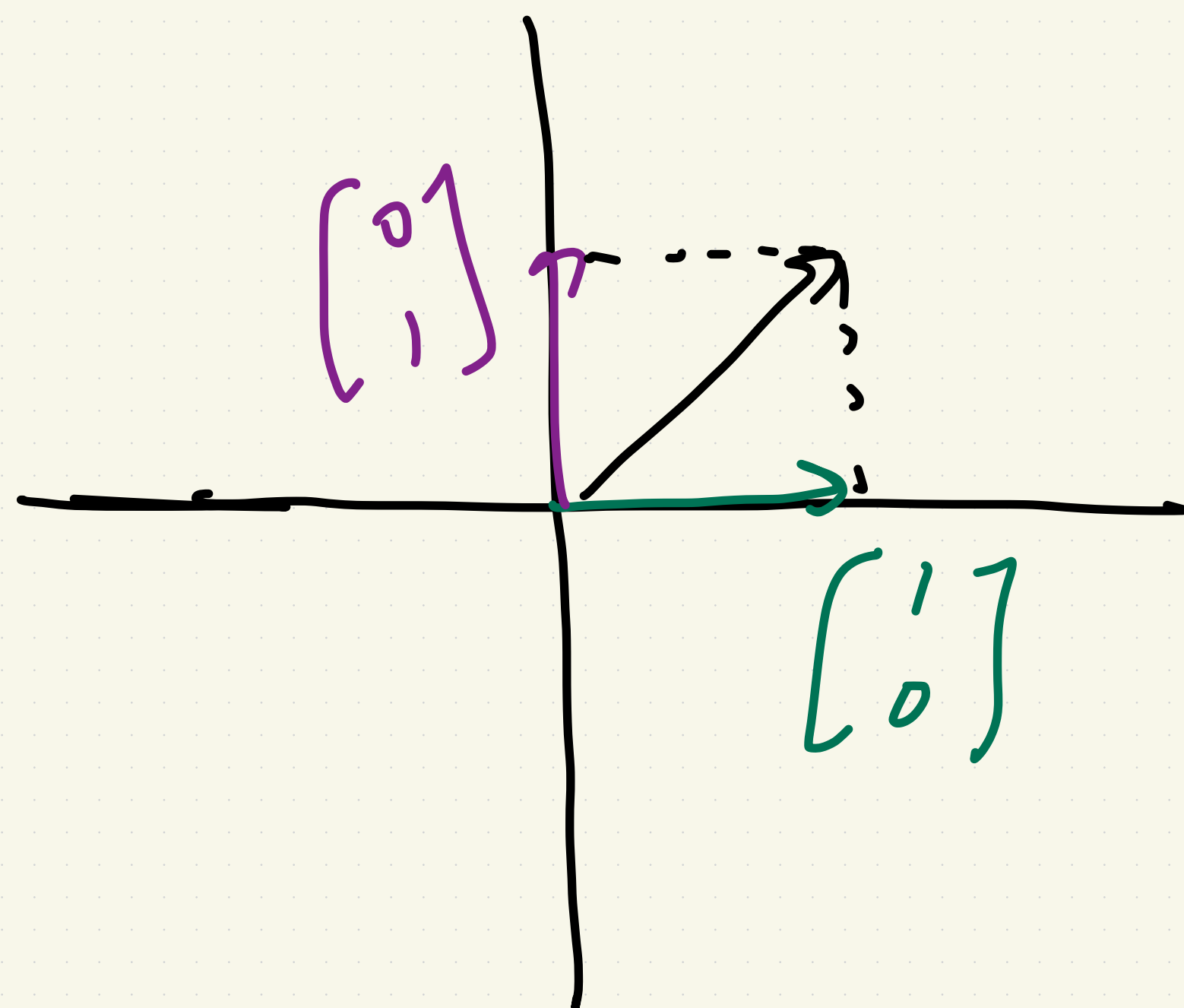


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



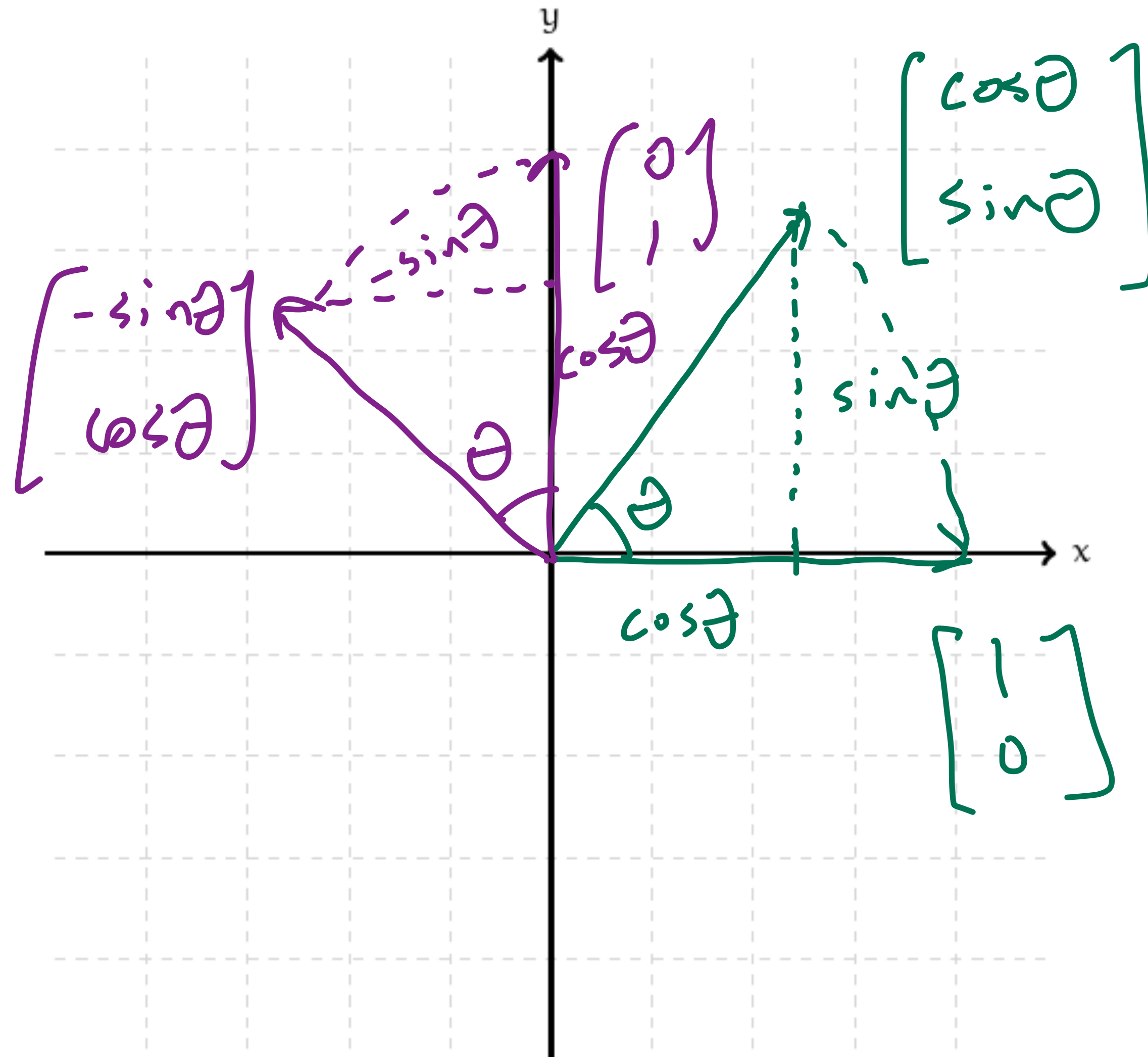
$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



General Rotation

How does rotation affect the standard basis?



$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Rotation Matrix

Rotation Matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

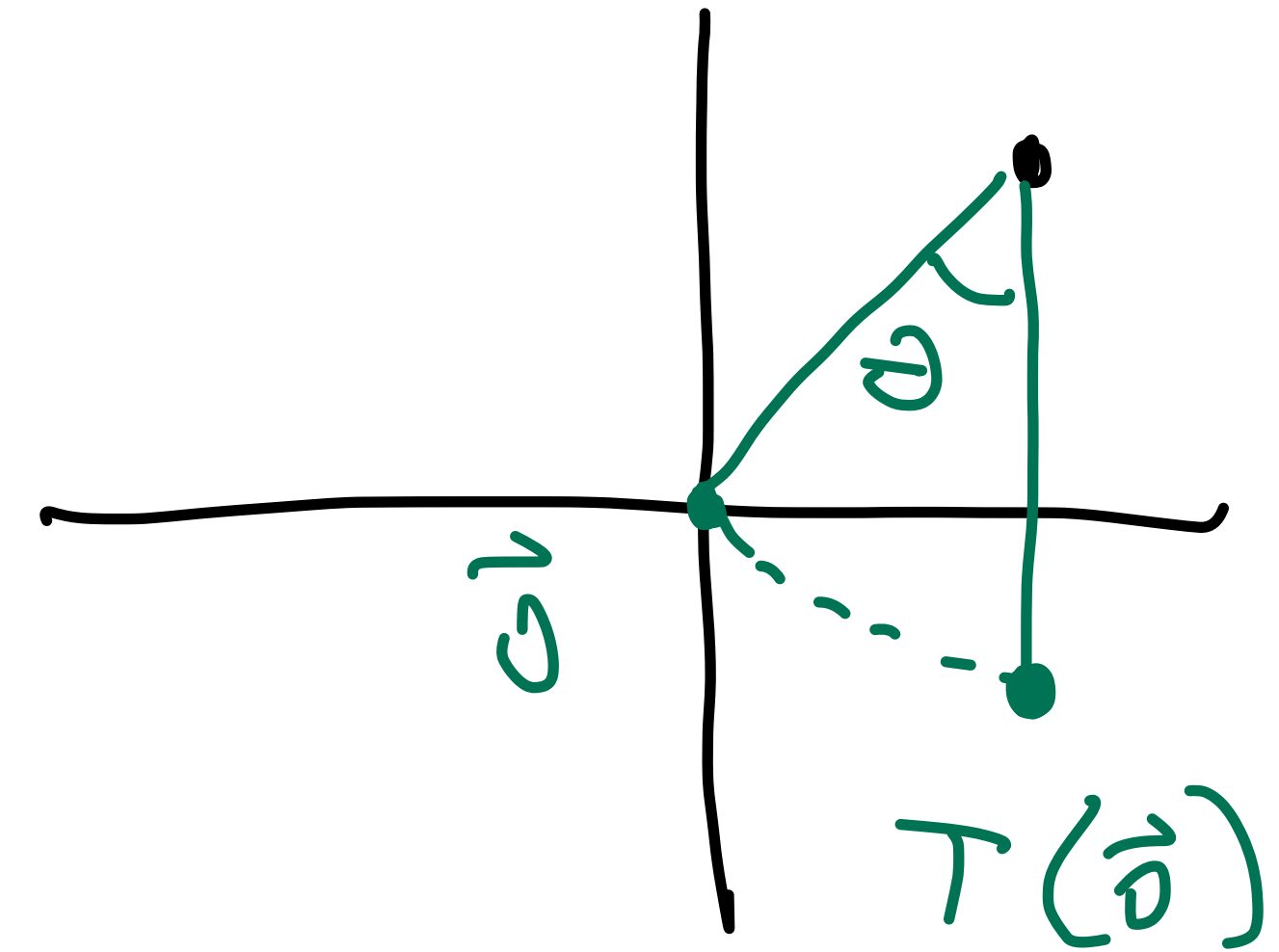
Rotation Matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Note: This is rotation about the origin.

Rotation Matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

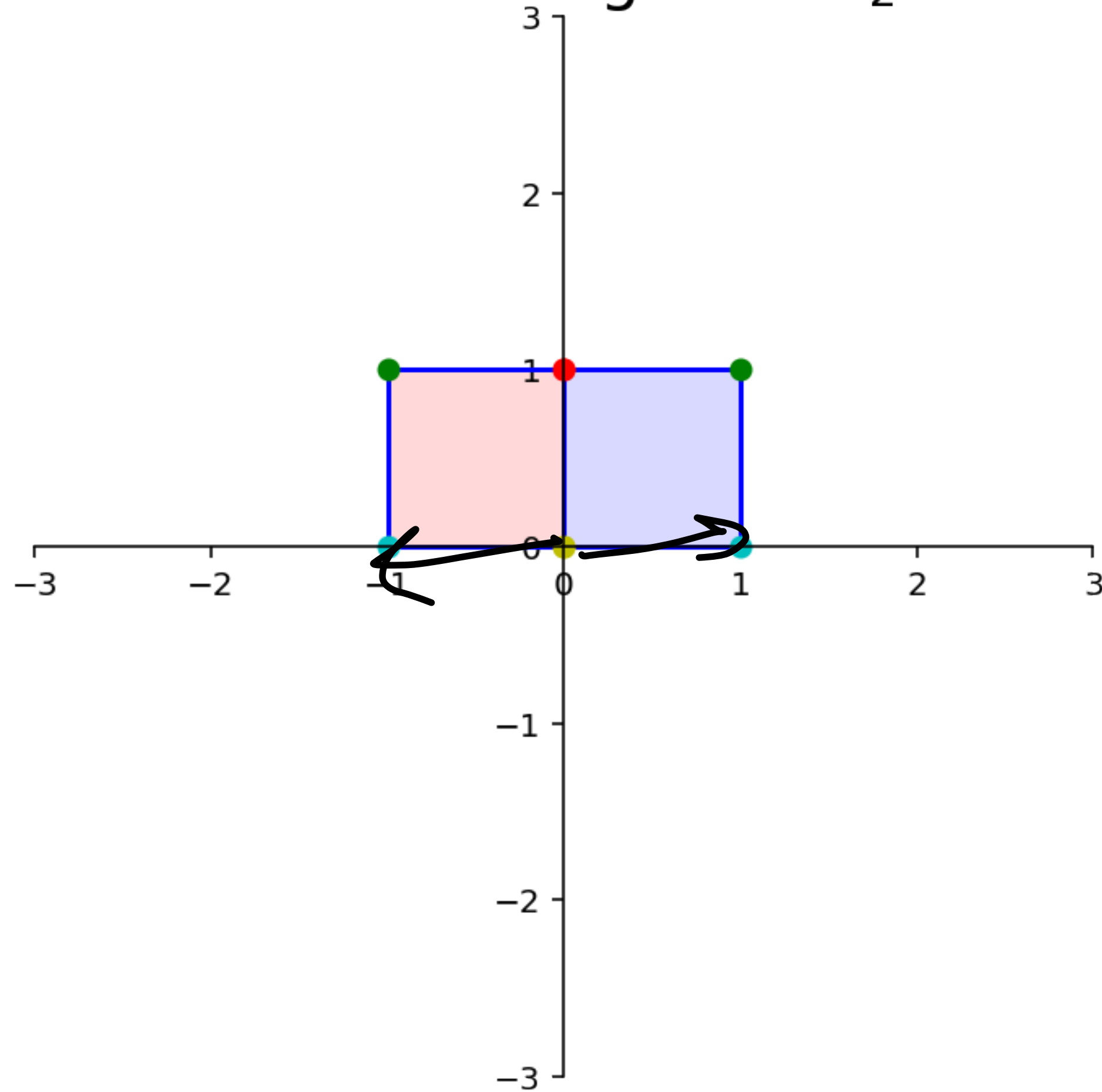


Note: This is rotation about the origin.

The Takeaway: We can figure out the matrices which implement complex linear transformations by understanding what they do to the standard basis.

Example: Reflection through the x_2 -axis

Reflection through the x_2 axis



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

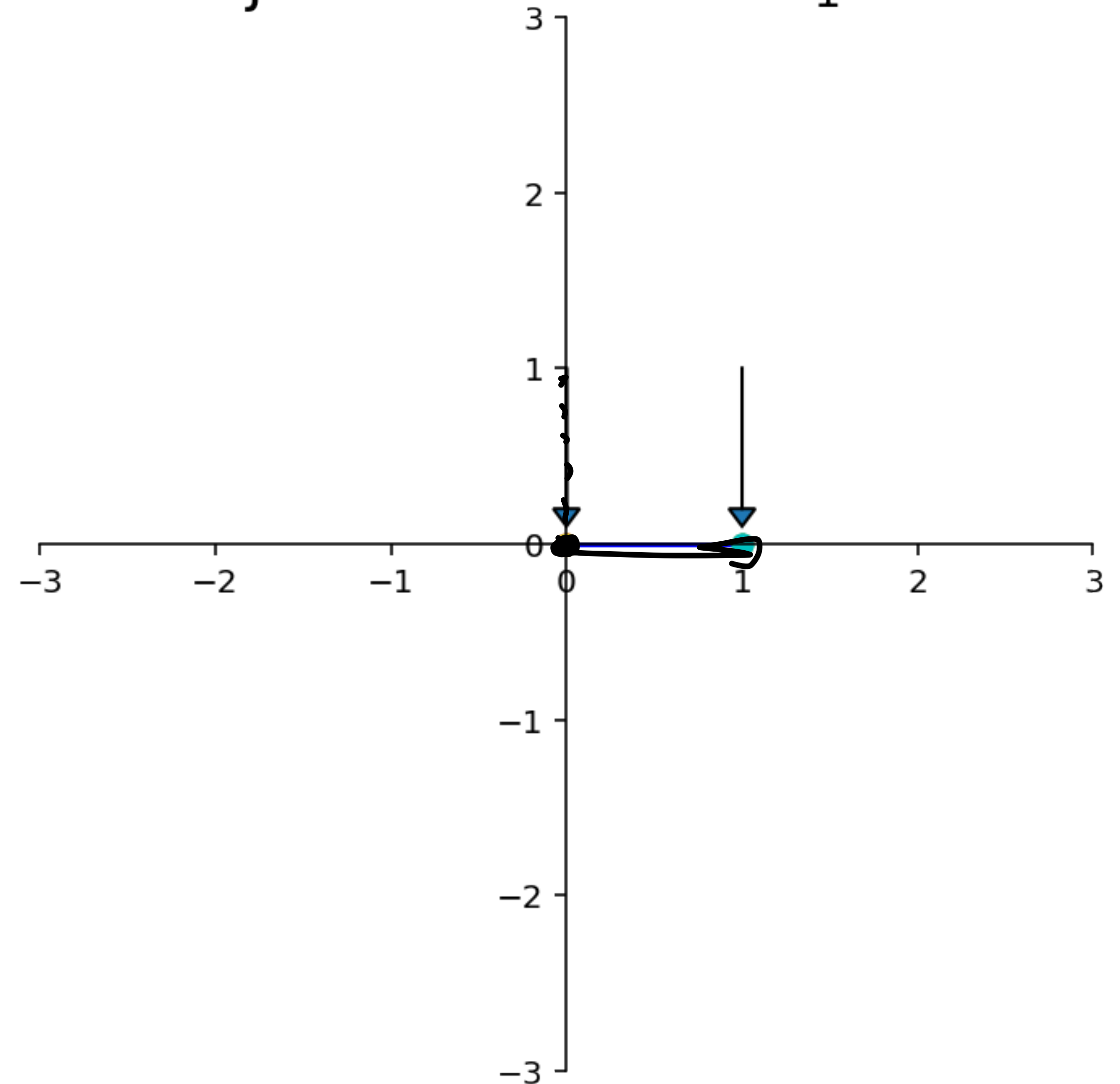
Example: Projections

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

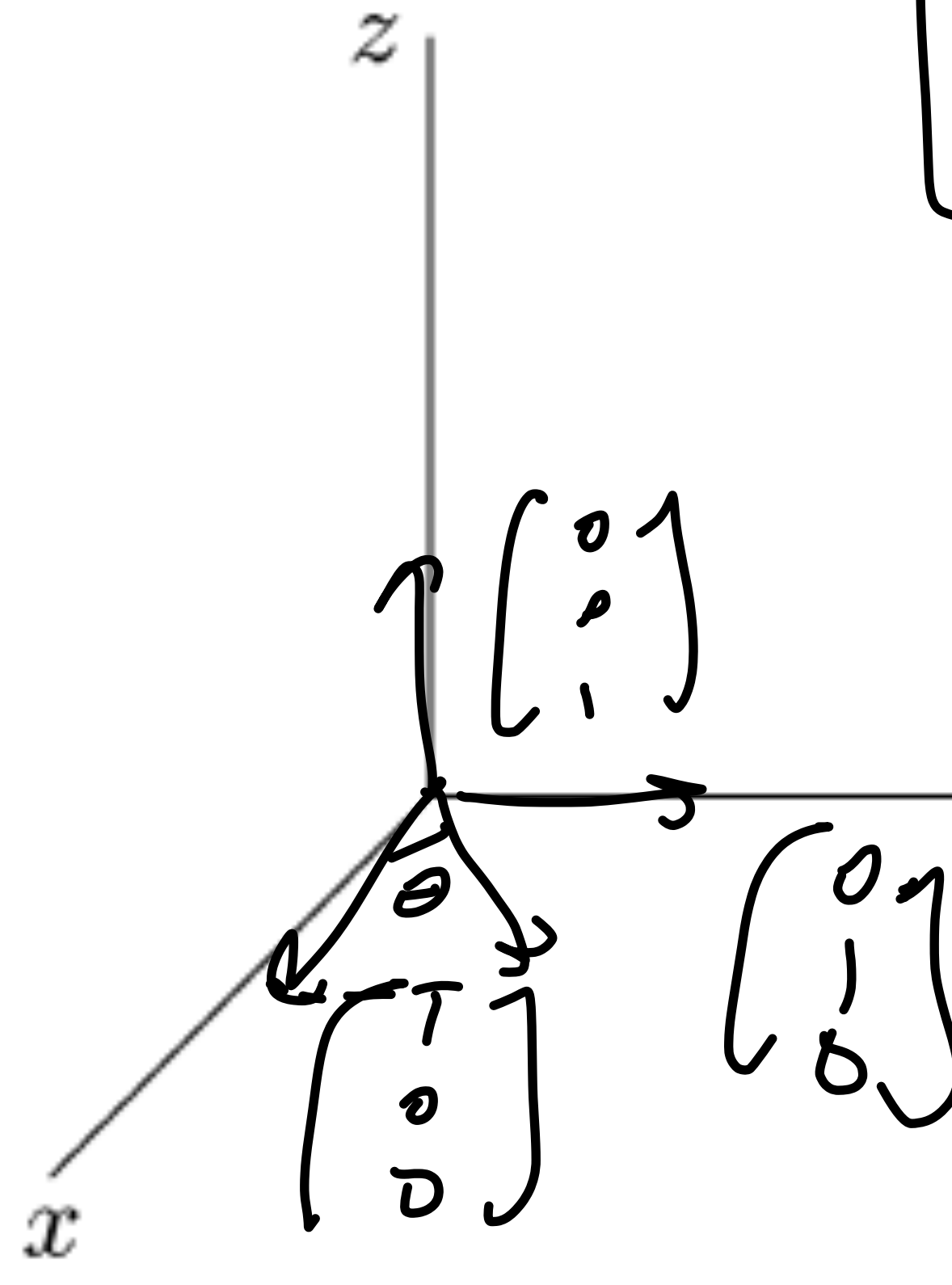
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Projection onto the x_1 axis

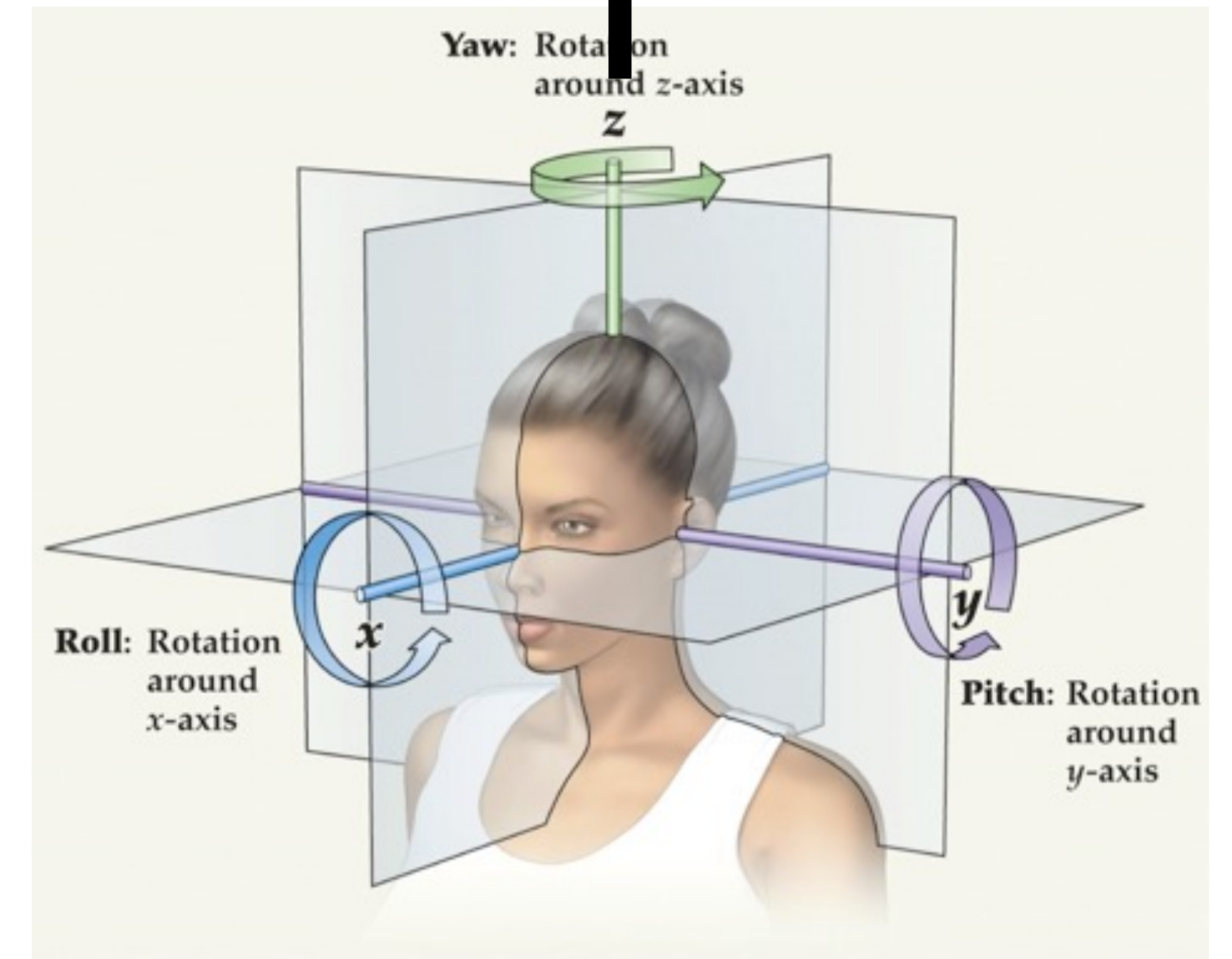


3D Example: Rotation about the x_3 -Axis (z-Axis)

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \theta \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

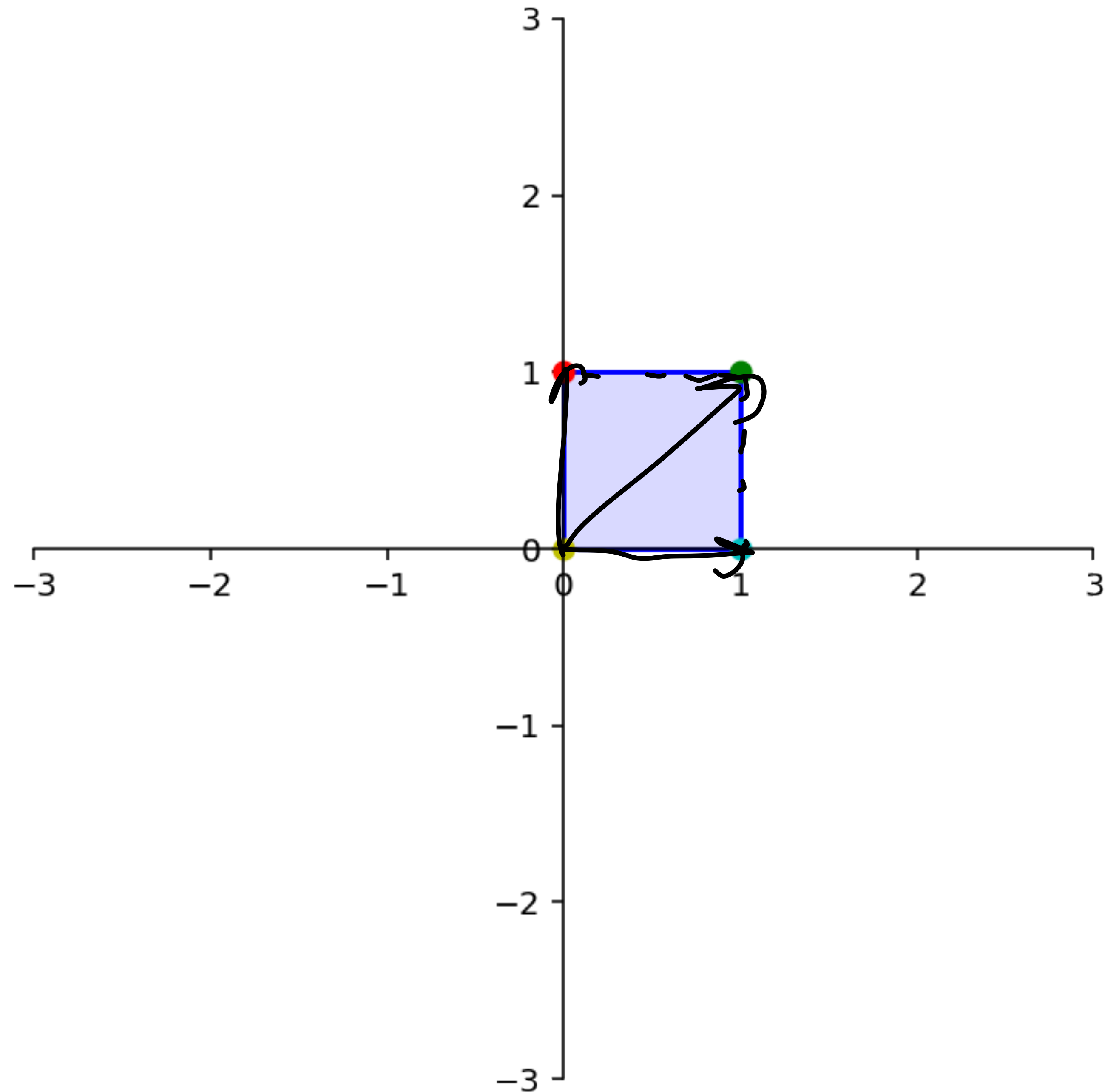


$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



The Unit Square

The *unit square* is the set of points in \mathbb{R}^2 enclosed by the points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.



How To: The Unit Square and Matrices

How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture.

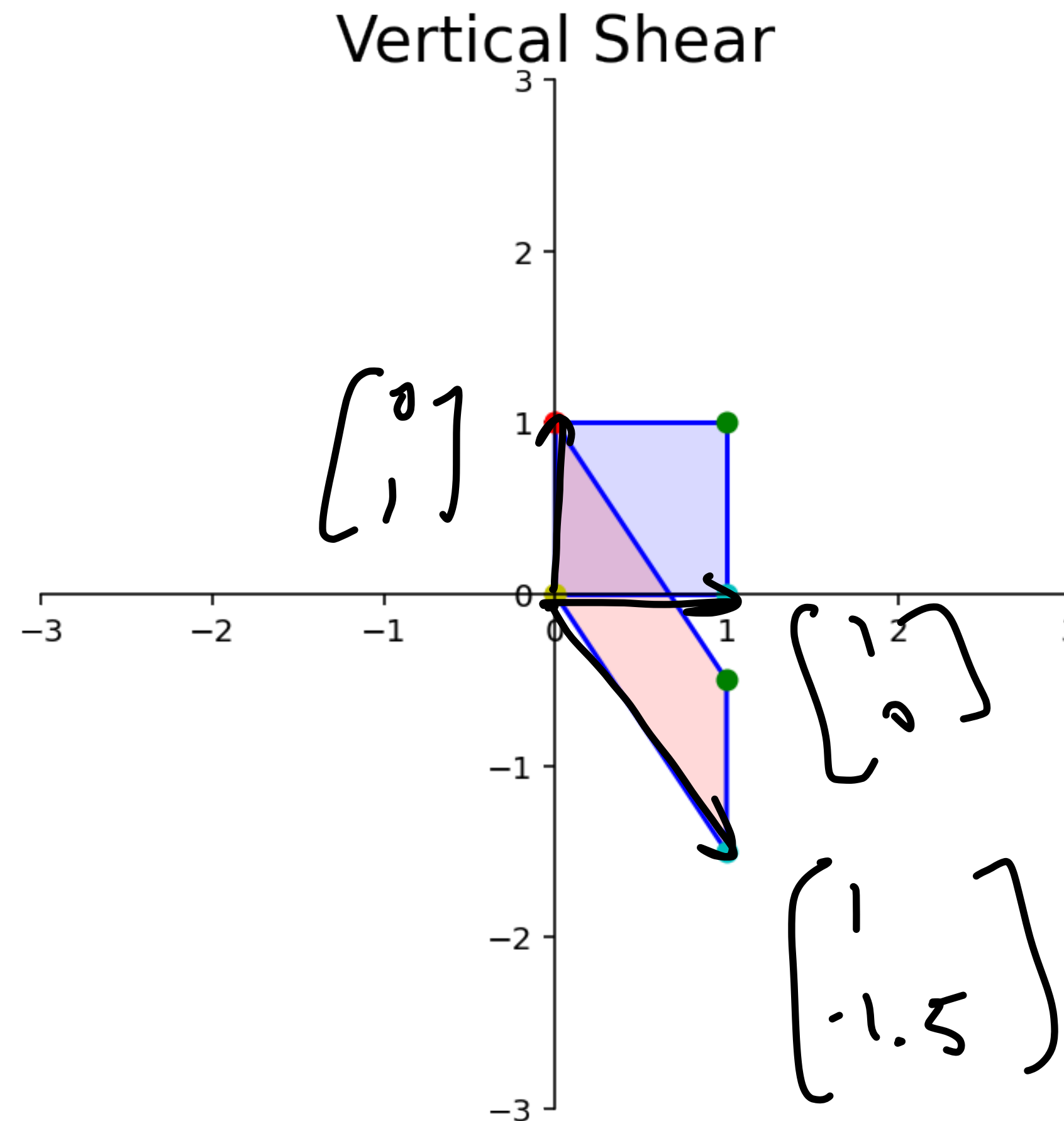
How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture.

Solution. Find where the standard basis vectors go.

Question

Write down the matrix for the following shearing operation using this method.

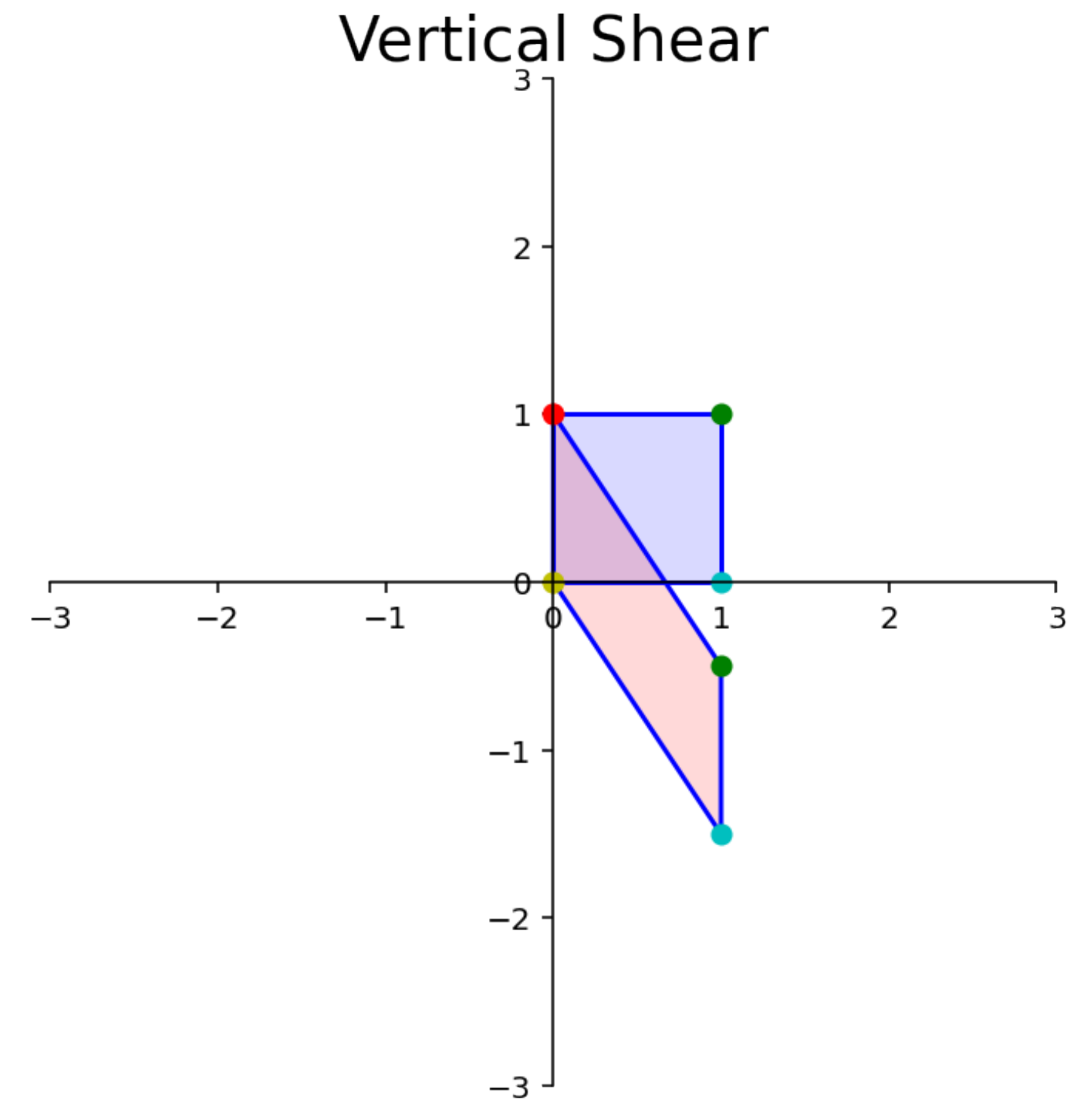


$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}$$

Answer



You need to **know** these matrices, but you don't need to memorize them.

Remember: What does this matrix do to the unit square? Then build the matrix from there.

List of Important 2D Linear Transformations

- » dilation, contraction
- » reflections
- » projections
- » horizontal/vertical contractions
- » horizontal/vertical shearing

Look through the notes for a comprehensive
collection of pictures or...

One-to-One and Onto

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

What about other questions?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Wait

$A\mathbf{x} = \mathbf{0}$ has a
unique solution

$\equiv A\mathbf{x} = \mathbf{b}$ has at most one
solution

why?:

$$\vec{r} \neq \vec{0}$$

$$A\vec{r} = \vec{0}$$

$$A\vec{w} = \vec{b}$$

$$A(\vec{w} + \vec{r}) = \vec{b}$$

$$A(\vec{w} + \vec{r}) = A\vec{w} + A\vec{r} = A\vec{w} + \vec{0} = \vec{b}$$

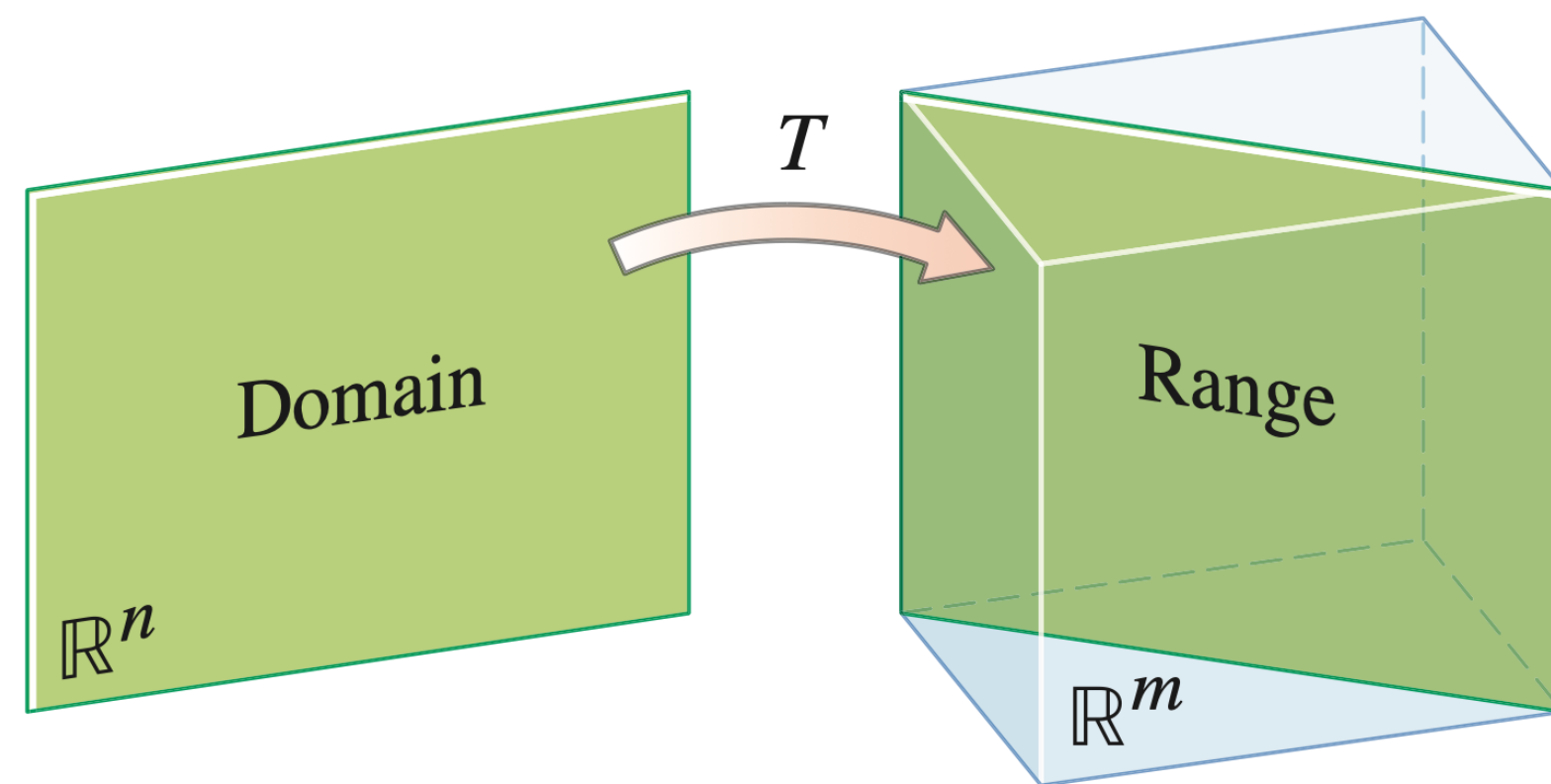
Onto Transformations

Onto Transformations

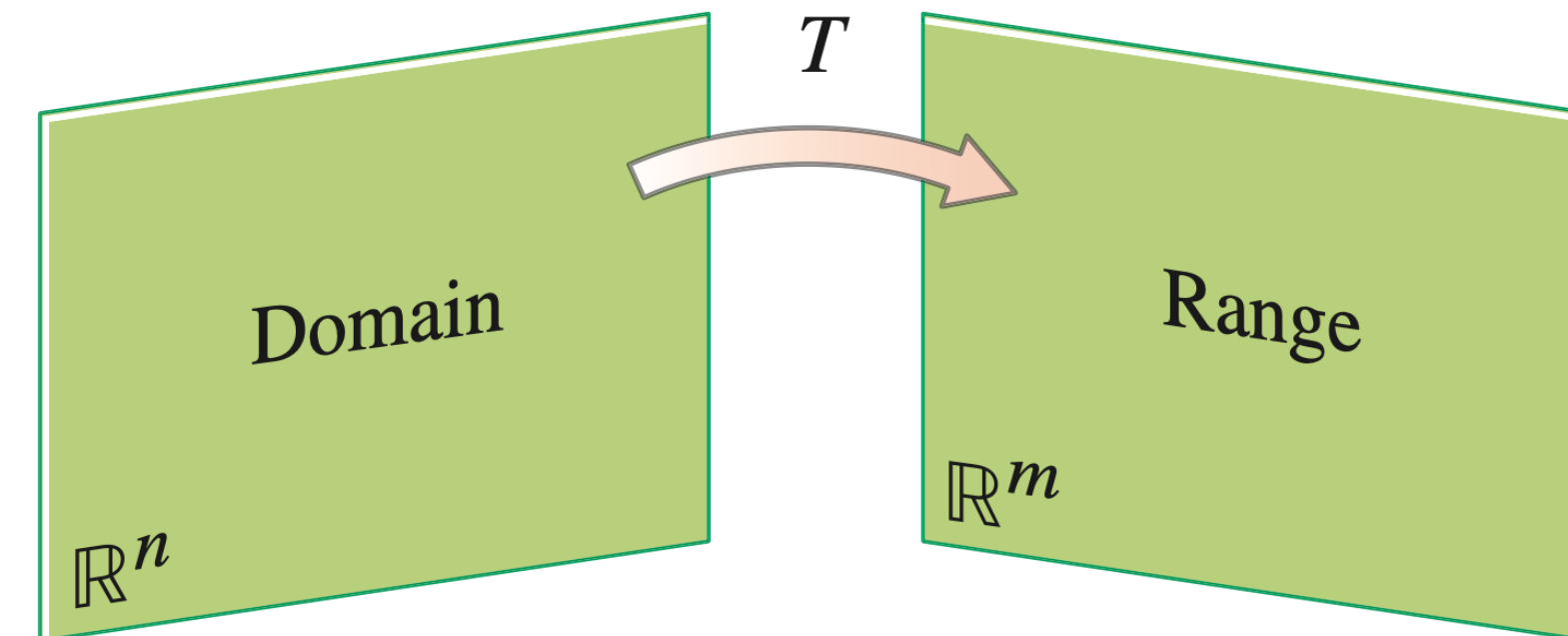
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

Onto Transformations

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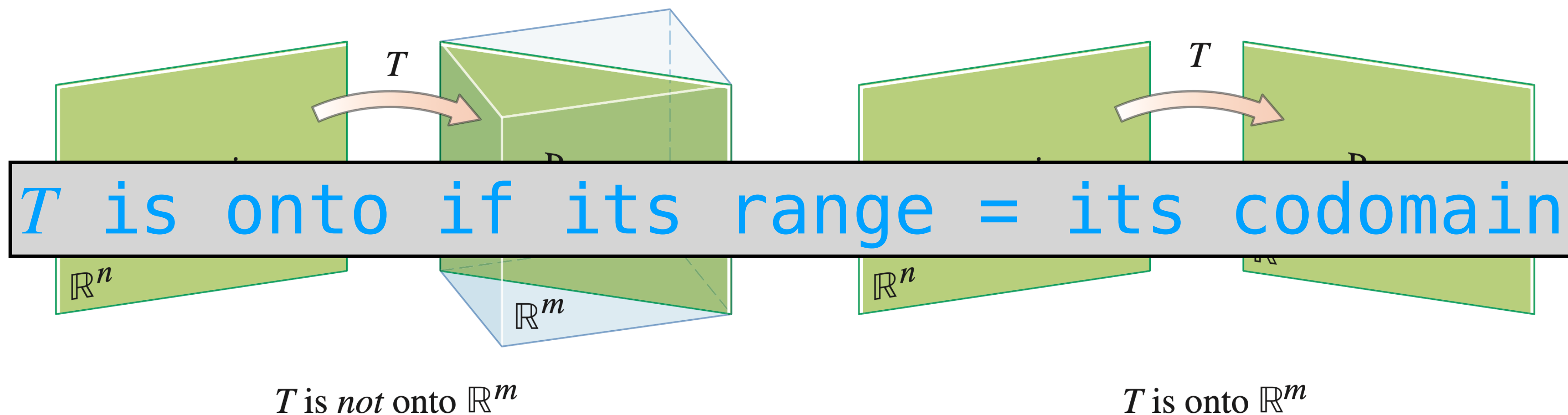
T is not onto \mathbb{R}^m



T is onto \mathbb{R}^m

Onto Transformations

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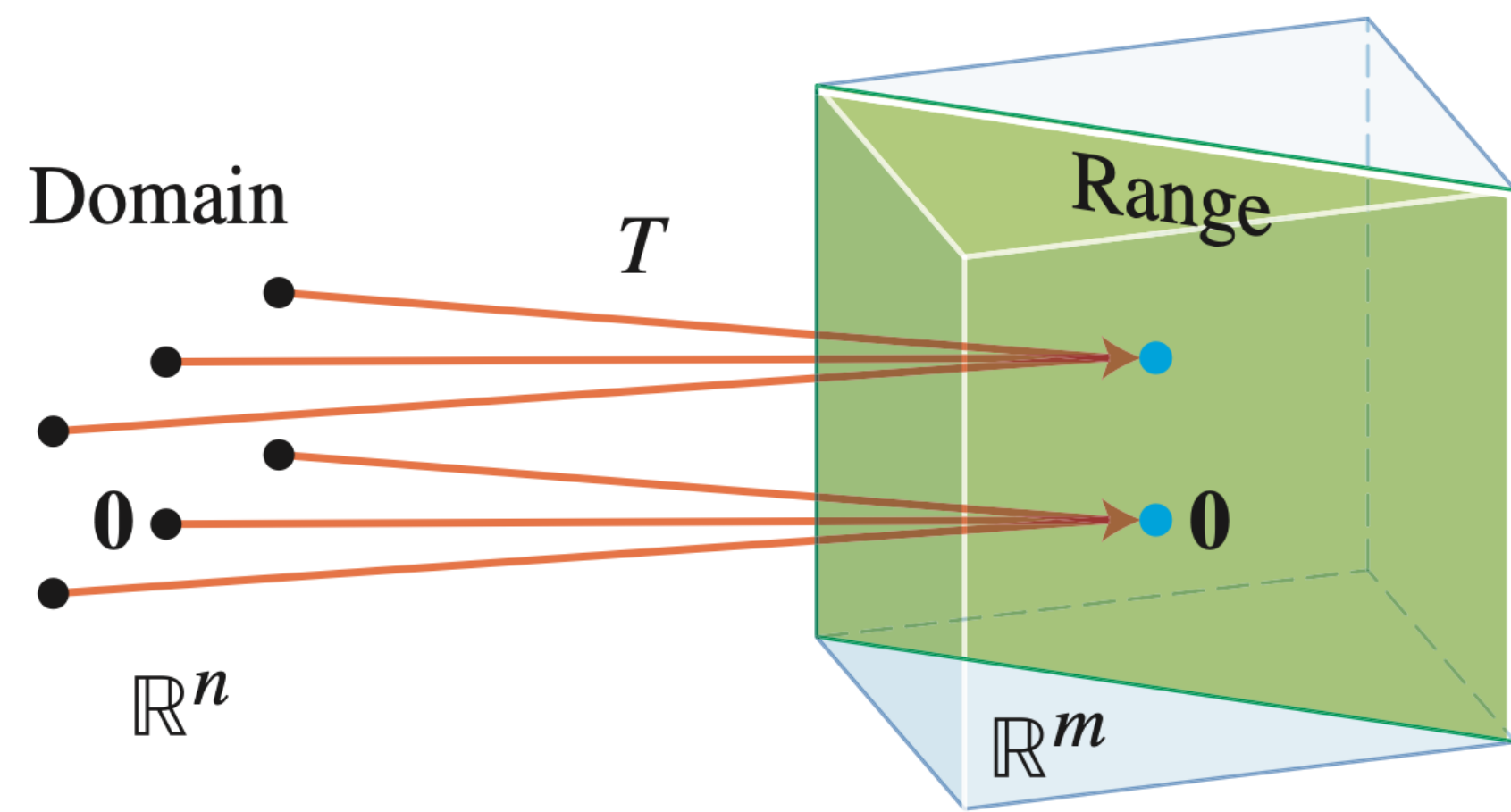
One-to-one Transformations

One-to-one Transformations

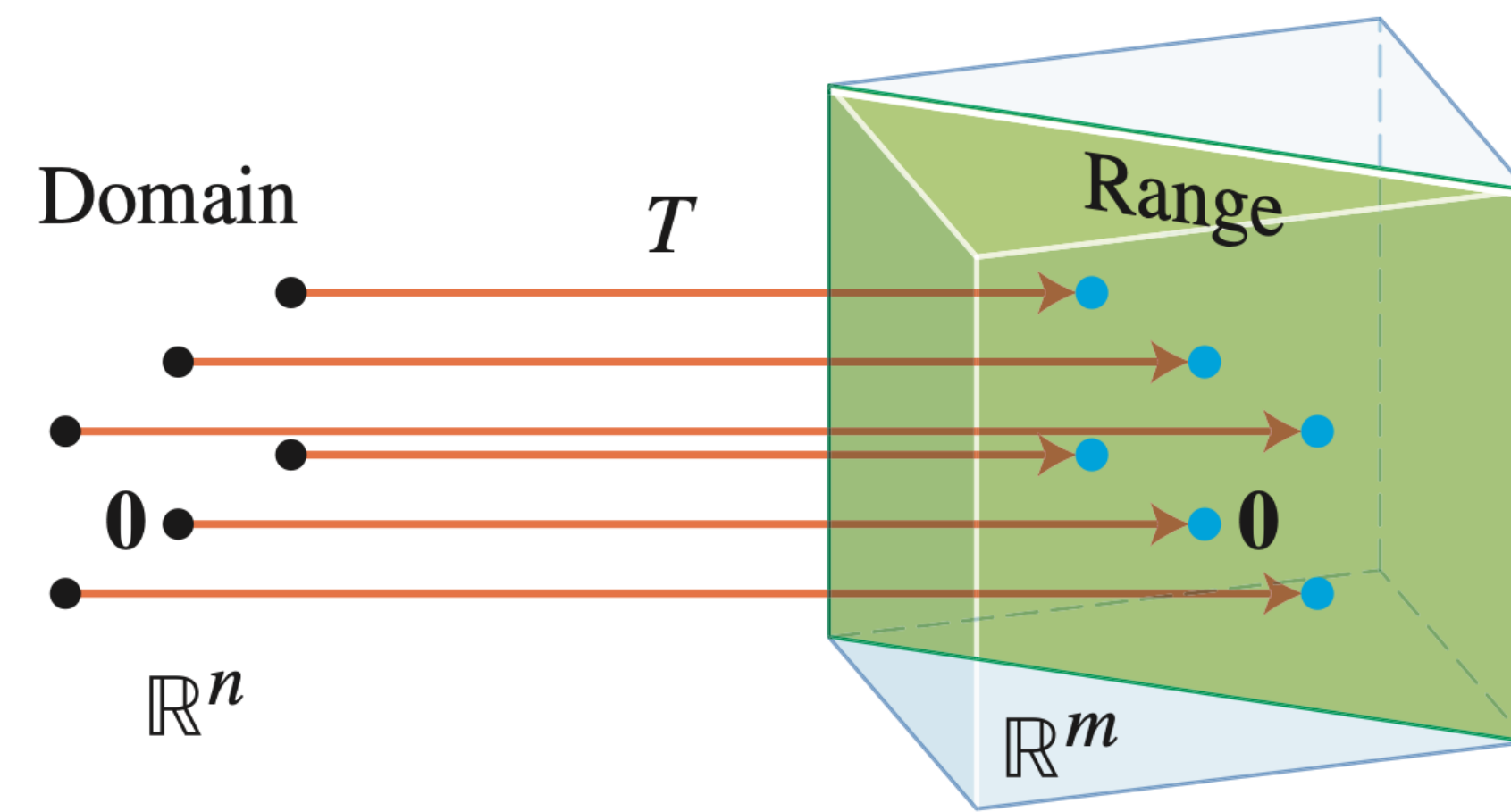
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

One-to-one Transformations

Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

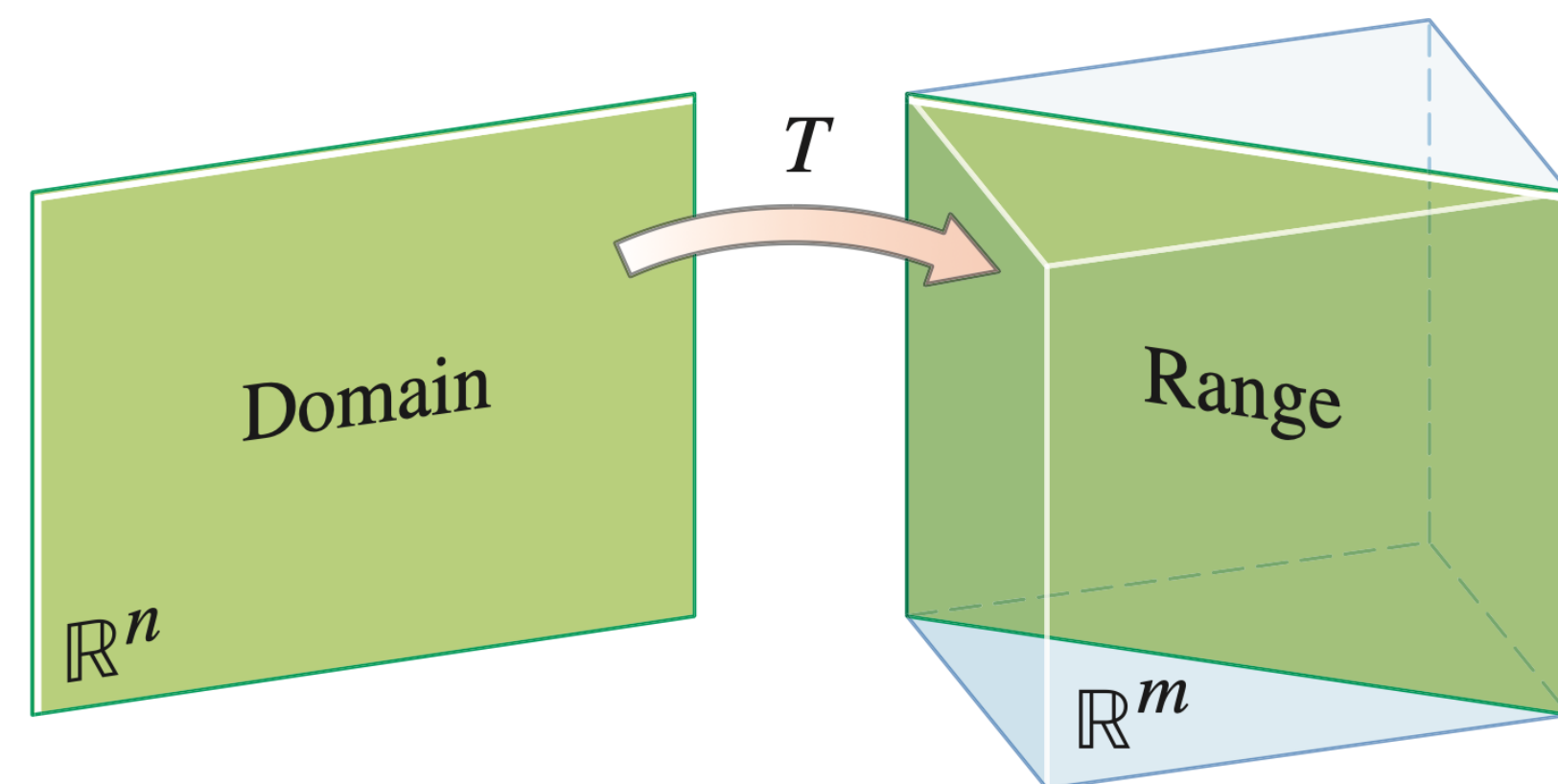


T is *not* one-to-one

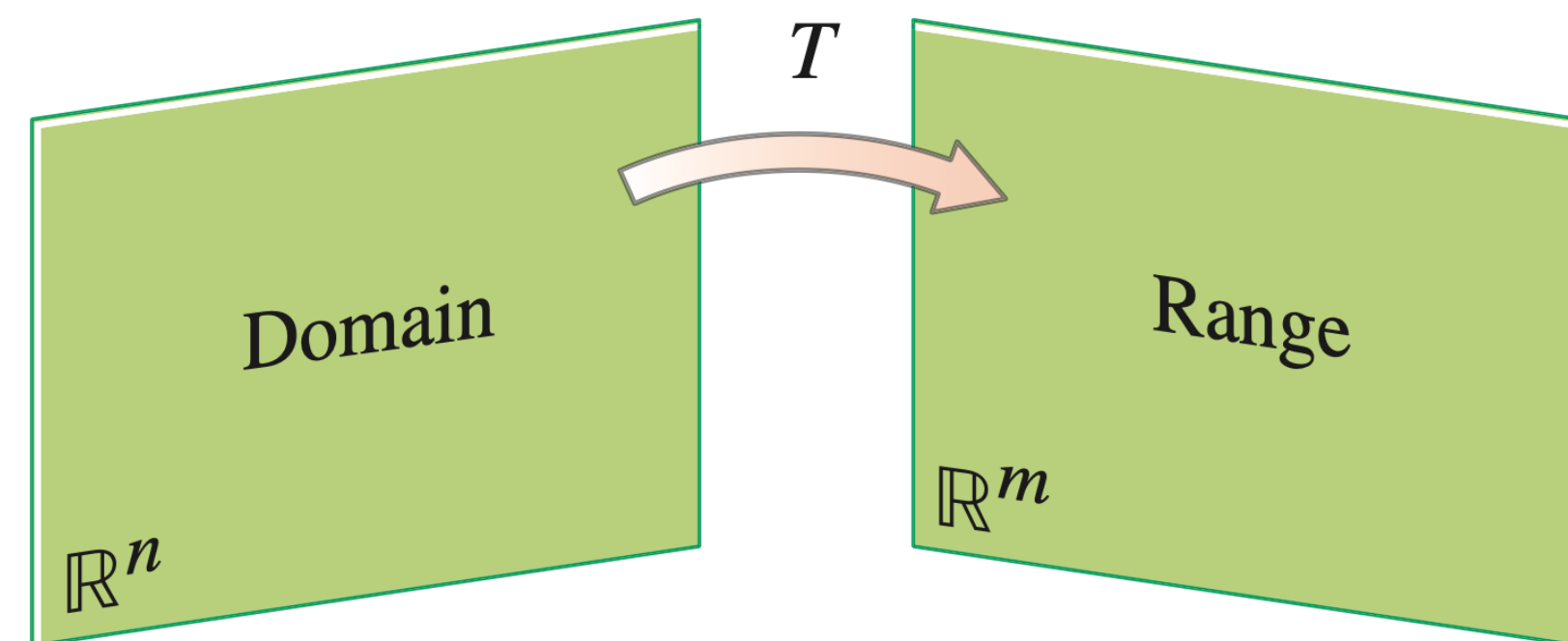


T is one-to-one

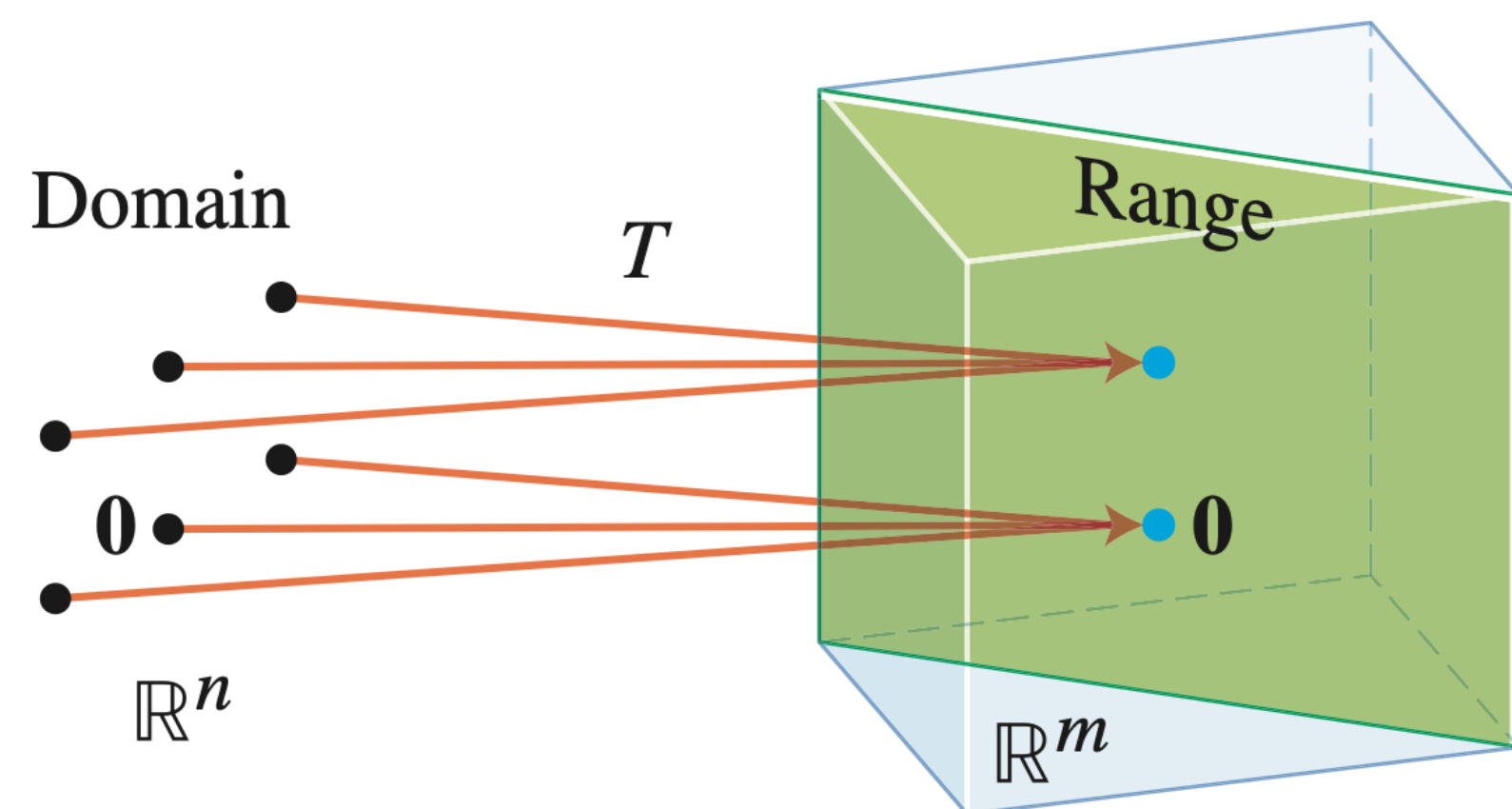
Comparing Pictures



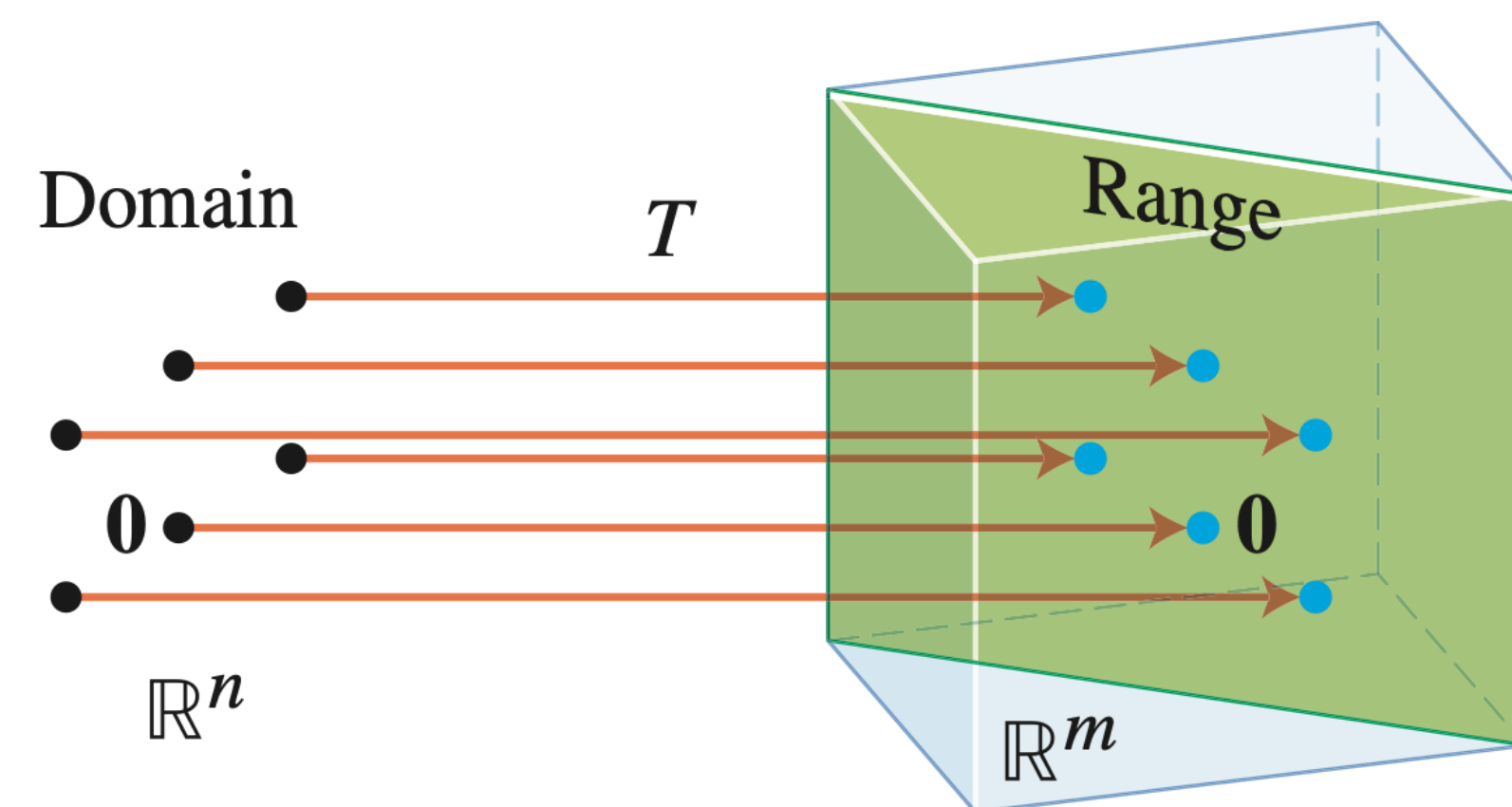
T is not onto \mathbb{R}^m



T is onto \mathbb{R}^m



T is not one-to-one



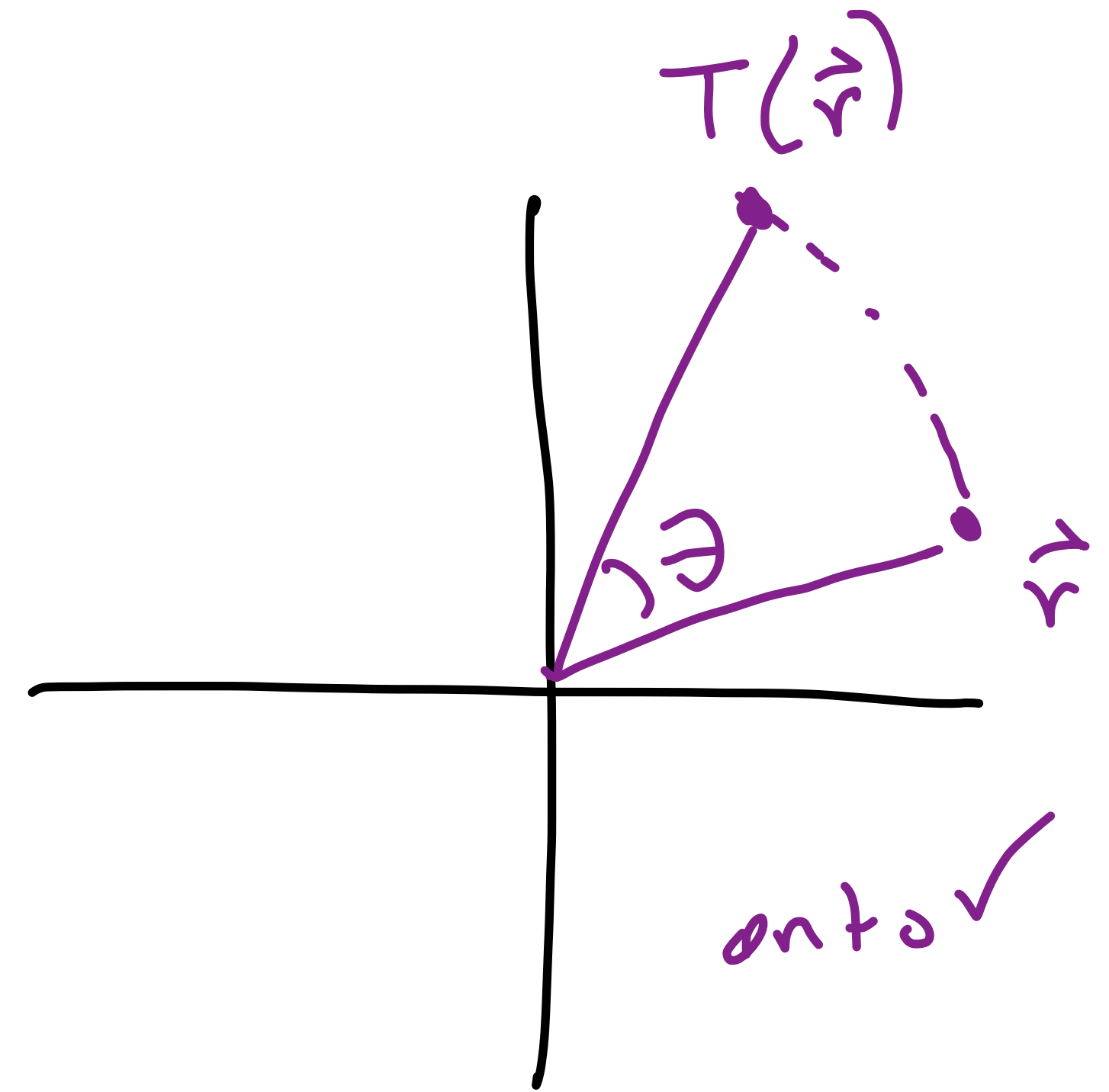
T is one-to-one

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :



Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \mapsto \begin{bmatrix} x_1' \\ x_2' \\ x_1' + x_2' \end{bmatrix}$$

why?: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$\notin \text{ran}(T)$
not onto

Example: not 1-1, not onto

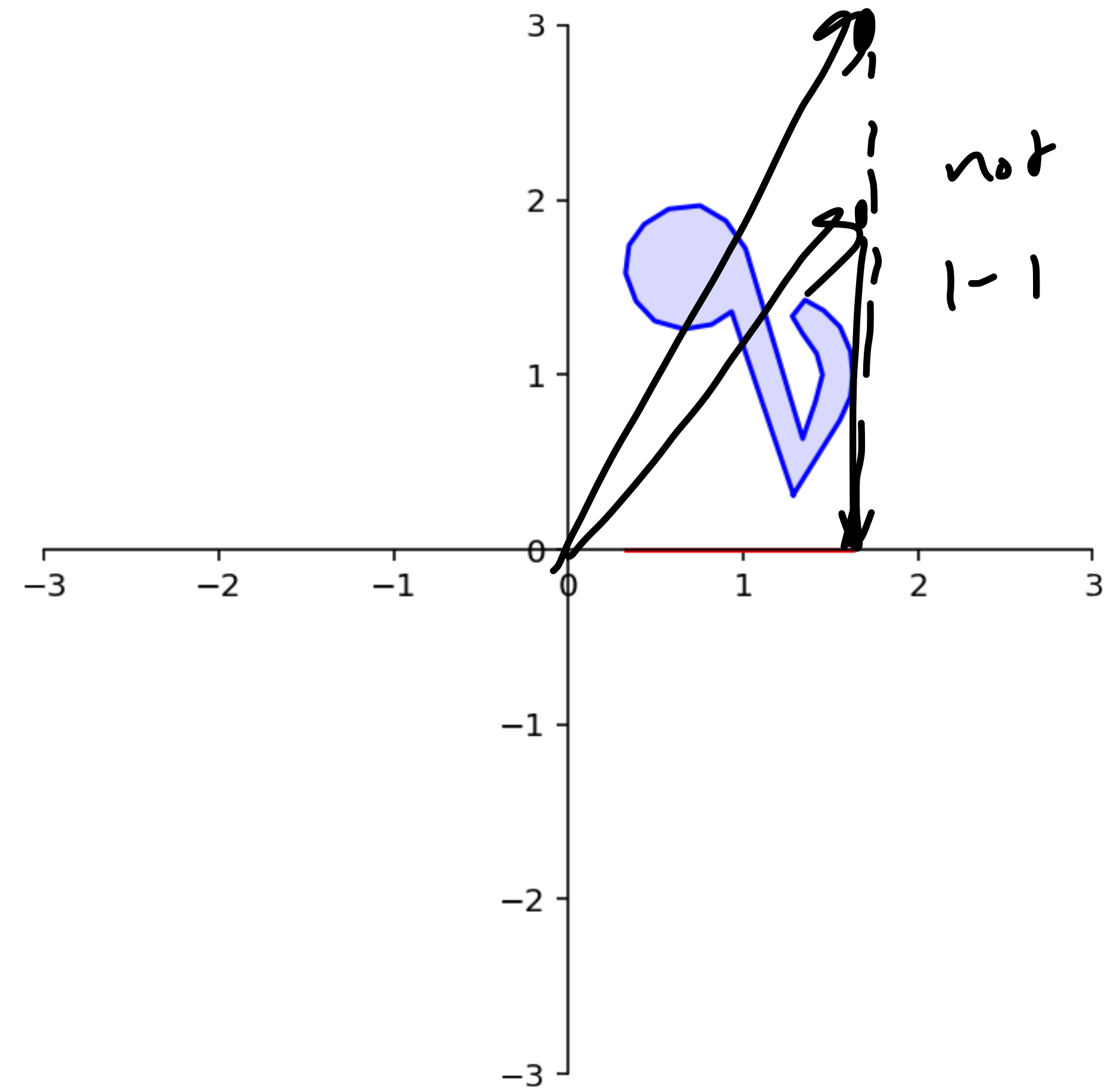
Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

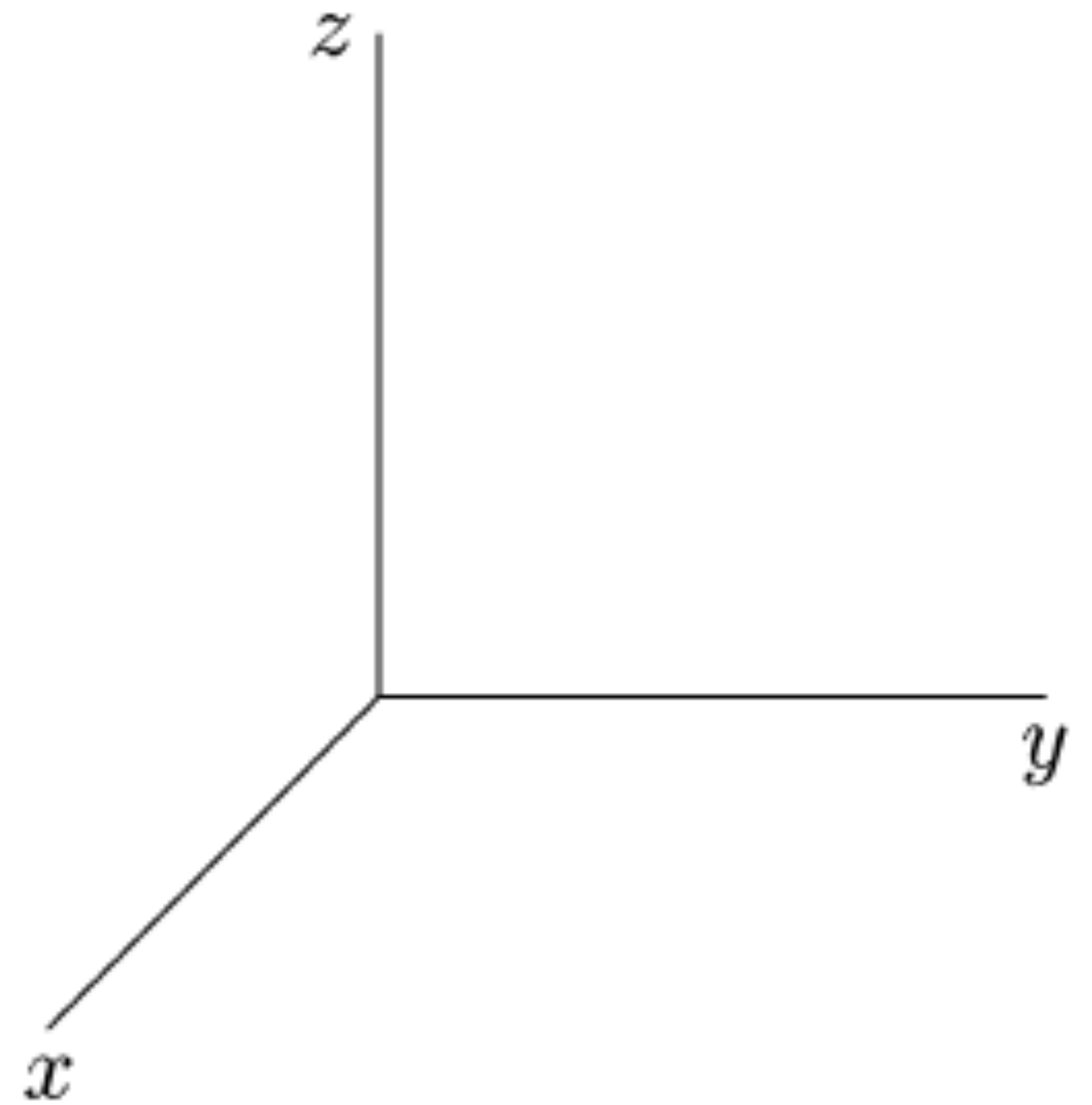
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \notin \text{row}(T)$$

not onto



Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .



$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

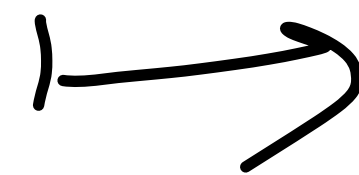
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why?: $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ ↗

not 1-1

collision.

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} x \\ y \end{bmatrix}$$

onto ✓

Taking Stock: Onto

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A .

- » T is onto
- » $A\mathbf{x} = \mathbf{b}$ has a solution for any choice of \mathbf{b}
- » $\text{range}(T) = \text{codomain}(T)$
- » the columns of A span \mathbb{R}^m
- » A has a pivot position in every row

Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A .

- » T is one-to-one
- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
- » $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- » The columns of A are linearly independent
- » A has a pivot position in every column

How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto.

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row.

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

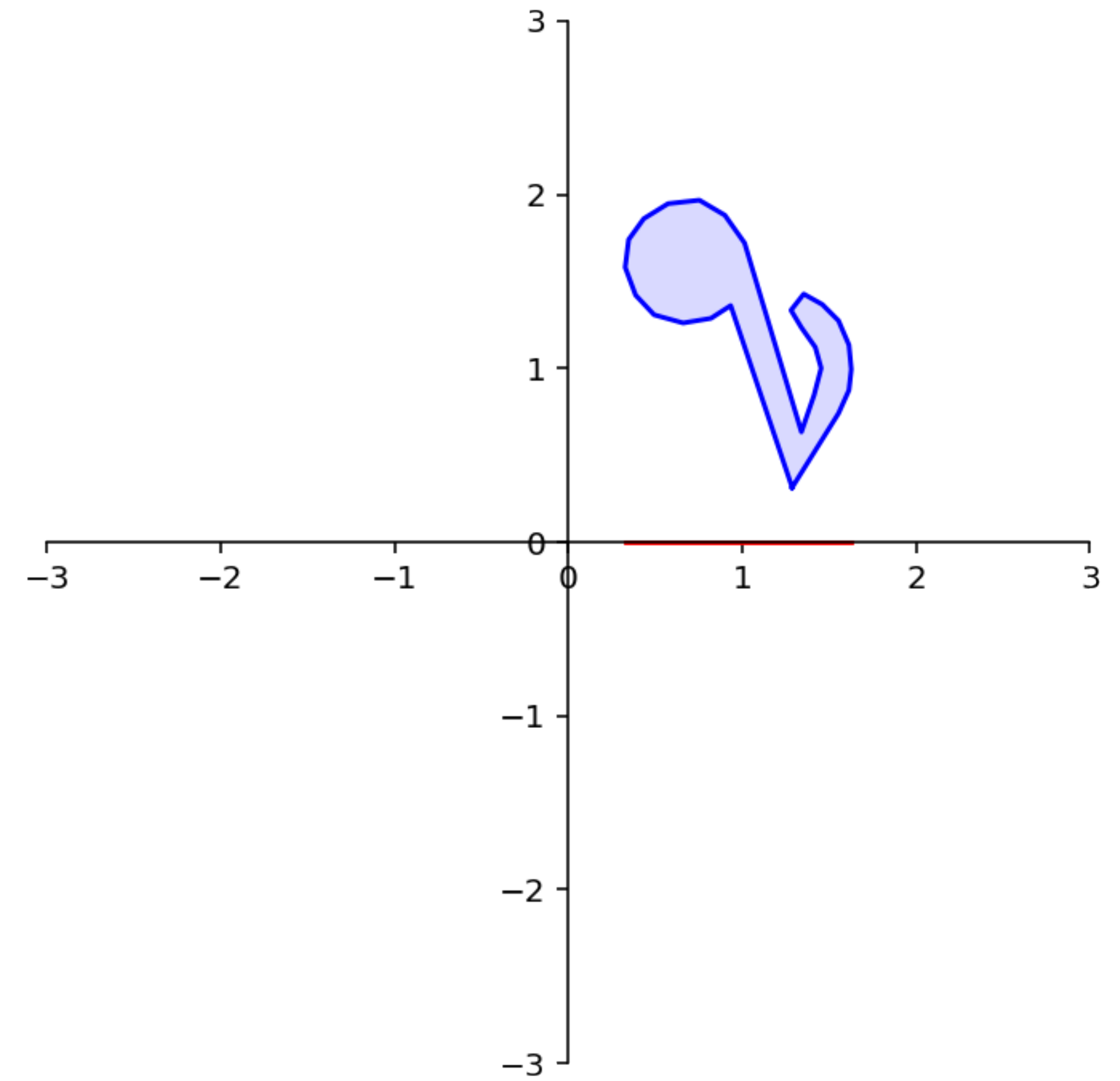
why? :

Example: not 1-1, not onto

Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

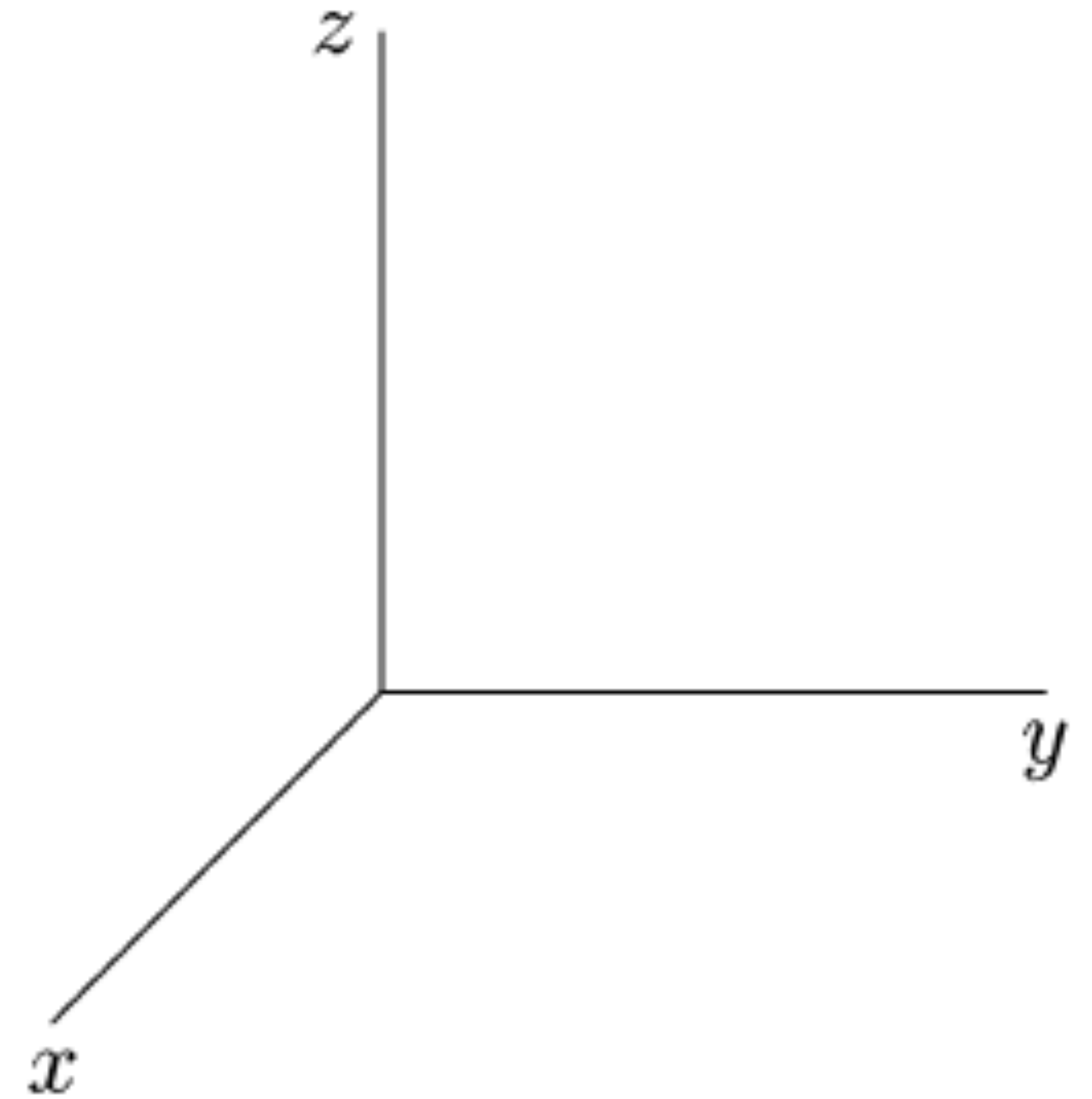


Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

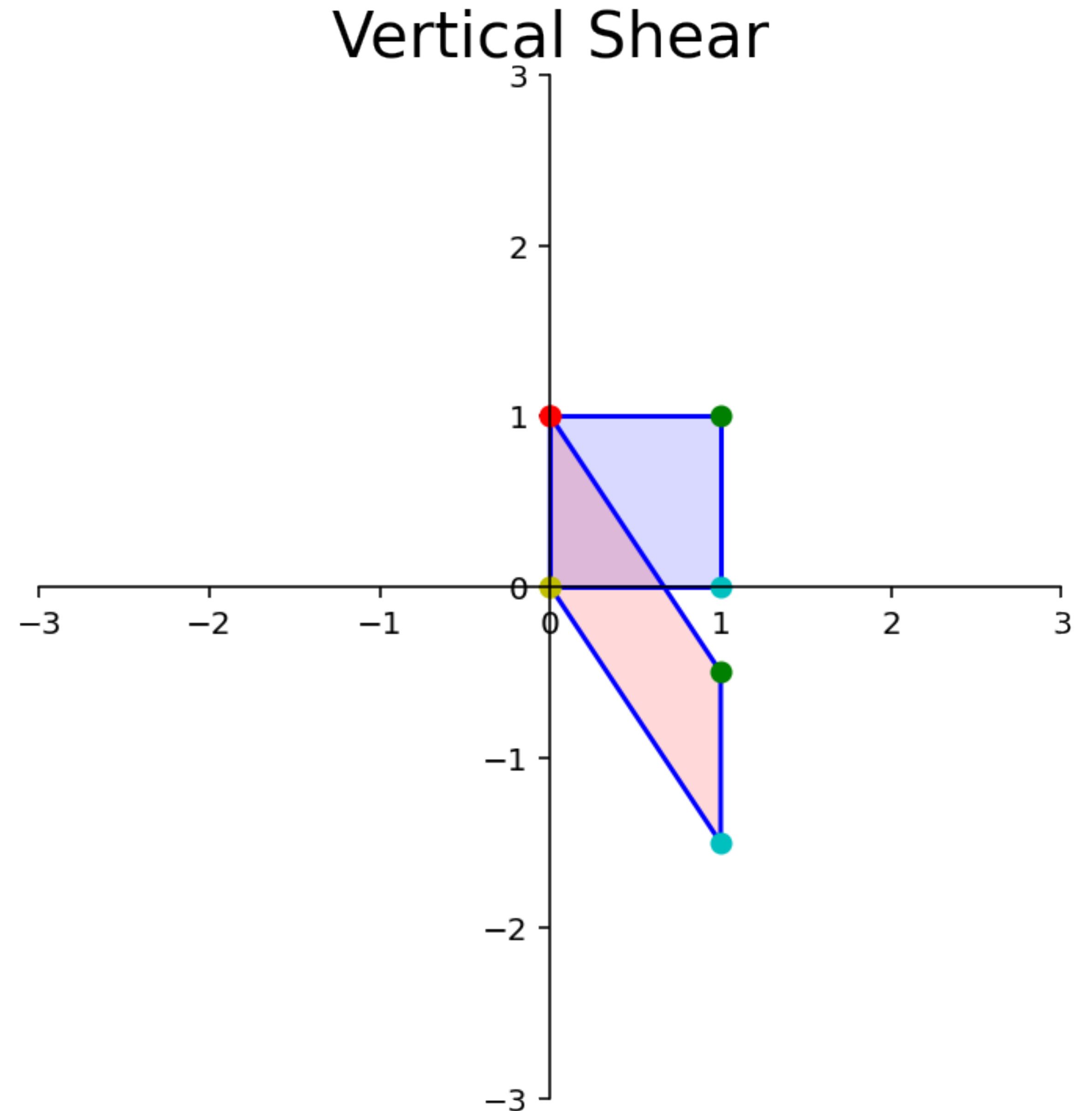
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why? :

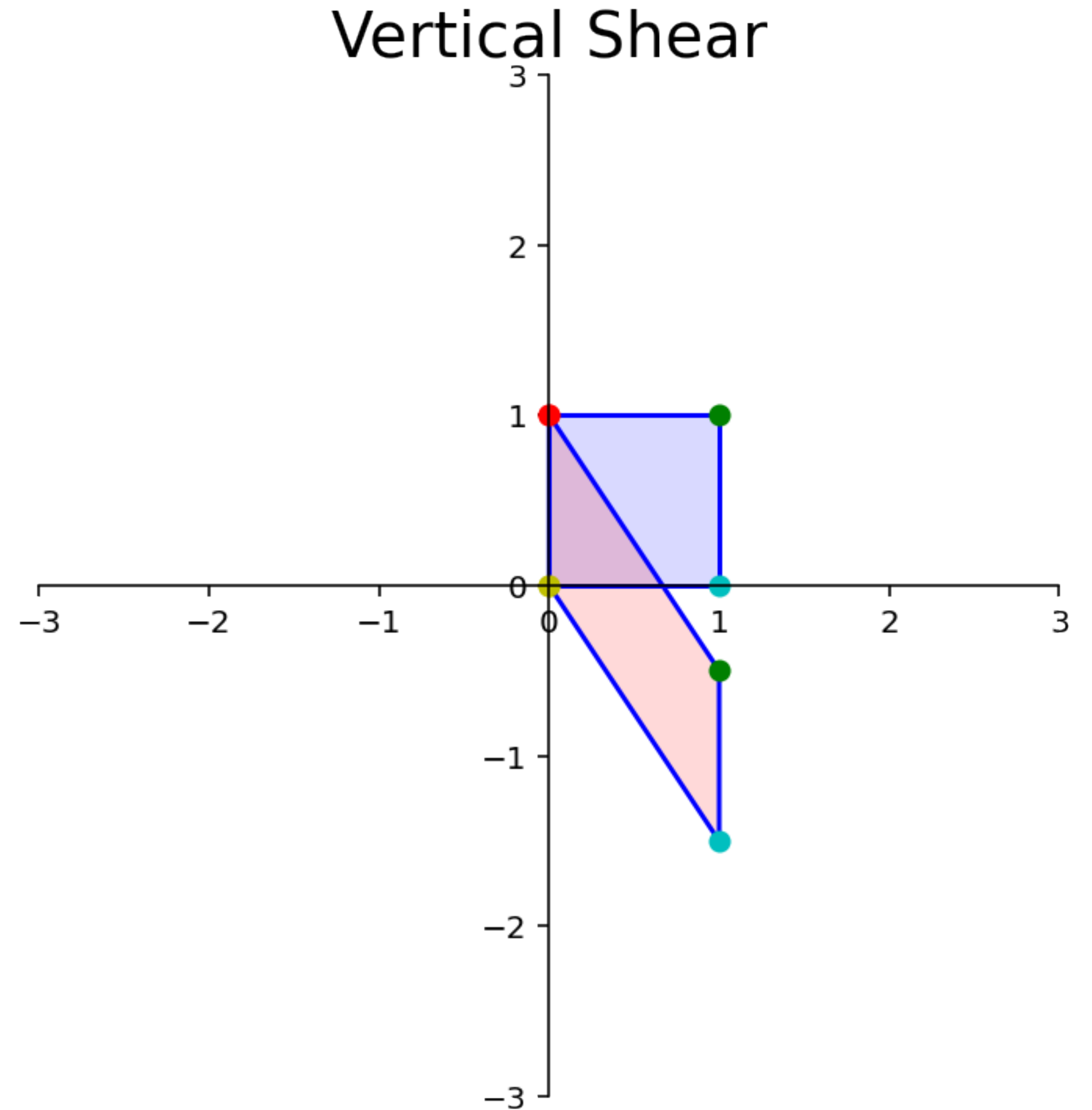


Question

*Is vertical shearing
a 1-1 transformation?
Justify your answer.*



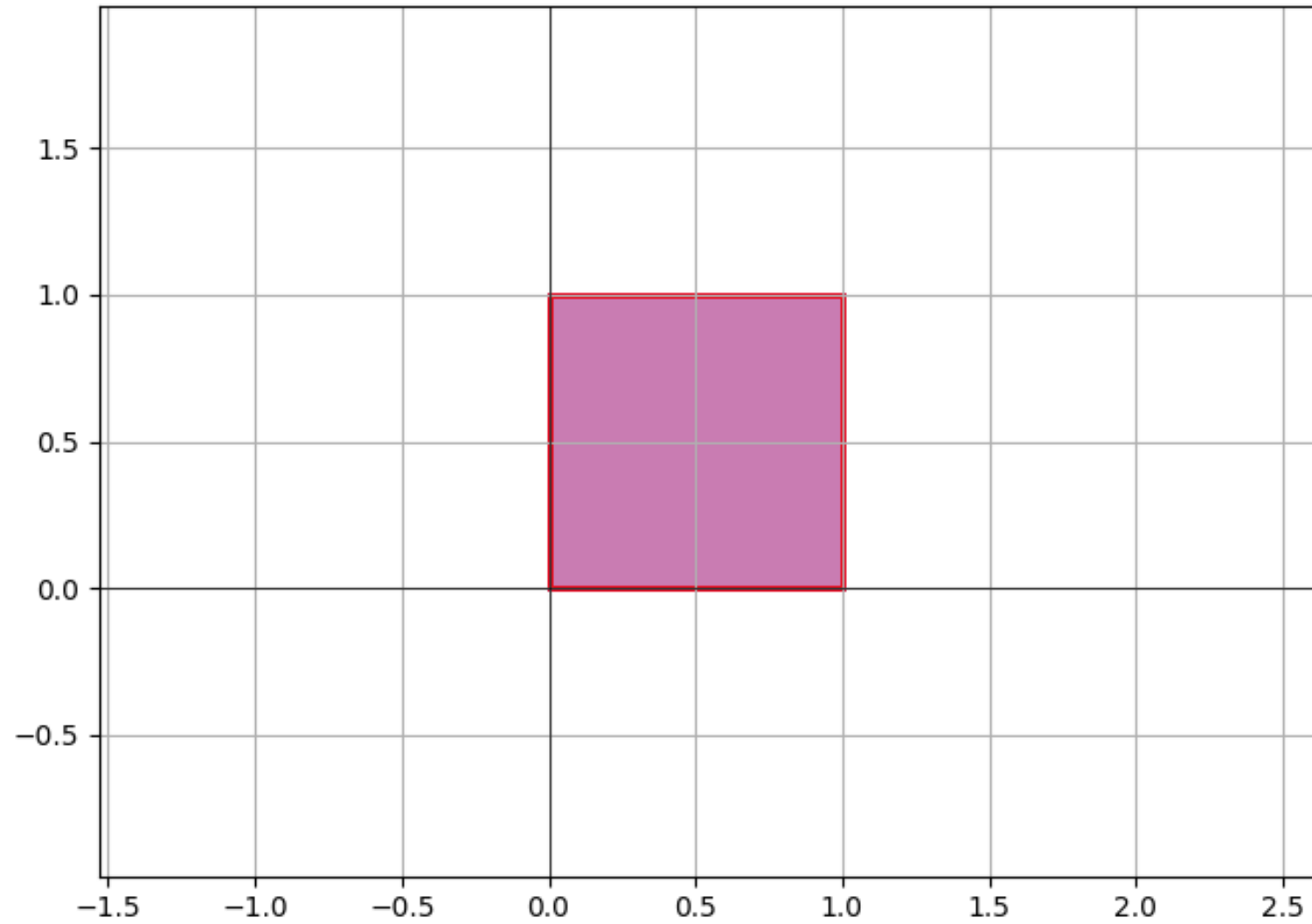
Answer: Yes



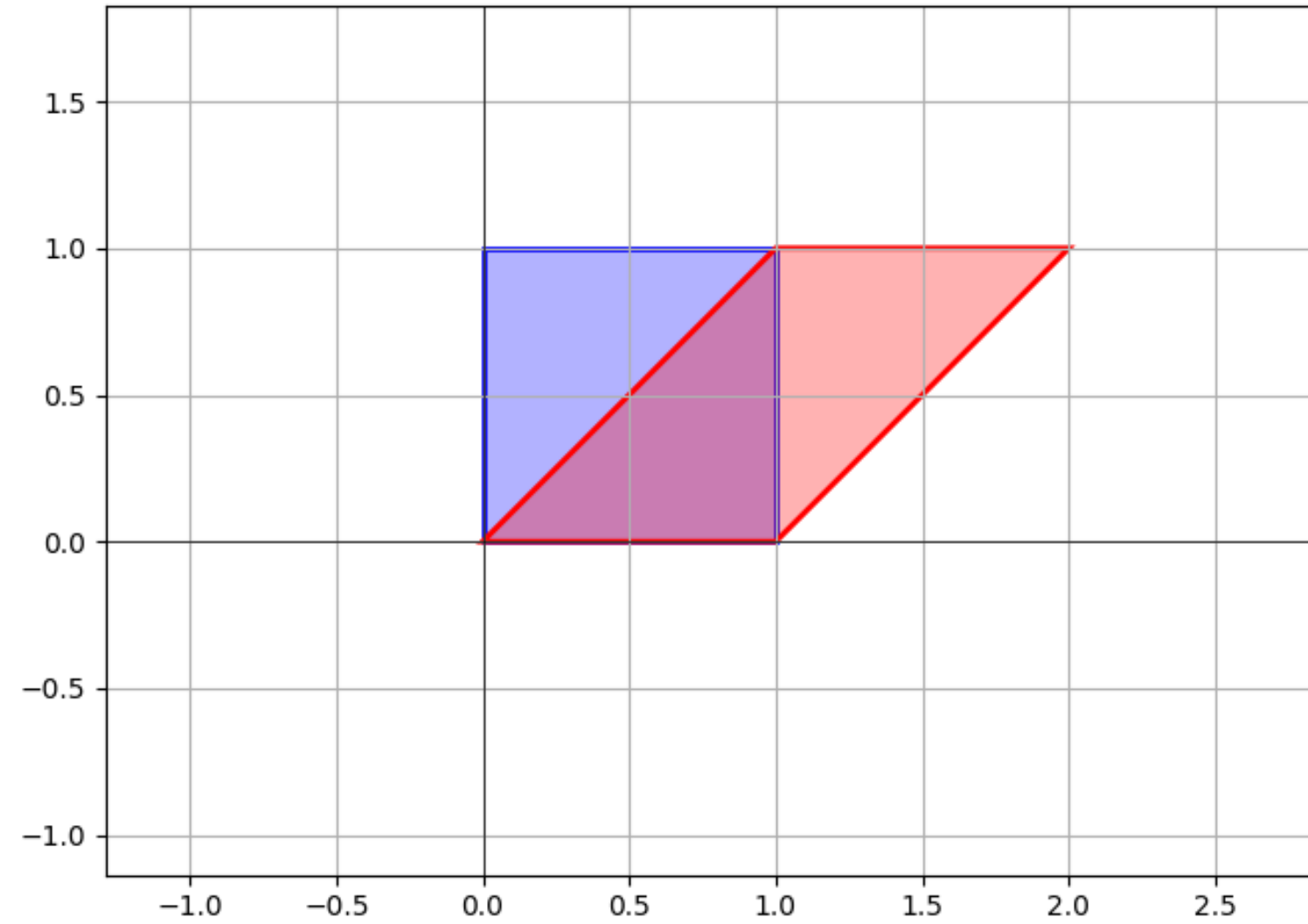
Composing Linear Transformations

Shearing and Reflecting (Geometrically)

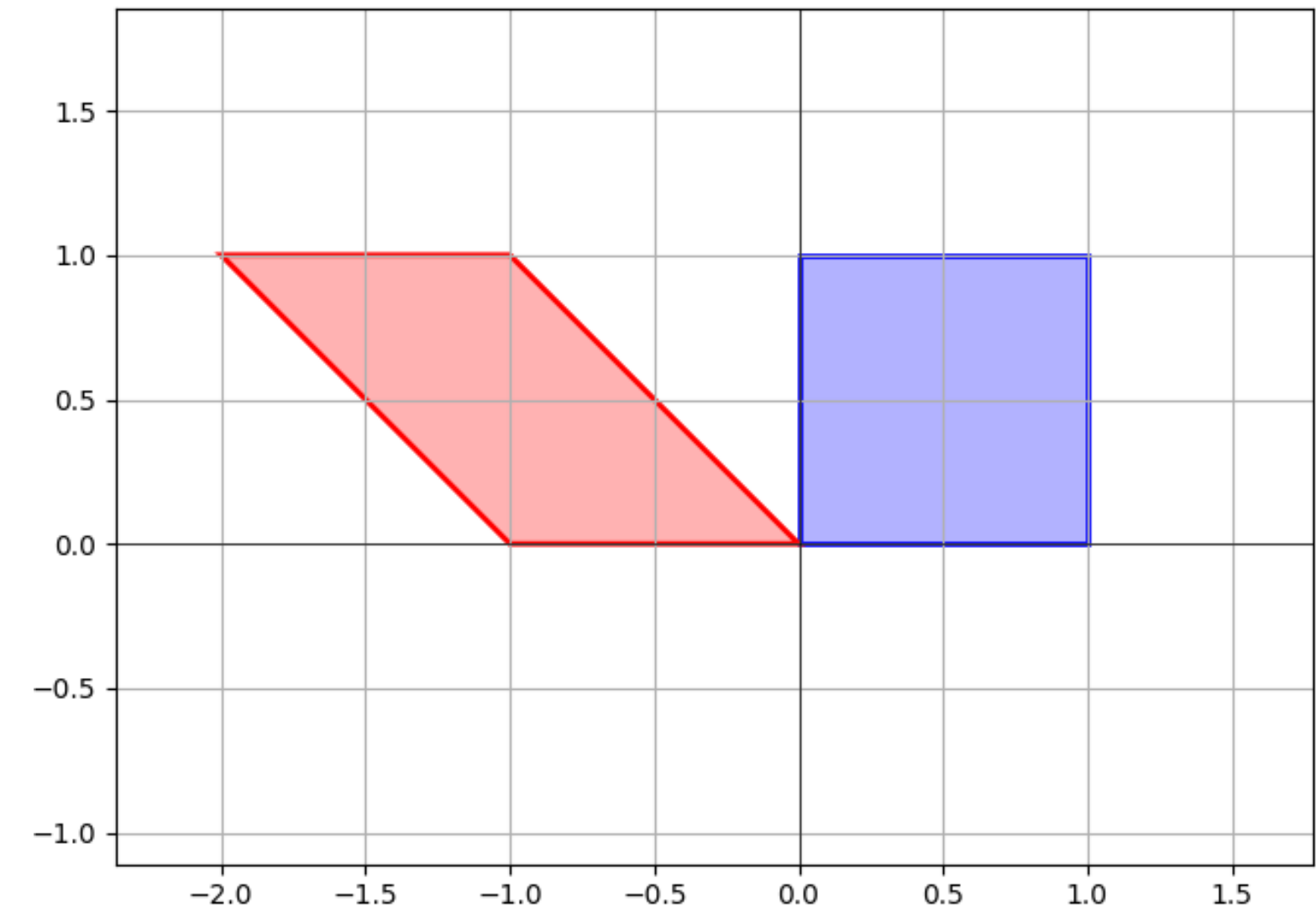
2D Matrix Transformations



2D Matrix Transformations



2D Matrix Transformations



shear



reflect

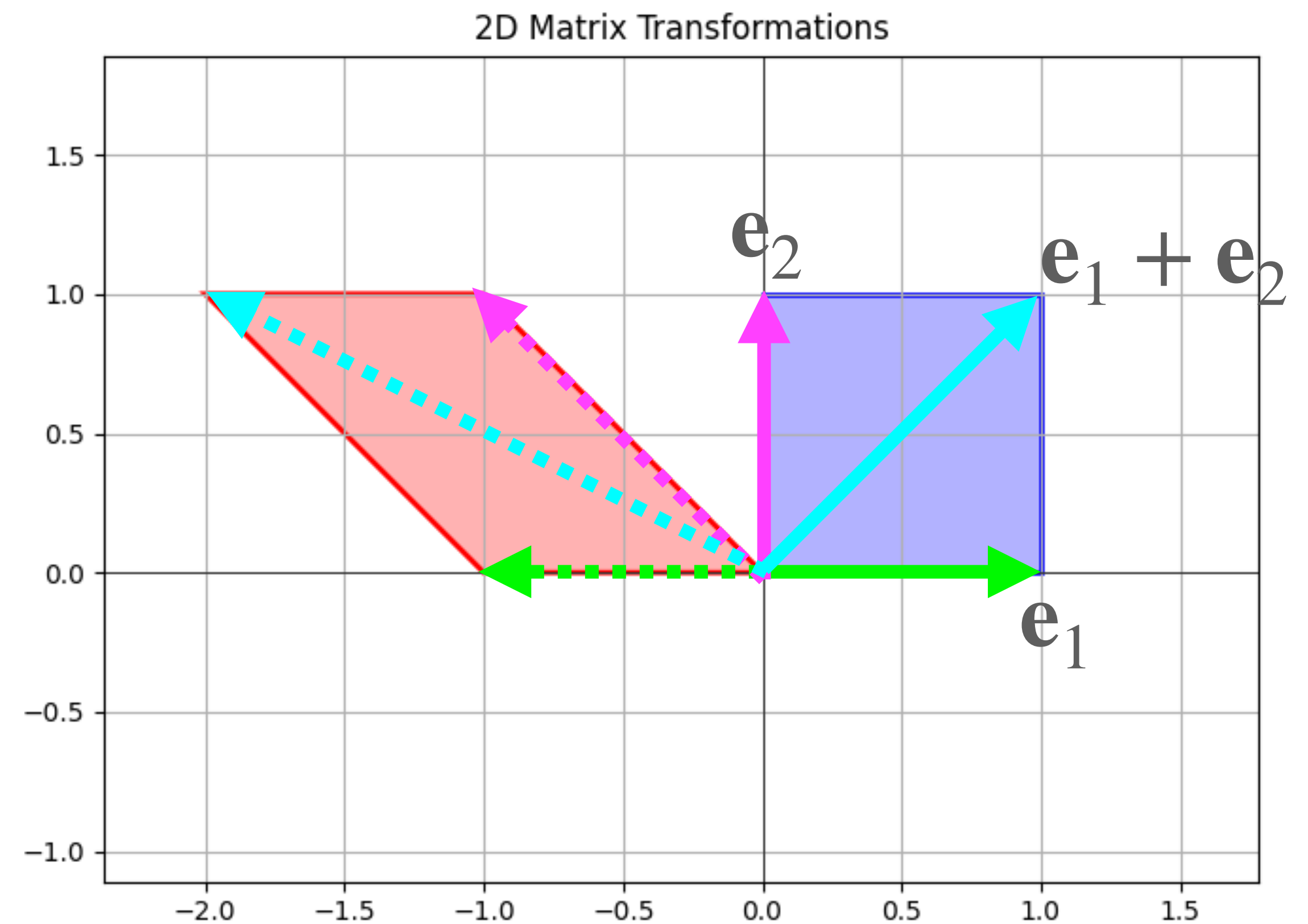
Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$



Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect shear

First multiply by shear matrix, then multiply
by reflection matrix

Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply
by reflection matrix

This gives us the same transformation.

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

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Fact. The composition of two linear transformations is a linear transformation.

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Verify:

$$\begin{aligned} T(S(\vec{u} + \vec{v})) &= T(S(\vec{u}) + S(\vec{v})) \\ &= T(S(\vec{u})) + T(S(\vec{v})) \\ &= (T \circ S)(\vec{u}) + (T \circ S)(\vec{v}) \end{aligned}$$

additivity

The Key Fact

Fact. The composition of two linear transformation is a linear transformation.

Verify:

This means the composition of two matrix transformation can be represented as a *single* matrix.

The Key Question

*Given two linear transformations,
how to we compute the matrix which
implements their composition?*

The Key Question

*Given two linear transformations,
how do we compute the matrix which
implements their composition?*

Matrix Multiplication

Matrix Multiplication

Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$x_1 \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} +$$

$$x_2 \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

$$A \left(x_1 \vec{b}_1 + x_2 \vec{b}_2 \right) =$$

$$x_1 A \vec{b}_1 + x_2 A \vec{b}_2 =$$

$$\boxed{\begin{bmatrix} A \vec{b}_1 & A \vec{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices and an equals sign:

- Left Matrix:** A 5x3 matrix with 5 rows and 3 columns. A blue vertical line to the left is labeled m . A red horizontal line above is labeled n . Below the matrix is the dimension notation $(m \times n)$, where m is in a light blue box and n is in a light red box.
- Multiplication:** An equals sign is placed between the second and third matrices.
- Middle Matrix:** A 3x4 matrix with 3 rows and 4 columns. A red vertical line to the left is labeled n . A purple horizontal line above is labeled k . Below the matrix is the dimension notation $(n \times k)$, where n is in a light red box and k is in a light purple box.
- Result Matrix:** A 5x4 matrix with 5 rows and 4 columns. A blue vertical line to the left is labeled m . A purple horizontal line above is labeled k . Below the matrix is the dimension notation $(m \times k)$, where m is in a light blue box and k is in a light purple box.

Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

Non-Example

The diagram illustrates two matrix multiplication attempts that are not defined. On the left, a 2x3 matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is multiplied by a 2x2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. On the right, the same 2x3 matrix is multiplied by a 3x2 matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Both attempts are marked with a large red 'X' and a semi-transparent red diamond, indicating they are not defined. The text 'These are not defined.' is written in red below the diagrams.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its top row highlighted in light blue. The second matrix is a 3x4 matrix with its first column highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with its top-left element highlighted in light purple. All elements in the matrices are represented by asterisks (*).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix (Matrix A) with its first row highlighted in light blue.
- A 3x4 matrix (Matrix B) with its second column highlighted in light red.
- A 5x4 matrix (Matrix C) with its first row and second column highlighted in light purple.

The equation is represented as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (left) with its first row highlighted in light blue.
- A 3x4 matrix B (middle) with its fourth column highlighted in light red.
- An equals sign ($=$) between matrices B and AB .
- A 5x4 matrix AB (right) with its top-right element highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The second row of A is highlighted in light blue, the second column of B is highlighted in light red, and the resulting second row of C is highlighted in light purple. The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication using three matrices of asterisks. The first matrix is a 5x3 grid with its second row highlighted in light blue. The second matrix is a 3x4 grid with its third column highlighted in light red. An equals sign follows, and then a 5x4 grid where the element at the intersection of the second row and third column is highlighted in light purple, representing the result of the dot product of the two highlighted rows and columns.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its second row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the second row and fourth column is highlighted in light purple, representing the result of the dot product of the two highlighted rows and columns.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The third row of A is highlighted in light blue, the third column of B is highlighted in light red, and the resulting third row of C is highlighted in light purple. The matrices are represented as grids of asterisks (*) enclosed in large square brackets.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The third matrix is a 5x4 matrix, also with all elements represented by asterisks (*). The first matrix has its third row highlighted in light blue. The second matrix has its third column highlighted in light red. The third matrix has its third column highlighted in light purple. An equals sign (=) is placed between the second and third matrices, indicating that the product of the first two matrices is the third matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices arranged in an equation:

- A 5x3 matrix with all elements marked with asterisks (*). The third row is highlighted in light blue.
- A 3x4 matrix with all elements marked with asterisks (*). The fourth column is highlighted in light red.
- An equals sign (=).
- A 5x4 matrix with all elements marked with asterisks (*). The element at the intersection of the third row and fourth column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices represented by grids of asterisks (*):

- A 5x3 matrix (Matrix A) with its bottom row highlighted in light blue.
- A 3x4 matrix (Matrix B) with its first column highlighted in light red.
- An equals sign (=) indicating the multiplication.
- A 5x4 matrix (Matrix AB) with its bottom-left element highlighted in light purple, representing the result of the dot product of the bottom row of A and the first column of B.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (left): The bottom row is highlighted in light blue.
- A 3x4 matrix B (middle): The second column is highlighted in light red.
- The resulting 5x4 matrix AB (right): The element at the intersection of the bottom row of A and the second column of B is highlighted in light purple.

The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product C , which is a 5x4 matrix.

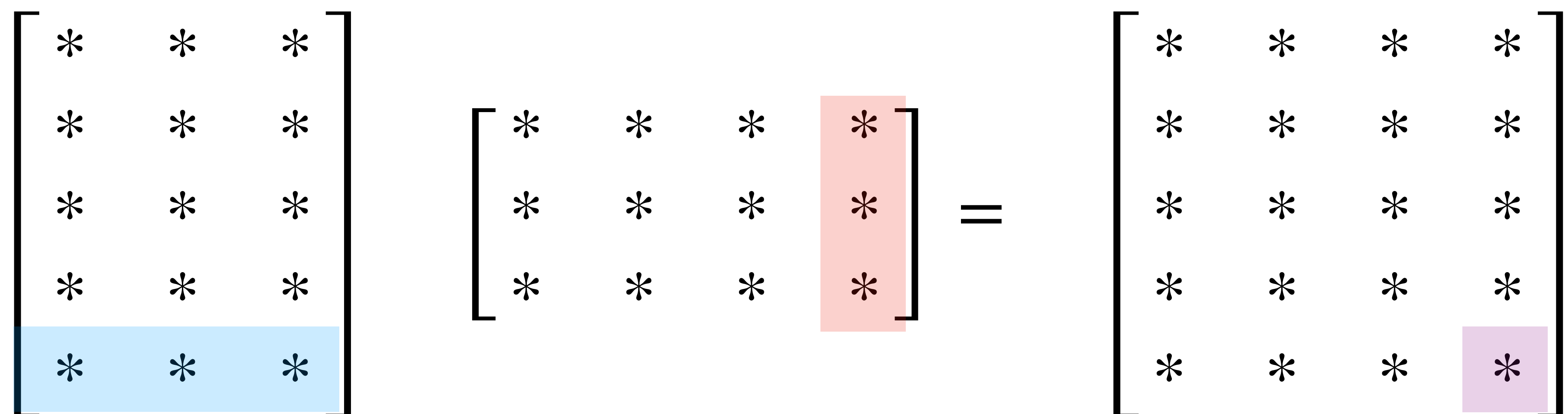
- Matrix A (5 rows, 3 columns) has its bottom row highlighted in light blue.
- Matrix B (3 rows, 4 columns) has its third column highlighted in light red.
- The product matrix C (5 rows, 4 columns) has its element at the intersection of the fifth row and third column highlighted in light purple.

The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)



$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

(also find a pair where they are the same)

Answer: Rotation and Reflection

Computational Aspects of Matrix Multiplication

Matrix Operations in Numpy

Let `a` and `b` be 2D numpy arrays and let `c` be a floating point number.

» `a @ b` (matrix multiplication)

» `a + b` (matrix addition)

» `c * a` (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more.

Analyzing Linear Algebra Algorithms

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We will not use $O(\cdot)$ notation!

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We will not use $O(\cdot)$ notation!

For numerics, we care about number of **F**loating-**o**int **O**perations (FLOPs):

- >> addition
- >> subtraction
- >> multiplication
- >> division
- >> square root

Analyzing Linear Algebra Algorithms

We will not use $O(\cdot)$ notation!

For numerics, we care about number of **F**loating-**o**int **O**perations (FLOPs):

- >> addition
- >> subtraction
- >> multiplication
- >> division
- >> square root

$2n$ vs. n is very different
when $n \sim 10^{20}$

Dominant Terms

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that said, we don't care about *exact* bounds

Dominant Terms

that said, we don't care about *exact* bounds

A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \rightarrow \infty} \frac{f(i)}{g(i)} = 1$$

Dominant Terms

that said, we don't care about *exact* bounds

A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \rightarrow \infty} \frac{f(i)}{g(i)} = 1$$

for polynomials, they are equivalent to their dominant term

Dominant Terms

the dominant term of a polynomial is the monomial with the highest degree

$$\lim_{i \rightarrow \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$ dominates the function even though the coefficient for x^2 is so large

A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Suppose A and B are $n \times n$ matrices.

This operations takes n multiplications and n divisions ($2n$ FLOPS total)

Repeating for each entry gives $\sim 2n^3$ FLOPS

A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

The main part of this procedure is highly parallelizable.

A Note on Parallelization

```
a = np.array(...)  
b = np.array(...)  
prod = np.zeros([a.shape[0], b.shape[1]])  
for i in range(a.shape[0]):  
    for j in range(b.shape[1]):  
        prod[i, j] = np.dot(a[i], b[:, j])
```

The main part of this procedure is highly parallelizable.

One processor per entry gets you to $\sim 2n$ FLOPS

A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers.

Best leave it to experts (or do research in the area).

LAPACK is the state of the art library for matrix operations.

numpy uses LAPACK

Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations.

Matrix multiplication coincides with composition of linear transformations.

There is an algebra of matrices which is consistent with the algebra of vectors.