

Matrix Inverses

Geometric Algorithms

Lecture 11

Practice Problem(s)

1. Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

2. Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Connect everything(!)

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1-1 Correspondence

`numpy.linalg.inv`

determinant

Invertible Matrix Theorem

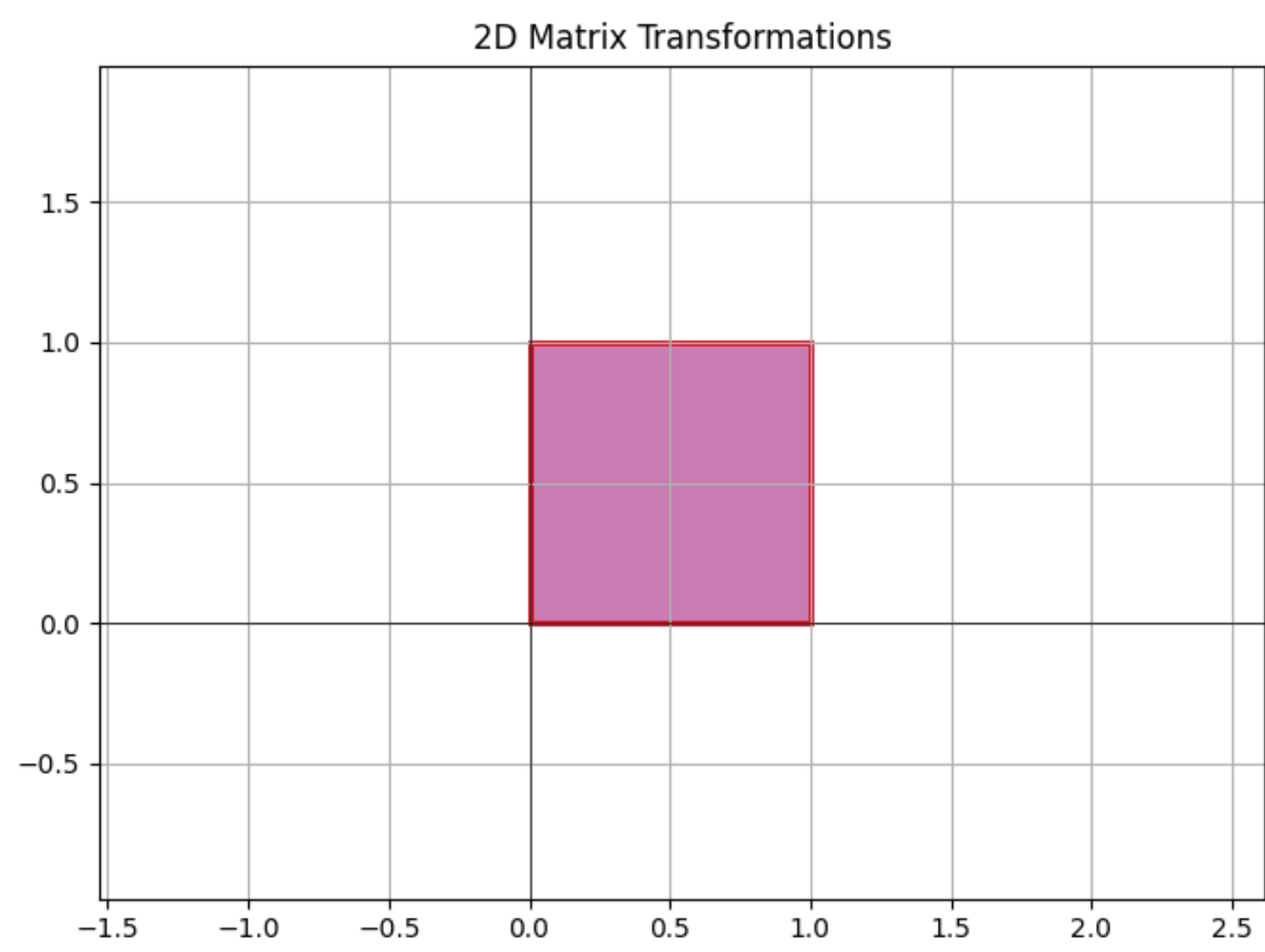
Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

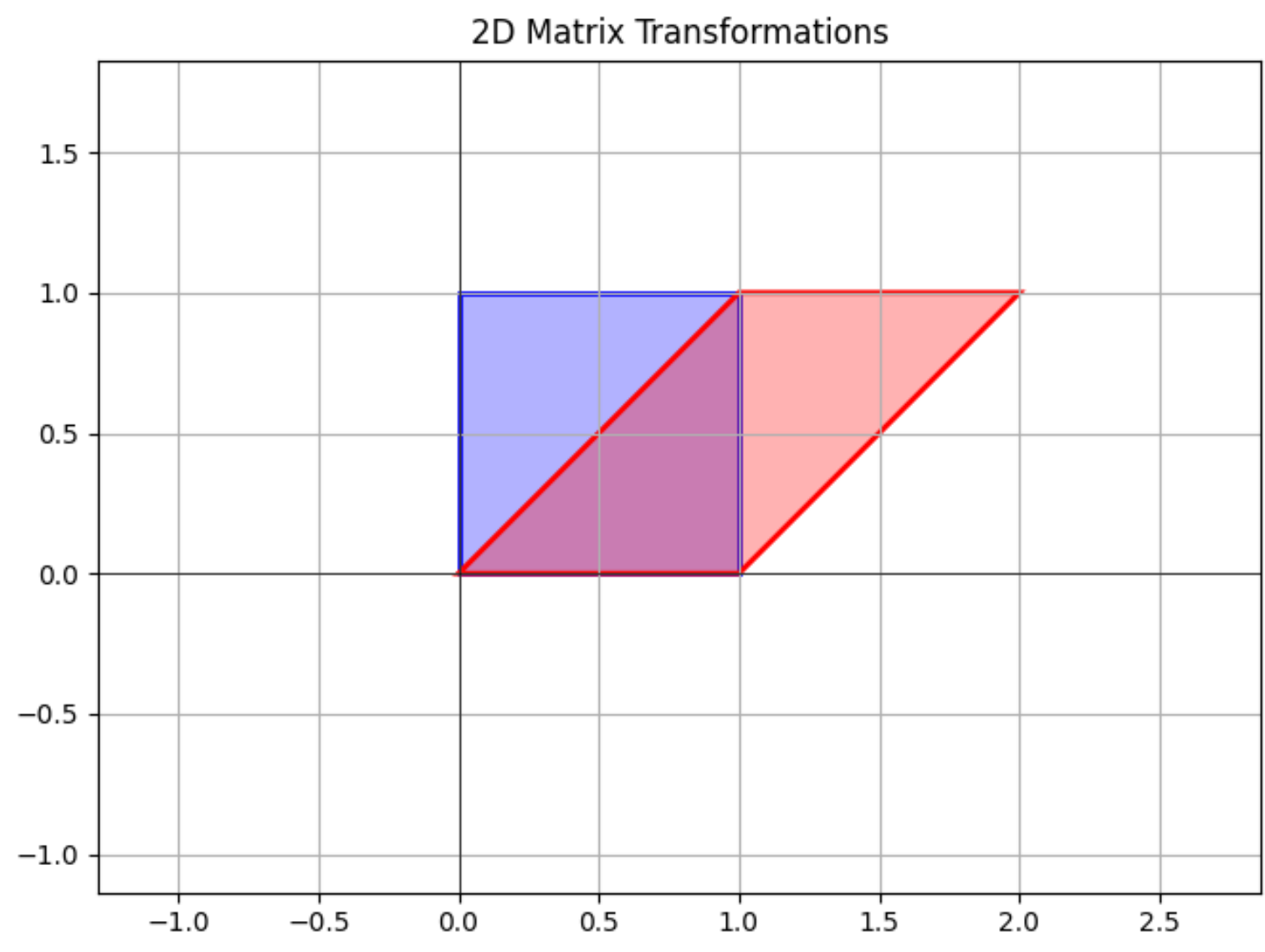
(also find a pair where they are the same)

Recap: Matrix Multiplication

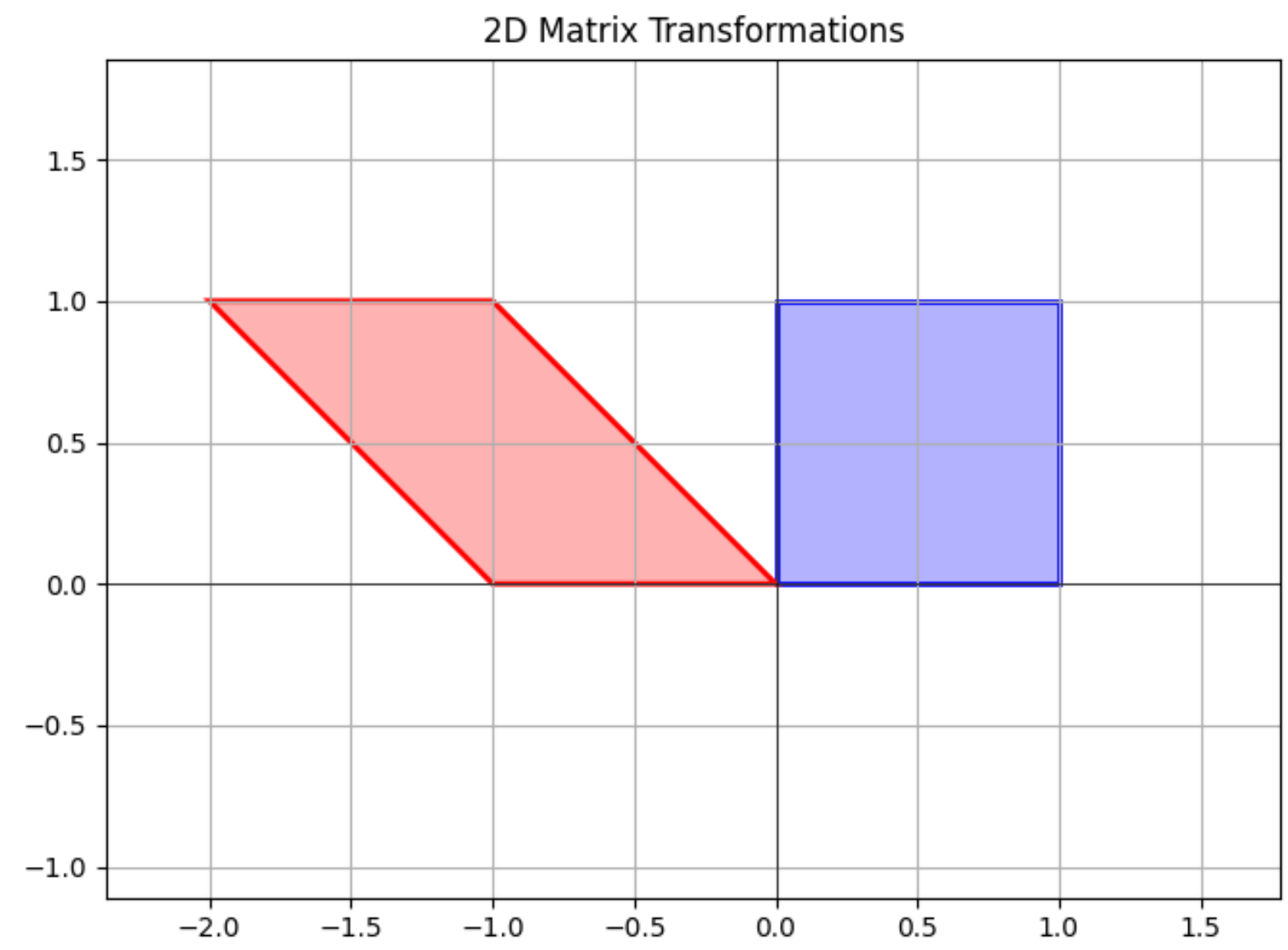
Recall: Composition



shear



reflect

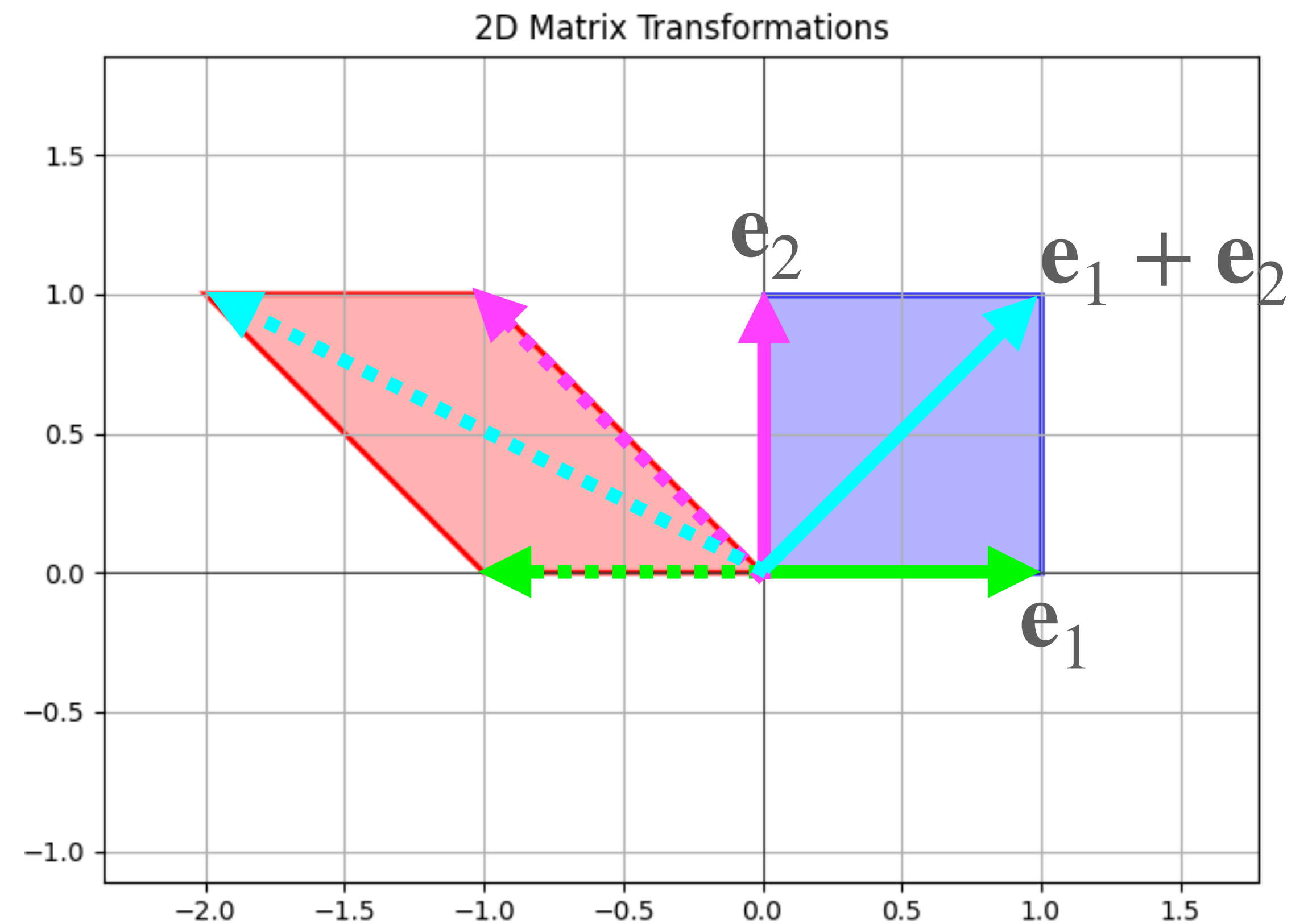


Recall: Composition

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices and an equals sign:

- Left Matrix:** A 5x3 matrix with 5 rows and 3 columns. A blue vertical line to its left is labeled m . A red horizontal line above it is labeled n . Below the matrix is the dimension notation $(m \times n)$, where m is in a light blue box and n is in a light red box.
- Multiplication:** A red vertical line is placed between the first and second matrices, with a red n label next to it.
- Middle Matrix:** A 3x4 matrix with 3 rows and 4 columns. A purple horizontal line above it is labeled k . Below the matrix is the dimension notation $(n \times k)$, where n is in a light red box and k is in a light purple box.
- Result:** An equals sign followed by a 5x4 matrix with 5 rows and 4 columns. A blue vertical line to its left is labeled m . A purple horizontal line above it is labeled k . Below the matrix is the dimension notation $(m \times k)$, where m is in a light blue box and k is in a light purple box.

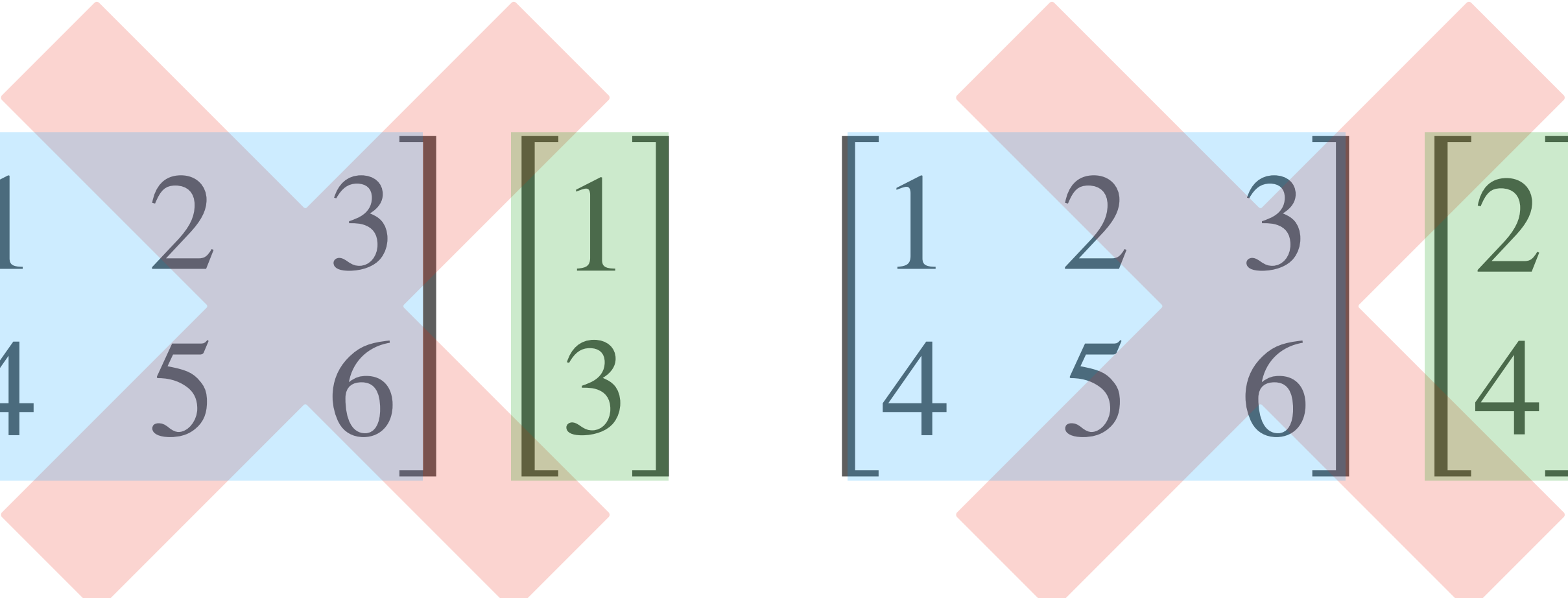
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$
The diagram shows two matrix-vector products, each crossed out with a large, semi-transparent red 'X'. The first product is a 2x3 matrix multiplied by a 2x1 vector. The second product is a 2x3 matrix multiplied by a 2x1 vector. The red 'X's are centered over each product, indicating that these operations are not defined.

These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A with its first row highlighted in light blue.
- A 3x4 matrix B with its first column highlighted in light red.
- The resulting 5x4 matrix C with its first row highlighted in light purple.

The first row of A is multiplied by the first column of B to produce the first row of C .

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix (Matrix A) with its first row highlighted in light blue.
- A 3x4 matrix (Matrix B) with its third column highlighted in light red.
- The resulting 5x4 matrix (Matrix AB) with its top-right element highlighted in light purple.

An equals sign is placed between the second and third matrices, indicating the multiplication operation.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (represented by asterisks) with its first row highlighted in light blue.
- A 3x4 matrix B (represented by asterisks) with its fourth column highlighted in light red.
- An equals sign ($=$) between matrices B and AB .
- A 5x4 matrix AB (represented by asterisks) with its top-right element highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its second row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the second row and fourth column is highlighted in light purple, representing the result of the dot product of the two highlighted rows and columns.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product C , which is a 5x4 matrix. The third row of A is highlighted in light blue, and the first column of B is highlighted in light red. The resulting element in the third row, first column of C is highlighted in light purple. The matrices are represented as grids of asterisks (*).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The third matrix is a 5x4 matrix, also with all elements represented by asterisks (*). The first matrix has its third row highlighted in light blue. The second matrix has its third column highlighted in light red. The third matrix has its third column highlighted in light purple. An equals sign (=) is placed between the second and third matrices, indicating that the product of the first two matrices (with the highlighted row and column) results in the third matrix (with the highlighted column).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The third matrix is a 5x4 matrix, also with all elements represented by asterisks (*). The third matrix is the result of the multiplication of the first two matrices. The third row of the first matrix is highlighted in light blue, and the fourth column of the second matrix is highlighted in light red. The element at the intersection of the third row and fourth column in the third matrix is highlighted in light purple, representing the dot product of the third row of the first matrix and the fourth column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices arranged horizontally, separated by an equals sign. The first matrix is a 5x3 matrix with asterisks in each cell; its fourth row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; its second column is highlighted in light red. The third matrix is a 5x4 matrix with asterisks; its fourth row is highlighted in light purple. This visualizes the calculation of the element in the fourth row and second column of the product matrix, which is the dot product of the fourth row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product C , which is a 5x4 matrix.

Matrix A is represented as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Matrix B is represented as:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

The third row of A and the fourth column of B are highlighted, indicating the specific row and column used in the calculation. The resulting element in the third row, fourth column of C is highlighted.

Matrix C is represented as:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C .

Matrix A is represented by a grid of asterisks. The bottom row is highlighted in light blue.

Matrix B is represented by a grid of asterisks. The first column is highlighted in light red.

Matrix C is represented by a grid of asterisks. The bottom-left element is highlighted in light purple.

The equation is shown as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product C , which is a 5x4 matrix.

- Matrix A is represented by a 5x3 grid of asterisks. The bottom row is highlighted in light blue.
- Matrix B is represented by a 3x4 grid of asterisks. The fourth column is highlighted in light red.
- The product matrix C is represented by a 5x4 grid of asterisks. The bottom-right element (row 5, column 4) is highlighted in light purple.

The equation is shown as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

Connection with Matrix-Vector Multiplication

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This is just vector multiplication.

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

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scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

More Matrix Operations

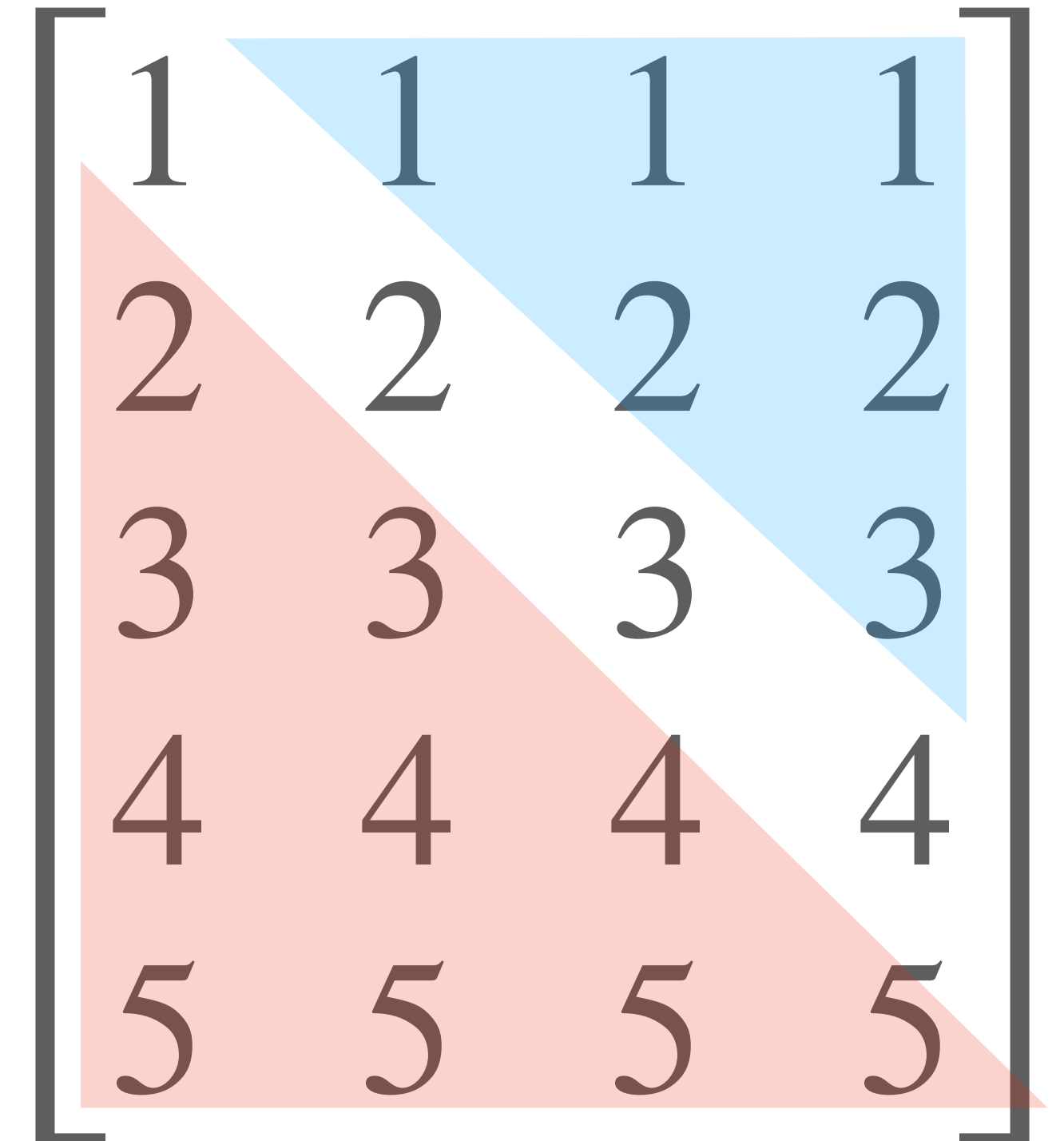
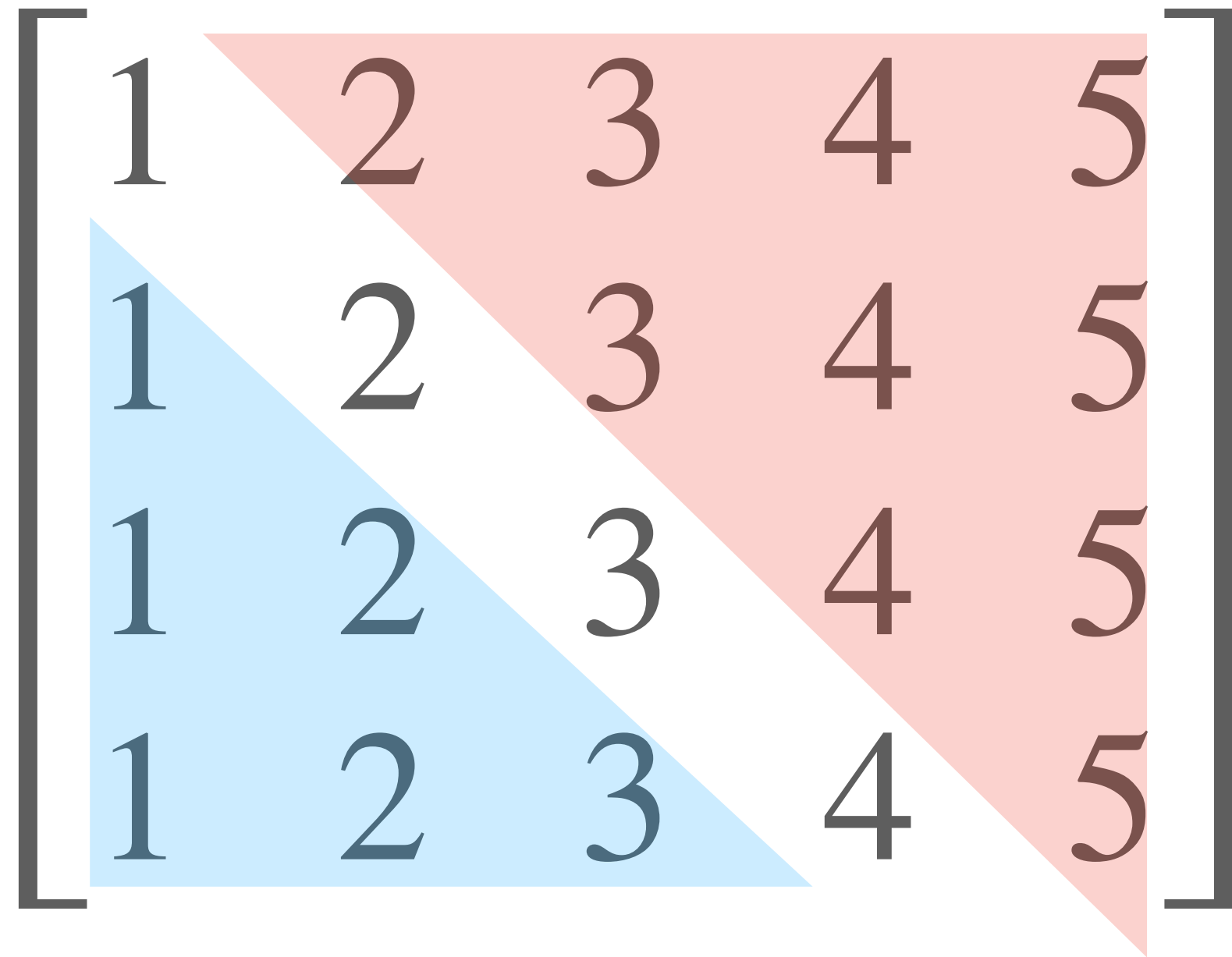
Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)



Transpose

Definition. For a $m \times n$ matrix A , the **transpose** of A , written A^T , is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = cA^T \text{ (where } c \text{ is a scalar)}$$

$$(AB)^T = B^T A^T$$

Algebraic Properties (Transpose)

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = cA^T \text{ (where } c \text{ is a scalar)}$$

$$(AB)^T = B^T A^T \quad \text{Important: the order reverses!}$$

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^T A^T$.

Example: $\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^T$

Transposes and Inner Products

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For a vector $\mathbf{v} \in \mathbb{R}^n$, what is \mathbf{v}^T ?

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For a vector $\mathbf{v} \in \mathbb{R}^n$, what is \mathbf{v}^T ?

It's a $1 \times n$ matrix.

Transposes and Inner Products

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It's a $1 \times n$ matrix.

For two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n ,
is $\mathbf{u}^T \mathbf{v}$ defined?

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is \mathbf{v}^T ?

It's a $1 \times n$ matrix.

$$1 \times n$$

$$n \times 1$$

$$1 \times 1$$

For two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n ,
is $\mathbf{u}^T \mathbf{v}$ defined?

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is \mathbf{v}^T ?

It's a $1 \times n$ matrix.

$$\boxed{1} \times \boxed{n}$$

$$\boxed{n} \times \boxed{1}$$

$$\boxed{1} \times \boxed{1}$$

For two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n ,
is $\mathbf{u}^T \mathbf{v}$ defined?

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

Transposes and Inner Products

Transposes and Inner Products

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

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Definition. The **inner product** of two vectors **u** and **v** in \mathbb{R}^n is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix Powers

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What should A^0 be?

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(we want $A^0 A^k = A^{0+k} = A^k$)

Matrix Powers (Computationally)

We can use `numpy.linalg.matrix_power`

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16}$$

Why? :

Final Warnings about Matrix Multiplication

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1. AB is not necessarily equal to BA , even if both are defined.
2. If $AB = AC$ then it is not necessary that $B = C$.
3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. Choose A and B such that they have all nonzero entries.

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

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transpose

A^T

So Far: Matrix Operations

transpose

$$A^T$$

scaling

$$cA$$

So Far: Matrix Operations

transpose

$$A^T$$

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addition (subtraction)

$$A + B$$

$$A + (-1)B = A - B$$

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So Far: Matrix Operations

transpose

$$A^T$$

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multiplication (powers)

$$AB$$

$$A^k$$

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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The identity matrix implements the "do nothing" transformation. For any \mathbf{v} ,

$$I\mathbf{v} = \mathbf{v}$$

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$$IA = AI = A$$

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$$IA = AI = A$$

These may be different sizes

Recall: The Identity Matrix

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \\ 2 \times 2 & 2 \times 4 & & 2 \times 4 & 4 \times 4 & & 2 \times 4 \end{array}$$

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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How do we solve this equation?

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$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

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Multiply each side by $\frac{1}{2}$ a.k.a. 2^{-1} .

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Basic Algebra

$$1x = 5$$

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$$x = 5$$

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$$\mathbf{Ax} = \mathbf{b}$$

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Wouldn't it be nice...

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

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Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

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A^{-1} is the **multiplicative inverse** of A

Do all matrices have
inverses?

Do all matrices have
inverses?

No. If they did, then every linear
system would have a solution.

When does a matrix have
an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

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They are the only kind of matrices...

- » that can have a pivot in every row and every column.
- » whose transformations can be both 1-1 and onto.
- » whose columns can have full span and be linearly independent.
- » that can have inverses.

Matrix Inverses

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Definition. For a $n \times n$ matrix A , an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1 -axis in \mathbb{R}^2 is it's own inverse.

Verify:

Example: No inverse

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

If A is invertible, then we write A^{-1}
for *the* inverse of A .

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A , if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of \mathbf{b} .

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» exactly one solution for any choice of \mathbf{b}

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

- » at least one solution for any choice of \mathbf{b}
- » at most one solution for any choice of \mathbf{b}

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» T is onto

» T is one-to-one

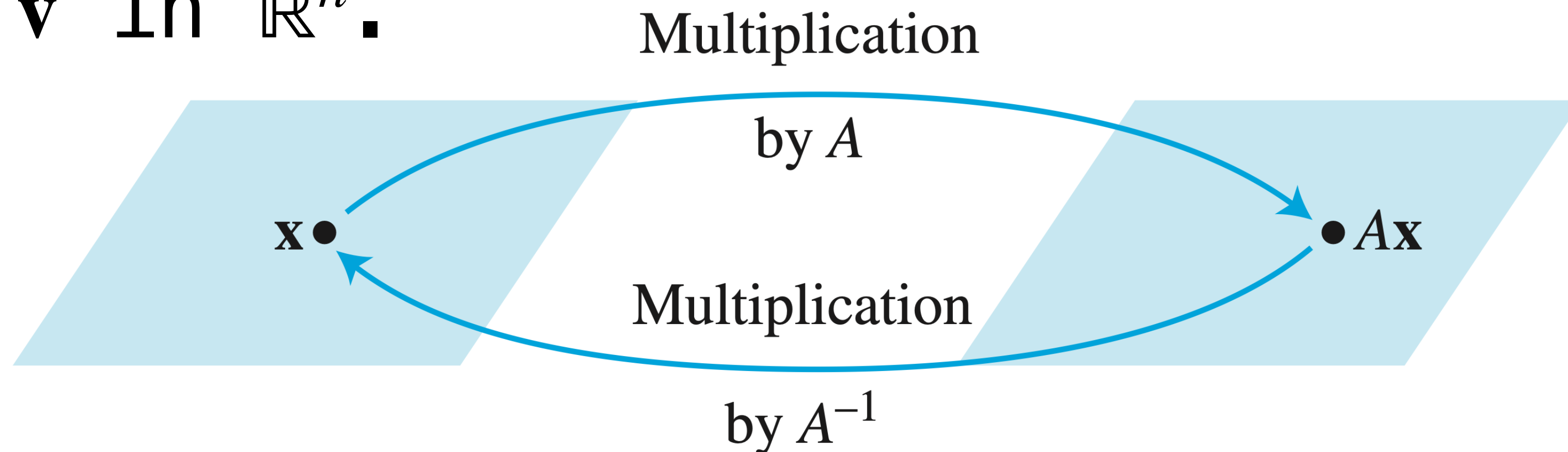
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n .



Connection to Transformations

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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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A matrix is invertible if it's possible to "undo" its transformation without "losing information".

Non-Example. Projection onto the x_1 -axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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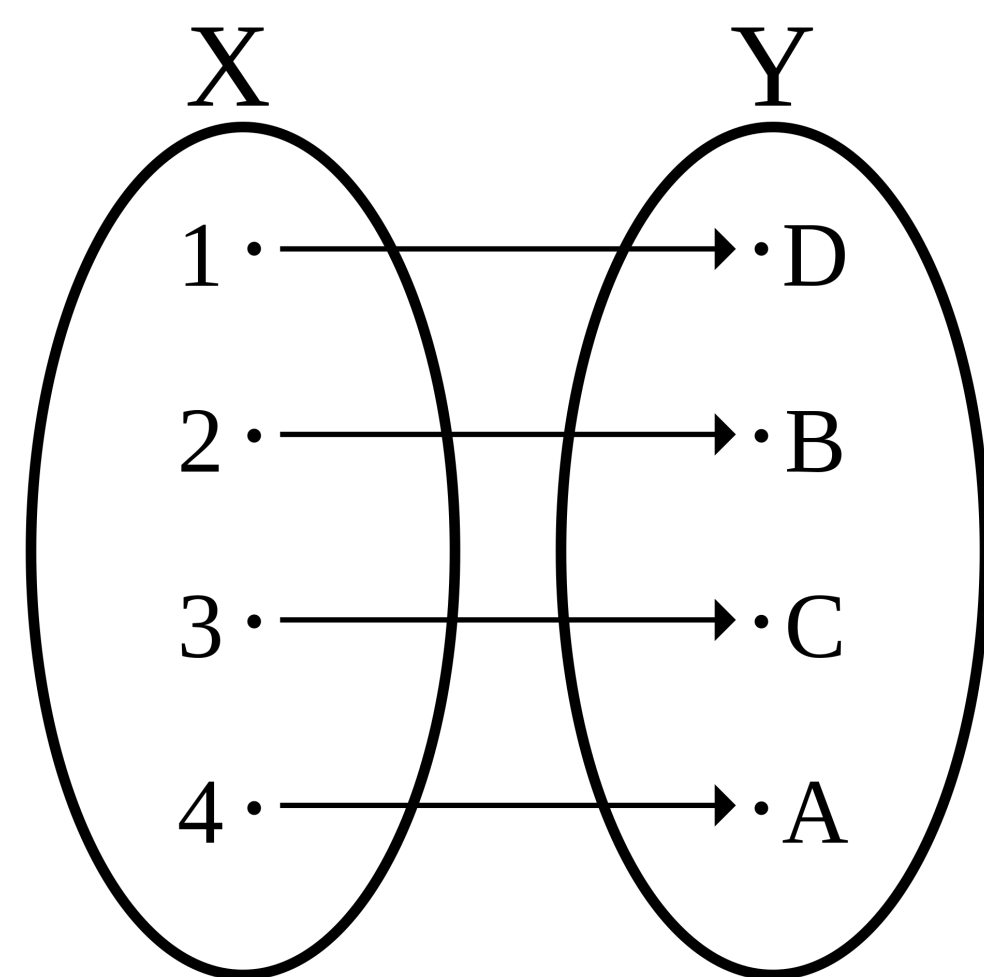
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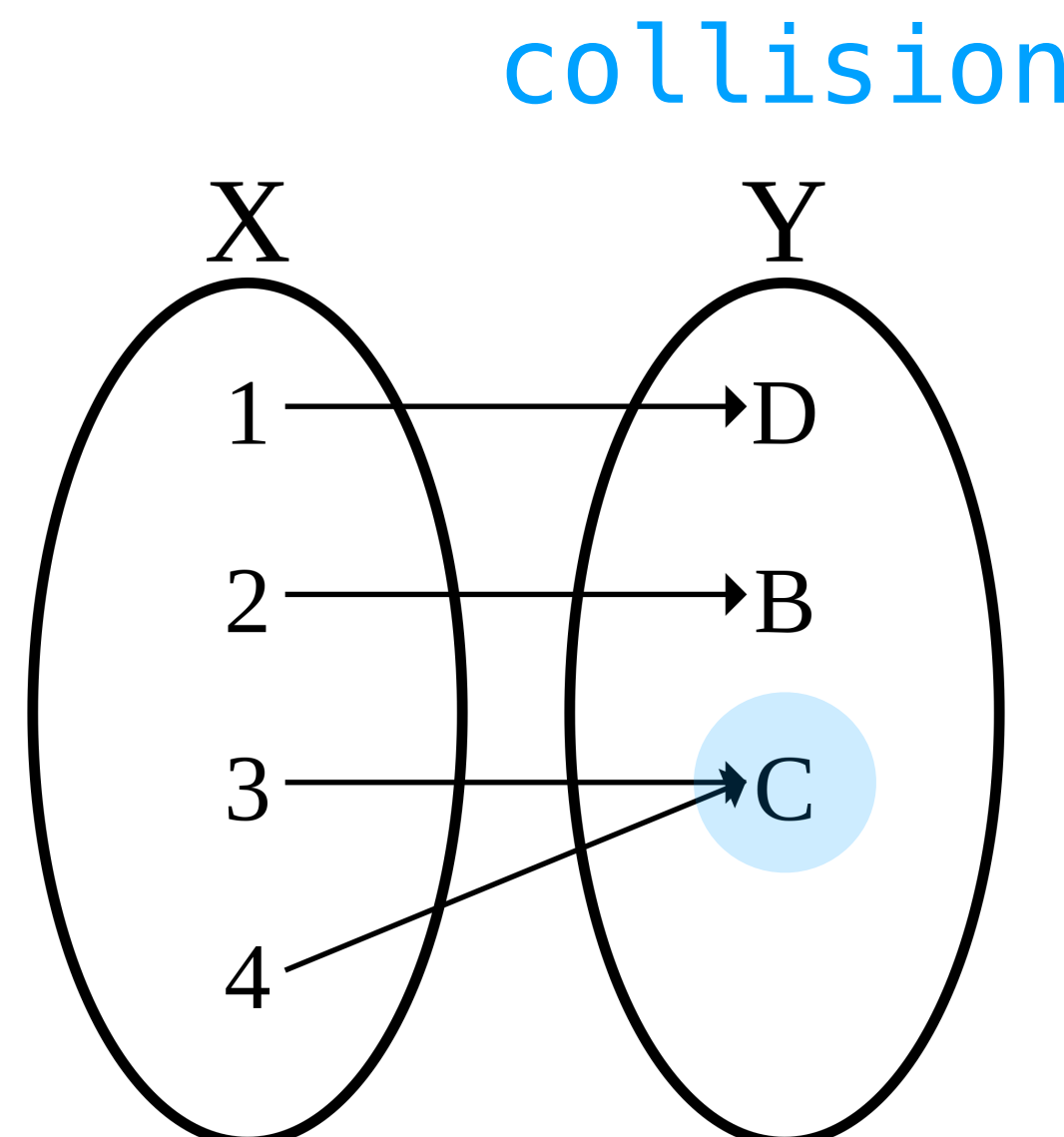
A transformation is a 1-1 correspondence if it is 1-1 and onto.

Invertible transformations are 1-1 correspondences.

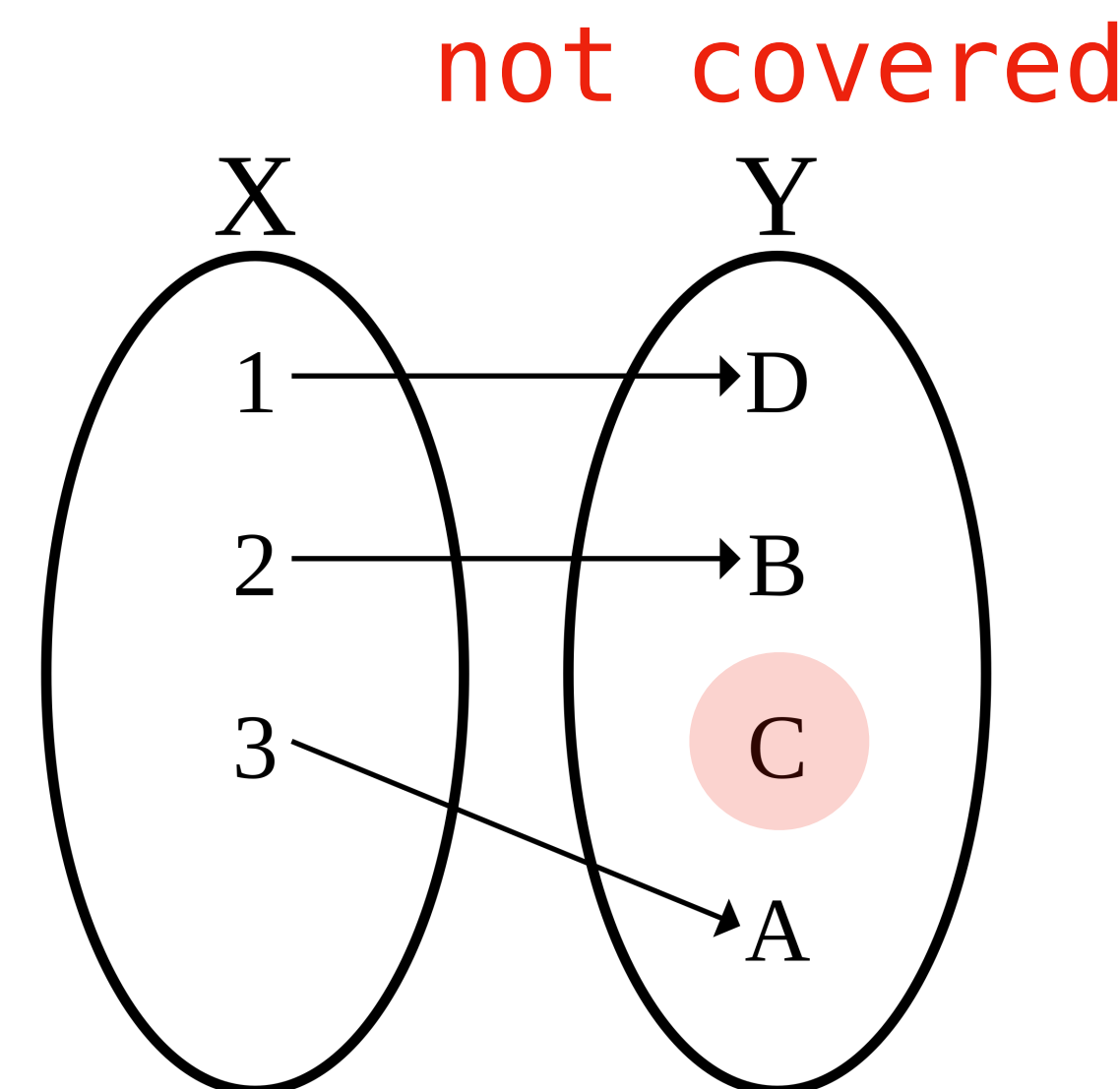
Kinds of Transformations (Pictorially)



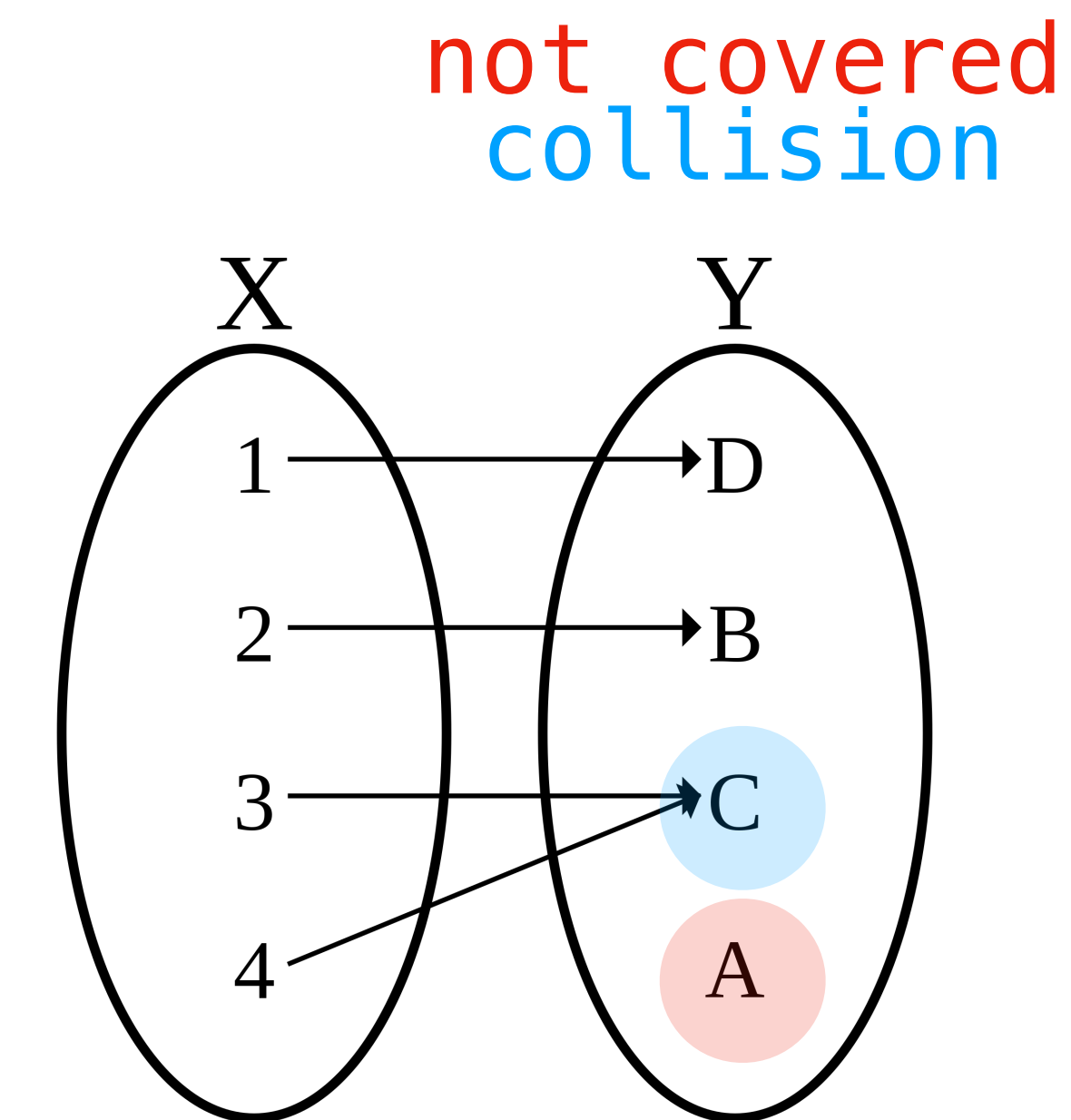
1-1 correspondence



onto, not 1-1



1-1 not onto



not 1-1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each \mathbf{b}_i ?

In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

Recall: In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \dots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Row reduce the matrix $[A \ I]$ to a matrix $[I \ B]$. Then B is the inverse of A .

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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The inverse is defined only if the determinant is nonzero.

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The **determinant** of a 2×2 matrix is the value $ad - bc$.

The inverse is defined only if the determinant is nonzero.

(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C , demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B , the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix.

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen.

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every \mathbf{b}

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

- 5. A has a pivot in every column
- 6. A has a pivot in every row
- 7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

- 8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- 9. The columns of A are linearly independent
- 10. The columns of A span \mathbb{R}^n

Verify:

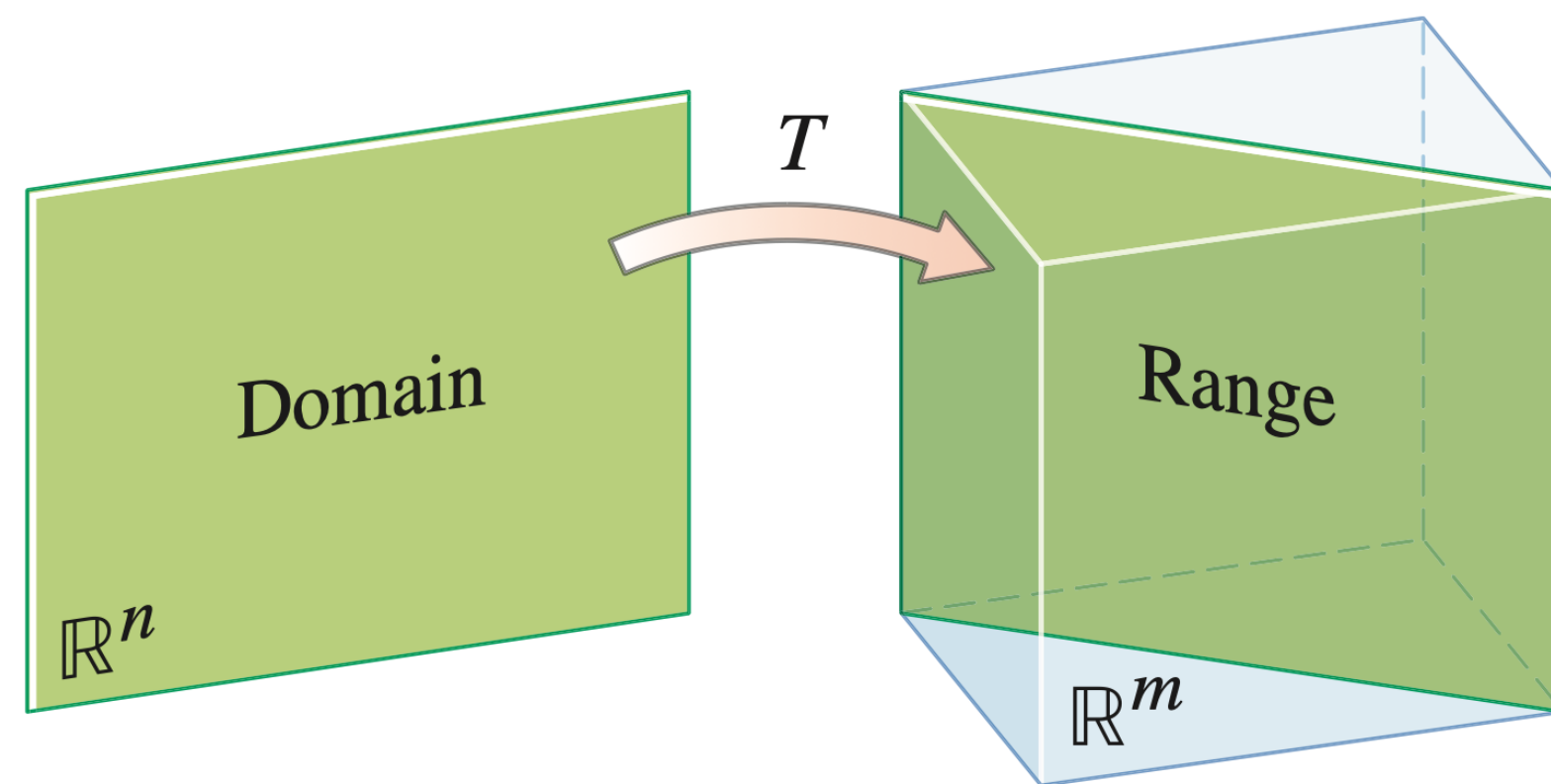
Recall: Onto Transformations

Recall: Onto Transformations

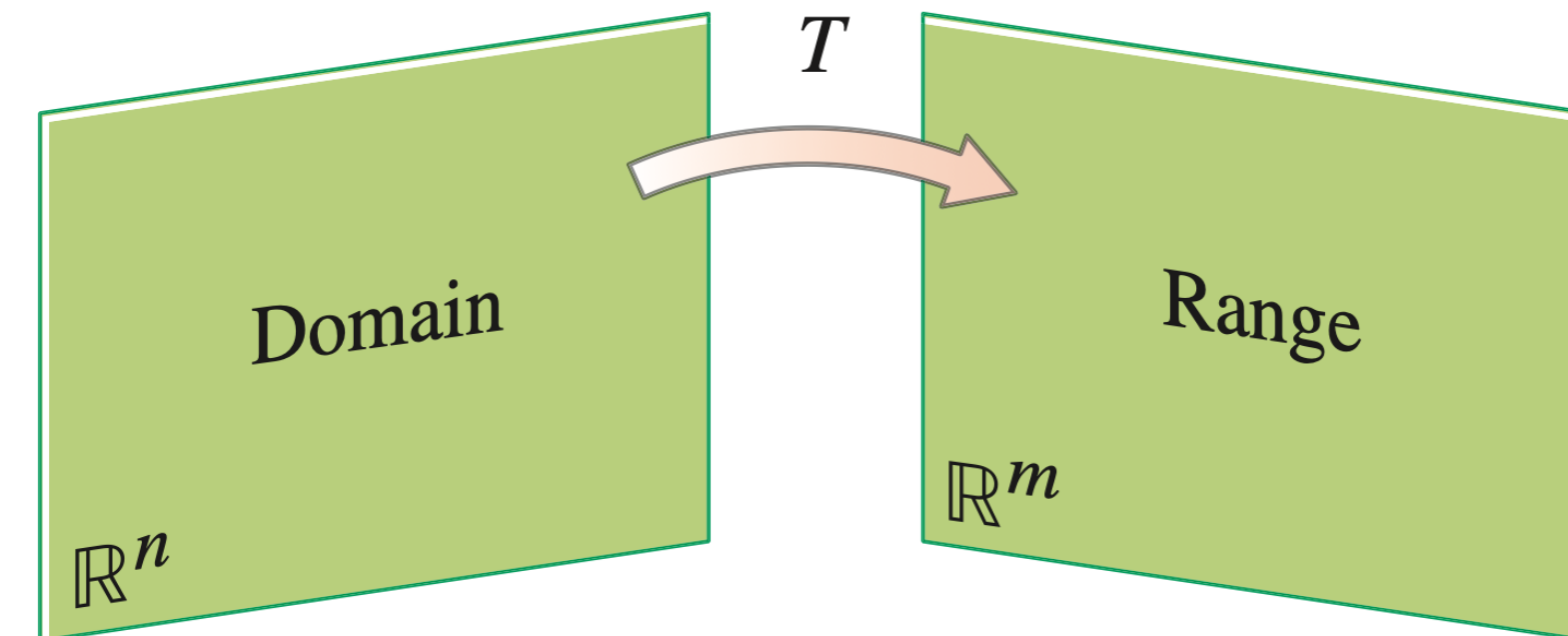
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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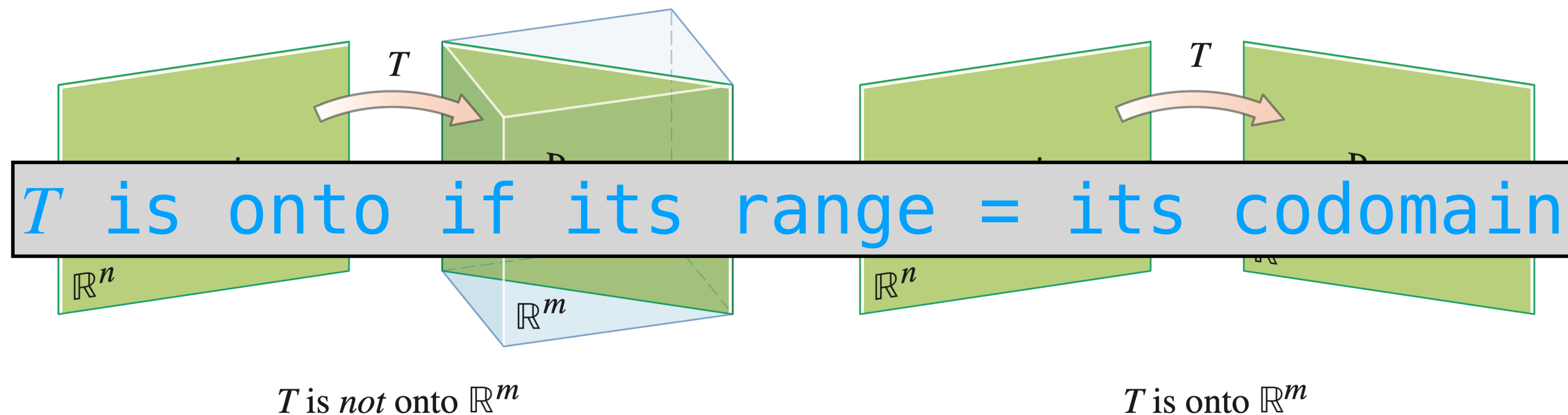
T is not onto \mathbb{R}^m



T is onto \mathbb{R}^m

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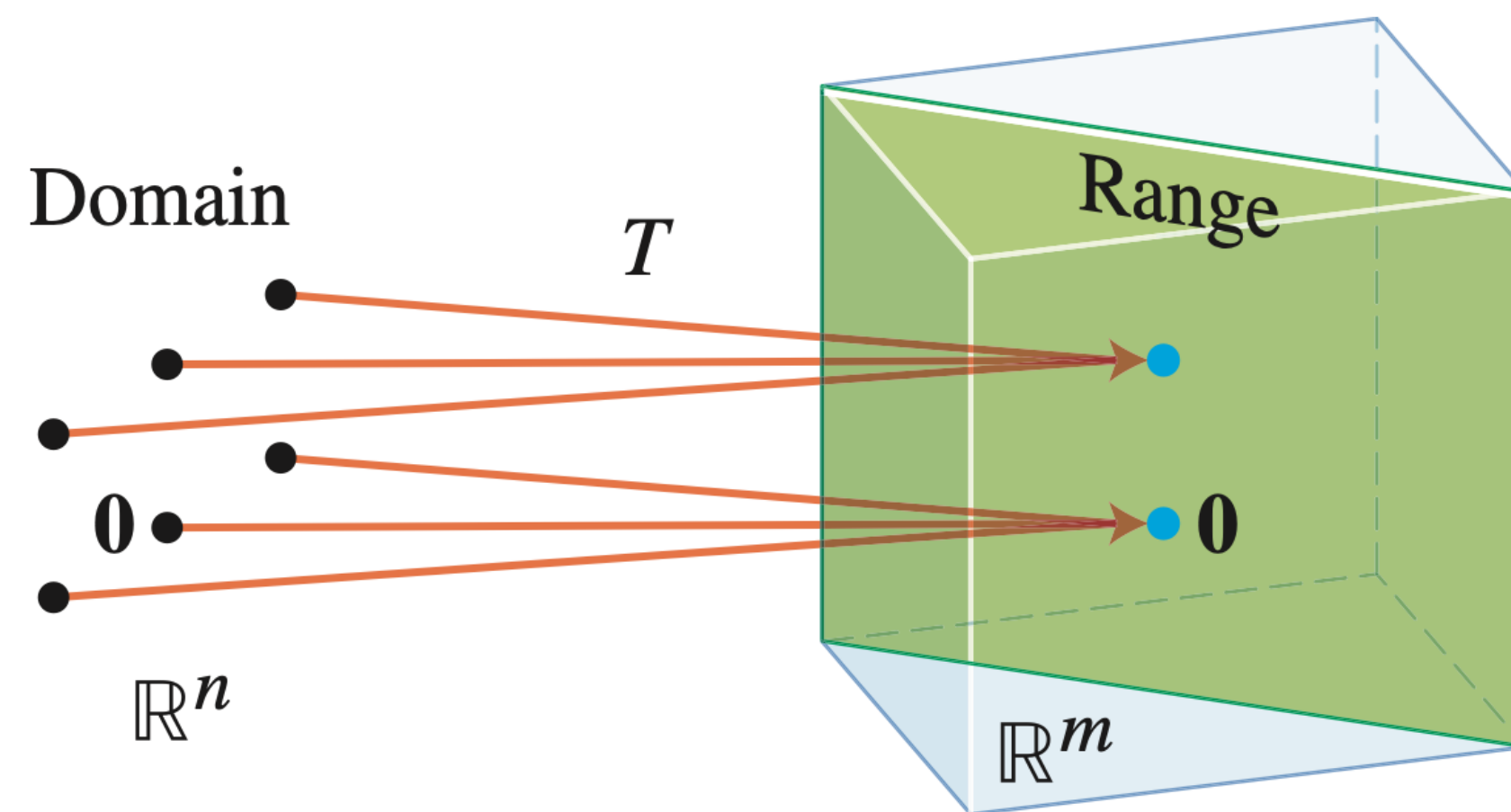
Recall: One-to-one Transformations

Recall: One-to-one Transformations

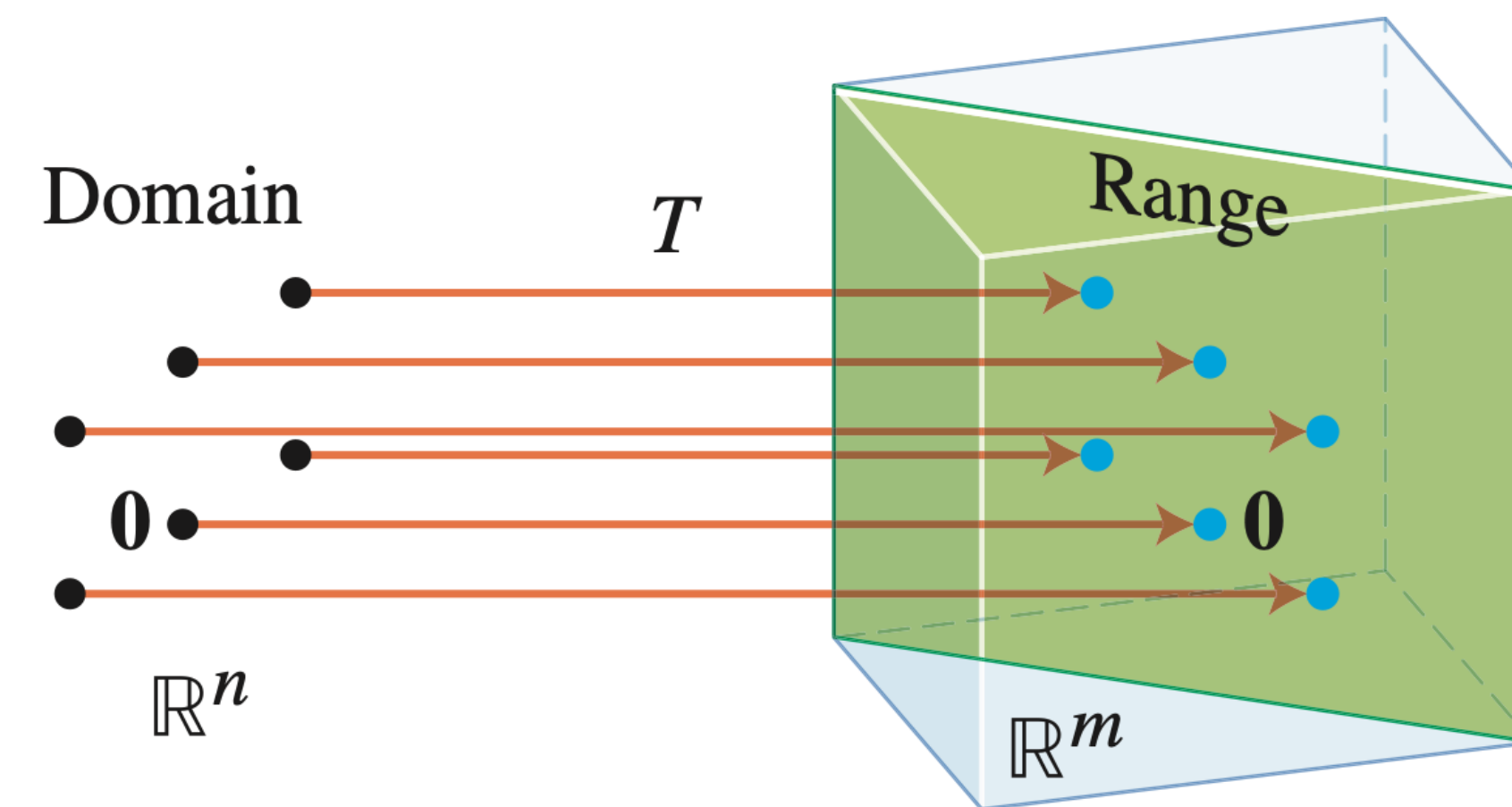
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

Recall: One-to-one Transformations

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T is *not* one-to-one



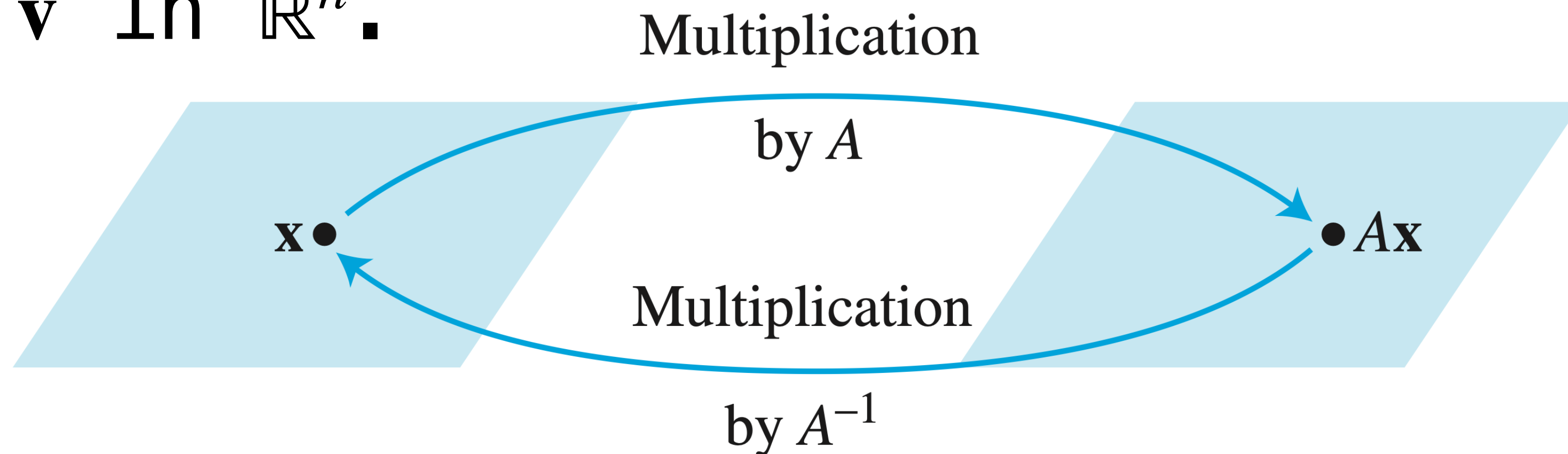
T is one-to-one

Recall: Invertible Transformations

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$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n .



Recall: One-to-One Correspondence

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Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

Recall: One-to-One Correspondence

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

Recall: One-to-One Correspondence

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Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

- 11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
- 12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
- 13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
- 14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any \mathbf{b}
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span \mathbb{R}^n
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1-1** if and only if A **is onto**

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Warning. Remember this only applies square matrices.

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Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True or False: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), then B is also invertible.

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ is invertible, then is $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]$ also invertible? Justify your answer.

Answer

Consider $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$. We can get to $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]^T$ by row operations

Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.