Assignment 1

`CAS CS 132`: *Geometric Algorithms*

Due September 11, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual.

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises.

Basic Problems

1. Determine the coefficient matrix and the augmented matrix of the following linear system.

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= 2 \\ 2x_1 - 3x_2 - 5x_3 &= 2 \\ -2x_1 + 2x_2 + 7x_3 &= -1 \end{aligned}$$

2. Determine the linear system whose augmented matrix is the following.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 7 \\ 1 & 3 & 0 & 2 & 15 \\ -2 & -6 & 0 & -3 & -27 \end{bmatrix}$$

3. Verify that $(1, 3, 2, 3)$ is a solution of the following linear system.

$$\begin{aligned} x_1 - 2x_2 + x_3 - 2x_4 &= -9 \\ x_1 - x_2 - x_3 - 2x_4 &= -10 \\ -3x_1 + 8x_2 - 6x_3 + 4x_4 &= 21 \\ 2x_2 - 7x_3 + 7x_4 &= 13 \end{aligned}$$

4. Demonstrate that the following linear system has a unique solution. Also determine the solution.

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= -7 \\ -x_1 + 3x_2 + 2x_3 &= 10 \\ 2x_1 - 6x_2 - 3x_3 &= -18 \end{aligned}$$

5. Apply the row operations:

$$\begin{aligned} R_4 &\leftarrow -R_4 \\ R_2 &\leftarrow R_2 + -2R_2 \\ R_2 &\leftarrow R_2 + -5R_4 \\ R_3 &\leftarrow R_3 + 3R_4 \\ R_3 &\leftrightarrow R_2 \end{aligned}$$

to the following matrix.

$$\begin{bmatrix} 9 & 5 & -7 & -5 & -9 \\ 5 & -7 & 1 & -2 & -9 \\ 5 & 1 & -10 & 6 & -5 \\ 5 & 7 & -5 & 2 & 1 \end{bmatrix}$$

6. Determine a general form solution for a linear system whose augmented matrix is row equivalent to the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 1 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Determine the reduced echelon form of the following matrix.

$$\begin{bmatrix} 1 & -1 & -2 & 1 \\ -1 & 2 & 4 & 0 \\ 2 & -3 & -6 & 2 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

1. Elementary row operations cannot change the solution set of a linear system.
2. There is a linear system with exactly three solutions.
3. If A is the augmented matrix of an inconsistent linear system, and B is a matrix such that $A \sim B$ (that is, A and B are row equivalent), then B is the augmented matrix of an inconsistent linear system.
4. If $A \sim B$ and $A \sim C$ and B and C are in reduced echelon form, then $B = C$.
5. There is a unique sequence of row operations that reduces a given matrix to reduced echelon form.
6. If a general form solution of a linear system has a free variable, then the system must have infinitely many solutions.

7. A matrix may have different pivot positions depending on the sequence of row operations used to attain a matrix in echelon form.

8. A linear system over 3 variables and 2 equations must be consistent.

9. If the coefficient matrix of a linear system as more rows than columns, then the system must have infinitely many solutions.

More Difficult Problems

1. For what values of the coefficient h is the following system inconsistent?

$$\begin{aligned} x + 4y &= -1 \\ 3x - hy &= 7 \end{aligned}$$

   Is there a value of h for which the above system has infinitely many solutions? Justify your answer.

2. Consider the following linear system with two unknown coefficients h and k.

$$\begin{aligned} hx + 2y &= 1 \\ 3x + 9y &= k \end{aligned}$$

   (a) Determine values of h and k so that the above linear system has no solutions.
   (b) Determine values of h and k so that the above linear system has exactly one solution.
   (c) Determine values of h and k so that the above linear system has infinitely many solutions.

Challenge Problems (Optional)

1. Consider the following general form solution.

$$\begin{aligned} &x_1 = -6 + 6x_3 + 2x_5 \\ &x_2 = 4 + 4x_3 + 6x_5 \\ &x_3 \text{ is free} \\ &x_4 = -4 + 5x_5 \\ &x_5 \text{ is free} \end{aligned}$$

   Determine a general form solution that describes the same solution set but in which x_1 is free.

2. Determine what must hold of a, b, c, d, f, and g so that the following system is inconsistent.

$$\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned}$$