Singular Value Decomposition

Geometric Algorithms
Lecture 26

CAS CS 132

Objectives

1. Introduce the **singular value decomposition** (probably the most important matrix decomposition for computer science)

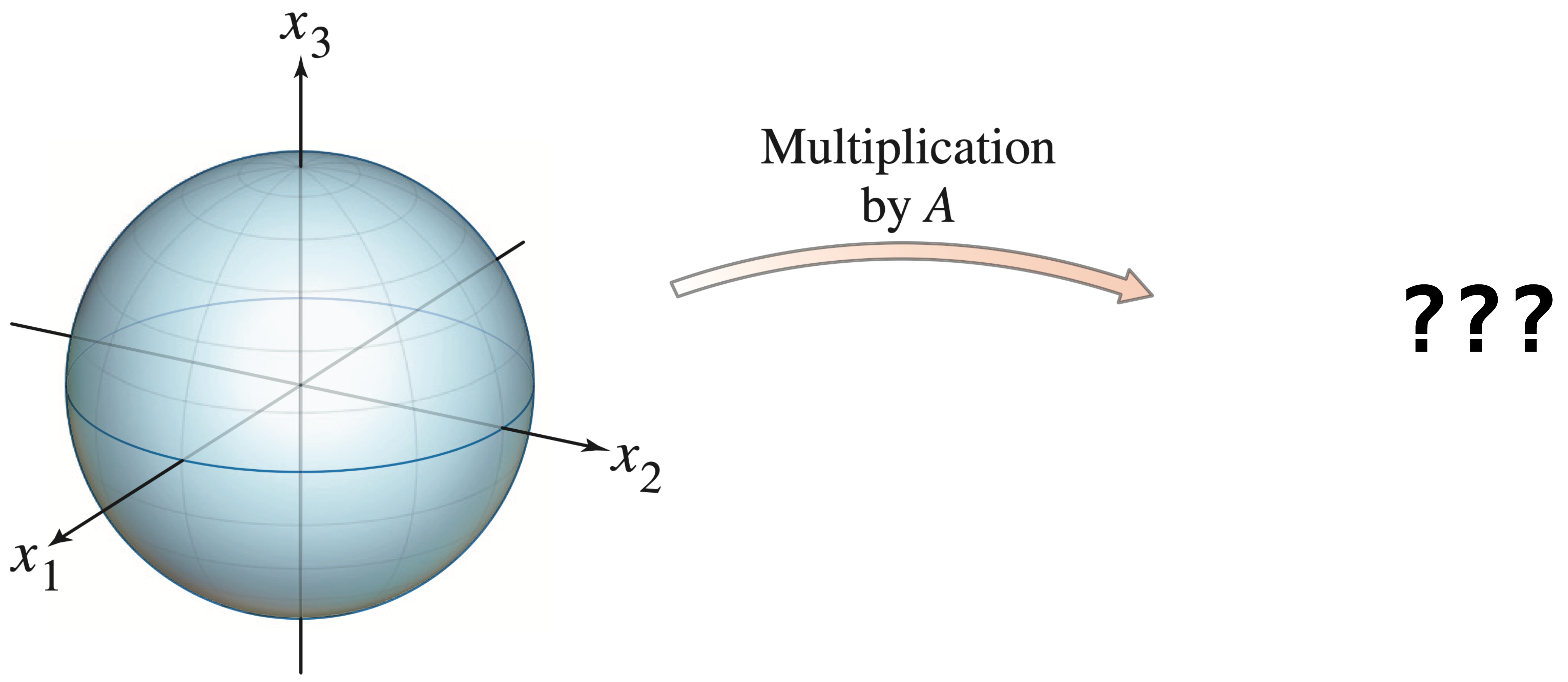
2. Talk very briefly about what to do after this course if you want (or have to) to see more linear algebra

3. ~~Fill out course evals(!)~~ Next time

Motivation

Question

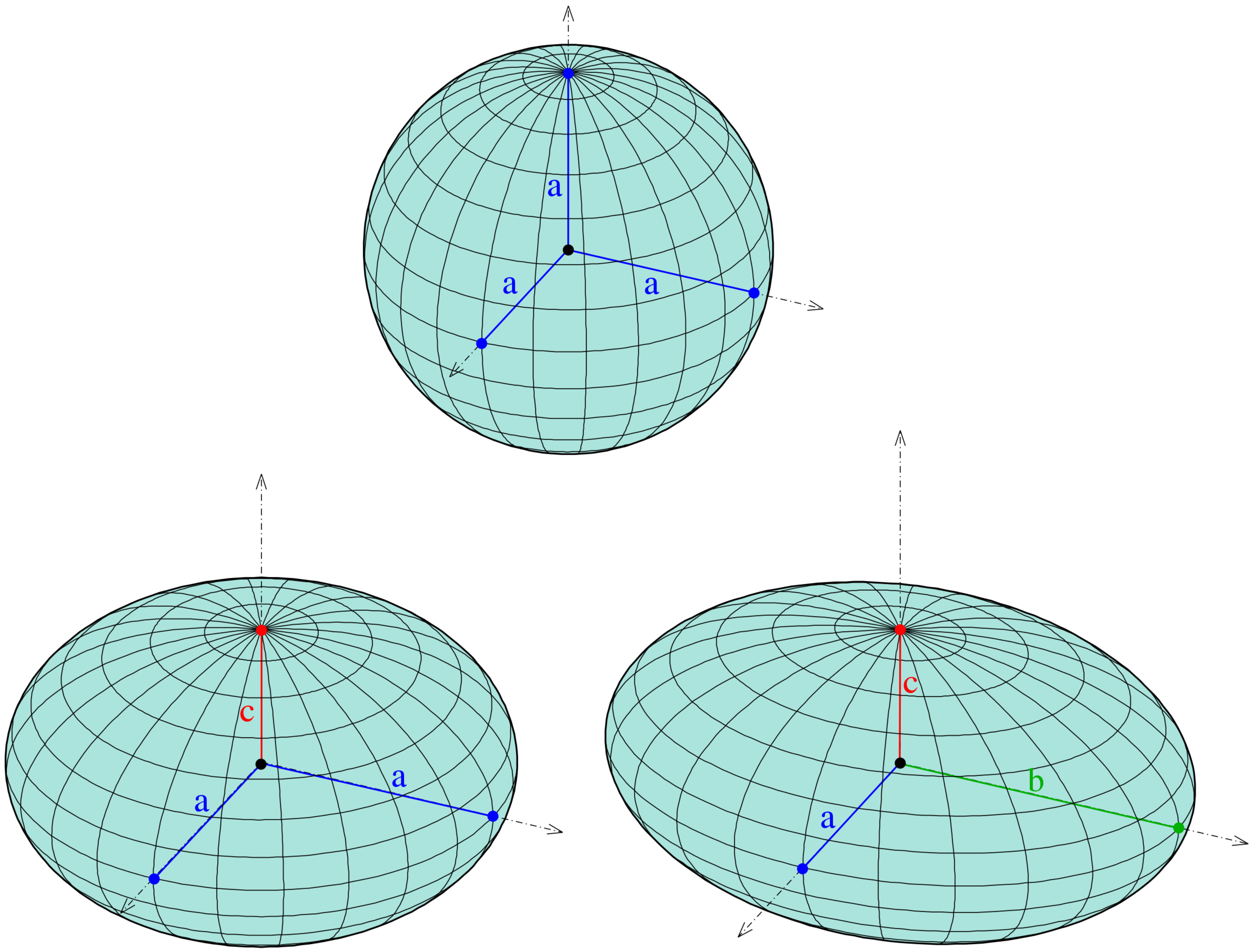
What shape is a the unit sphere after a linear transformation?



Ellipsoids

Ellipsoids are spheres "stretched" in orthogonal directions called the **axes of symmetry** or the **principle axes.**

Linear transformations maps <u>spheres</u> to <u>ellipsoids</u>.

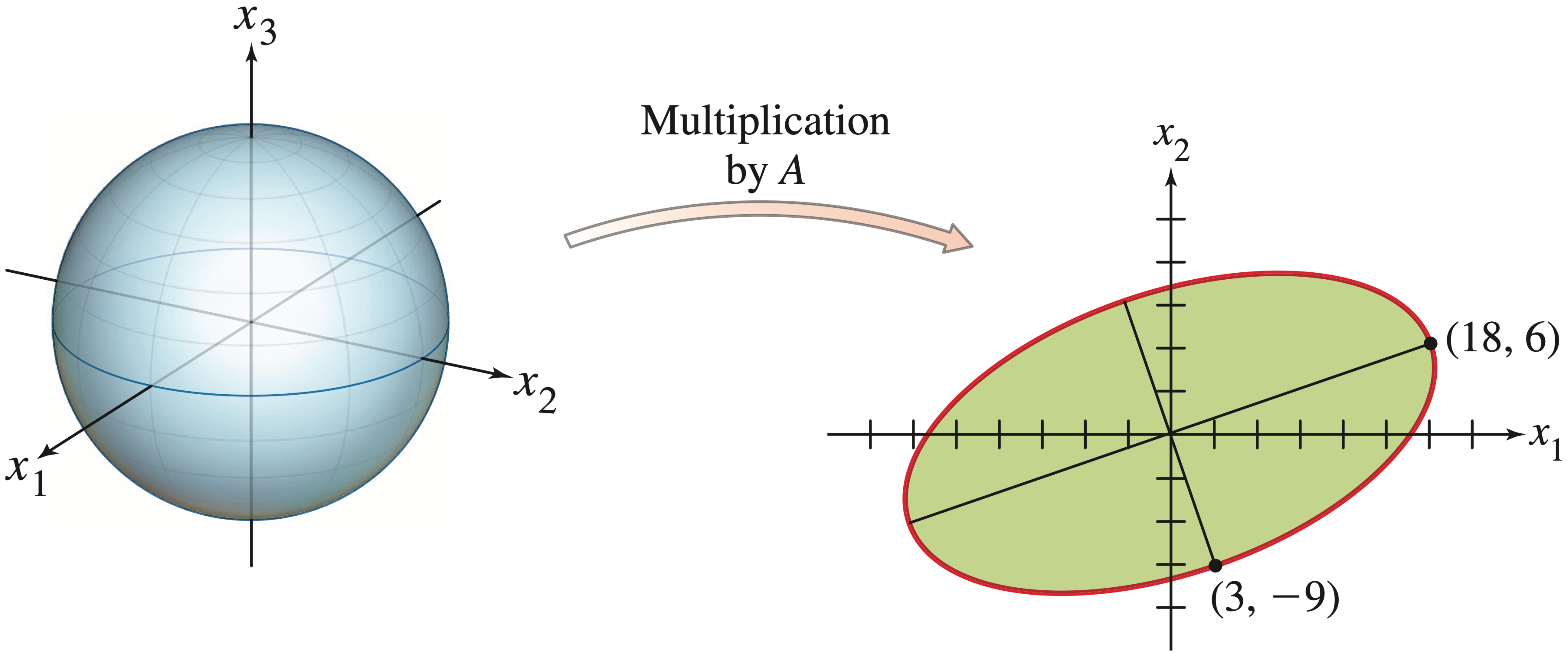


https://commons.wikimedia.org/wiki/File:Ellipsoide.svg

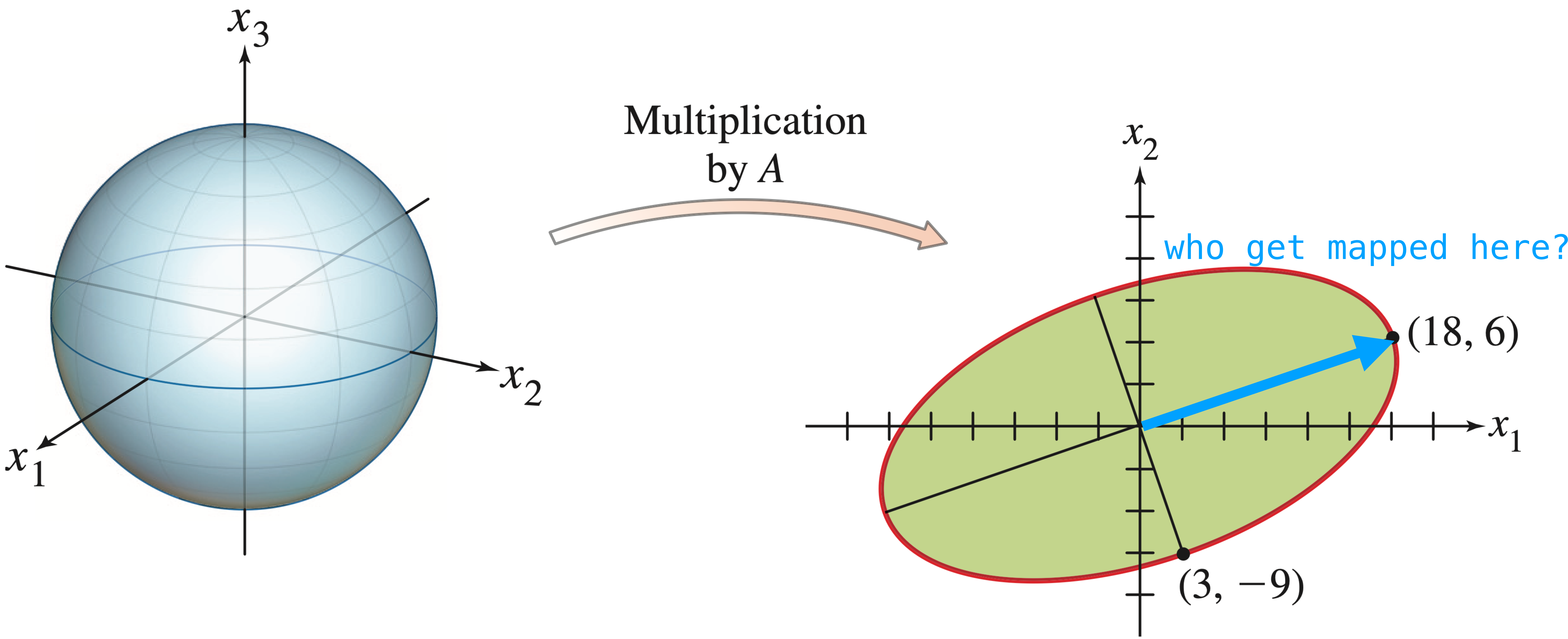
Simple Example : Scaling Matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

The Picture

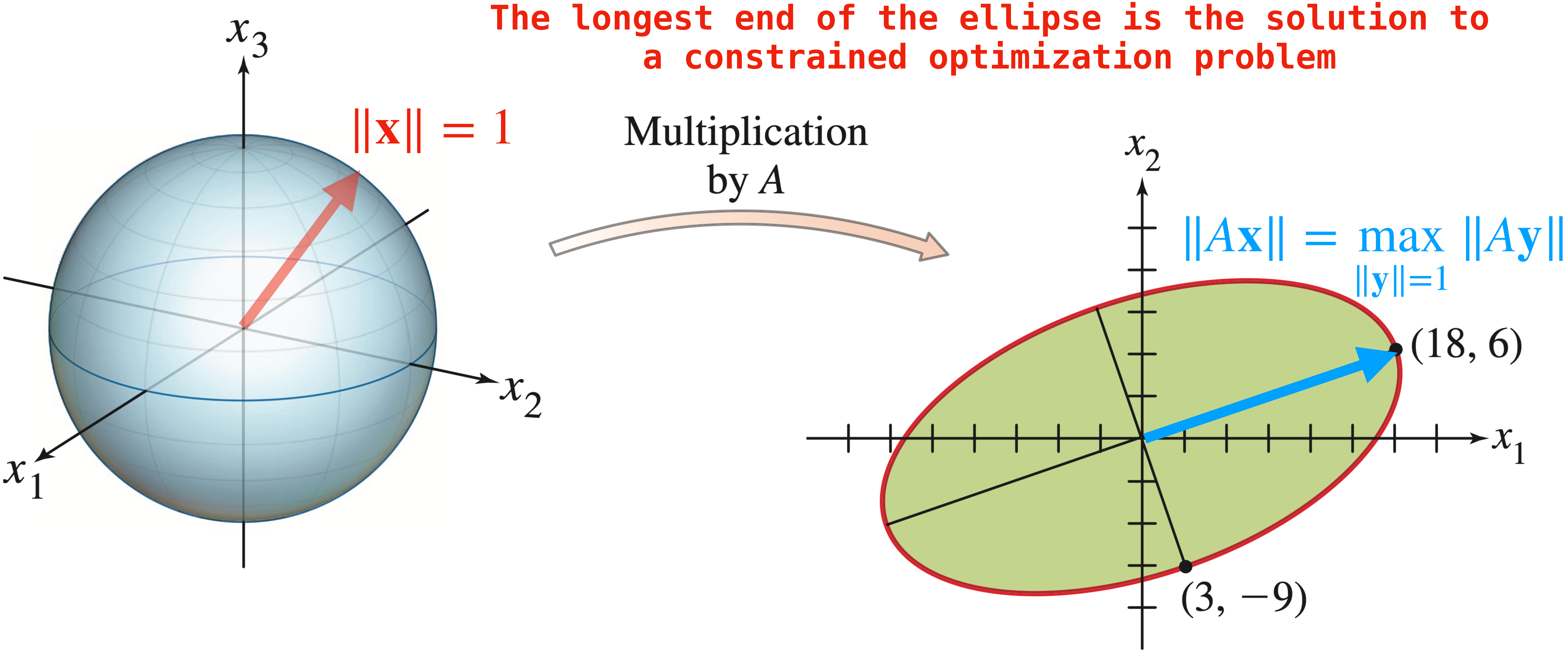


The Picture

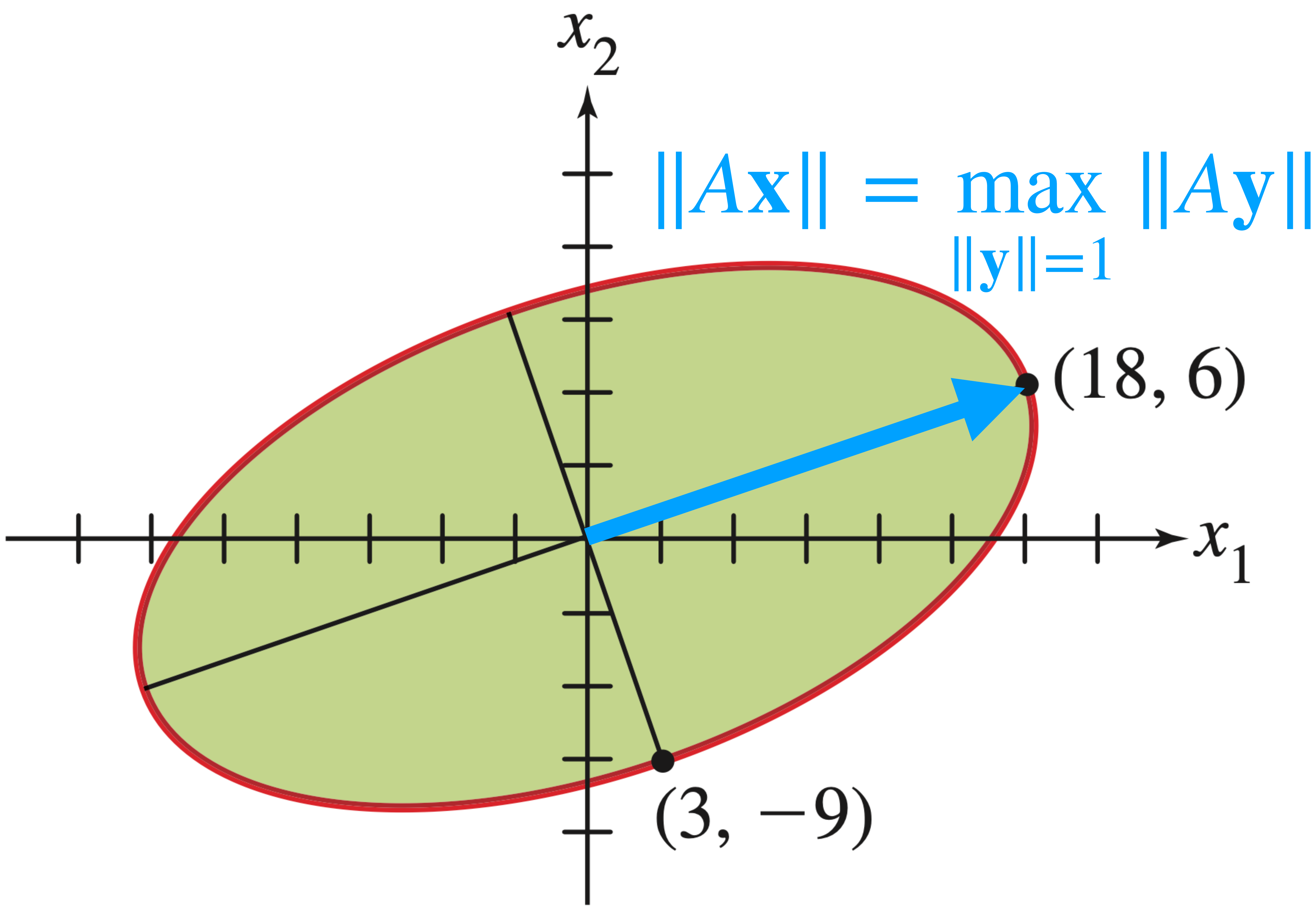


The Picture

The longest end of the ellipse is the solution to a constrained optimization problem

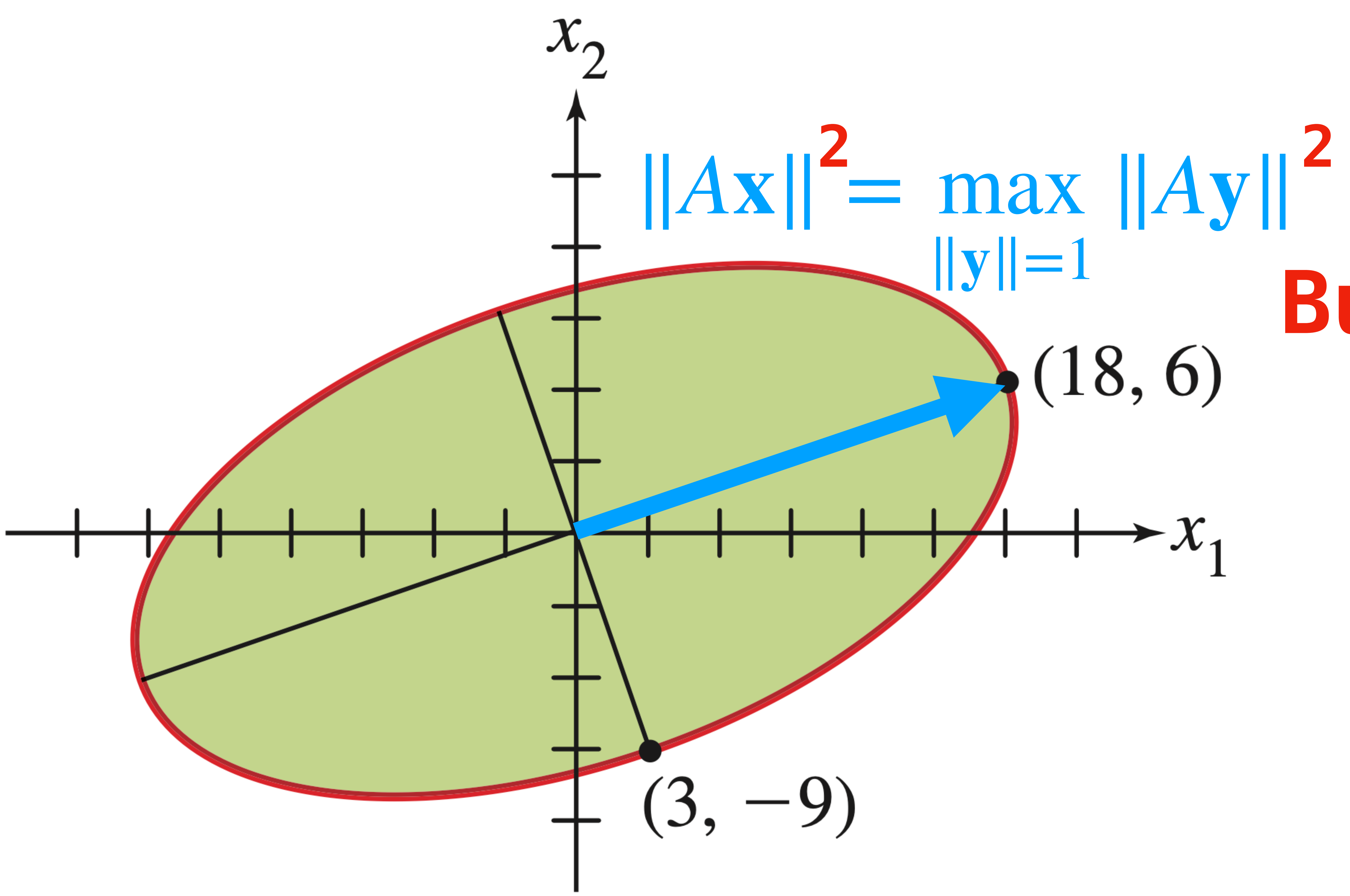


The Picture



This is not a quadratic form...

The Picture



But this is.

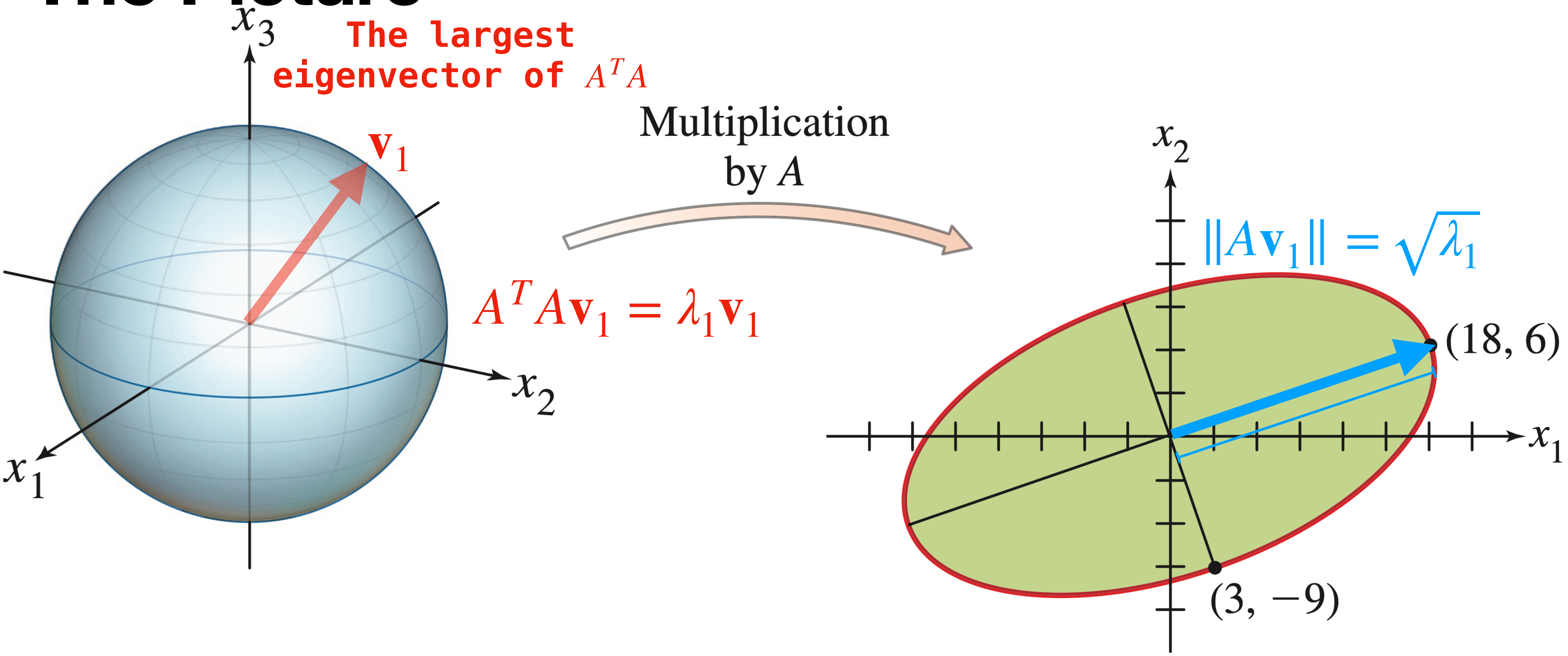
This is not a quadratic form...

A Quadratic Form

What does $\|A\mathbf{x}\|^2$ look like?:

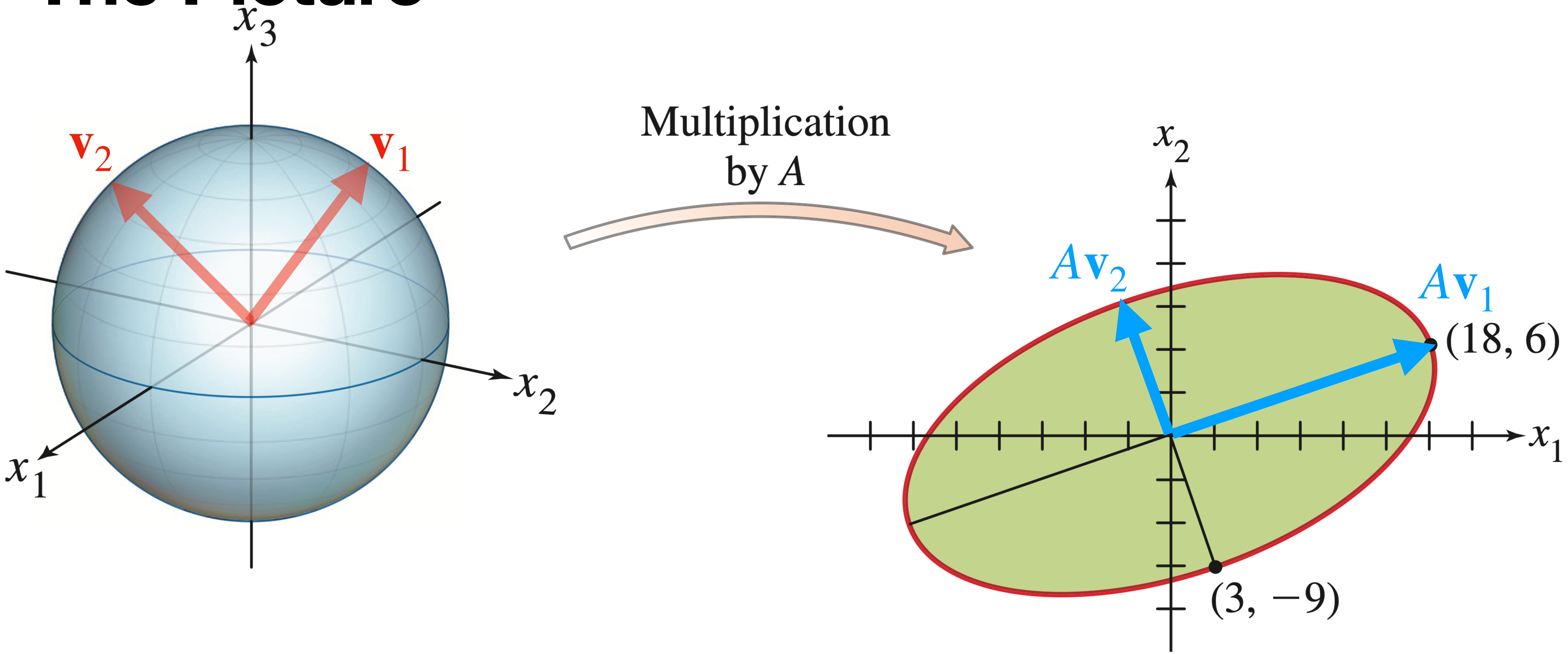
$$(Ax)^T(Ax) = \vec{x}^T \underbrace{A^T A}_{\text{symmetric}} \vec{x}$$

The Picture



$\mathbf{v}_1$ solves the constrained optimization problem.

The Picture



The second eigenvector is sent to the *minimum* principle axis

Properties of $A^T A$

Properties of $A^T A$

$(A^T A)^T = A^T A$

» It's symmetric

Properties of $A^T A$

» It's symmetric

» So its <u>orthogonally diagonalizable</u>

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- **There is an orthogonal basis of eigenvectors**

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Properties of $A^T A$

- It's symmetric
- So its <u>orthogonally diagonalizable</u>
- **There is an orthogonal basis of eigenvectors**
- It's eigenvalues are nonnegative
- **It's positive semidefinite**, why?

if $\vec{x} \neq 0$, $\vec{x}^T A^T A \vec{x} = \|A\vec{x}\|^2 \geq 0$

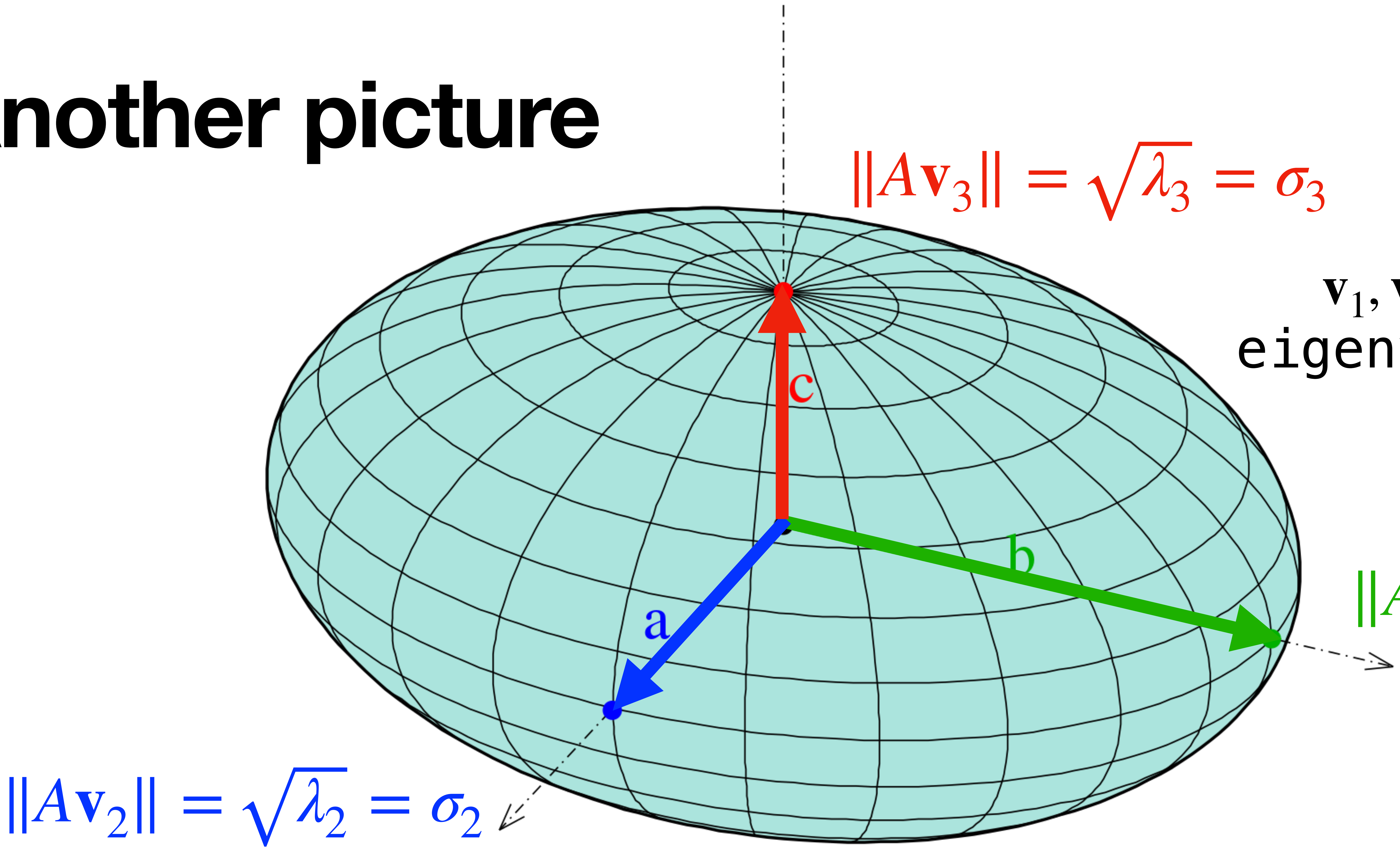
Singular Values

Definition. For an $m \times n$ matrix A, the **singular values** of A are the n values

$$\sigma_1 \geq \sigma_2 \ldots \geq \sigma_n \geq 0$$

where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is an eigenvalue of $A^T A$.

Another picture



$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the eigenvectors of A^TA

The **singular values** are the <u>lengths</u> of the *axes of symmetry* of the ellipsoid after transforming the unit sphere.

https://commons.wikimedia.org/wiki/File:Ellipsoide.svg

Every $m \times n$ matrix transforms the unit $\cancel{m}$−sphere into an $\cancel{n}$−ellipsoid

n m

So every $m \times n$ matrix has
n m singular values

What else can we say?

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an **orthogonal** eigenbasis of $\mathbb{R}^n$ for $A^T A$ and suppose A has r <u>nonzero</u> singular values

Theorem. $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ is an orthogonal basis of $\mathrm{Col}(A)$

What else can we say?

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = \dots = \lambda_n$

$\lambda_{r+1} = 0$, $\lambda_n = 0$

$\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_r \quad \underbrace{\vec{v}_{r+1} \quad \dots \quad \vec{v}_n}_{\text{Null}(A^TA) \text{ basis}}$

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an **orthogonal** eigenbasis of $\mathbb{R}^n$ for A^TA and suppose A has r nonzero singular values

Theorem. $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ is an orthogonal basis of Col(A)

This is the most important theorem for SVD

Verifying it

$\lambda_i \neq \lambda_j$

Let's show $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ are orthogonal (and linearly independent):

$$\vec{v}_j^{\,T} A^T A \vec{v}_i = \vec{v}_j^{\,T} \lambda_i \vec{v}_i = \lambda_i \vec{v}_j^{\,T} \vec{v}_i = 0$$

$$\cancel{(A^T A \vec{v}_j)^T \vec{v}_i = \lambda_j \vec{v}_j^{\,T} \vec{v}_i}$$

$$\cancel{(\lambda_i - \lambda_j) \vec{v}_j^{\,T} \vec{v}_i = 0} \quad \neq 0$$

Verifying it

Recall $\vec{v}_1, \dots, \vec{v}_n$ orthonormal eigenbasis

& $\vec{v}_{r+1}, \dots, \vec{v}_n$ basis for $\mathrm{Nul}\, A^TA$

Let's show $A\mathbf{v}_1, \dots, A\mathbf{v}_r$ span $\mathrm{Col}(A)$:

For any $A\vec{x}$, want

$A\vec{x} = c_1 A\vec{v}_1 + \cdots + c_r A\vec{v}_r$

for some constants $c_1, \dots, c_r$

$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

$\Rightarrow A\vec{x} = c_1 A\vec{v}_1 + \cdots + c_n A\vec{v}_n$

Q: Can we lop off the last $n-r$ terms?

Yes!, so $A\vec{x} = c_1 A\vec{v}_1 + \cdots + c_r A\vec{v}_r$

$\boxed{\text{DONE}}$

Fact: $\mathrm{Nul}(A) = \mathrm{Nul}(A^TA)$

Pf: $\vec{x} \in \mathrm{Nul}\, A \Rightarrow A\vec{x} = \vec{0} \Rightarrow A^TA\vec{x} = \vec{0}$

$\Rightarrow \vec{x} \in \mathrm{Nul}\, A^TA$

so $\mathrm{Nul}\, A \subseteq \mathrm{Nul}\, A^TA$

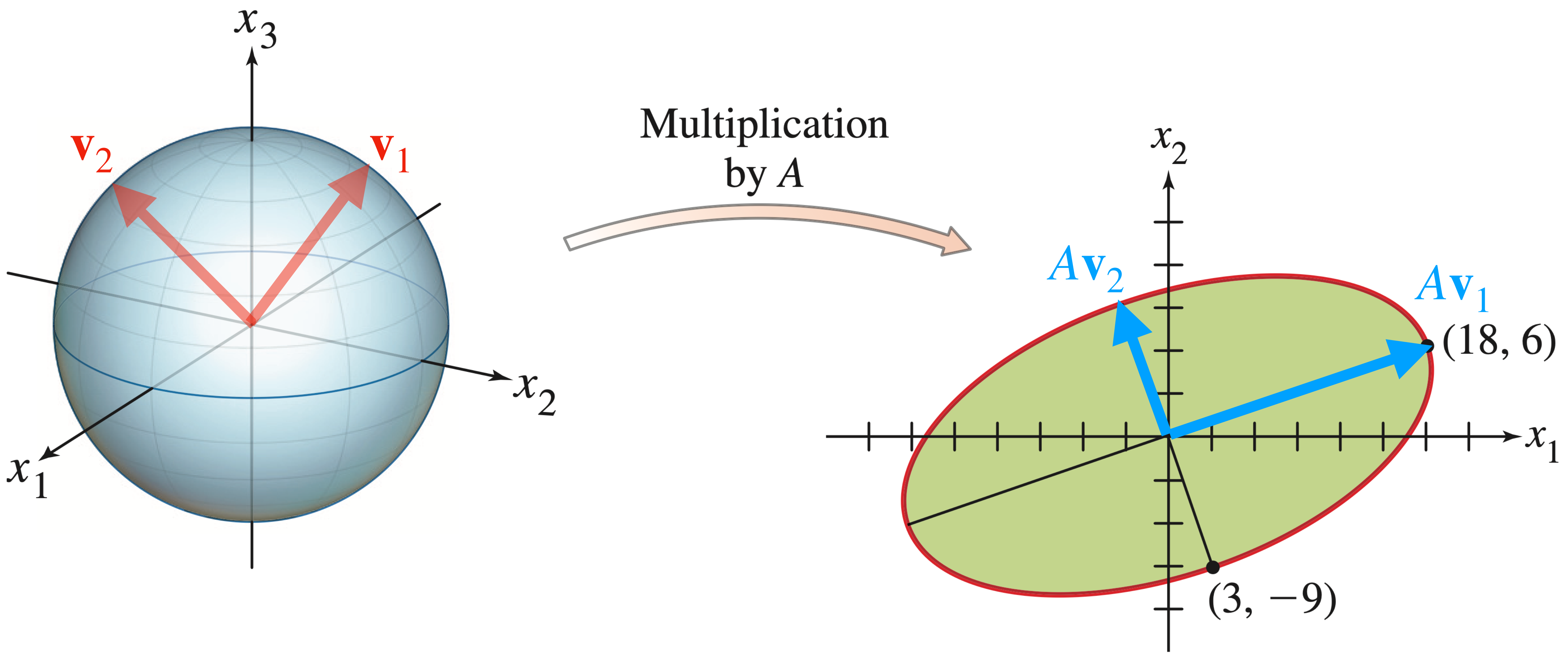
$\vec{x} \in \mathrm{Nul}\, A^TA \Rightarrow A^TA\vec{x} = \vec{0}$

$\Rightarrow \vec{x}^T A^T A\vec{x} = 0 \Rightarrow \|A\vec{x}\|^2 = 0$

$\Rightarrow A\vec{x} = 0 \Rightarrow \vec{x} \in \mathrm{Nul}(A)$

so $\mathrm{Nul}\, A^TA \subseteq \mathrm{Nul}\, A$

Putting it all together



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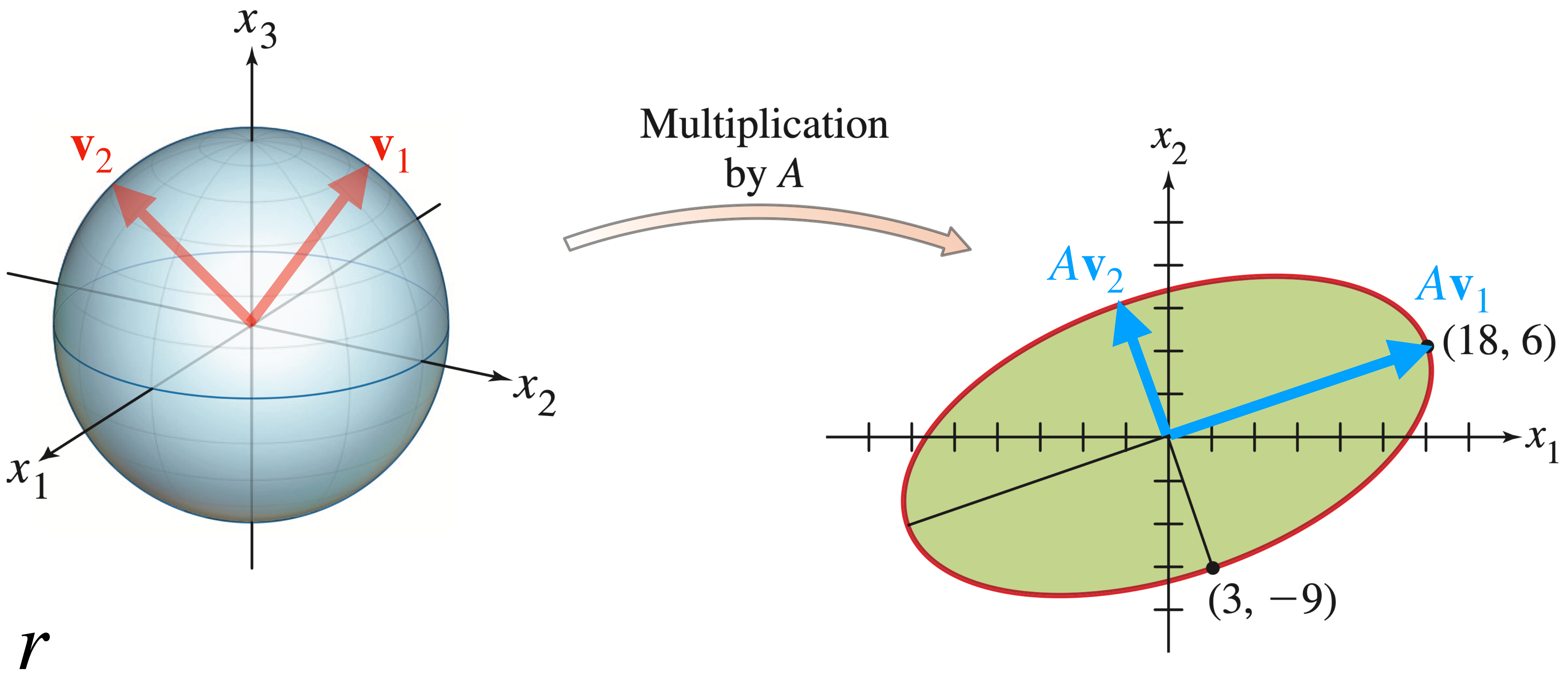
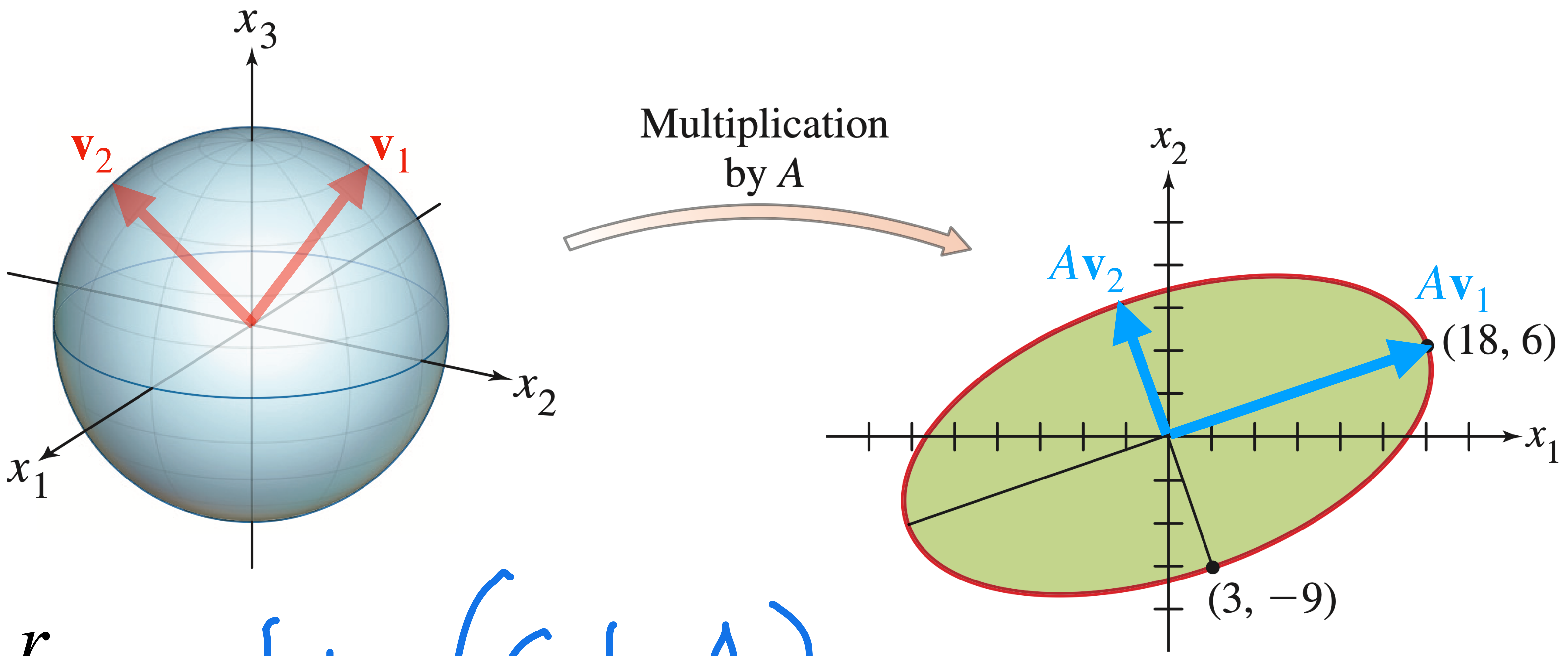


Let A be an $m \times n$ matrix of rank r

Putting it all together

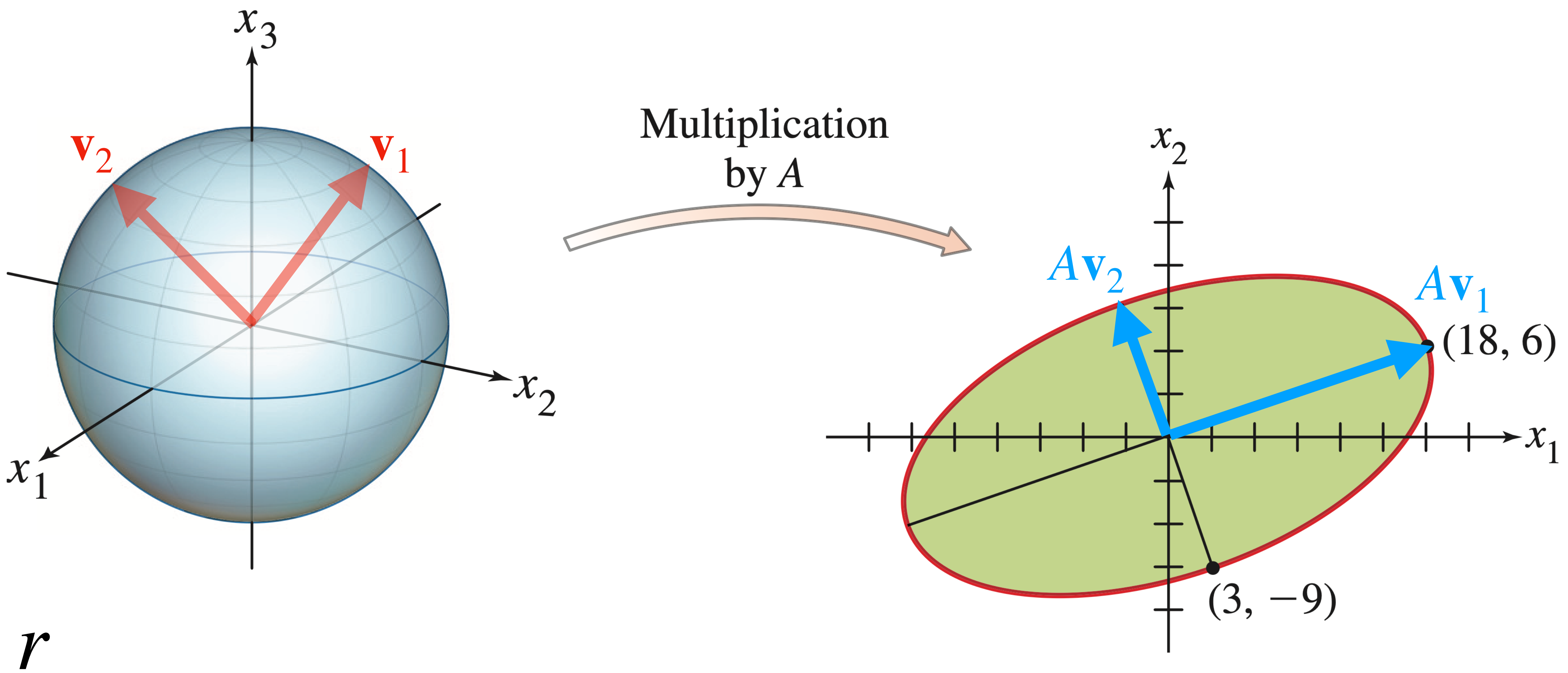


Let A be an $m \times n$ matrix of rank r

dim(Col A)

<u>What we know:</u>

Putting it all together

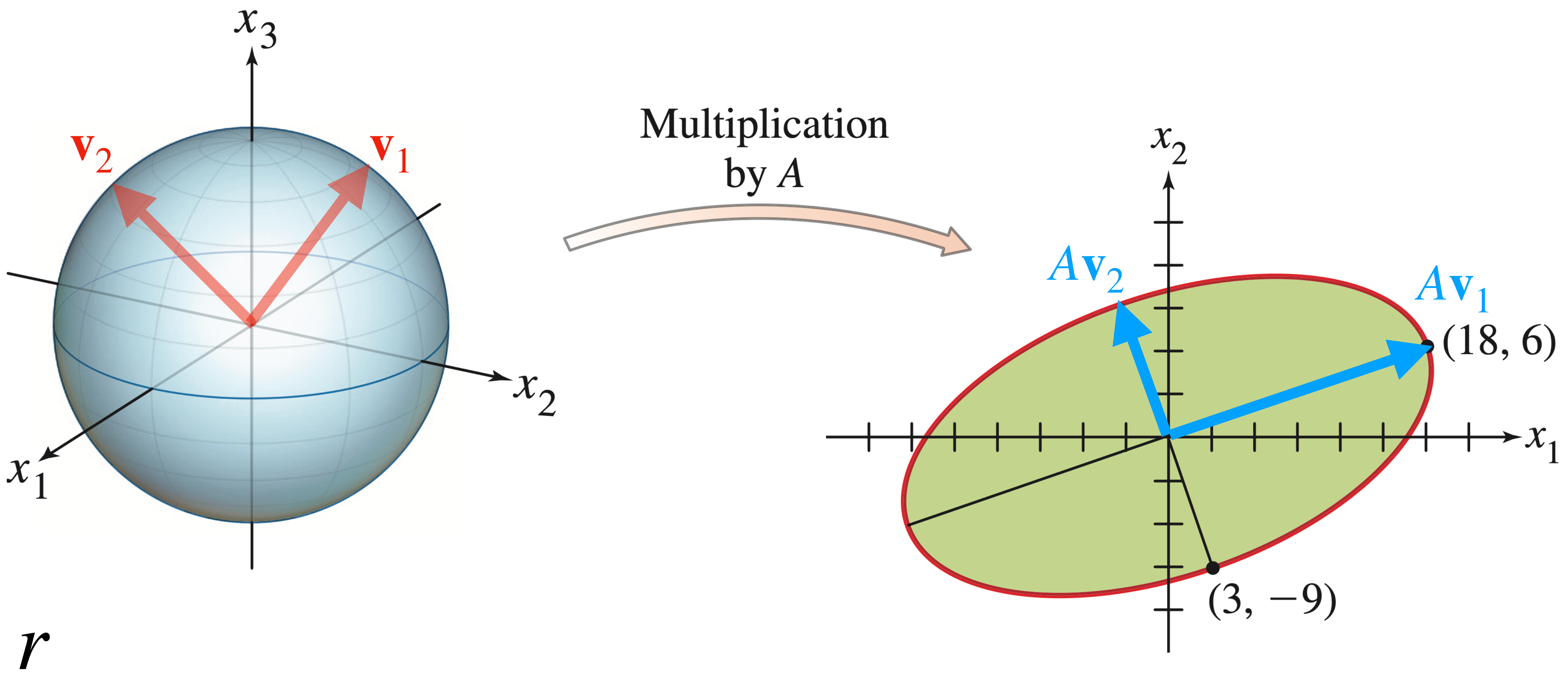


Let A be an $m \times n$ matrix of rank r

<u>What we know:</u>

» We can find orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ such that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ in $\mathbb{R}^m$ form an orthogonal basis for $\text{Col}(A)$

Putting it all together



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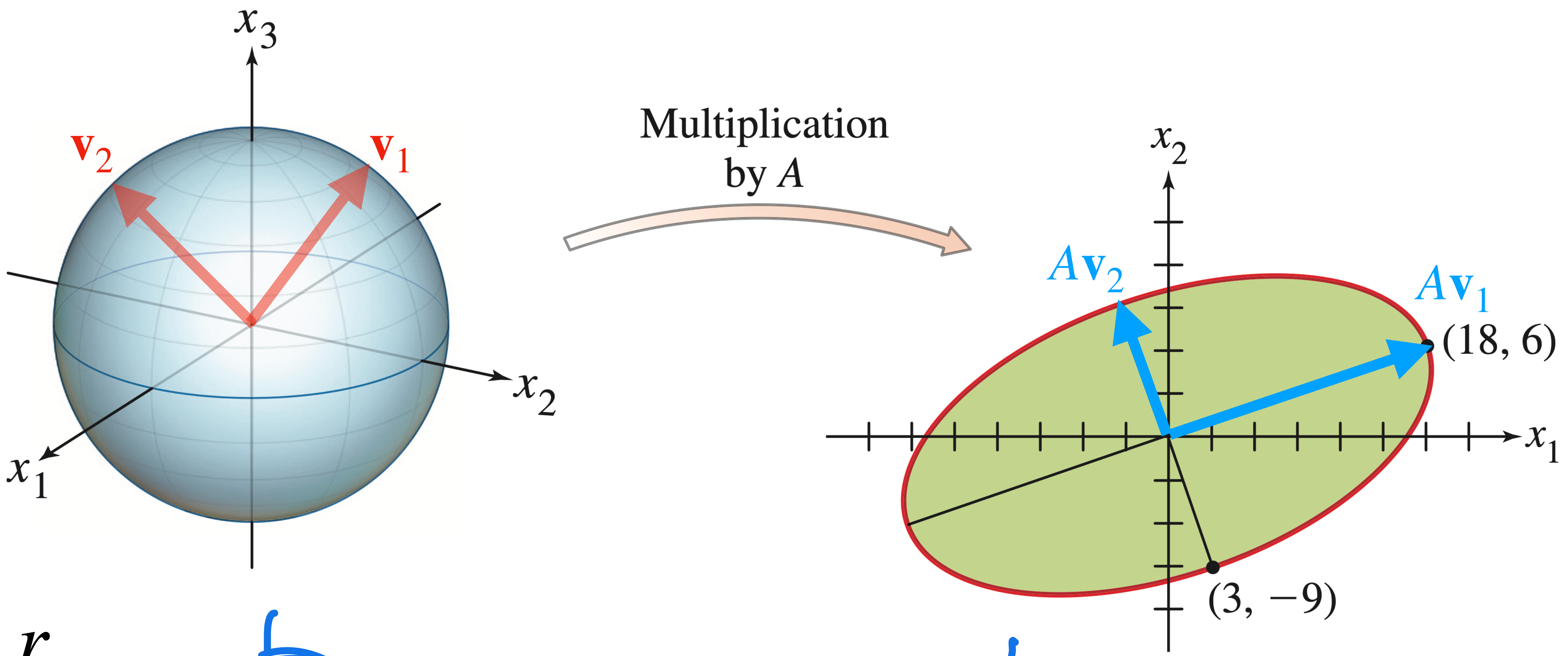
» So if we take $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$, we get an **orthonormal** basis of $\text{Col}(A)$

Putting it all together



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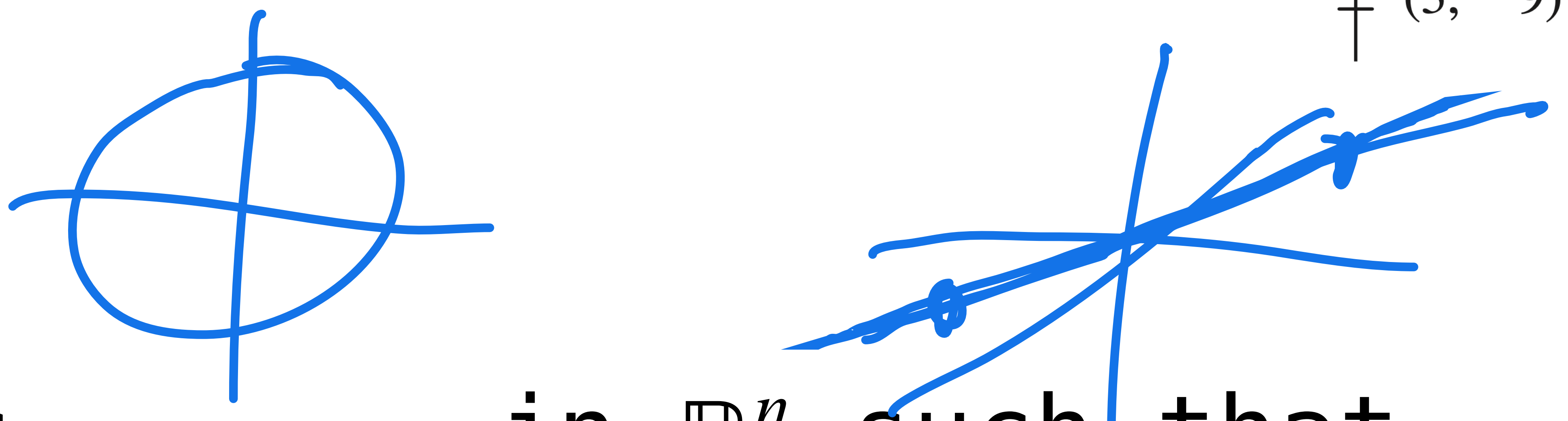


<u>What we know:</u>

» We can find orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ such that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ in $\mathbb{R}^m$ form an orthogonal basis for $\mathrm{Col}(A)$

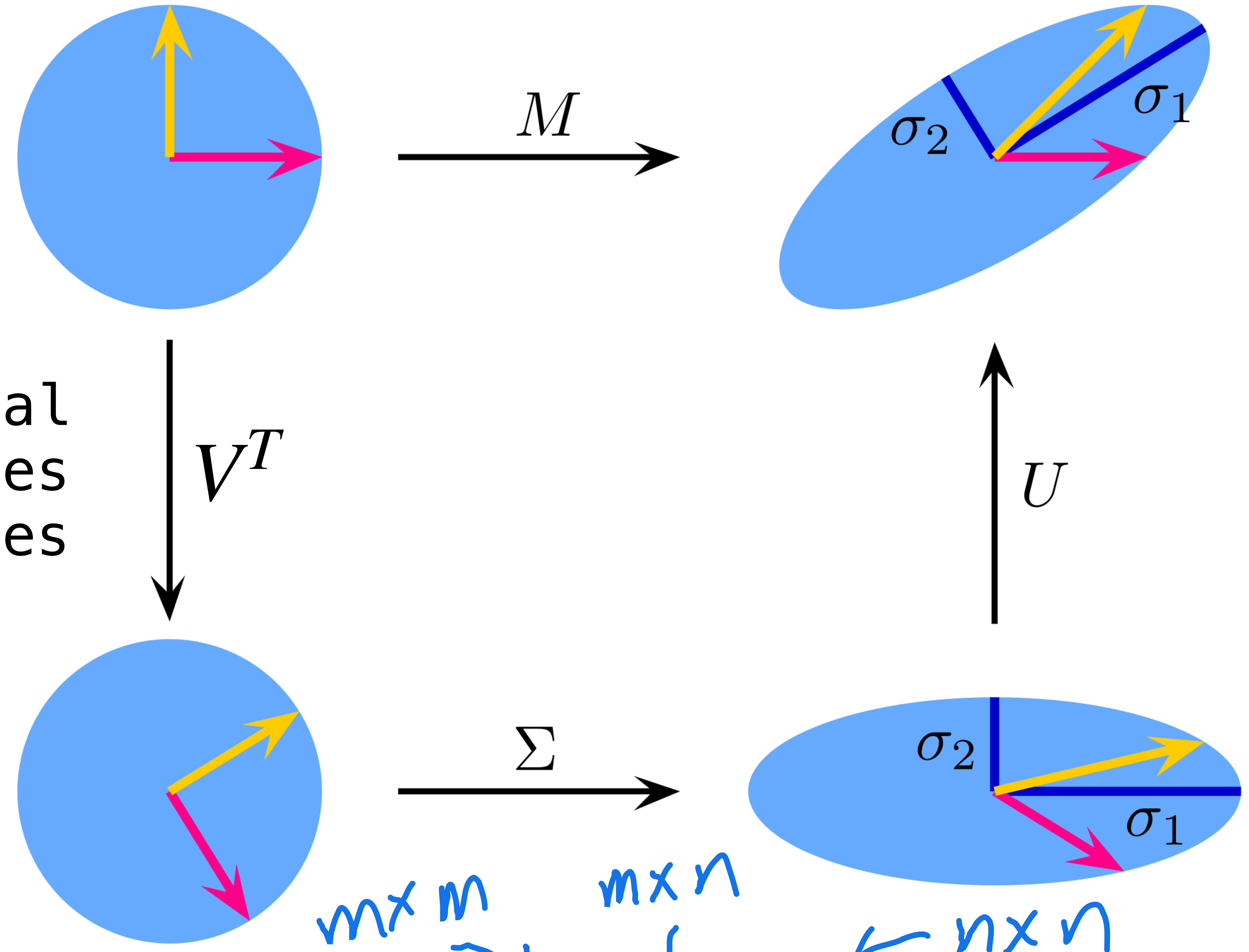
» So if we take $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$, we get an **orthonormal** basis of $\mathrm{Col}(A)$

» And we can extend this to $\mathbf{u}_1, \ldots, \mathbf{u}_m$ an orthonormal basis of $\mathbb{R}^m$ (via Gram–Schmidt).

(didn't cover technically)

Singular Value Decomposition

High Level View of the Decomposition



$$\begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix}$$

recall: Orthogonal matrices preserves lengths and angles

$$M = \overset{m \times m}{U} \cdot \overset{m \times n}{\Sigma} \cdot \overset{n \times n}{V^T}$$

https://commons.wikimedia.org/wiki/File:Singular-Value-Decomposition.svg

The Important Equality

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$$

$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

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Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$

The Important Equality

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$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$

What happens when we write this in matrix form?

The Important Equality

$$A[\mathbf{v}_1 \ \ \dots \ \ \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \ \ \dots \sigma_n \mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

The Important Equality

$$A[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \ \ldots\sigma_n\mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \ldots \ \mathbf{u}_m]$ and

The Important Equality

$$A[\mathbf{v}_1 \ \ \dots \ \ \mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \ \ \dots\sigma_n\mathbf{u}_n]$$

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Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

$$\underset{m>n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}} \text{ or } \underset{m<n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}} \text{ or } \underset{m=n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}}$$

The Important Equality

$$A[\mathbf{v}_1 \ \ \dots \ \ \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \ \ \dots \sigma_n \mathbf{u}_n]$$

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remember: U is orthonormal

$$\overset{m>n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}} \text{ or } \overset{m<n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}} \text{ or } \overset{m=n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}}$$

The Important Equality

$$AV = U\Sigma$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

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$m > n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

$m < n$

or $$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

$m = n$

or $$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

$$\underset{m \times n}{A}\,\underset{n \times n}{V} = \underset{m \times m}{U}\,\underset{m \times n}{\Sigma}$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

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$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

or

$m = n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

$$AVV^T = U\Sigma V^T$$

$\| \; Id$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \; \dots \; \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \; \dots \; \mathbf{u}_m]$ and

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The Important Equality

$$A = U\Sigma V^T$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

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or

$m = n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

singular value decomposition

$$A = \boxed{U\Sigma V^T}$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

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$m = n$

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Singular Value Decomposition

Theorem. For a $m \times n$ matrix A, there are *orthogonal* matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = \overset{m \times m}{U} \underset{m \times n}{\Sigma} \overset{n \times n}{V^T}$$

where diagonal entries* of Σ are $\sigma_1, \ldots, \sigma_n$ the singular values of A.

* these are diagonal entries in a <u>non-square</u> matrix.

Singular Value Decomposition

Theorem. For a $m \times n$ matrix A, there are *orthogonal* matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

left singular vectors

right singular vectors

$$A = \overset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \overset{n\times n}{V^T}$$

where diagonal entries* of Σ are $\sigma_1, \ldots, \sigma_n$ the singular values of A.

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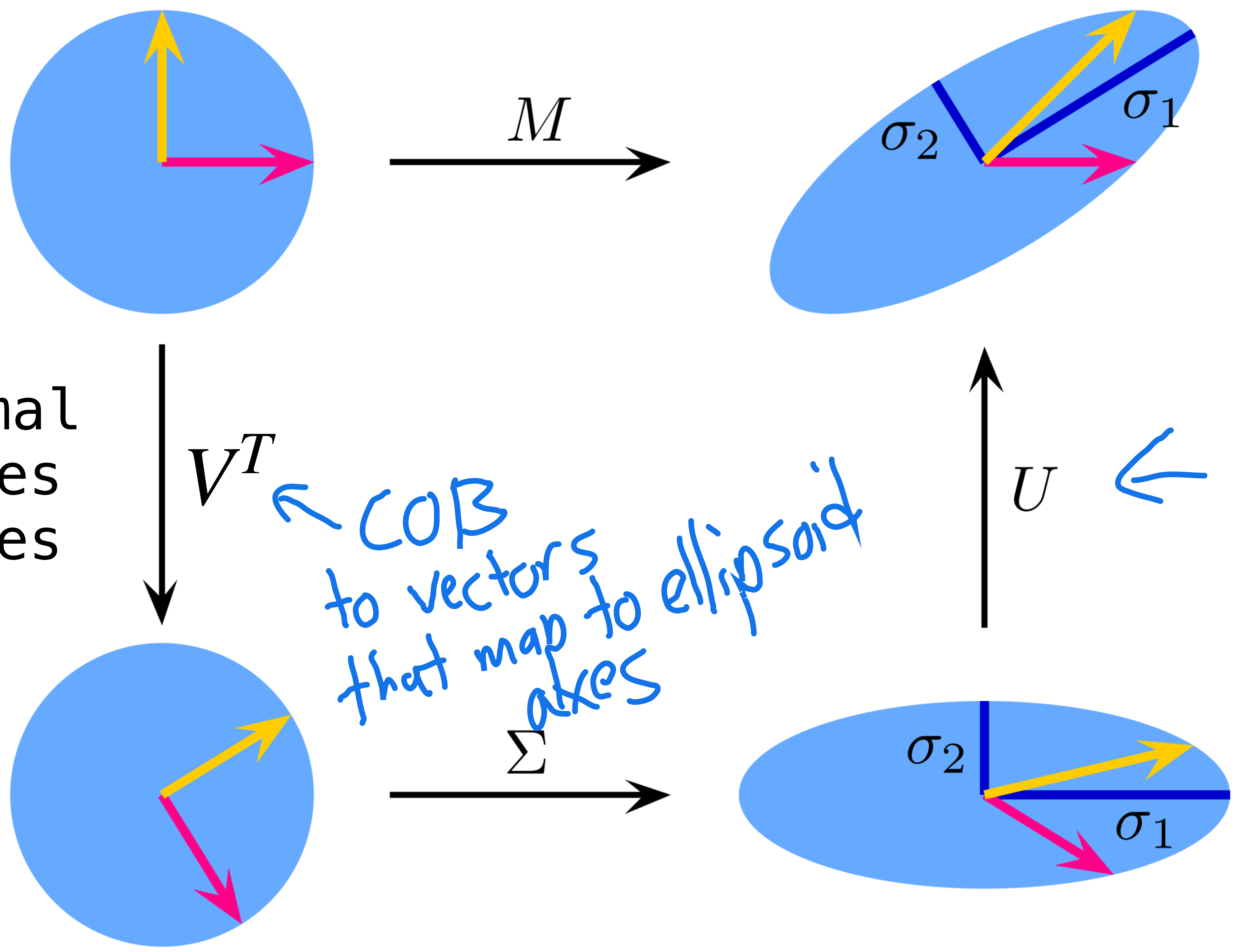
The Picture (Again)



<u>recall</u>: Orthonormal matrices preserves lengths and angles

COB to vectors that map to ellipsoid axes

$\vec{u}_i = \frac{A\vec{v}_i}{\|A\vec{v}_i\|}$

COB to standard basis from basis of ellipsoid axes

(really subsets if one considers nullspaces)

$$M = U \cdot \Sigma \cdot V^T$$

https://commons.wikimedia.org/wiki/File:Singular-Value-Decomposition.svg

How To: Finding a SVD

Step 1: Set up Σ

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

The **singular values** are the square roots of the eigenvalues of A^TA (or AA^T):

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = (9-\lambda)^2 - 81$$

$$= \lambda^2 - 18\lambda + 81 - 81 = \lambda^2 - 18\lambda = \lambda(\lambda - 18)$$

$$\lambda_1 = 18 \geq \lambda_2 = 0$$

$$\sigma_1 = \sqrt{18} \quad \sigma_2 = 0$$

Step 2: Set up V

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

Find an orthonormal eigenbasis for A^TA:

$\lambda_1 = 18 \qquad A^TA - 18I = \left[\begin{array}{cc|c} -9 & -9 & 0 \\ -9 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0 \qquad \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step 3: Set up U (Part 1)

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

If $\mathbf{v}_1 \ldots, \mathbf{v}_n$ is an eigenbasis of $\mathbb{R}^n$ (in decreasing order of eigenvalue), then $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ is an eigenbasis of $\mathrm{Col}(A)$ (where r is the rank of A). These vectors can be normalized and made the first r columns of U:

$r = 1$

$$A\vec{v}_1 = A\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \propto \cancel{\frac{1}{\sqrt{2}}} A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -4 \end{bmatrix} \propto \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Step 4: Set up U (Part 2)

$$\underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}}_{m=3,\ n=2}$$

U $m \times m$

If $m > r$, then extend $\mathbf{u}_1, \ldots, \mathbf{u}_r$ until it has m orthonormal vectors:

V $n \times n$

2×2

$\vec{v}_1 \quad \vec{v}_2$

$\vec{x} \cdot \vec{v}_1 = 0$

$\vec{x} \cdot \vec{v}_2 = 0$

$$\begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 4/\sqrt{18} \\ 1/\sqrt{18} \\ -1/\sqrt{18} \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 2(1)-(-2)(1) \\ -(-1) \\ -1 \end{bmatrix}$$

Step 5: Put everything together

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

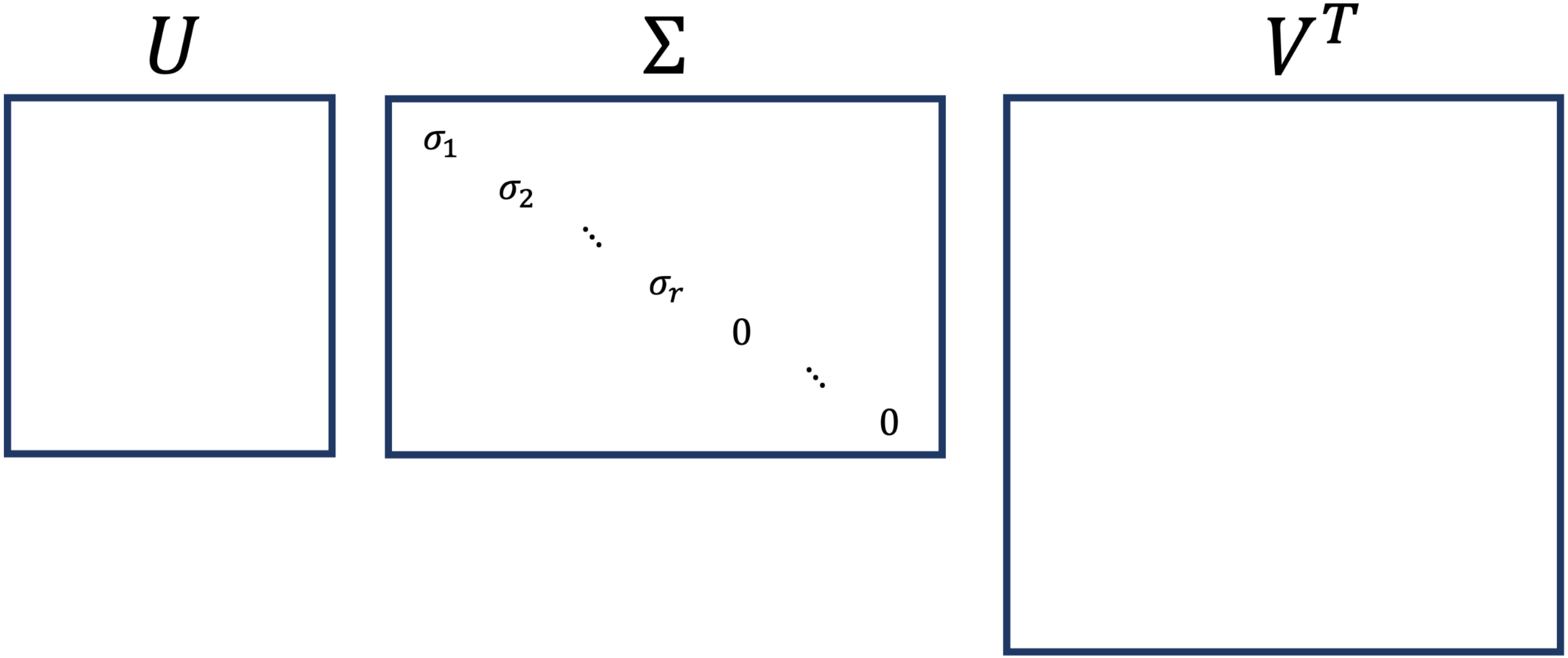
SVD in NumPy

In reality, we will almost never build SVDs by hand. We can use:

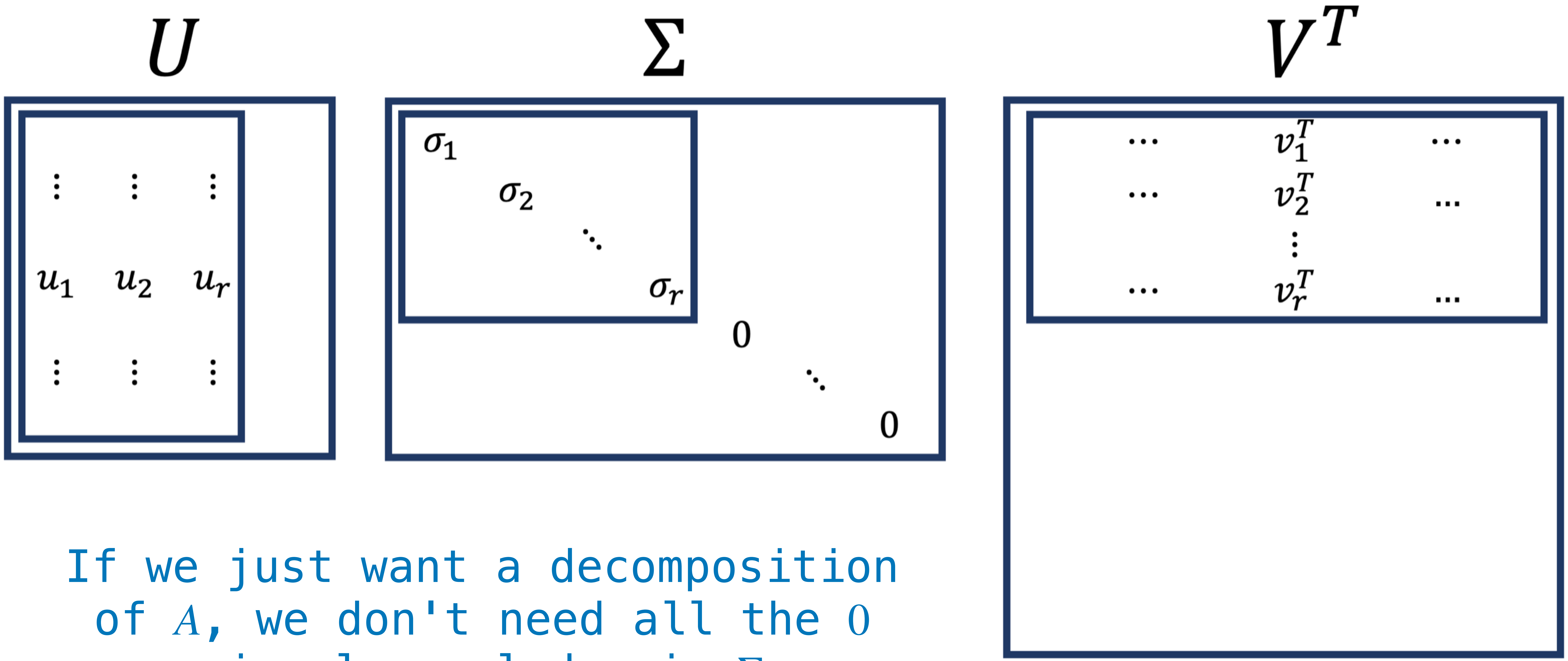
numpy.linalg.svd

Pseudoinverses

SVD (The Picture)



Reduced SVD (The Picture)



If we just want a decomposition of A, we don't need all the 0 singular valudes in Σ

The Reduced SVD

Theorem. For every matrix A of rank r, there is an orthonormal matrix $U \in \mathbb{R}^{m\times r}$, a diagonal matrix $\Sigma \in \mathbb{R}^{r\times r}$ with **positive** entries on the diagonal, and an orthonormal matrix $V \in \mathbb{R}^{n\times r}$ such that

$$A = U\Sigma V^T$$

The Pseudoinverse

Definition. Given a reduced SVD $A = U\Sigma V^T$, the ***pseudoinverse*** of A is $A^+ = V\Sigma^{-1}U^T$

$(U\Sigma_1 V^T)^{-1} = (V^T)^{-1}\Sigma_1^{-1}U^{-1}$

$= V\Sigma_1^{-1}U^T$

Theorem. $A^+\mathbf{b}$ is the *minimum length* **least squares solution** of $A\mathbf{x} = \mathbf{b}$

$A^+A = V\Sigma_1^{-1}U^TU\Sigma_1V^T$

$= V\Sigma_1^{-1}\Sigma_1V^T = VV^T = Id$

*(in Python we have **numpy.linalg.pinv**)*

Recall: Least Squares in NumPy

numpy.linalg.lstsq

`linalg.lstsq(a, b, rcond='warn')` [source]

Return the least-squares solution to a linear matrix equation.

Computes the vector *x* that approximately solves the equation `a @ x = b`. The equation may be under-, well-, or over-determined (i.e., the number of linearly independent rows of *a* can be less than, equal to, or greater than its number of linearly independent columns). If *a* is square and of full rank, then *x* (but for round-off error) is the "exact" solution of the equation. Else, *x* minimizes the Euclidean 2-norm $||b - ax||$. If there are multiple minimizing solutions, the one with the smallest 2-norm $||x||$ is returned.

Parameters:

a : *(M, N) array_like*

"Coefficient" matrix.

b : *{(M,), (M, K)} array_like*

Ordinate or "dependent variable" values. If *b* is two-dimensional, the least-squares solution is calculated for each of the *K* columns of *b*.

rcond : *float, optional*

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(why?...)

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(why?...)

because they use SVD!

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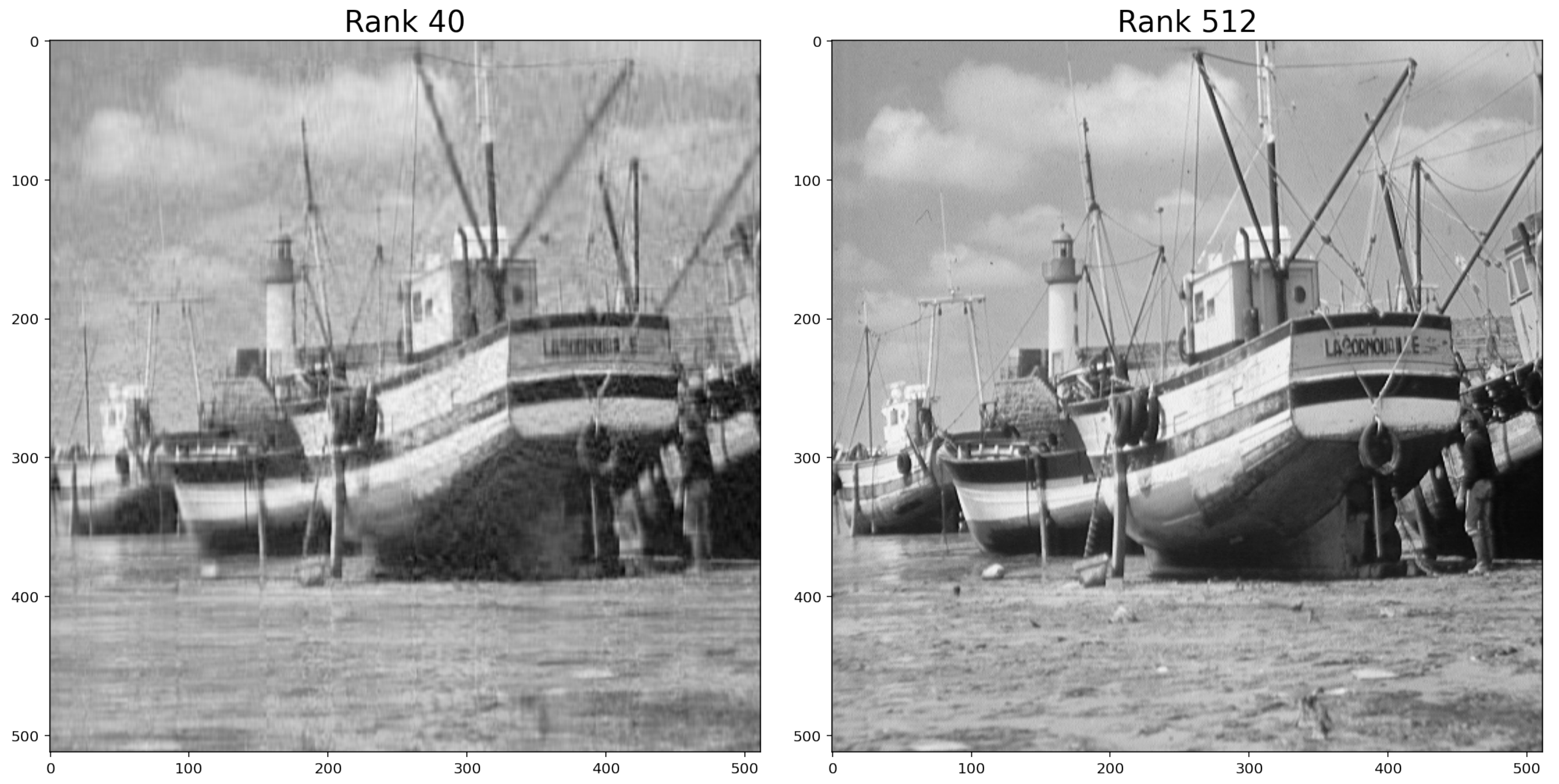
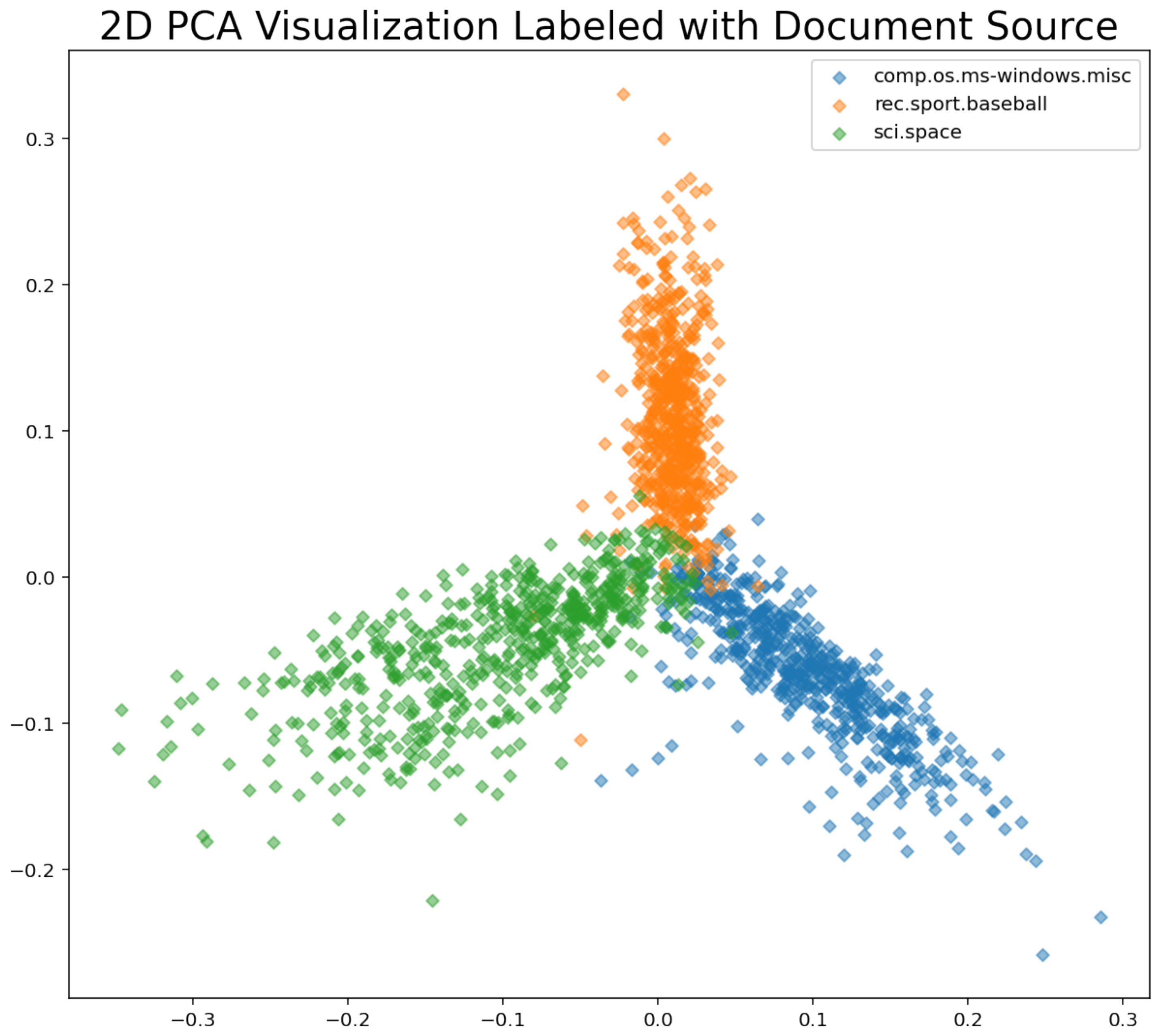
What's next?
A couple final thoughts

Applications of SVD

image compression



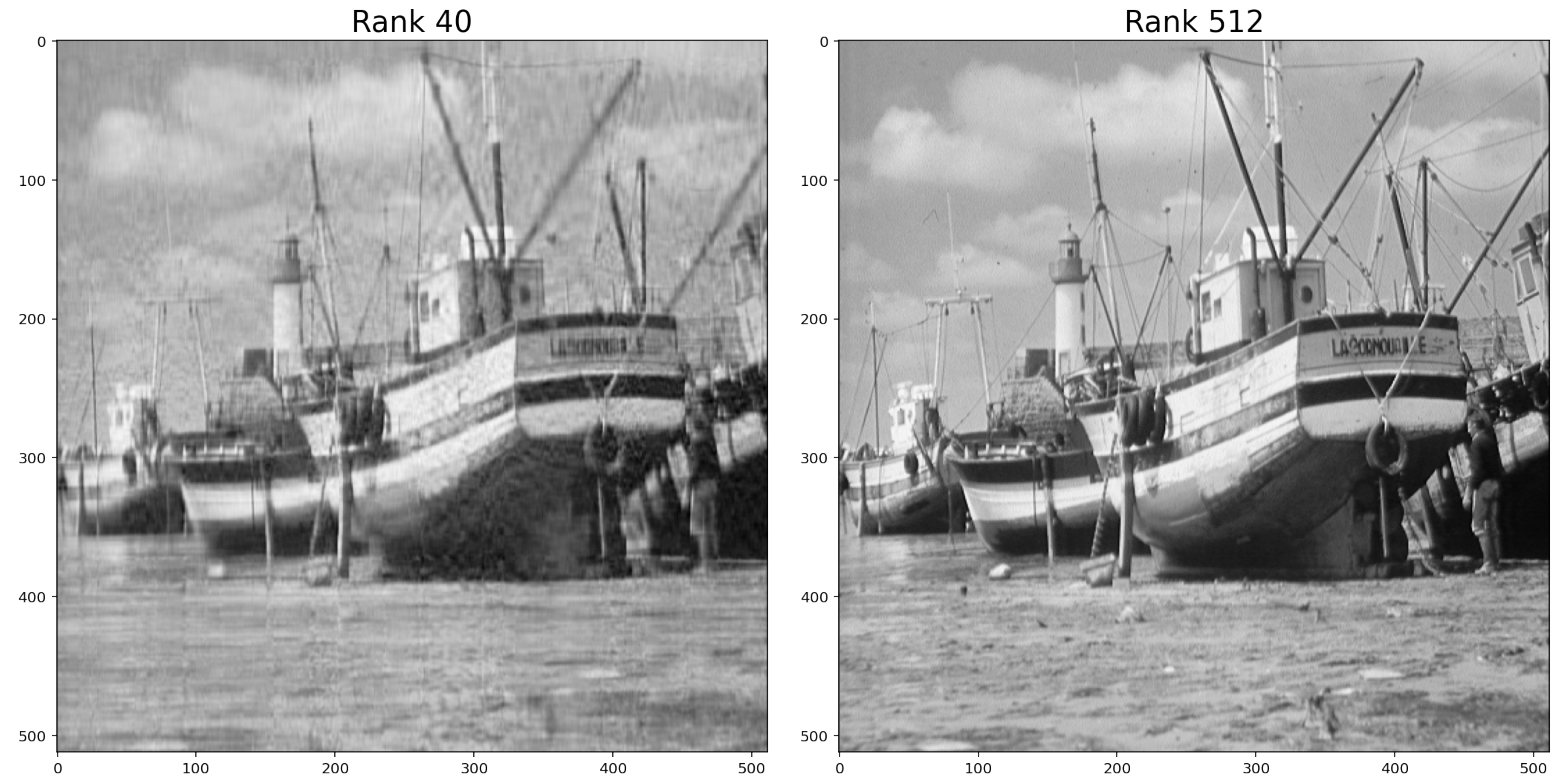


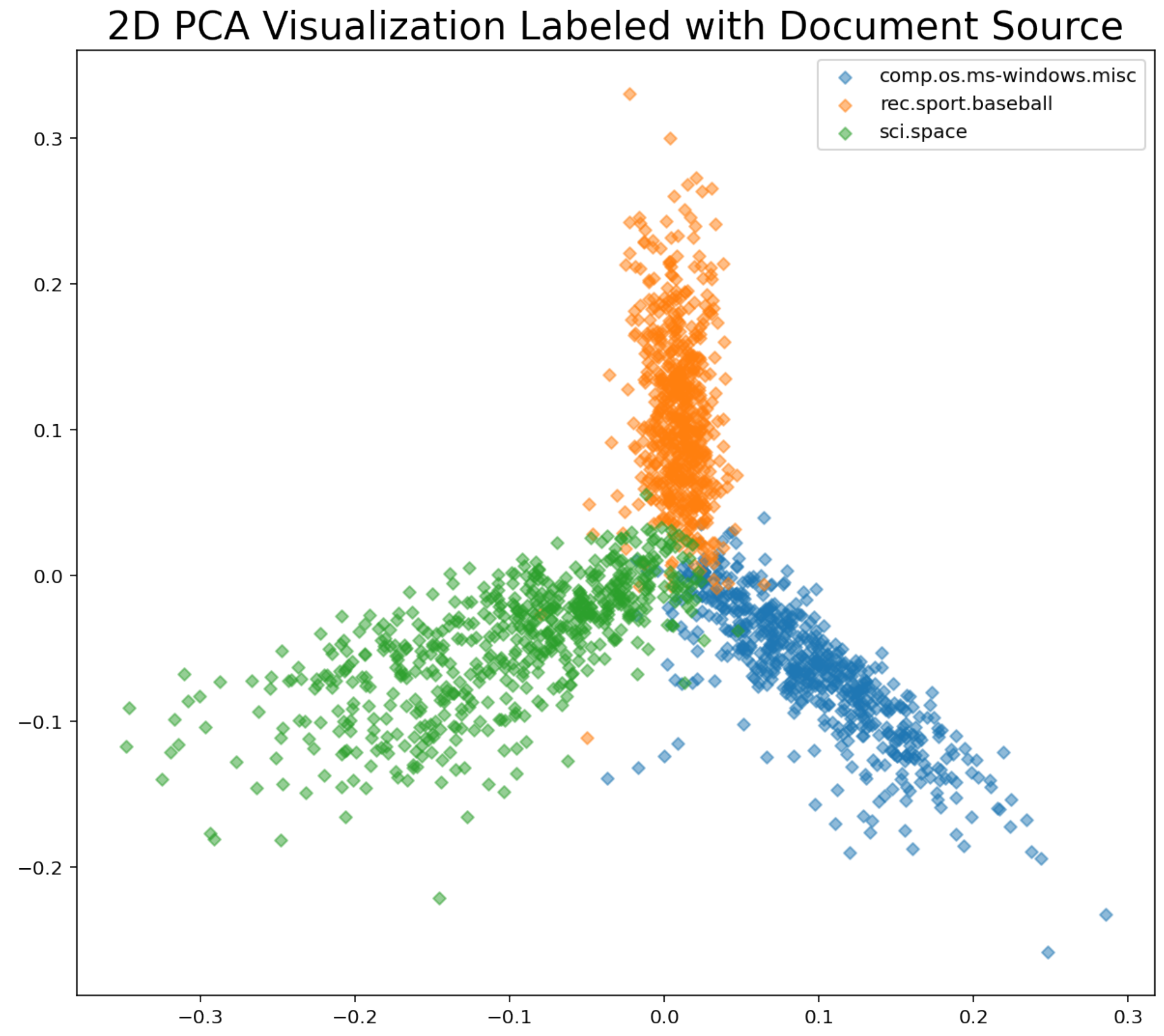
document classification

Applications of SVD

- Reduced SVD, pseudoinverses and least squares

image compression





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Applications of SVD

- Reduced SVD, pseudoinverses and least squares
    - If $A^+ = V\Sigma^{-1}U^T$, then $A^+\mathbf{b}$ is a least squares solution of minimum length

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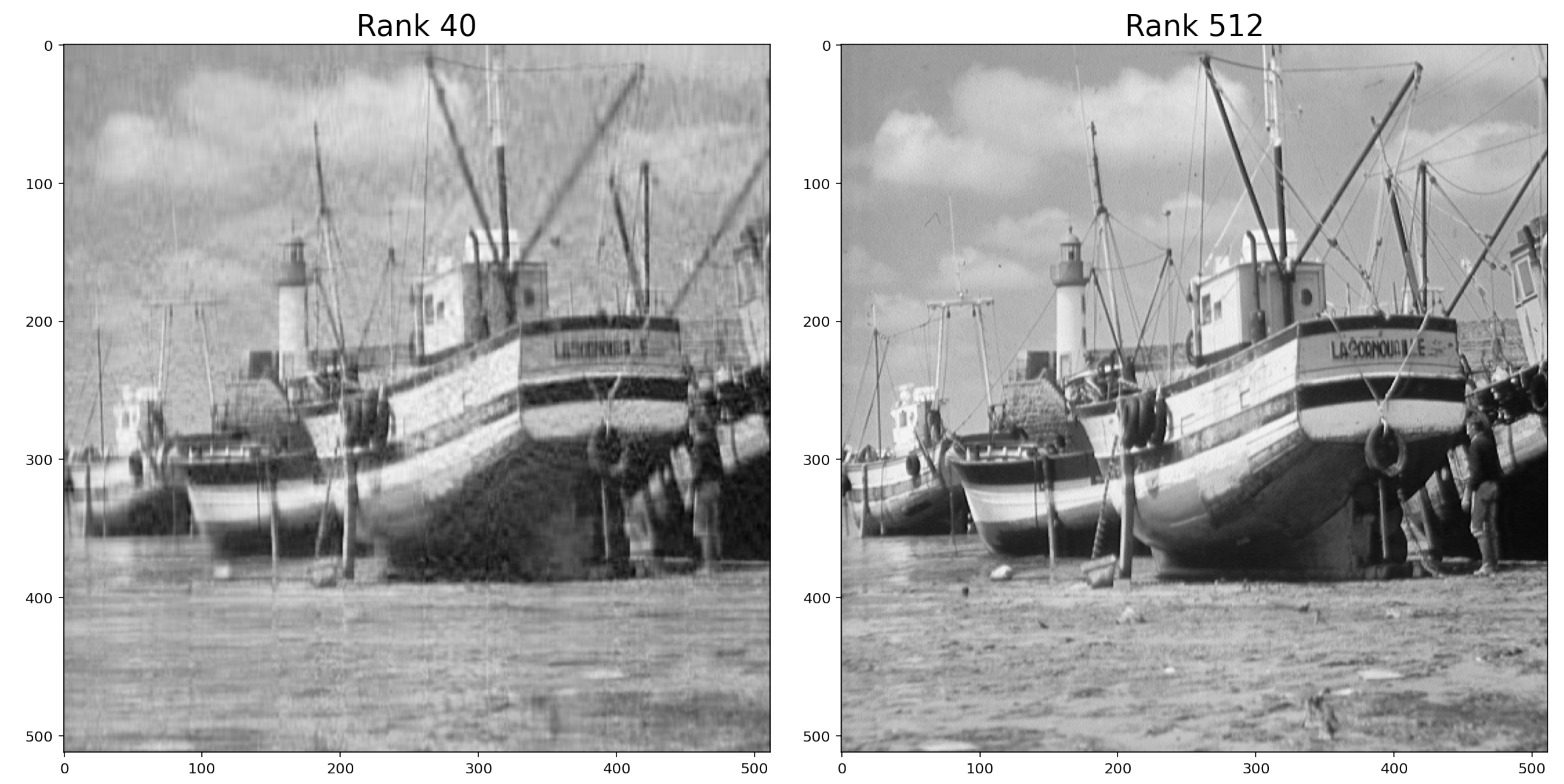

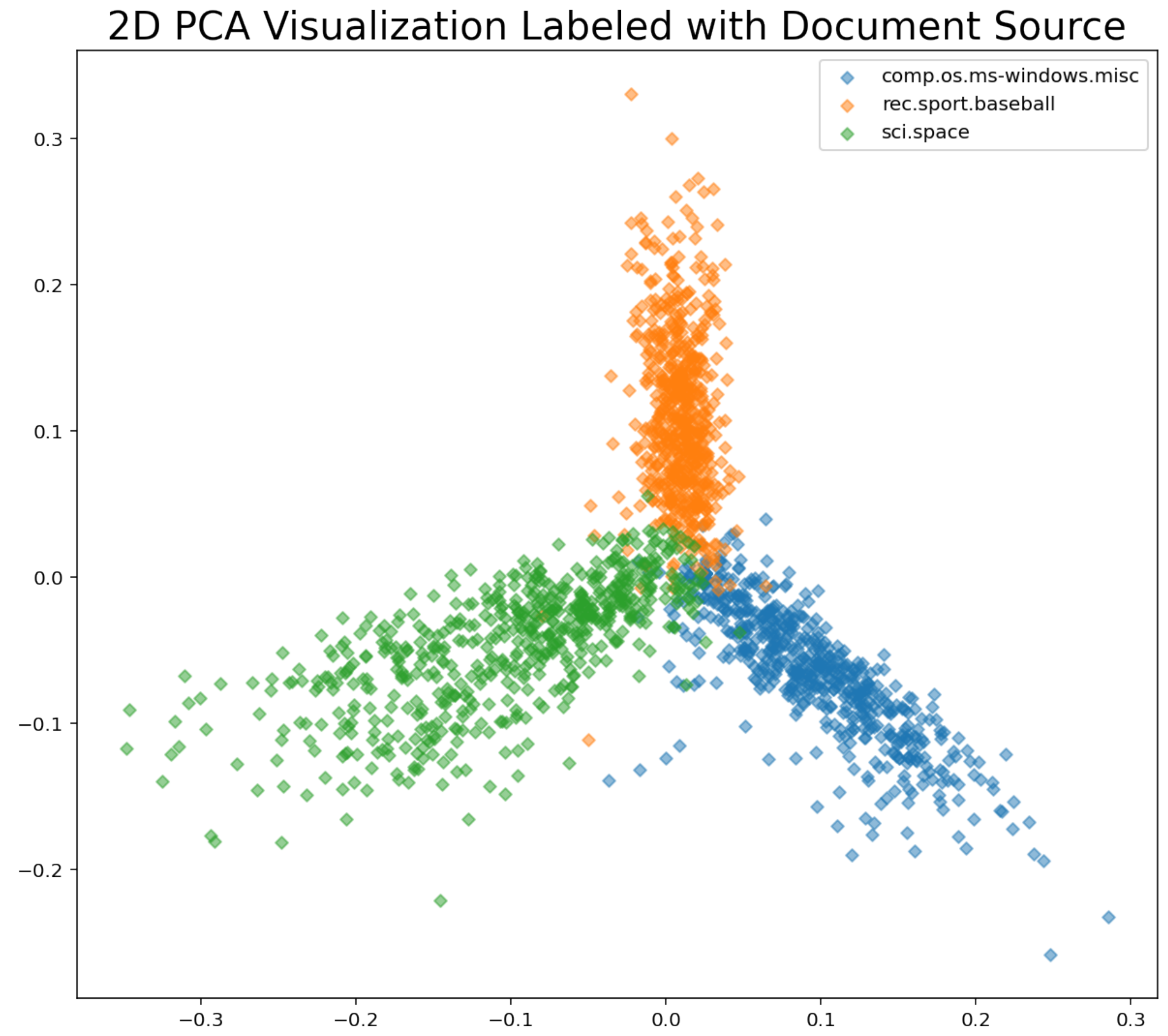

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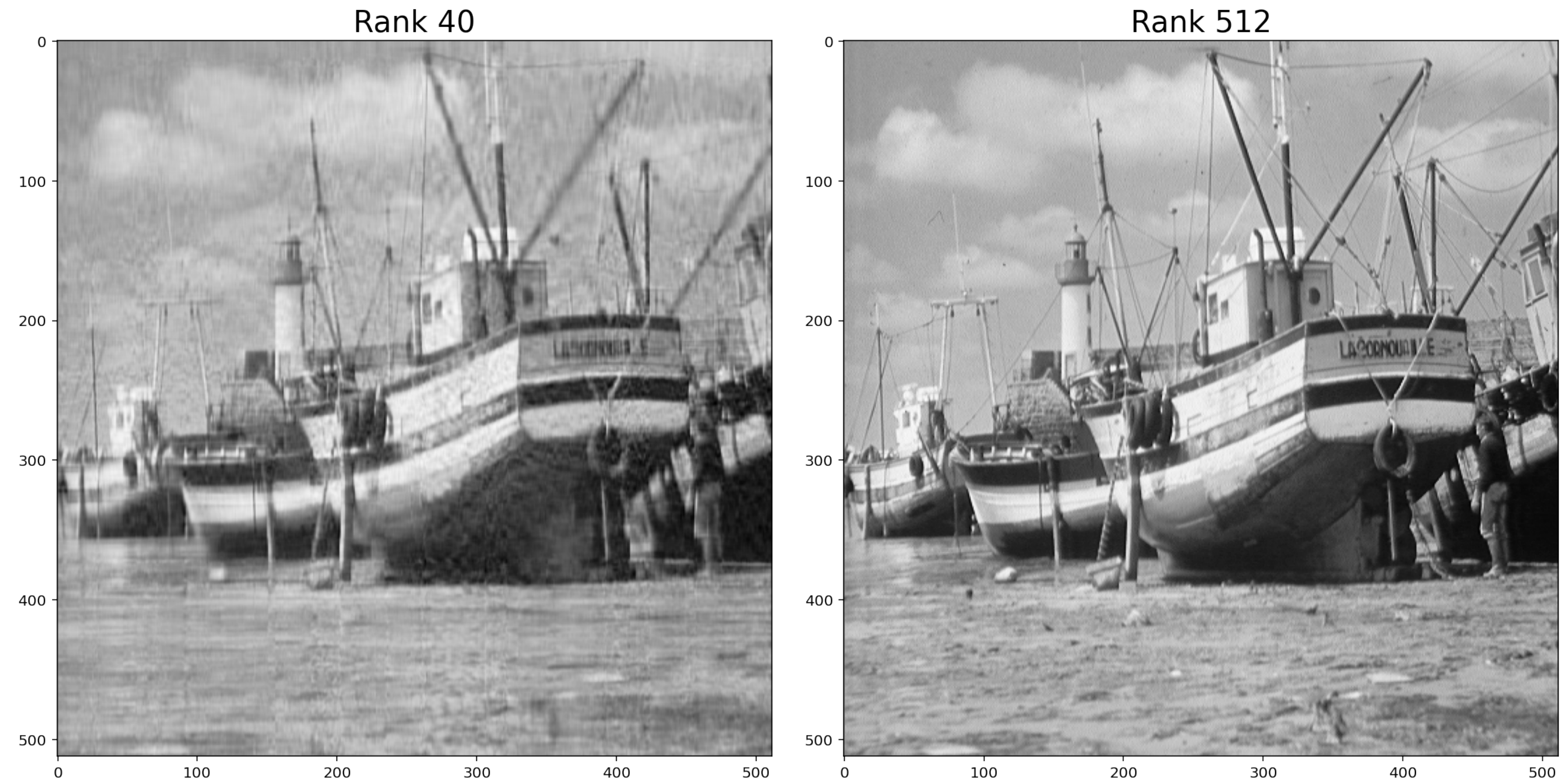

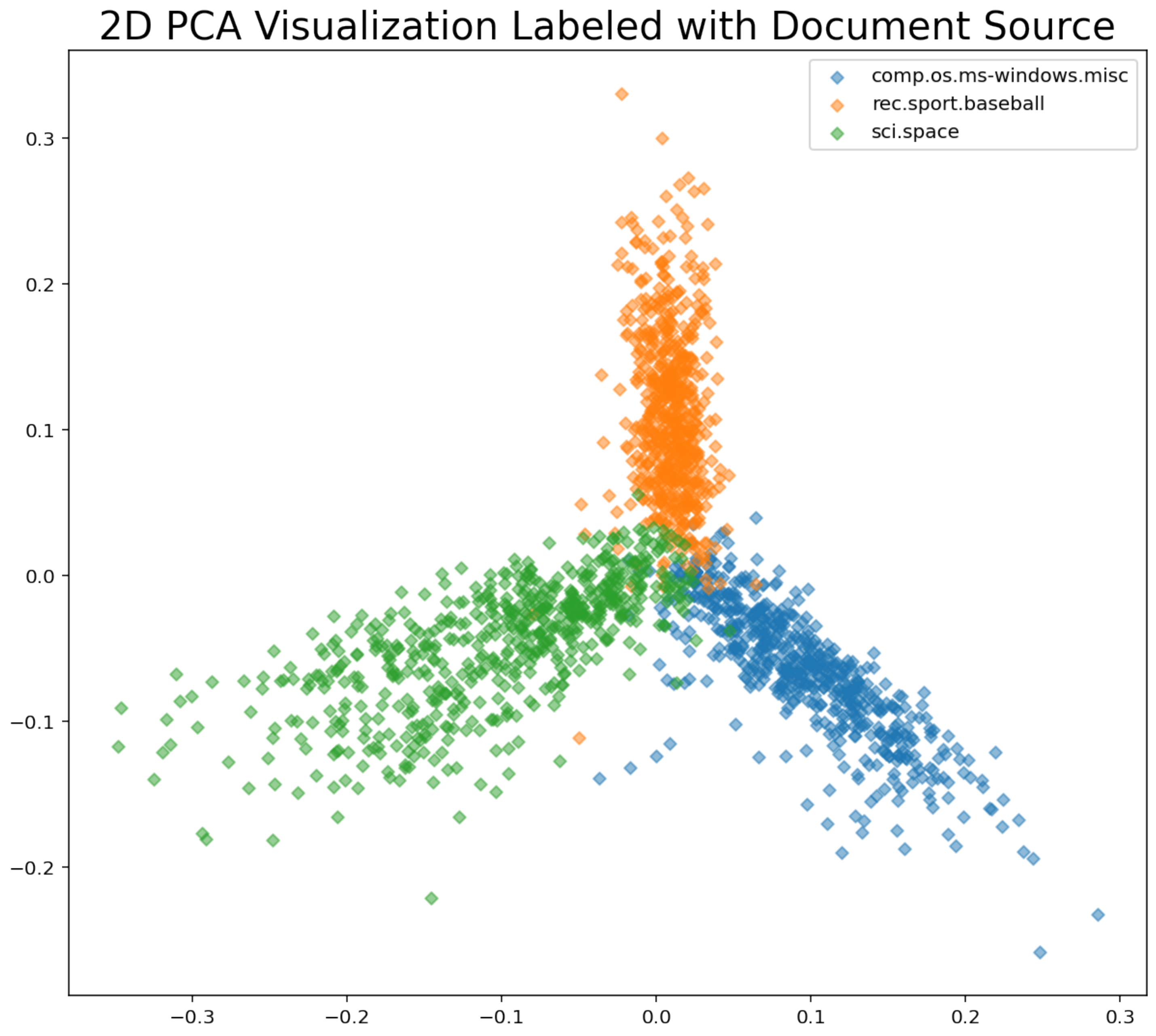

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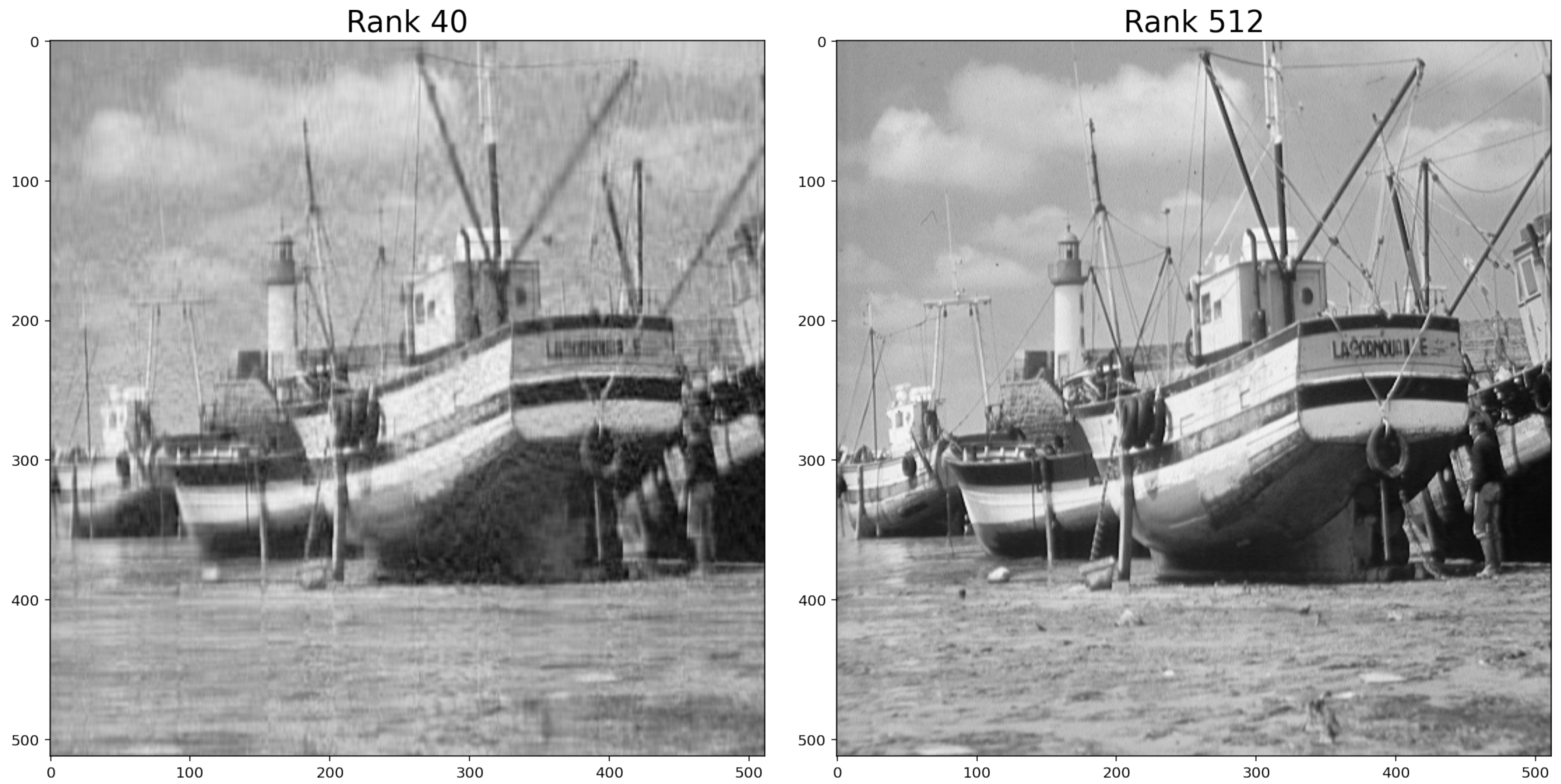

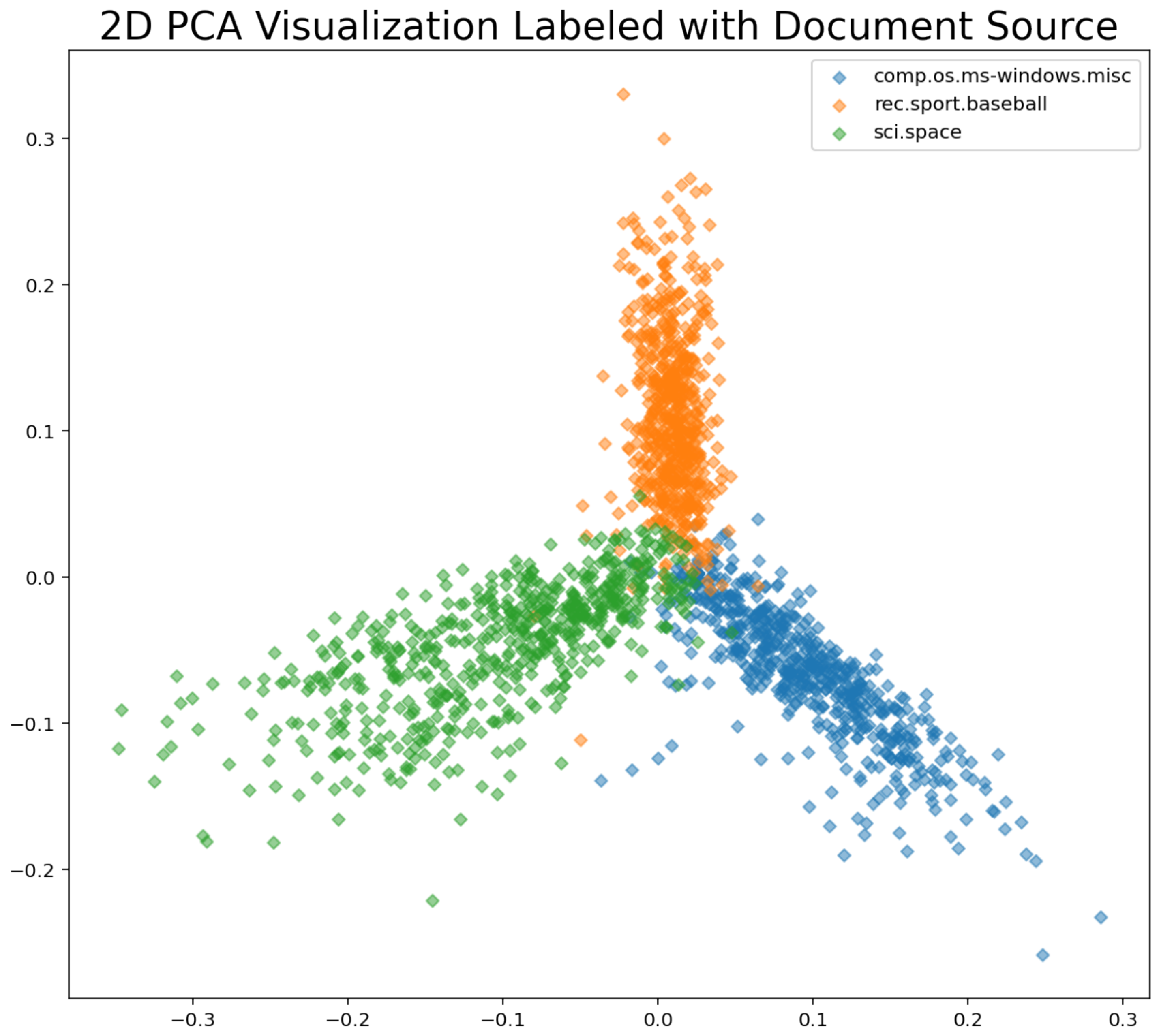

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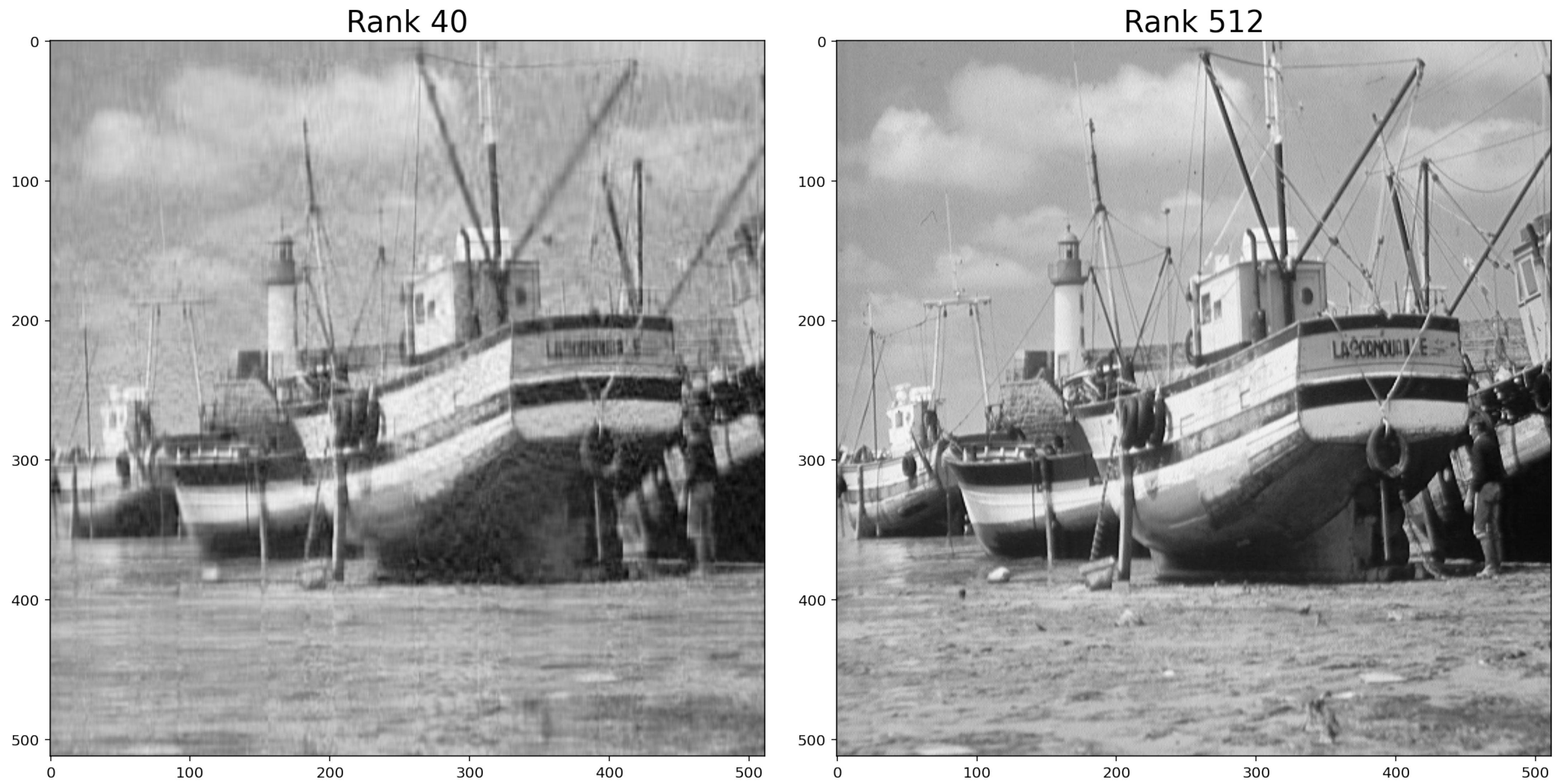

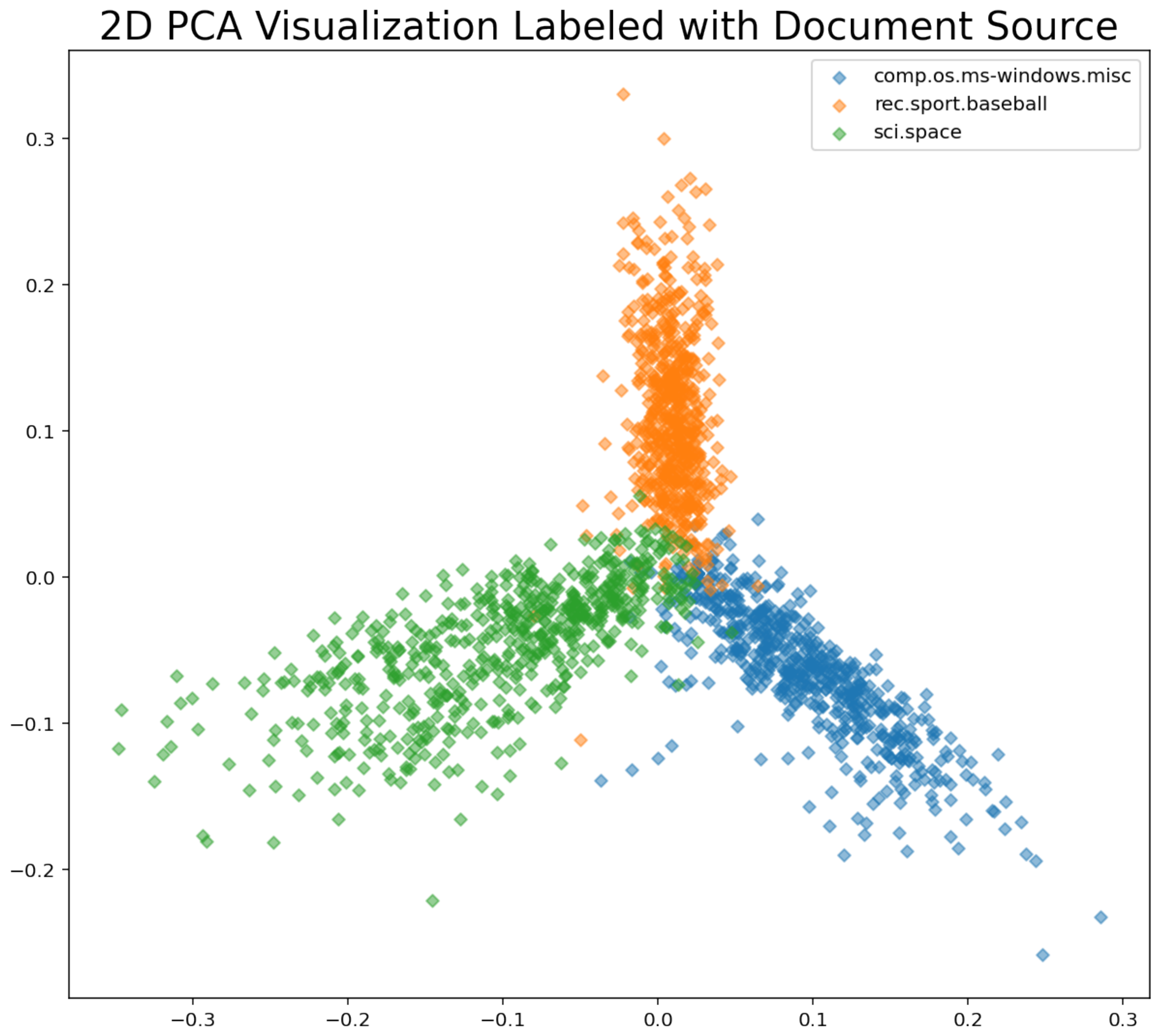

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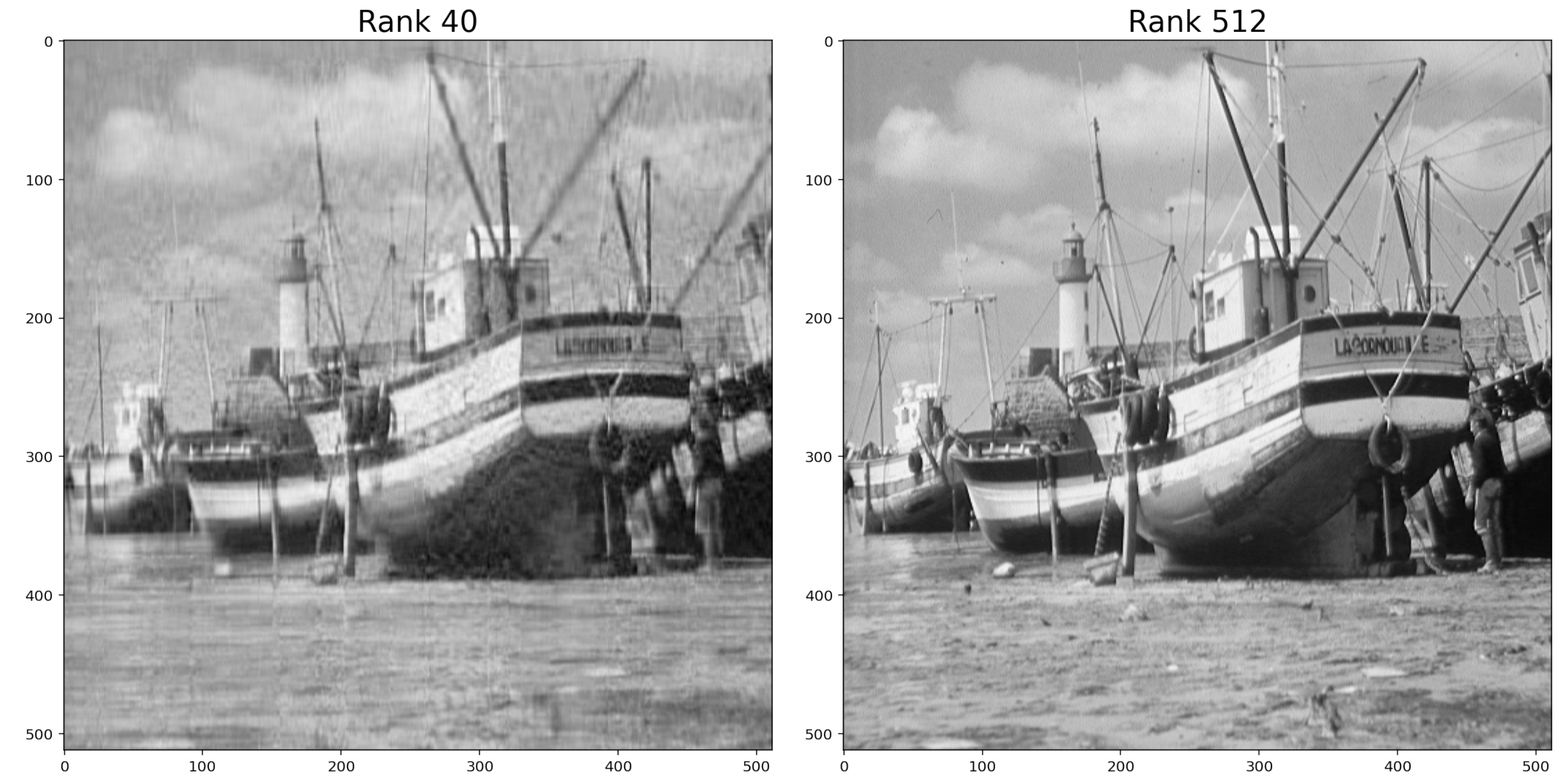
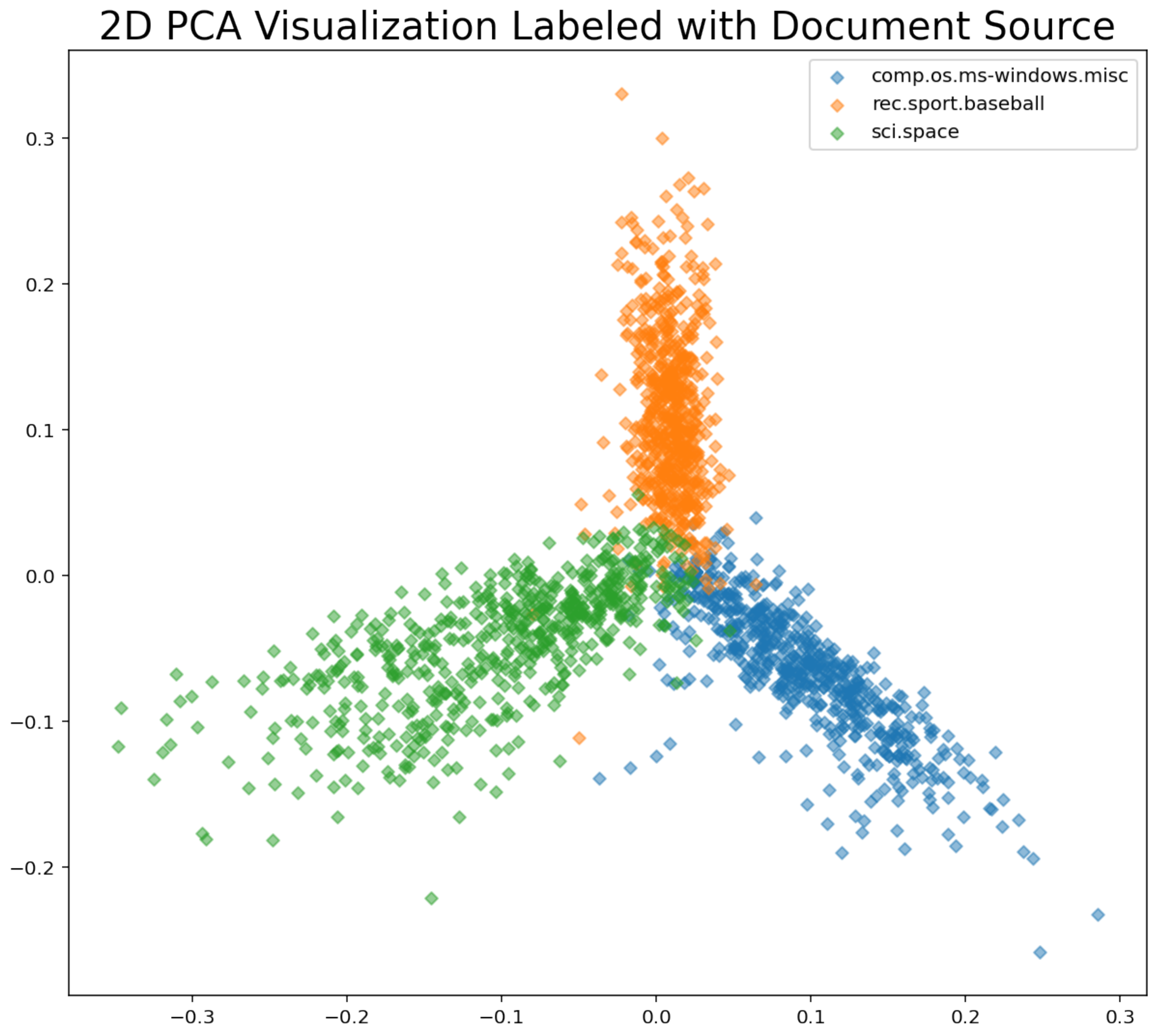
image compression





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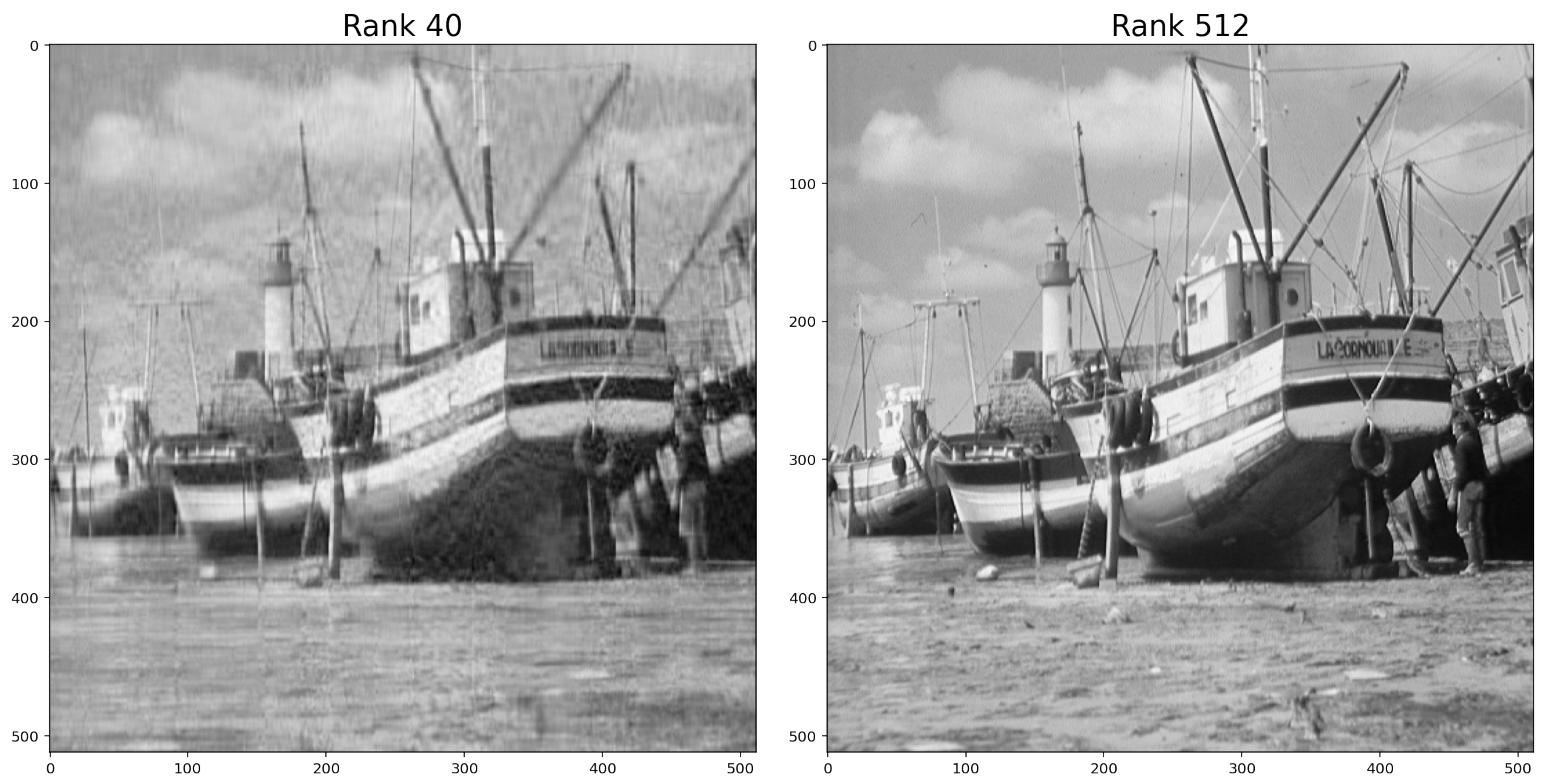
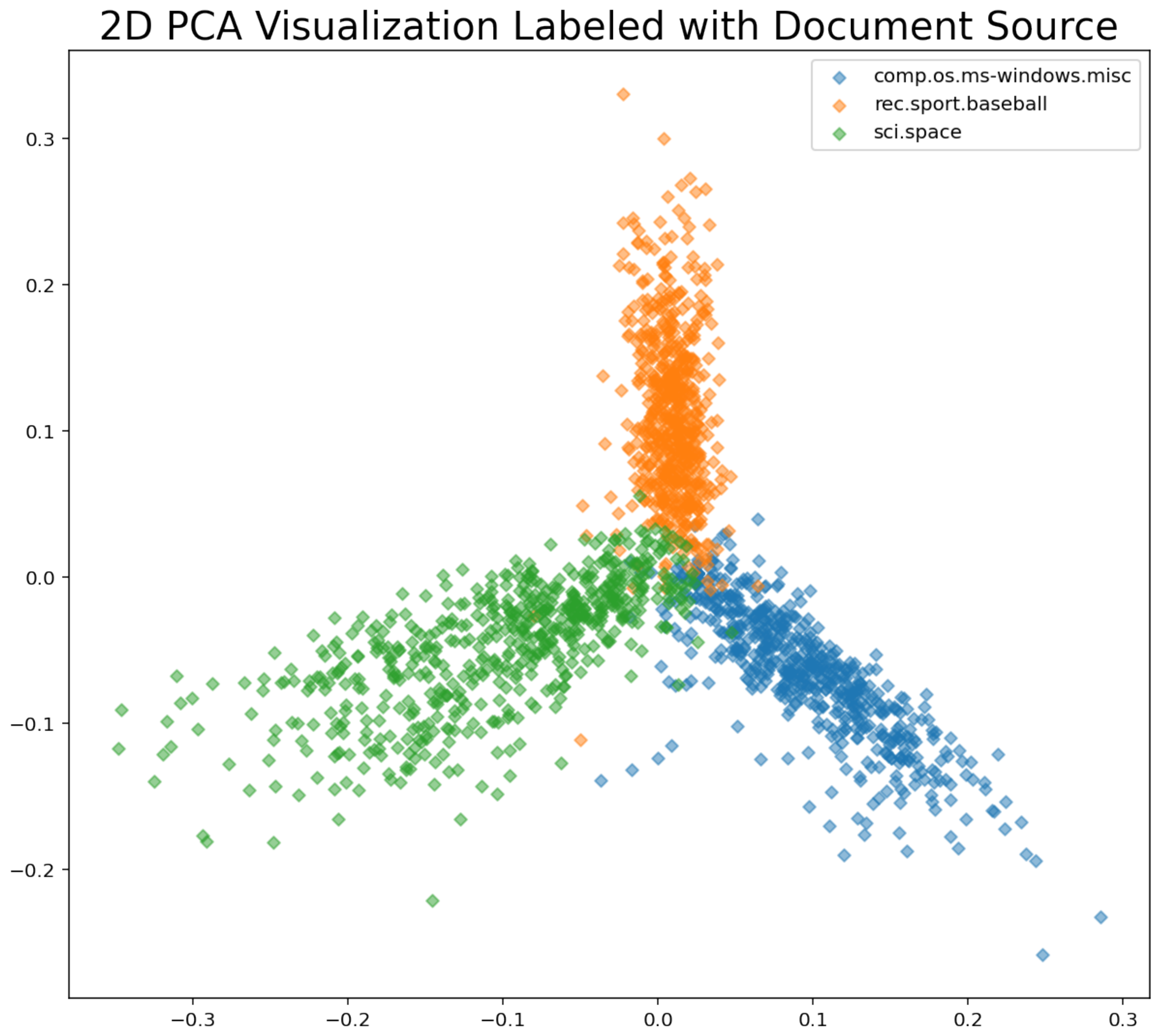
  - Large singular vectors are "most affected."

image compression





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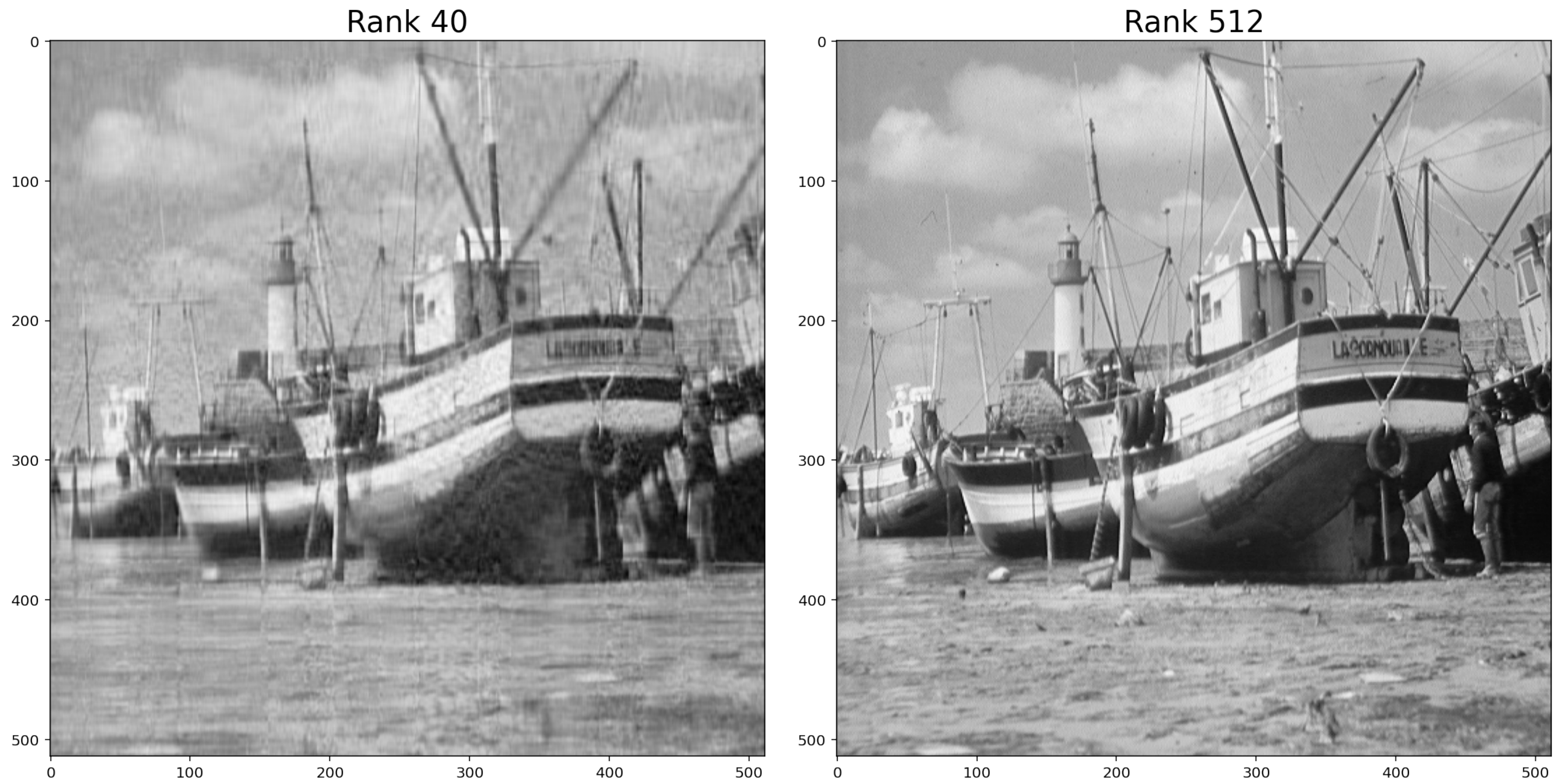
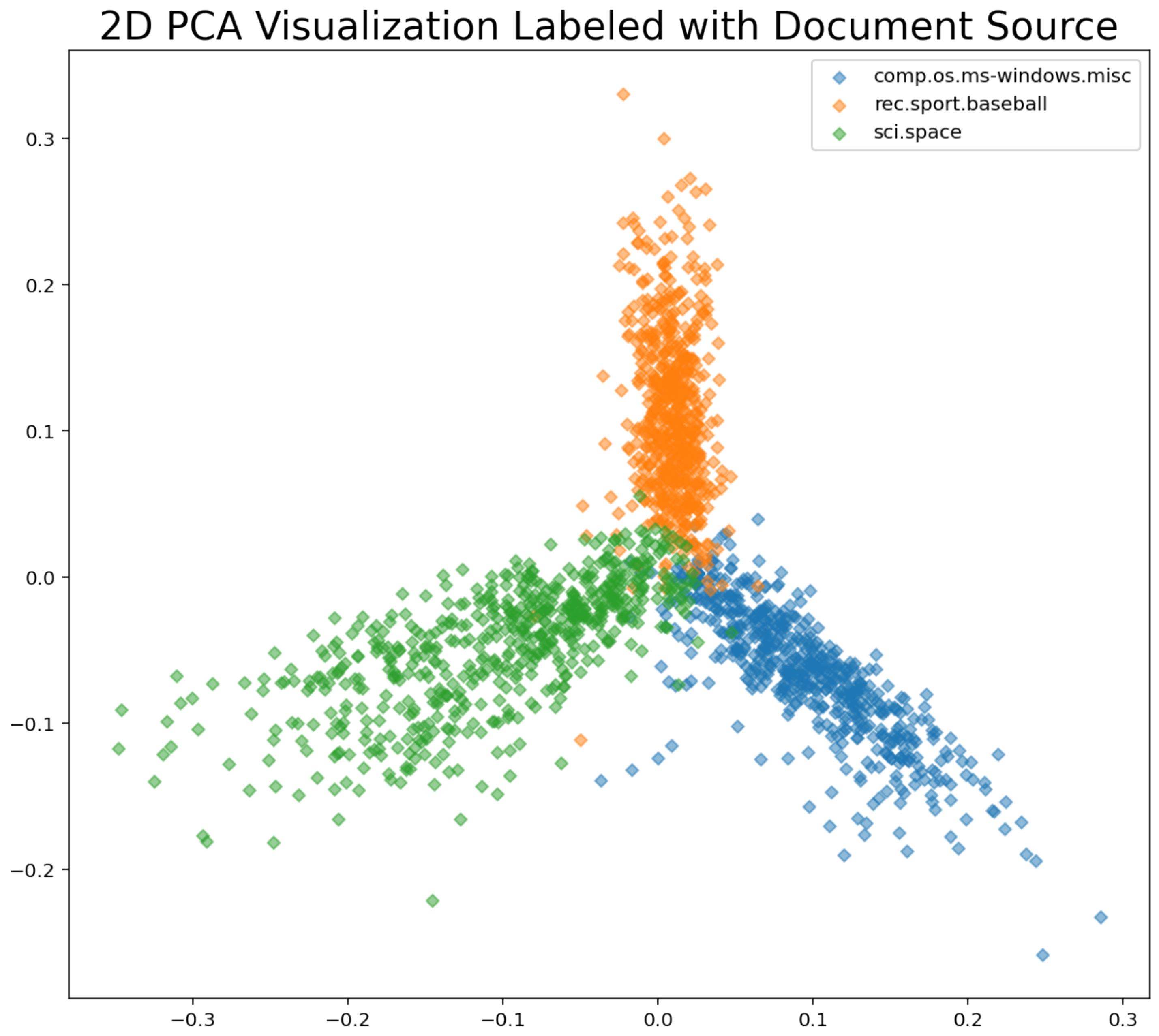
- Principle Component Analysis
  - Large singular vectors are "most affected."
  - These are good vectors to look at for classifying data

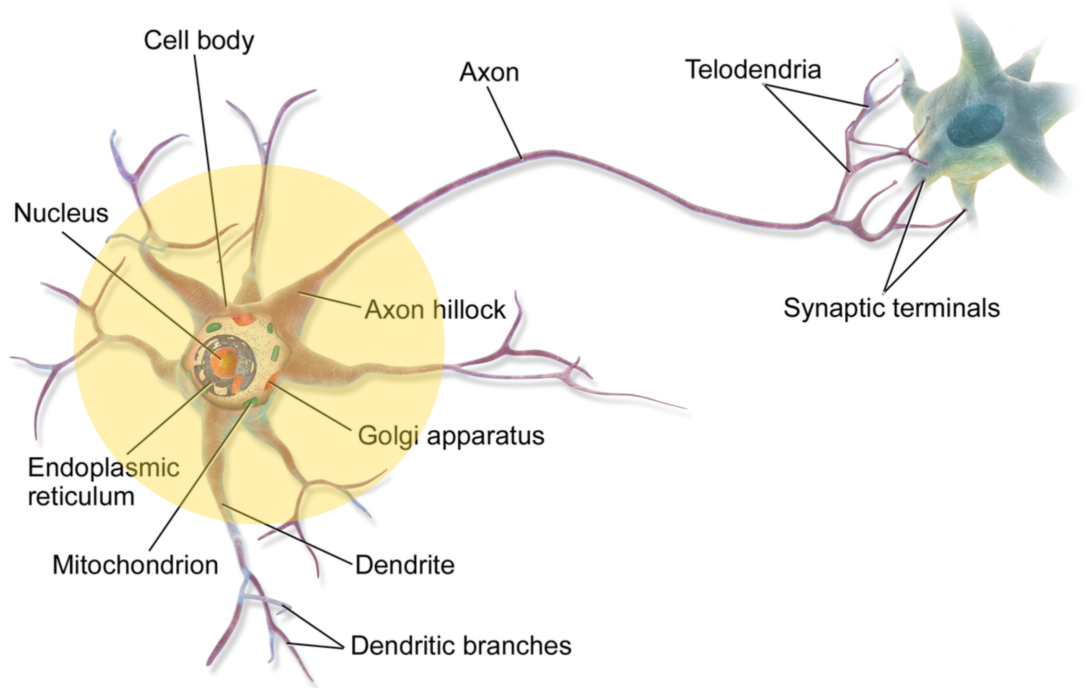
image compression

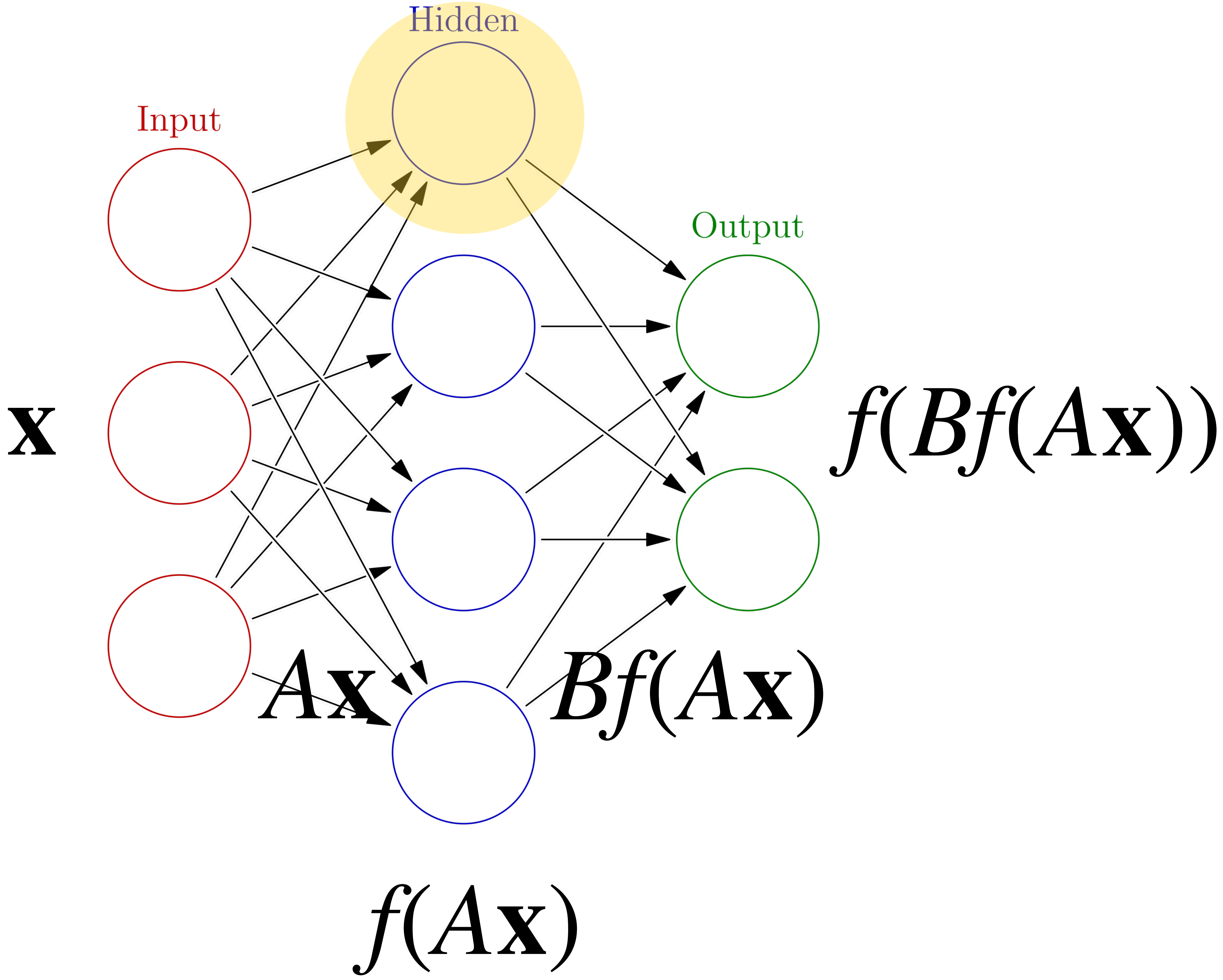




document classification

Neural Networks (Non-Linearity)

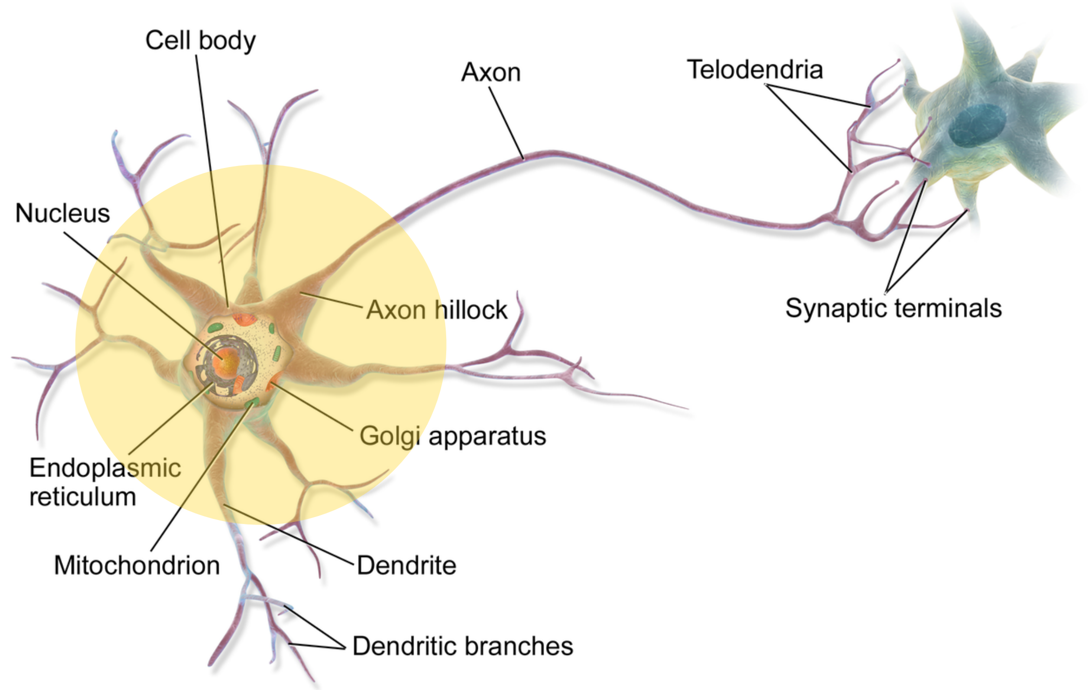


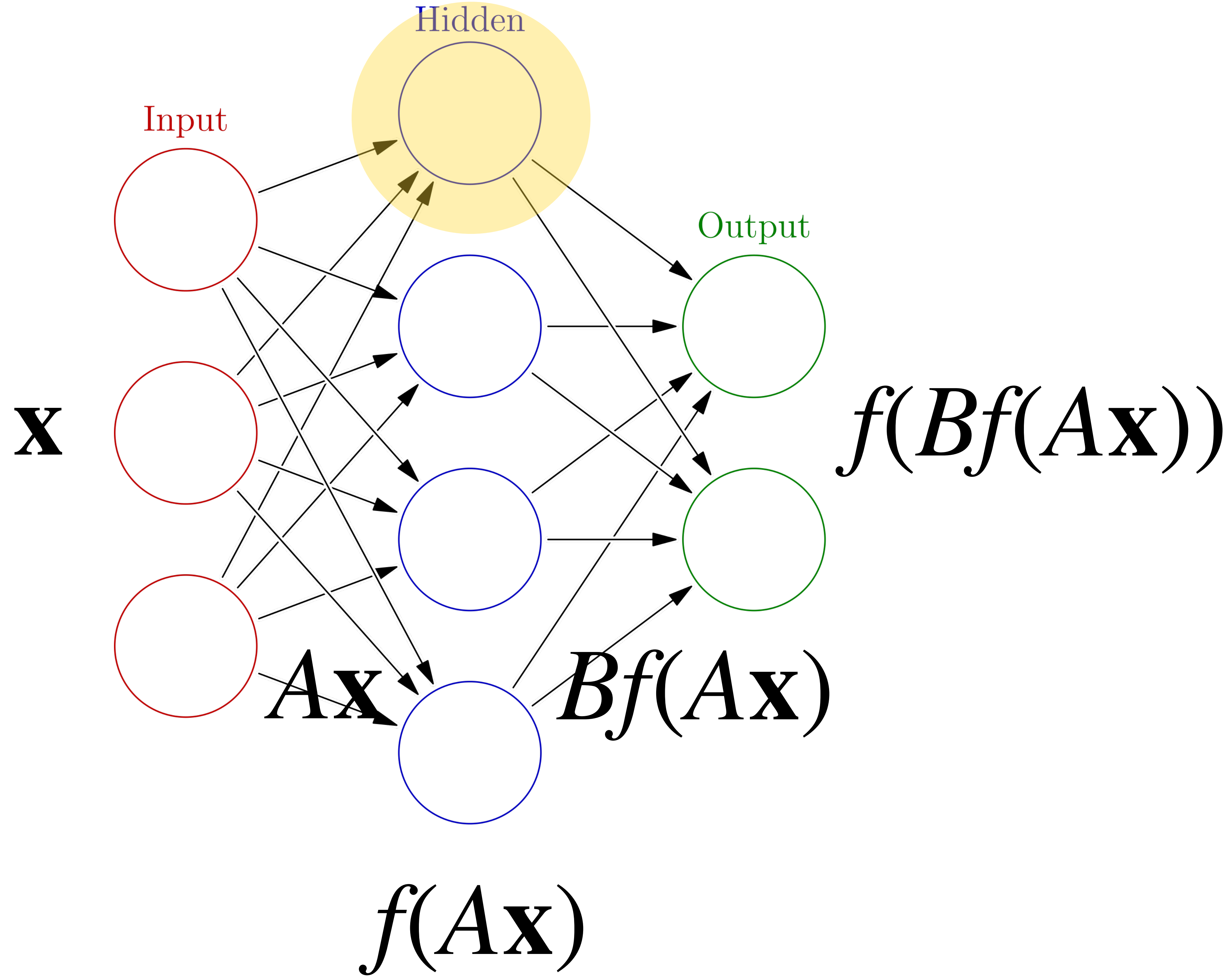


https://commons.wikimedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png
https://commons.wikimedia.org/wiki/File:Colored_neural_network.svg

Neural Networks (Non-Linearity)

Neural networks are models of artificial neurons bundles.





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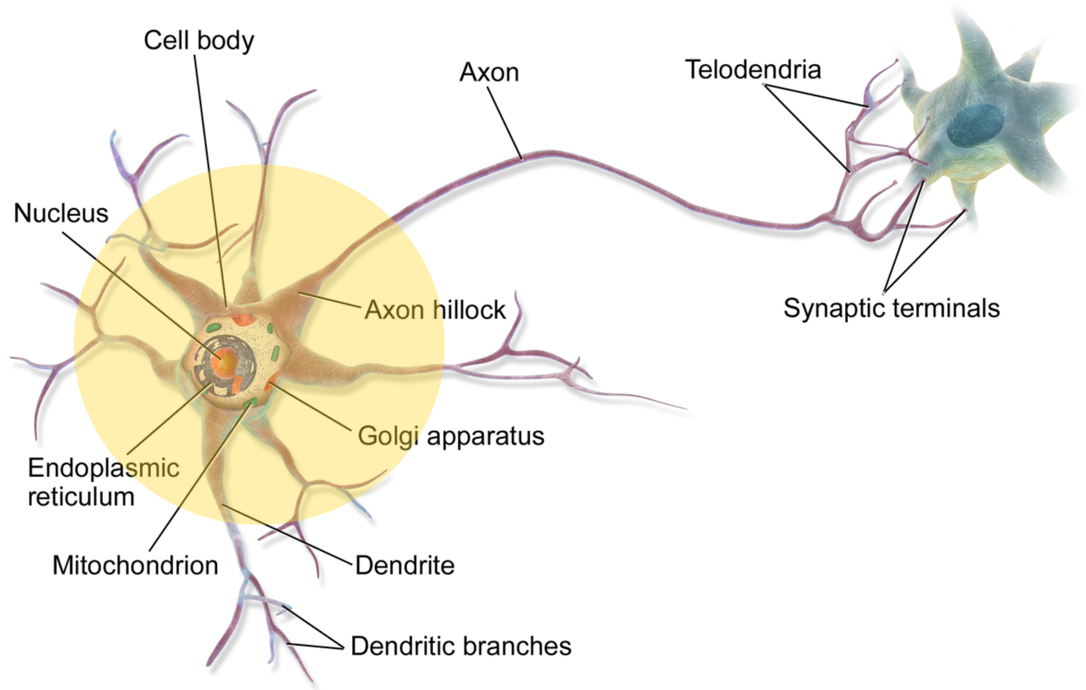
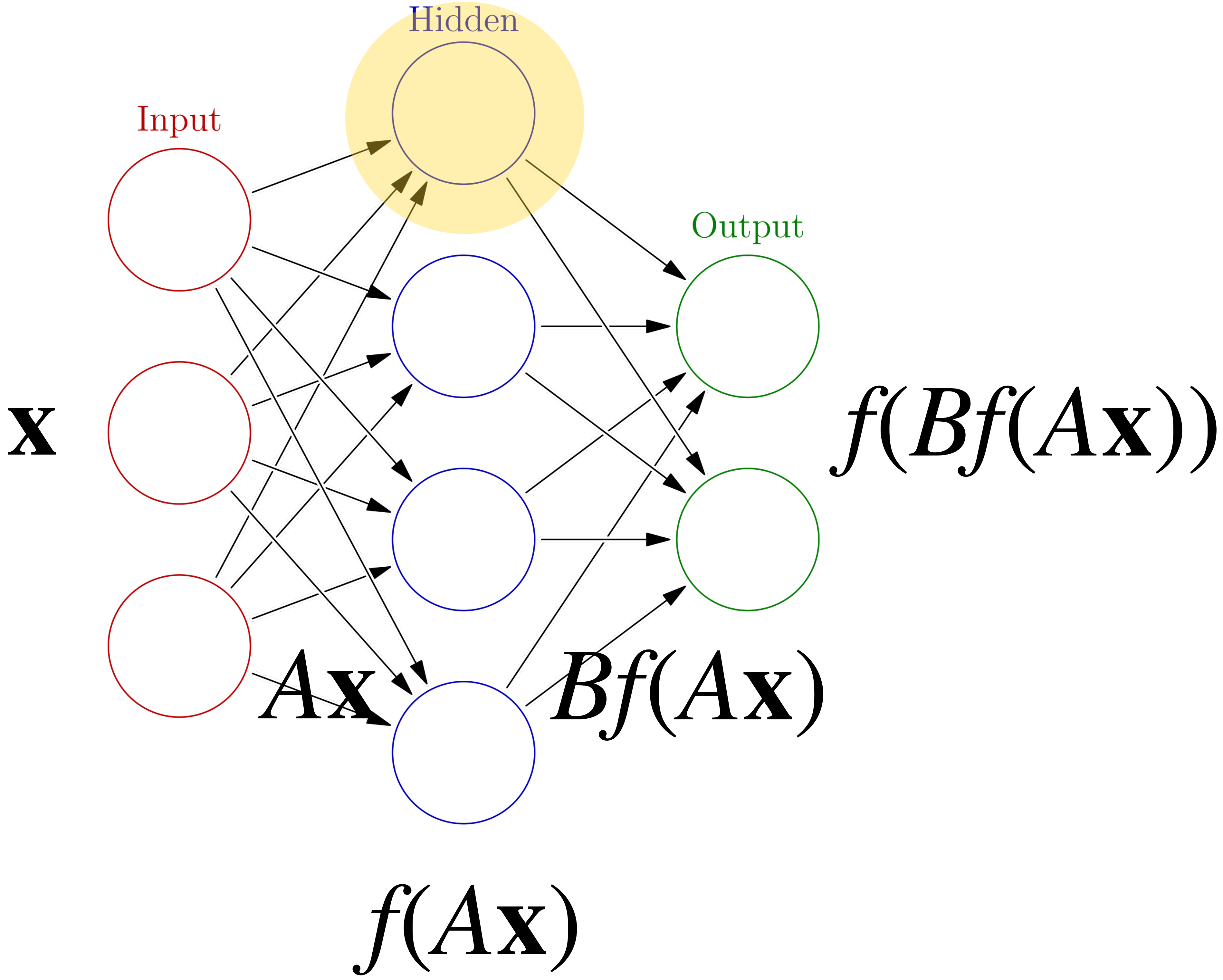
Given an input vector $\mathbf{x}$, it is transformed into a *hidden* vector $A\mathbf{x}$ by a linear transformation, and then an *activation function* f is applied to the result.





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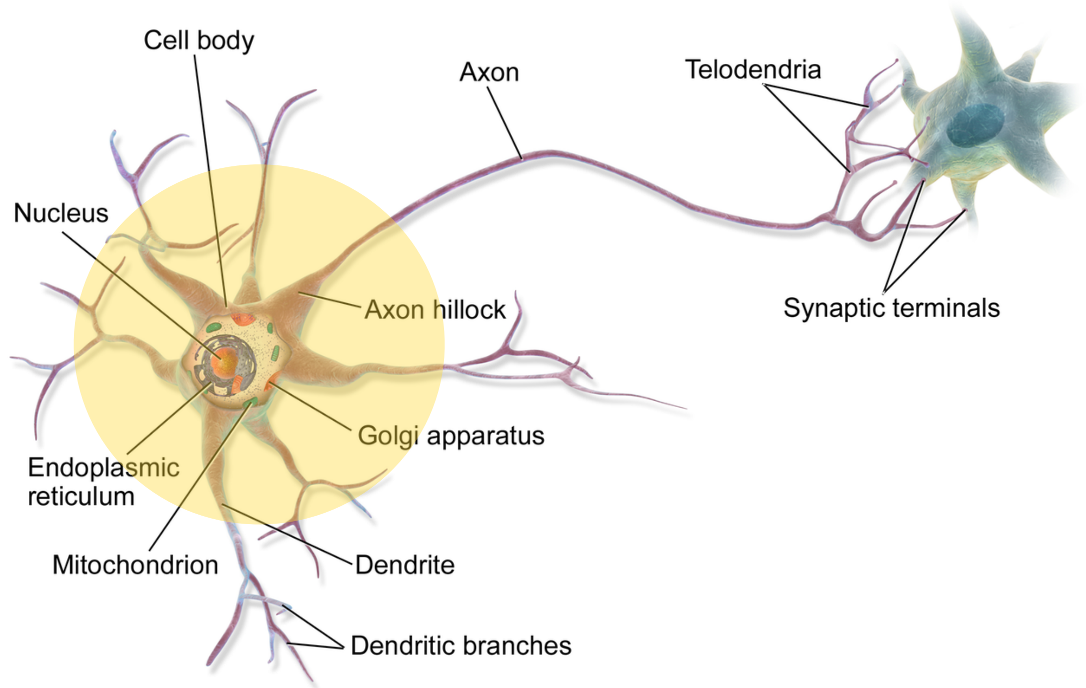

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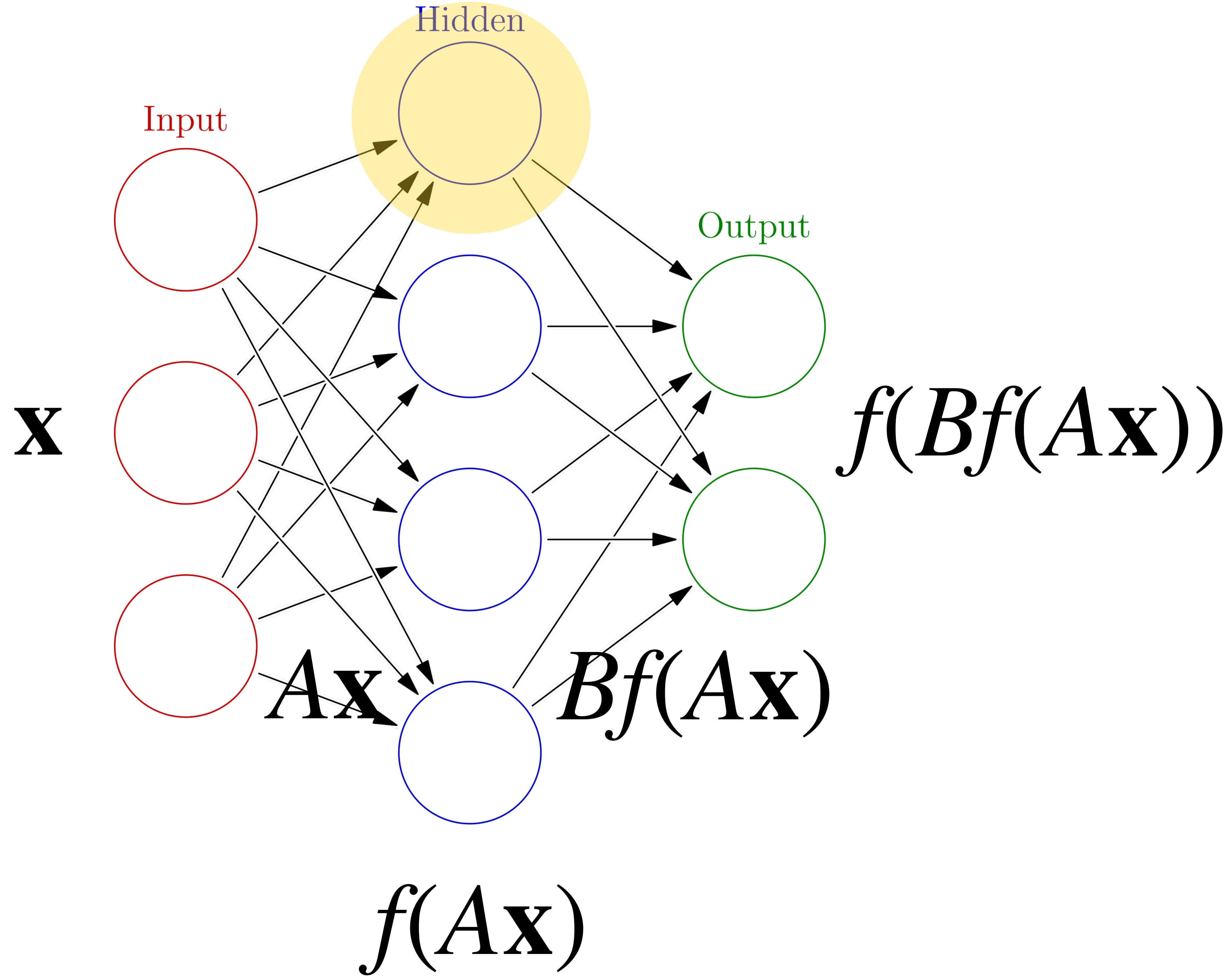

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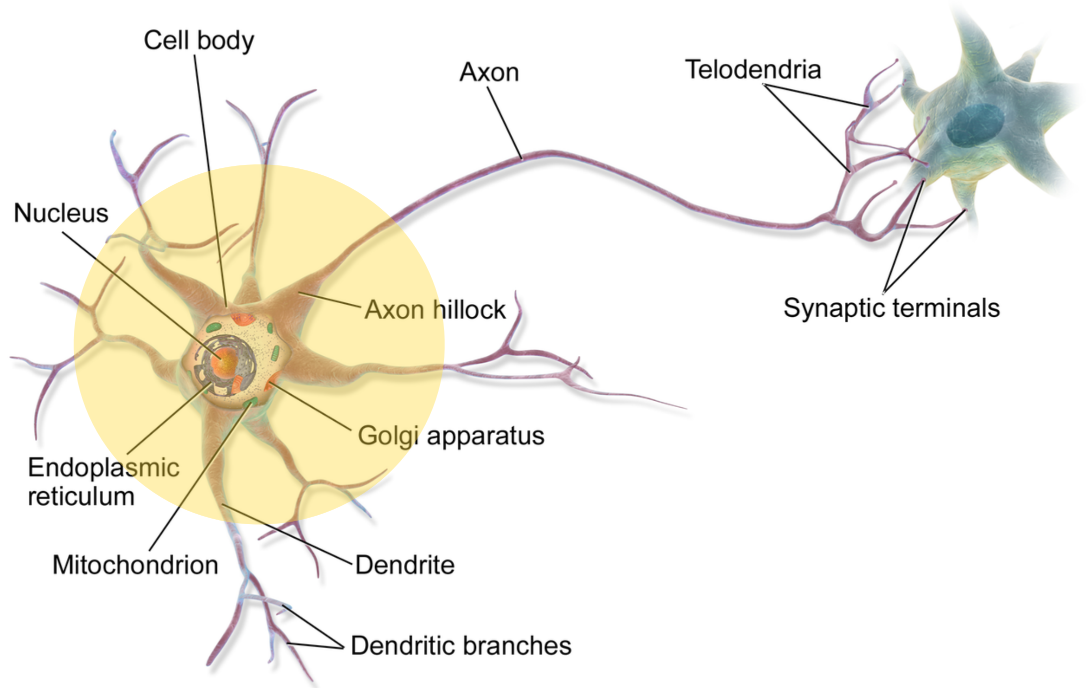

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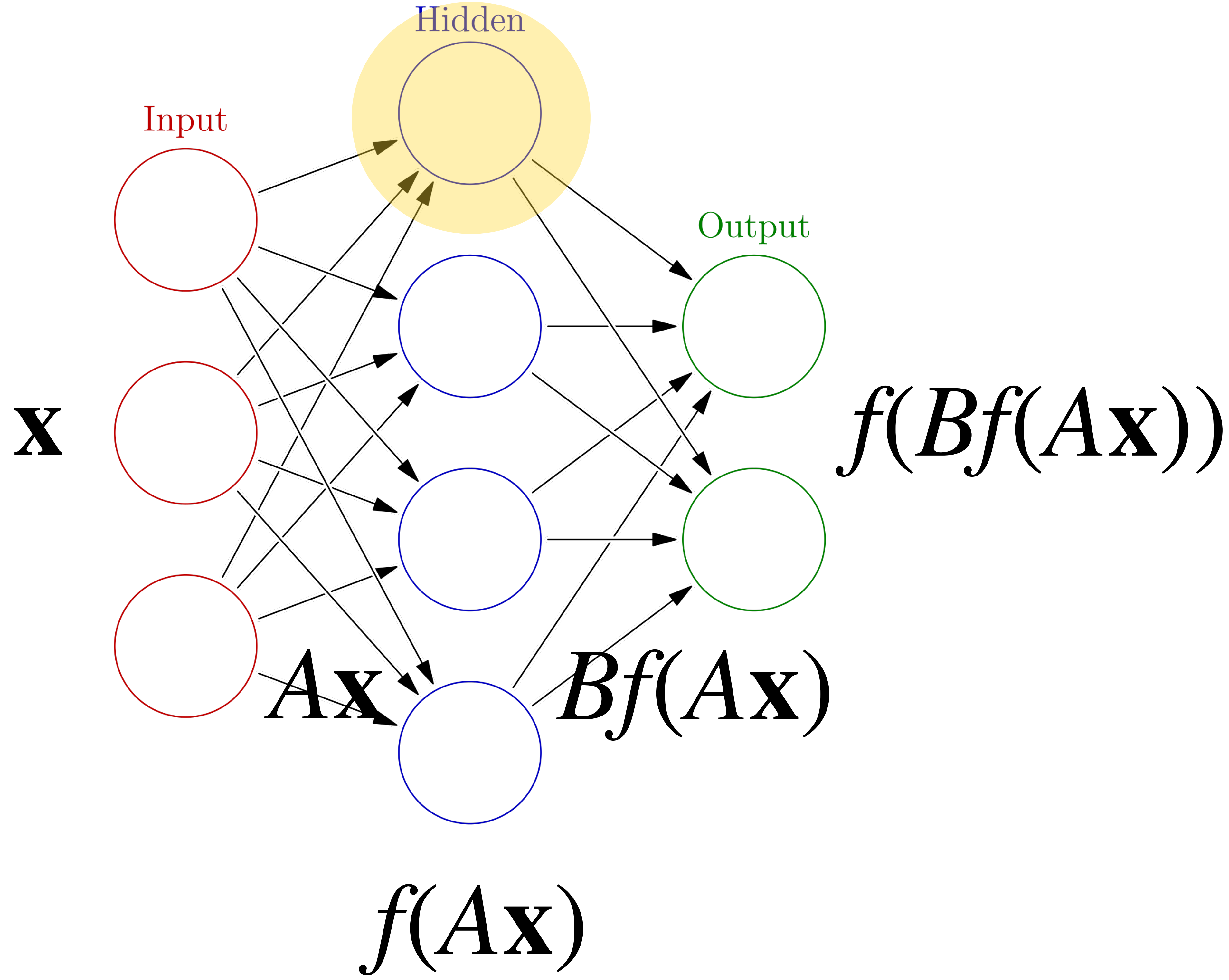
$$\mathrm{NN}(\mathbf{x}) = f(A_k(f(A_{k-1} \ldots f(A_1\mathbf{x}))$$



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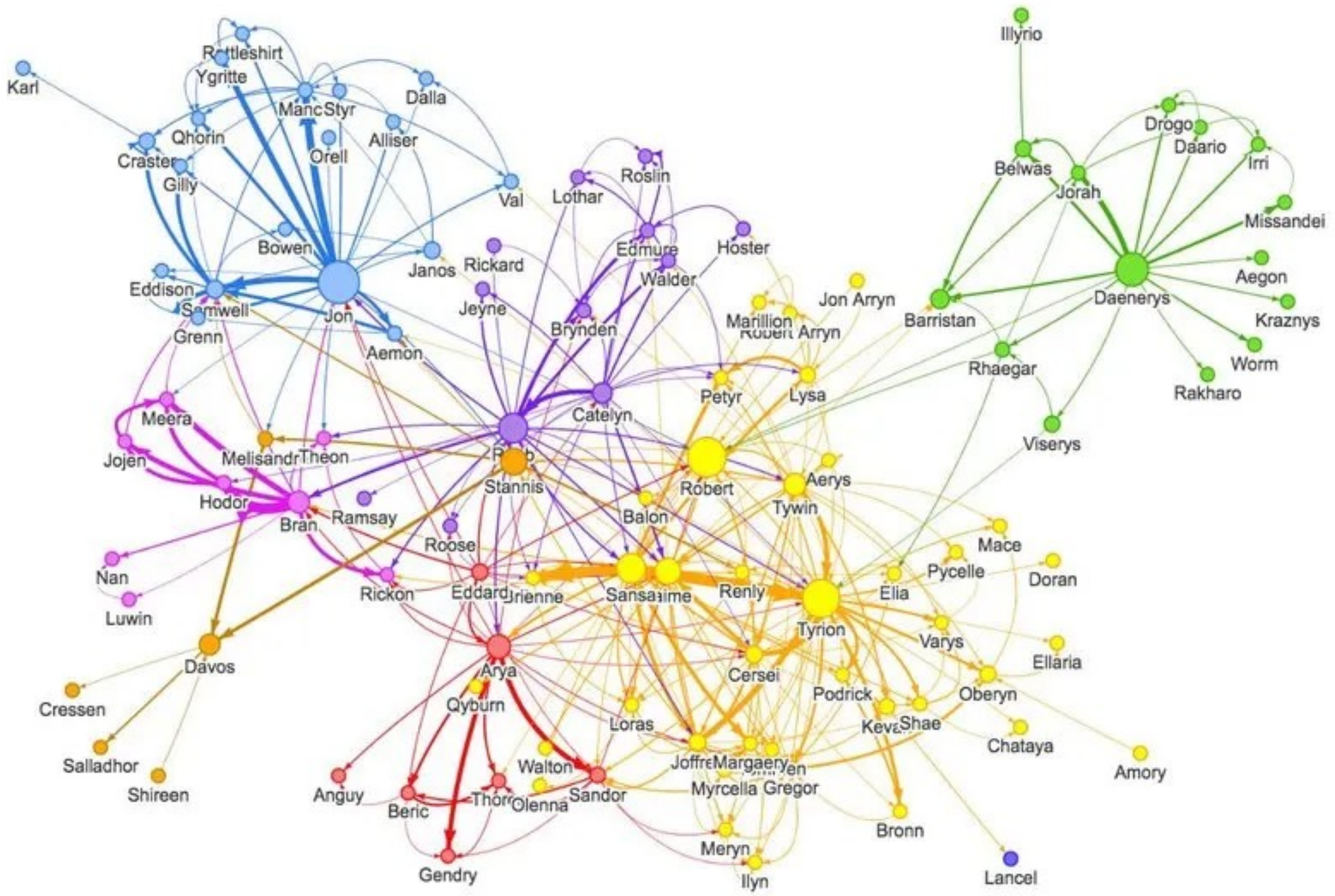
Spectral/Algebraic Graph Theory

Graphs can be viewed as matrices.

The finding eigenvalues in graphs can gives use better clustering and cutting algorithms.



https://medium.com/@n.rajadhyaksha/a-gentle-introduction-to-graph-spectral-filtering-df03ddc9d3f7

Abstract Algebra

$$\frac{U}{\mathrm{Nul}(f)} \cong \mathrm{Range}(f)$$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & V \\ U/\mathrm{Nul}(f) \arrow[ur] & \end{tikzcd}$$

There's a lot of beautiful structure in the algebra we've done in this course.

And there are lots of directions to go from here (infinite dimensional spaces, less restrictive settings like groups and modules,...)

Course List

- CS 365 Foundations of Data Science
- CS 440 Intro to Artificial Intelligence
- CS 480 Intro to Computer Graphics
- CS 505 Intro to Natural Language Processing
- CS 506 Tools for Data Science
- CS 507 Intro to Optimization in ML
- CS 523 Deep Learning
- CS 530 Advanced Algorithms
- CS 531 Advanced Optimization Algorithms
- CS 542 Machine Learning
- CS 565 Algorithmic Data Mining
- CS 581 Computational Fabrication
- CS 583 Audio Computation

- CS 582 Geometry Processing

Some of these may not exist anymore...

Appreciations

The Course Staff

I'd like to thank:

Abhinit Sati, ~~Vishesh Jain, Ieva Sagaitis, Kevin Wrenn, Jin Zhang~~, Sohan Atluri, ~~Fynn Buesnel, Aseef Imran, Eugene Jung, Chris Min~~, Wyatt Napier, ~~Kyle Yung~~

If you see them around you should thank them as well

Rahul Mitra, Gor Matcakian, Helen Zhou, Ian Sun

The CS Department Staff

If you're ever in the CS Department office, be kind to the people who work there. They work very hard to keep all our courses running

The Students of CS132

Thanks for sticking with it

Thanks for giving feedback

Thanks for participating

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