

Matrix Operations

Geometric Algorithms
Lecture 10

Practice Problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_2 \\ 4x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$$

Determine if the above transformation is onto, one-to-one, both, or neither

Answer

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_2 \\ 4x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$$

Objectives

- » Define several important matrix operations
- » Motivate and define matrix multiplication and inverses

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1-1 Correspondence

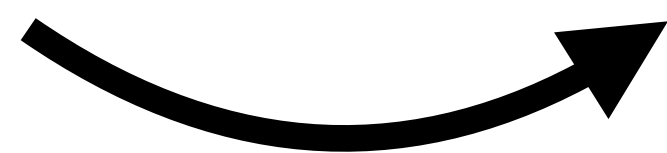
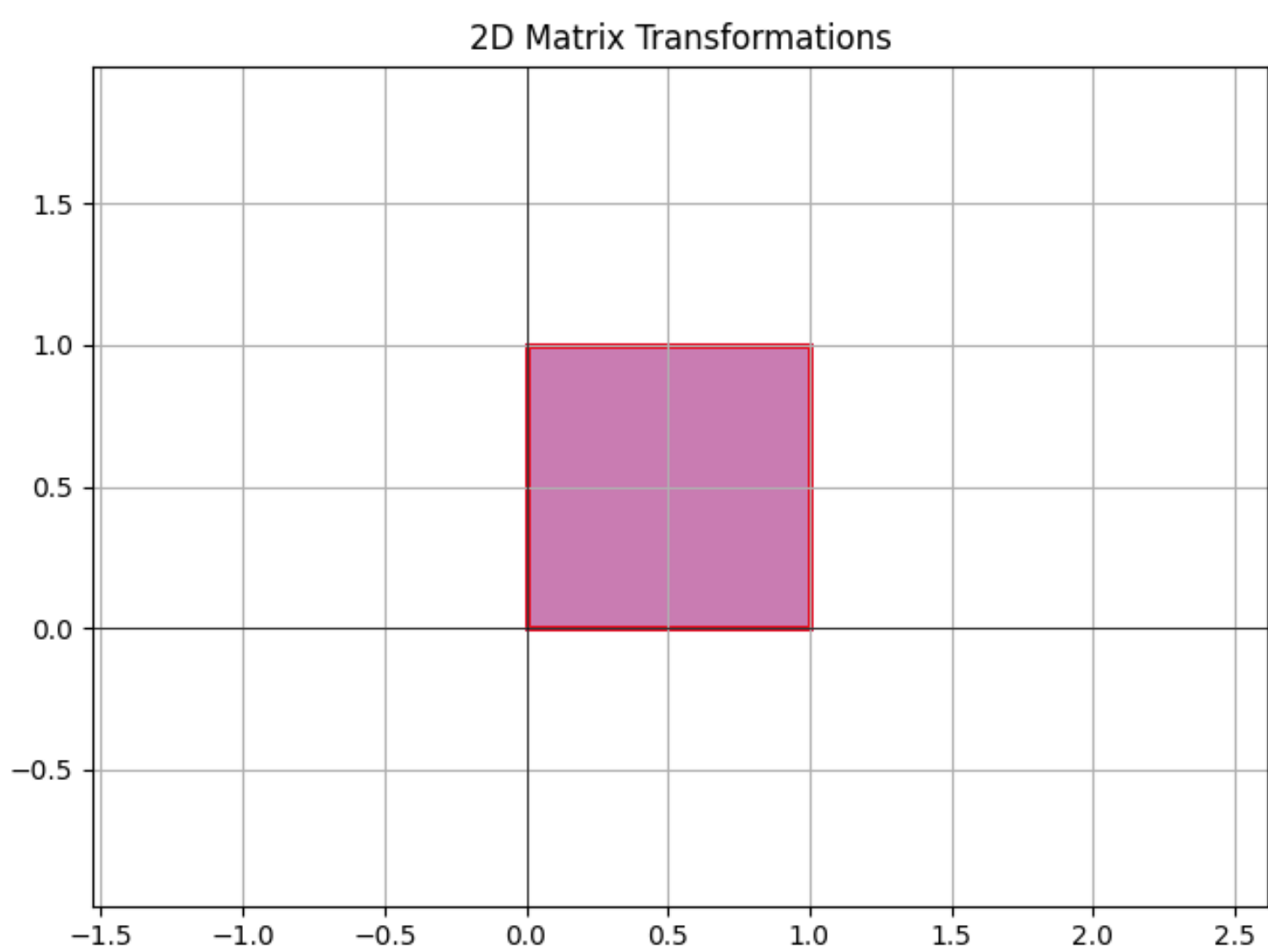
`numpy.linalg.inv`

determinant

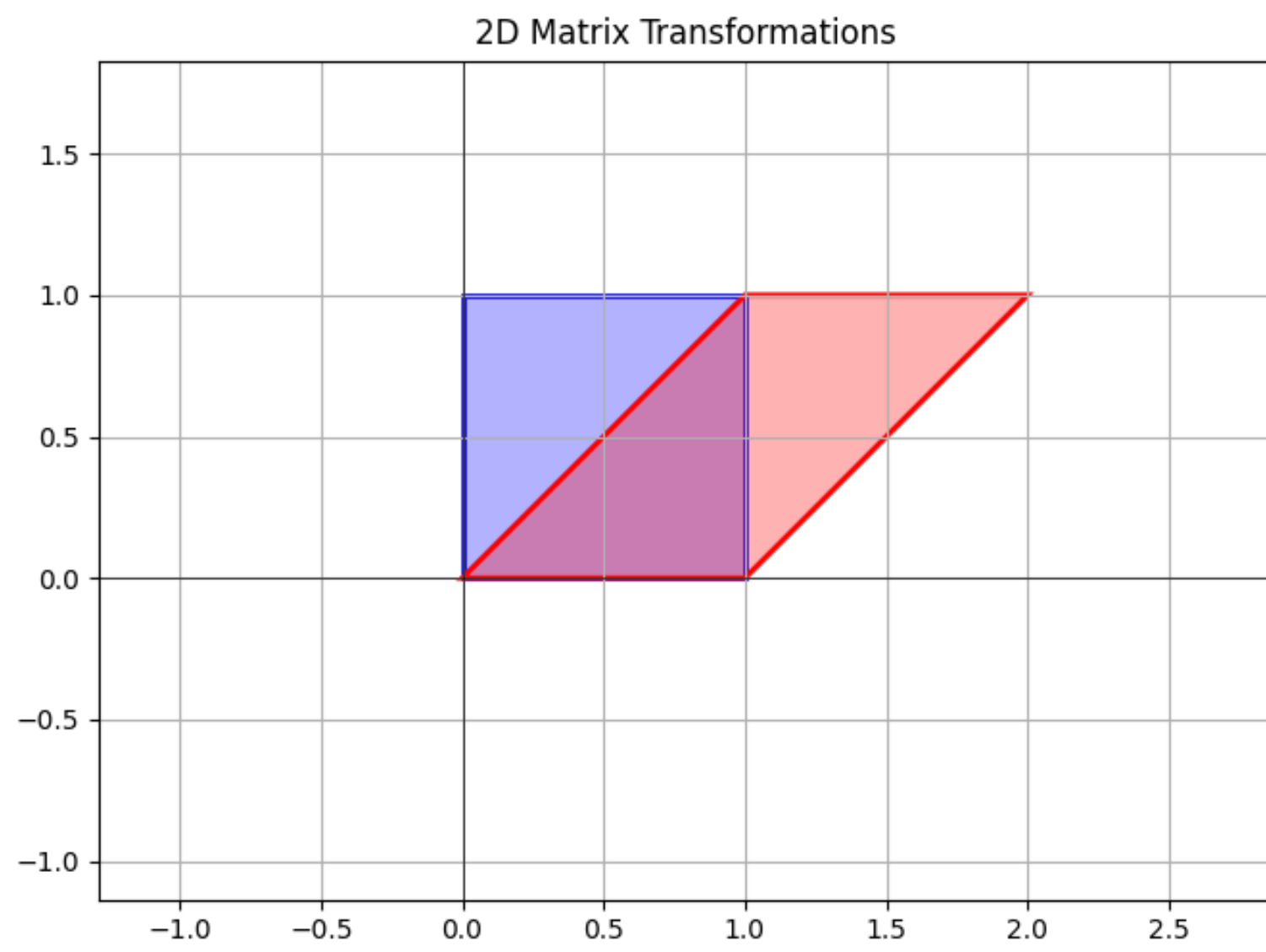
Invertible Matrix Theorem

Composing Linear Transformations

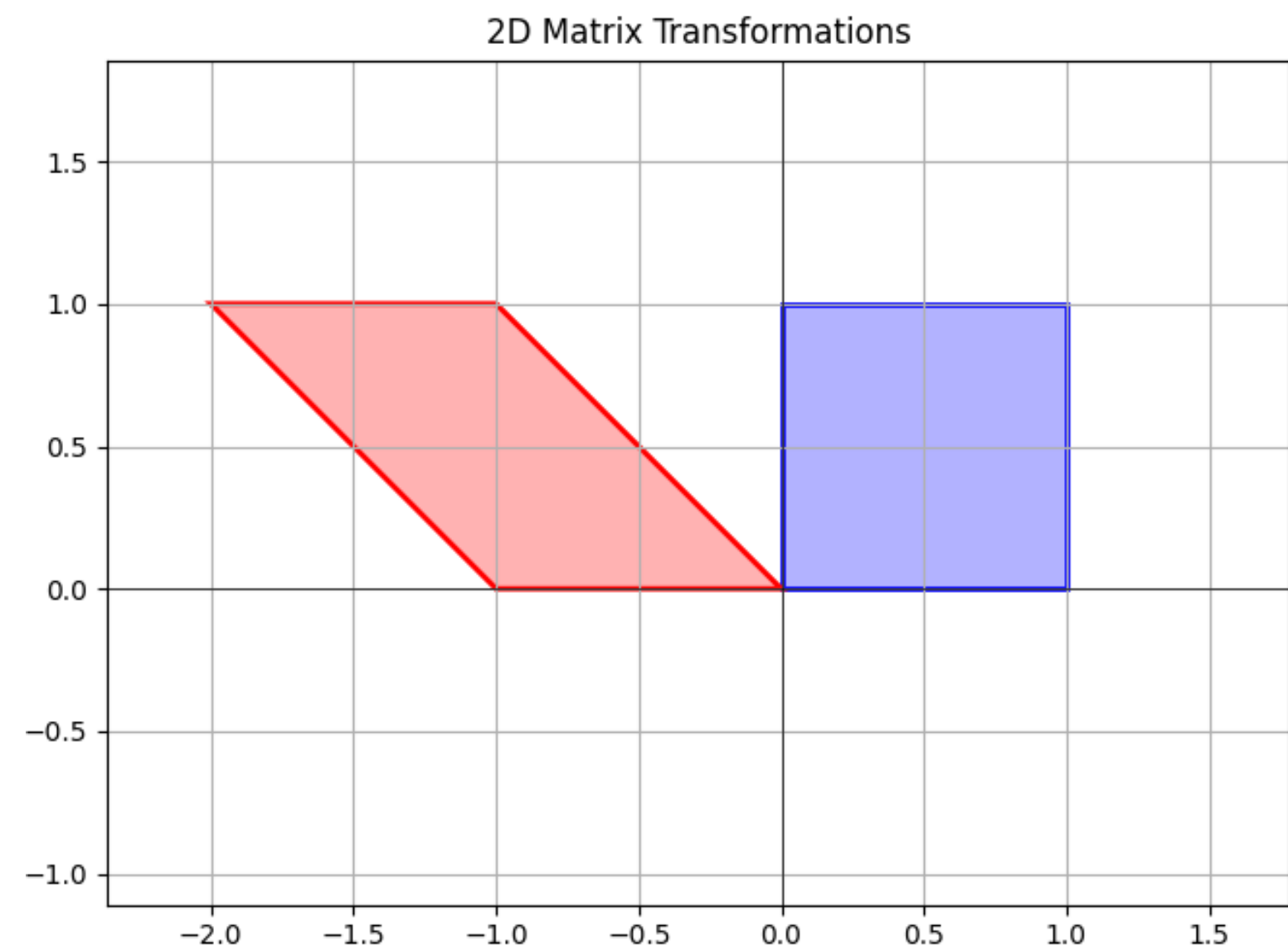
Shearing and Reflecting (Geometrically)



shear



reflect

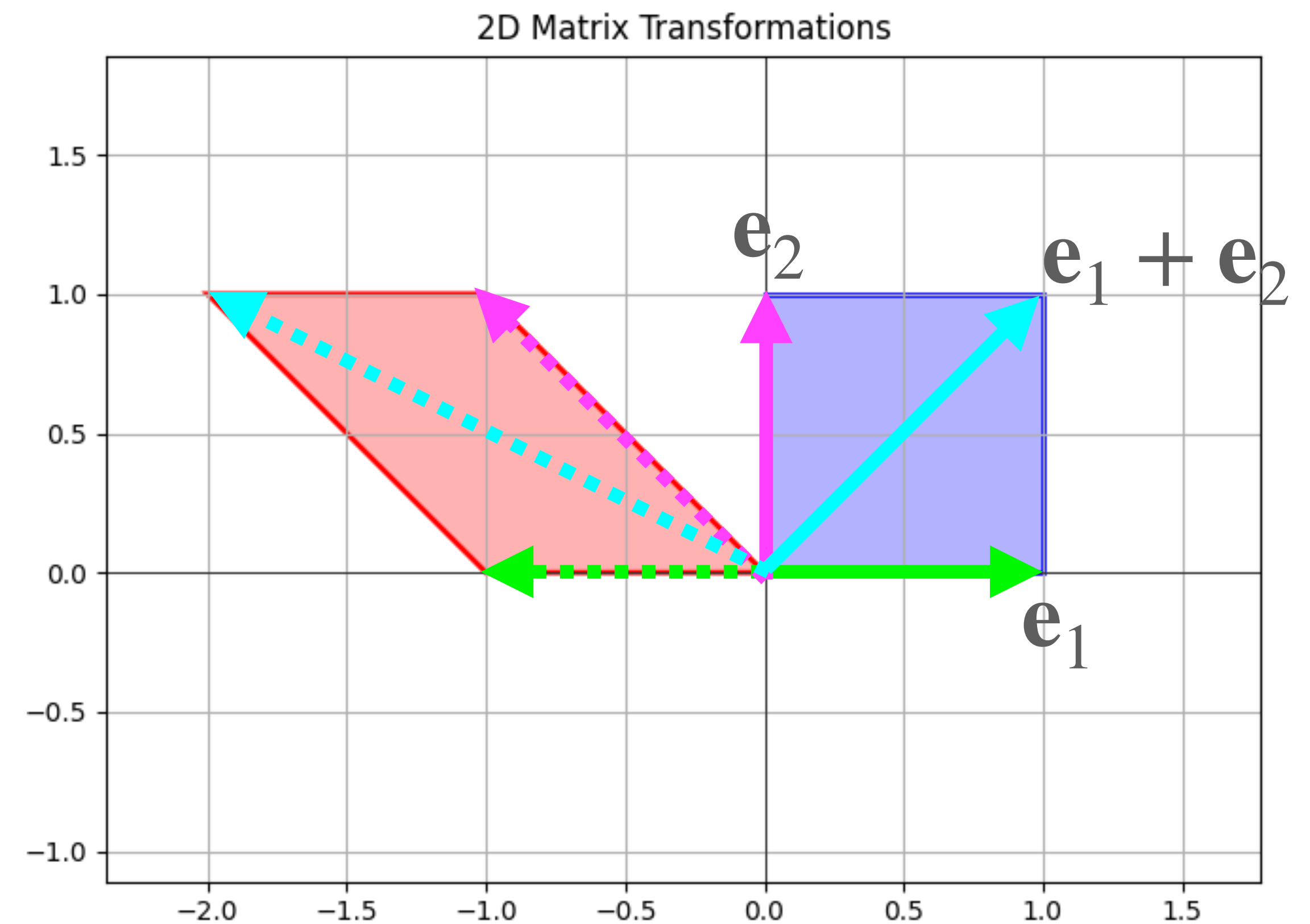


Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect shear

First multiply by shear matrix, then multiply
by reflection matrix

Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect shear

First multiply by shear matrix, then multiply
by reflection matrix

This gives us the same transformation

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

The Key Fact

Fact. The composition of two linear transformations is a linear transformation

The Key Fact

Fact. The composition of two linear transformations is a linear transformation

Verify:

The Key Fact

Fact. The composition of two linear transformations is a linear transformation

Verify:

This means the composition of two matrix transformations can be represented as a *single* matrix

The Key Question

*Given two linear transformations,
how to we compute the matrix which
implements their composition?*

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*Given two linear transformations,
how do we compute the matrix which
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Matrix Multiplication

Matrix Multiplication

Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column

Tracking Dimensions

This only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices arranged in a sequence separated by an equals sign. The first matrix is a 5x3 matrix, represented by a blue vertical line on the left labeled m and a red horizontal line on top labeled n . The second matrix is a 3x4 matrix, represented by a red vertical line on the left labeled n and a purple horizontal line on top labeled k . The third matrix is a 5x4 matrix, represented by a blue vertical line on the left labeled m and a purple horizontal line on top labeled k . Each matrix contains asterisks representing elements. Below each matrix, its dimensions are written in a colored box: $(m \times n)$ for the first, $(n \times k)$ for the second, and $(m \times k)$ for the third.

$$\begin{matrix} m \\ \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{array} \right] \end{matrix} \begin{matrix} n \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix} = \begin{matrix} m \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix}$$

$(m \times n)$ $(n \times k)$ $(m \times k)$

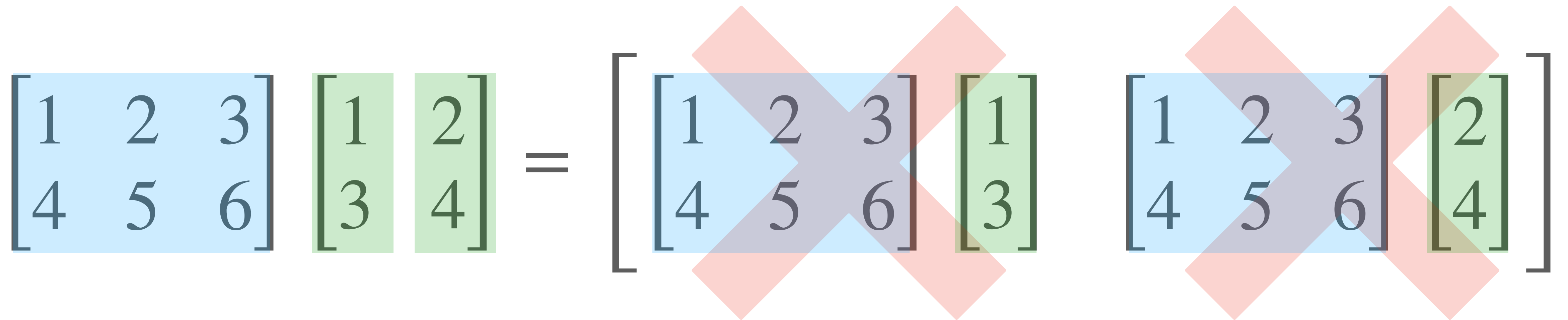
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$
The diagram illustrates two matrix-vector products that are not defined. On the left, a 2x3 matrix is multiplied by a 2x1 vector, and on the right, a 2x3 matrix is multiplied by a 3x1 vector. Both operations are crossed out with large red X's, indicating they are not defined.

These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$ and any vector $\mathbf{v} \in \mathbb{R}^k$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its first column highlighted in light red. These two matrices are multiplied together, as indicated by an equals sign, to produce a 5x4 matrix. The first row of the resulting matrix is highlighted in light purple, corresponding to the row from the first matrix and the column from the second matrix that were highlighted in the previous matrices.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The top row of this matrix is highlighted with a light blue background. The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The second column of this matrix is highlighted with a light red background. An equals sign (=) follows. The third matrix is a 5x4 matrix with all elements represented by asterisks (*). The element in the first row and second column of this matrix is highlighted with a light purple background, representing the result of the dot product of the first row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element in the first row and fourth column is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the fourth column of the second matrix.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks. The third matrix is a 5x4 matrix, also with all elements represented by asterisks. An equals sign (=) is placed between the second and third matrices. The first matrix has its second row highlighted in light blue. The second matrix has its first column highlighted in light red. The third matrix has its first row highlighted in light purple. This highlights the specific row and column used in the calculation of a single element in the product matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The third matrix is a 5x4 matrix, also with all elements represented by asterisks (*). The second matrix is highlighted with a light red background, and the third matrix is highlighted with a light purple background. The first matrix is highlighted with a light blue background. The second and third matrices are separated by an equals sign (=).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements marked with asterisks (*). The second matrix is a 3x4 matrix, also with all elements marked with asterisks. The third matrix is a 5x4 matrix, also with all elements marked with asterisks. An equals sign (=) is placed between the second and third matrices. In the first matrix, the second row is highlighted with a light blue background. In the second matrix, the third column is highlighted with a light red background. In the third matrix, the element at the intersection of the second row and third column is highlighted with a light purple background, representing the result of the dot product of the second row of the first matrix and the third column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

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$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; its third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in each cell; its first column is highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element in the third row and first column is highlighted in light purple, representing the result of the dot product of the third row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; its third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; its second column is highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the third row and second column is highlighted in light purple, representing the result of the dot product of the third row of the first matrix and the second column of the second matrix.

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$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its third row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the third row and fourth column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the fourth row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the first column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the fourth row and first column is highlighted in light purple, representing the result of the dot product of the fourth row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with a light blue horizontal band highlighting its fourth row. The second matrix is a 3x4 matrix with a light red vertical band highlighting its third column. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the fourth row and third column is highlighted with a light purple square, representing the result of the dot product of the highlighted row and column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its fourth row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks with its fourth column highlighted in light purple. This illustrates that the elements in the fourth row of the product matrix are calculated by multiplying the fourth row of the first matrix by each of the four columns of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted with a light blue background. The second matrix is a 3x4 matrix with asterisks; the first column is highlighted with a light red background. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the bottom-left cell is highlighted with a light purple background.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in all cells; its bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in all cells; its second column is highlighted in light red. An equals sign follows, and then a 5x4 matrix with asterisks in all cells; its second column (aligned with the red column of the second matrix) is highlighted in light purple. This visualizes that the element in the bottom row, second column of the product matrix is the sum of the products of the bottom row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the third column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the bottom row and third column is highlighted in light purple, representing the result of the dot product of the first matrix's bottom row and the second matrix's third column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in each cell; the fourth column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in each cell; the bottom-right element is highlighted in light purple, representing the result of the dot product of the first matrix's bottom row and the second matrix's fourth column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently,
element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

This is exactly the same as vector addition, but for matrices

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices

Algebraic Properties (Addition and Scaling)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

We need to know/memorize these

Algebraic Properties (Addition and Scaling)

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

We need to know/memorize these

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1

(also find a pair where they are the same)

One Answer: Rotation and Reflection

More Matrix Operations

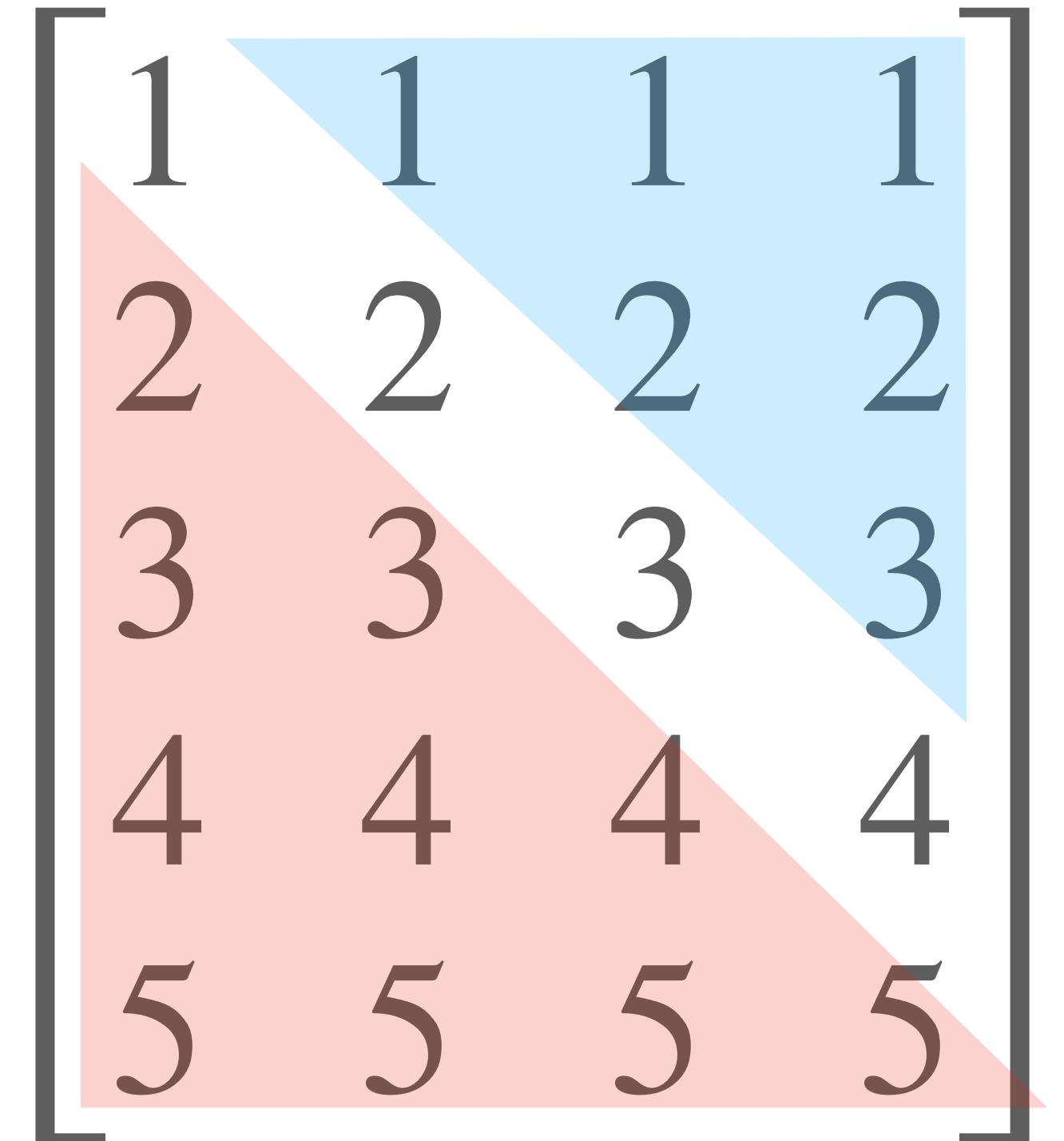
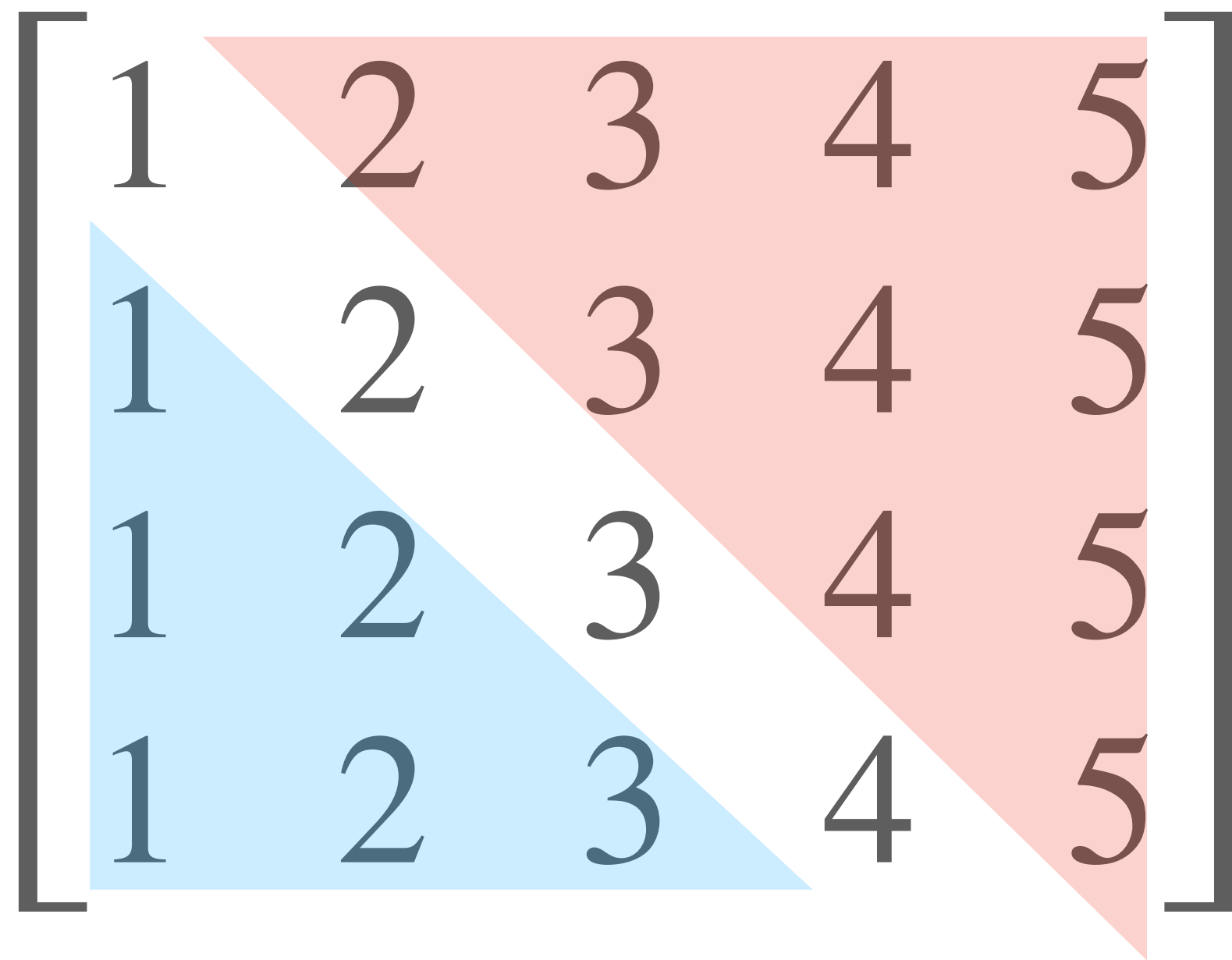
Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)



Transpose

Definition. For a $m \times n$ matrix A , the **transpose** of A , written A^T , is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = cA^T \text{ (where } c \text{ is a scalar)}$$

$$(AB)^T = B^T A^T$$

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$$(AB)^T = B^T A^T \text{ Important: the order reverses!}$$

Challenge Problem

Demonstrate that $(AB)^T = B^T A^T$ in general.

Transposes and Inner Products

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$$n \times 1$$

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For two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n , is $\mathbf{u}^T \mathbf{v}$ defined?

$$[u_1 \ u_2 \ u_3 \ u_4]$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$=$$

?

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$$[u_1 \ u_2 \ u_3 \ u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = ?$$

Transposes and Inner Products

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$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Definition. The **inner product** of two vectors **u** and **v** in \mathbb{R}^n is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix Powers

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What should A^0 be? (we want $A^0 A^k = A^{0+k} = A^k$)

$10^0 = 1$, so it stands to reason that $A^0 = I$

Matrix Powers (Computationally)

We can use `numpy.linalg.matrix_power`

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16}$$

Why? :

Final Warnings about Matrix Multiplication

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1. AB is not necessarily equal to BA , even if both are defined.
2. If $AB = AC$ then it is not necessary that $B = C$.
3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$

Challenge. Choose A and B such that they have all nonzero entries

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

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transpose

A^T

So Far: Matrix Operations

transpose

$$A^T$$

scaling

$$cA$$

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transpose

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addition (subtraction)

$$A + B$$

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$$AB$$

$$A^k$$

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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$$I\mathbf{v} = \mathbf{v}$$

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These may be different sizes

Recall: The Identity Matrix

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \\ 2 \times 2 & 2 \times 4 & & 2 \times 4 & 4 \times 4 & & 2 \times 4 \end{array}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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How do we solve this equation?

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Divide on both sides by 2 to get $x = 5$.

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Do all matrices have
inverses?

Do all matrices have
inverses?

No. If they did, then every linear
system would have a solution

When does a matrix have
an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

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- » whose transformations can be both 1-1 and onto
- » whose columns can have full span and be linearly independent
- » that can have inverses

Matrix Inverses

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Definition. For a $n \times n$ matrix A , an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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A is **invertible** if it has an inverse. Otherwise it is **singular**.

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Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1 -axis in \mathbb{R}^2 is its own inverse.

Verify:

Example: No inverse

Verify:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

If A is invertible, then we write A^{-1}
for *the* inverse of A .

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A , if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of \mathbf{b} .

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» exactly one solution for any choice of \mathbf{b}

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If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» at least one solution for any choice of \mathbf{b}

» at most one solution for any choice of \mathbf{b}

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» T is onto

» T is one-to-one

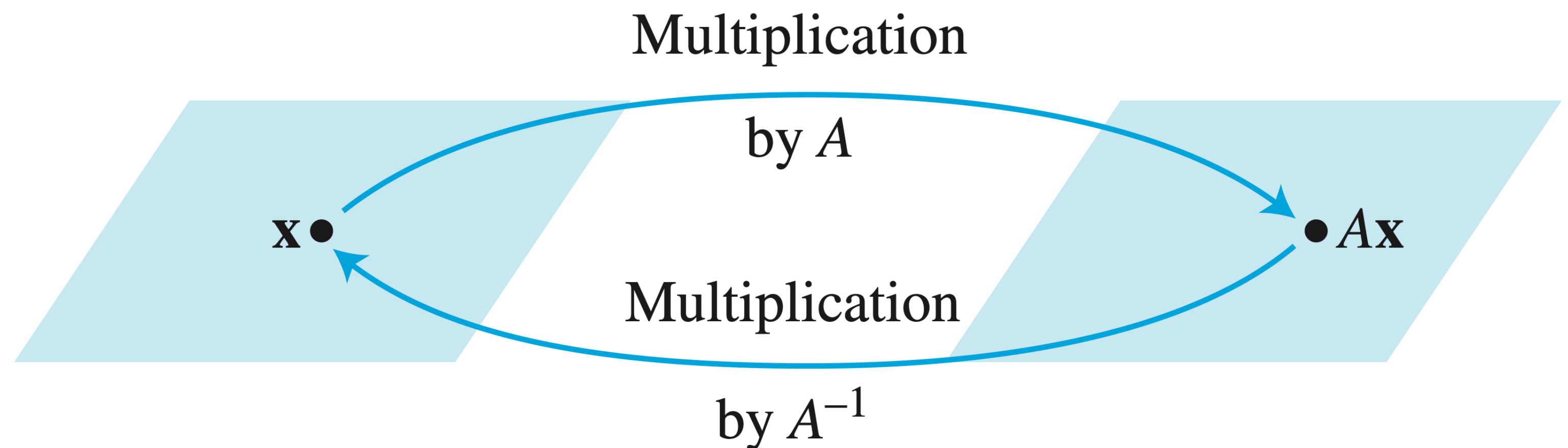
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n



Connection to Transformations

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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

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Non-Example. Projection onto the x_1 -axis

Connection to Transformations

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Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

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A transformation is a 1-1 correspondence if it is 1-1 and onto

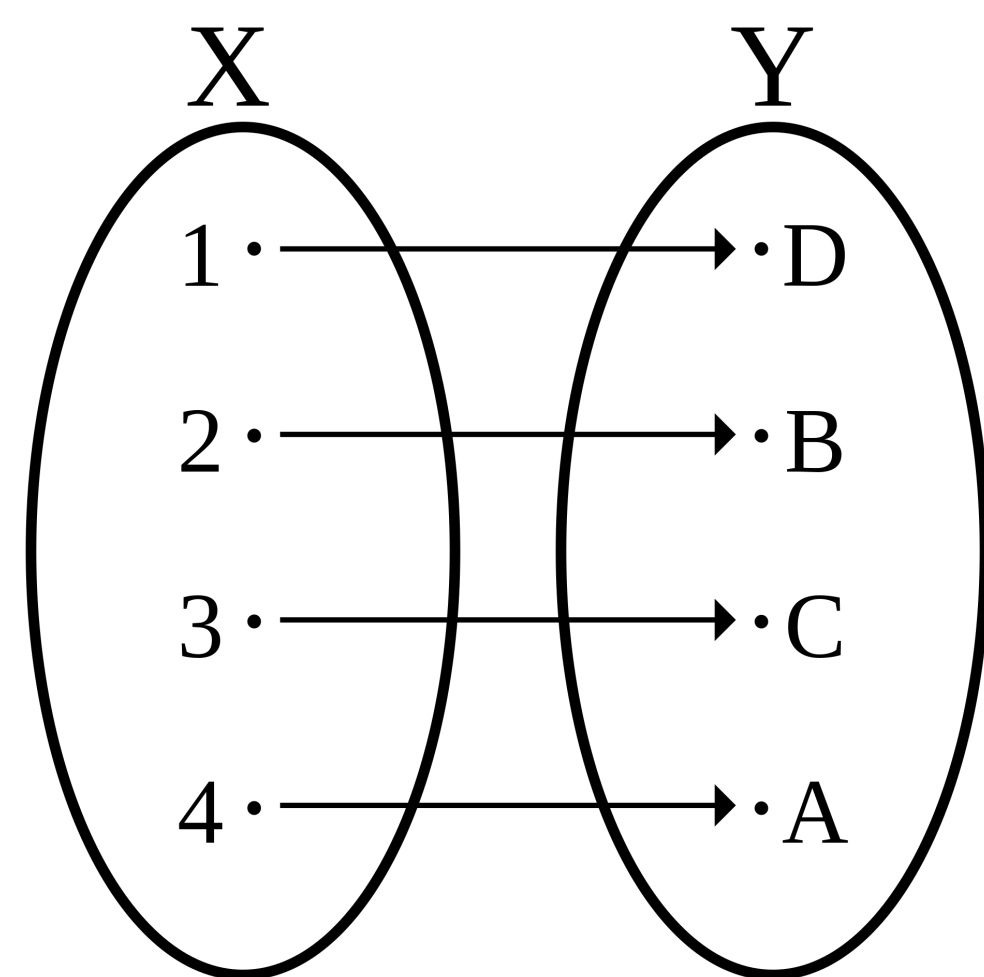
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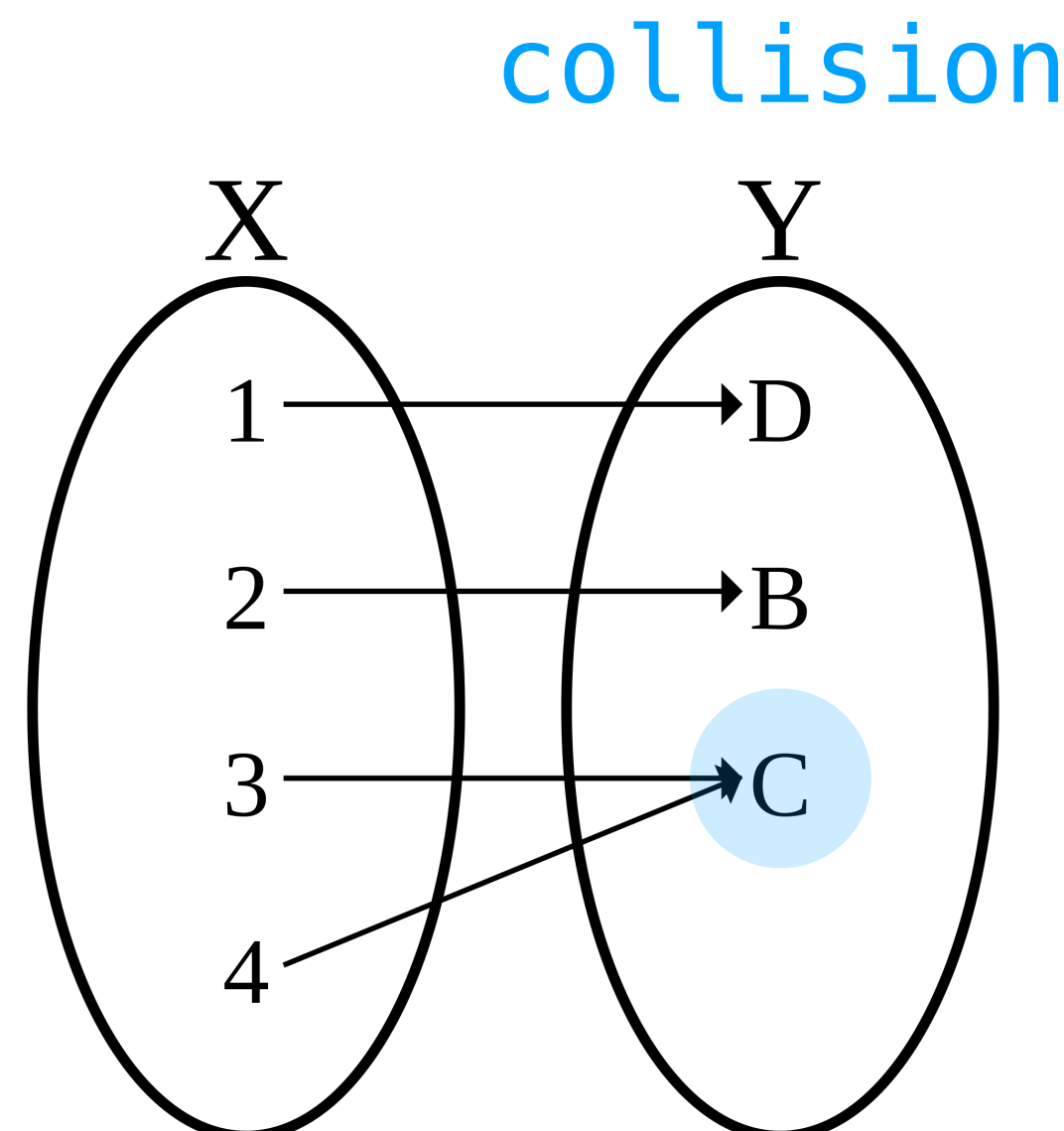
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Invertible transformations are 1-1 correspondences

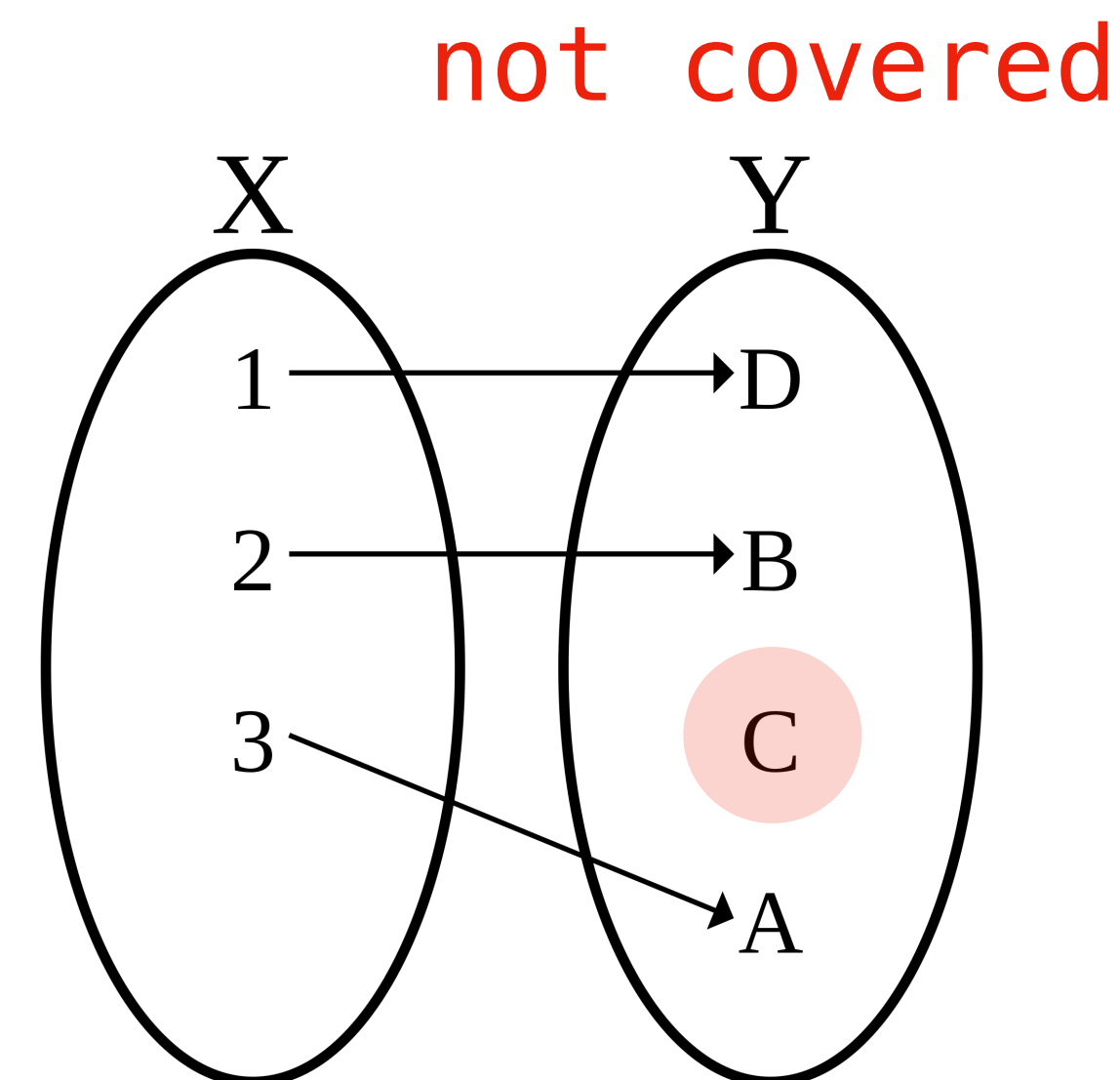
Kinds of Transformations (Pictorially)



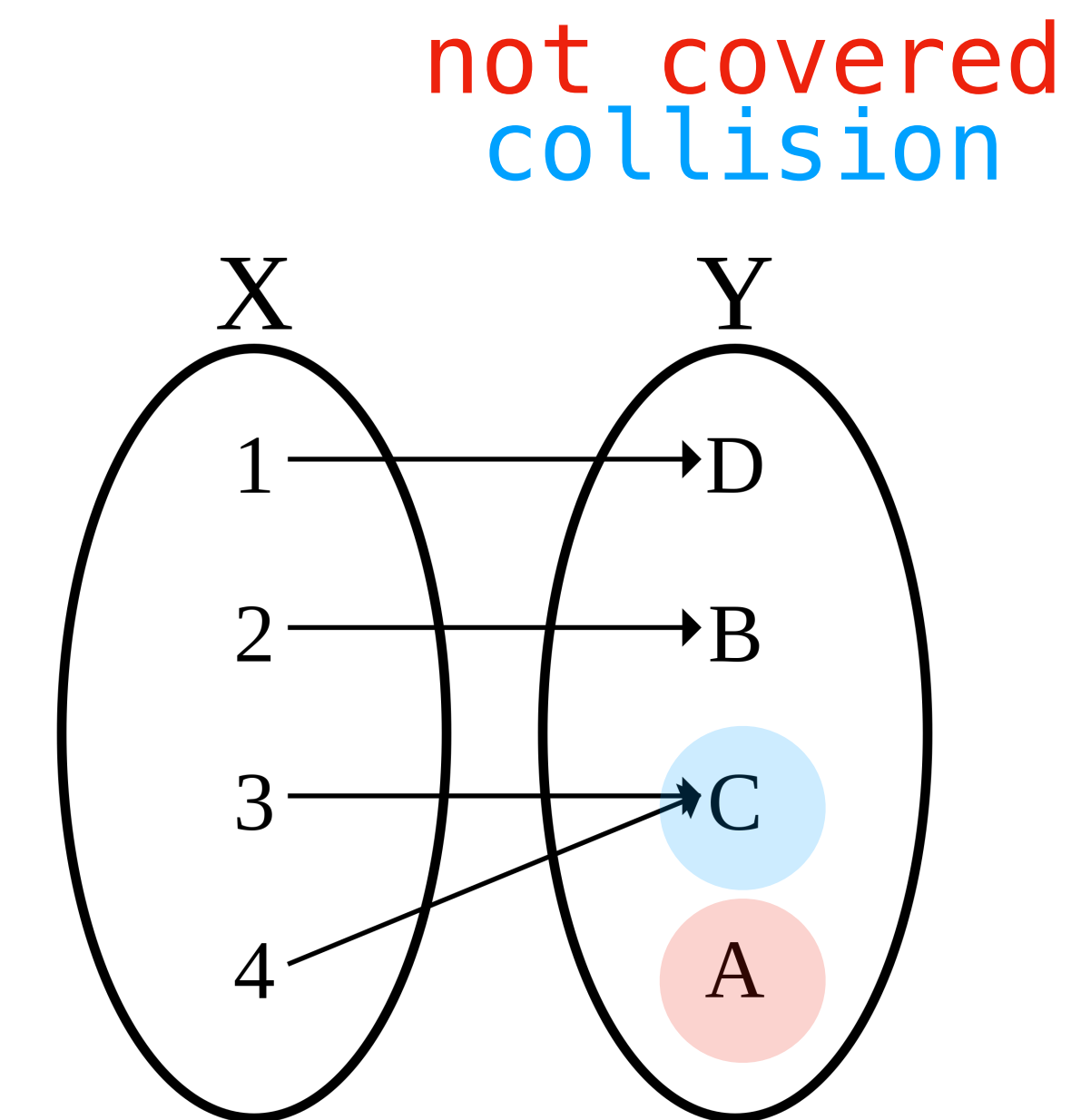
1-1 correspondence



onto, not 1-1



1-1 not onto



not 1-1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

How can we determine if a matrix has an inverse?

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How can we determine if a matrix has an inverse?

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each \mathbf{b}_i ?

In General

$$A\mathbf{b}_1 = \mathbf{e}_1$$

$$A\mathbf{b}_2 = \mathbf{e}_2$$

$$A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns)

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$$A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns)

We need to solve 3 matrix equations.

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector \mathbf{e}_i . Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \dots \quad \mathbf{s}_n]$$

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A

Solution. Row reduce the matrix $[A \ I]$ to a matrix $[I \ B]$. Then B is the inverse of A

This is really the same thing. It's a simultaneous reduction

demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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The **determinant** of a 2×2 matrix is the value $ad - bc$

The inverse is defined only if the determinant is nonzero

(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C , demonstrate that $B^{-1} = C$

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both)

This works because inverses are unique

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B , the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every \mathbf{b}

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

- 5. A has a pivot in every column
- 6. A has a pivot in every row
- 7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

- 8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- 9. The columns of A are linearly independent
- 10. The columns of A span \mathbb{R}^n

Verify:

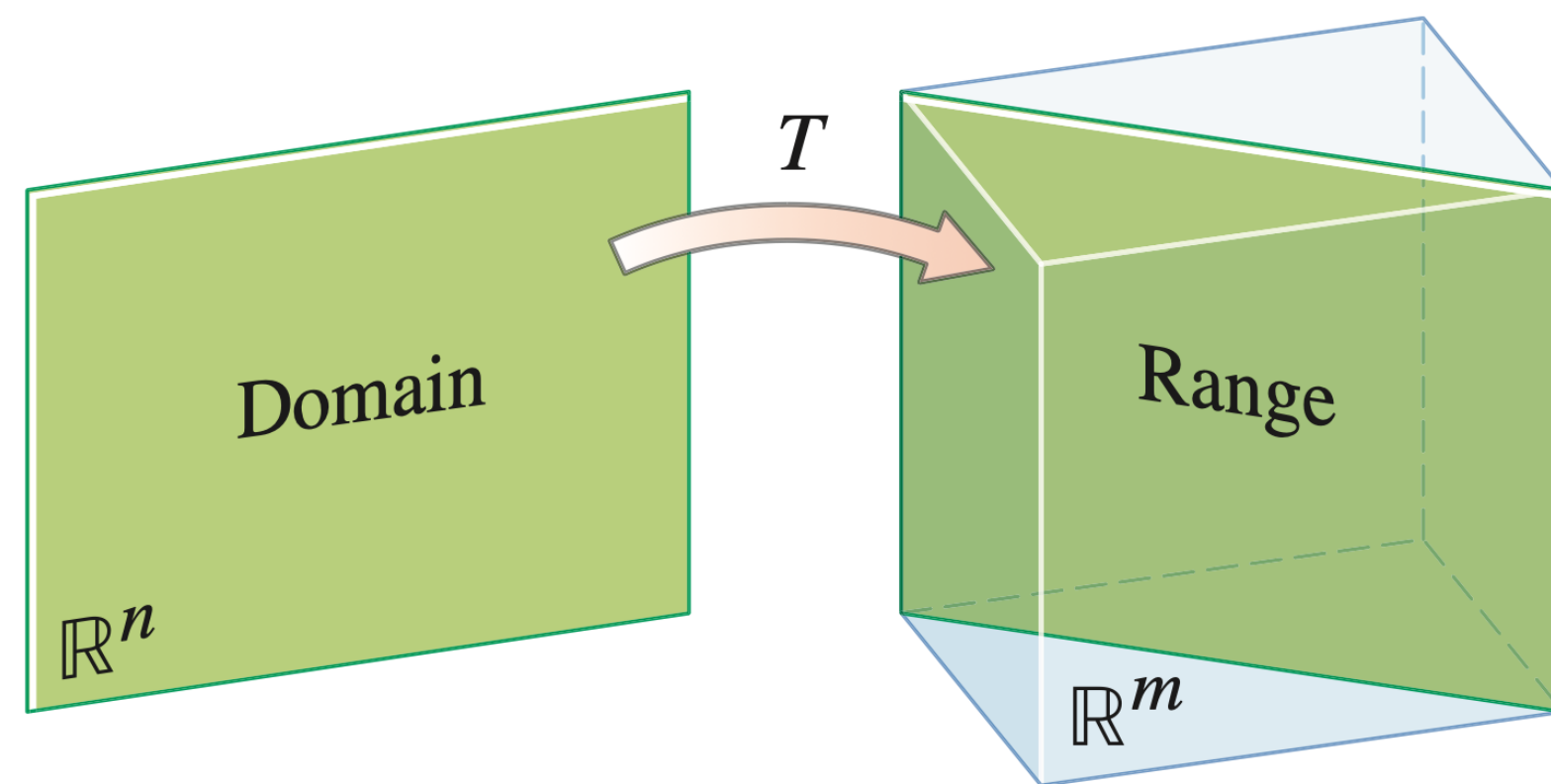
Recall: Onto Transformations

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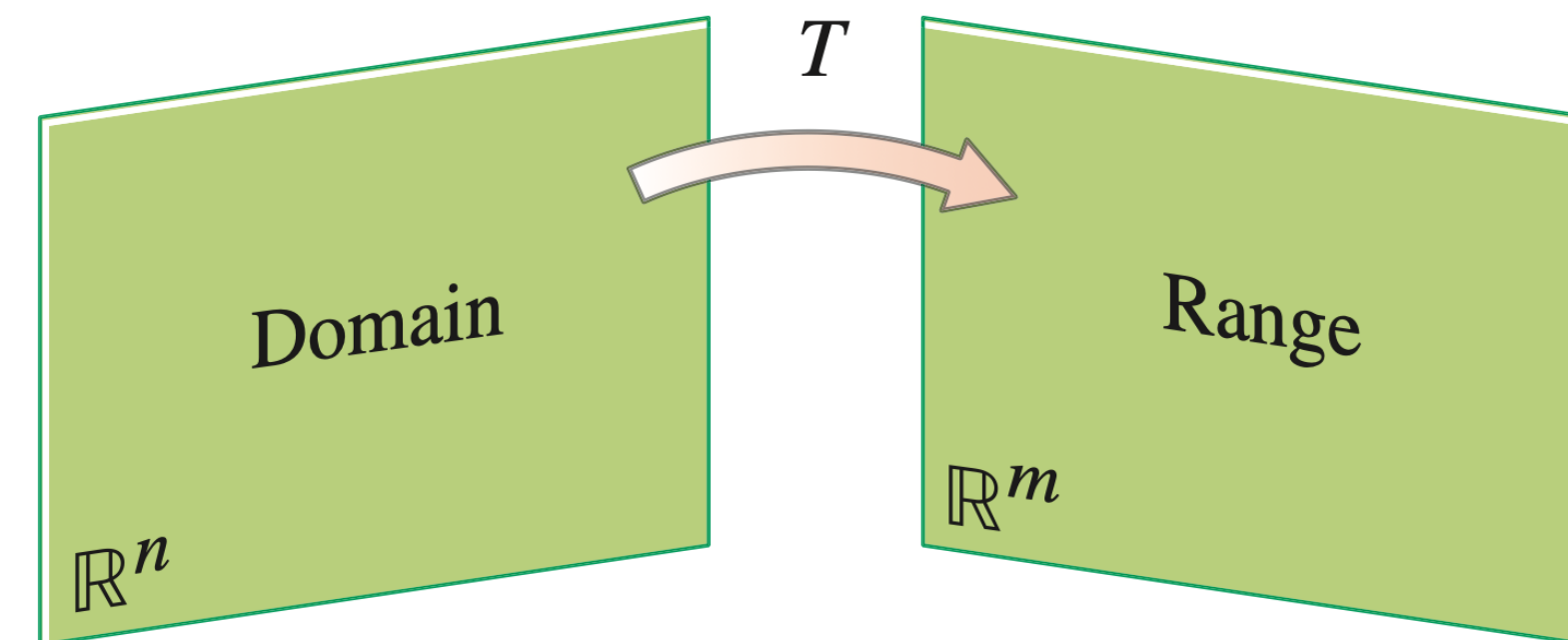
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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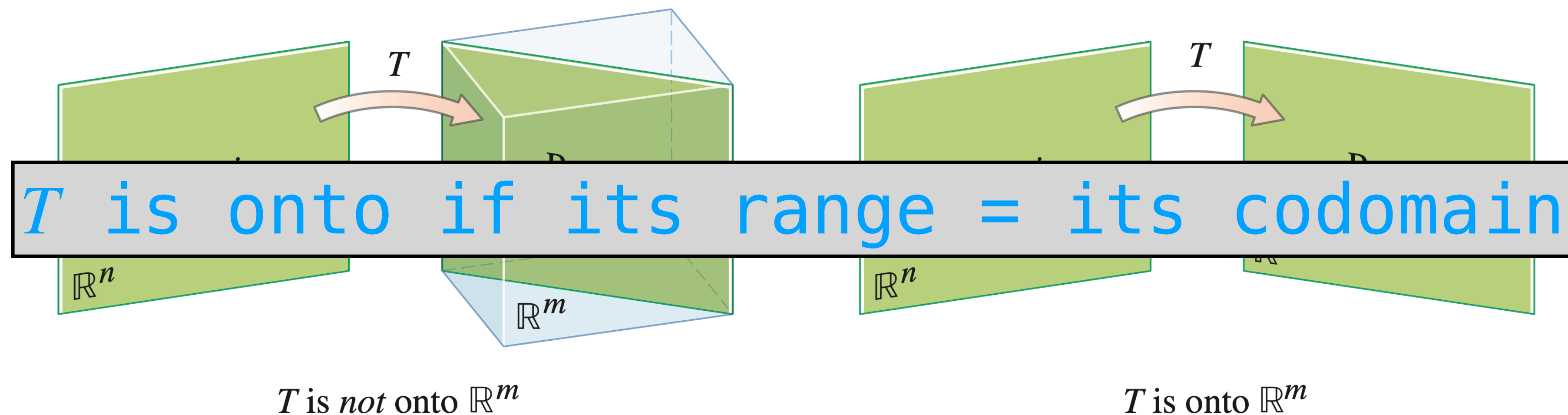
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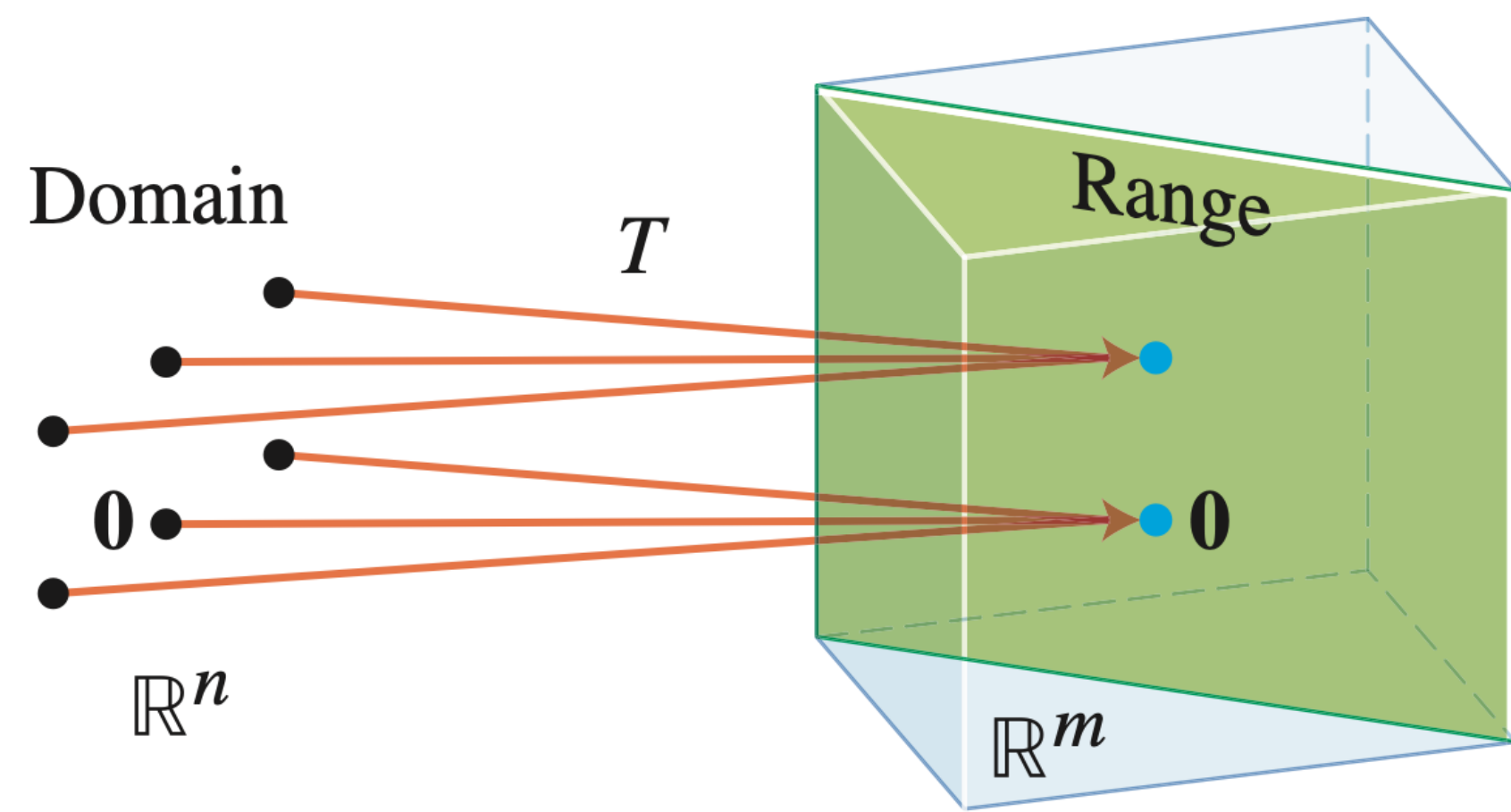
Recall: One-to-one Transformations

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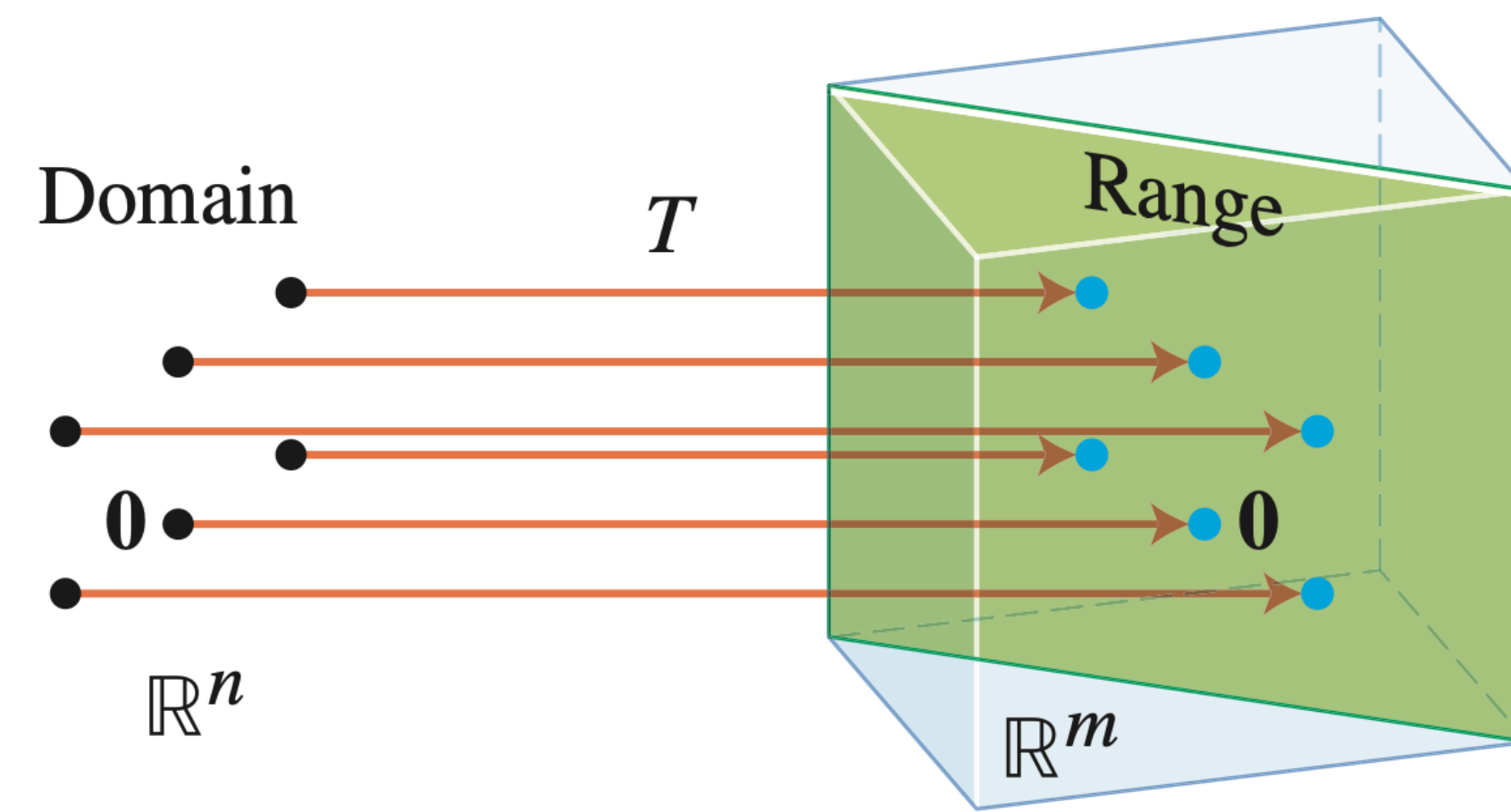
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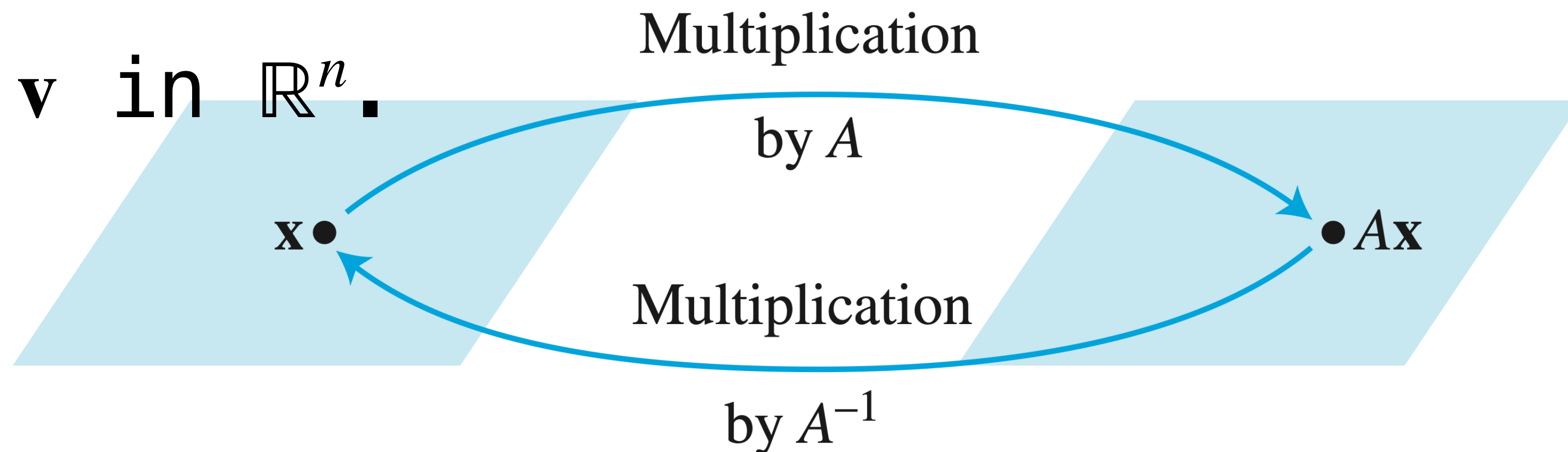
T is one-to-one

Recall: Invertible Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n .



Recall: One-to-One Correspondence

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Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

- 11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
- 12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
- 13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
- 14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any \mathbf{b}
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
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(this is a stronger statement than
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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1-1** if and only if A **is onto**

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Warning. Remember this only applies square matrices.

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Invertibility is completely determined by how A behaves on 0 .

Question (Conceptual)

True or False: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), then B is also invertible.

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ is invertible, then is $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]$ also invertible? Justify your answer.

Answer

Consider $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$. We can get to $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]^T$ by row operations

Summary

The algebra of matrices can help us simplify matrix expressions

The invertible matrix theorem connects all the perspectives we've taken so far