Linear Independence

Geometric Algorithms
Lecture 6

CAS CS 132

Practice Problem

Do these three vectors span all of $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Answer

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

full span $\equiv$ for any $\vec{v} \in \mathbb{R}^3$ there $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v} + \alpha_3 \vec{v}_3 = \vec{v}$$

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 2 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 4^{-4} & 6^{+6} & 8^{+4} \\ -4^{+4} & -3^{-6} & -5^{-4} \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 0 & 12 & 12 \\ 0 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 1^{-1} & 1^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{2} & -3 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots

3 rows

$\boxed{\text{NO}}$

Answer: No

Consider the matrix

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 2 & -3 & -2 \end{bmatrix}$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 4 & -6 & -4 \end{bmatrix}$$

$$R_3 \leftarrow 2R_3$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & -9 & -9 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$
$$R_3 \leftarrow R_3 + R_1$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 3R_2$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Third row has no pivot

Outline

» Motivate and define **linear independence**

» See several perspectives on linear independence

» If there's time: see an application of linear systems to **network flows**

Keywords

linear independence

linear dependence

homogenous systems of linear equations

trivial and nontrivial solutions

Homogeneous Linear Systems

Recall: The Zero Vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

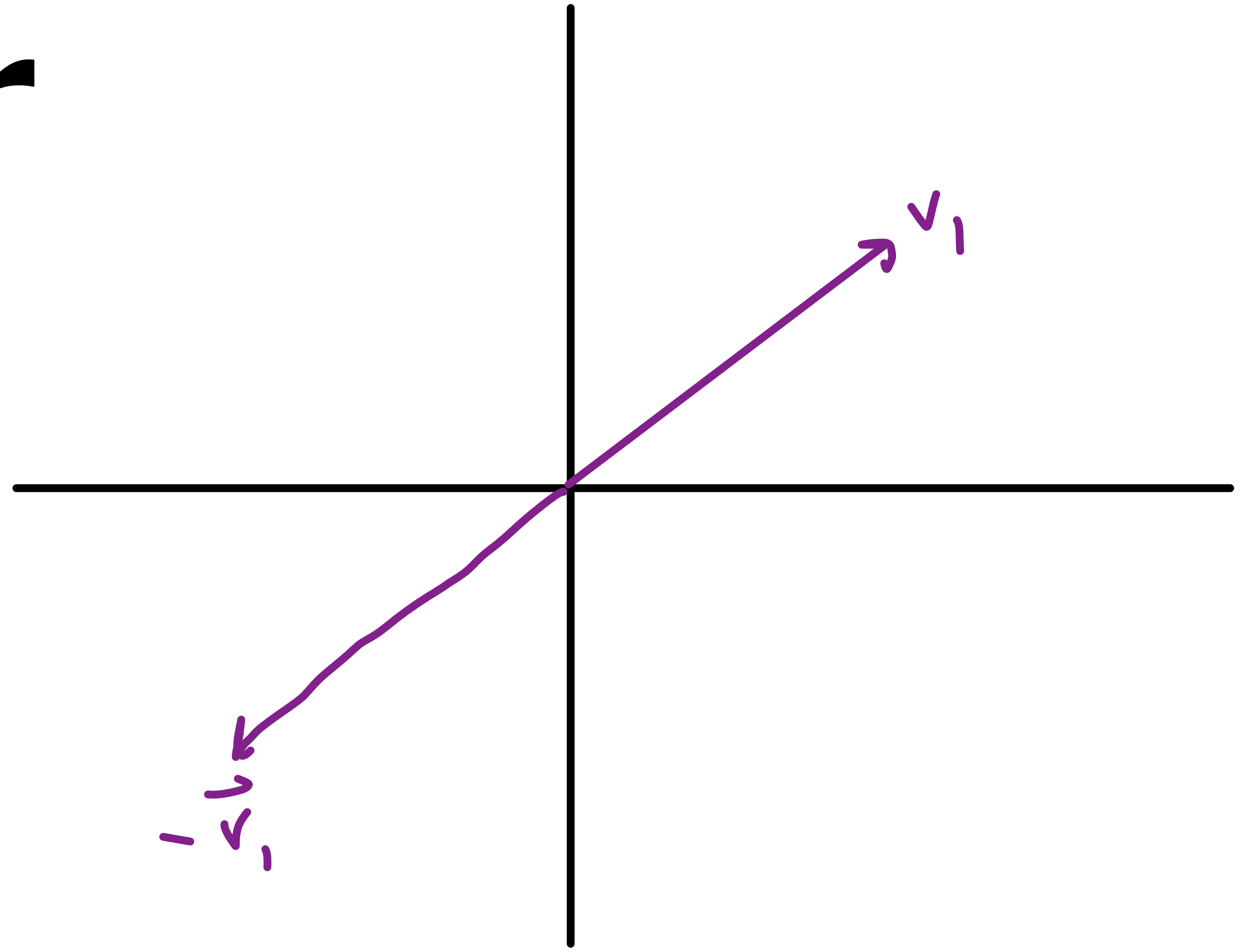
Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

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$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$\vec{0}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} n$$

the dimension is implicit in the notation

Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

Homogenous Linear Systems

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$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

Trivial Solutions

$$A\vec{0} = \vec{0}$$

Definition. For the matrix equation $A\mathbf{x} = \mathbf{0}$ the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***

Any other solution is called ***nontrivial***

Trivial Solutions

Definition. For the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***

Any other solution is called ***nontrivial***

Trivial Solutions

Definition. For the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***

Any other solution is called ***nontrivial***

Questions about Homogeneous Systems

When does $A\mathbf{x} = \mathbf{0}$ have only the trivial solution?

When does $A\mathbf{x} = \mathbf{0}$ have nontrivial solutions?

What does it mean *geometrically* in each case?

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

An Important Feature of Homogenous Systems

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What do we know about the covered column?

An Important Feature of Homogenous Systems

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What do we know about the covered column?

It has to be all zeros

Linear Independence

Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution)

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The columns of A are linearly independent if $A\mathrm{x} = 0$ has exactly one solution

Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

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has a nontrivial solution

A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals $\mathbf{0}$

Linear Dependence (Alternative)

Definition. A set of vectors is **linearly dependent** if it is <u>not</u> linearly independent

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Definition. A set of vectors is **linearly dependent** if it is <u>not</u> linearly independent

$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

$\equiv$

$A\mathbf{x} = \mathbf{0}$ does <u>not</u> have only the trivial solution

Examples

$$\{\ \} \quad \text{(empty set)}$$

$$\vec{0} = \vec{0}$$

$$\boxed{\text{Lin. ind.}}$$

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \vec{0}$$

$$2x_1 = 0$$
$$3x_1 = 0 \Rightarrow x_1 = 0$$
$$-x_1 = 0$$

$\boxed{\text{Lin. Ind.}}$

Examples

$$\left\{ \begin{array}{ccc} \vec{v} & & 2\vec{v} \\ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} & , & \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} \end{array} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \vec{0}$$

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \end{aligned}}$$

$\boxed{\text{Lin. Dep.}}$

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 2x_1 = 0 \\ 3x_1 = 0 \end{array}\right] \Rightarrow x_1 = 0$$

$$\left. x_2 - x_1 = 0 \right] \Rightarrow x_2 = 0$$

$\boxed{\text{Lin. Ind.}}$

Another Interpretation of Linear Dependence

demo

(from ILA)

Three Vectors in $\mathbb{R}^3$

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It's possible for three vectors in $\mathbb{R}^3$ to span all of $\mathbb{R}^3$, but it's <u>not</u> guaranteed

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Three Vectors in $\mathbb{R}^3$

It's possible for three vectors in $\mathbb{R}^3$ to span all of $\mathbb{R}^3$, but it's <u>not</u> guaranteed

There may be vectors which lies in the plane spanned by two other vectors

Or even two vectors which lie in the span of one of the others

Fundamental Concern

How do we classify when a set of vectors does <u>not</u> span as much as it possibly can? When it is "*smaller*" *than it could be?*

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How do we classify when a set of vectors does <u>not</u> span as much as it possibly can? When it is "smaller" than it could be?

This is the role of linear dependence

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Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself)

Linear Dependence (Another Alternative)

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e.g., $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$

(the recap problem)

The Linear Combination Perspective

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is lin. dep.

Suppose we have four vectors such that

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \qquad + 5\mathbf{v}_4$$

what do we know about the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

$(2, 3, -1, 5)$

The Linear Combination Perspective

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \qquad + 5\mathbf{v}_4$$

Implies $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution:

$$(2,3,-1,5)$$

The Vector Equation Perspective

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

The Vector Equation Perspective

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

We can turn this into a linear combination

The Vector Equation Perspective

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}$$

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

The Vector Equation Perspective

$$\alpha_1 \mathbf{v}_1 + \qquad\qquad \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = -\alpha_2 \mathbf{v}_2$$

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

The Vector Equation Perspective

$$\frac{-\alpha_1}{\alpha_2}\mathbf{v}_1 + \qquad\qquad \frac{-\alpha_3}{\alpha_2}\mathbf{v}_3 + \frac{-\alpha_4}{\alpha_2}\mathbf{v}_4 = \qquad \mathbf{v}_2$$

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

We get one vector as a linear combination of the others

The Vector Equation Perspective

This division only works because $\alpha_2 \neq 0$

$$\frac{-\alpha_1}{\alpha_2}\mathbf{v}_1 + \qquad \frac{-\alpha_3}{\alpha_2}\mathbf{v}_3 + \frac{-\alpha_4}{\alpha_2}\mathbf{v}_4 = \qquad \mathbf{v}_2$$

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

We get one vector as a linear combination of the others

In All

Theorem. A set of vectors is linearly dependent <u>if and only if</u> it is nonempty and *at least* one of its vectors can be written as a linear combination of the others

P if and only if Q means
P implies Q and Q implies P

Linear Dependence Relation

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

A linear dependence relation *witnesses* the linear dependence

How To: Linear Dependence Relation

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Question. Write down a linear dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$

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$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]\,\mathbf{x} = \mathbf{0}$$

How To: Linear Dependence Relation

Question. Write down a linear dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$

Solution. Find a nontrivial solution to the equation

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]\,\mathbf{x} = \mathbf{0}$$

(there will be a free variable you can choose to be nonzero)

Example

Write down the linear dependence relation for the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Added $\mathbf{0}$ column

Where we left off

Answer

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2/3$$

Answer

$$\begin{bmatrix} -4 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

Answer

$$\begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1/(-4)$$

Answer

$$x_1 = -(0.5)x_3$$
$$x_2 = -x_3$$
$$x_3 \text{ is free}$$

Answer

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2 \quad \text{(plug in any value)}$$

Answer

$$\begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} - 2\begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} - 2 \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note there are other solutions as well...

Simple Cases

The Empty Set

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$\{\}$ (a.k.a. $\varnothing$) is linearly independent

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We stretch the definition a bit: there is no nontrivial linear combination of the vectors equaling $\mathbf{0}$

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There are none at all...

The Empty Set

$\text{span}\{\ \} = \{\vec{0}\}$

$\{\}$ (a.k.a. $\varnothing$) is linearly independent

We stretch the definition a bit: there is no nontrivial linear combination of the vectors equaling $\mathbf{0}$

There are none at all...

$\mathbf{0}$ is in every span, even the span of the empty set

One Vector

A single vector $\mathbf{v}$ is linearly independent if and only if it $\mathbf{v} \neq \mathbf{0}$

(Note that $x_1\mathbf{0} = \mathbf{0}$ *has many nontrivial solutions)*

The Zero Vector and Linear Dependence

If a set of vectors V contains the $\mathbf{0}$, then it is linearly dependent

The Zero Vector and Linear Dependence

If a set of vectors V contains the $\mathbf{0}$, then it is linearly dependent

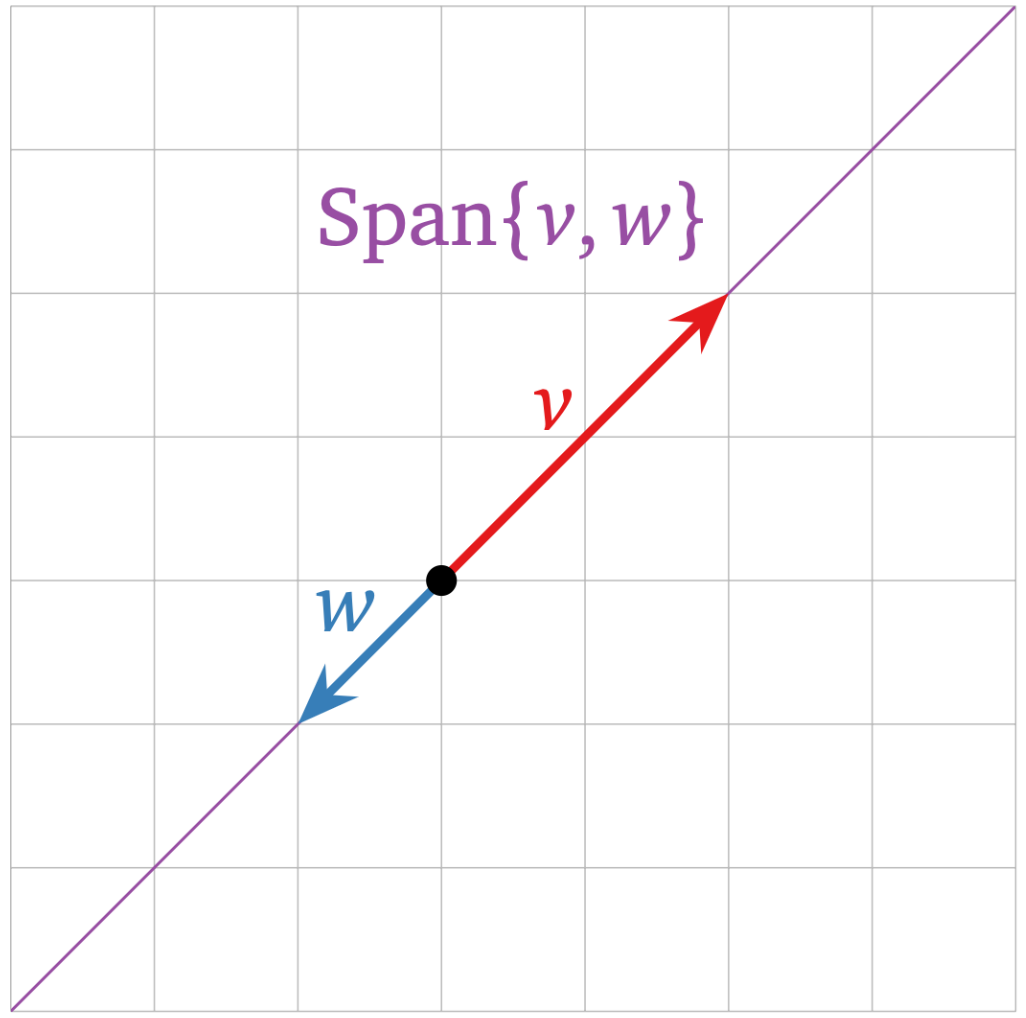
$$(1)\mathbf{0} + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \ldots + 0\mathbf{v}_n = \mathbf{0}$$

There is a very simple nontrivial solution

Two Vectors

Definition. Two vectors are ***colinear*** if they are scalar multiples of each other

e.g., $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1.5\\1.5\\3\end{bmatrix}$ or $\begin{bmatrix}2\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$ $\longrightarrow$



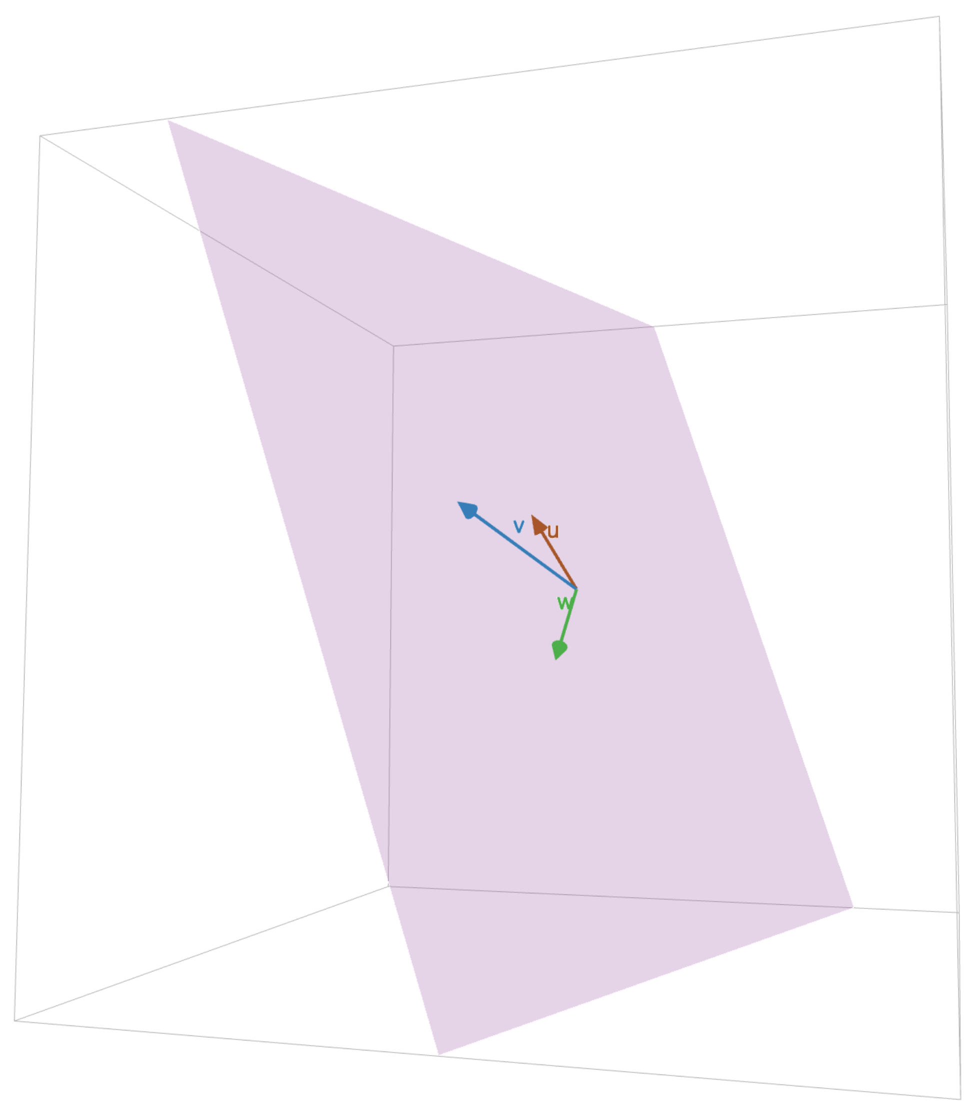
Two vectors are linearly dependent if and only if they are colinear

image source

Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane

Three vectors are linearly dependent if an only if they are colinear or coplanar



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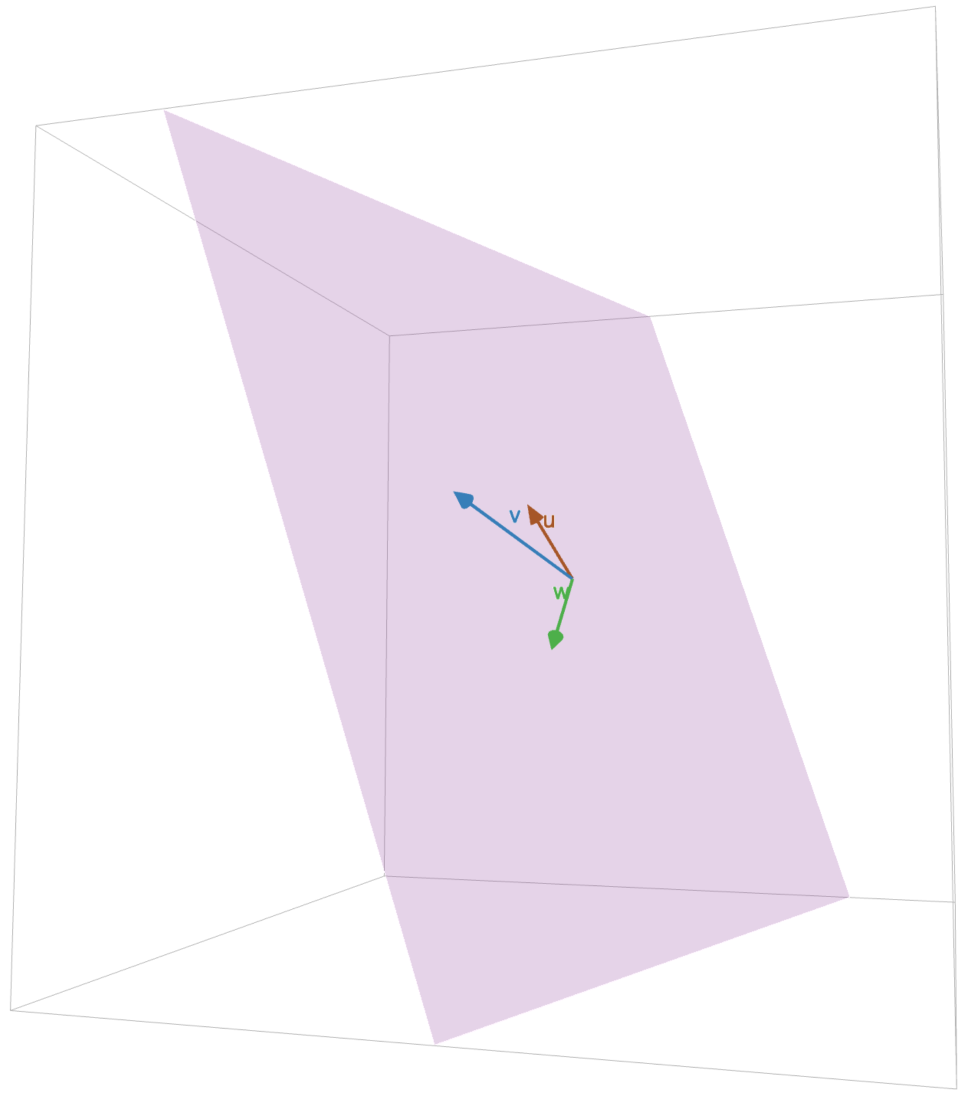
Three vectors are linearly dependent if an only if they are colinear or coplanar

This reasoning can be extended to more vectors, but we run out of terminology



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Yet Another Interpretation

Increasing Span Criterion

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly *independent* then we cannot write one of it's vectors as a linear combination of the others

But we get something stronger

Increasing Span Criterion

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if for all $i \leq n$,

$$\mathbf{v}_i \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

Increasing Span Criterion

$v_1 \notin \text{span}\{\}$
$v_2 \notin \text{span}\{v_1\}$
$v_3 \notin \text{span}\{v_1, v_2\}$

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if for all $i \leq n$,

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As we add vectors, the span gets larger

Increasing Span Criterion

So in this case, our span keeps getting "bigger"

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$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

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$\text{span}\{\mathbf{v}_1\}$ is a line

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane

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So in this case, our span keeps getting "bigger"

$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

$\text{span}\{\mathbf{v}_1\}$ is a line

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a 3d–hyperplane

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a 4d-hyperlane

...

Characterization of Linear Dependence

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only if there is an $i \leq n$,

$$\mathbf{v}_i \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

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$$\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

As we add vectors, we'll eventually find one in the span of the preceding ones.

Characterization of Linear Dependence

$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

$\text{span}\{\mathbf{v}_1\}$ is a line

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane

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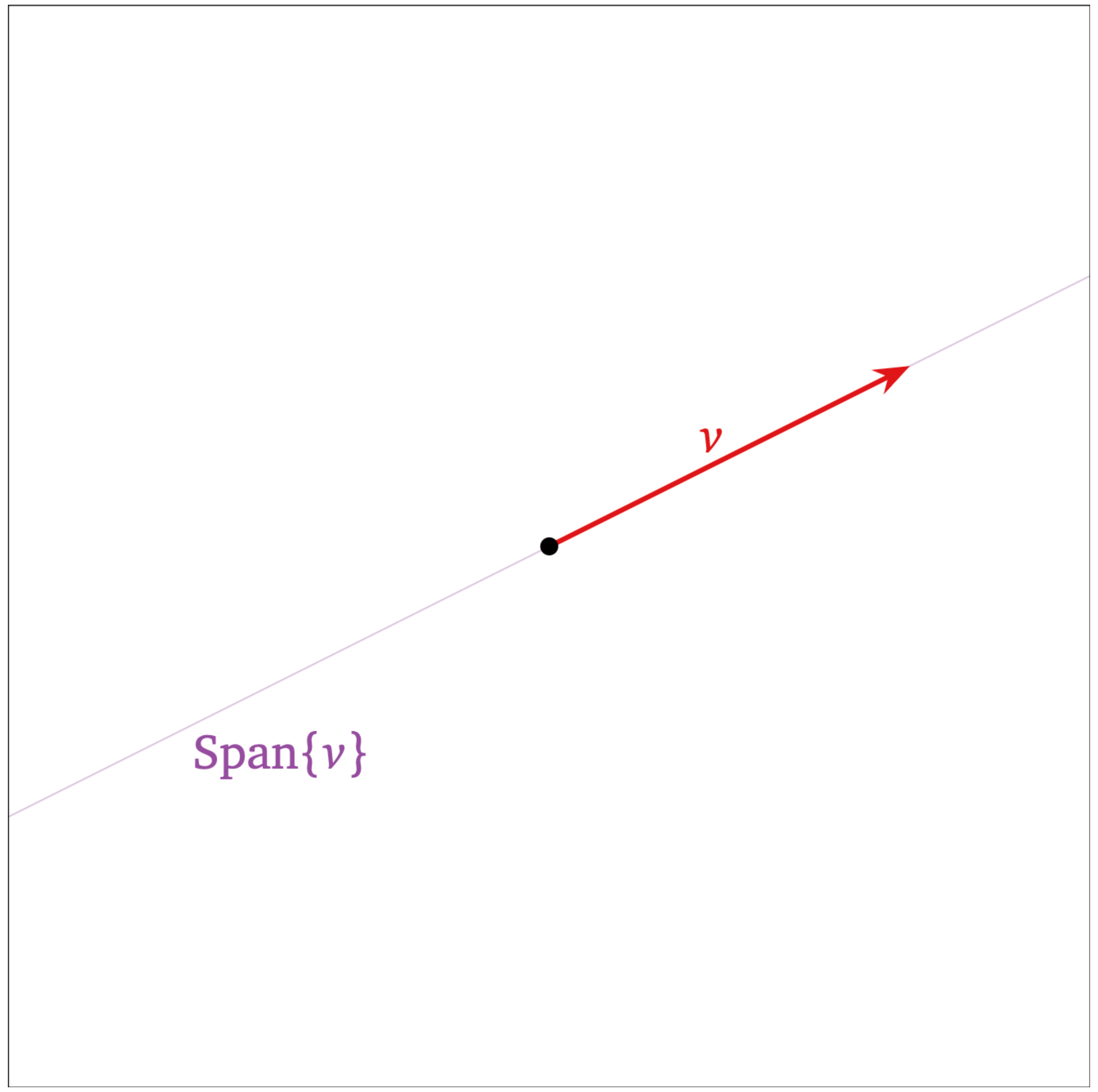
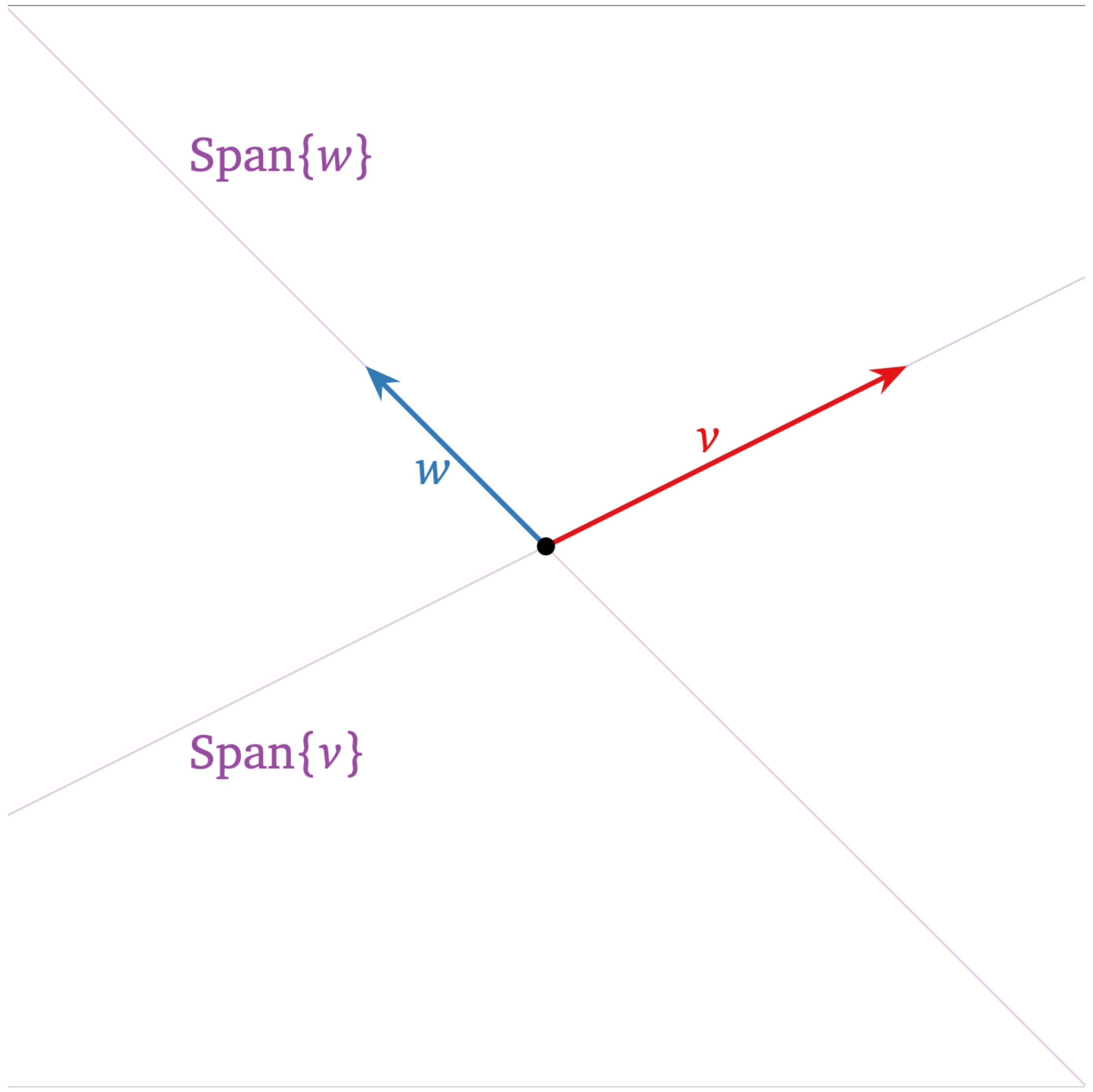
$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is still a plane

...   (this is an example, it may take a lot more vectors before we find one in the span of the preceding vectors)

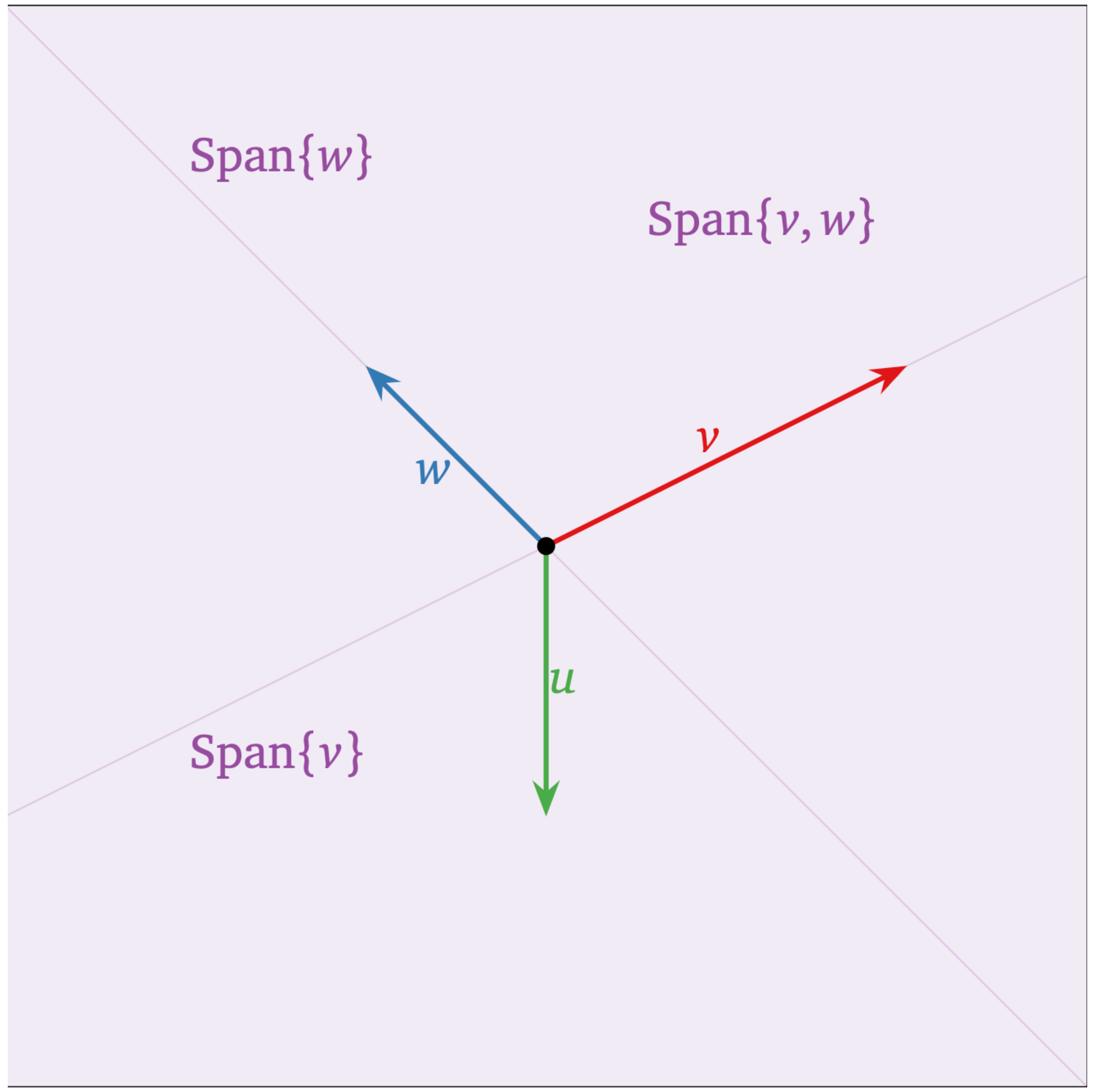
As a Picture



span of 1 vector
a line



span of 2 vector
a plane



span of 3 vector
still a plane

image source

Characterization of Linear Dependence

Corollary. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent, then for any vector $\mathbf{v}_{k+1}$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly dependent

If we add a vector to a linearly dependent set, it remains linearly dependent

Question

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Answer: No

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Any three vectors can at most span a plane

The first two are not colinear, so they span a plane ($\mathbb{R}^2$)

Linear Independence and Free Variables

Linear Dependence Relations (Again)

When finding a linear dependence relation, we came across a system which a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take x_3 to be free

Pivots and Linear Dependence

Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every <u>column</u>

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Remember that we choose our free variables to be the ones whose columns don't have pivots

Pivots and Linear Dependence

$$\begin{bmatrix} 2 & 6 & 5 & 8 \\ 2 & 2 & 2 & 8 \\ 2 & 8 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every <u>column</u>

Remember that we choose our free variables to be the ones whose columns don't have pivots

Free variables allow for infinitely many (nontrivial) solution

Recall: General Form Solutions

$$x_1 = -(0.5)x_3$$
$$x_2 = -x_3$$
$$x_3 \text{ is free}$$

Recall: General Form Solutions

$$x_1 = -0.5$$
$$x_2 = -1$$
$$x_3 = 1$$

Recall: General Form Solutions

$$x_1 = 0.5$$
$$x_2 = 1$$
$$x_3 = -1$$

Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

Recall: General Form Solutions

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = -2$$

The point: the solution is not unique

How To: Linear Independence

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

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Solution. Check if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$ has a unique solution

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{0}$ has a unique solution

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if the general form solution of $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0} \end{bmatrix}$ has any free variables

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_n]$ to echelon form and check if has a pivot position in every column

Example: Recap Problem Again

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column without a pivot

Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every <u>row</u>

The columns of a matrix are linearly independent if there is a pivot in every <u>column</u>

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}$$

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 rows

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} * & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \end{bmatrix}$$

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

This matrix as at most 3 pivots, but 4 columns

A Warning

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every <u>row</u>

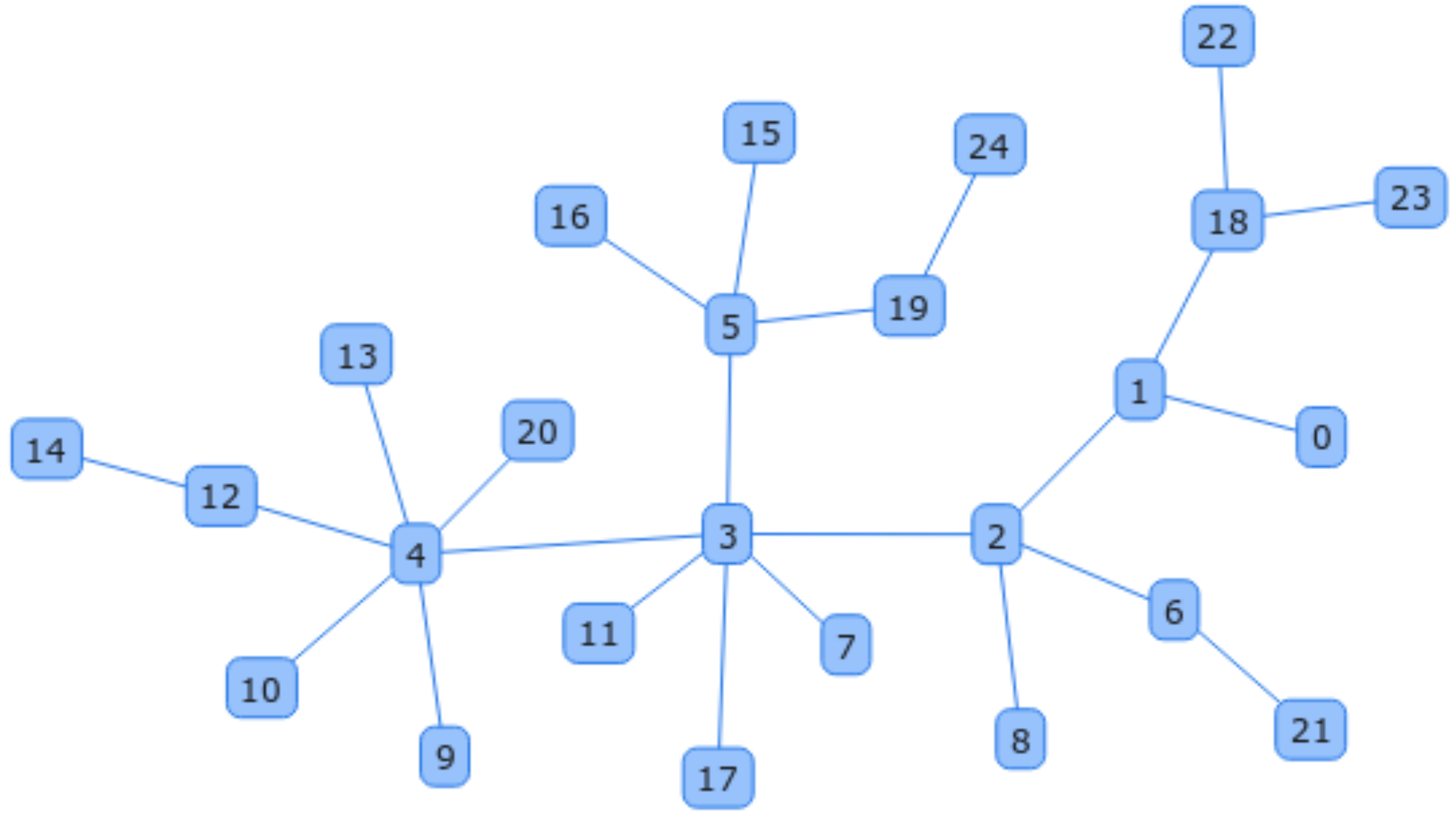
The columns of a matrix are linearly independent if there is a pivot in every <u>column</u>

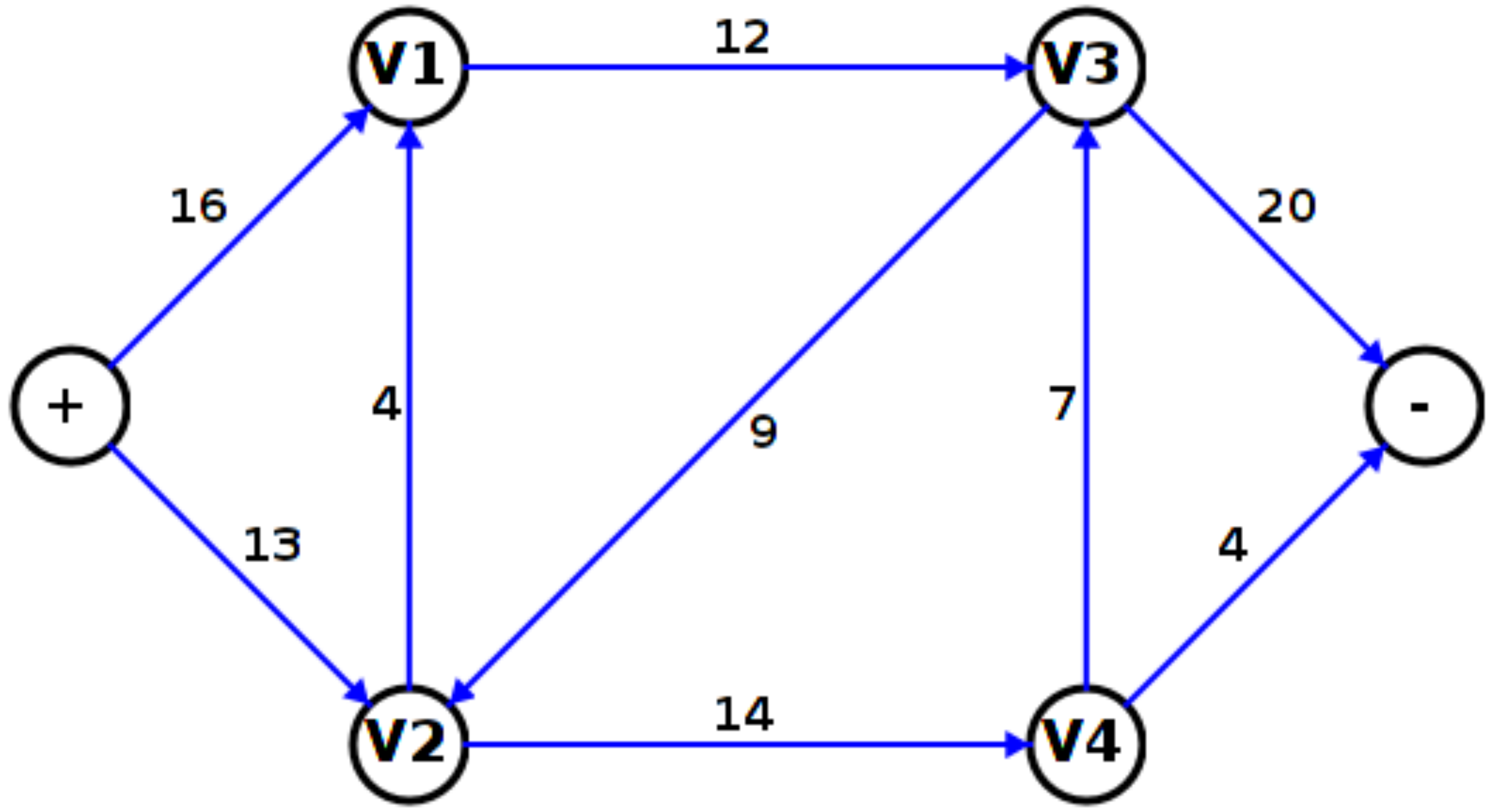
Don't confuse these!

Application: Networks and Flow

Graphs/Networks

A ***graph/network*** is a mathematical object representing collection of *nodes* and *edges* connecting them





Graphs/Networks

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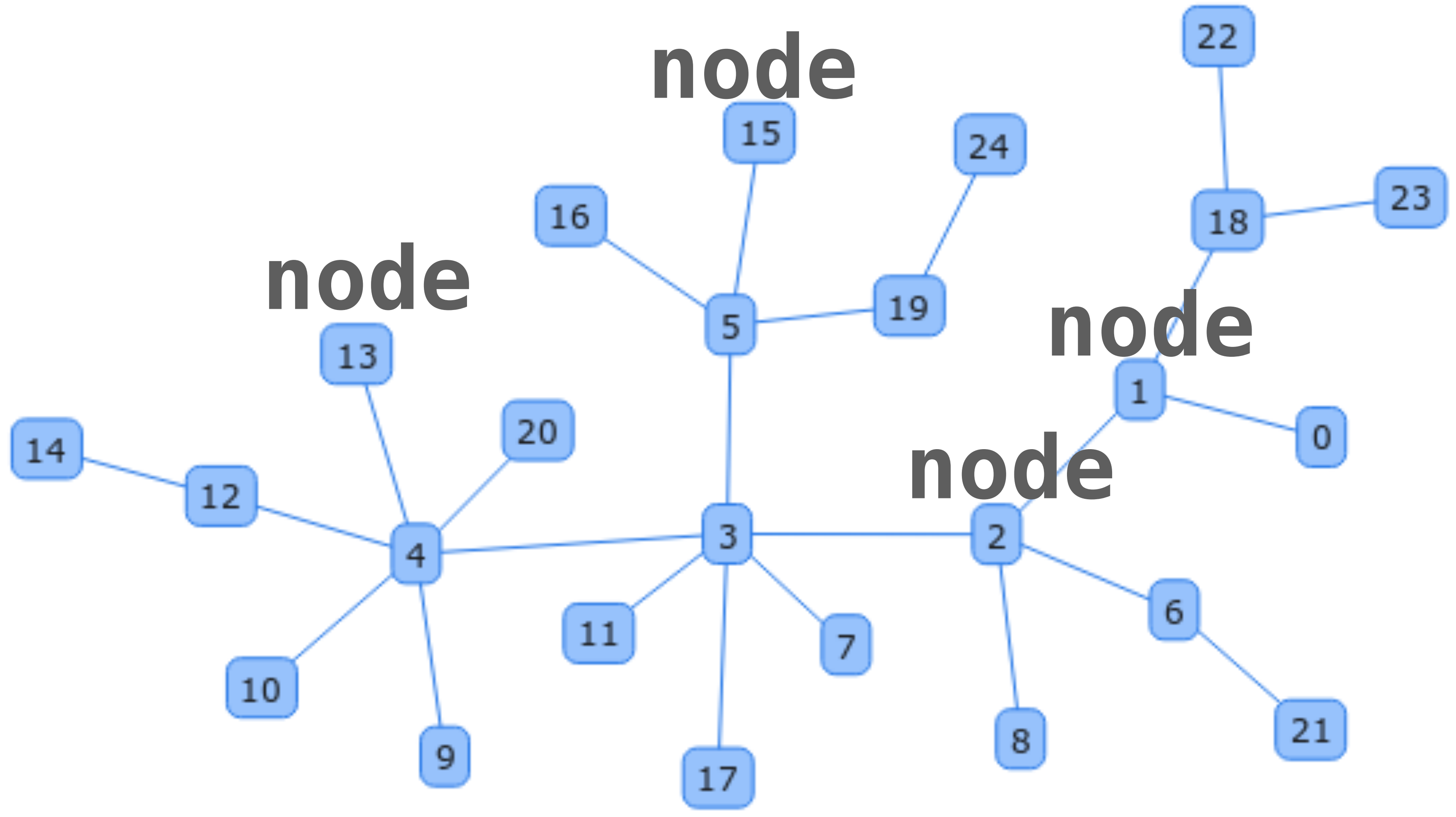
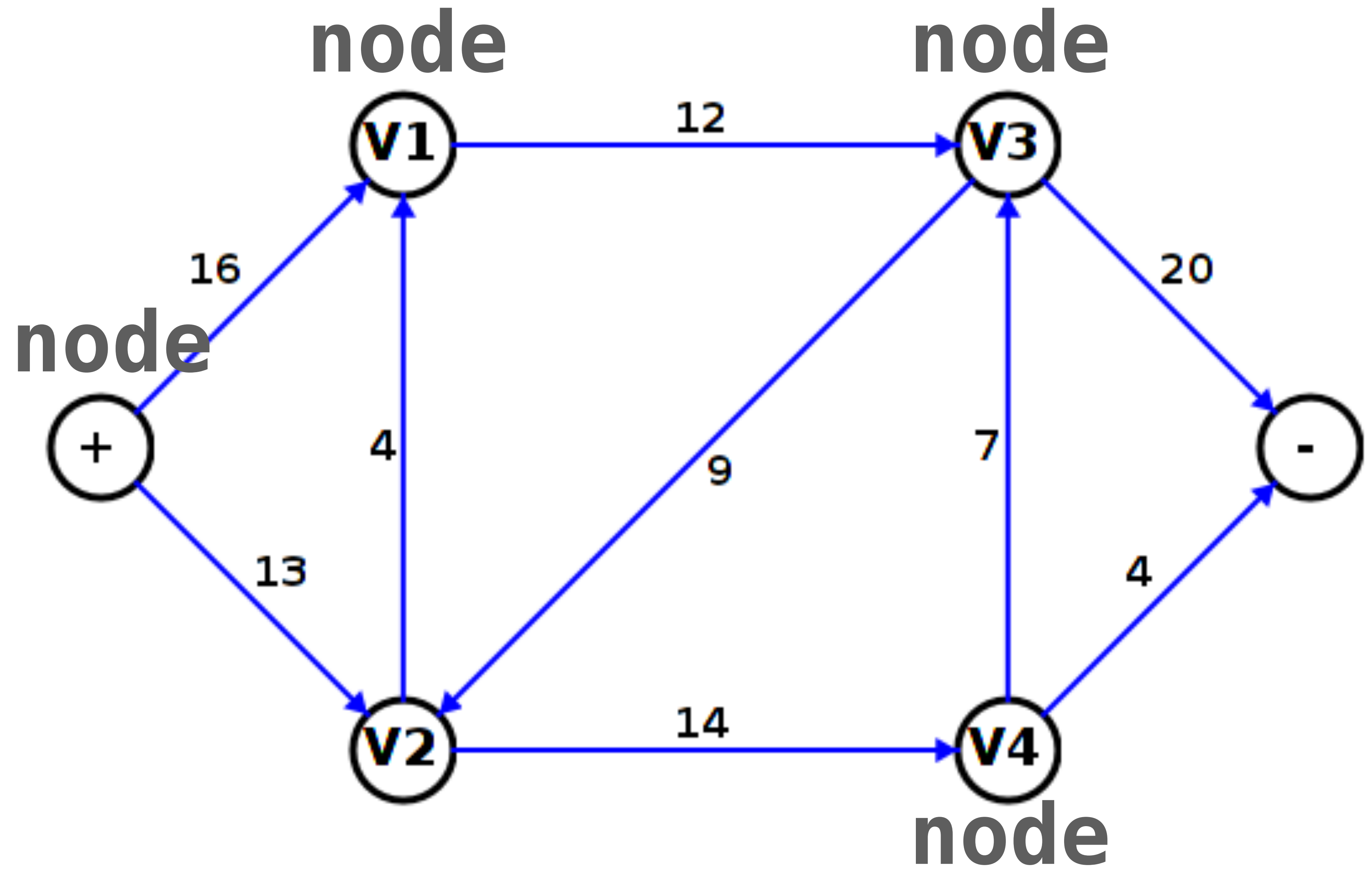
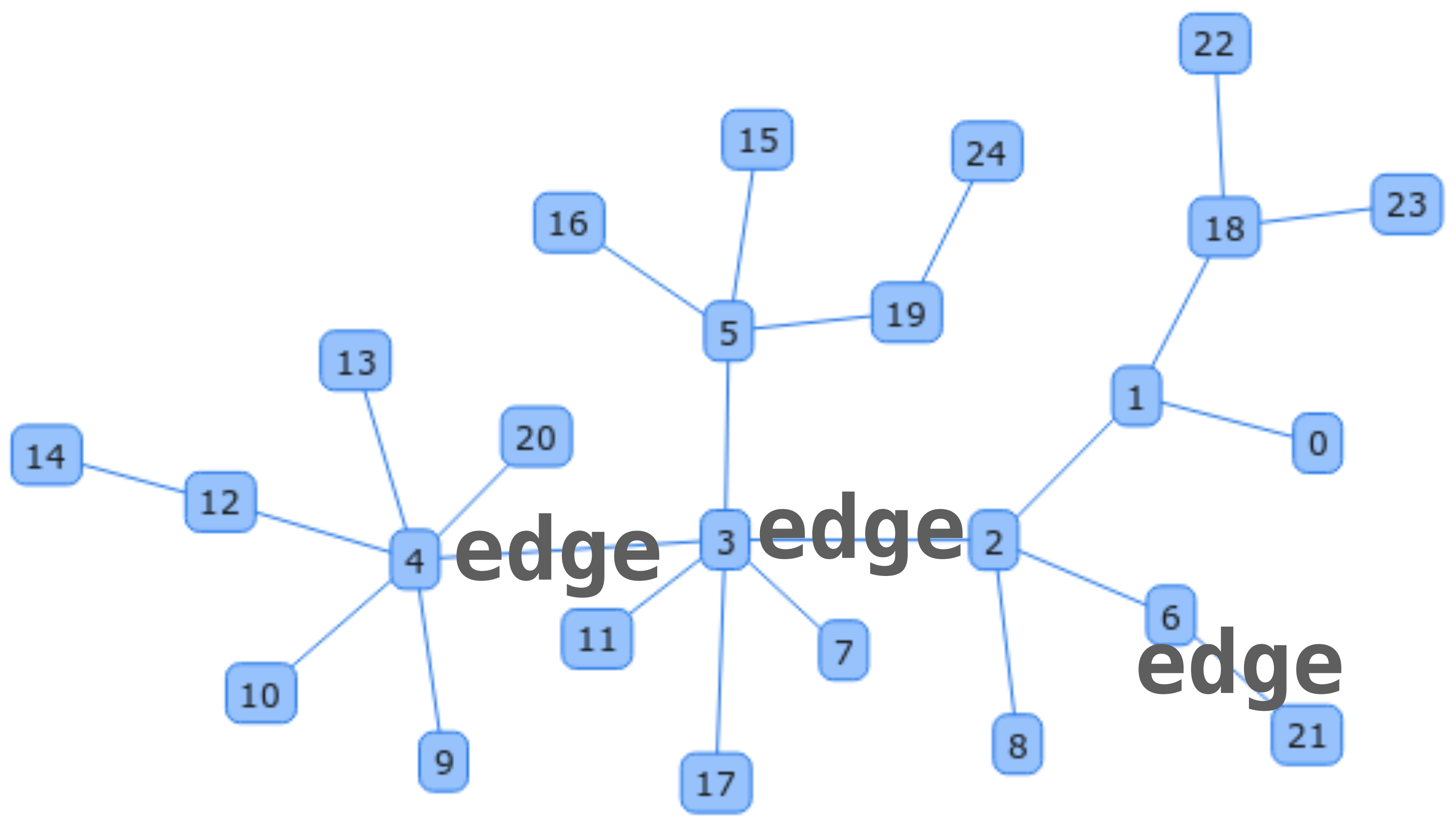
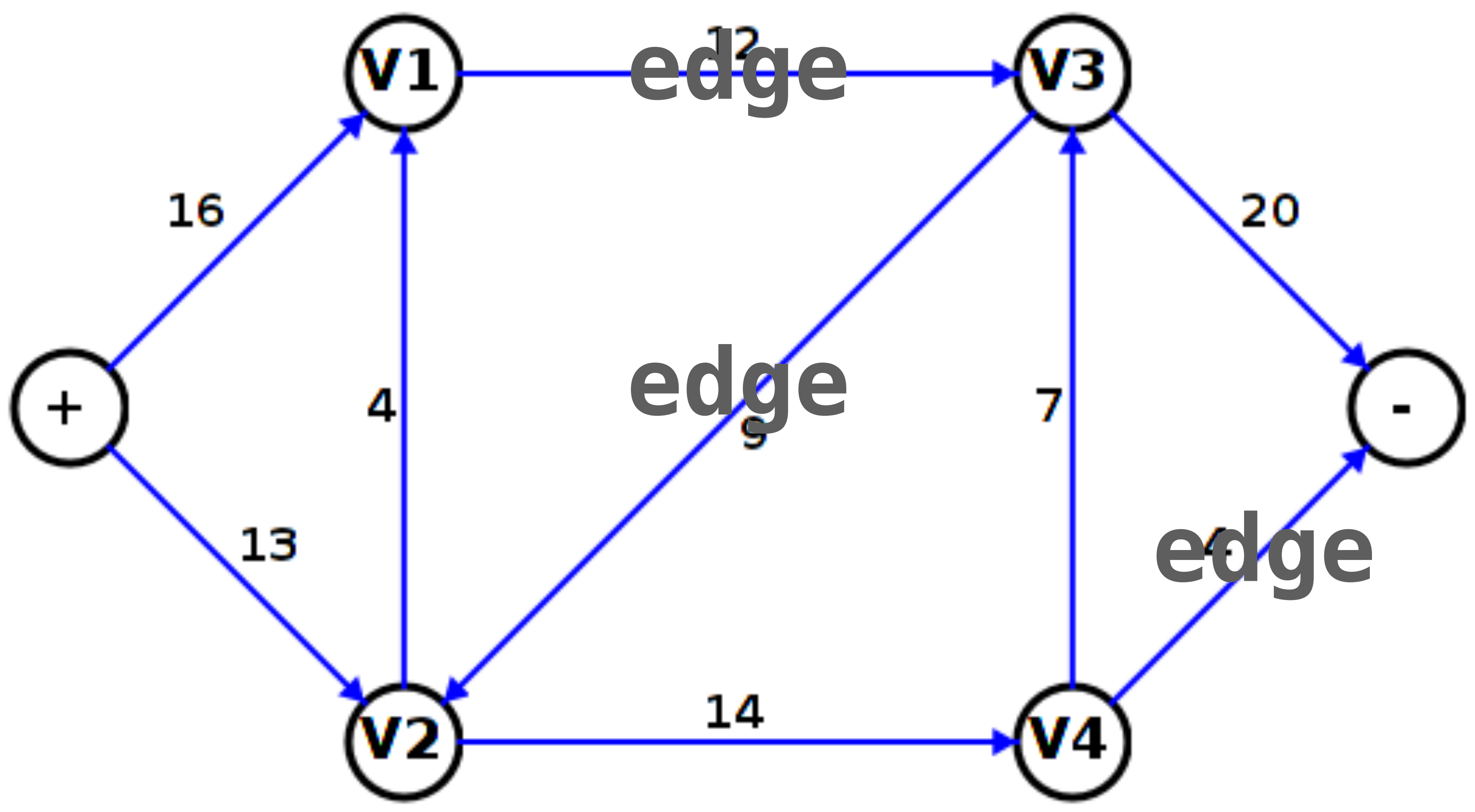




Graphs/Networks

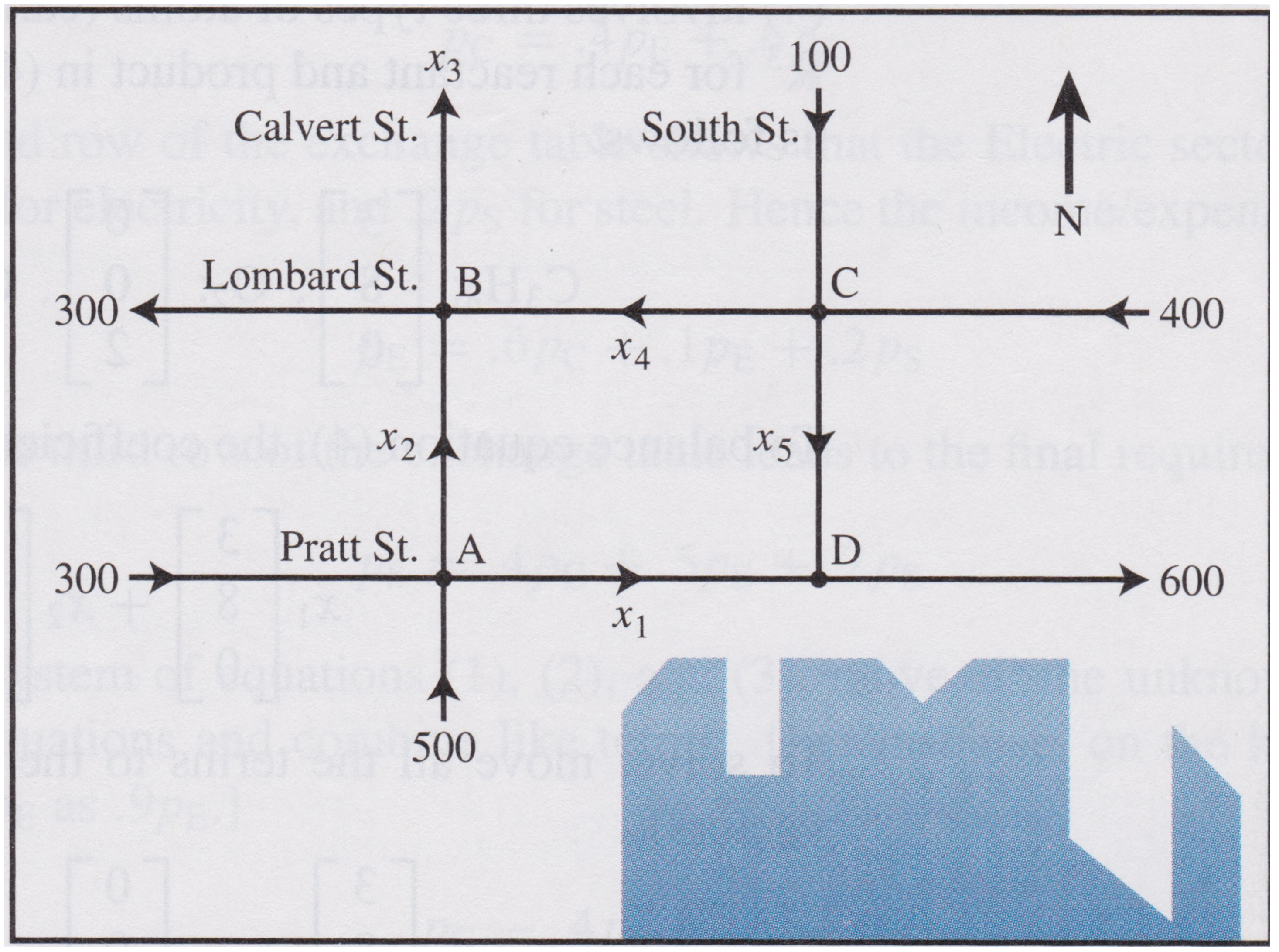
A ***graph/network*** is a mathematical object representing collection of *nodes* and *edges* connecting them





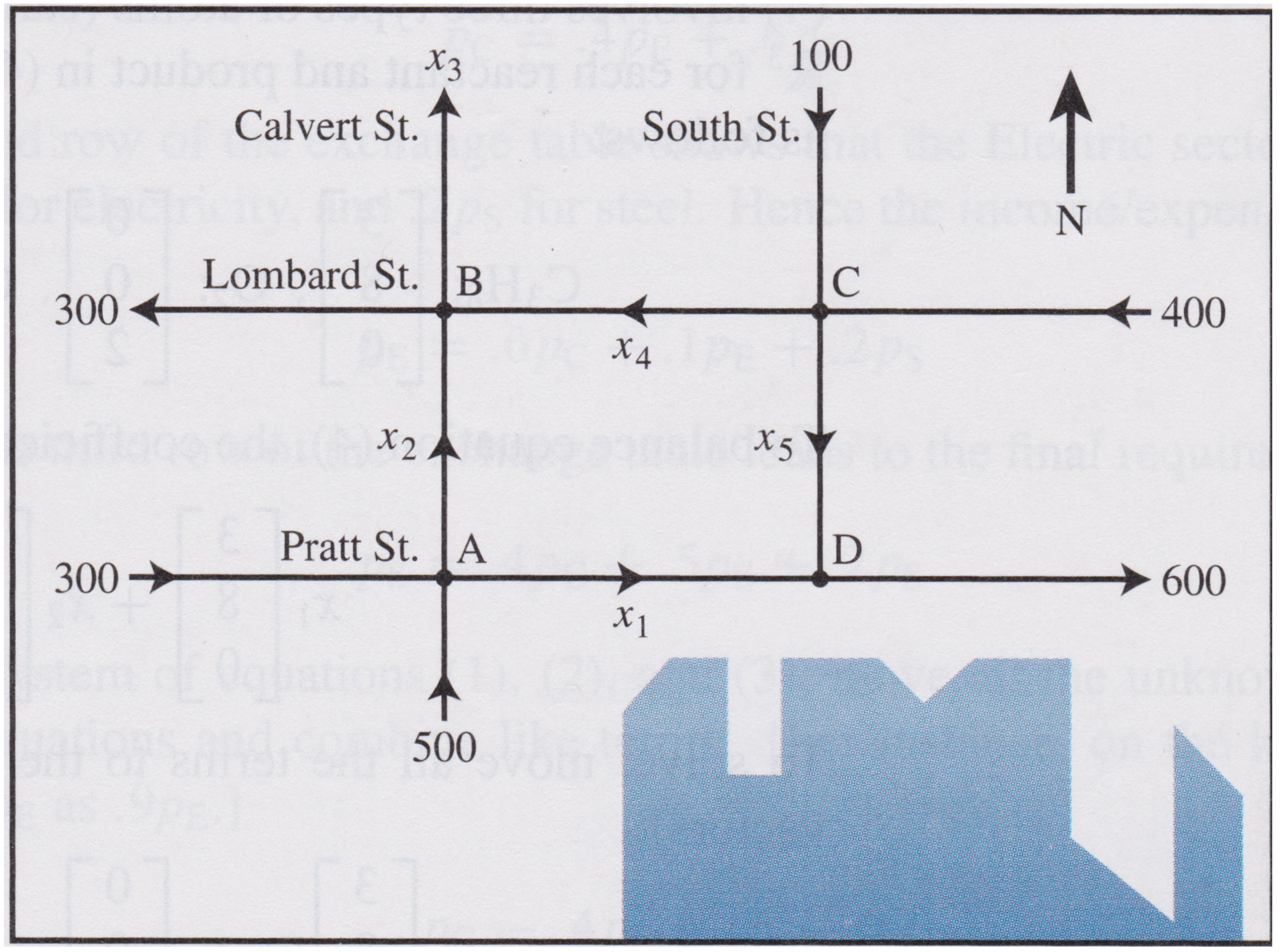
Directed Graphs

Today we focus on ***directed*** graphs, in which edges have a specified direction



Think of these as one-way streets

Flow



We are often interested in how much "stuff" we can push through the edges

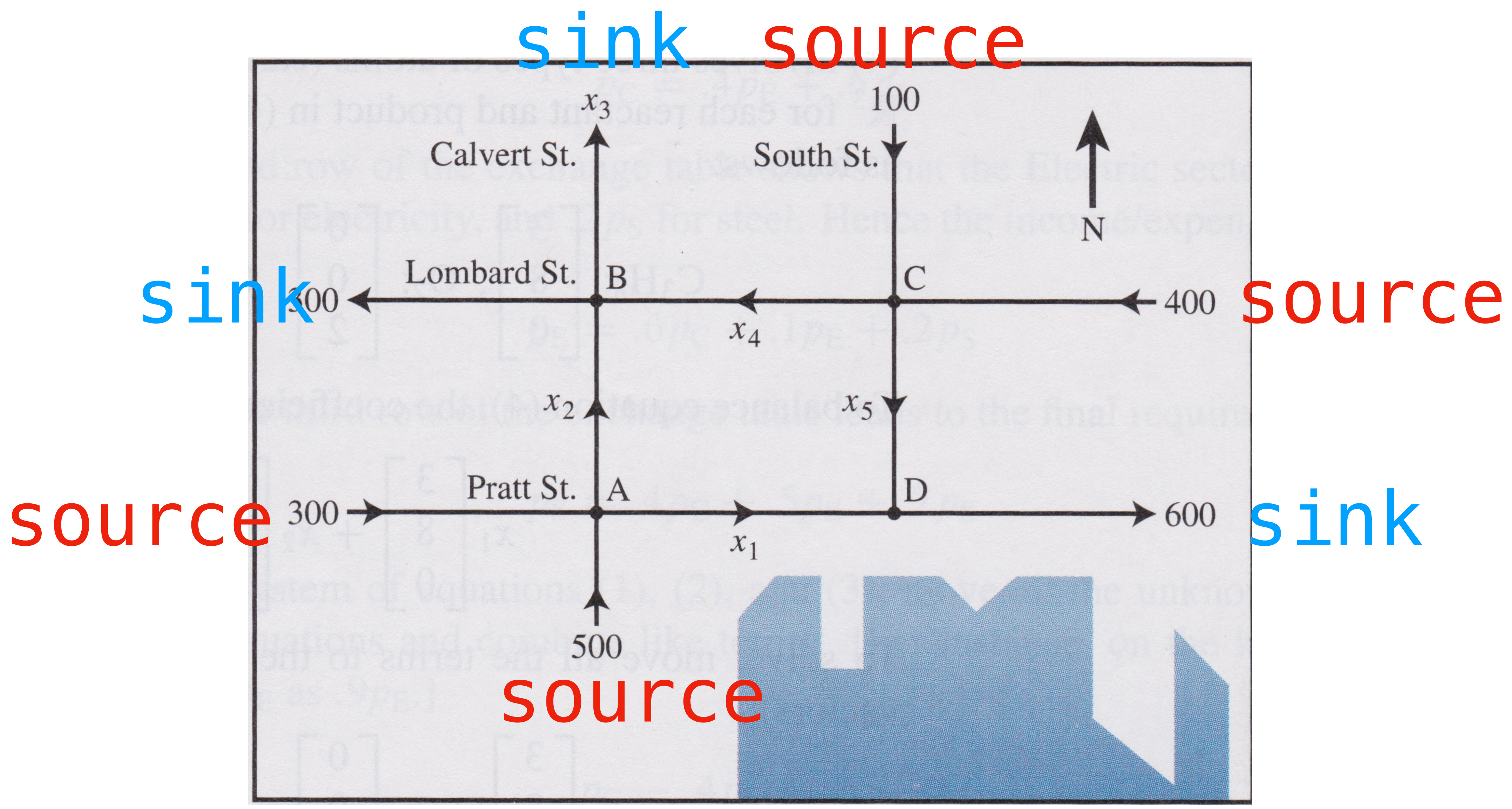
In the above example, the "stuff" is cars/hr

I like to imagine water moving through a pipe, and splitting an joints in the pipe

Flow Network

A ***flow network*** is a directed graph with specified **source and sink nodes**

Flow <u>comes out of</u> and <u>goes into</u> sources and sinks. They are assigned a flow value (or variable)



Flow

Flow

Definition. The ***flow*** of a graph is an assignment of nonnegative values to the edges so that the following holds

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conservation: flow into a node = flow out of a node

Flow

Definition. The ***flow*** of a graph is an assignment of nonnegative values to the edges so that the following holds

conservation: flow into a node = flow out of a node

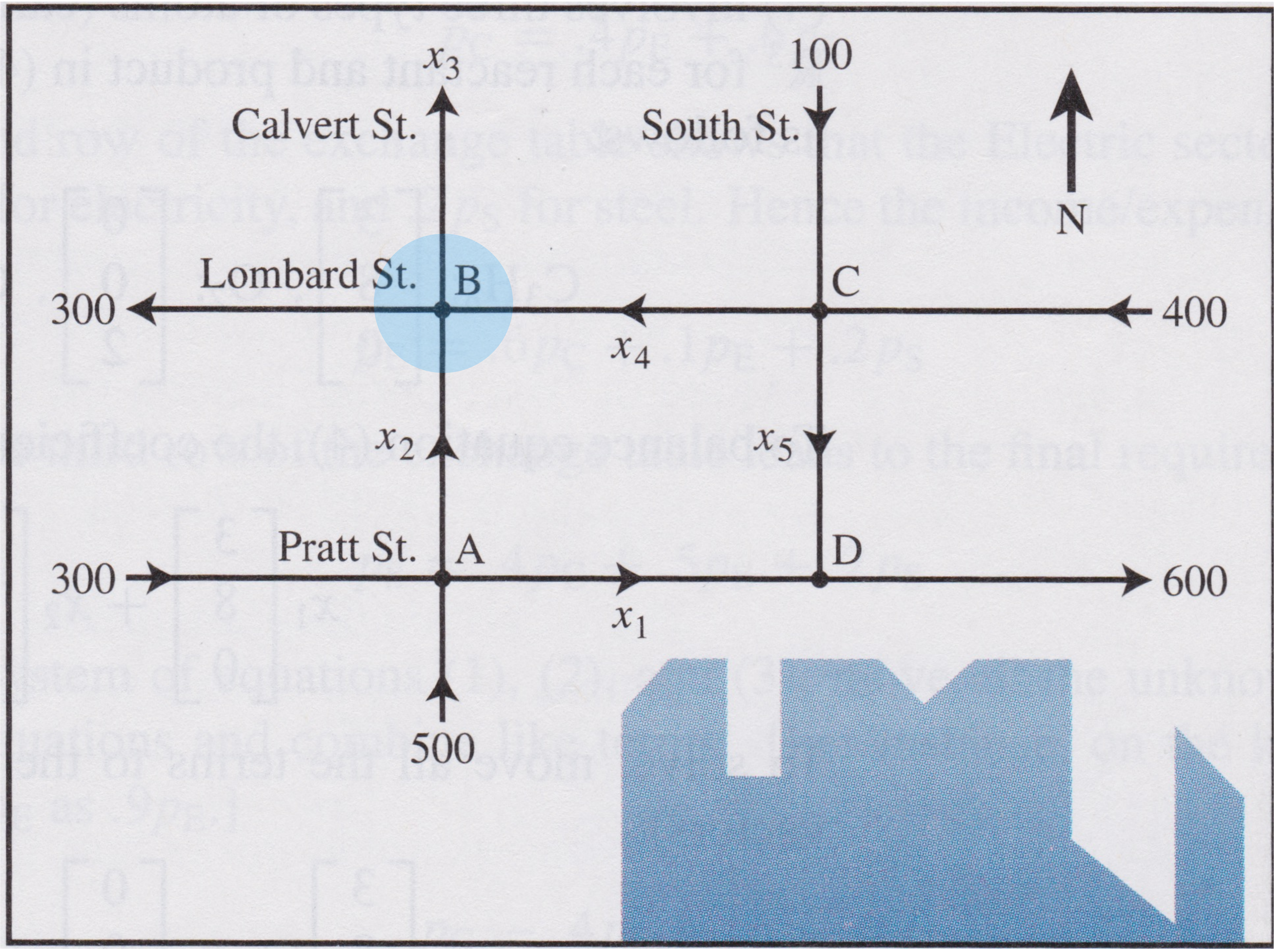
source/sink constraint: flow into a source/out of a sink is nonnegative

Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$



Flow Conservation

Flow in = Flow out

e.g.,

$$x_2 + x_4 = 300 + x_3$$

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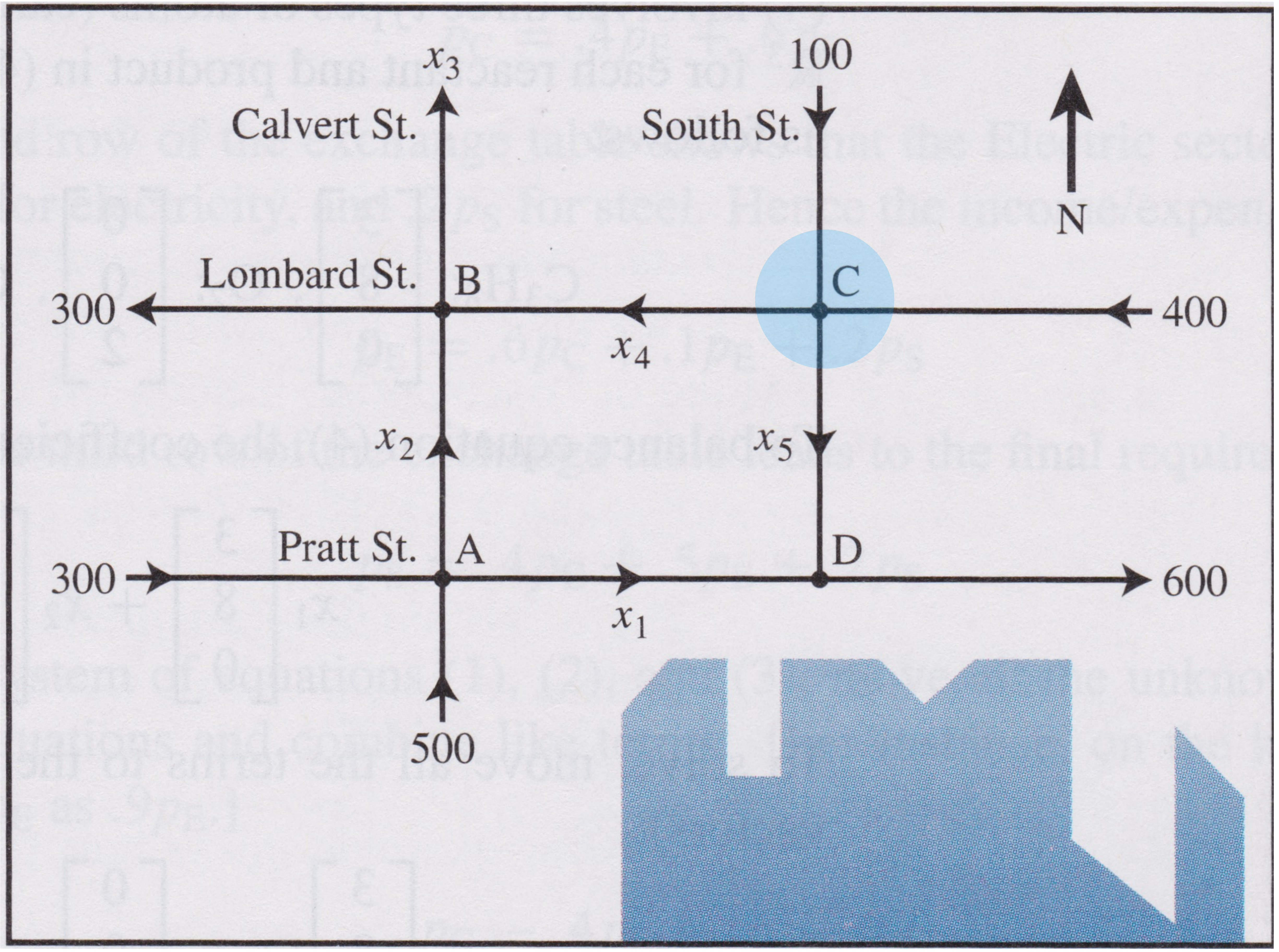

Flow Conservation

Flow in = Flow out

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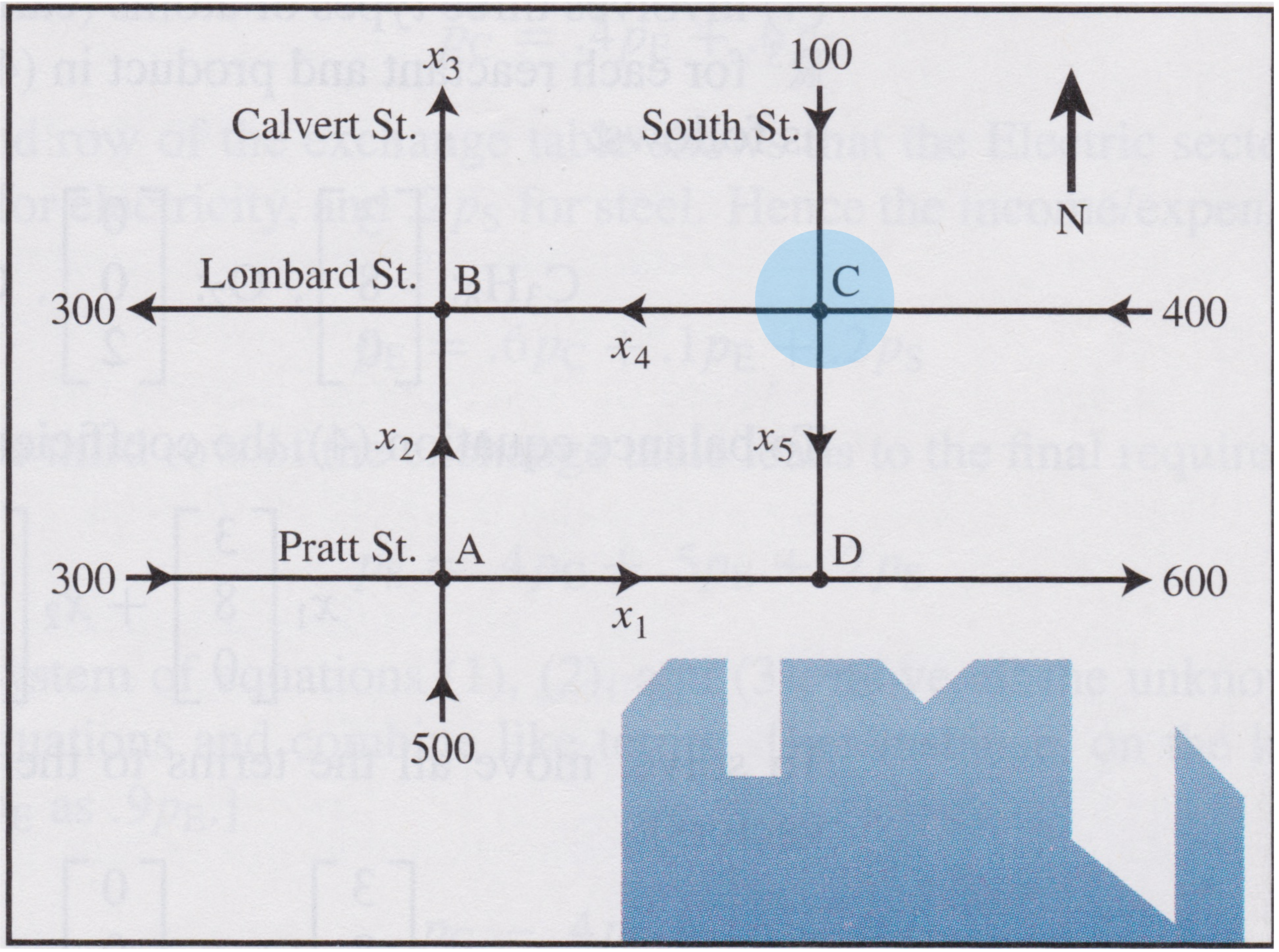


Every node determines a linear equation

How To: Network Flow

How To: Network Flow

Question. Find a general solution for the flow of a given graph

How To: Network Flow

Question. Find a general solution for the flow of a given graph

Solution. Write down the linear equations determined by flow conservation at non-source and non-sink nodes, and then solve

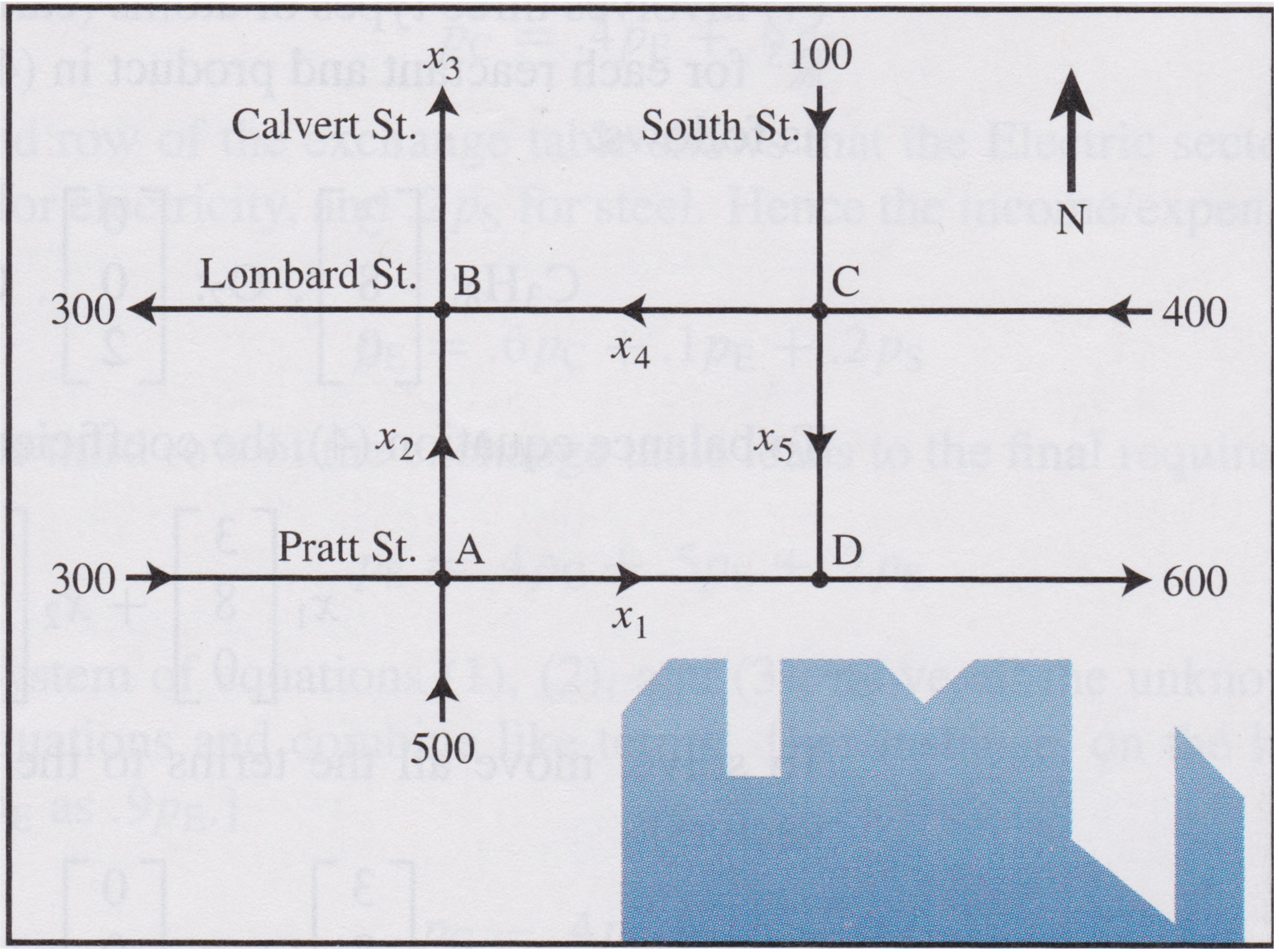
Example

(A) $500 + 300 = x_1 + x_2$

(B) $x_2 + x_4 = 300 + x_3$

(C) $100 + 400 = x_4 + x_5$

(D) $x_1 + x_5 = 600$



Example

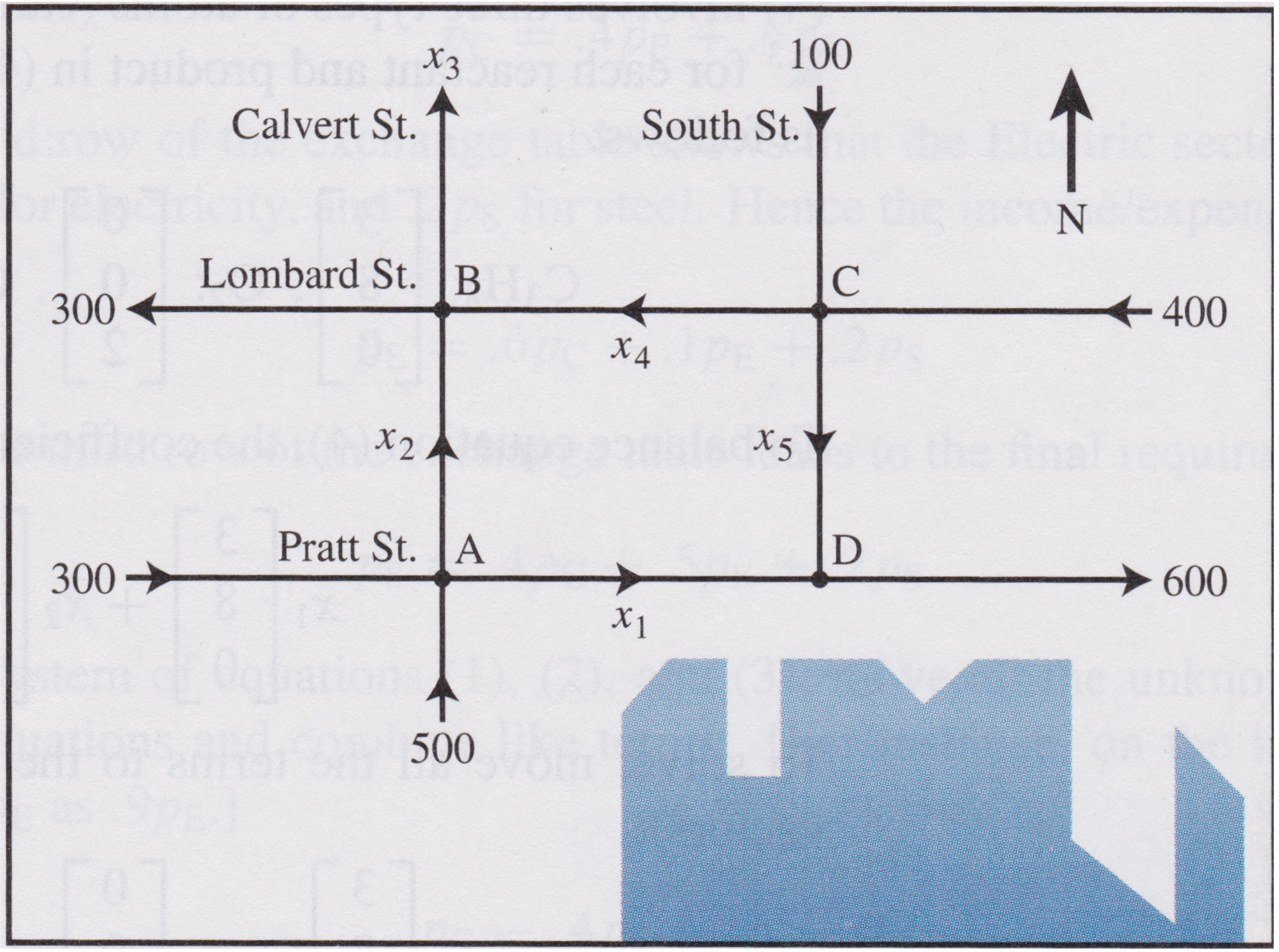
(A) $500 + 300 = x_1 + x_2$

(B) $x_2 + x_4 = 300 + x_3$

(C) $100 + 400 = x_4 + x_5$

(D) $x_1 + x_5 = 600$

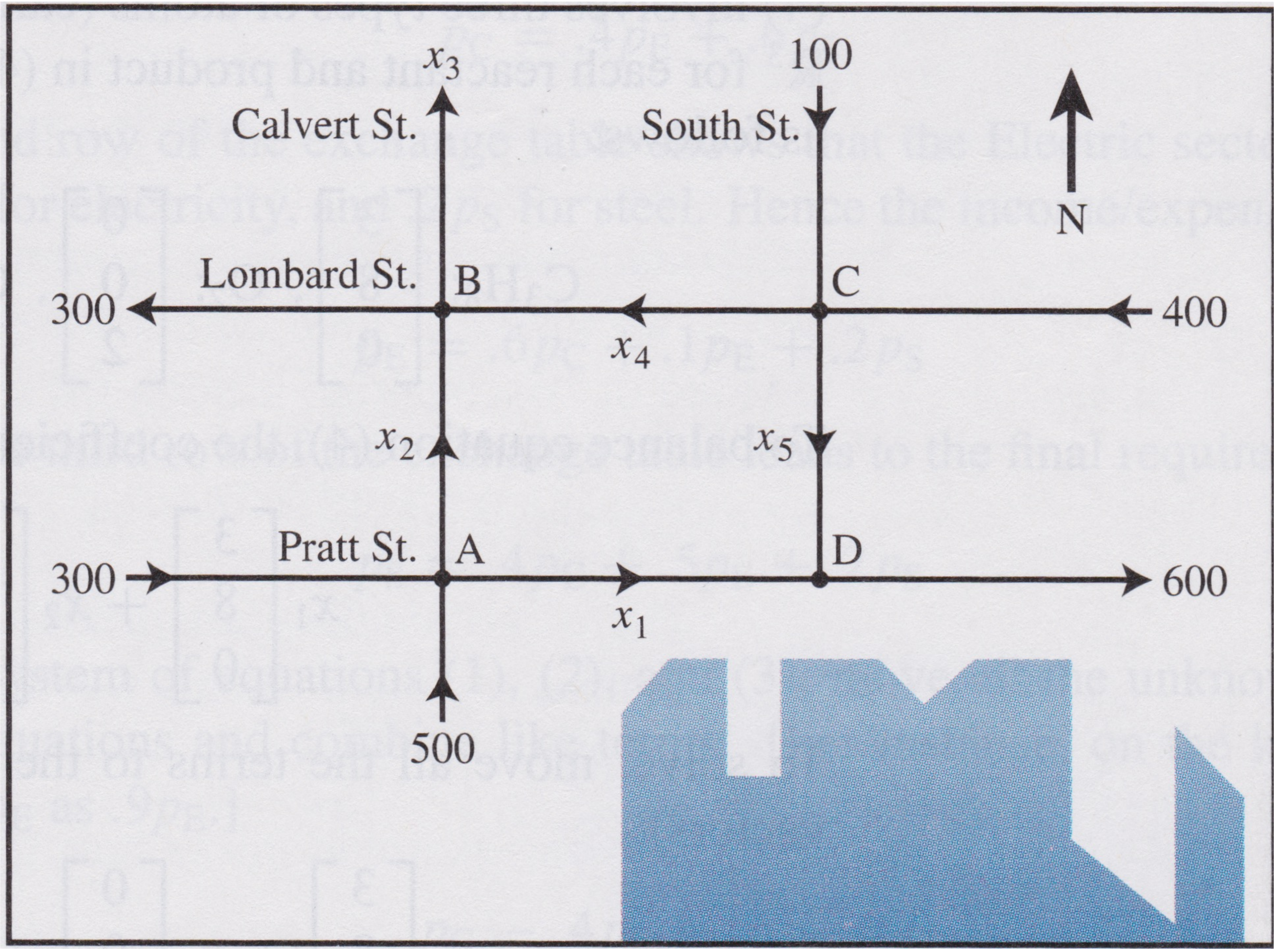
System of Linear Equations



Example

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

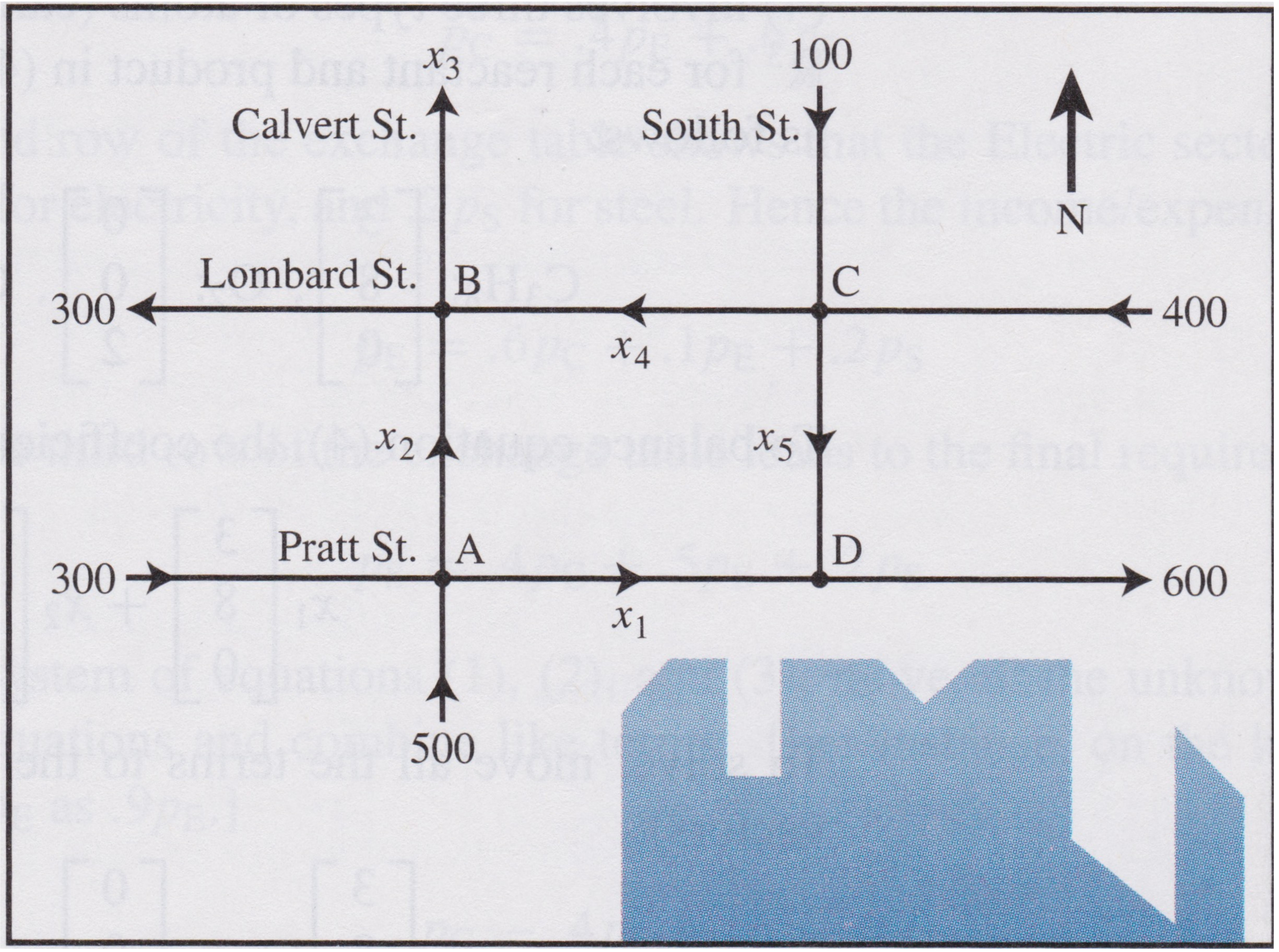
Augmented Matrix



Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{bmatrix}$$

Reduced Echelon Form

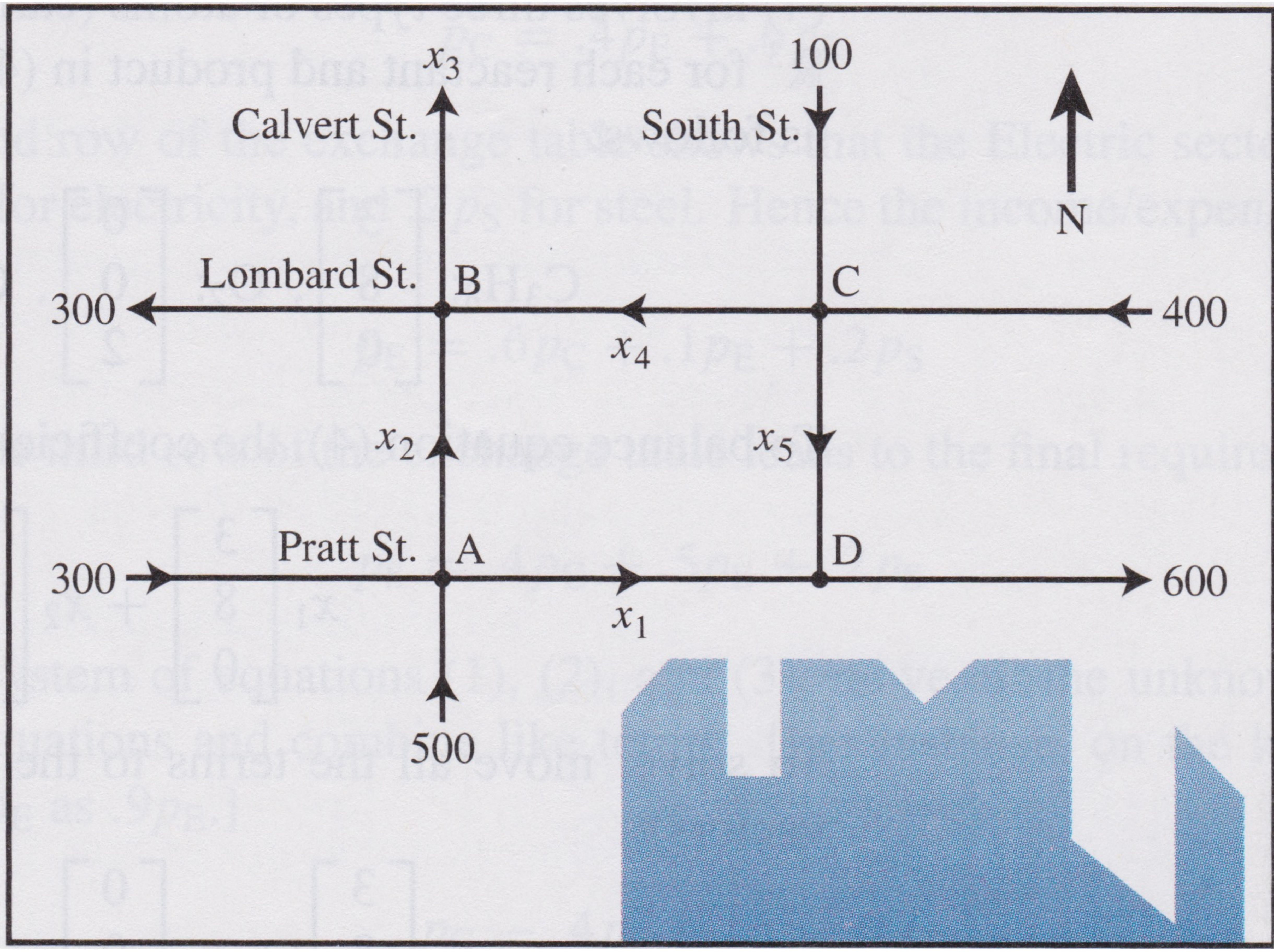


Note that global flow is conserved.

Example

$$x_1 = 600 - x_5$$
$$x_2 = 200 + x_5$$
$$x_3 = 400$$
$$x_4 = 500 - x_5$$
$$x_5 \text{ is free}$$

General Solution



How To: Max Flow Value for an Edge

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Question. Find the maximum value of a flow variable in a given flow network

How To: Max Flow Value for an Edge

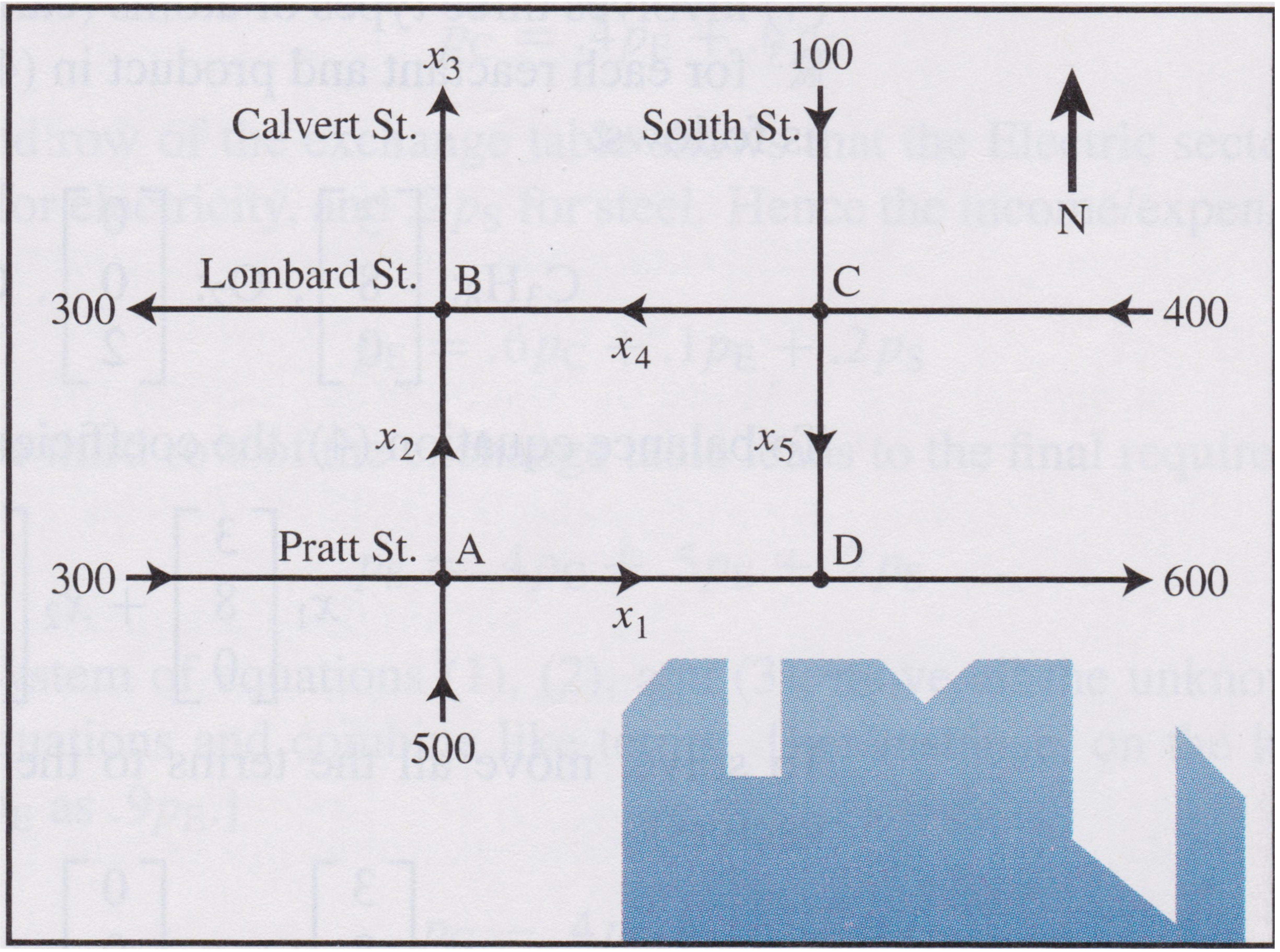
Question. Find the maximum value of a flow variable in a given flow network

Solution. Remember that flow values must be positive. Look at the general form solution and see what makes this hold

Example

$$x_1 = 600 - x_5$$
$$x_2 = 200 + x_5$$
$$x_3 = 400$$
$$x_4 = 500 - x_5$$
x_5 is free

$x_4 \geq 0$ implies $x_5 \leq 500$
$x_1 \geq 0$ implies $x_5 \leq 600$



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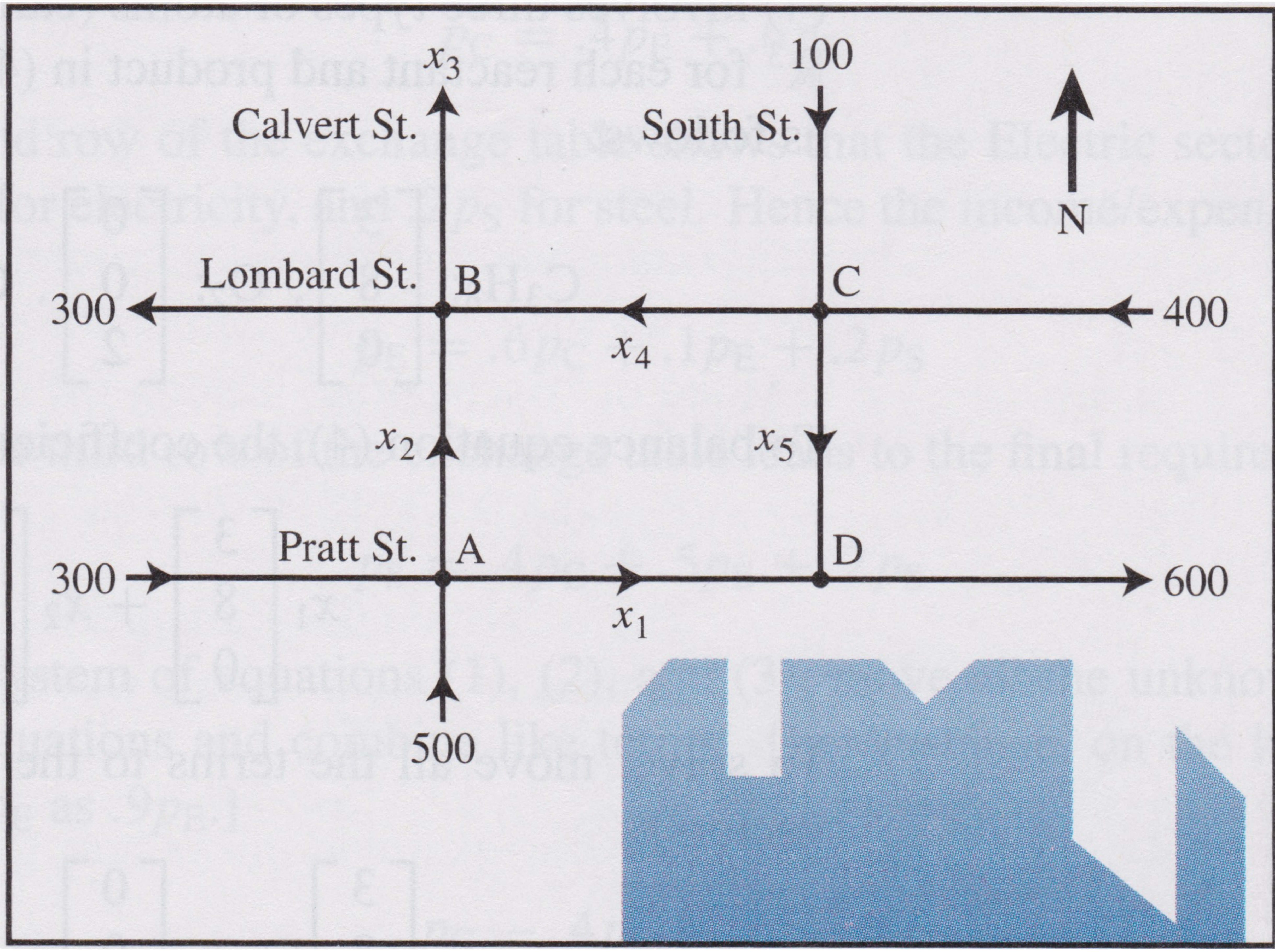

Example

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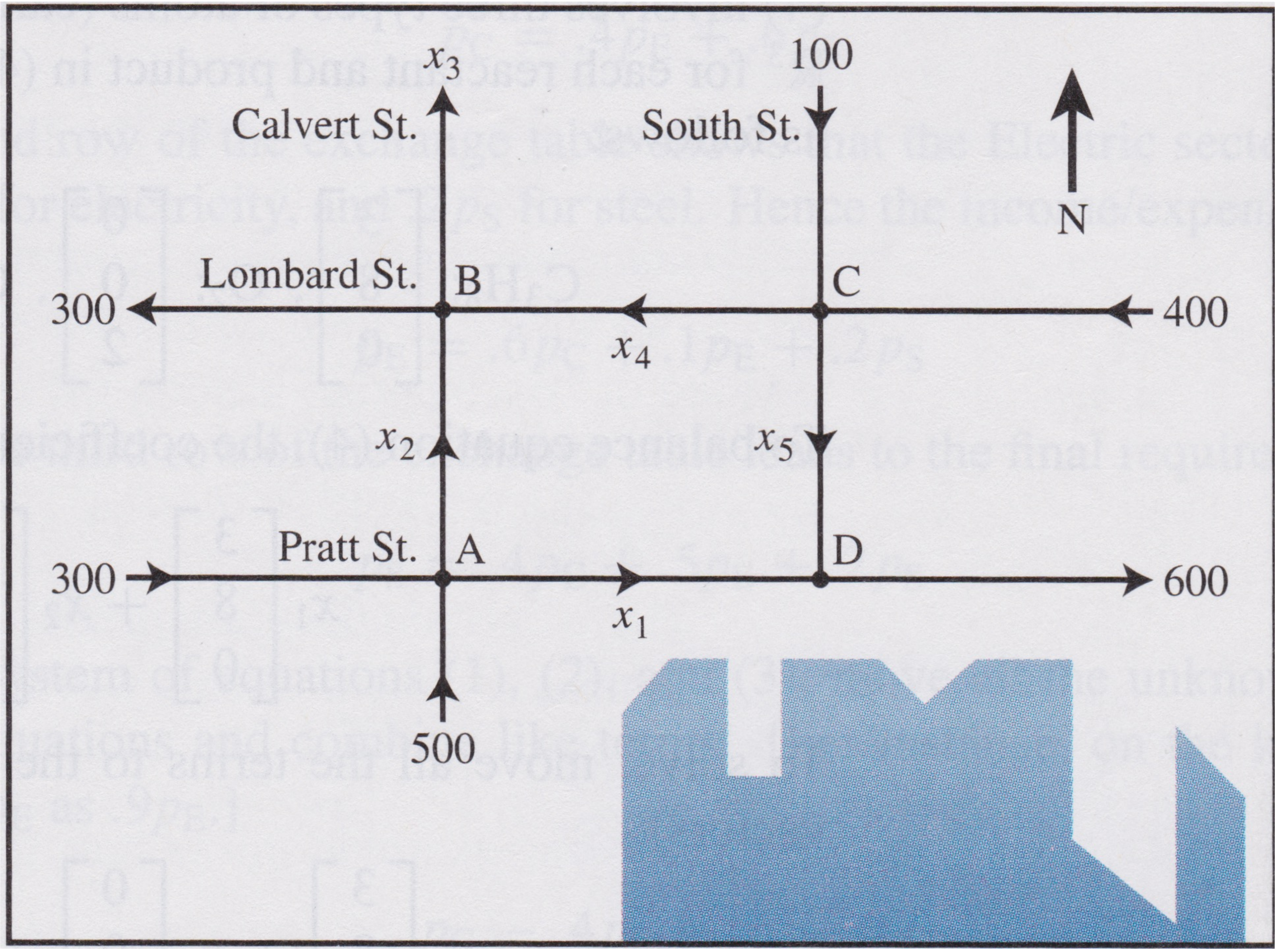
$x_1 \geq 0$ implies $x_5 \leq 600$



The maximum value of x_5 is 500

Summary

Linear independence helps us understand when a span is "as large as it can be"

We can reduce this seeing if a single homogeneous equation has a **unique solution**

Network flows define linear systems we can solve