

The Characteristic Equation

Geometric Algorithms

Lecture 19

Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Determine the dimension of the eigenspace of A for the eigenvalue 4.

(try not to do any row reductions)

Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

See A1 for answer

Objectives

1. Briefly recap eigenvalues and eigenvectors
2. Get a primer on determinants
3. Determine how to find eigenvalues (not just verify them)

Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

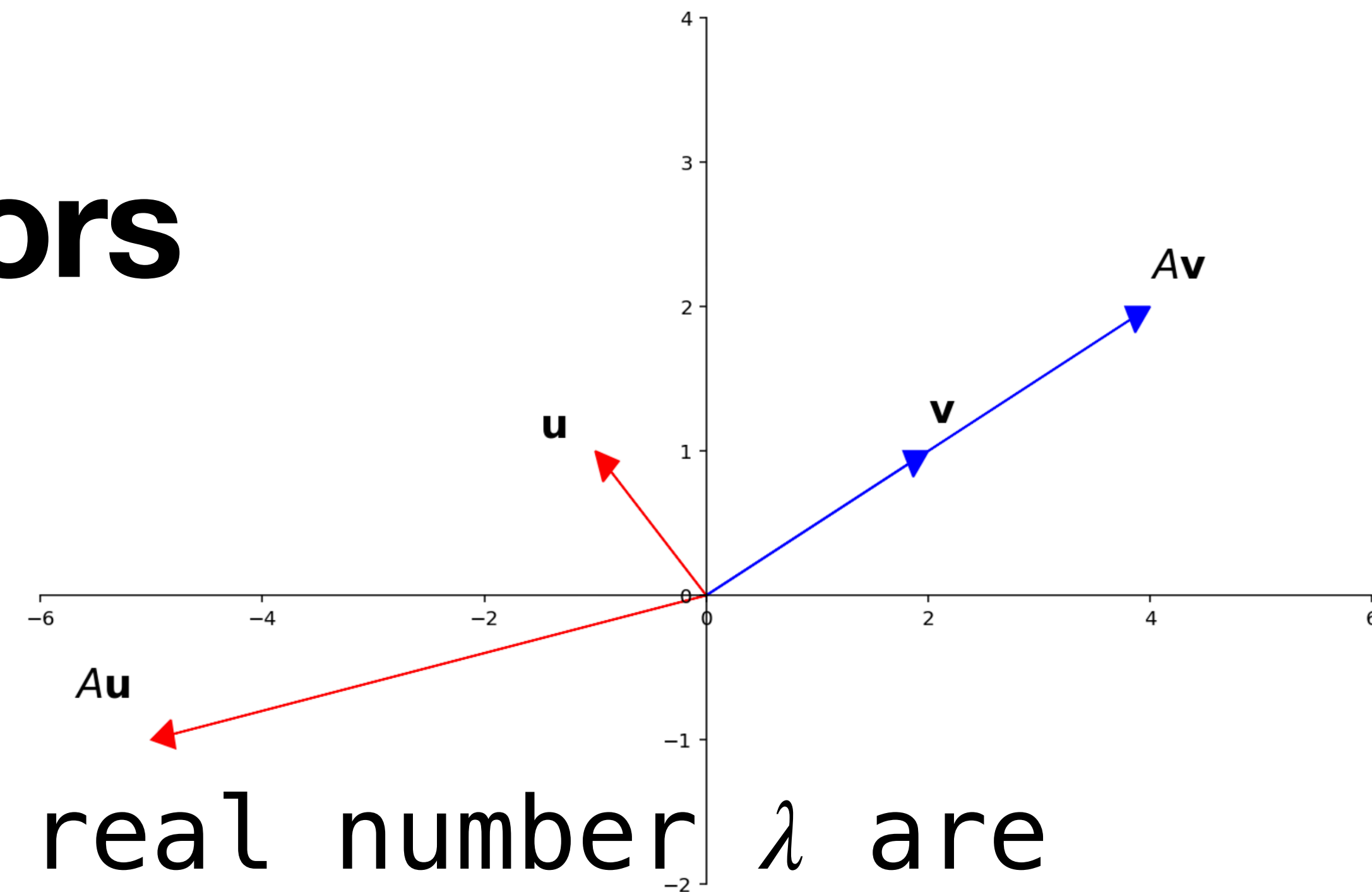
polynomial roots

triangular matrices

multiplicity

Recap

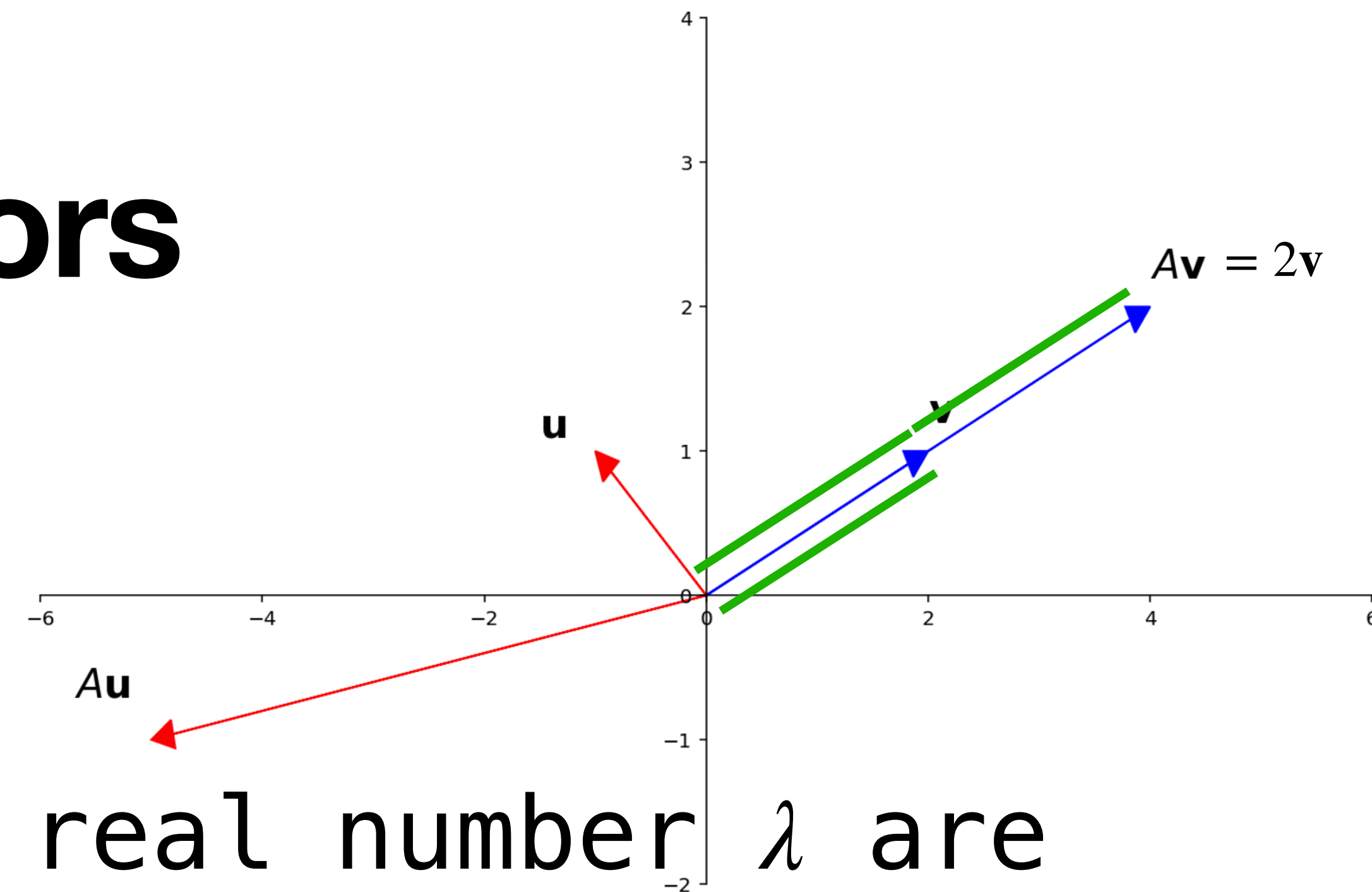
Recall: Eigenvalues/vectors



A *nonzero* vector \mathbf{v} in \mathbb{R}^n and real number λ are an **eigenvector** and **eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

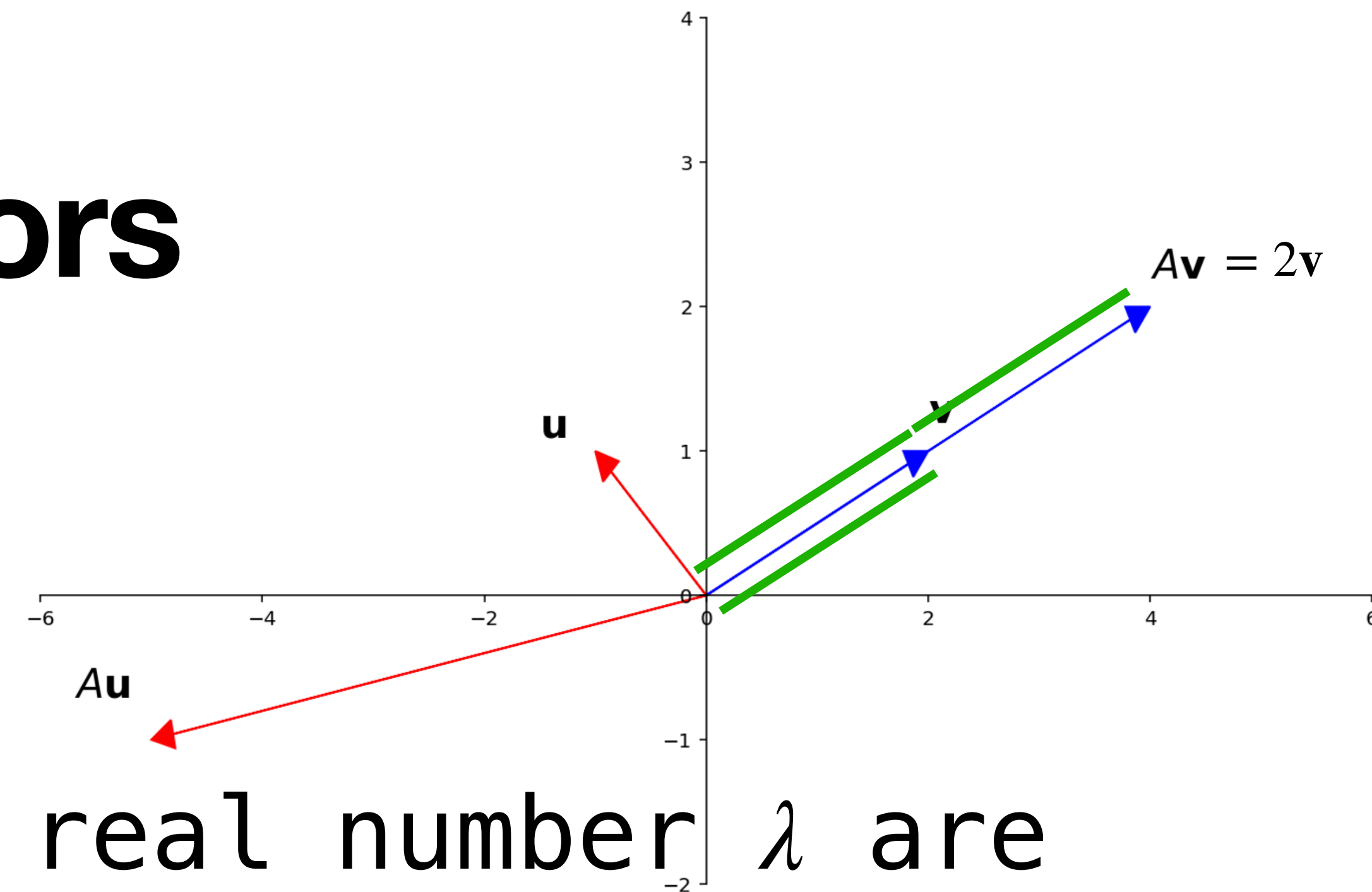
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\mathbf{v} is "just scaled" by A , not rotated

Recall: Verifying Eigenvectors

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Question. Determine if \mathbf{v} is an eigenvector of A and determine the corresponding eigenvalues.

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Solution. Easy. Work out the matrix–vector multiplication.

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix} \quad \times$$

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*If we don't need the vector we can just show that $A - \lambda I$ is **not** invertible (by IMT).*

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Solution. Find a basis for $\text{Nul}(A - \lambda I)$.

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(we did this for our recap problem)

How do eigenvectors relate
to linear dynamical systems?

Recall: (Closed-Form) Solutions

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A (closed-form) solution of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for \mathbf{v}_k which is does **not** contain A^k or previously defined terms

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for \mathbf{v}_k which is does **not** contain A^k or previously defined terms

In other word, it does not depend on A^k and is not **recursive**

Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed form for the above linear dynamical system.

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

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The Key Point. This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k \mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of \mathbb{R}^2 :

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state \mathbf{v}_0 , if \mathbf{v}_0 can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \dots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows exponentially in λ_1).

Verify:

Eigenbases

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*We can represent vectors as **unique** linear combinations of eigenvectors.*

Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state \mathbf{v}_0 , if A has an eigenbasis $\mathbf{b}_1, \dots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant c_1 , where λ_1 is the **largest eigenvalue** of A and \mathbf{b}_1 is its **eigenvector**.

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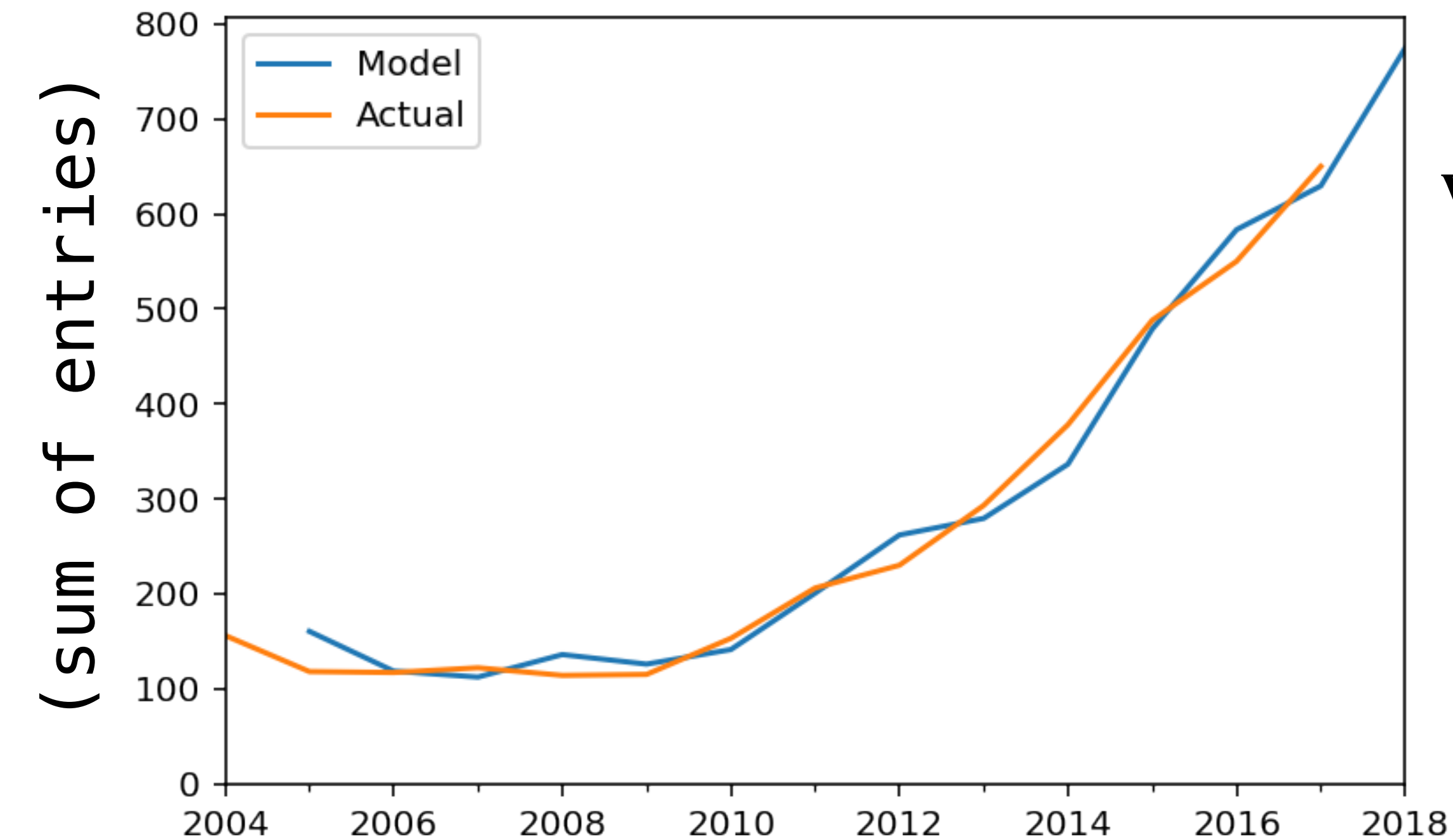
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for some constant c_1 , where λ_1 is the **largest eigenvalue of A** and \mathbf{b}_1 **is its eigenvector**.

The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \# \text{ of year 1 students enrolled in 2024} \\ \# \text{ of year 2 students enrolled in 2024} \\ \# \text{ of year 3 students enrolled in 2024} \\ \# \text{ of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the largest eigenvalue.

moving on...

Finding Eigenvalues

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Question. Determine the eigenvalues of A , along with their associated eigenspaces.

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Solution (Idea). Can we somehow "solve for λ " in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

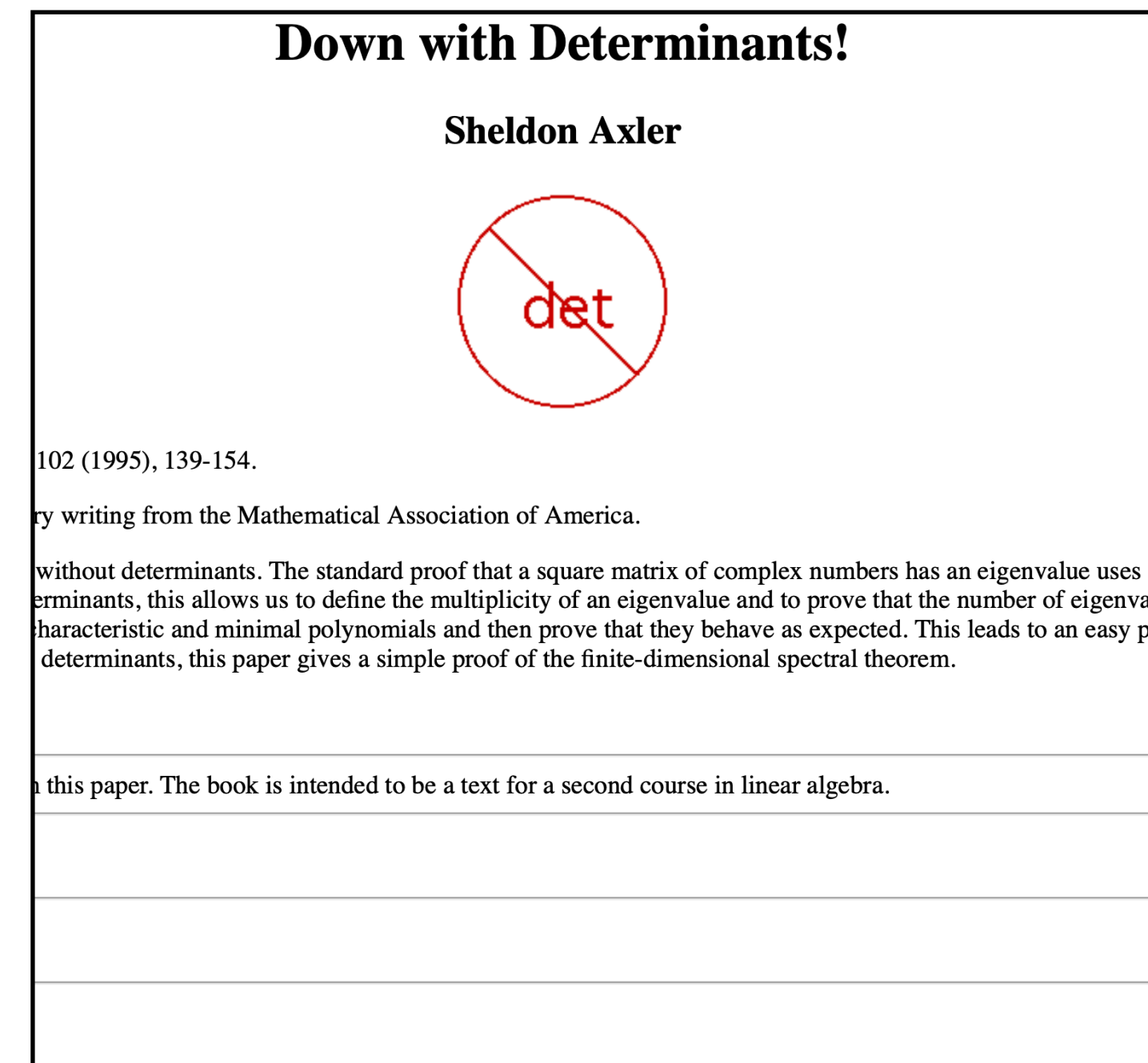
Determinants

An Aside: Determinants are Mysterious

Determinants are
strangely polarizing

Some people love them,
some people hate them

We'll only scratch the
surface...



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In broad strokes, it's a big sum of products of entries of A .

A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
1 FUNCTION det(A):  
2   total = 0  
3   FOR all matrix B we can get by swapping a bunch of rows of A:  
4     s = 1 IF (# of swaps necessary) is even ELSE -1  
5     total += s * (product of the diagonal entries of B)  
6   RETURN total
```

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$(-1)^0 ad$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow^1 \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$(-1)^1 cb$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow^2 \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

$$(-1)^2 gbf$$

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$$(-1)^1 ahf$$

Another Perspective

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's row reduce an arbitrary 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b & | & 0 \\ 0 & ad-bc & | & 0 \end{bmatrix}$$

$$A\vec{x} = 0$$

sol'n set

\uparrow if $ad-bc = 0 \Rightarrow$ free variable

$\Rightarrow A\vec{x} = 0$ has nontrivial sol'n's

$$\det(A) = ad-bc = 0 \Leftrightarrow A \text{ not invertible}$$

$\Rightarrow A$ not invertible

Another Perspective

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Let's row reduce an arbitrary 3×3 matrix:

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 0 & \cancel{1} & \cancel{*} \\ 0 & 0 & a(\det A) \end{bmatrix}$$

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So we can yet again extend the IMT:

- » A is invertible
- » $\det(A) \neq 0$
- » 0 is not an eigenvalue

$A\vec{v} = \vec{0}$ has some nontrivial sol'n

so $\text{Nul } A \neq \{\vec{0}\}$

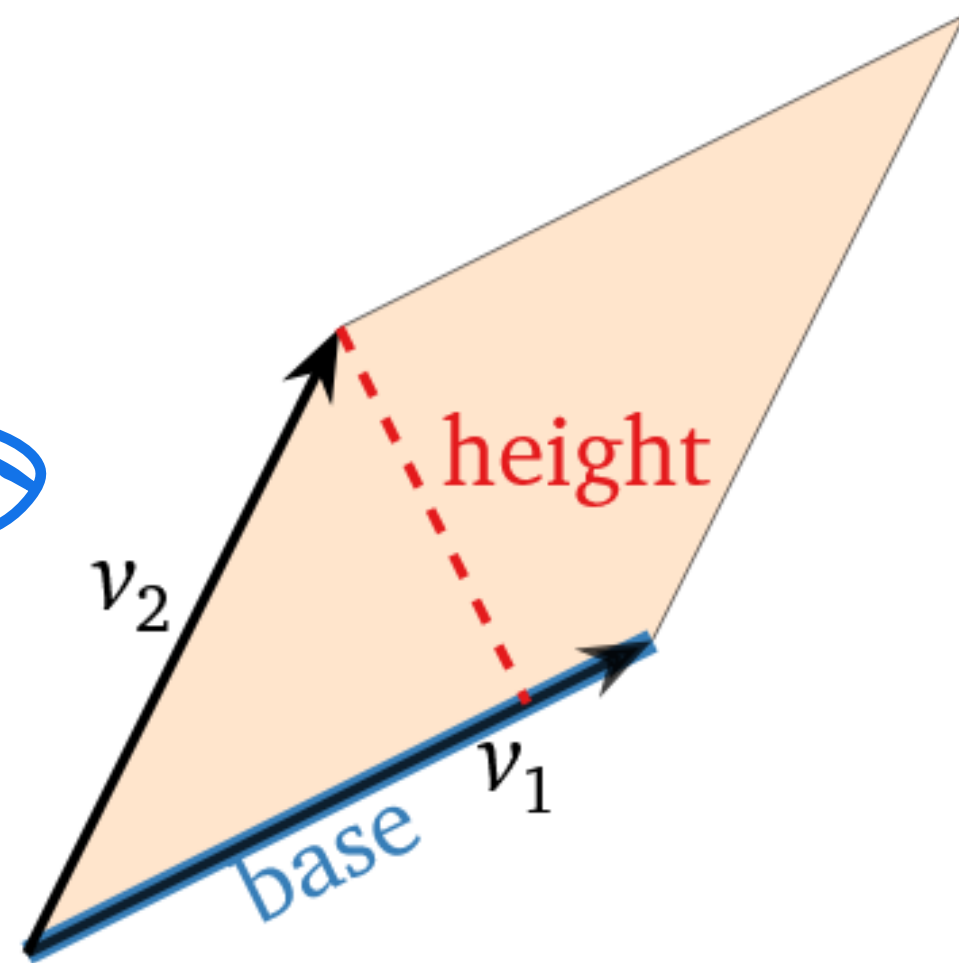
These must be all true or all false.

so A not invertible

A Geometric Interpretation: Volume

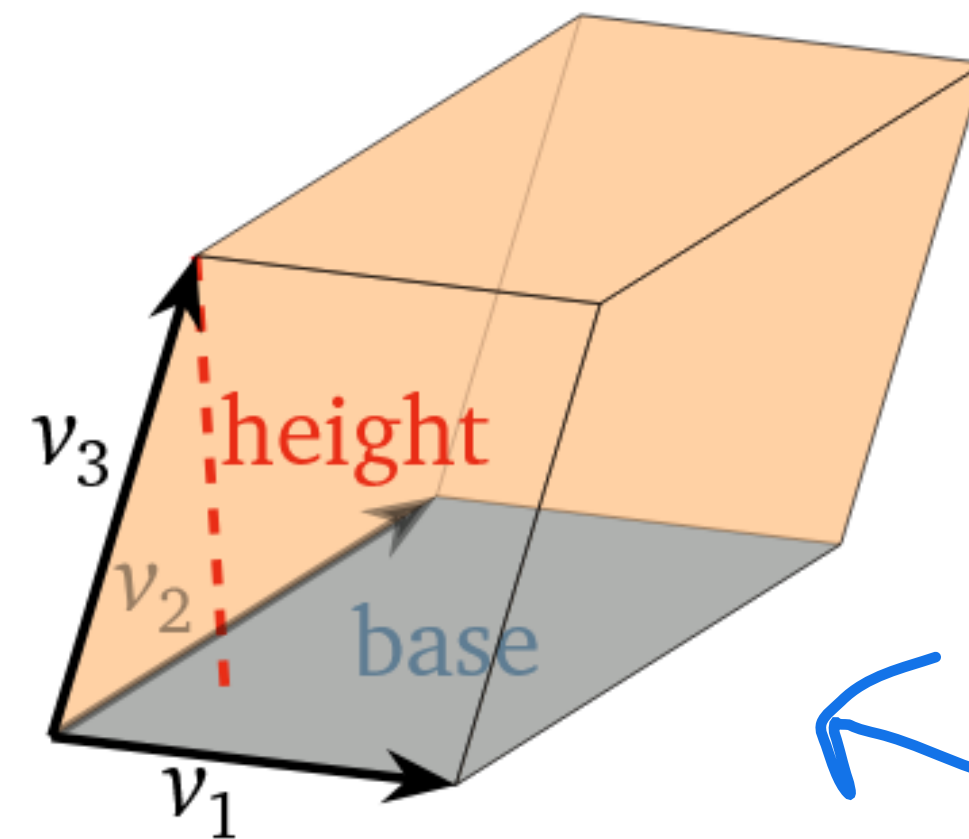
$$|\det[\vec{v}_1, \vec{v}_2]|$$

$$\parallel \text{vol}(P)$$



$$|\det[\vec{v}_1, \vec{v}_2, \vec{v}_3]|$$

$$\parallel \text{vol}(P)$$



A not invertible $\Rightarrow \dim(\text{col } A) < n \Rightarrow P$ has zero volume
 $\Rightarrow \det(A) = 0$

(look up cofactor expansion also)

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \cdots U_{nn}$$

↙ echelon form

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$$\det(A) = \frac{(-1)^s}{c} \text{product of diagonal entries } U_{11}U_{22}\cdots U_{nn}$$

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$c = 0$ if A is not invertible

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Example

$$s = 0 + 1$$
$$c = 1 \cdot (-1/2)$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow[\text{② } R_2 \leftarrow -1/2 R_2]{\text{① swap } R_2, R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & -1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 6R_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A) = \frac{(-1)^1}{-1/2} (1)(1)(-1) = -2$$

Example (Again)

$$S = 0 + 1 + 1$$

$$C = 1$$

$$R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\underline{R_2 \leftarrow R_2 - \frac{1}{2}R_1}$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & \frac{1}{2} \\ 0 & -2 & 0 \end{bmatrix}$$

$$\underline{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 3 & \frac{1}{2} \end{bmatrix}$$

$$\} R_3 + \frac{3}{2}R_2$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\det A = \frac{(-1)^2}{1} (2)(-2)\left(\frac{1}{2}\right) = -2$$

The definition holds no matter
which sequence of row
operations you use.

How To: Determinants

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2. Keep track of the number of row swaps you used, call this s , and the product of all scalings, call this c

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Question. Determine the determinant of a matrix A .

Solution.

1. Convert A to an echelon form U .
2. Keep track of the number of row swaps you used, call this s , and the product of all scalings, call this c .
3. Determine the product of entries along the diagonal of U , call this P .
4. The determinant of A is $\frac{(-1)^s P}{c}$.

Question

$$S = 1$$
$$C = 1$$

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

$$\det A = \frac{(-1)^1}{1} (1)(2)(1) = -2$$

Find the determinant of the above matrix.

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

The Shorter Version

Beyond small matrices, we'll just use a computer

With NumPy:

`numpy.linalg.det(A)`

Properties of Determinants

Properties of Determinants (1)

$$\det(AB) = \det(A) \det(B)$$

$$\det(E_k \cdots E_1 A) = \det(E_k) \cdots \det(E_1) \det(A)$$

It follows that AB is invertible if and only if A and B are invertible

(we won't verify this)

Example Question

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$$

Use the fact that $\det(AB) = \det(A)\det(B)$ to give an expression for $\det(A^{-1})$ in terms of $\det(A)$.

Hint. What is $\det(I)$? $\rightarrow 1$

$$\det(I) = \det(A \cdot A^{-1}) = \det(A)\det(A^{-1}) = 1$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that A^T is invertible if and only if A is invertible.

(we also won't verify this)

Example Question

orthogonal matrices

If $A^{-1} = A^T$, then what are the possible values of $\det(A)$?

$$\det(A^T) = \det(A^{-1}) = \frac{1}{\det A}$$

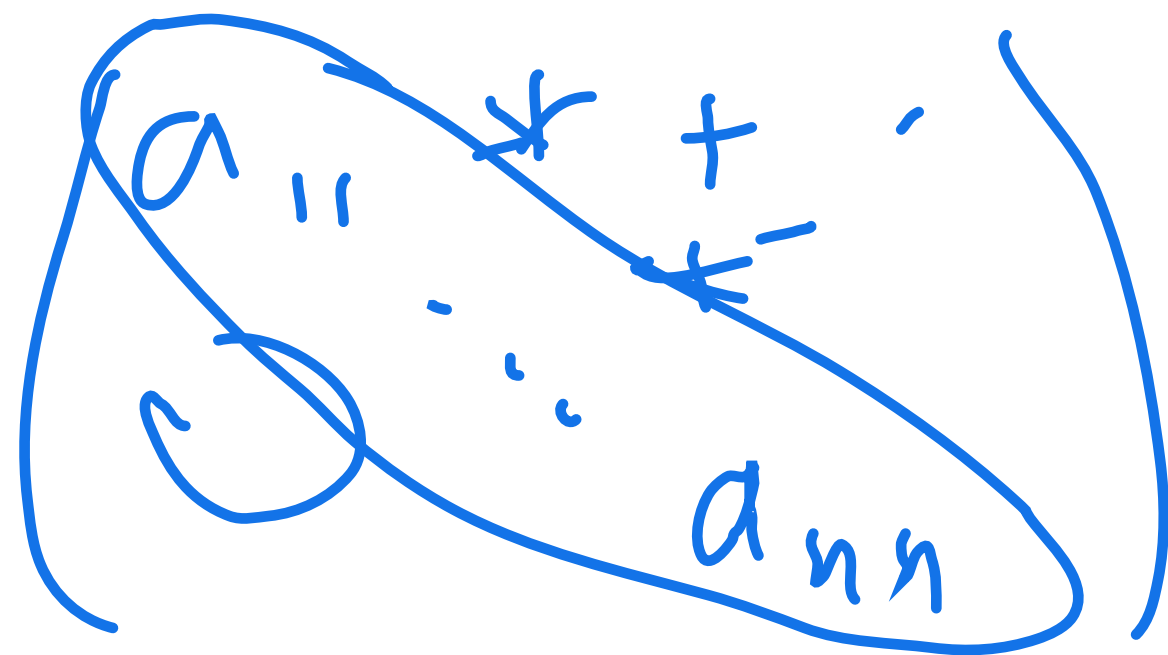
$\det(A^T) \stackrel{||}{=} \det(A)$

$$\frac{1}{\det A} = \det A \Rightarrow (\det A)^2 = 1$$
$$\det A = \pm 1$$

Properties of Determinants (3)

Theorem. If A is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:



$$A = \begin{pmatrix} a_{11} & & 0 \\ * & \ddots & \\ & & a_{nn} \end{pmatrix}$$

$$\det(A) = \det(A^T)$$

turns it into an upper triangular matrix

Answer

Characteristic Equation

What kind of thing is the determinant, really?

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But a matrix may not have numbers as entries.

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The determinant of a matrix A is an arithmetic expression written in terms of the entries of A .

But a matrix may not have numbers as entries.

We might think of the matrix $A - \lambda I$ as having *polynomials* as entries.

What kind of thing is the determinant, really?

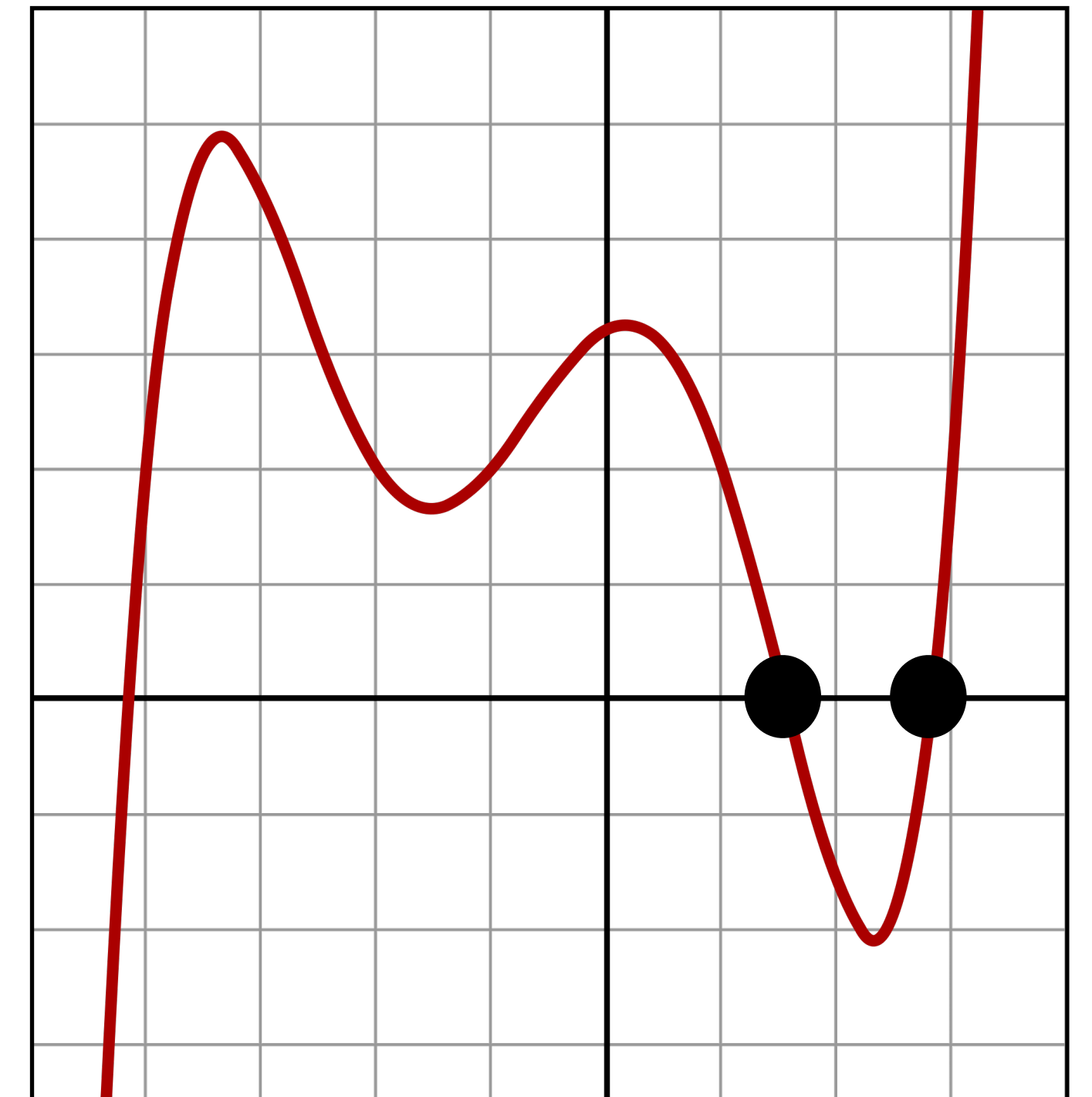
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We might think of the matrix $A - \lambda I$ as having *polynomials* as entries.

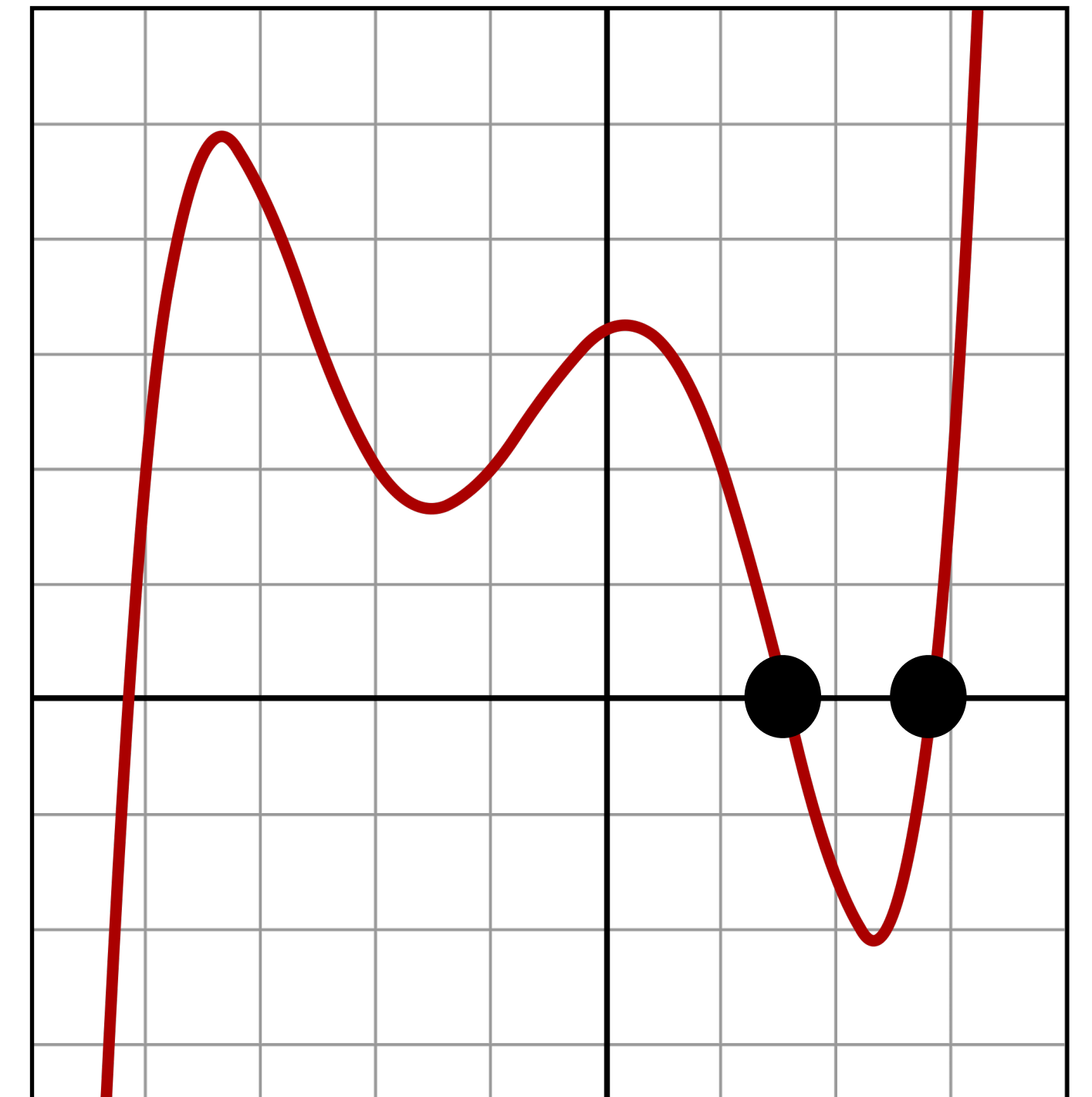
Then $\det(A - \lambda I)$ is a **polynomial**.

Reminder: Polynomial Roots



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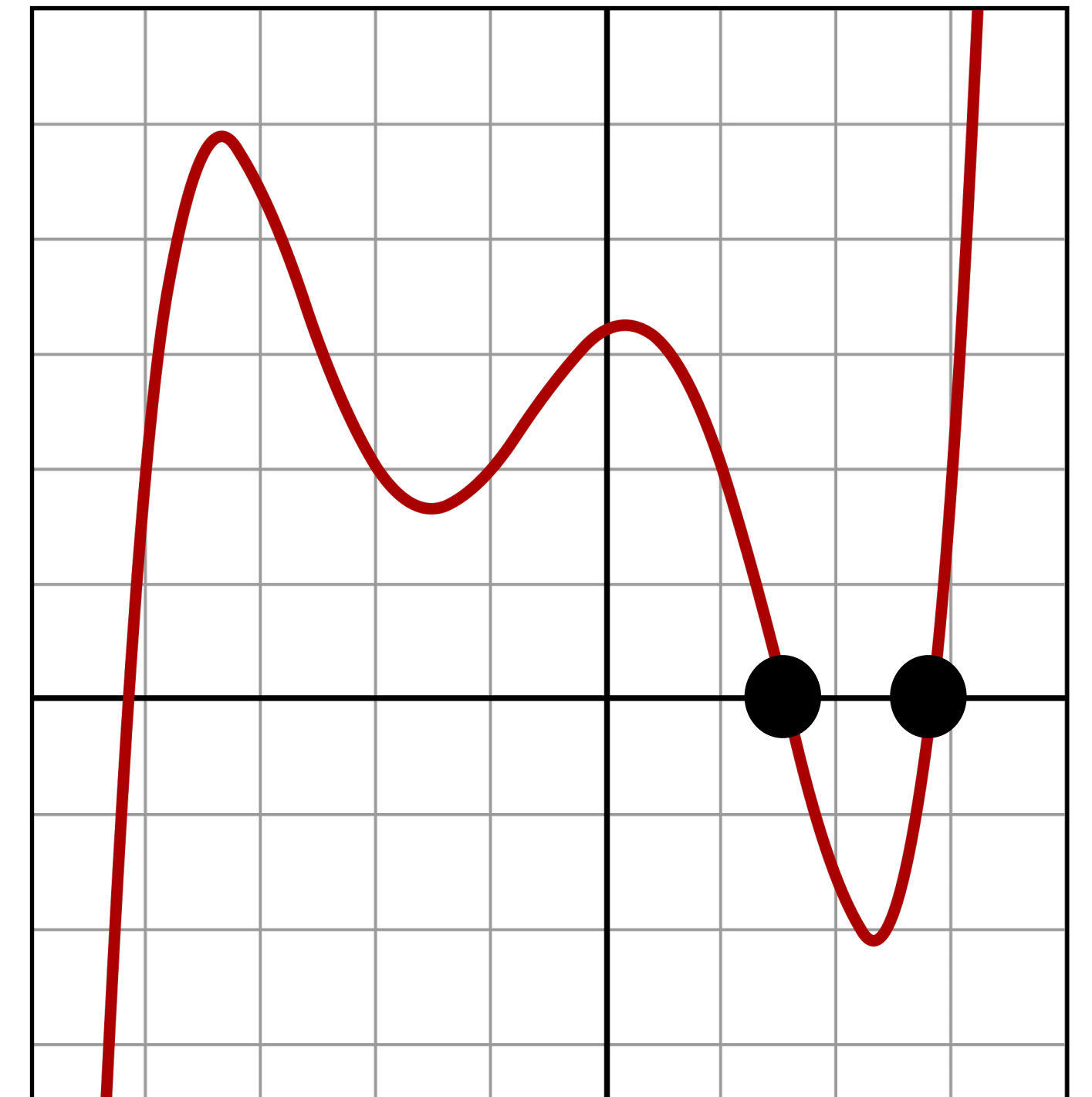
A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.



Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

(A polynomial may have many roots)



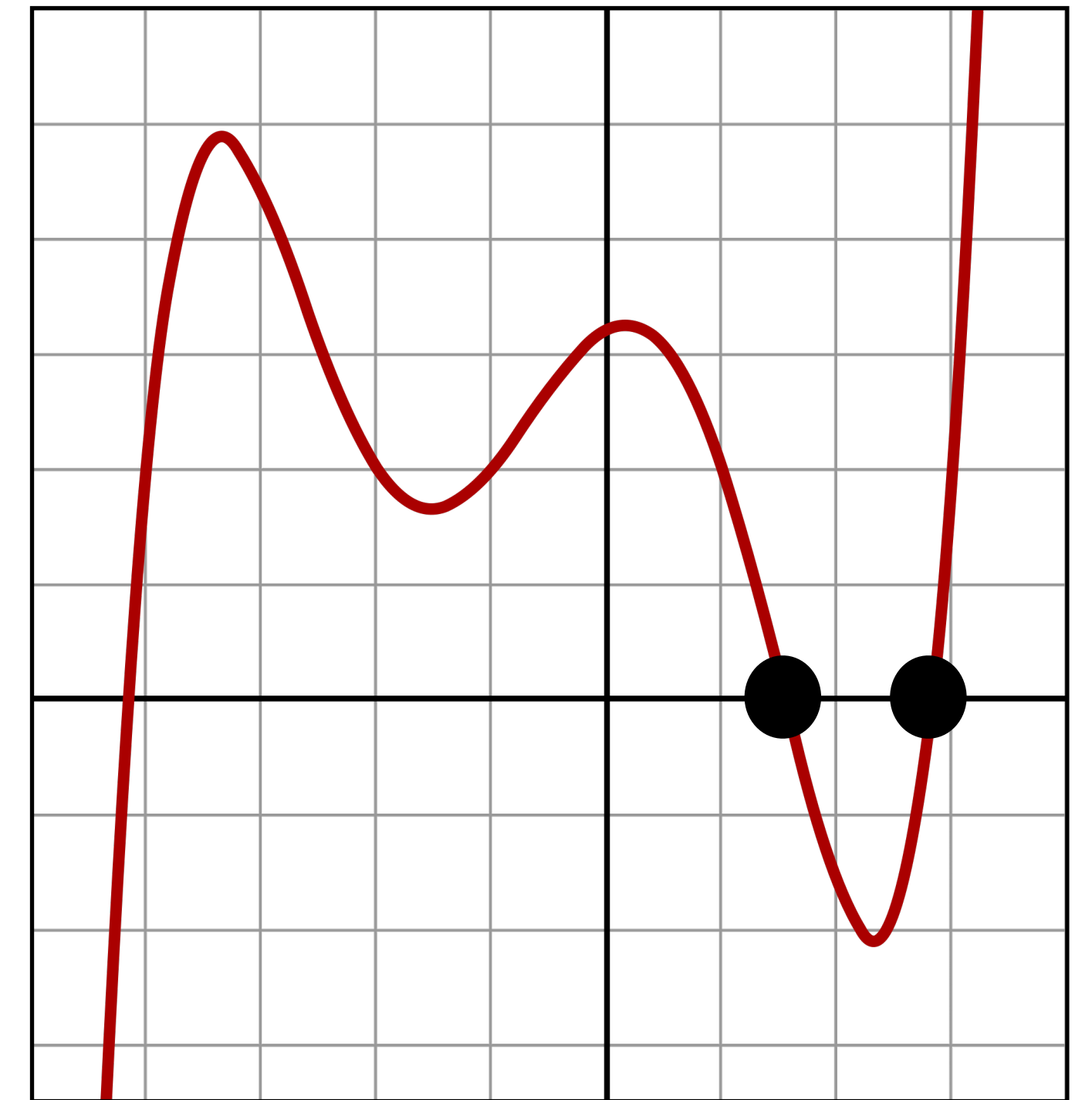
Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

(A polynomial may have many roots)

If r is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



Characteristic Polynomial

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So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

"Deriving" the characteristic polynomial

Q: When is λ an eigenvalue for A ?

A: When $(A - \lambda I)\vec{v} = 0$ has nontrivial solutions.

\Downarrow ($A - \lambda I$ not invertible)

$$\underline{\underline{\det(A - \lambda I) = 0}}$$

Hence, the characteristic polynomial

Example: 2×2 Matrix

$$rk \left(A - \frac{1+\sqrt{5}}{2} I \right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = (1-\lambda)(-\lambda) - (1)(1) \\ = \boxed{\lambda^2 - \lambda - 1}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} \\ = \frac{1 \pm \sqrt{5}}{2}$$

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 0 & 6 \\ 0 & -\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = (1-\lambda)^2 (-\lambda) (4-\lambda)$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$

$$= (\lambda-1)^2 \lambda (\lambda-4)$$

How To: Finding Eigenvalues

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Question. Find all eigenvalues of the matrix A .

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Solution. Find the roots of the characteristic polynomial of A .

An Observation: Multiplicity

$$\lambda^1(\lambda - 1)^2(\lambda - 4)^1$$

multiplicities

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$.

The multiplicity is an upper bound on
"how large" the eigenspace is.

dimensionality

"Eigenspace $_{\lambda}$ " = $\text{Nul}(A - \lambda I)$
not notation

Example

$$A\vec{v} = 0\vec{v} = \vec{0}$$

Since x not present
as a factor
 $\Rightarrow \dim(\text{Nu}(A)) = 0$
 $\Rightarrow \text{rank } A = 5$

Let A be a 5×5 matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$.

» What is $\text{rank}(A)$?

~~$\text{rk } A \geq 3$~~

distinct nonzero
eigenvalues

» What is the minimum possible rank of $A - I$?

$$\text{rk } A - I + \text{nullity } A - I = 5$$

$$\boxed{\text{rk } A - I \geq 2}$$

$$\leq 3$$

$$1 \leq \dim(\text{Nu}(A - I)) \leq 3$$