Assignment 8

CAS CS 132: *Geometric Algorithms*

Due November 6, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 4.1-4.5.

Basic Problems

1. For the following matrix A, determine **(1)** a basis for $\mathrm{Col}(A)$, **(2)** a basis for $\mathrm{Nul}(A)$, **(3)** $\mathrm{rank}(A)$, and **(4)** $\dim(\mathrm{Nul}(A))$.

$$A = \begin{bmatrix} 1 & -5 \\ -2 & 10 \end{bmatrix}$$

2. For the following matrix A, determine **(1)** a basis for $\mathrm{Col}(A)$, **(2)** a basis for $\mathrm{Nul}(A)$, **(3)** $\mathrm{rank}(A)$, and **(4)** $\dim(\mathrm{Nul}(A))$.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. For the following matrix A, determine **(1)** a basis for $\mathrm{Col}(A)$, **(2)** a basis for $\mathrm{Nul}(A)$, **(3)** $\mathrm{rank}(A)$, and **(4)** $\dim(\mathrm{Nul}(A))$.

$$A = \begin{bmatrix} 1 & -4 & 3 & -3 \\ -2 & 8 & -6 & 7 \end{bmatrix}$$

4. For the following matrix A, determine **(1)** a basis for $\mathrm{Col}(A)$, **(2)** a basis for $\mathrm{Nul}(A)$, **(3)** $\mathrm{rank}(A)$, and **(4)** $\dim(\mathrm{Nul}(A))$.

$$A = \begin{bmatrix} 1 & -4 & -3 \\ -3 & 12 & 10 \\ -2 & 8 & 8 \\ -1 & 4 & 2 \end{bmatrix}$$

5. Determine a basis for the following subspace.

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -8 \end{bmatrix} \right\}$$

6. Determine a basis for the following subspace.

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 18 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix} \right\}$$

7. Determine the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ where $\mathbf{u}$ and $\mathcal{B}$ are defined below.

$$\mathbf{u} = \begin{bmatrix} 8 \\ -12 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$$

8. Determine the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ where $\mathbf{u}$ and $\mathcal{B}$ are defined below.

$$\mathbf{u} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \right\}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. There is a unique basis for any subspace.
2. A 7×4 matrix A may have $\dim(\text{Nul}(A)) = 5$.
3. A 3×6 matrix A may have $\dim(\text{Nul}(A)) = 2$.
4. For any matrix $A \in \mathbb{R}^{n \times n}$, if A is invertible then $\text{rank}(A) = n$.
5. For any matrix $A \in \mathbb{R}^{m \times n}$, $\text{Col}(A)$ is the same as the set of vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.
6. A basis is a spanning set that is as large as possible.
7. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps $\mathbb{R}^3$ to a plane has a trivial kernel (i.e., if $T(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$).

More Difficult Problems

1. For a matrix $A \in \mathbb{R}^{m \times n}$, consider the subset of vectors $\mathbf{x} \in \mathbb{R}^n$ that are solutions to $A\mathbf{x} = \mathbf{e_1}$ (*Recall:* $\mathbf{e_1}$ is the first standard basis vector). Is this subset closed under addition? Is it closed under scaling? Is this a subspace of $\mathbb{R}^n$? Justify your answers.

2. Consider the following vectors.

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

   List all possible subsets of the above vectors that form a basis of $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. *Note:* We've taught you how to come up with one choice, but there will be multiple choices that are suitable.

3. Consider the following vectors.

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

   List all possible subsets of the above vectors that form a basis of $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. *Note:* We've taught you how to come up with one choice, but there will be multiple choices that are suitable.

4. Consider the 3-dimensional vector space of all quadratic polynomials $Q = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ and consider the linear derivative map $\frac{d}{dx} : Q \to Q$ defined by $\frac{d}{dx}(ax^2 + bx + c) = 0x^2 + 2ax + b$. Using the standard basis given by $\{x^2, x, 1\}$, determine a 3×3 matrix A that implements $\frac{d}{dx}$. Then determine **(1)** a basis for $\text{Col}(A)$, **(2)** a basis for $\text{Nul}(A)$, **(3)** $\text{rank}(A)$, and **(4)** $\dim(\text{Nul}(A))$.

Challenge Problems (Optional)

1. The row space of a matrix $A \in \mathbb{R}^{m \times n}$ is the span of the rows of A, denoted $\text{Row}(A)$. Show that $\dim(\text{Row}(A)) + \dim(\text{Nul}(A)) = n$.

2. In the vector space of all real-valued functions, find a basis for the subspaced spanned by $\{\sin t,\ \sin 2t,\ \sin t \cos t\}$.