

# Matrix Algebra

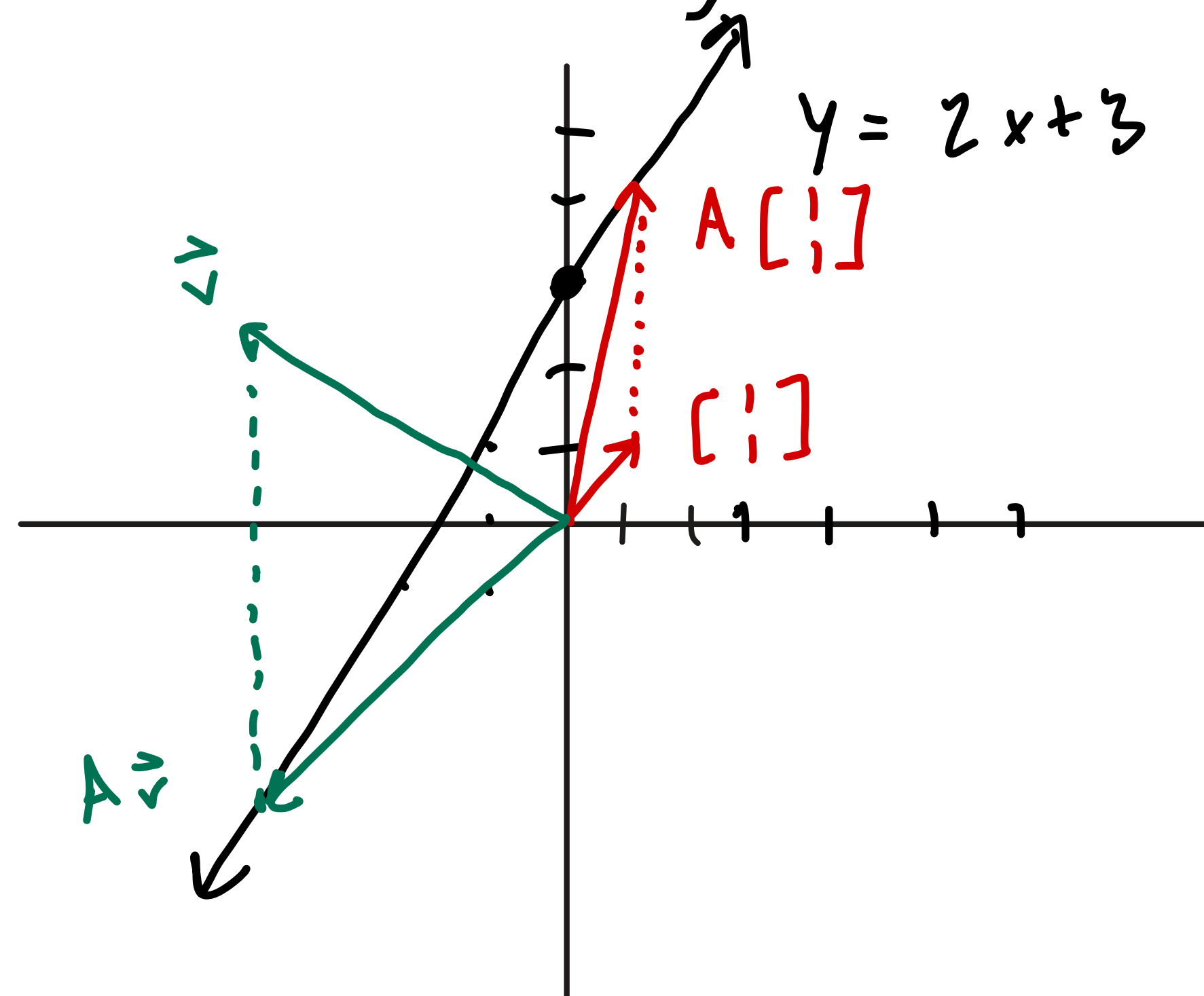
## Geometric Algorithms Lecture 9

# Practice Problem

PROBLEM: THIS IS NOT LINEAR

Write the matrix for the transformation which projects vectors in  $\mathbb{R}^2$  vertically onto the line  $y = 2x + 3$  in  $\mathbb{R}^2$

$$T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

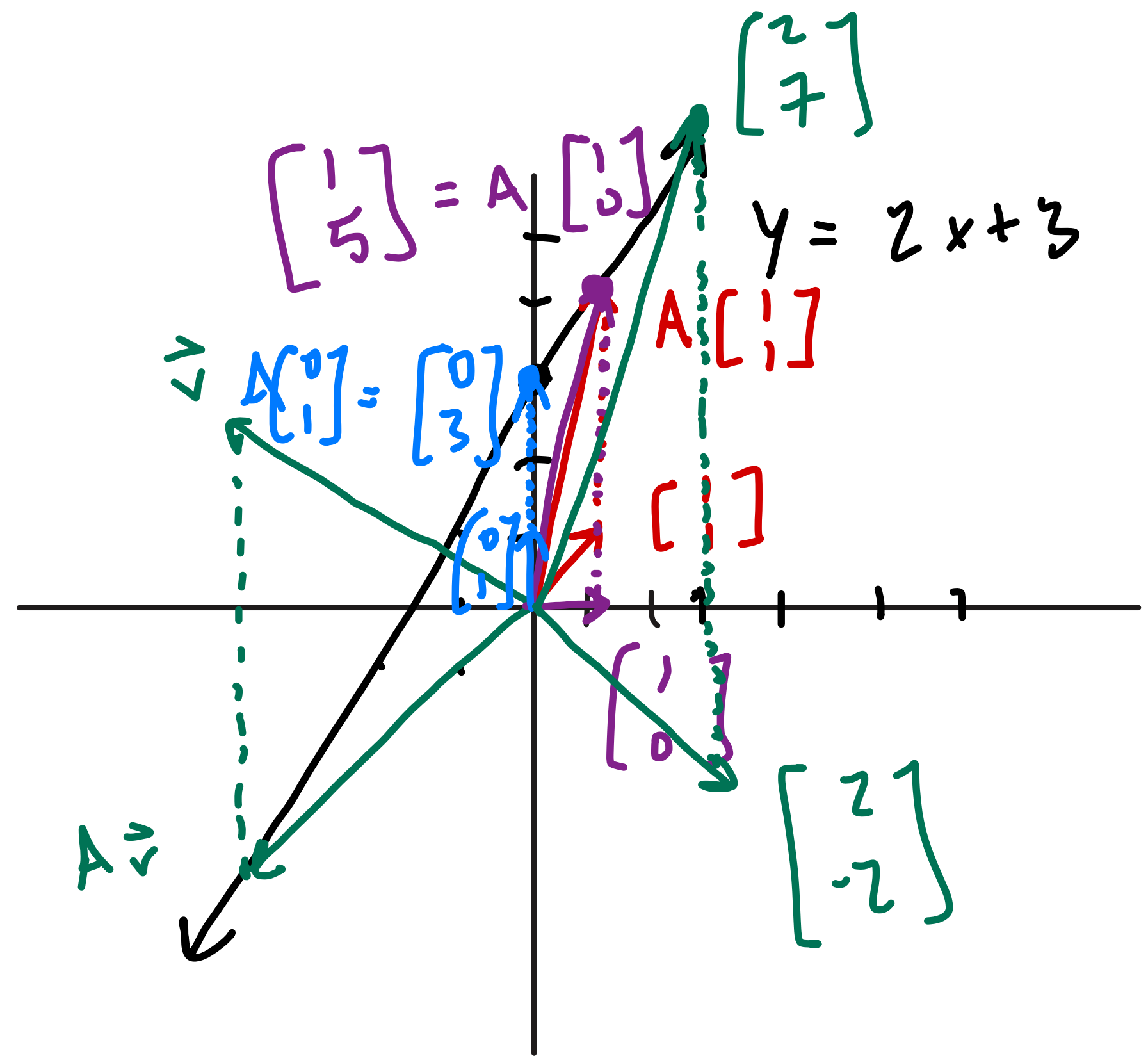


# Answer

$$\begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} =$$

$$2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

~~$$\begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$~~

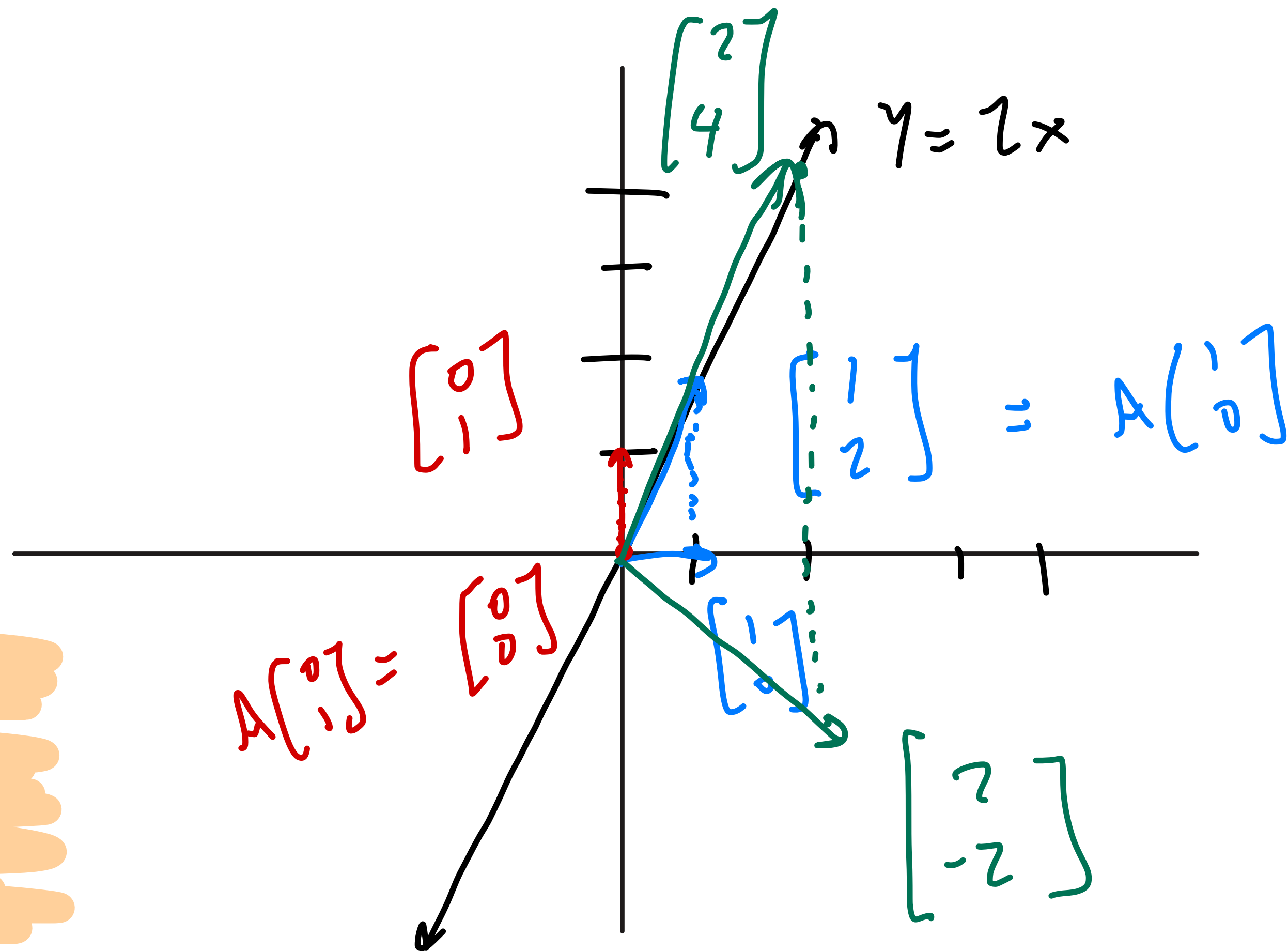


**Answer**

$$y = 2x$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} =$$

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



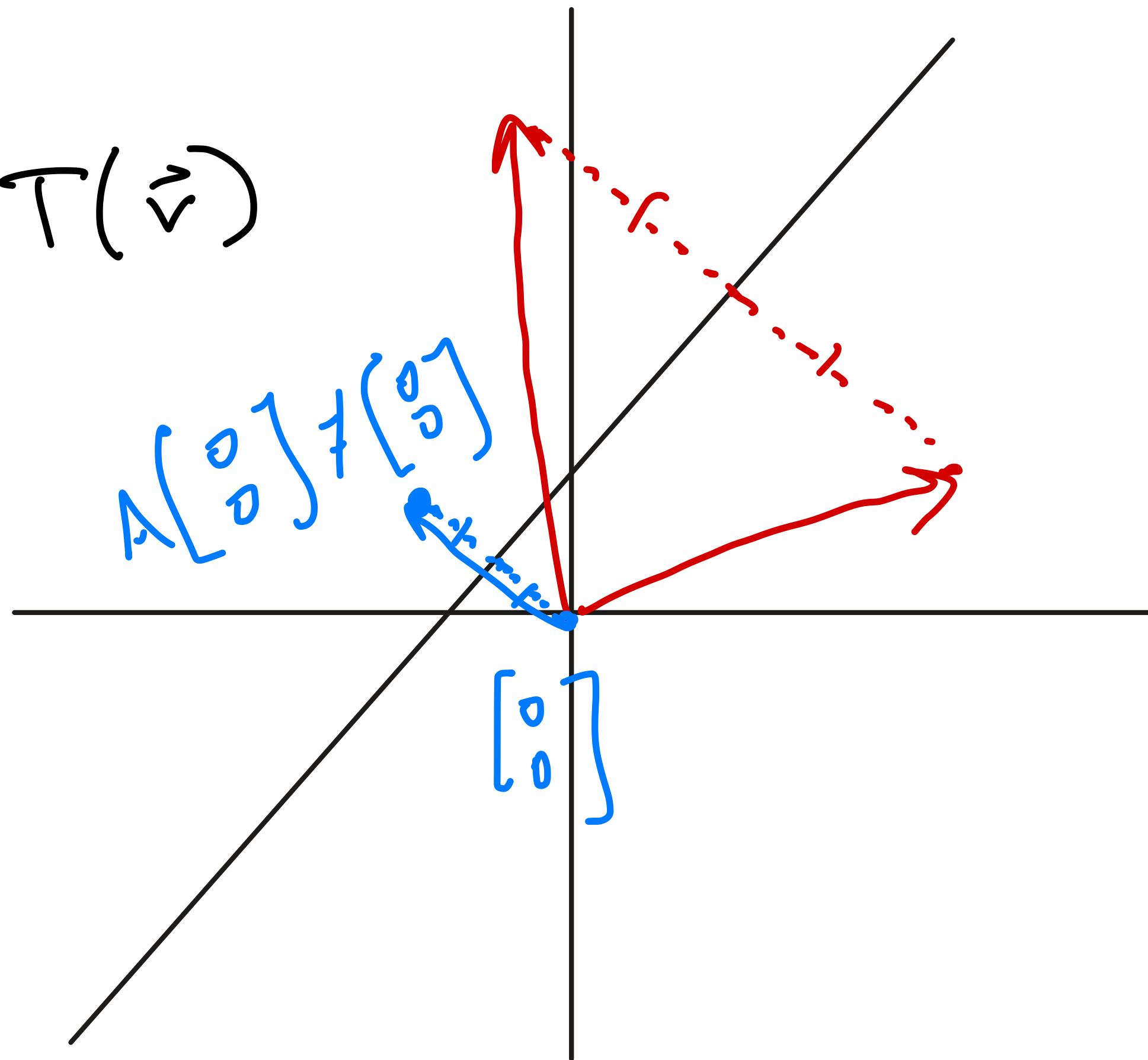
# Answer

$$\textcircled{1} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\textcircled{2} T(c\vec{v}) = c T(\vec{v})$$

$$T(\vec{0}) = T(0[1])$$
$$= 0 T([1])$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



# Objectives

- » Connect questions about **matrix equations** and **linear transformations**
- » Motivate **matrix multiplication**
- » Define matrix multiplication
- » Look at the **algebra** of matrix multiplication

# Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

# **Recap: Geometry of Linear Transformations**



# Recall: Matrices as Transformations

Matrices allow us to *transform* vectors

The transformed vector lies in the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector  $\mathbf{x}$  to the vector  $A\mathbf{x}$

# Recall: Motivating Questions

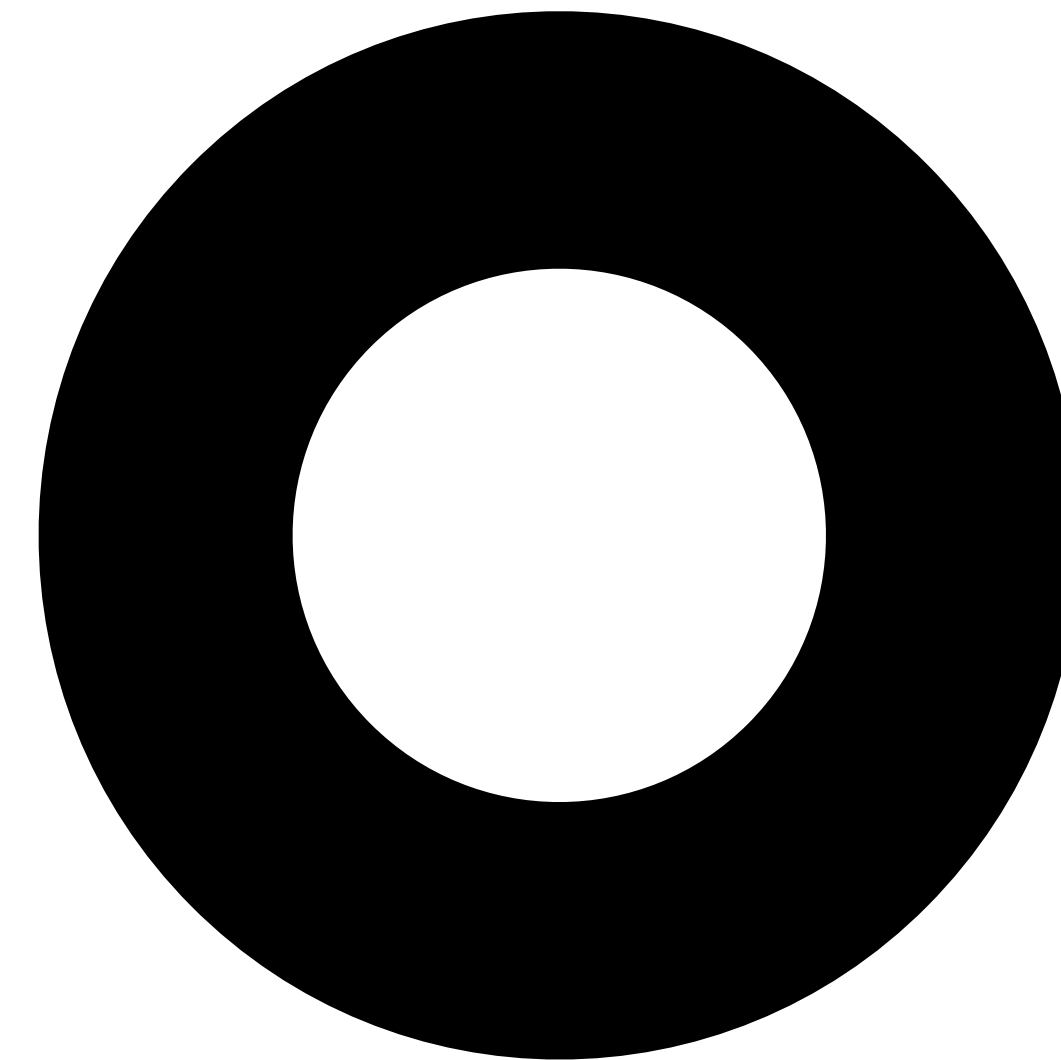
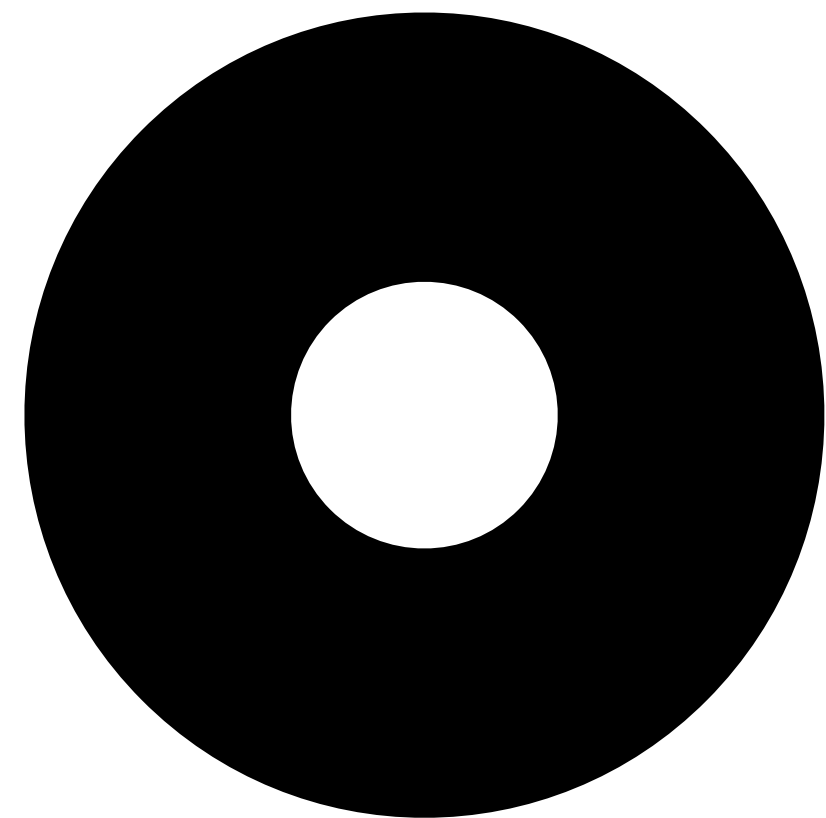
What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

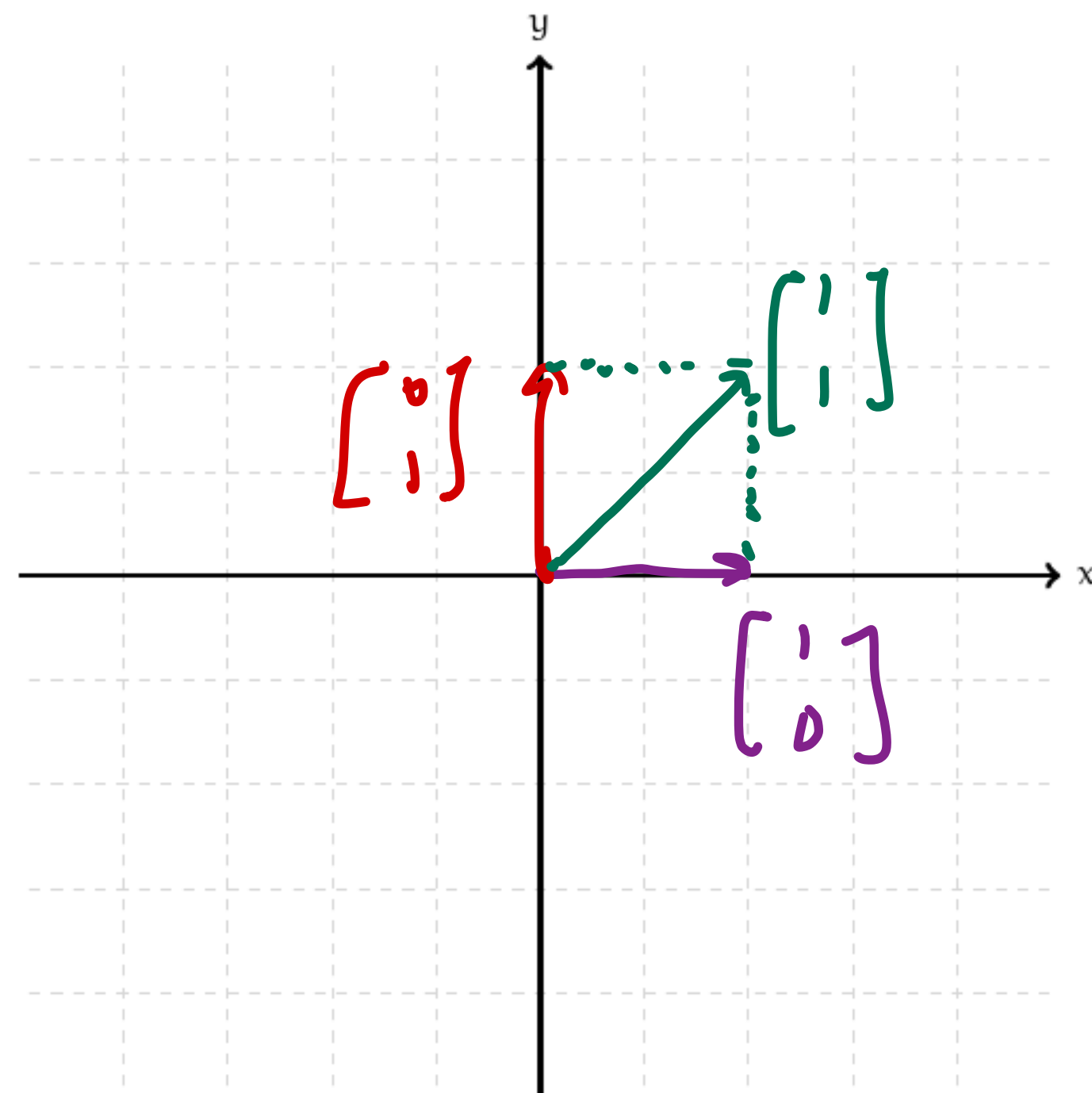
Matrix transformations change the  
"shape" of a set of set of  
vectors (points)

# Example: Dilation



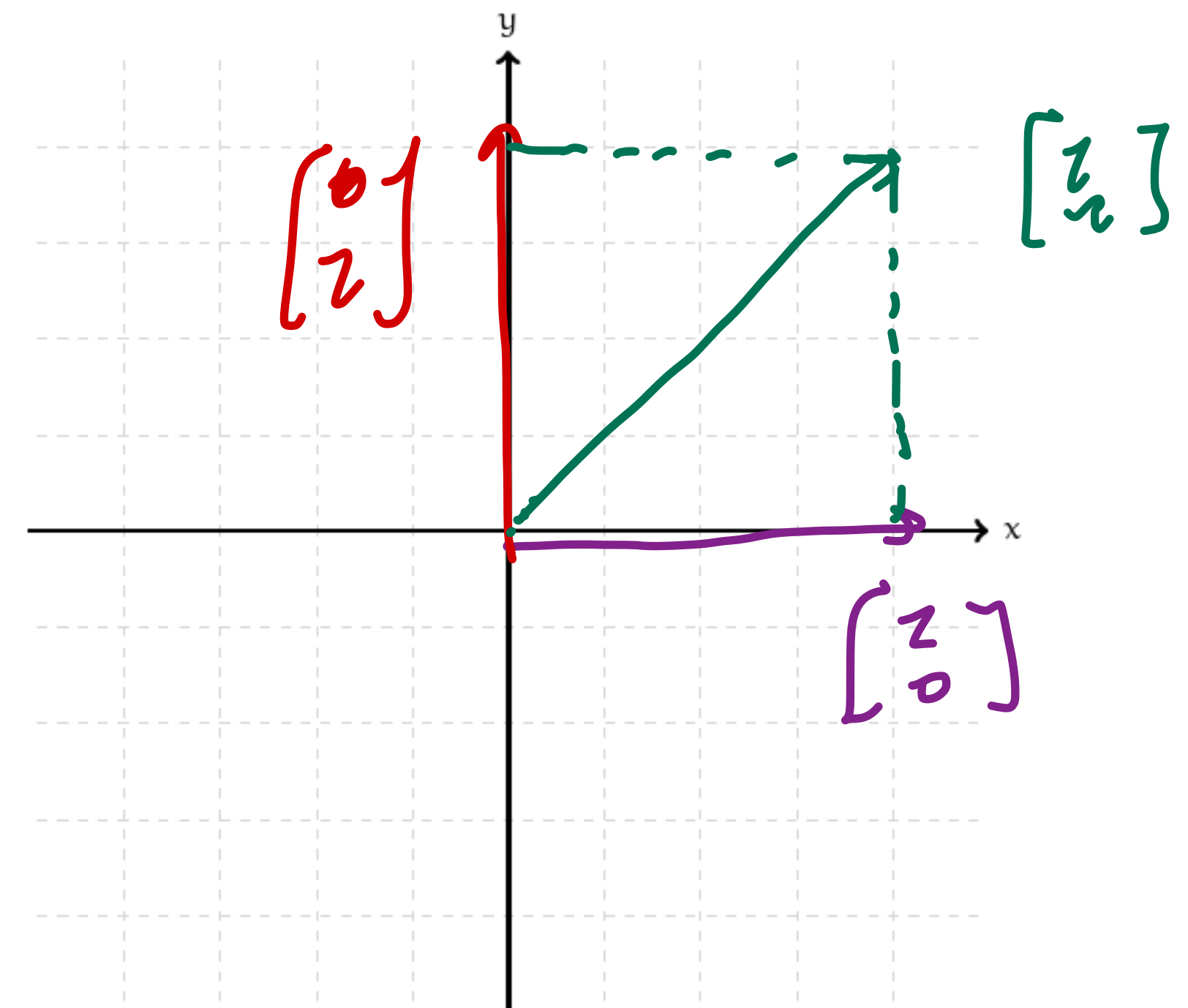
# Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



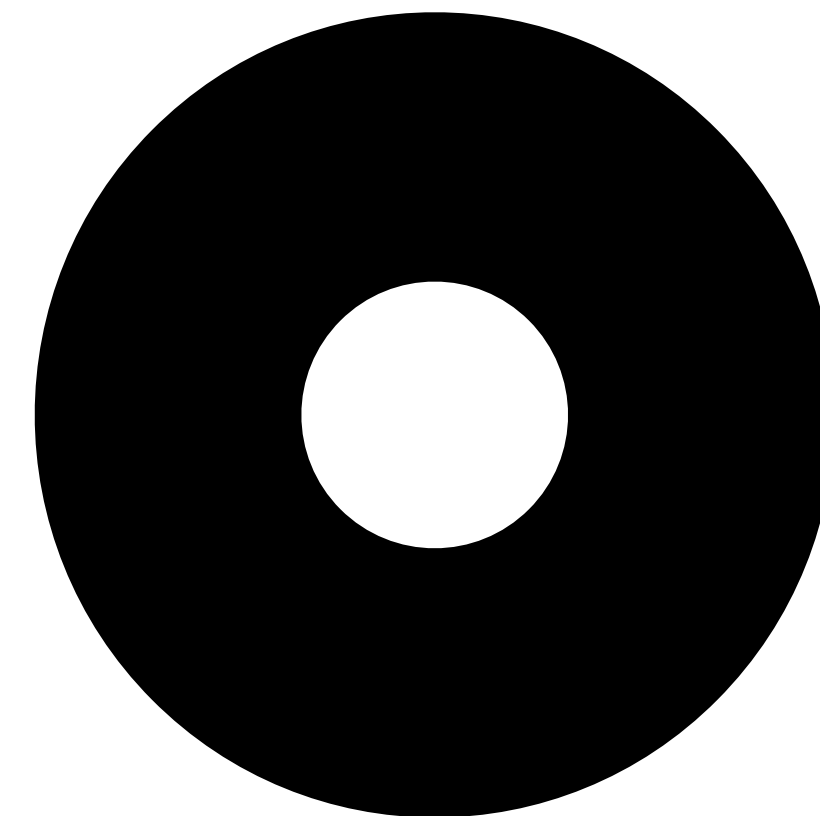
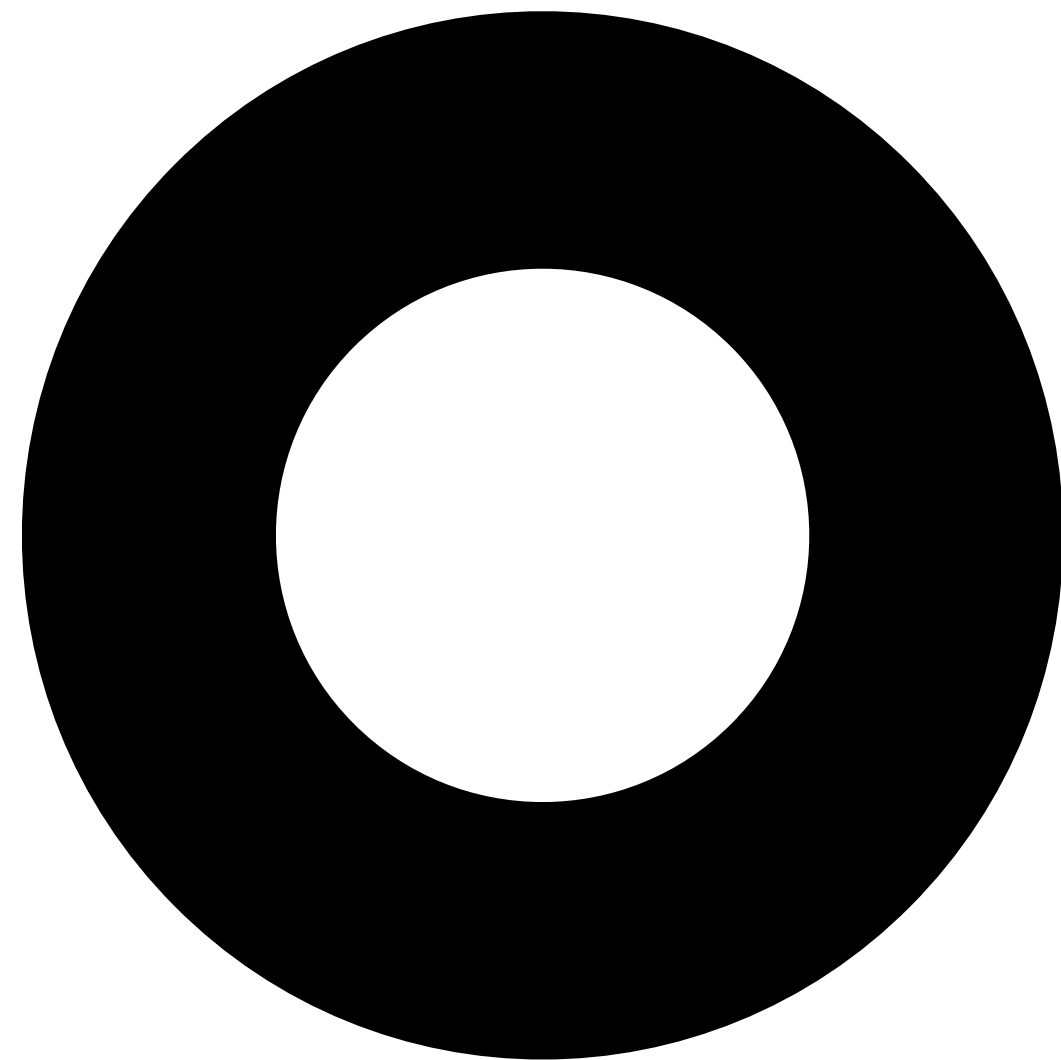
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

→



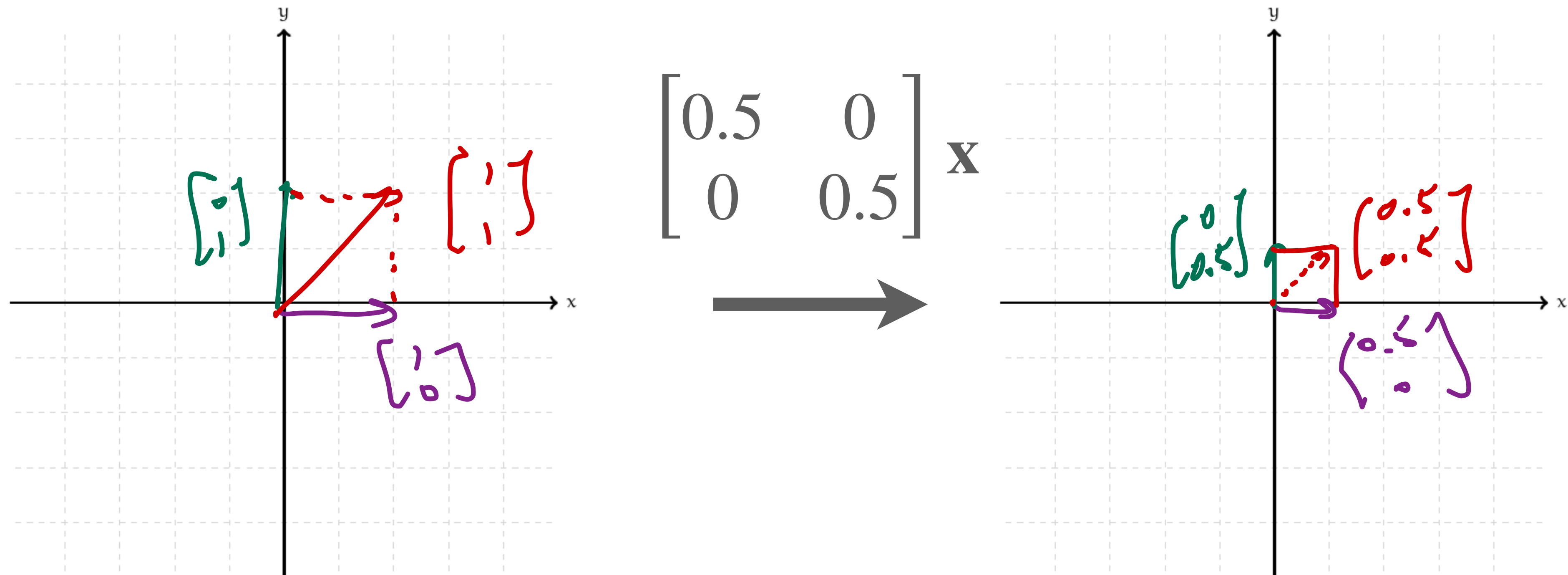
if  $r > 1$ , then the transformation pushes points away from the origin

# Example: Contraction



# Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



if  $0 \leq r \leq 1$ , then the transformation  
pulls points towards the origin



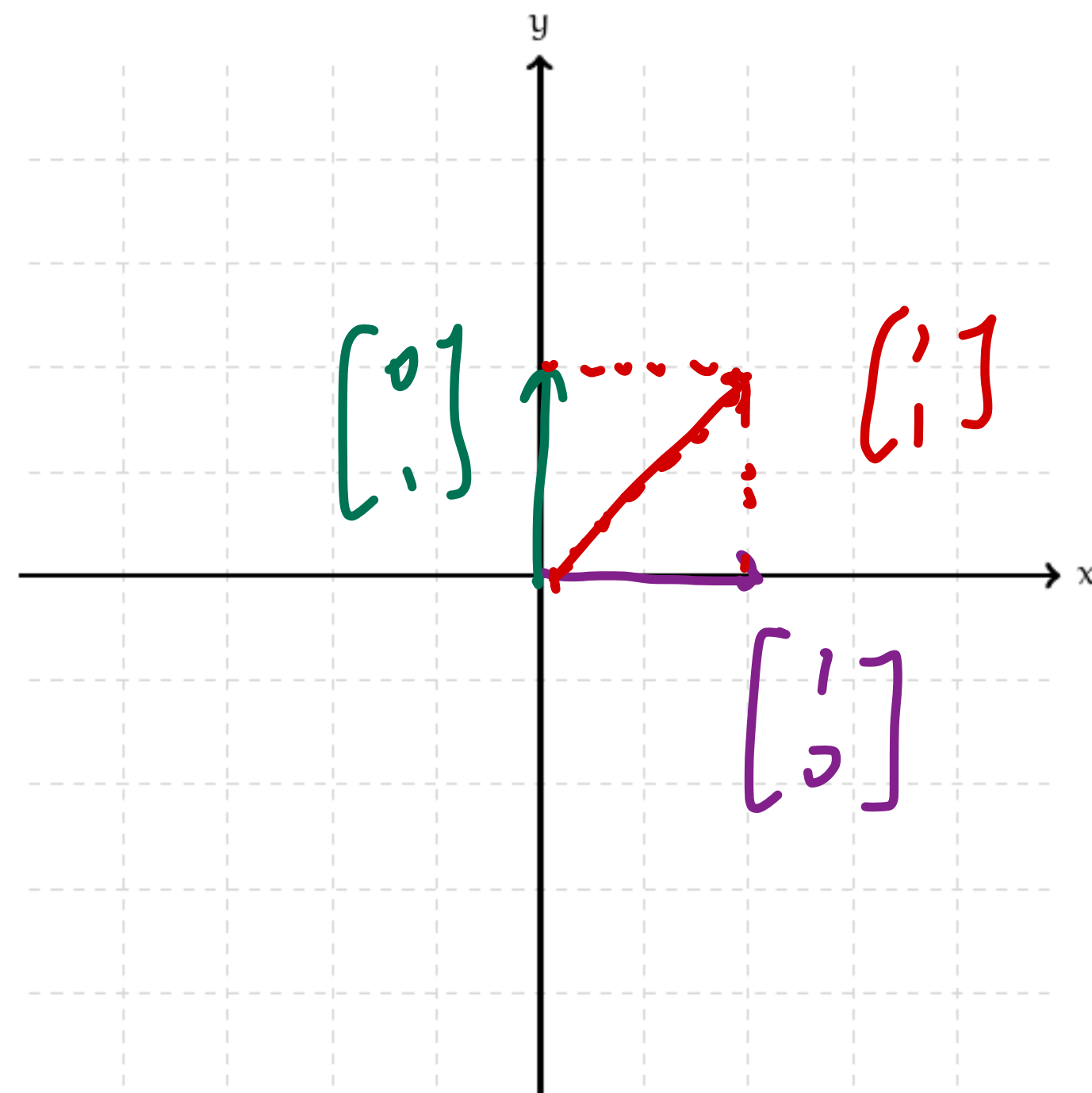
# Example: Shearing





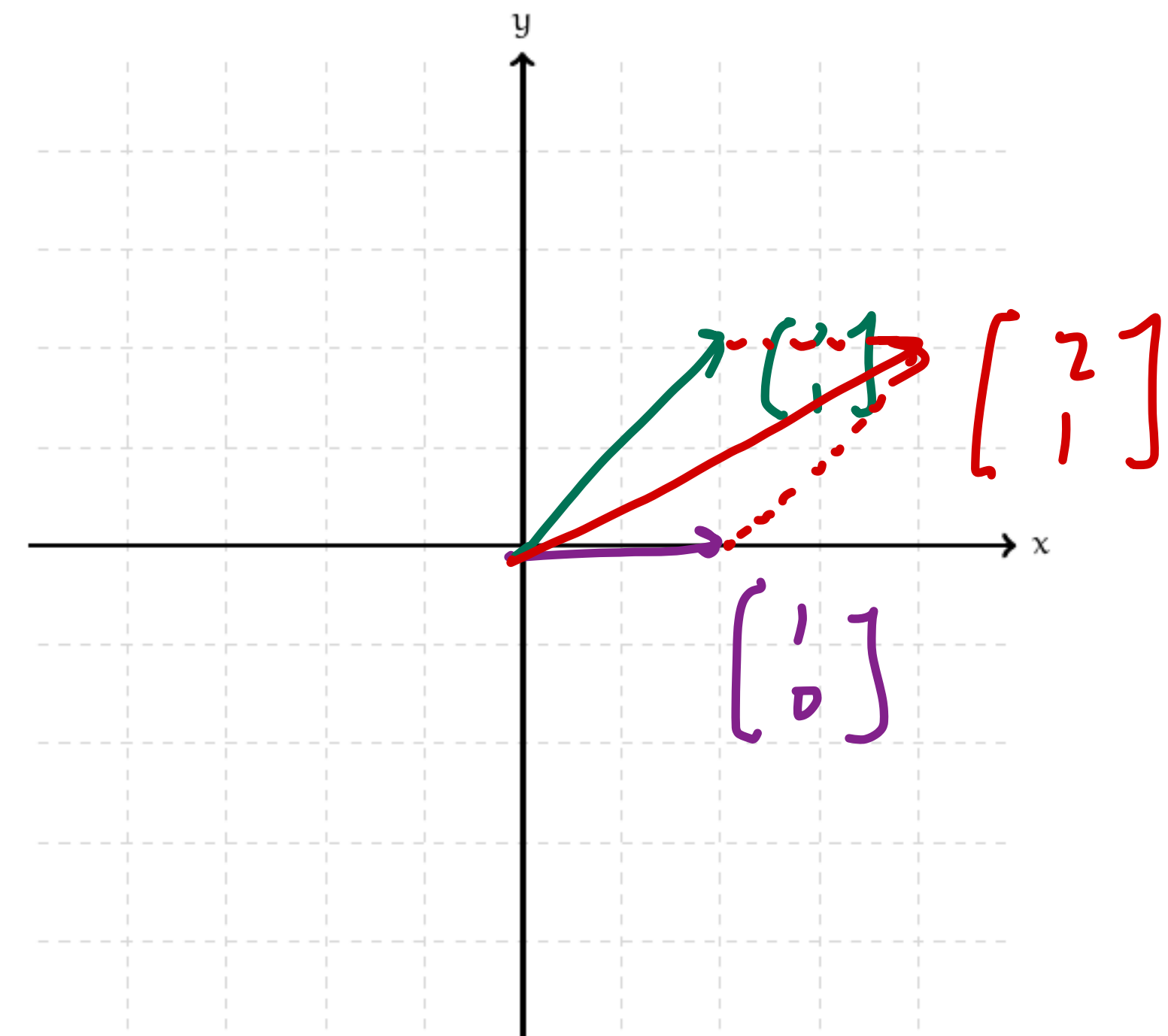
# Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



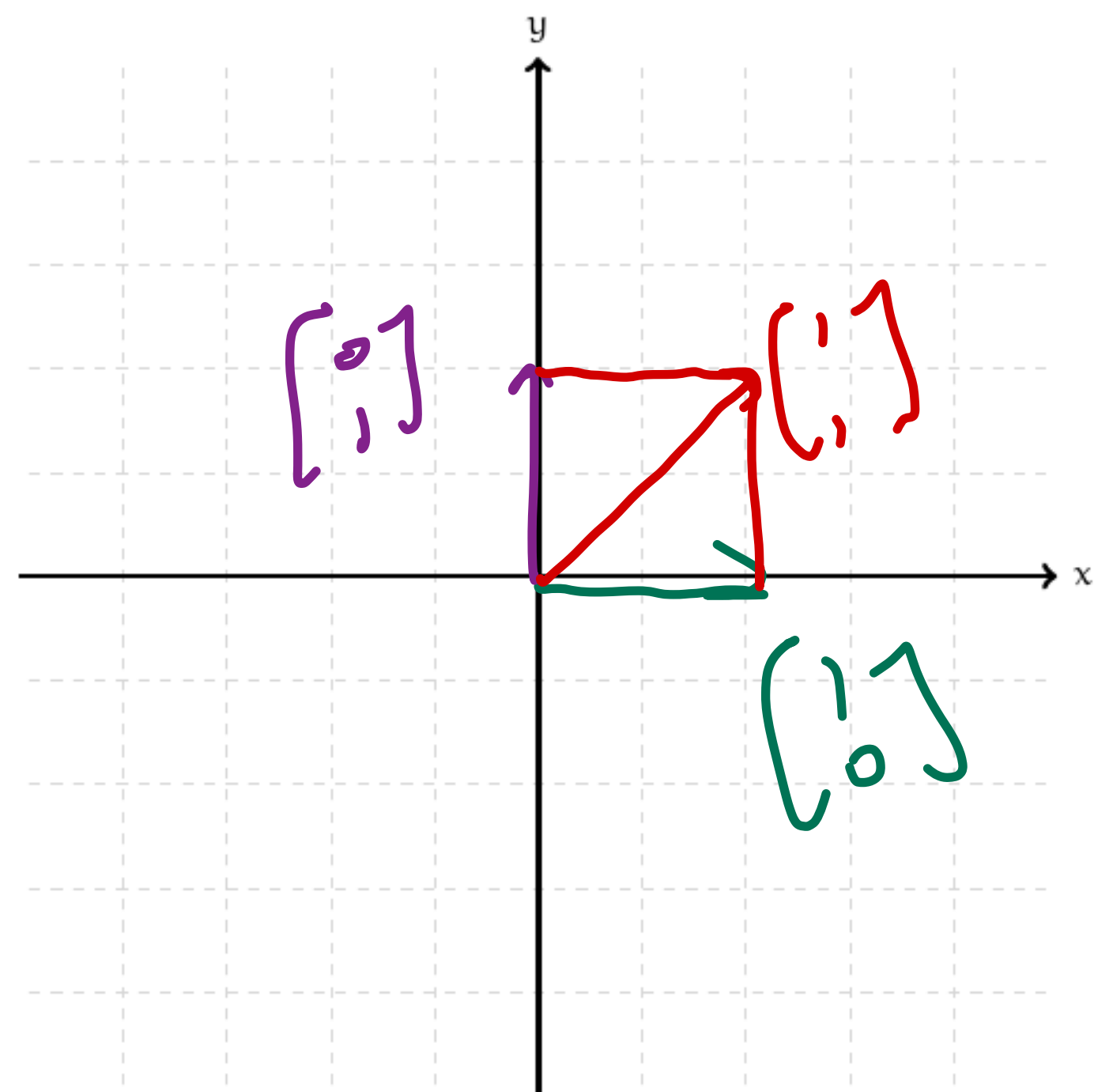
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

A large gray arrow points from the left coordinate system to the right coordinate system, indicating a transformation.

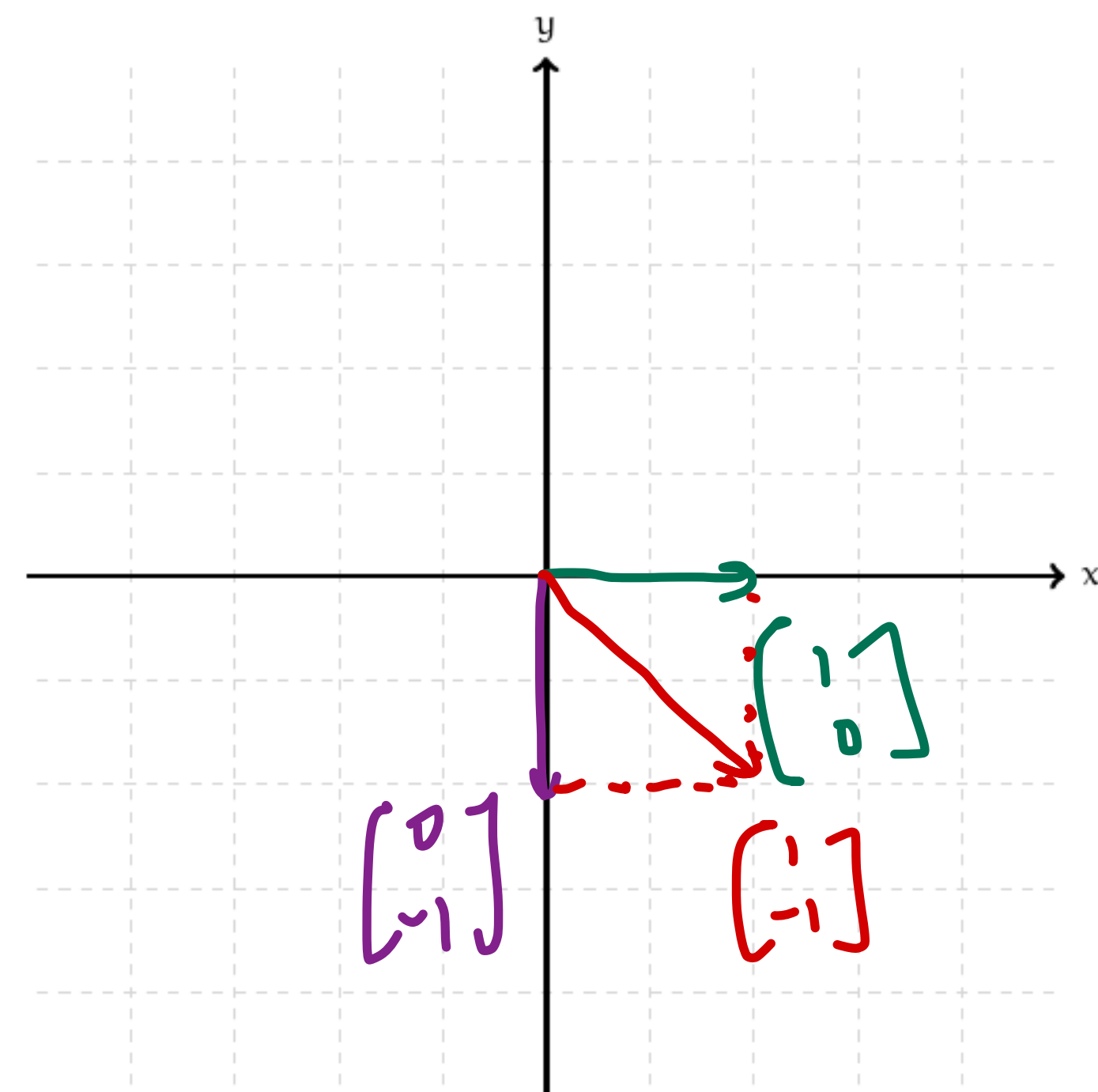


Imagine shearing like with rocks or metal

# Example: Reflection

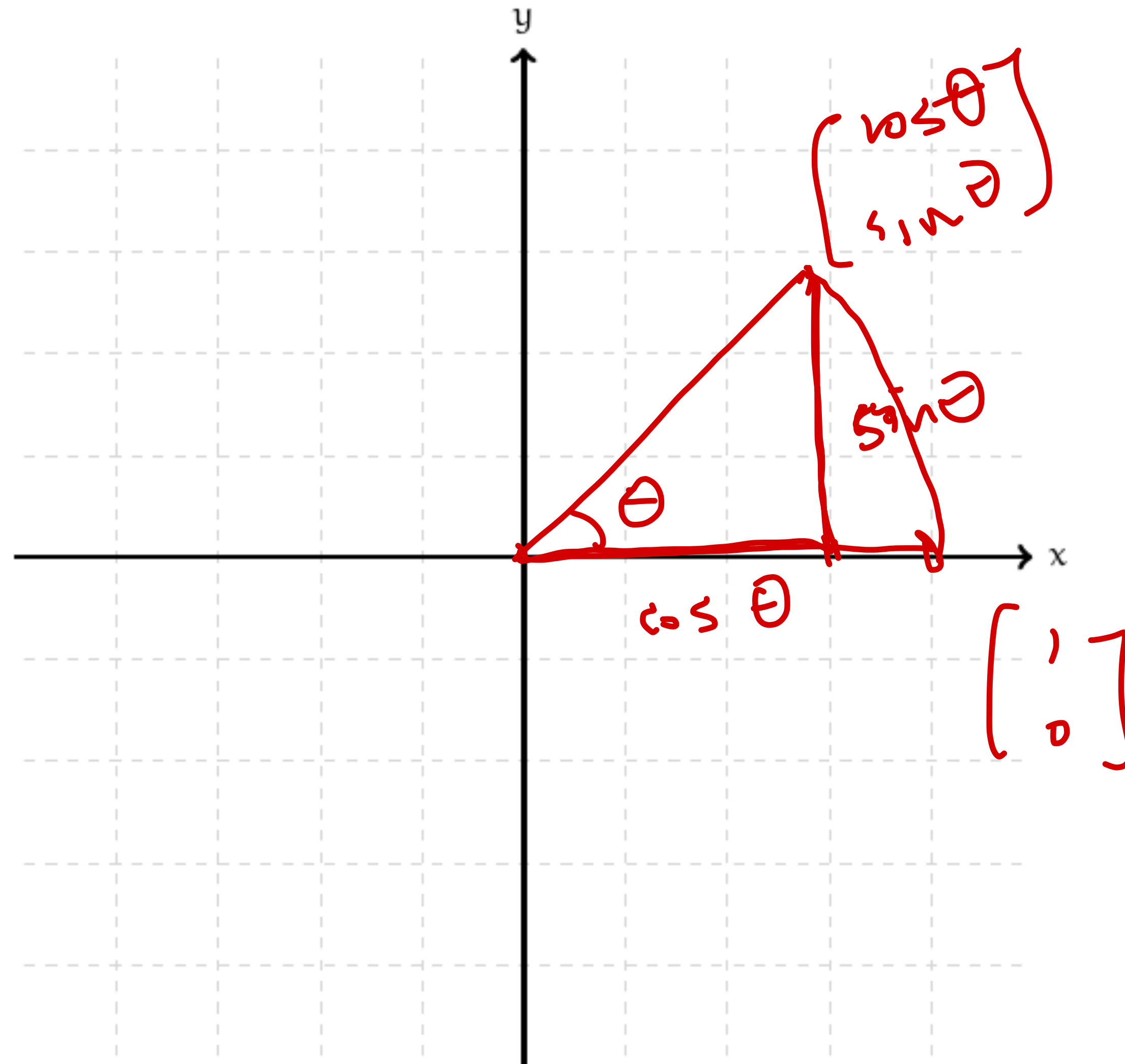


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



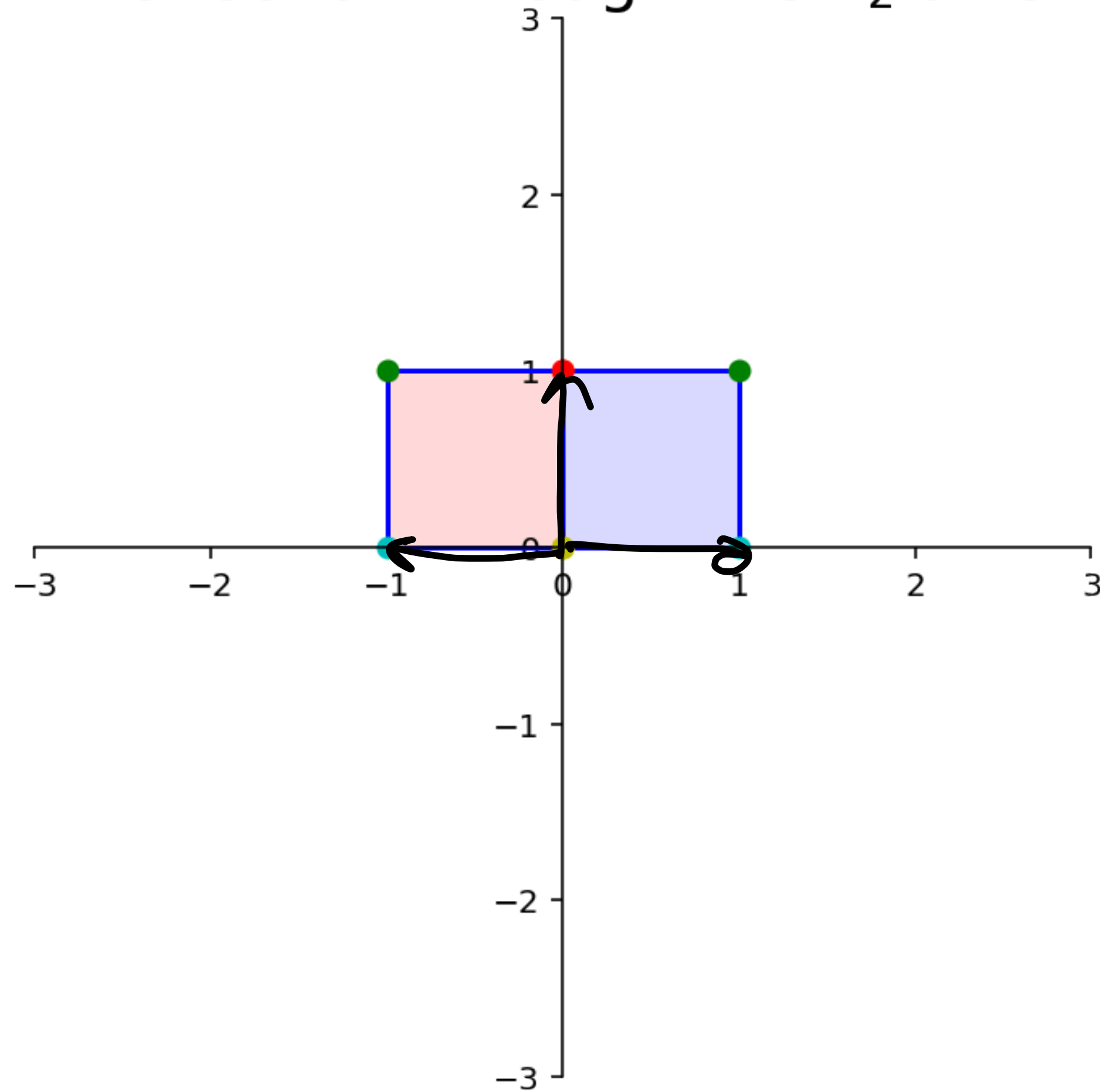
# General Rotation

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



# Example: Reflection through the $x_2$ -axis

Reflection through the  $x_2$  axis

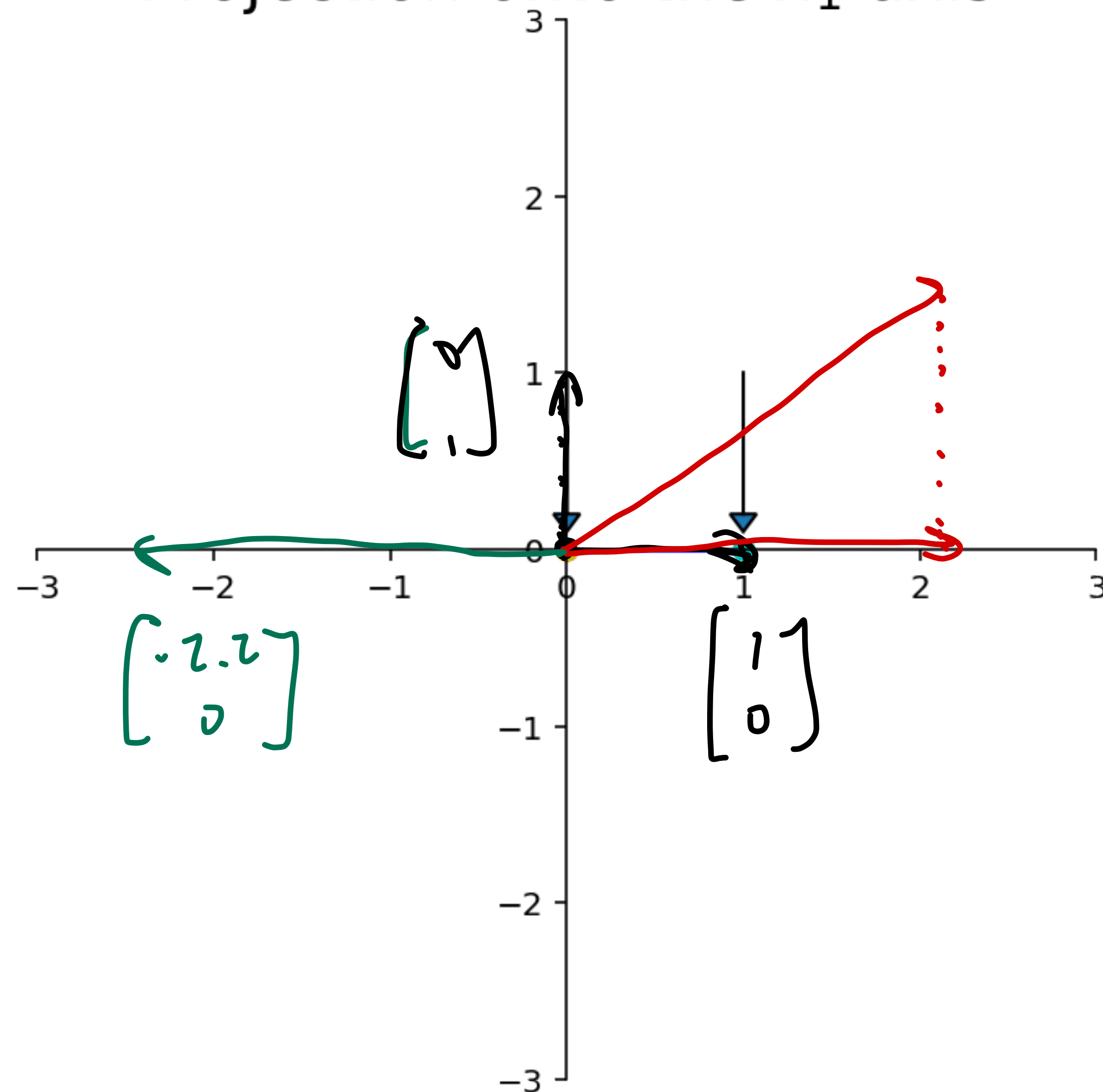


$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

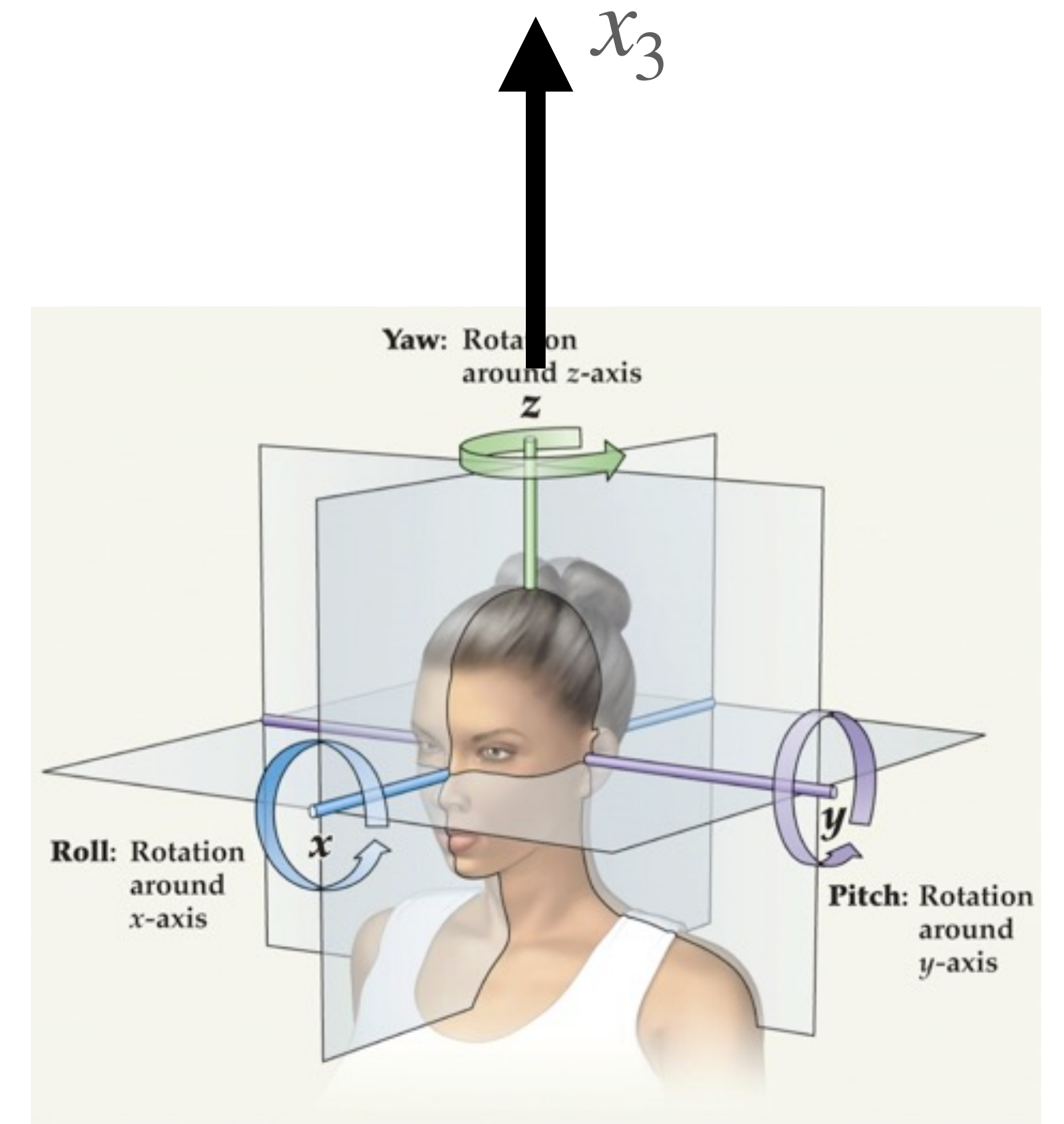
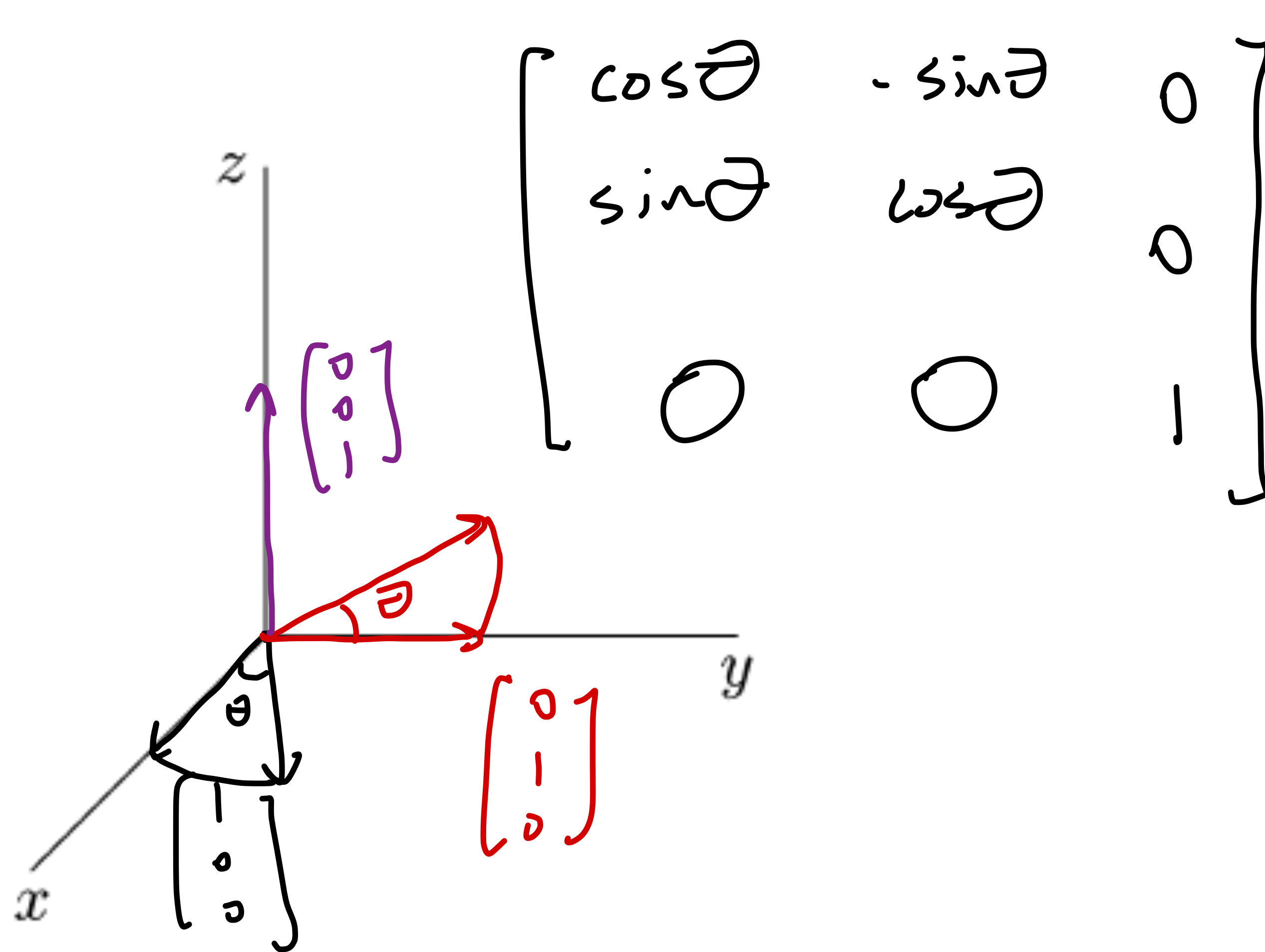
# Example: Projections

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Projection onto the  $x_1$  axis

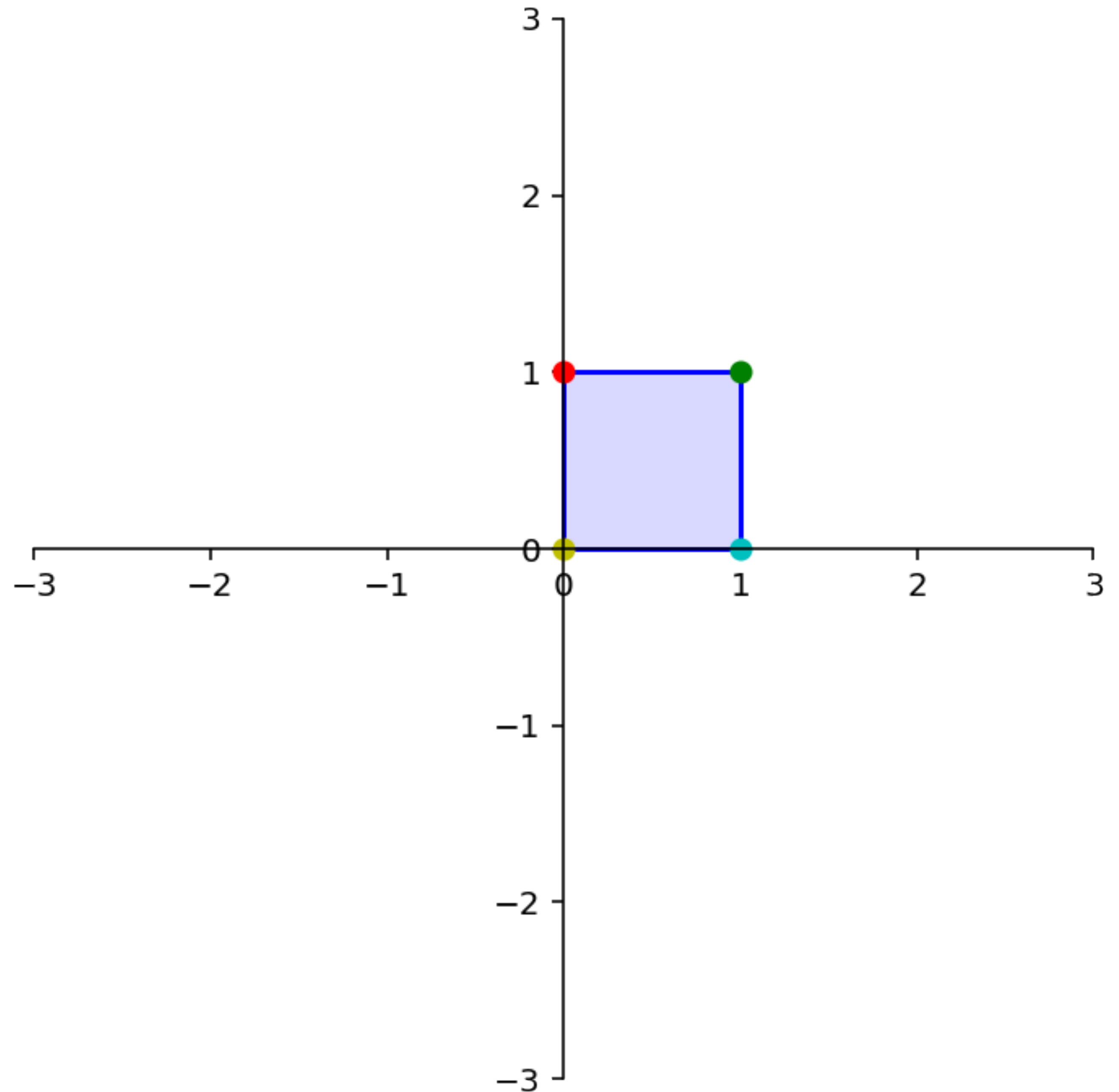


# 3D Example: Rotation about the $x_3$ -Axis ( $z$ -Axis)



# The Unit Square

The *unit square* is the enclosed by the points



# How To: The Unit Square and Matrices



# How To: The Unit Square and Matrices

**Question.** Find the matrix which implements the linear transformation which is represented geometrically in the following picture

# How To: The Unit Square and Matrices

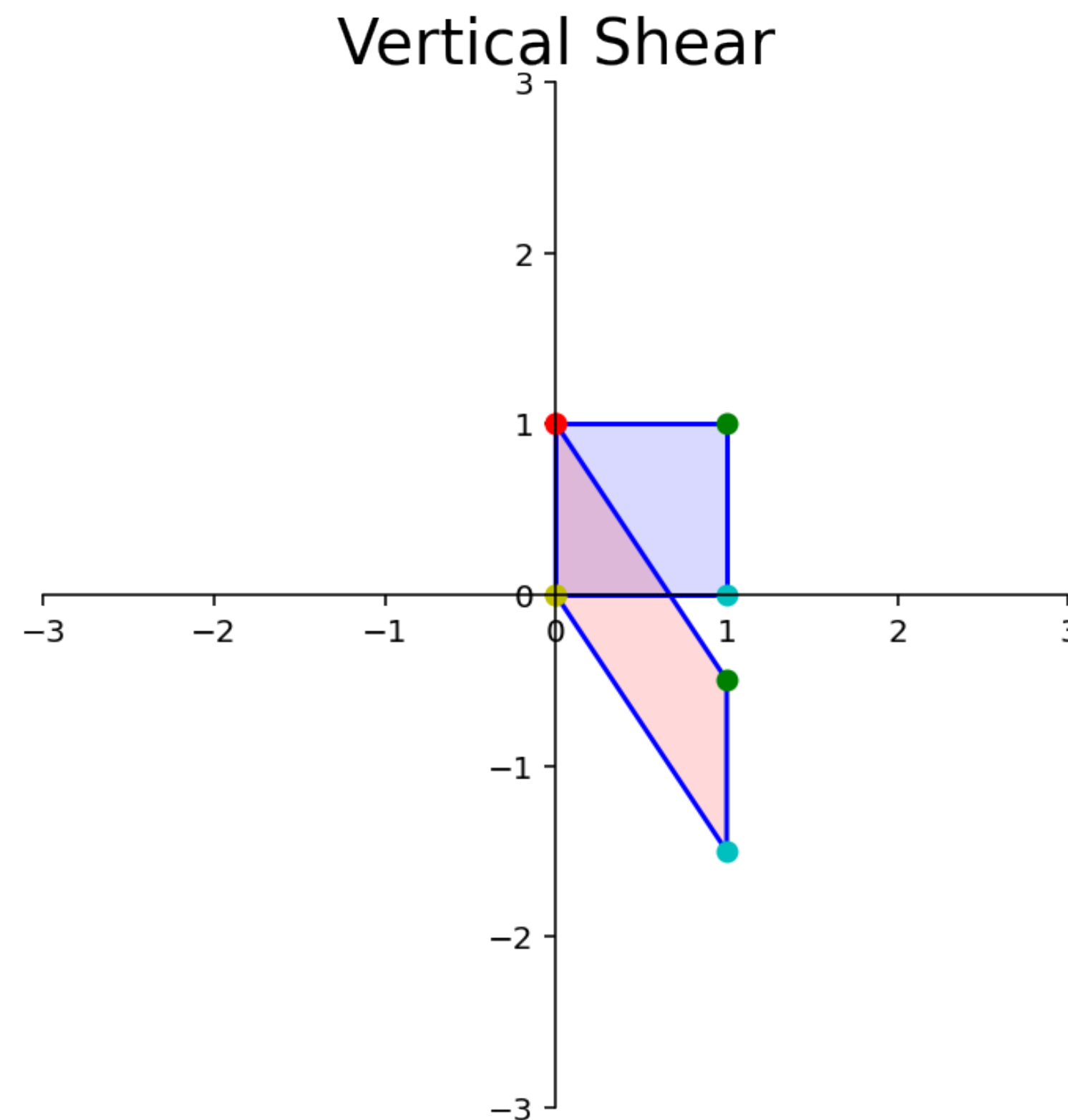
**Question.** Find the matrix which implements the linear transformation which is represented geometrically in the following picture

**Solution.** Find where the standard basis vectors go

# Example

Write down the matrix for the following shearing operation using this method

Exercise:



You need to know these matrices, but you don't need to  
memorize them

Remember: What does this matrix do to the unit square?  
Then build the matrix from there

# List of Important 2D Linear Transformations

- » dilation, contraction
- » reflections
- » projections
- » horizontal/vertical contractions
- » horizontal/vertical shearing

Look through the notes for a comprehensive  
collection of pictures or... *(demo)*

# One-to-One and Onto

# Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

# Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$   $\equiv$  is there a vector which  $A$   
transforms into  $\mathbf{b}$ ?

Solve  $A\mathbf{x} = \mathbf{b}$   $\equiv$  find a vector which  $A$   
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What about other questions?

# Other Questions Like...

Does  $A\mathbf{x} = \mathbf{b}$  have a solution for any choice of  $\mathbf{b}$ ?

Does  $A\mathbf{x} = \mathbf{0}$  have a unique solution?

# Other Questions Like...

Does  $A\mathbf{x} = \mathbf{b}$  have at least one solution for any choice of  $\mathbf{b}$ ?

Does  $A\mathbf{x} = \mathbf{b}$  have at most one solution for any choice of  $\mathbf{b}$ ?

# Wait

$A\mathbf{x} = \mathbf{0}$  has a  
unique solution

$\equiv$

$A\mathbf{x} = \mathbf{b}$  has at most one  
solution

why?:

$$A\vec{v}_1 = \vec{b}$$

$$A\vec{v}_2 = \vec{b}$$

$$\vec{v}_1 \neq \vec{v}_2$$

$$A(\vec{v}_1 - \vec{v}_2) = A\vec{v}_1 - A\vec{v}_2 = \vec{b} - \vec{b} = \vec{0}$$

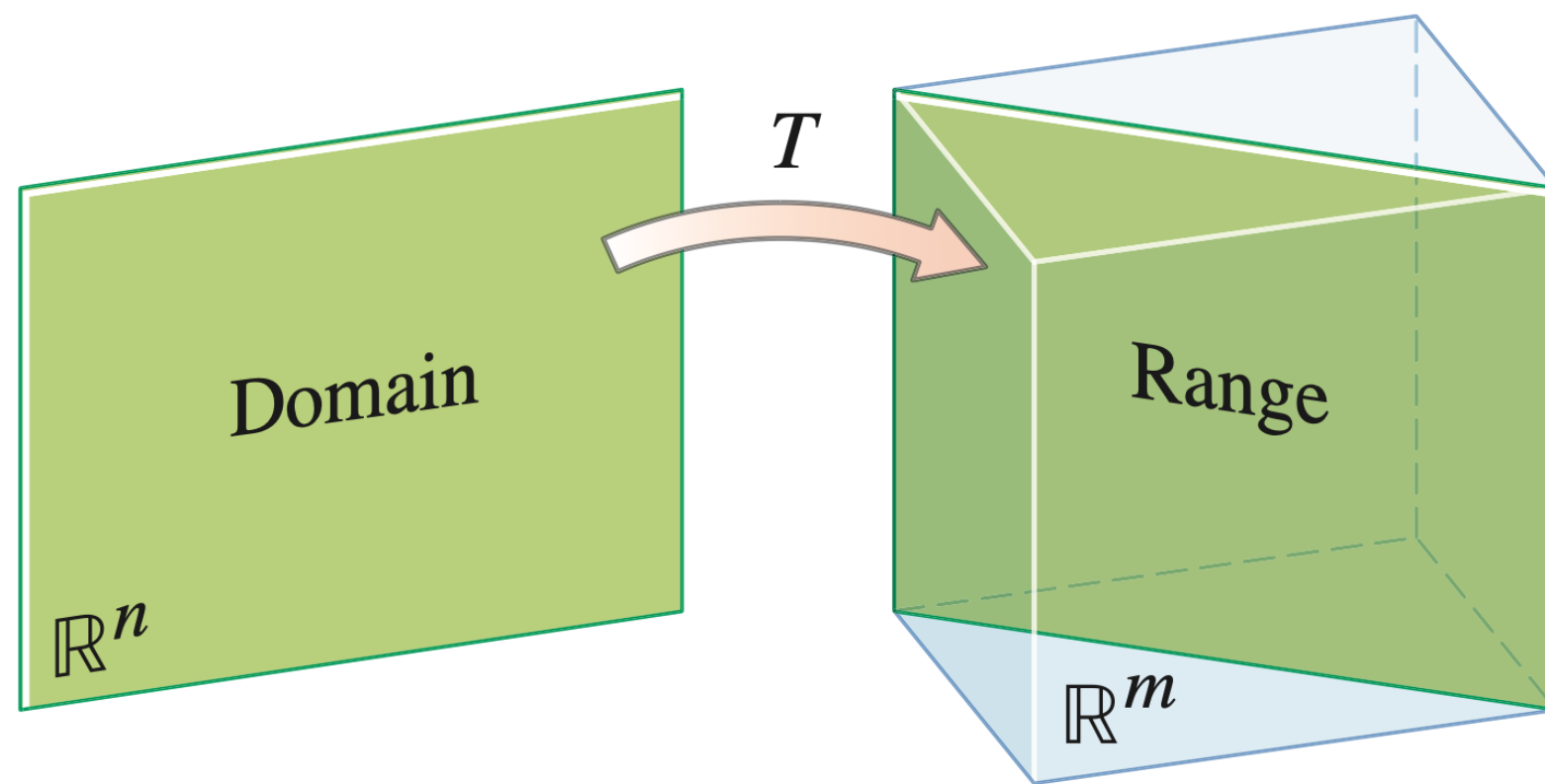
# Onto Transformations

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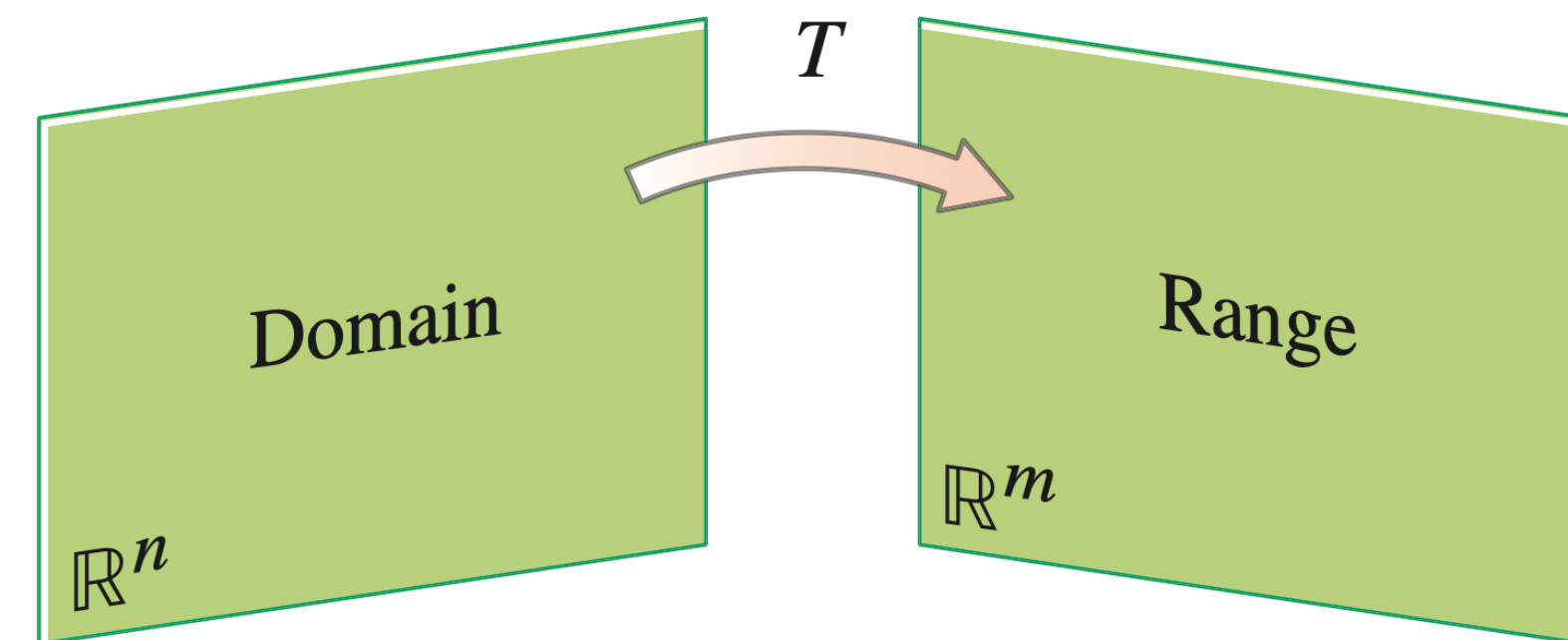
**Definition.** A transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is ***onto*** if any vector  $\mathbf{b}$  in  $\mathbb{R}^m$  is the **image of at least one vector**  $\mathbf{v}$  in  $\mathbb{R}^n$  (where  $T(\mathbf{v}) = \mathbf{b}$ )

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$T$  is *not* onto  $\mathbb{R}^m$

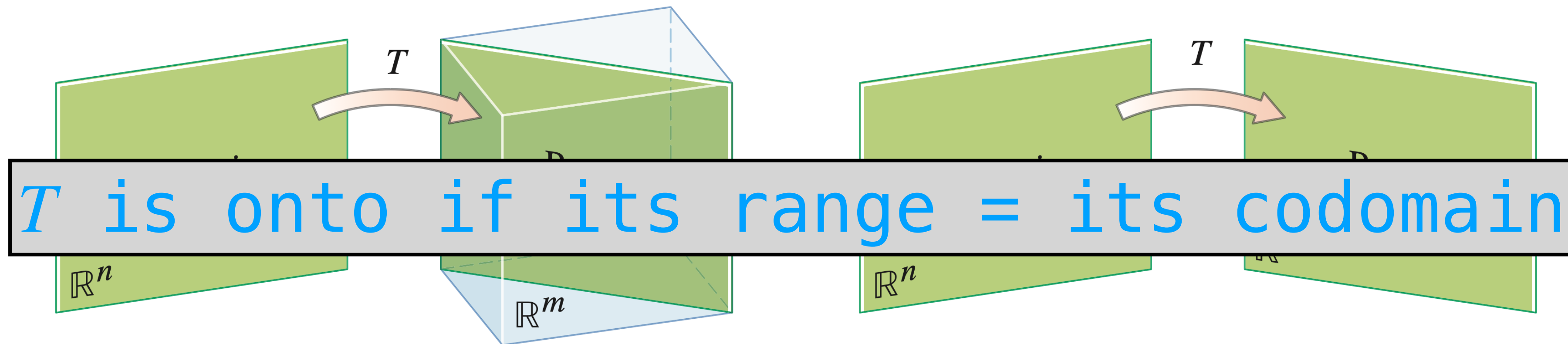


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# Onto Transformations

(surjective)

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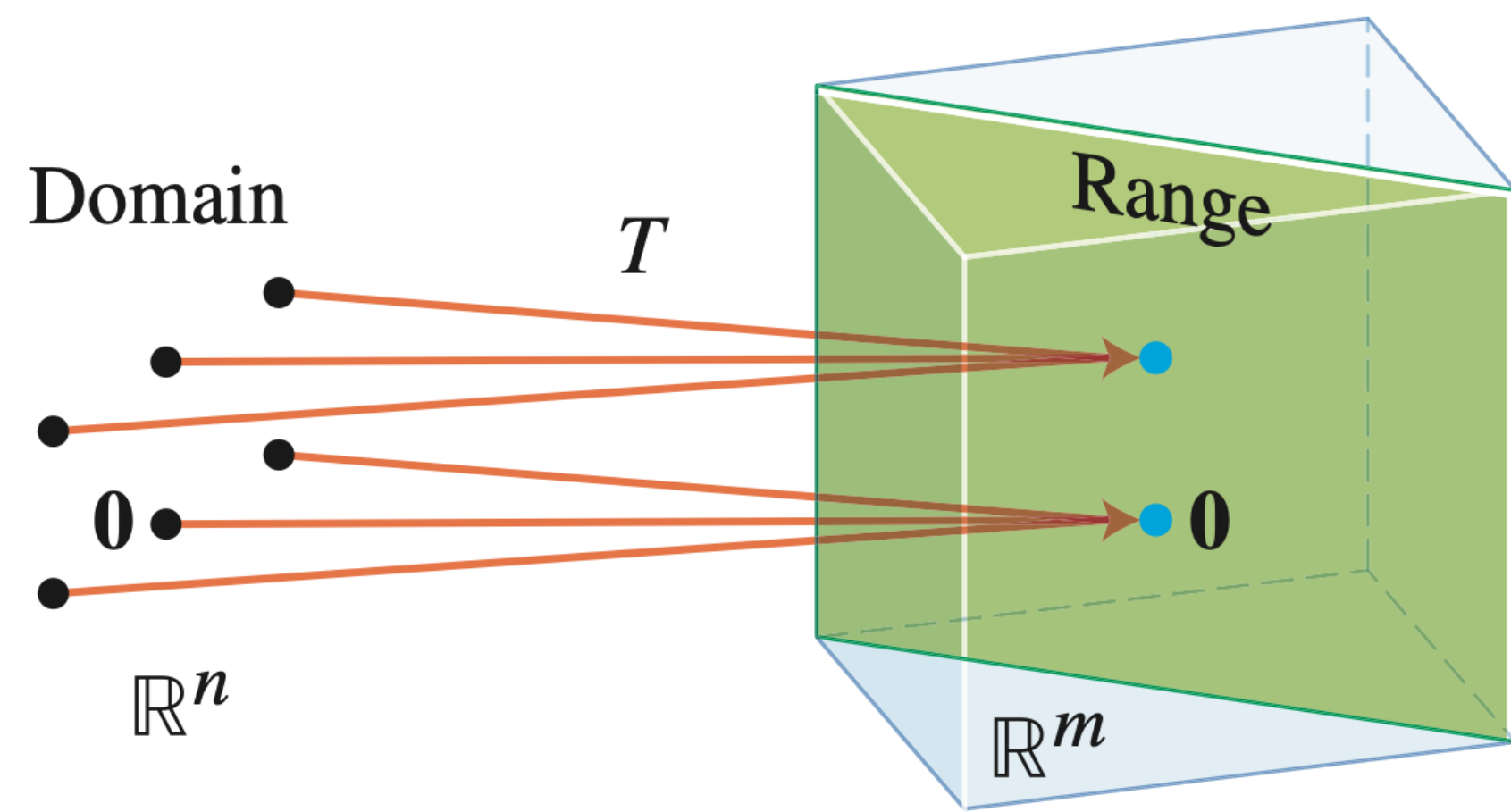
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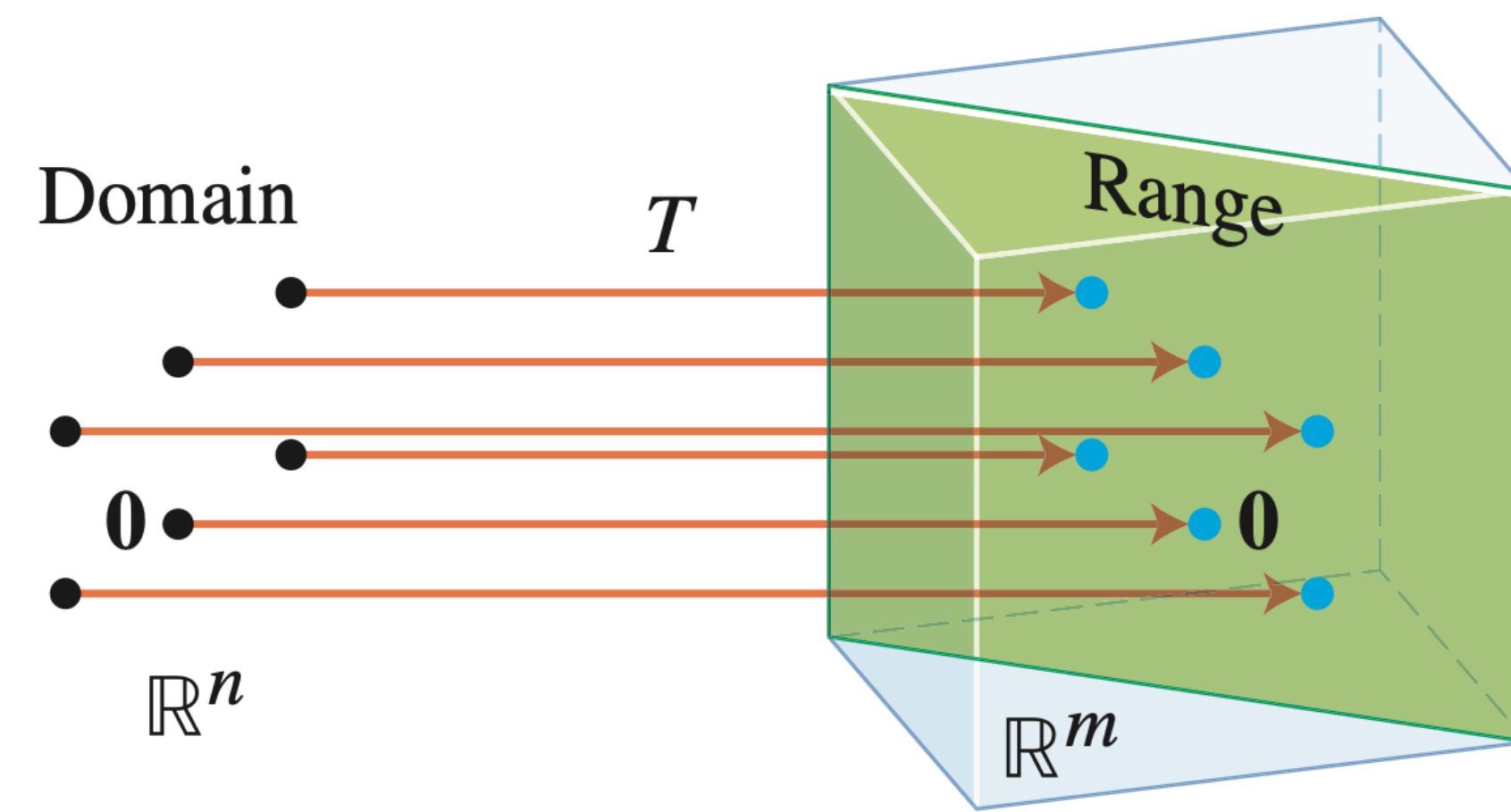
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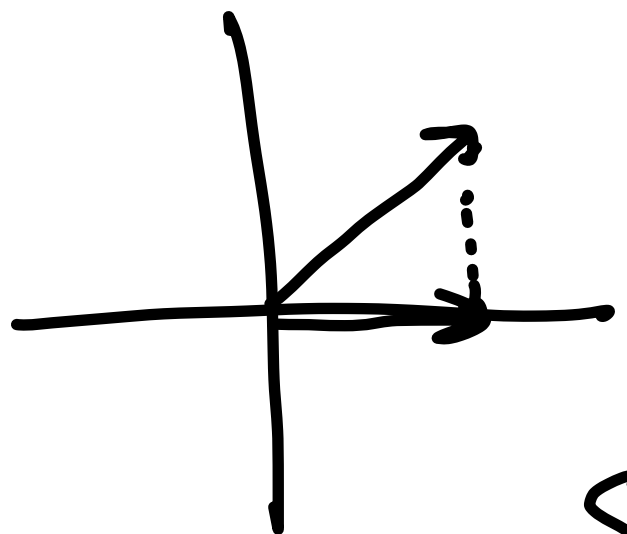


$T$  is *not* one-to-one

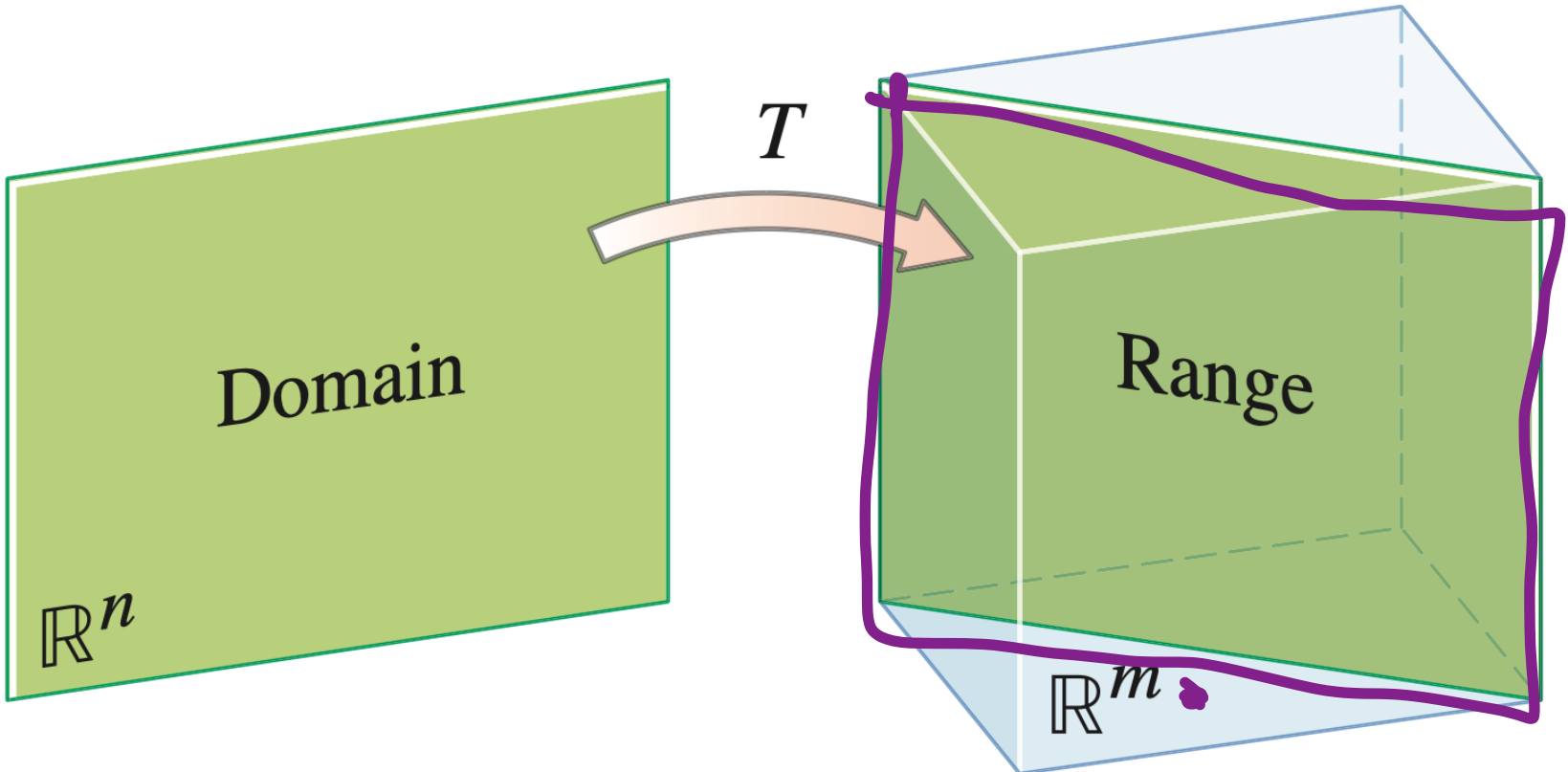


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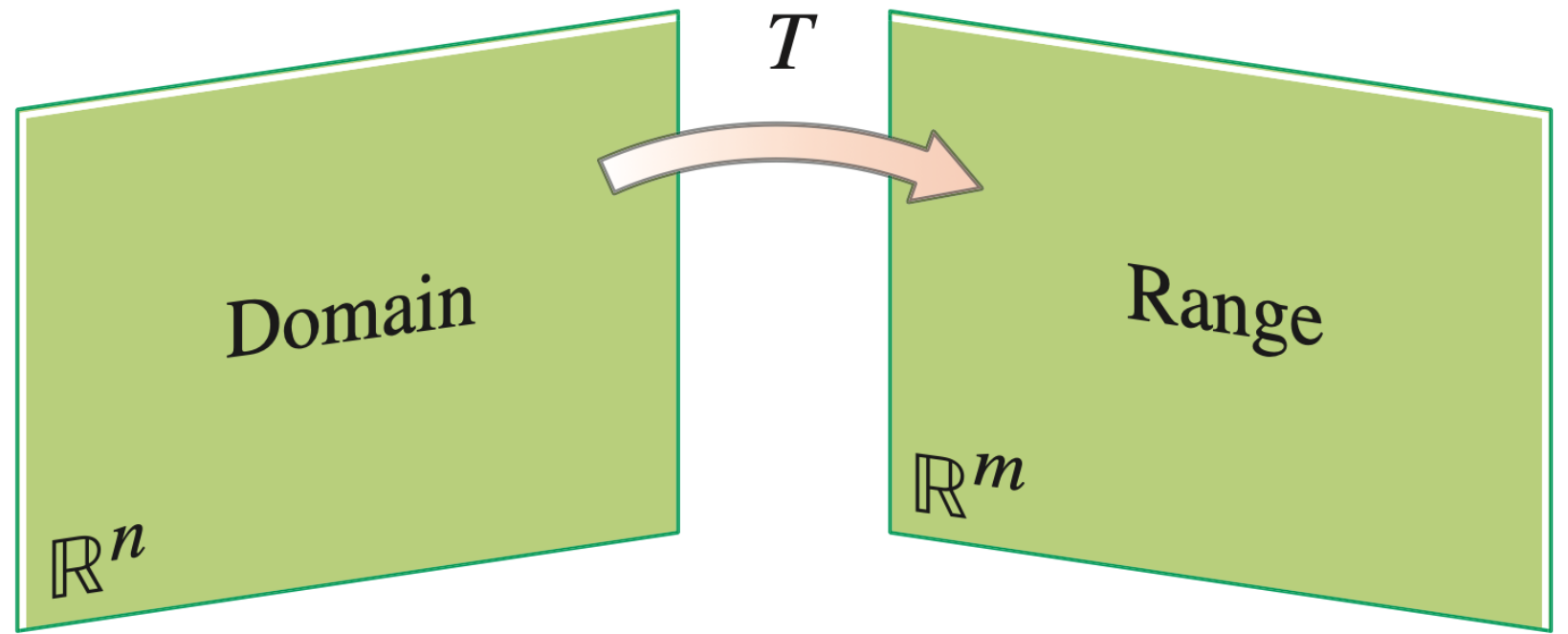
# Comparing Pictures



$$\text{ran}(T) = \{T(\vec{v}) : v \in \text{dom}(T)\}$$

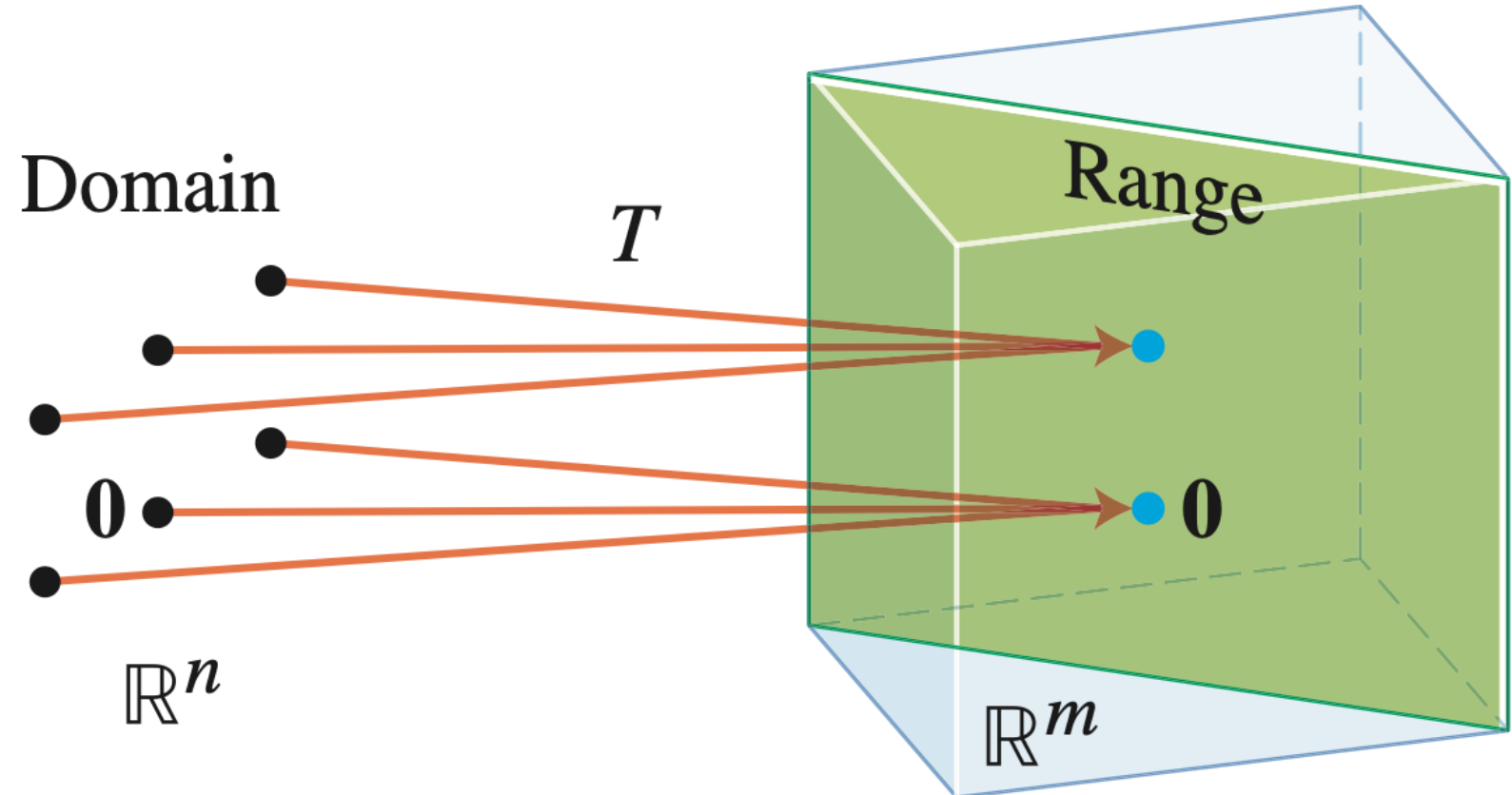


$T$  is not onto  $\mathbb{R}^m$

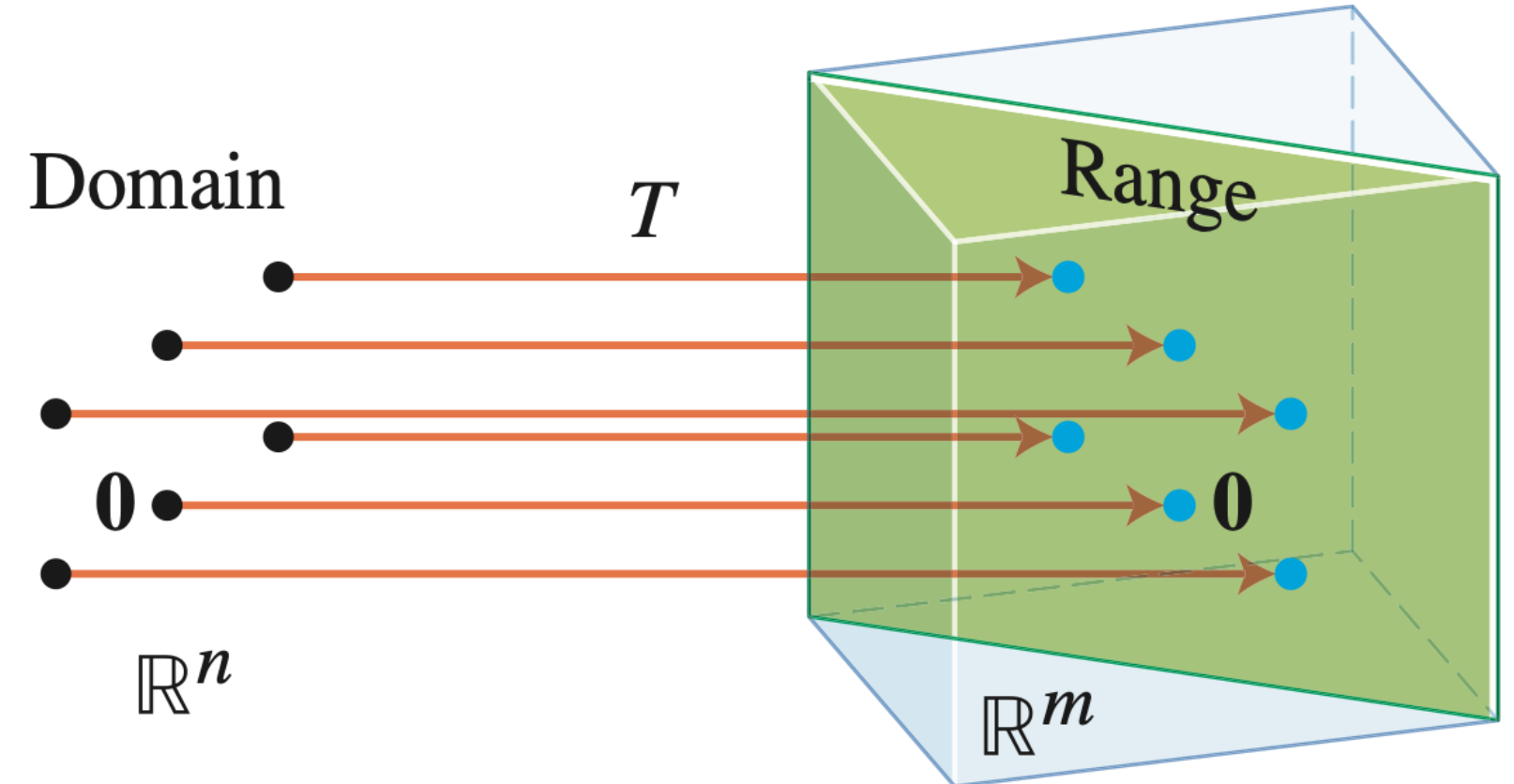


$T$  is onto  $\mathbb{R}^m$

$$\mathbb{R}^m$$



$T$  is not one-to-one



$T$  is one-to-one

$$\text{range}(T) = \{ \underbrace{T(\vec{v})}_{\substack{\text{the} \\ \text{set of all} \\ \text{output of} \\ T \text{ on } \vec{v}}} : \underbrace{\vec{v} \in \text{dom}(T)}_{\substack{\vec{v} \text{ is in the} \\ \text{domain of } T}} \}$$

the  
set of all  
output of  
 $T$  on  $\vec{v}$

$\vec{v}$  is in the  
domain of  $T$

such that

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \in \text{ran}(T)$$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \in \text{ran}(T)$$

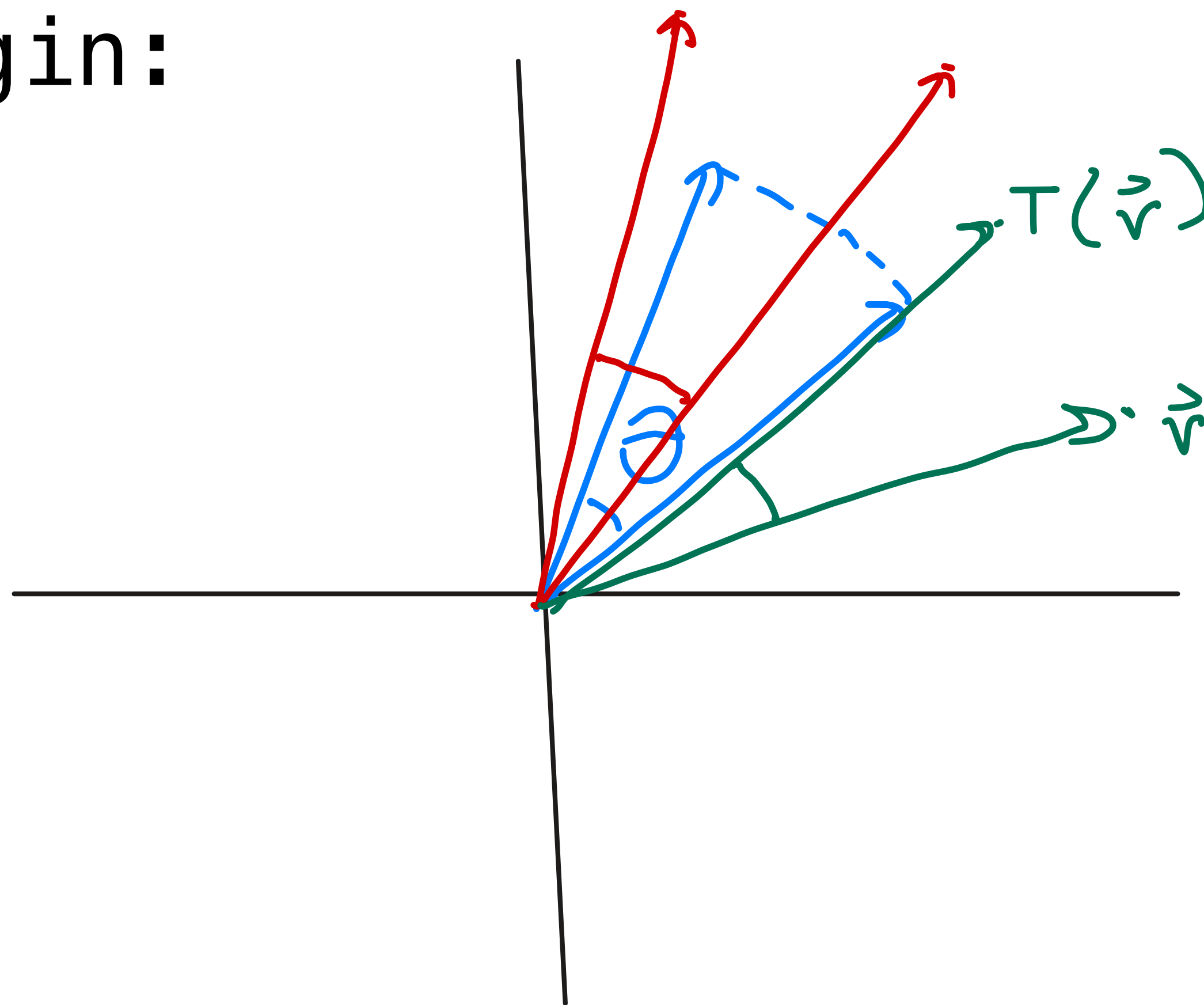
set of all  
images of  $T$

# Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :



# Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

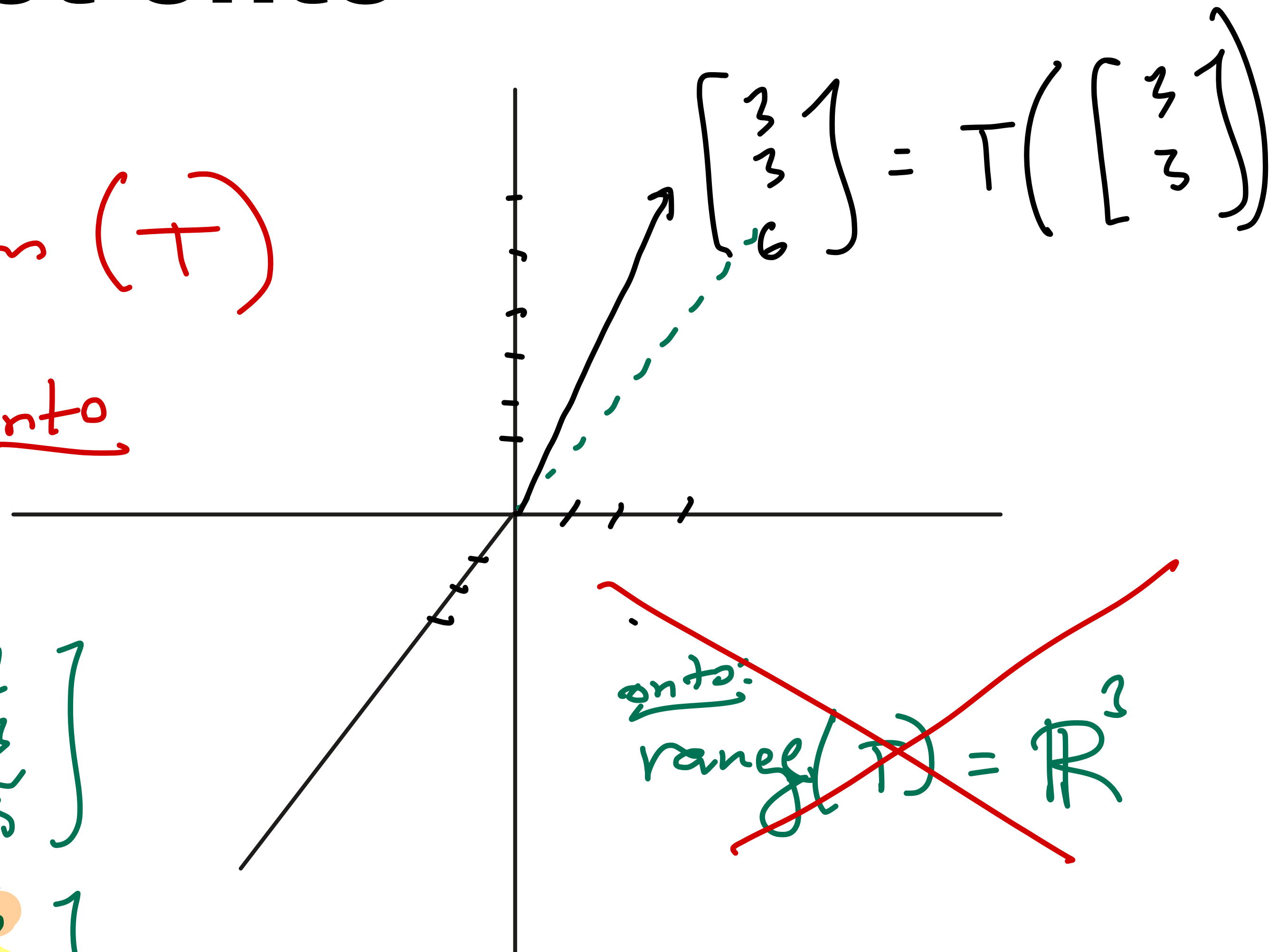
why? :

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \notin \text{ran}(T)$$

not onto

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

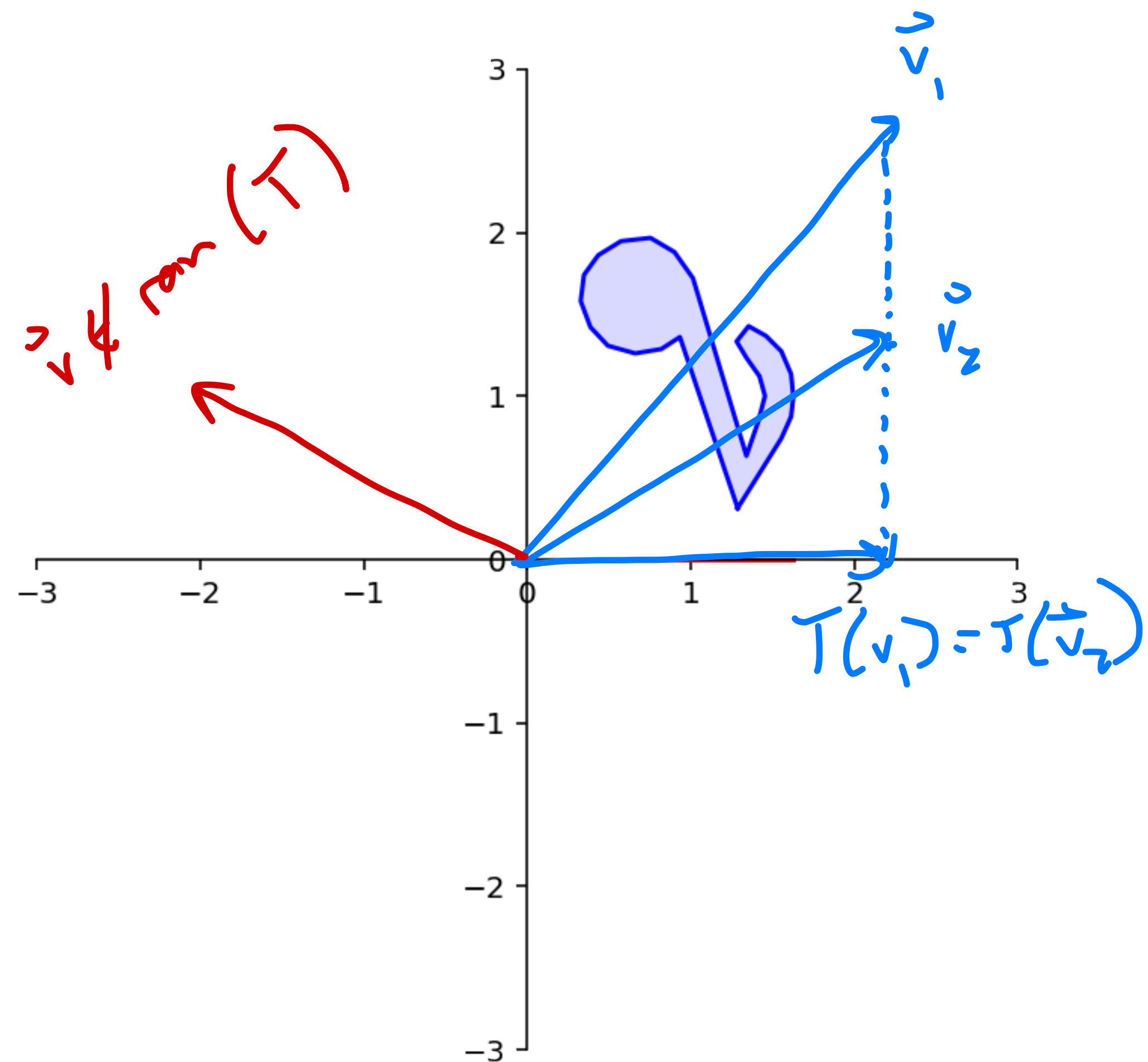


# Example: not 1-1, not onto

Projection onto the  $x_1$  axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :





# Example: onto, not 1-1

Projection from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

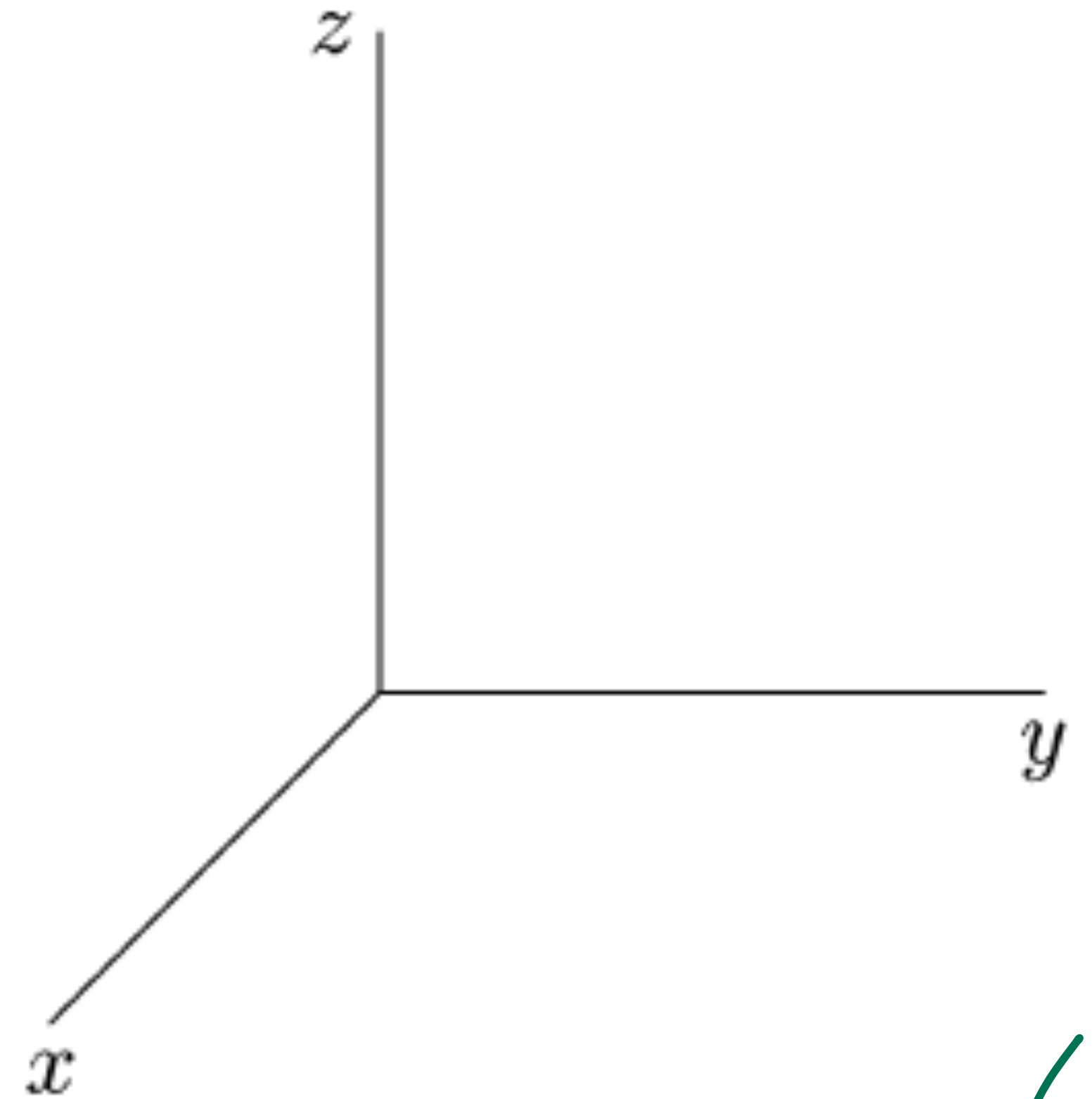
$$T\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

onto ✓

why? :

$$T\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \neq \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

not 1-1 ✓



# Taking Stock: Onto

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**Theorem.** The following are logically equivalent for the linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  implemented by the matrix  $A$

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- » the columns of  $A$  span  $\mathbb{R}^m$  *codomain*
- »  $A$  has a pivot position in every row



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- »  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution

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- »  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution
- » The columns of  $A$  are linearly independent
- »  $A$  has a pivot position in every column

# How To: One-to-One and Onto

**Question.** Show that the linear transformation  $T$  is one-to-one/onto

**Solution.** (one approach) Find the matrix which implements  $T$  and see if it has a pivot in every column/row

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives



# Example: both 1-1 and onto

Rotation about the origin:

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why? :

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Lifting:

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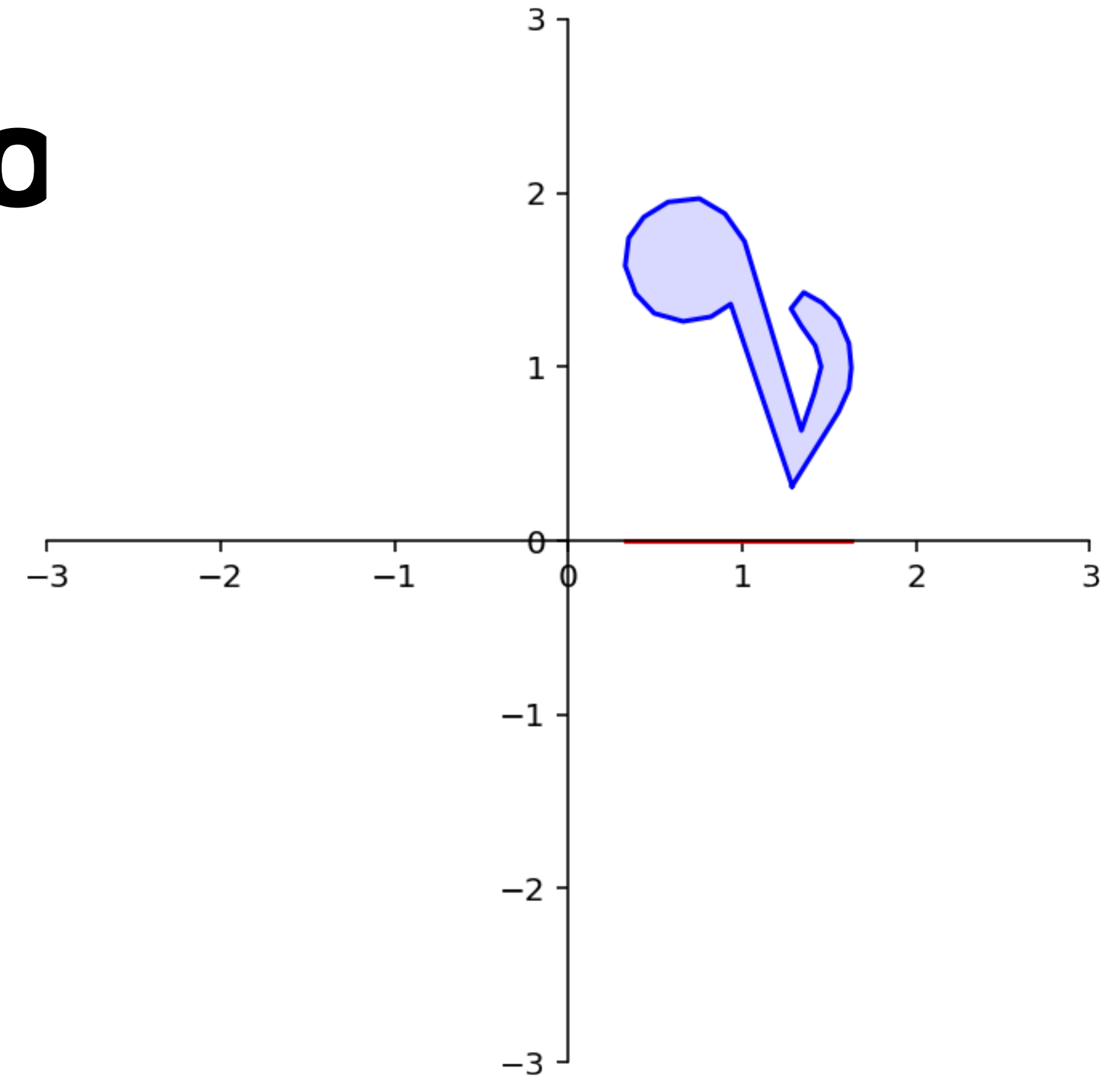
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

# Example: not 1-1, not o

Projection onto the  $x_1$  axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

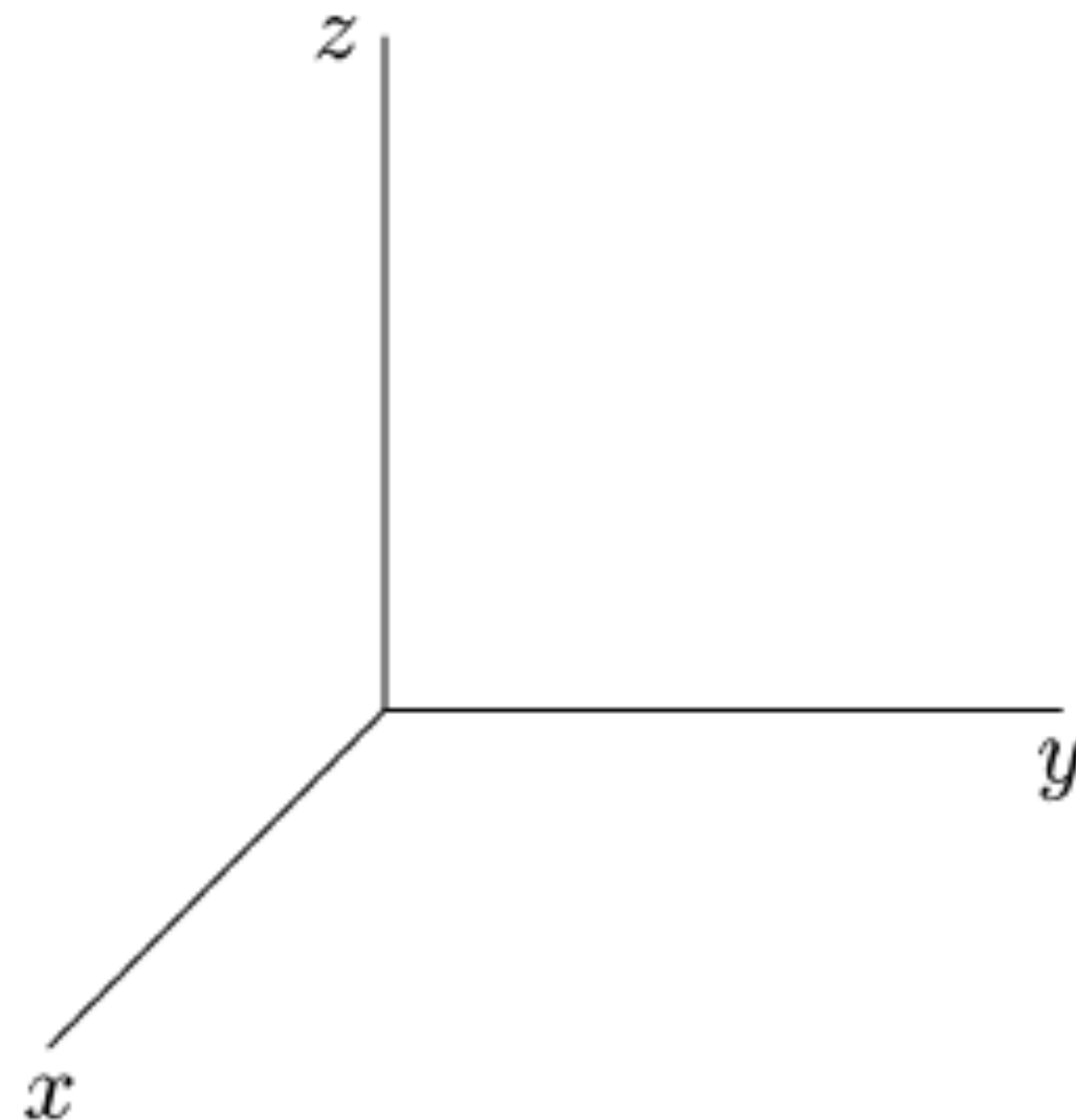


# Example: onto, not 1-1

Projection from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

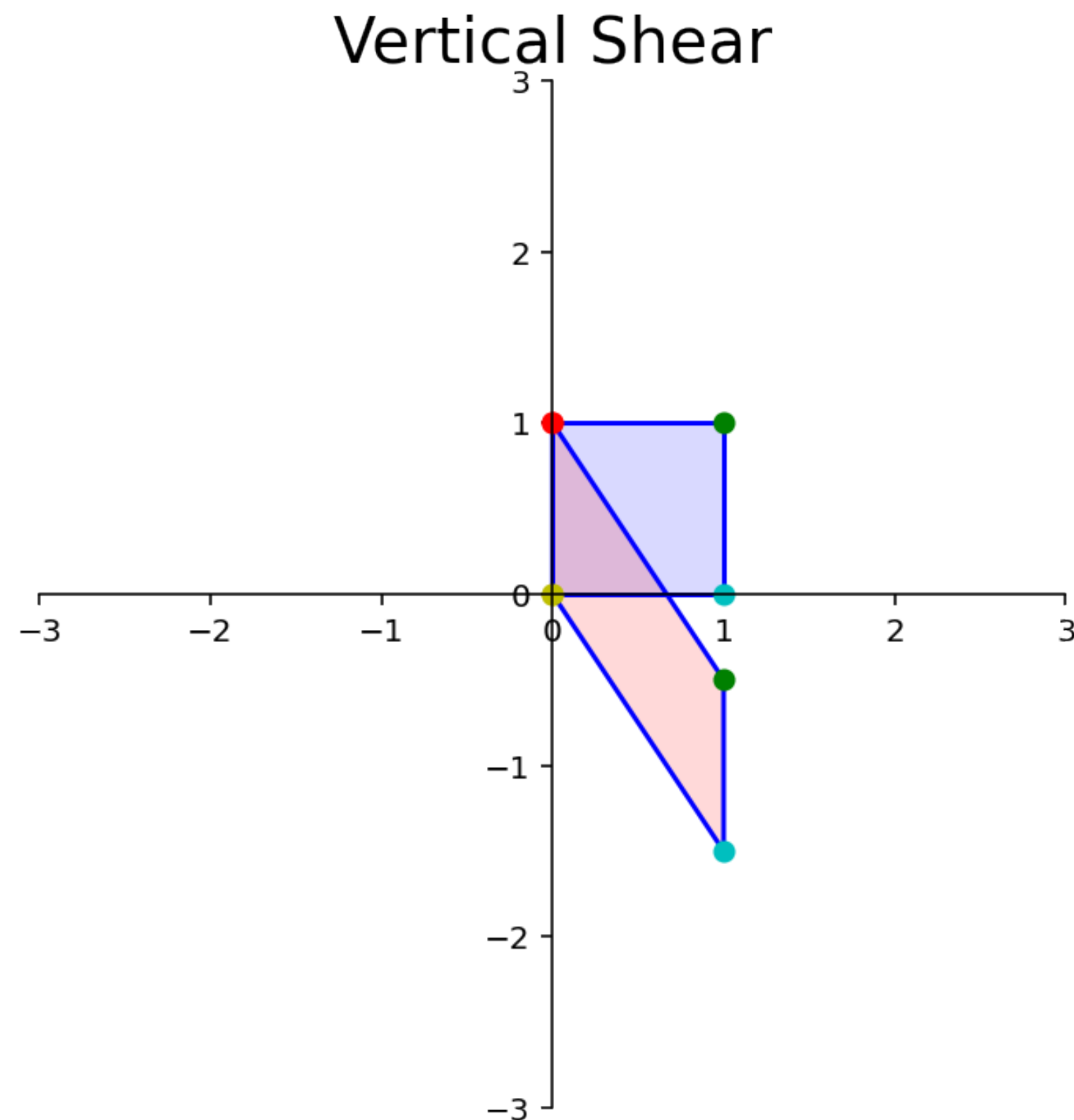


why? :

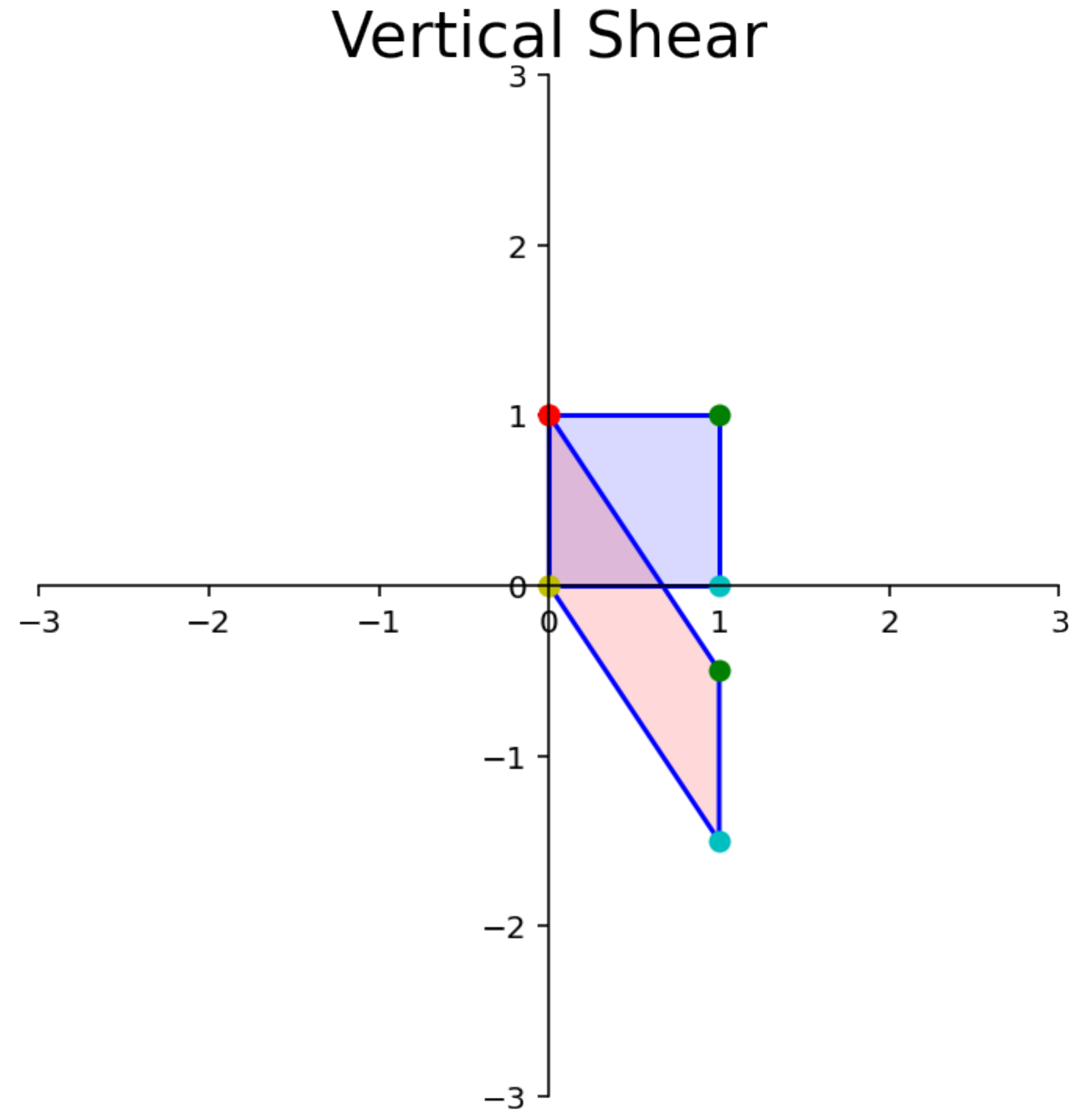
# Question

Is vertical shearing a 1-1 transformation?  
Justify your answer

Example:

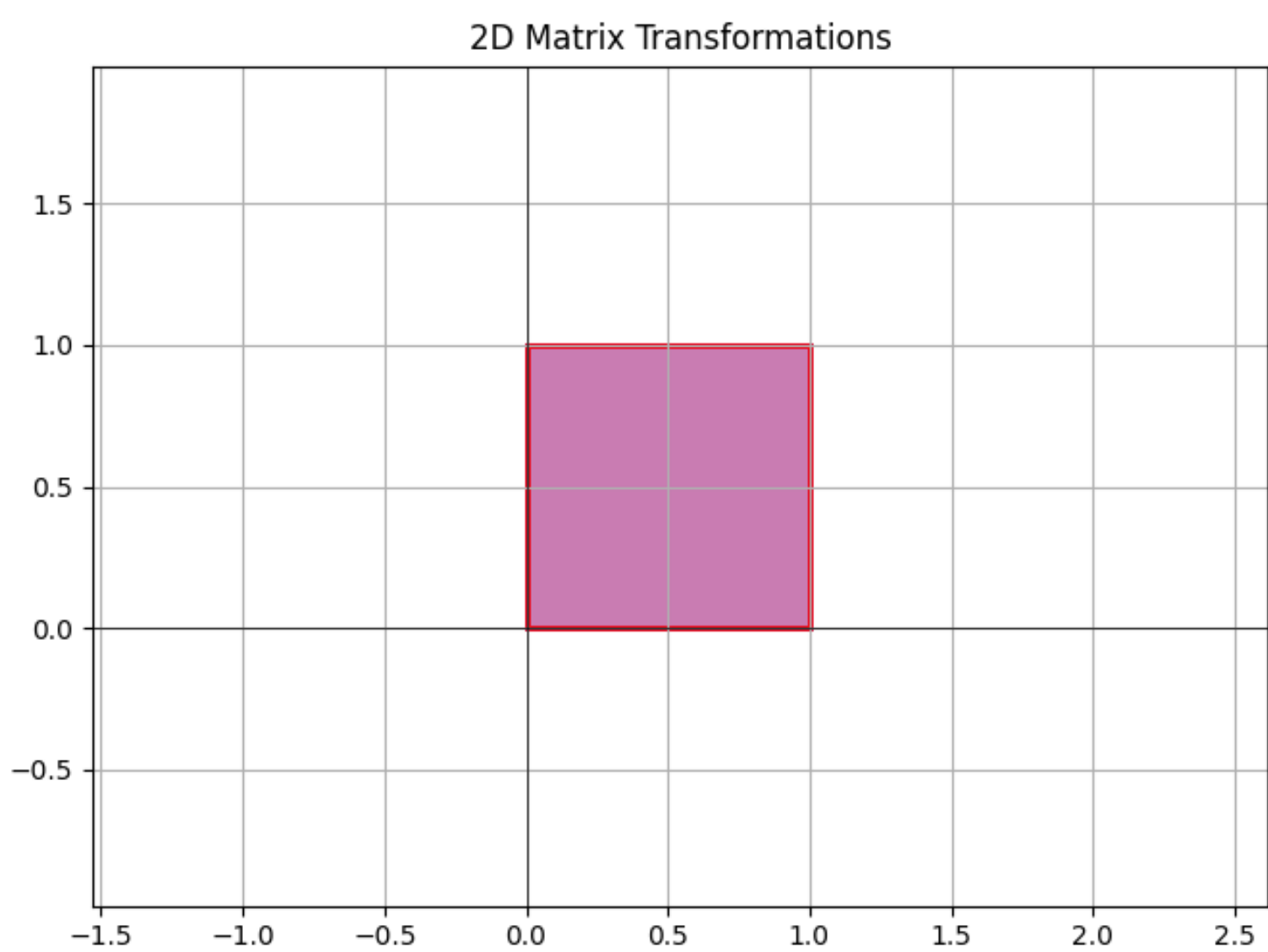


**Answer: Yes**

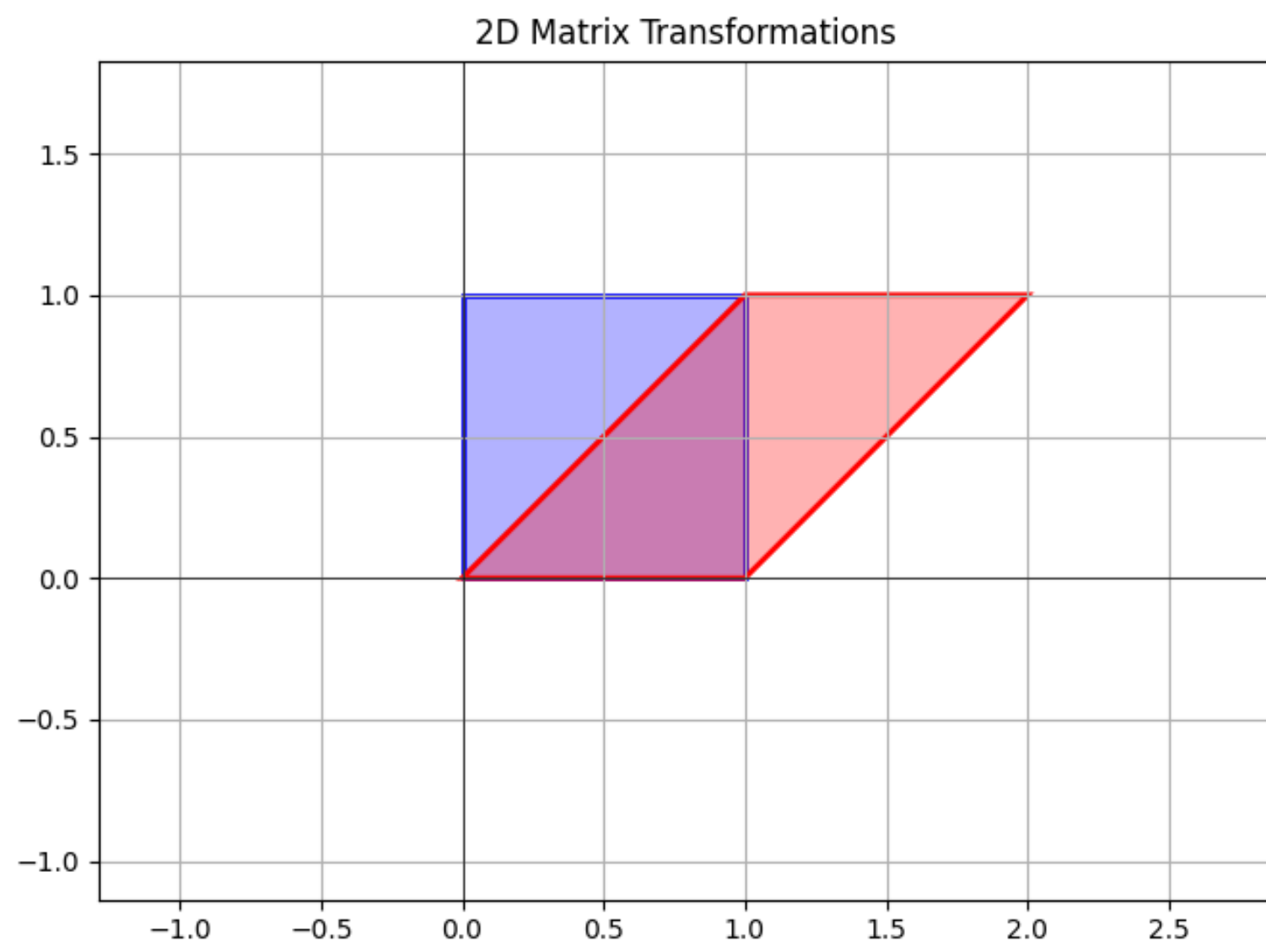


# Composing Linear Transformations

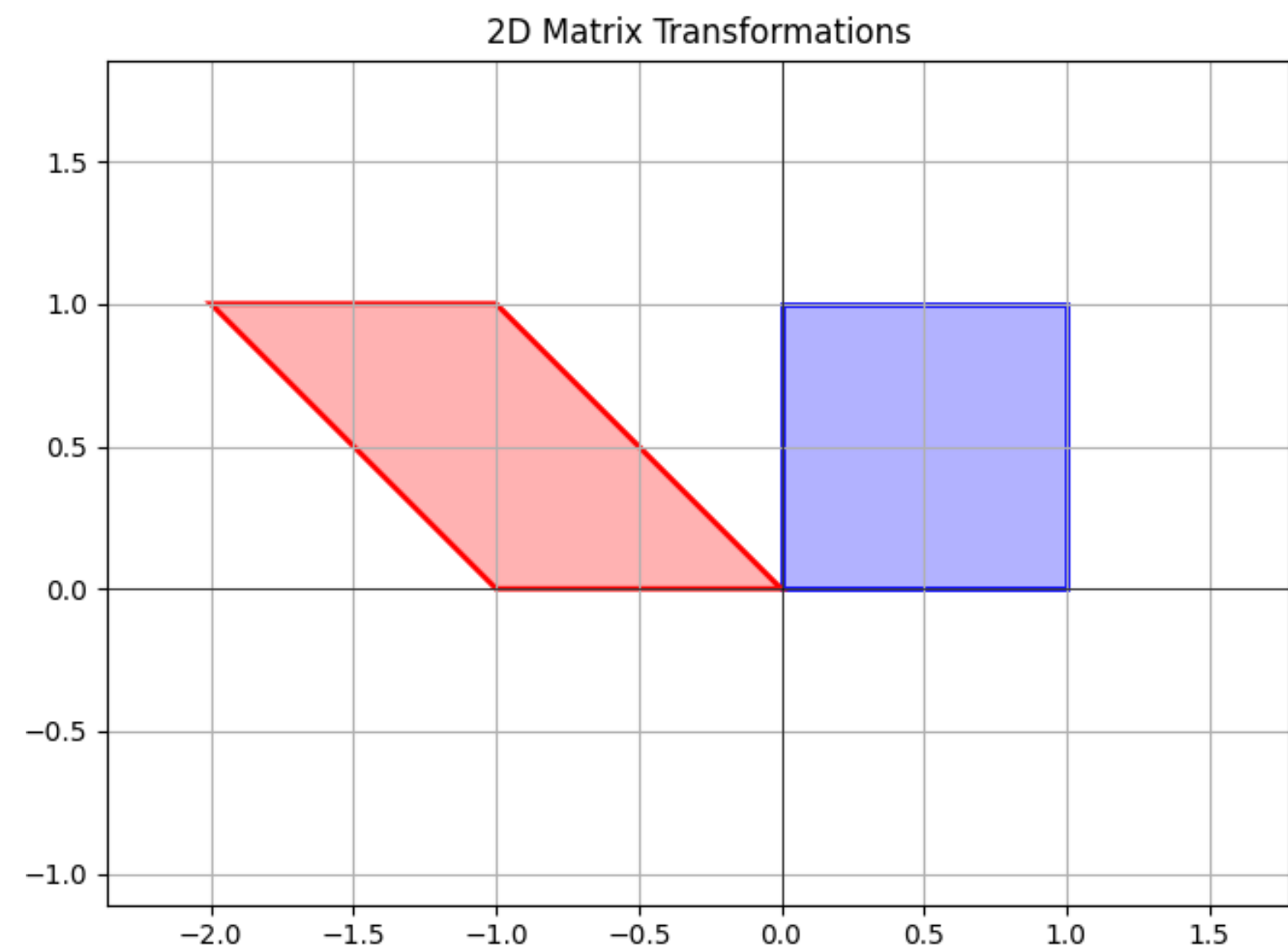
# Shearing and Reflecting (Geometrically)



shear



reflect



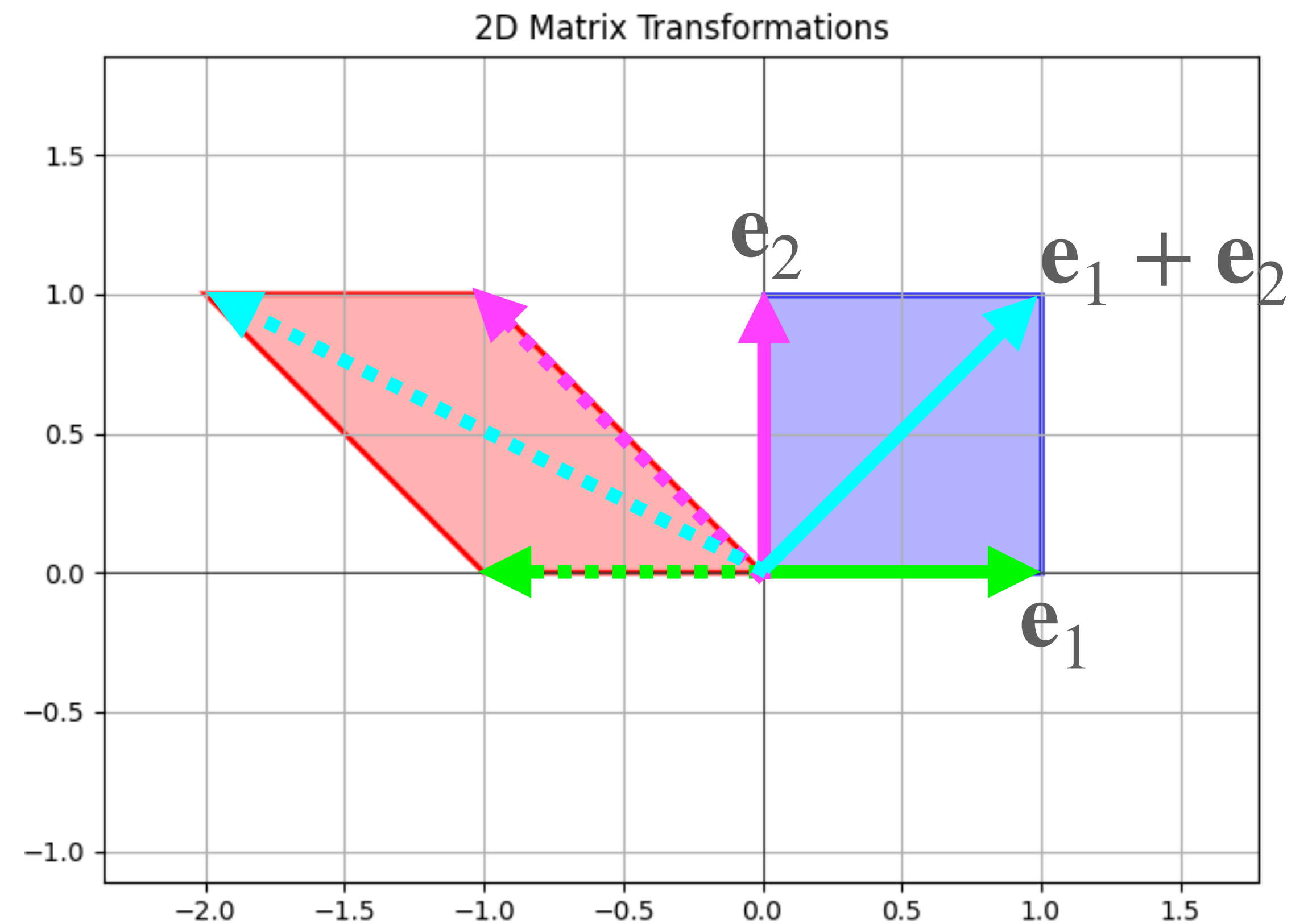


# Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



# Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect                  shear

First multiply by shear matrix, then multiply  
by reflection matrix

# Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply  
by reflection matrix

This gives us the same transformation

# Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

# The Key Fact

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**Fact.** The composition of two linear transformations is a linear transformation

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Verify:

# The Key Fact

**Fact.** The composition of two linear transformations is a linear transformation

Verify:

This means the composition of two matrix transformations can be represented as a *single* matrix



# The Key Question

*Given two linear transformations,  
how to we compute the matrix which  
implements their composition?*

# The Key Question

*Given two linear transformations,  
how to we compute the matrix which  
implements their composition?*

Matrix Multiplication

# Matrix Multiplication

# Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

# General Composition (2D)

$$A \left( \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

# Matrix Multiplication

**Definition.** For a  $m \times n$  matrix  $A$  and a  $n \times p$  matrix  $B$  with columns  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$  the product  $AB$  is the  $m \times p$  matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

*Replace each column of  $B$  with  $A$  multiplied by that column*

# Tracking Dimensions

This only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices arranged in a sequence separated by an equals sign. The first matrix is a 5x3 matrix, represented by a blue vertical line on the left labeled  $m$  and a red horizontal line on top labeled  $n$ . The second matrix is a 3x4 matrix, represented by a red vertical line on the left labeled  $n$  and a purple horizontal line on top labeled  $k$ . The third matrix is a 5x4 matrix, represented by a blue vertical line on the left labeled  $m$  and a purple horizontal line on top labeled  $k$ . Each matrix contains asterisks representing elements. Below each matrix, its dimensions are written in a colored box:  $(m \times n)$  for the first,  $(n \times k)$  for the second, and  $(m \times k)$  for the third.

$$\begin{matrix} m \\ \left[ \begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{array} \right] \end{matrix} \begin{matrix} n \\ \left[ \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix} = \begin{matrix} m \\ \left[ \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix}$$

$(m \times n)$        $(n \times k)$        $(m \times k)$

# Important Note

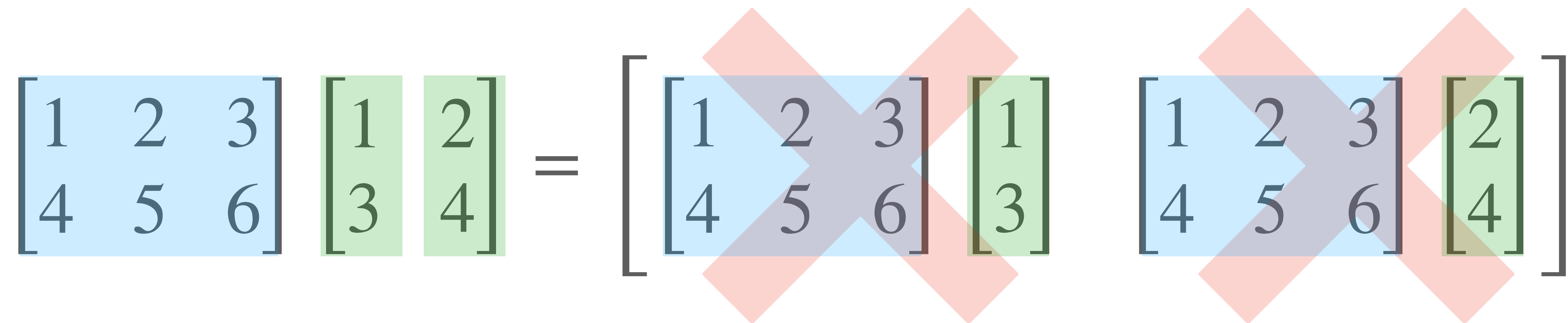
Even if  $AB$  is defined, it may be that  $BA$  is not defined



# Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

# Non-Example



The diagram illustrates a non-example of matrix multiplication. On the left, a 2x3 matrix (blue) is multiplied by a 2x2 matrix (green). This is followed by an equals sign and a large bracketed expression. Inside the bracket, two possible products are shown, each crossed out with a large red 'X'. The first crossed-out product shows the 2x3 matrix multiplied by the first column of the 2x2 matrix (a 2x1 vector). The second crossed-out product shows the 2x3 matrix multiplied by the second column of the 2x2 matrix (a 2x1 vector). The red 'X' marks indicate that these operations are not defined because the number of columns in the first matrix (3) does not match the number of rows in the second matrix (2).

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

These are not defined.

# Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

# The Key Fact (Restated)

For any matrices  $A$  and  $B$  (such that  $AB$  is defined) and any vector  $\mathbf{v}$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

*The matrix implementing the composition is the product of the two underlying matrices*

# Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a  $m \times n$  matrix  $A$  and a  $n \times p$  matrix  $B$ , the entry in row  $i$  and column  $j$  of  $AB$  is defined above

# Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

# Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices: a 5x3 matrix  $A$ , a 3x4 matrix  $B$ , and their product  $C$ , which is a 5x4 matrix. The first row of  $A$  is highlighted in light blue, the first column of  $B$  is highlighted in light red, and the first row of  $C$  is highlighted in light purple. The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix  $A$  (left) with its first row highlighted in light blue.
- A 3x4 matrix  $B$  (middle) with its second column highlighted in light red.
- The resulting 5x4 matrix  $AB$  (right) with its first row and second column highlighted in light purple.

The matrices are separated by an equals sign, indicating the multiplication operation.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$



# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its third column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the first row and third column is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the third column of the second matrix.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element in the first row and fourth column is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the fourth column of the second matrix.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (\*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks. The third matrix is a 5x4 matrix, also with all elements represented by asterisks. An equals sign (=) is placed between the second and third matrices. The first matrix has its second row highlighted in light blue. The second matrix has its first column highlighted in light red. The third matrix has its first row highlighted in light purple. This highlights the specific row and column used in the calculation of a single element in the product matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (\*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks. The third matrix is a 5x4 matrix, also with all elements represented by asterisks. An equals sign (=) is placed between the second and third matrices. The first matrix has its second row highlighted in light blue. The second matrix has its second column highlighted in light red. The third matrix has its second column highlighted in light purple. This visualizes the calculation of the element in the second row and second column of the product matrix AB, which is the dot product of the second row of A and the second column of B.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements marked with asterisks (\*). The second matrix is a 3x4 matrix, also with all elements marked with asterisks. The third matrix is a 5x4 matrix, also with all elements marked with asterisks. An equals sign (=) is placed between the second and third matrices. The first matrix has its second row highlighted in light blue. The second matrix has its third column highlighted in light red. The third matrix has its third column highlighted in light purple, indicating the result of the dot product of the second row of the first matrix and the third column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all cells containing an asterisk (\*). The second matrix is a 3x4 matrix, also with all cells containing an asterisk (\*). The third matrix is a 5x4 matrix with all cells containing an asterisk (\*). The second matrix is positioned between the first and third matrices, with an equals sign (=) to its right. The first matrix has its second row highlighted in light blue. The second matrix has its fourth column highlighted in light red. The third matrix has its fourth column highlighted in light purple. This visualizes the calculation of the element in the second row and fourth column of the product matrix, which is the dot product of the second row of the first matrix and the fourth column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; its third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in each cell; its first column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in each cell; its first row is highlighted in light purple. This visualizes the calculation of the element in the third row and first column of the product matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (\*). The second matrix is a 3x4 matrix, also with all elements as asterisks, but its second column is highlighted with a light red background. An equals sign (=) follows. The third matrix is a 5x4 matrix with all elements as asterisks, but its second column is highlighted with a light purple background. This visualizes the calculation of a single element in the product matrix by taking the dot product of a row from the first matrix and a column from the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$



# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in each cell; the third column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in each cell; the element in the third row and third column is highlighted in light purple, representing the dot product of the third row of the first matrix and the third column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix of asterisks with its third row highlighted in light blue. The second matrix is a 3x4 matrix of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 matrix of asterisks with its second row and fourth column highlighted in light purple, representing the resulting element.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its fourth row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its first column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks with its fourth row and first column highlighted in light purple, representing the resulting matrix element.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix  $A$  with elements represented by asterisks. The fourth row is highlighted in light blue.
- A 3x4 matrix  $B$  with elements represented by asterisks. The second column is highlighted in light red.
- An equals sign followed by a 5x4 matrix  $C$ , which is the result of the multiplication. The element at the intersection of the fourth row and second column of  $C$  is highlighted in light purple, representing the dot product of the fourth row of  $A$  and the second column of  $B$ .

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with a light blue horizontal band highlighting its fourth row. The second matrix is a 4x4 matrix with a light red vertical band highlighting its fourth column. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the fourth row and fourth column is highlighted with a light purple square, representing the result of the dot product of the highlighted row and column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted with a light blue background. The second matrix is a 3x4 matrix with asterisks; the first column is highlighted with a light red background. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the bottom-left cell is highlighted with a light purple background, representing the dot product of the highlighted row and column from the first two matrices.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the second column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the bottom row and second column is highlighted in light purple, representing the result of the dot product of the first row and second column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$



# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the third column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the bottom row and third column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in all cells; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in all cells; the fourth column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in all cells; the bottom-right element is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

# Question

Compute  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

**Answer**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

# Matrix Operations

# Connection with Matrix-Vector Multiplication

# Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

# Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$



# Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

**This is just vector multiplication**

# Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

**This is just vector multiplication**

We can think of  $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$  as collection of simultaneous matrix-vector multiplications

# Matrix "Interface"

multiplication

what does  $AB$  mean when  $A$  and  $B$  are matrices?

addition

what does  $A + B$  mean when  $A$  and  $B$  are matrices?

scaling

what does  $cA$  mean when  $A$  is matrix and  $c$  is a real number?

# Matrix "Interface"

multiplication

what does  $AB$  mean when  $A$  and  $B$  are matrices?

addition

what does  $A + B$  mean when  $A$  and  $B$  are matrices?

scaling

what does  $cA$  mean when  $A$  is matrix and  $c$  is a real number?

These should be consistent with matrix-vector interface and vector interface

# Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

# Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

**This is exactly the same as vector addition, but for matrices**

# Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

# Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

**This is exactly the same as vector scaling, but for matrices**



# Algebraic Properties (Addition and Scaling)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

In these properties  $A$ ,  $B$ , and  $C$  are matrices of the same size and  $r$  and  $s$  are scalars ( $\mathbb{R}$ )

*We need to know/memorize these*

# Algebraic Properties (Addition and Scaling)

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

In these properties  $A$ ,  $B$ , and  $C$  are matrices of the appropriate size so that everything is defined, and  $r$  is a scalar

*We need to know/memorize these*

# Matrix Multiplication is not Commutative

**Important.**  $AB$  may not be the same as  $BA$

(it may not even be defined)

# Question (Conceptual)

Find a pair of 2D linear transformations  $T_1$  and  $T_2$  such that  $T_1$  followed by  $T_2$  is not the same as  $T_2$  followed by  $T_1$

(also find a pair where they are the same)

**Answer: Rotation and Reflection**

# **Computational Aspects of Matrix Multiplication**

# Matrix Operations in Numpy

Let `a` and `b` be 2D numpy arrays and let `c` be a floating point number

» `a @ b` (matrix multiplication)

» `a + b` (matrix addition)

» `c * a` (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more

# Analyzing Linear Algebra Algorithms



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For numerics, we care about number of **F**loating-oint  
**O**perations (FLOPs):

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$2n$  vs.  $n$  is very different  
when  $n \sim 10^{20}$

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for polynomials, they are equivalent to their dominant term

# Dominant Terms

the dominant term of a polynomial is the monomial with the highest degree

$$\lim_{i \rightarrow \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$  dominates the function even though the coefficient for  $x^2$  is so large



# A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Suppose  $A$  and  $B$  are  $n \times n$  matrices

This operations takes  $n$  multiplications and  $n$  divisions ( $2n$  FLOPS total)

Repeating for each entry gives  $\sim 2n^3$  FLOPS

# A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

The main part of this procedure is highly parallelizable

# A Note on Parallelization

```
a = np.array(...)  
b = np.array(...)  
prod = np.zeros([a.shape[0], b.shape[1]])  
for i in range(a.shape[0]):  
    for j in range(b.shape[1]):  
        prod[i, j] = np.dot(a[i], b[:,j])
```

The main part of this procedure is highly parallelizable

One processor per entry gets you to  $\sim 2n$  FLOPS

# A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers

Best leave it to experts (or do research in the area)

**LAPACK** is the state of the art library for matrix operations

**numpy uses LAPACK**

# Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations

Matrix multiplication coincides with composition of linear transformations

There is an algebra of matrices which is consistent with the algebra of vectors