Dimension and Rank

Geometric Algorithms
Lecture 17

CAS CS 132

Practice Problem

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Consider the subspace H generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that $\mathbf{v}_3$ and $\mathbf{v}_4$ form a basis for H.

H is 2-dim'l

v_3, v_4 clearly linearly ind.

aim: show $v_3, v_4 \in \text{span}\{v_1, v_2\}$

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Hint. Show that $\mathbf{v}_{\cancel{1}\,3}$ *and* $\mathbf{v}_{\cancel{2}\,4}$ *are in the span of* $\mathbf{v}_{\cancel{3}\,1}$ *and* $\mathbf{v}_{\cancel{4}\,2}$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad\qquad 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$v_3 = v_1 + v_2 \qquad\qquad 2v_1 + v_2 = v_4$$

Objectives

1. Discuss the coordinate systems.

2. Introduce the fundamental notion of dimension, which quantifies how "large" a space is

3. Relate the dimension of the column space and the null space of a matrix

Keywords

basis

column space

null space

coordinate system

change of basis

dimension

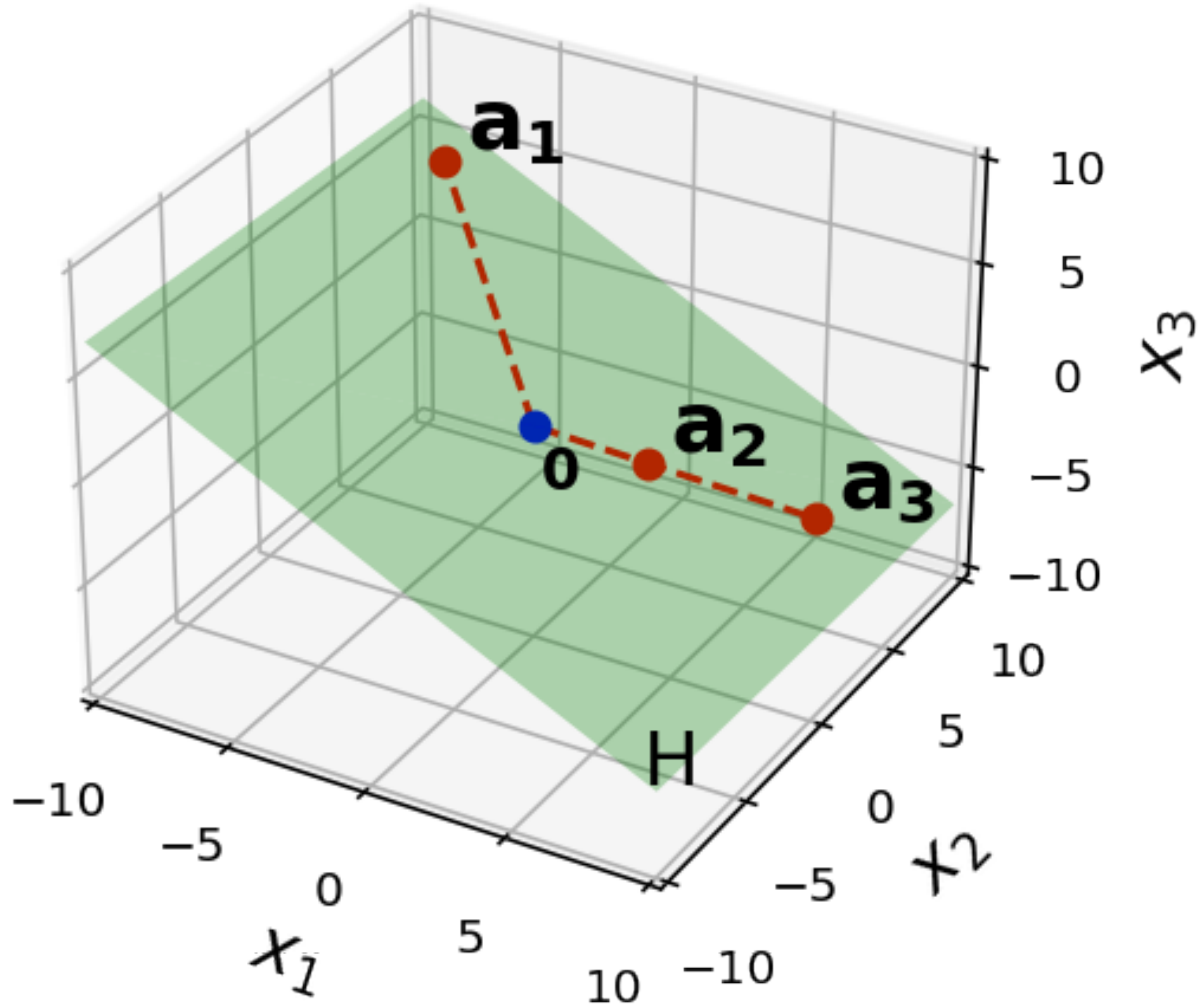
rank

rank theorem

invertible matrix theorem (extended)

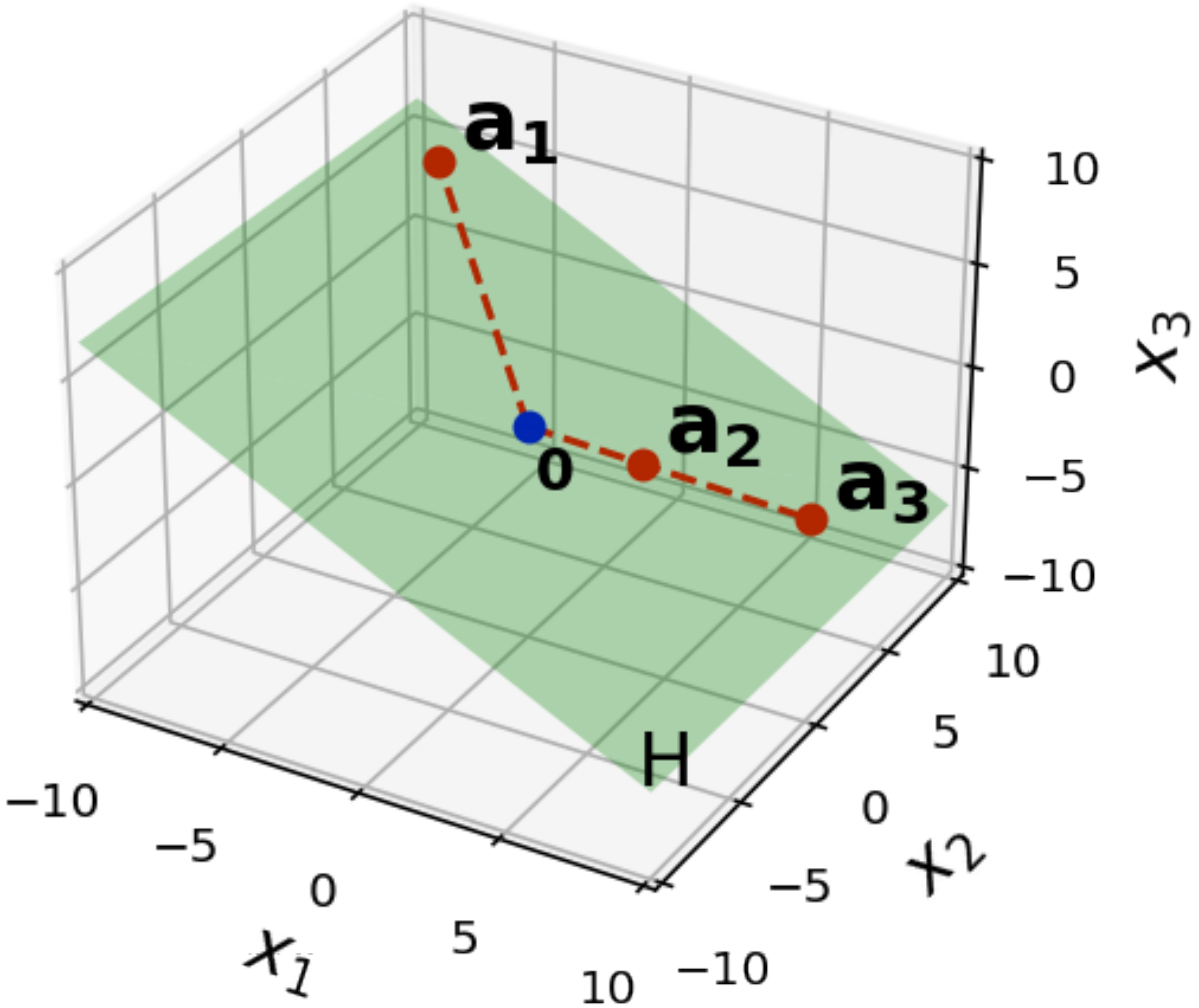
Recap

Recall: The Idea Behind Subspaces



Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"



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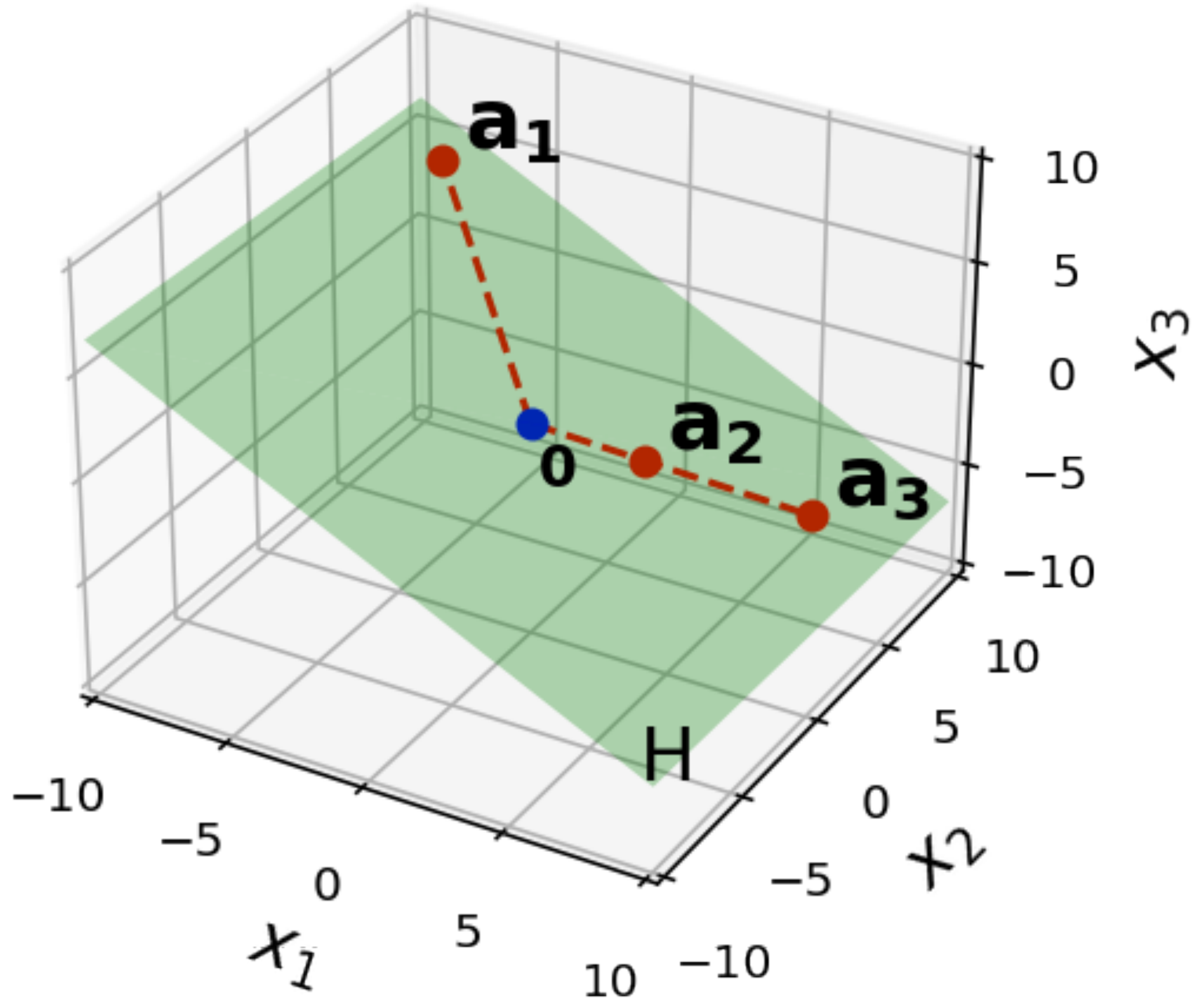
A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$



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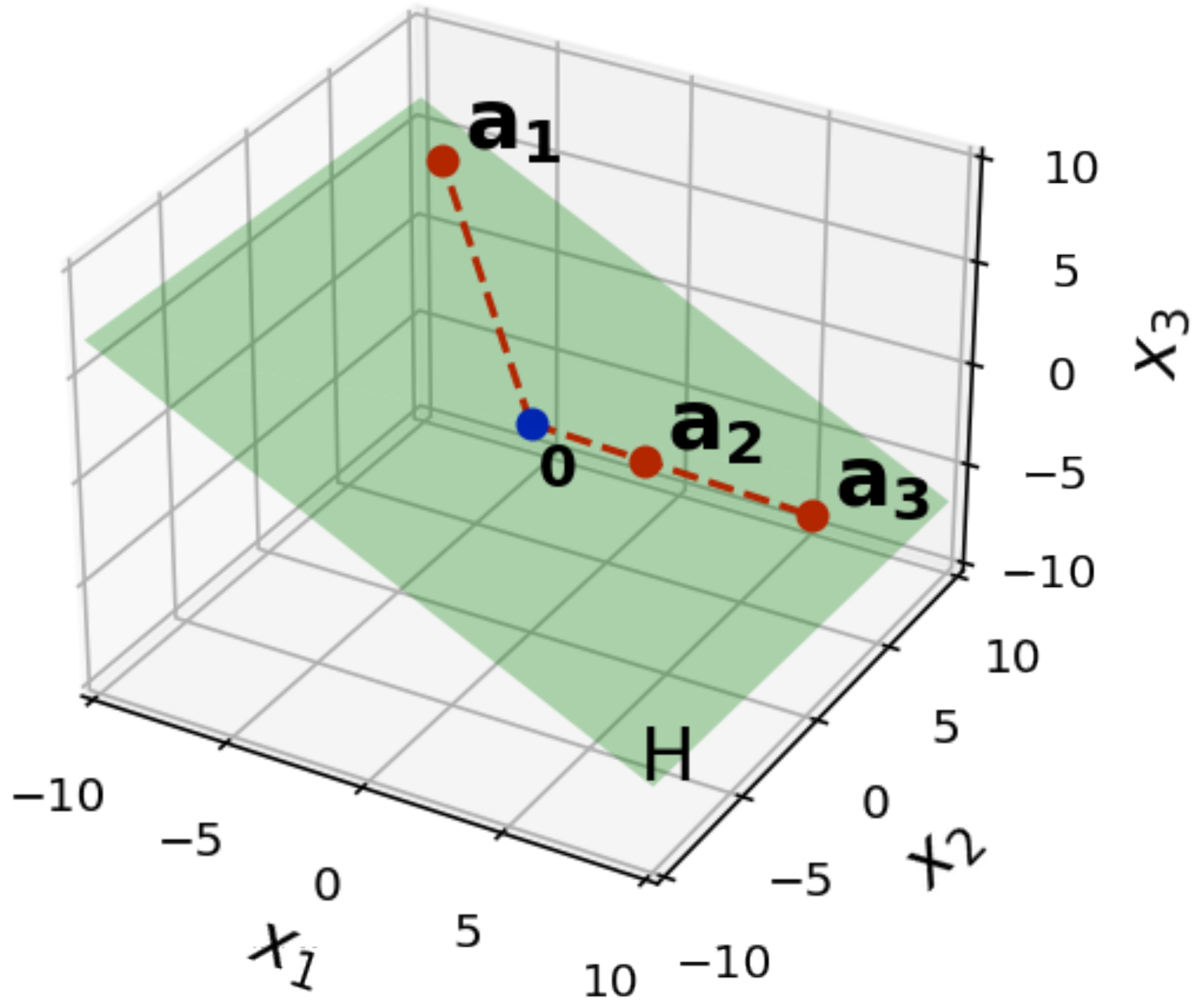
Subspaces *generalize* of this idea.



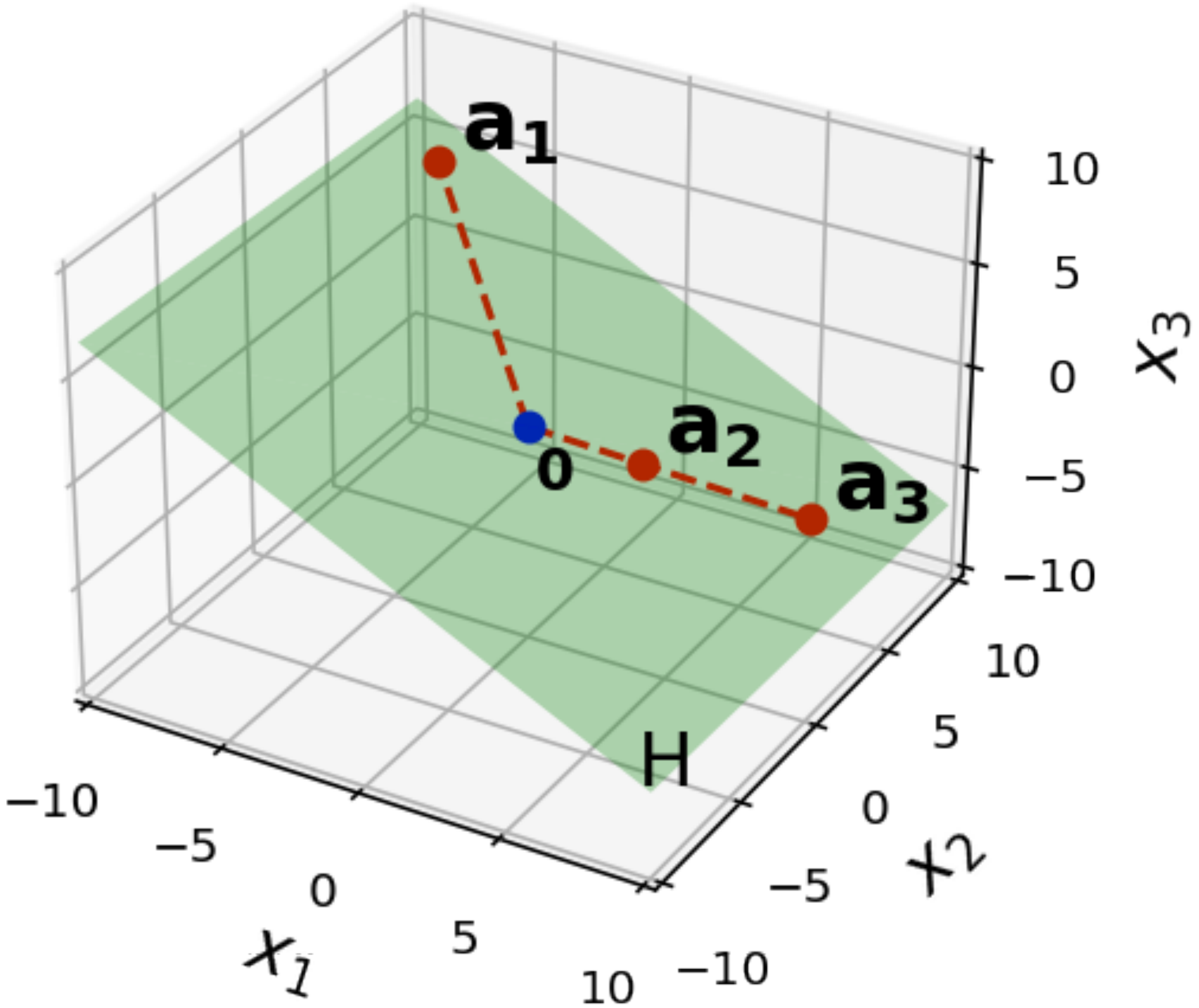
Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"

A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$



Recall: Subspace (Algebraic Definition)

Definition. A **subspace** of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $\mathbf{u} + \mathbf{v}$ is in H
2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H

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!! Subspaces must "live" somewhere !!

Column Space

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Definition. The **column space** of a matrix A, written $\text{Col}(A)$ or $\text{Col}\, A$, is the set of all linear combinations of the columns of A.

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The column space of a matrix is the range of the linear transformation it implements.

Subspace of What?

$$\begin{array}{c|c} & n \\ \hline m & \begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \cdots & | & | \end{bmatrix} \end{array}$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\text{Col}(A)$

is a subspace of

$\mathbb{R}^m$

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The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A.

Subspace of What?

n columns

m rows

$$\mathbf{A}\mathbf{v} = \mathbf{0}$$

$m \times n \quad n \times 1 \quad m \times 1$

$\mathbf{v}$ is a vector in $\mathbb{R}^n$

$\text{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

Recall: Basis

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Definition. A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$).

A basis is a *minimal* set of vectors which spans all of H.

Recall: Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

$$x_1 = 2x_2 + x_4 - 3x_5$$

$\boxed{x_2}$ is free

$$x_3 = (-2)x_4 + 2x_5$$

$\boxed{x_4}$ is free

$\boxed{x_5}$ is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

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!! this transformation is only linear !!
!! in the case of homogeneous equations !!

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Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.
3. The resulting vectors form a basis for $\mathrm{Nul}(A)$.

Recall: Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The idea. What if we cover up the non-pivot columns?

Recall: Column Space and Reduced Echelon form

$$\begin{bmatrix}\mathbf{a}_1 & \text{[redacted]} & \mathbf{a}_3 & \text{[redacted]} & \text{[redacted]}\end{bmatrix} \sim \begin{bmatrix}1 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]} \\ 0 & \text{[redacted]} & 1 & \text{[redacted]} & \text{[redacted]} \\ 0 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]}\end{bmatrix}$$

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Then we see $[\mathbf{a}_1 \ \ \mathbf{a}_3]$ has 2 pivots.

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Then we see $[\mathbf{a}_1 \ \ \mathbf{a}_3]$ has 2 pivots.

So the pivot columns are linearly independent.

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Recall: Column Space and Reduced Echelon form

$$\begin{array}{ccc} 2 & 1 & \\ \end{array}$$

$$\left[\begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \text{[redacted]} \end{array}\right] \sim \begin{array}{ccc} 2 & 1 & \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} -2 \\ 0 \\ 0 \end{array} & \left.\text{[redacted]}\right] \end{array}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

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So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

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So $\mathbf{2a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> $\text{Col}(A)$.

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The takeaway. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

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!! IMPORTANT !!

Choose the columns <u>of A</u>.

($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\mathrm{Col}(A)$)

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1. Find the pivot columns in an echelon form of A.
2. The associated columns <u>in A</u> form a basis for $\mathrm{Col}(A)$.

Example

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A.

Answer

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

moving on...

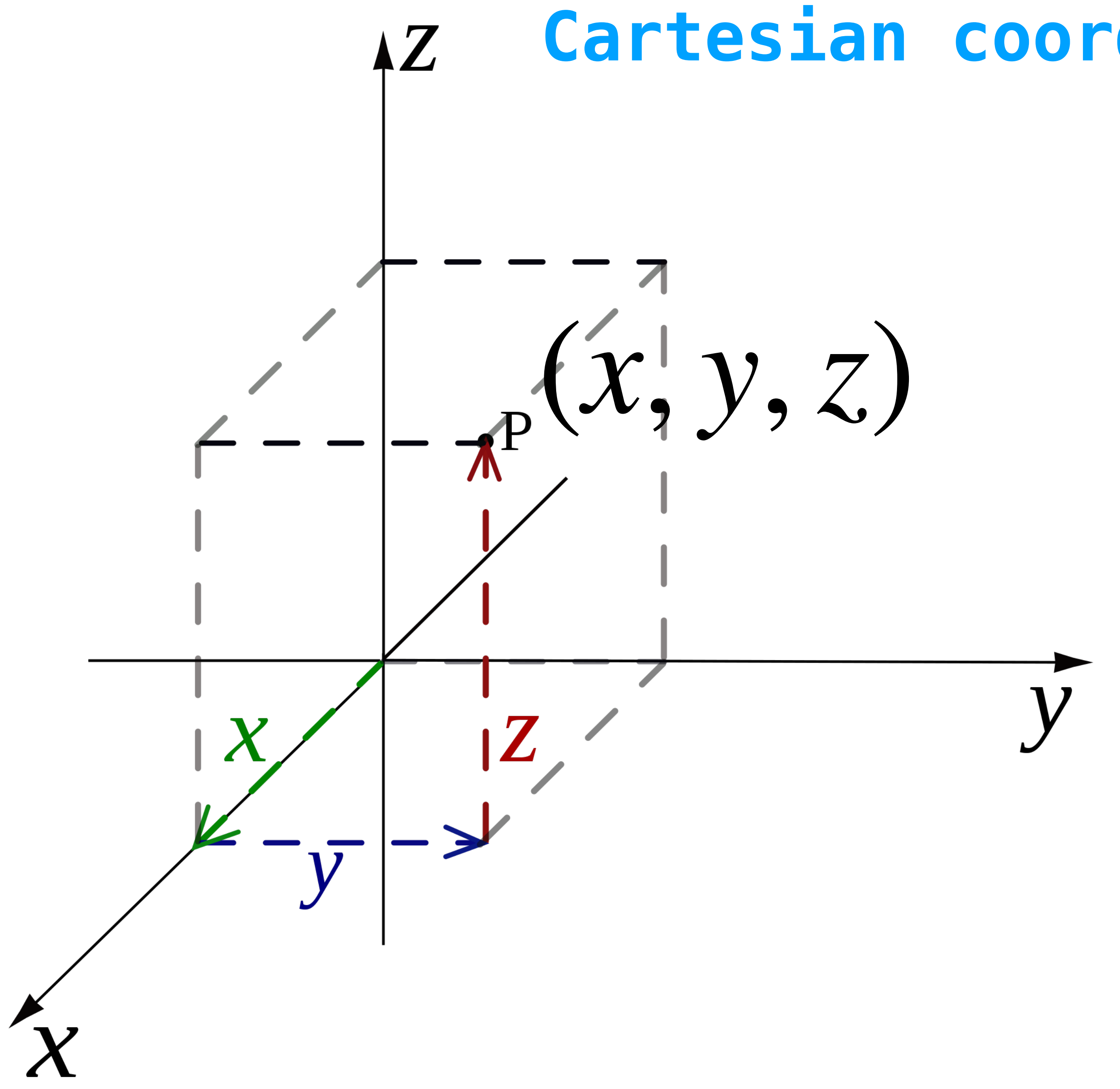
Coordinate Systems

At a High Level

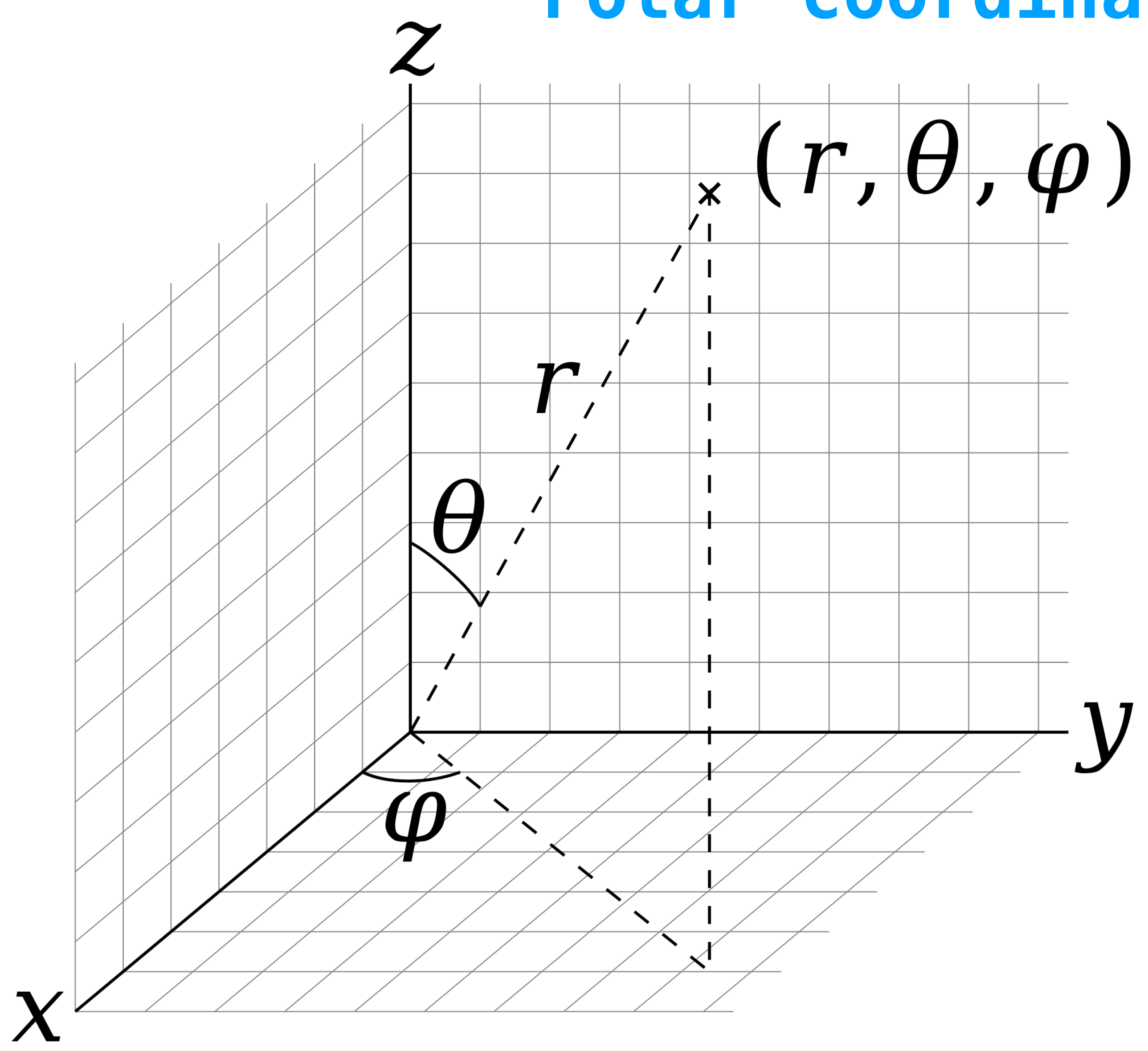
`A coordinate system is a way of representing positions in terms of a sequence of numbers.`

Examples.

Cartesian coordinates



Polar coordinates



Question (Conceptual)*

*And a bit of a trick question

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Is $(2.3, 0.01, 5)$ *a polar coordinate or a cartesian coordinate?*

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This question is non-sensical.

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Question (Conceptual)*

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This question is non-sensical.

It's <u>just a sequence of numbers</u>. We need to be *told* if it should be interpreted in the **polar** coordinate system or the **Cartesian** coordinate system.

*And a bit of a trick question

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Verify:

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Every basis provides a way to write down *coordinates* of a vector.

And every time we write down a vector, we are **assuming a coordinate system.**

what do we mean by this?

A Thought Experiment

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Imagine doing this whole class from the beginning, but never saying *what vectors are.*

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Then one day, you get tired of talking about "abstract" vectors, you want to work with *numbers.*

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Because we've learned everything up to now, we know that there is a basis $\mathbf{b}_1$, $\mathbf{b}_2$,..., $\mathbf{b}_n$ for the space $\mathbb{R}^n$.

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \\ -0.1 \end{bmatrix}$$

and then choose those weights as a representation of v as a sequence of numbers

But wait...

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`This` *`depends`* `on the choice of basis.`

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2, \ldots,$ $\mathbf{c}_n$ then we would get some other representation.

$$\mathbf{v} = (-10)\mathbf{c}_1 + (4.3)\mathbf{c}_2 + \ldots + 0\mathbf{c}_n = \begin{bmatrix} -10 \\ 4.3 \\ \vdots \\ 0 \end{bmatrix}$$

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Every basis defined a different coordinate system

Standard Basis

The standard basis defines the Cartesian coordinate system for $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

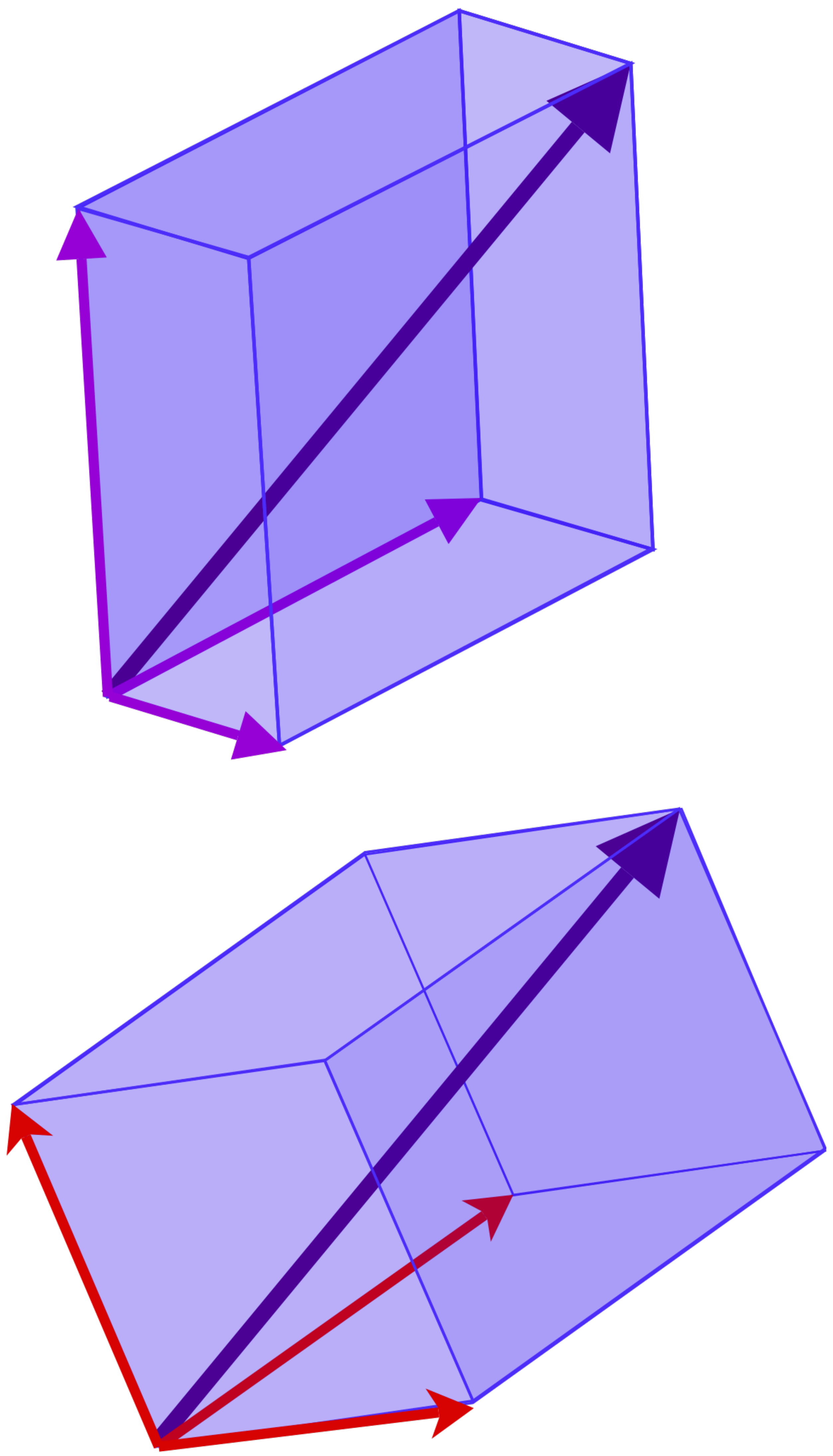
Column vectors are just weights for a linear combination of the standard basis

but we can also use
different coordinate systems

How to think about this

Changing the coordinate system "warps space".

The question is: how do we represent a vector $\mathbf{v}$ in the warped space if we wanted it to "be in the same place"?



https://commons.wikimedia.org/wiki/File:3d_two_bases_same_vector.svg

Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a subspace H of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis of H where

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Definition. The **coordinate vector of $\mathbf{v}$ relative to $\mathscr{B}$** is

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

Coordinate Vectors and the Standard Basis

When we write down a vector $\mathbf{v}$ in $\mathbb{R}^n$, we're really writing down a coordinate vector **relative to the standard basis** $\mathscr{E}$.

$$[\mathbf{v}]_{\mathscr{E}} = \mathbf{v}$$

How do we find coordinate vectors?

For an arbitrary basis $\mathscr{B}$, to determine $[\mathbf{v}]_{\mathscr{B}}$, we need to find weights $a_1, \ldots, a_k$ such that

$$a_1\mathbf{b}_1 + \ldots + a_k\mathbf{b}_k = \mathbf{v}$$

$$\begin{bmatrix} \vec{B} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} \vec{v} \end{bmatrix}$$

This is just <u>solving a vector equation</u>.

Example: 2D Case

Write the coordinate vector for $\vec{v} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$$

$a_1 = -2$

$a_2 = 3$

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} \stackrel{?\checkmark}{=} -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example: 2D Case (Geometrically)

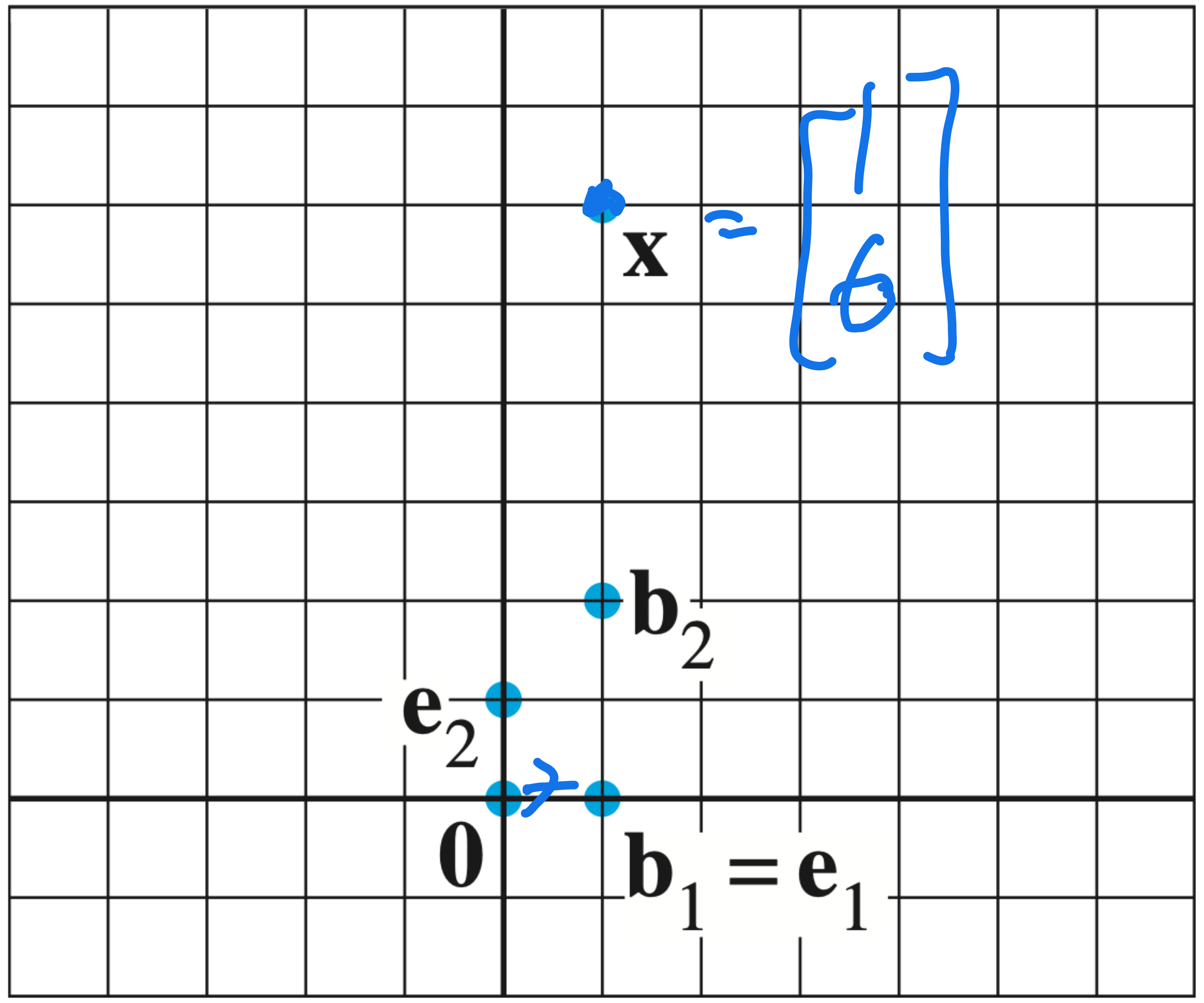


FIGURE 1 Standard graph paper.

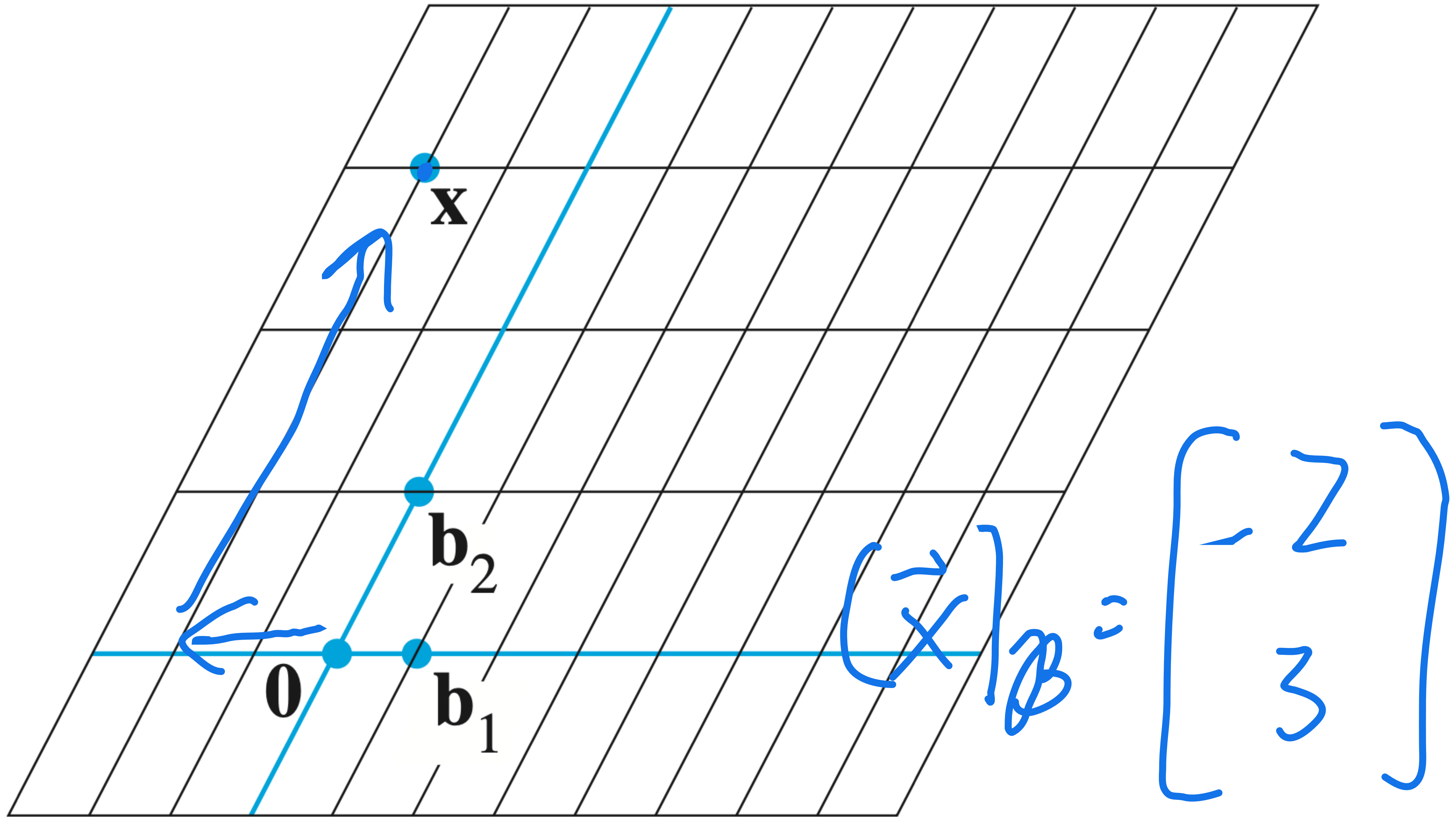


FIGURE 2 $\mathcal{B}$-graph paper.

$\mathcal{B}$ defines a "different grid for our graph paper"

How To: Coordinate Vectors

Question. Find the coordinate vector for $\mathbf{v}$ in the subspace H relative to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$.

Solution. Solve the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_k\mathbf{b}_k = \mathbf{v}$$

A solution $(a_1, \ldots, a_k)$ means

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

Example: 3D Case

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 6 & 0 & 0 \end{array}\right]$$

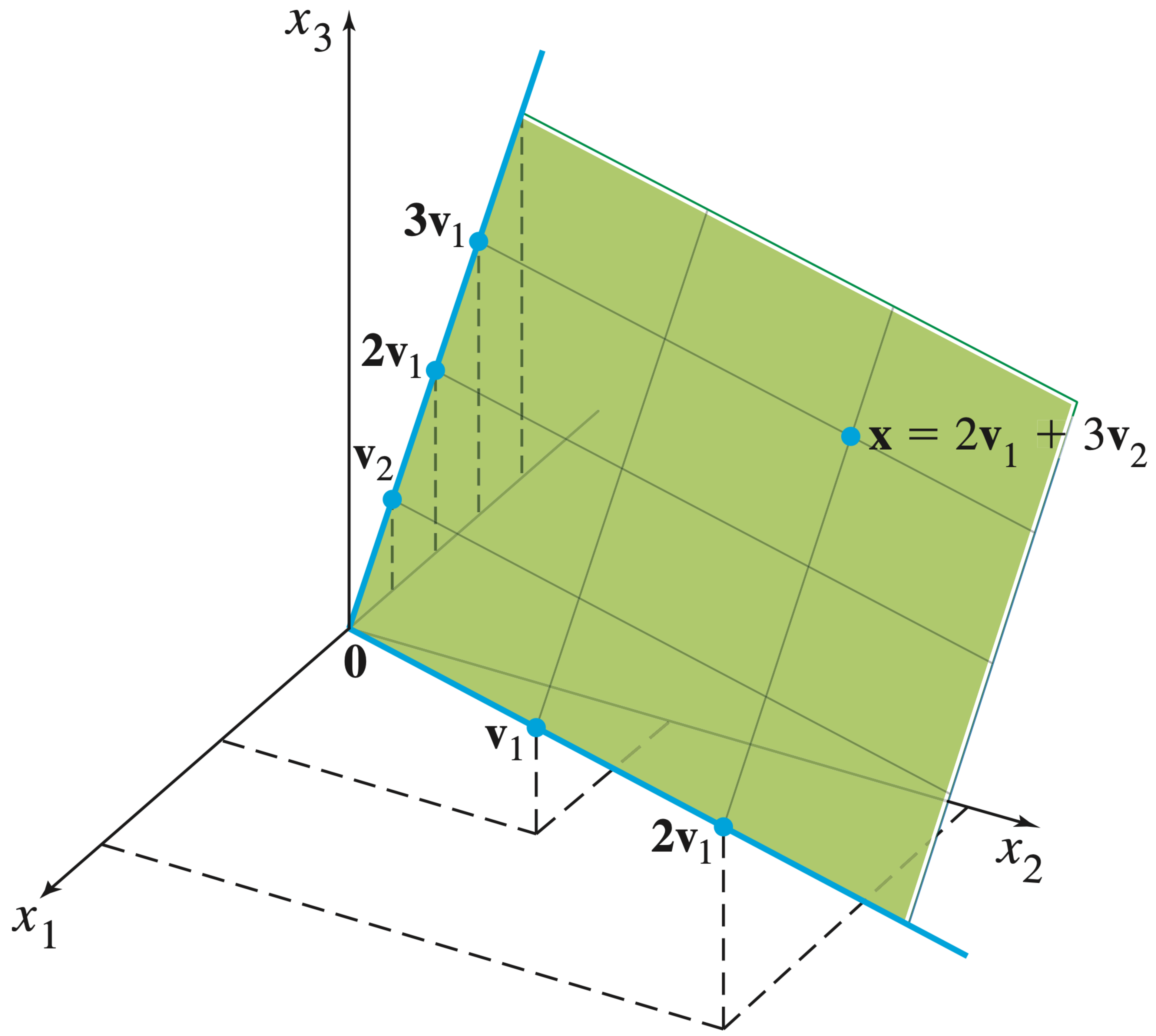
Find the coordinate vector for $\mathbf{u}$ relative to the basis $\mathcal{B}$ $\{\mathbf{v}_1, \mathbf{v}_2\}$ of a subspace H (of $\mathbb{R}^3$): $[\vec{u}]_{\mathcal{B}}$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \begin{bmatrix} 1 & -2 & -4 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 12 & 36 \\ 0 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3×2 (2×1)

$$[u]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

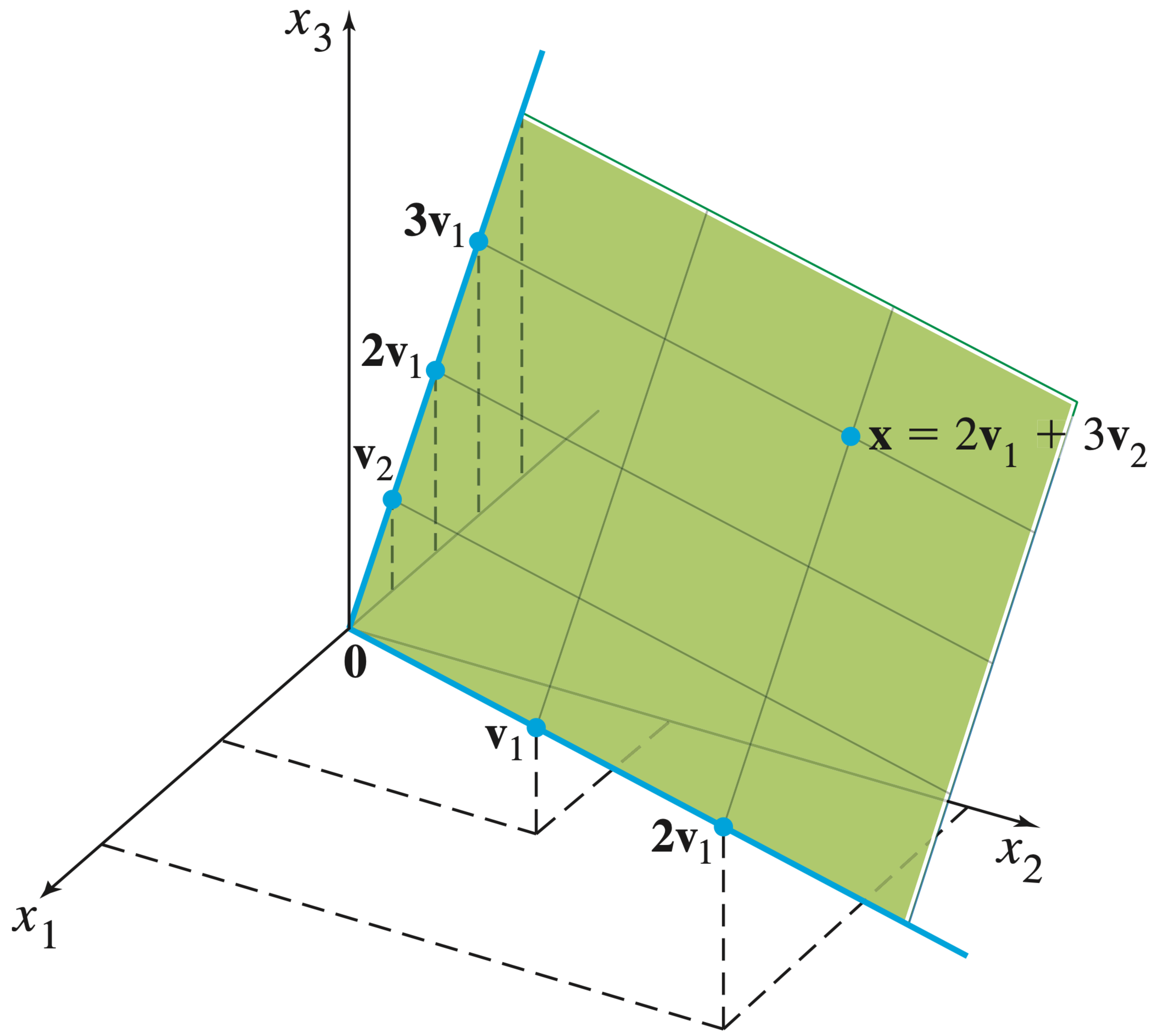
An Aside: Coordinates and one-to-one correspondences



Linear Algebra and its Applications (Lay, Lay, McDonald)

An Aside: Coordinates and one-to-one correspondences

In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a <u>one-to-one correspondence</u> from H to $\mathbb{R}^2$. This is also sometimes called an **isomorphism.**



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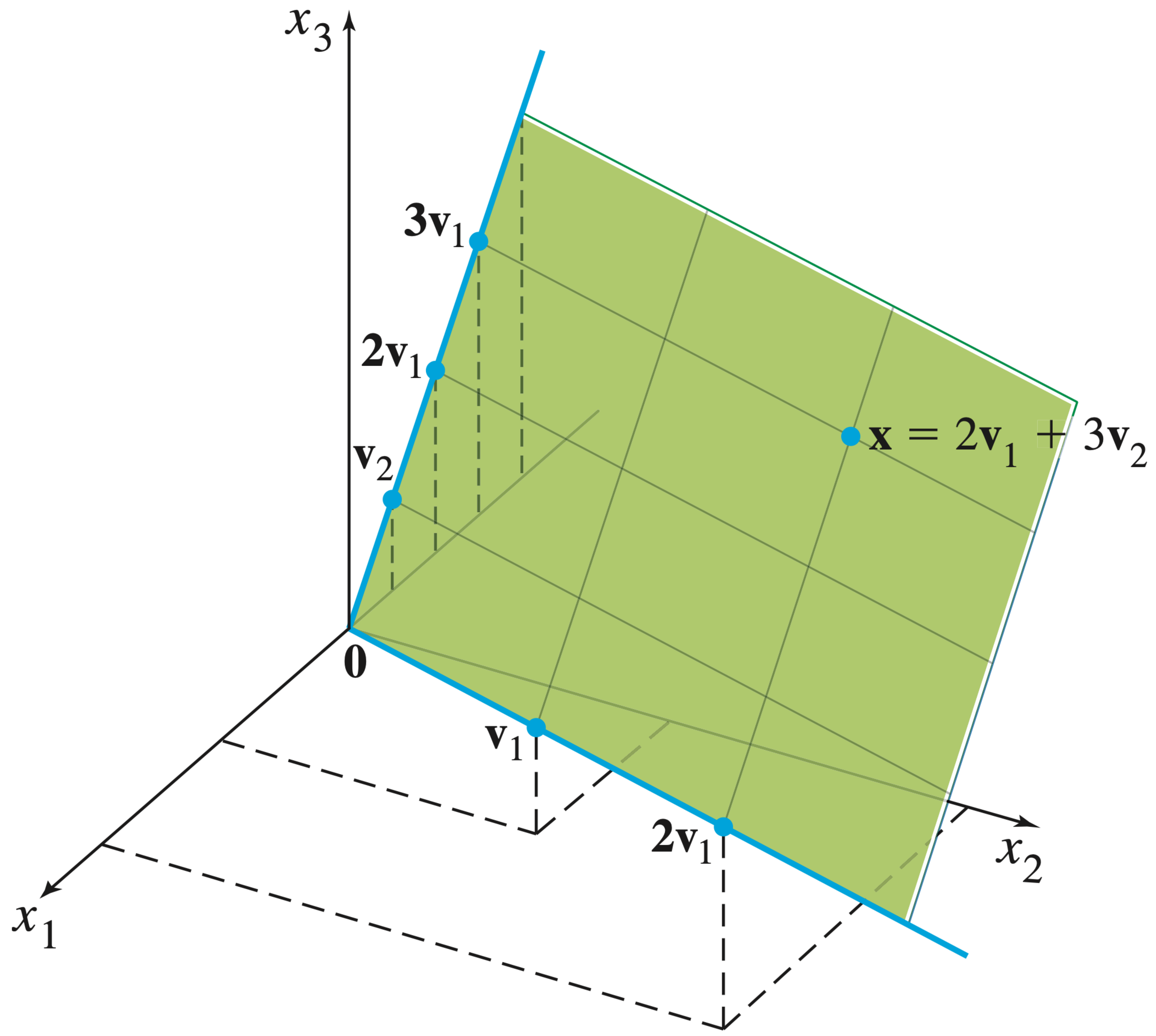

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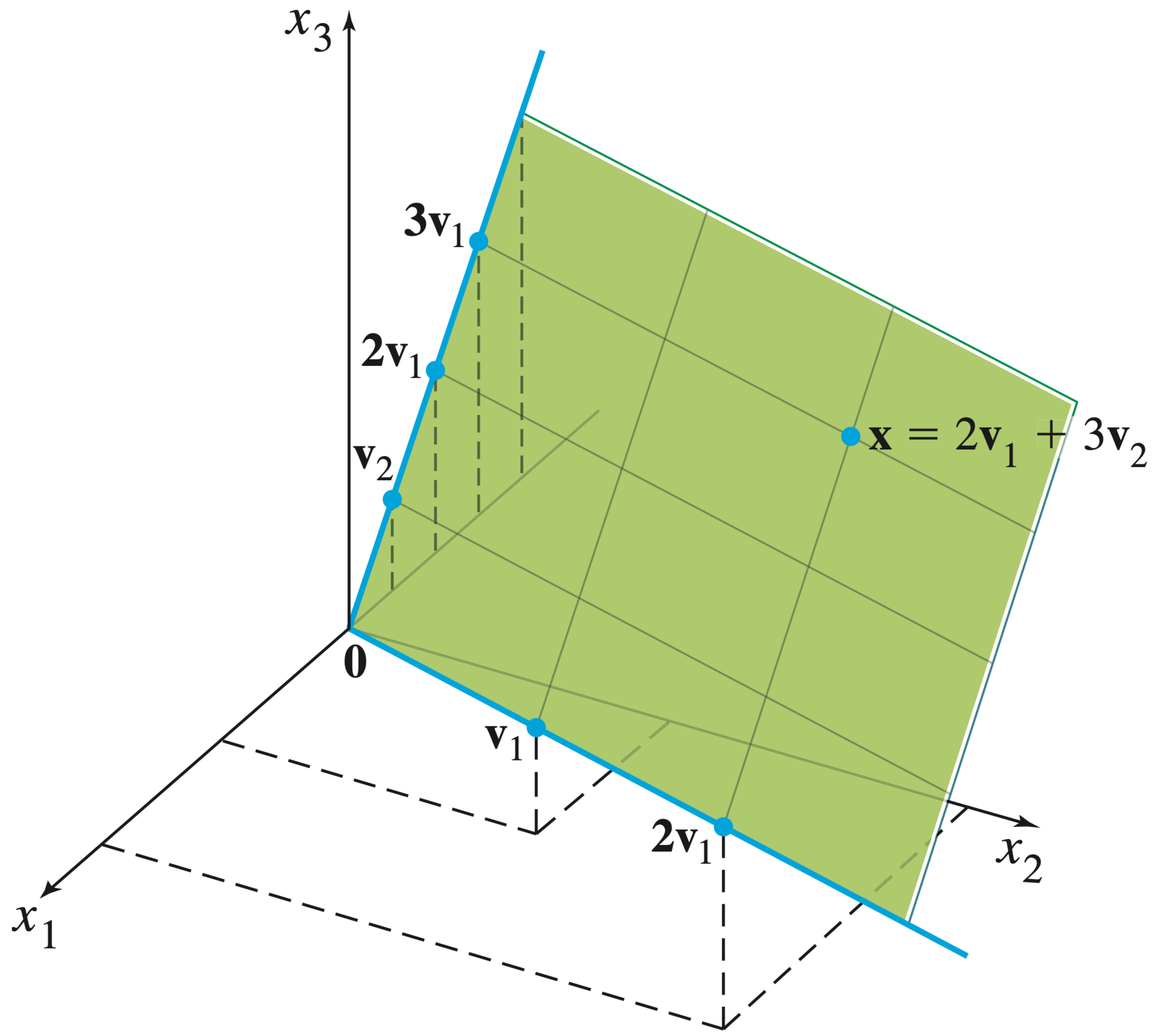

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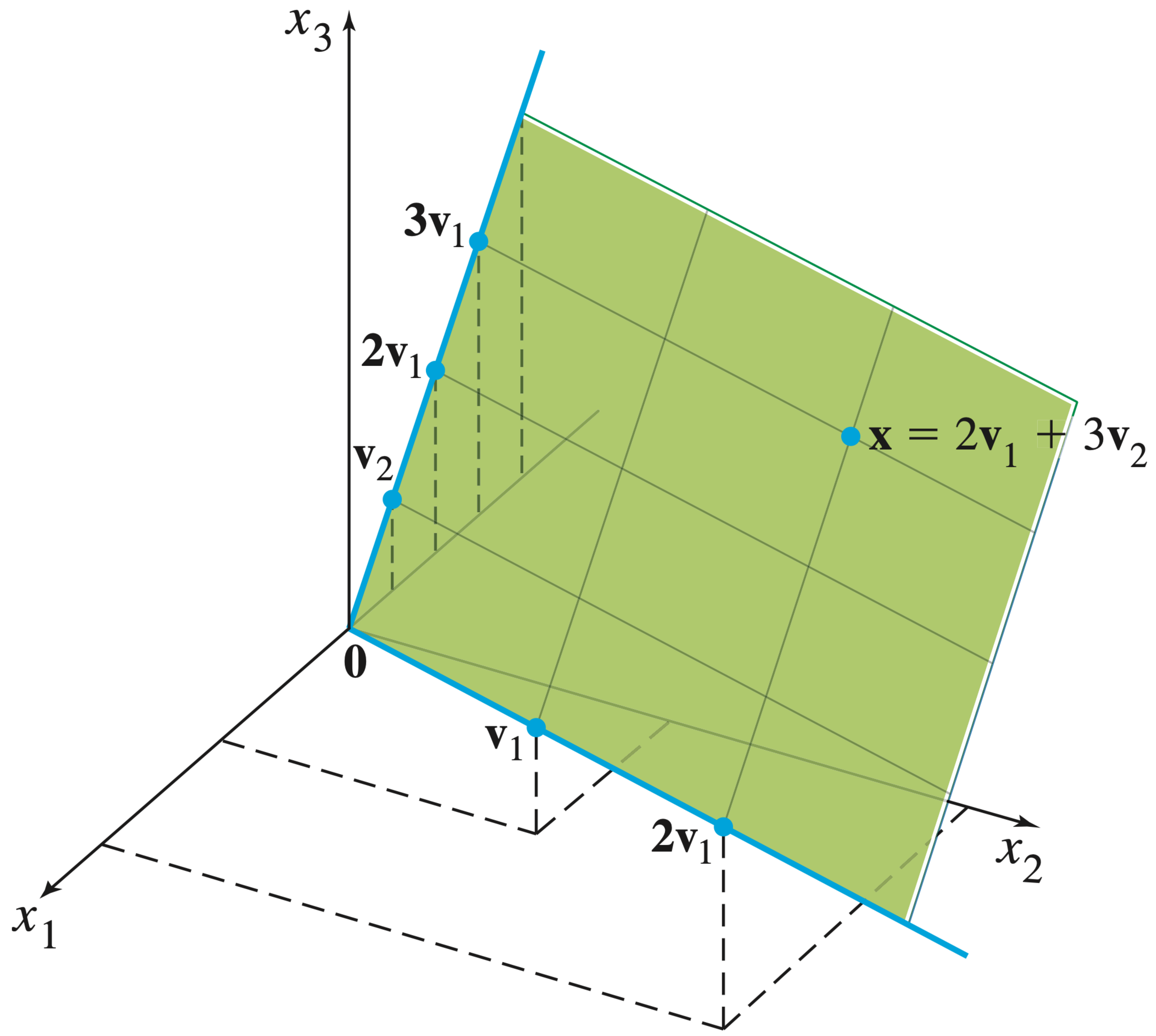
So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to $\mathbb{R}^2$.

This is a formal way of saying that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a "copy of $\mathbb{R}^2$."



Linear Algebra and its Applications (Lay, Lay, McDonald)

Question

$$\mathbf{v}_1 = \begin{bmatrix}3\\6\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$\vec{u} = 2\vec{v}_1 - 2\vec{v}_2$

Suppose $[\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix}2\\-2\end{bmatrix}$, *where* $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. *Find* $\mathbf{u}$.

$$2v_1 - 2v_2 = 2\begin{bmatrix}3\\6\\2\end{bmatrix} - 2\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}8\\12\\2\end{bmatrix}$$

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad [\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Dimension and Rank

The Idea of Dimension

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This number is a measure of how "large" H is.

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Definition. The **dimension** of a subspace H of $\mathbb{R}^n$, written $\dim(H)$ or $\dim H$, is the *number* of vectors in <u>any</u> basis of H.

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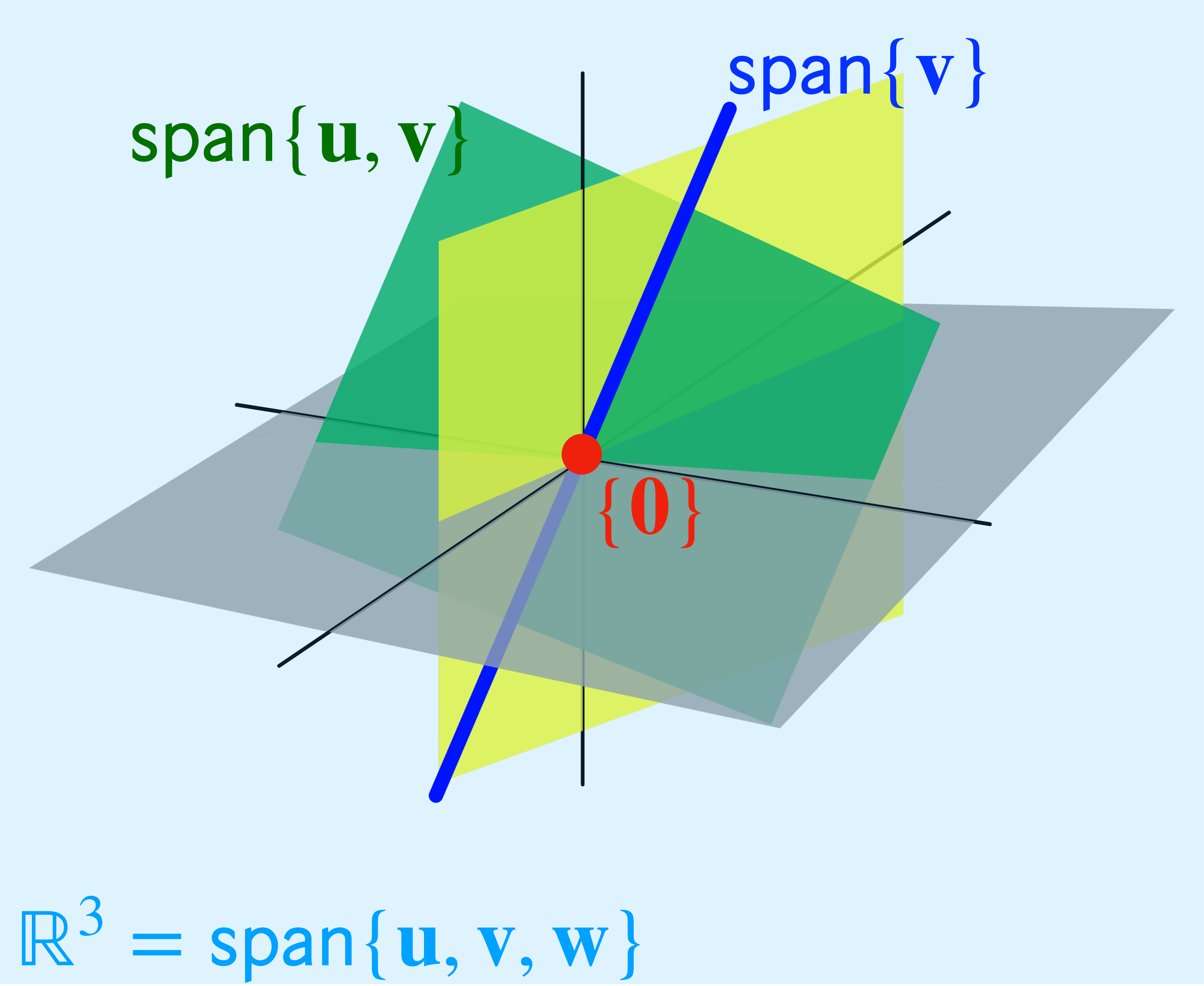
This should confirm our intuitions:

- a plane (through the origin) is a 2D subspace
- a line (through the origin) is a 1D subspace

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of $\mathbb{R}^3$

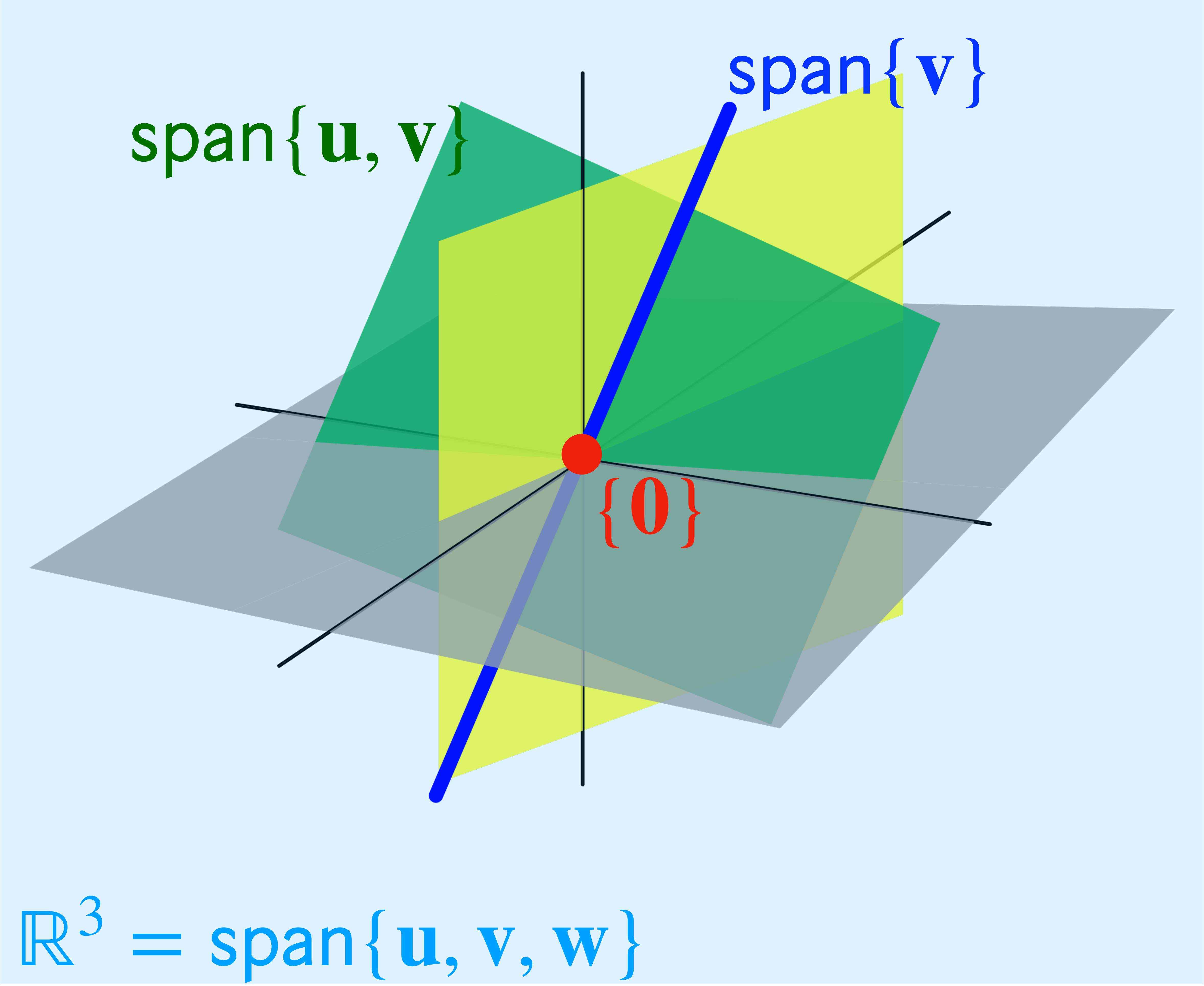


https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. 0-dimensional subspace
2. 1-dimensional subspaces
3. 2-dimensional subspaces
4. 3-dimensional subspace



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How does this connect to null space and column space?

Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of <u>free variables</u> in a general form solution.

$$x_1 = 2x_2 + x_4 - 3x_5$$

$\boxed{x_2}$ is free

$$x_3 = (-2)x_4 + 2x_5$$

$\boxed{x_4}$ is free

$\boxed{x_5}$ is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Dimension of the Null Space

The **dimension of** $\mathrm{Nul}(A)$ is the number of <u>free variables</u> in a general form solution to $A\mathbf{x} = \mathbf{0}$.

$x_1 = 2x_2 + x_4 - 3x_5$

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$x_3 = (-2)x_4 + 2x_5$

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x_5 is free

$$\equiv$$

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Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

dim(Col A)

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(Nul A)

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Rank

Definition. The **rank** of a matrix A, written $\operatorname{rank}(A)$ or $\operatorname{rank} A$, is the dimension of $\operatorname{Col}(A)$.

This is just terminology.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A,

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n$$

Verify:

This is incredibly important.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A,

$$\underbrace{\dim(\mathrm{Col}(A))}_{\text{rank}} + \underbrace{\dim(\mathrm{Nul}(A))}_{\text{nullity}} = n$$

Verify:

This is incredibly important.

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For a $m \times n$ matrix A, its columns space $\text{Col}(A)$ *could* have n dimensions.

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Example. If a "line's worth of stuff" is pulled into the null space (and mapped to $\mathbf{0}$) then

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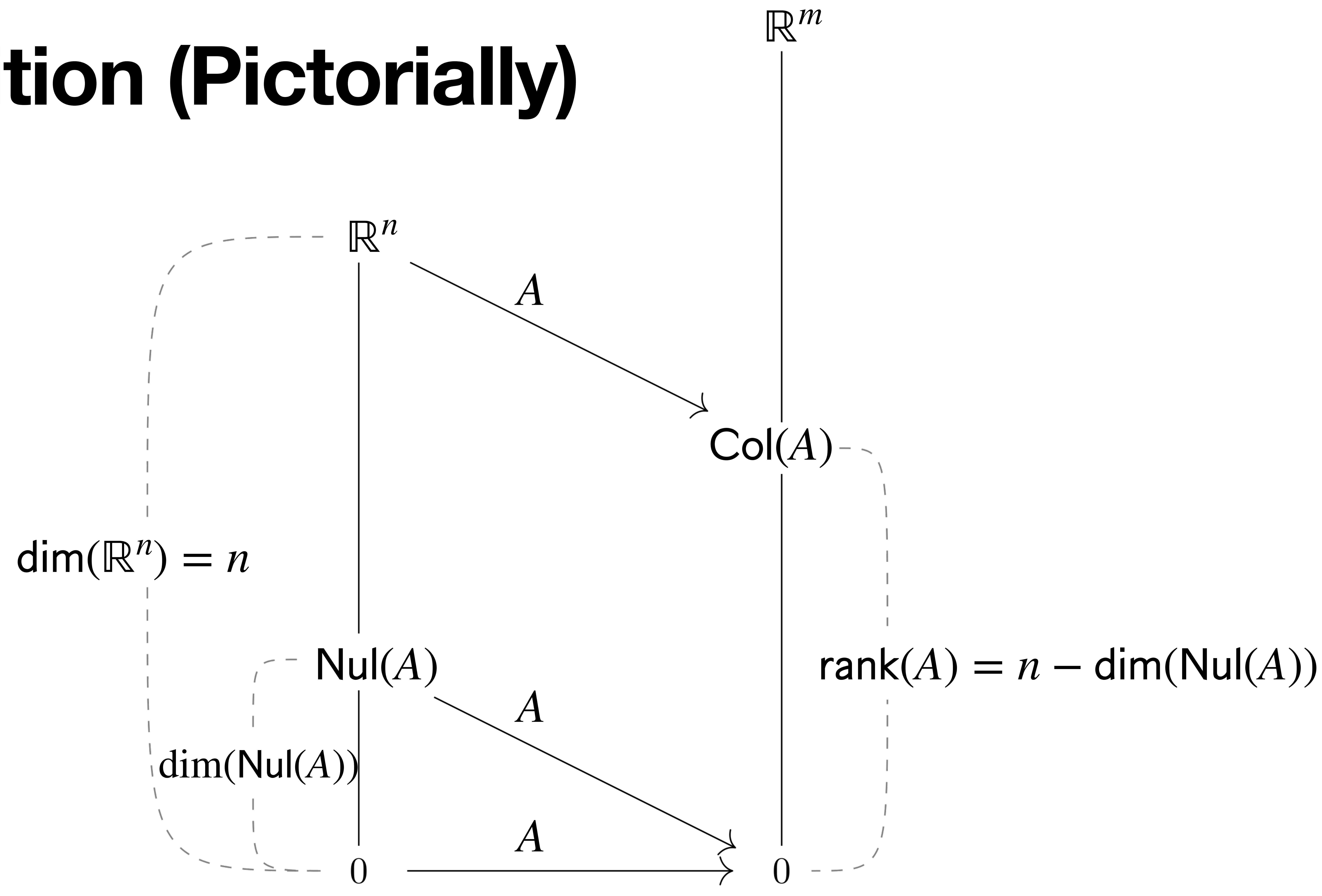
The null space "takes away" some of the dimensions of the column space.

The Intuition (Pictorially)



https://commons.wikimedia.org/wiki/File:Rank-nullity.svg

Question (Conceptual)

Let A *be a* 5×7 *matrix such that* $\dim(\mathrm{Nul}(A)) = 3$. *What is the dimension of* $\mathrm{Col}(A)$?

$$\underbrace{\dim(\mathrm{Col}\,A)}_{4} + \dim(\mathrm{Nul}\,A) = 7$$

$$5\left\{ \overbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}^{7} \right.$$

max 5 pivot columns

$\Rightarrow \dim(\mathrm{Col}\,A) \leq 5$

$\Rightarrow \dim(\mathrm{Nul}\,A) \geq 2$

Answer: 4

Extending the IMT

Theorem. For an $n \times n$ invertible matrix A, the following are logically equivalent (they must all by true or all by false.

- $\text{Col}(A) = \mathbb{R}^n$
- $\dim(\text{Col}(A)) = n$
- $\text{rank}(A) = n$
- $\text{Nul}(A) = \{\mathbf{0}\}$
- $\dim(\text{Nul}(A)) = 0$

Summary

We can find bases for the column space and null space by looking at the reduced echelon form of a matrix.

Column vectors are written in terms of a coordinate system, which we can change.

Dimension is a measure of how large a space is.