

# **The Characteristic Equation**

**Geometric Algorithms**

**Lecture 19**

CAS CS 132

# Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

*Determine the dimension of the eigenspace of A for the eigenvalue 4.*

*(try not to do any row reductions)*

# Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

See A1 for answer

# Objectives

1. Briefly recap eigenvalues and eigenvectors
2. Get a primer on determinants
3. Determine how to find eigenvalues (not just verify them)

# Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

polynomial roots

triangular matrices

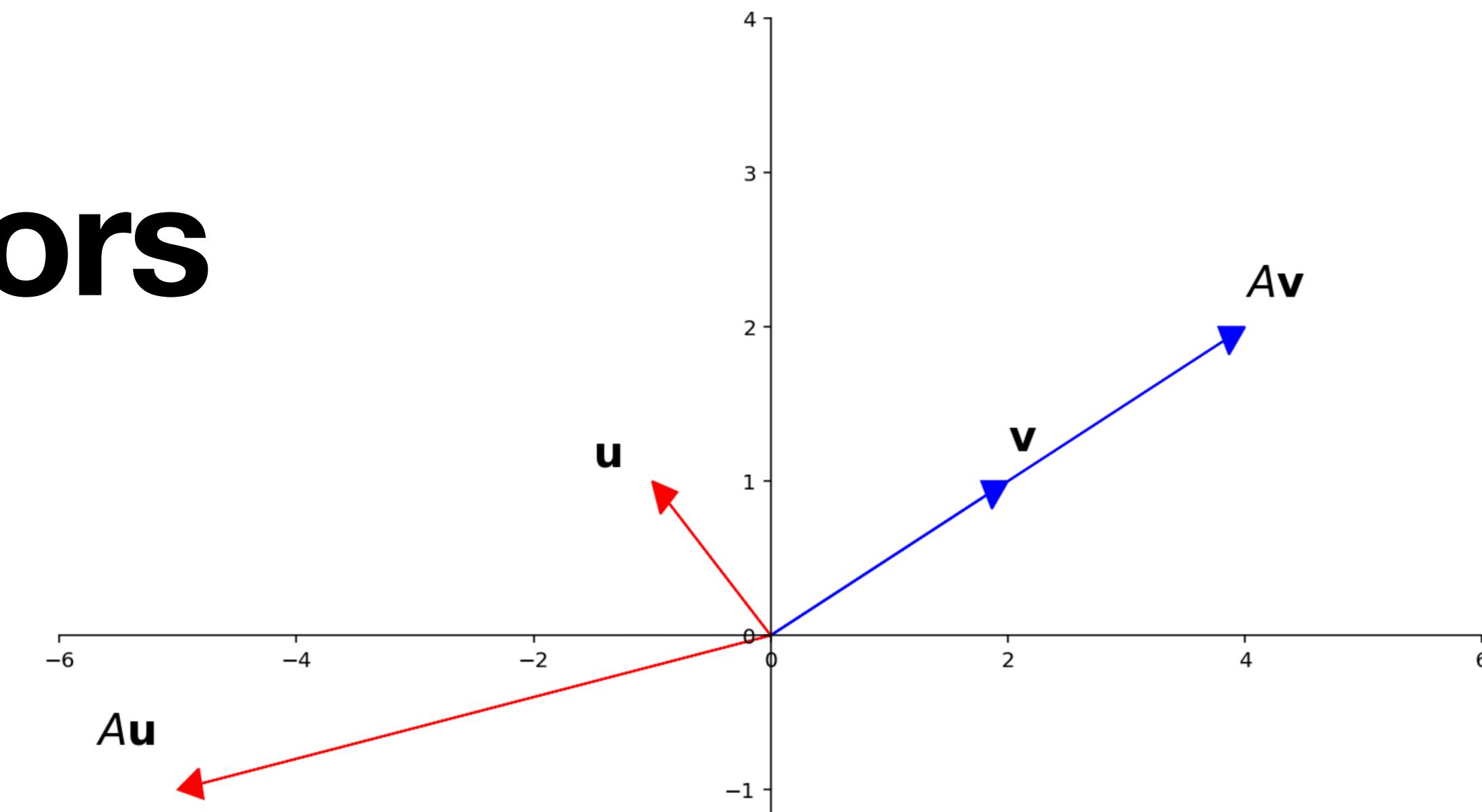
multiplicity

# Recap

# Recall: Eigenvalues/vectors

A nonzero vector  $v$  in  $\mathbb{R}^n$  and real number  $\lambda$  are an **eigenvector** and **eigenvalue** for a  $n \times n$  matrix  $A$  if

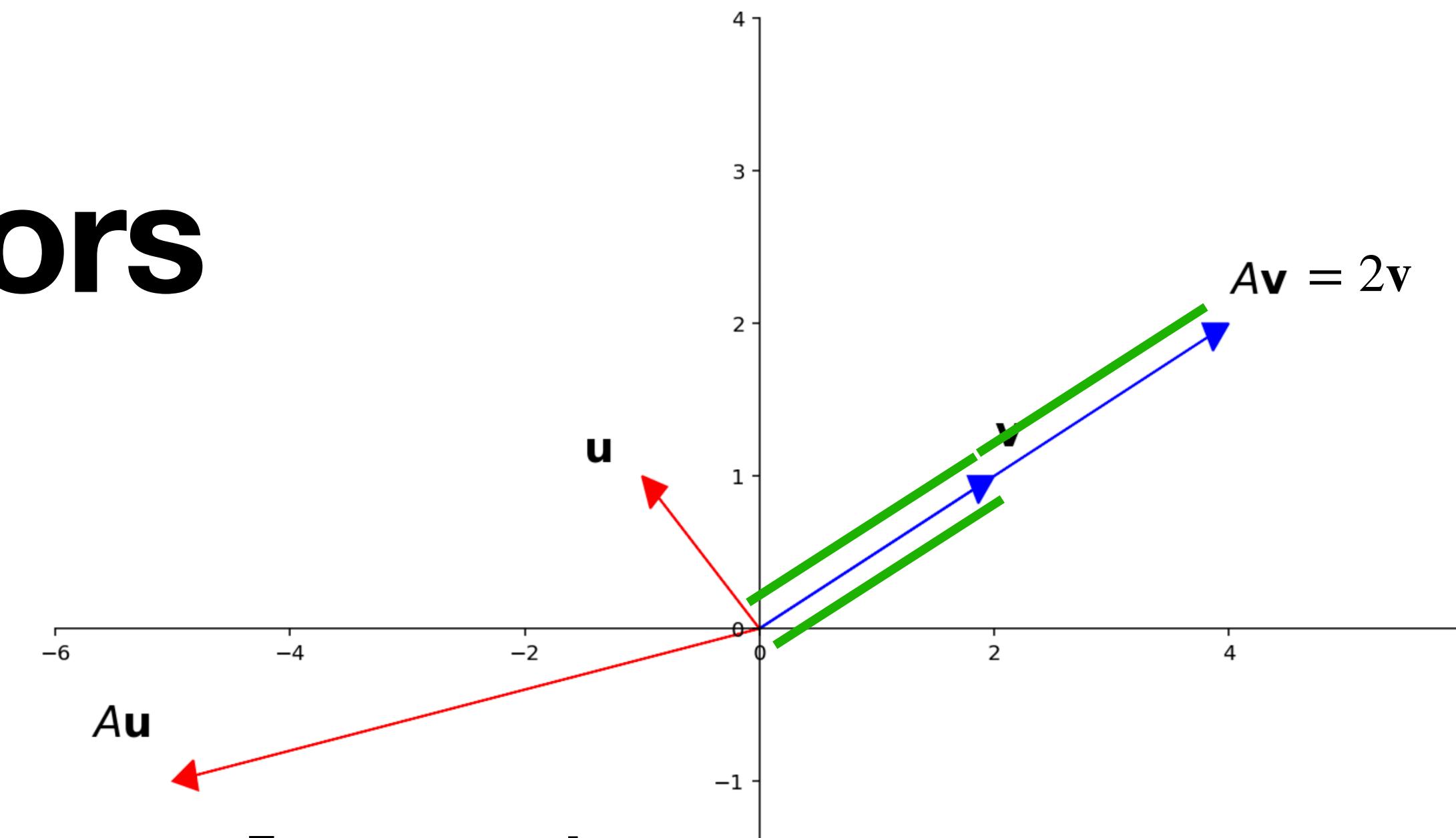
$$Av = \lambda v$$



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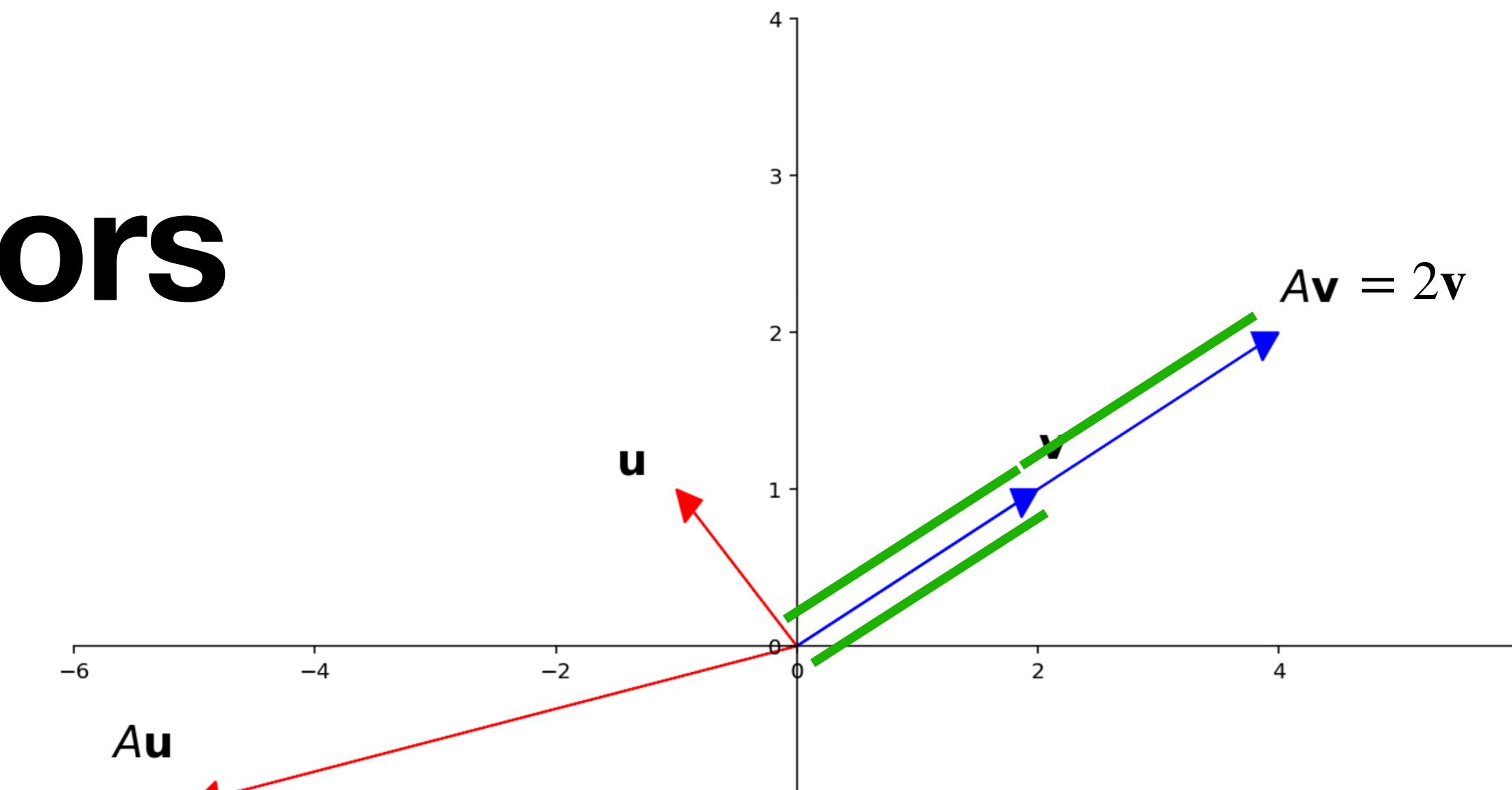
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$v$  is "just scaled" by  $A$ , not rotated

# **Recall: Verifying Eigenvectors**

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix} \times$$

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*If we don't need the vector we can just show that  $A - \lambda I$  is **not** invertible (by IMT).*

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(we did this for our recap problem)

How do eigenvectors relate  
to linear dynamical systems?

# **Recall: (Closed-Form) Solutions**

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A (**closed-form**) **solution** of a linear dynamical system  $v_{i+1} = Av_i$  is an expression for  $v_k$  which is does **not** contain  $A^k$  or previously defined terms

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In other word, it does not depend on  $A^k$  and is not **recursive**

# Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed form for the above linear dynamical system.

# **Solutions with Eigenvectors as Initial States**

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It's easy to give a closed-form solution if the initial state is an eigenvector:

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The Key Point. This is still true of sums of eigenvectors.

# Solutions in terms of eigenvectors

Let's simplify  $A^k \mathbf{v}$ , given we have eigenvectors  $\mathbf{b}_1, \mathbf{b}_2$  for  $A$  which span all of  $\mathbb{R}^2$ :

# Eigenvectors and Growth in the Limit

**Theorem.** For a linear dynamical system  $A$  with initial state  $\mathbf{v}_0$ , if  $\mathbf{v}_0$  can be written in terms of eigenvectors  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$  of  $A$  with eigenvalues

$$\lambda_1 > \lambda_2 \dots \geq \lambda_k$$

then  $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$  for some constant  $c_1$  (in other words, in the long term, the system grows exponentially in  $\lambda_1$ ).

Verify:

# Eigenbases

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**Definition.** An **eigenbasis** of  $\mathbb{R}^n$  for a  $n \times n$  matrix  $A$  is a basis of  $\mathbb{R}^n$  made up entirely of eigenvectors of  $A$ .

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*We can represent vectors as unique linear combinations of eigenvectors.*

***Not all matrices have eigenbases.***

# Eigenbases and Growth in the Limit

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$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant  $c_1$ , where where  $\lambda_1$  is the largest eigenvalue of  $A$  and  $\mathbf{b}_1$  is its eigenvalue.

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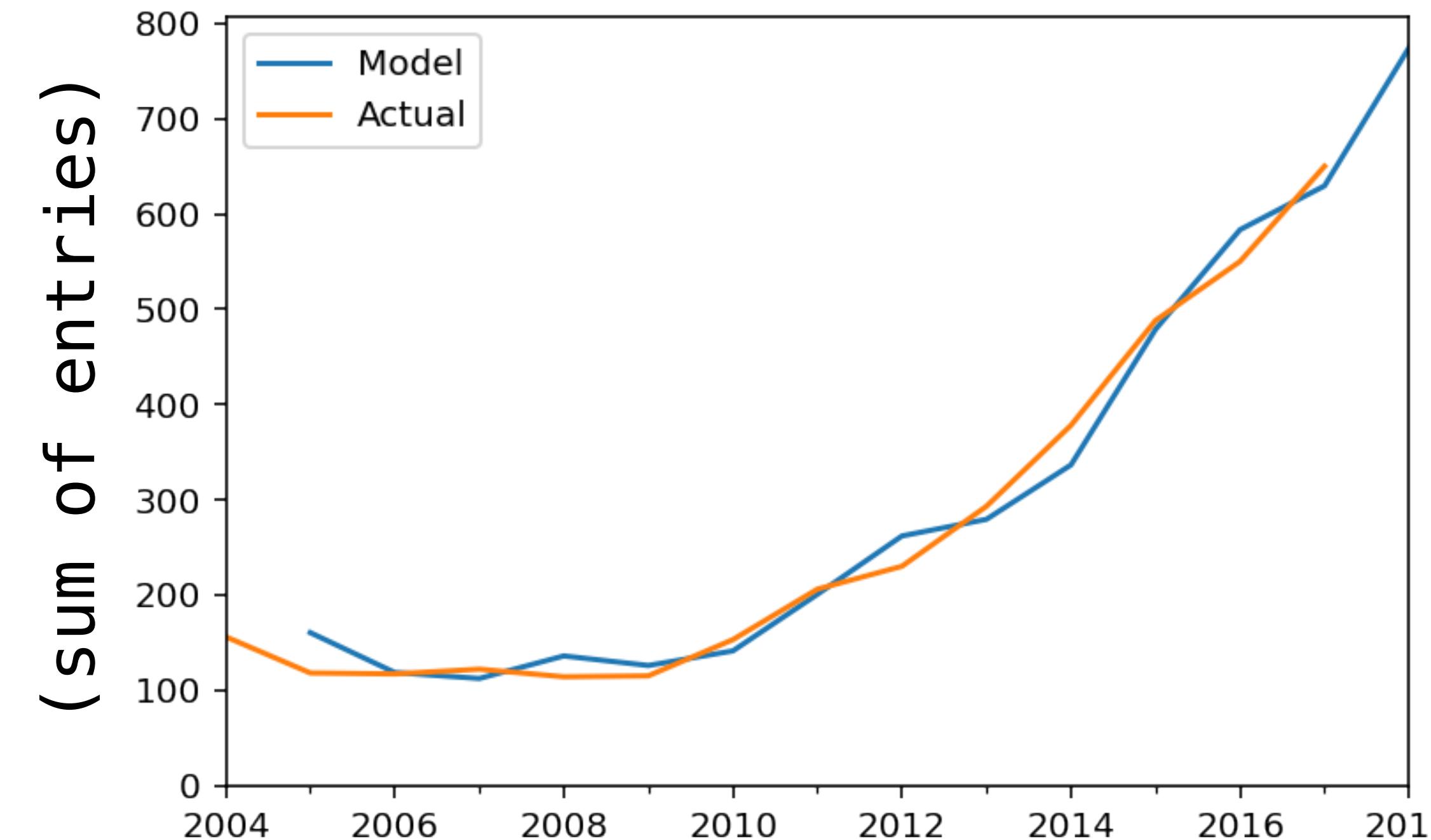
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for some constant  $c_1$ , where where  $\lambda_1$  is the **largest eigenvalue of  $A$  and  $\mathbf{b}_1$  is its eigenvalue**.

The largest eigenvalue describes the long-term exponential behavior of the system.

# Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \# \text{ of year 1 students enrolled in 2024} \\ \# \text{ of year 2 students enrolled in 2024} \\ \# \text{ of year 3 students enrolled in 2024} \\ \# \text{ of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

( $A$  is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the largest eigenvalue.

moving on. . .

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**Question.** Determine the eigenvalues of  $A$ , along with their associated eigenspaces.

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**Solution (Idea).** Can we somehow "solve for  $\lambda$ " in the equation

$$(A - \lambda I)\mathbf{x} = 0$$

# Determinants

# An Aside: Determinants are Mysterious

Determinants are  
strangely polarizing

Some people love them,  
some people hate them

We'll only scratch the  
surface...

## Down with Determinants!

Sheldon Axler



102 (1995), 139-154.

try writing from the Mathematical Association of America.

without determinants. The standard proof that a square matrix of complex numbers has an eigenvalue uses determinants, this allows us to define the multiplicity of an eigenvalue and to prove that the number of eigenvalues equals the dimension of the space. Using characteristic and minimal polynomials and then prove that they behave as expected. This leads to an easy proof of the spectral theorem. Without determinants, this paper gives a simple proof of the finite-dimensional spectral theorem.

In this paper. The book is intended to be a text for a second course in linear algebra.

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In broad strokes, it's a big sum of products of entries of  $A$ .

# A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
1 FUNCTION det(A):
2     total = 0
3     FOR all matrix B we can get by swapping a bunch of rows of A:
4         s = 1 IF (# of swaps necessary) is even ELSE -1
5         total += s * (product of the diagonal entries of B)
6     RETURN total
```

# The Determinant of $2 \times 2$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$(-1)^0 ad$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{1} \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$(-1)^1 cb$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# The Determinant of $3 \times 3$ matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{2} \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

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$$(-1)^1 ahf$$

# Another Perspective

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's row reduce an arbitrary  $2 \times 2$  matrix:

$$\begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim \begin{bmatrix} a & b & 1 & 0 \\ 0 & ad-bc & 0 & 0 \end{bmatrix}$$

$$Ax = 0$$

sol'n set

If  $ad-bc=0 \Rightarrow$  free variable

$\Rightarrow Ax = 0$  has nontrivial sol'n's

$\Rightarrow A$  not invertible

$$\det(A) = ad - bc = 0 \Leftrightarrow A \text{ not invertible}$$

# Another Perspective

Let's row reduce an arbitrary  $3 \times 3$  matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\left[ \begin{array}{ccc} \cancel{f} & \cancel{f} & \cancel{f} \\ 0 & \cancel{f} & \cancel{*} \\ 0 & 0 & a(\det A) \end{array} \right]$$

# **Determinants and Invertibility**

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So we can yet again extend the IMT:

- »  $A$  is invertible
- »  $\det(A) \neq 0$
- » 0 is not an eigenvalue

$A\vec{v} = \vec{0}$  has some nontrivial sol'n

so  $\text{Nul } A \neq \{\vec{0}\}$

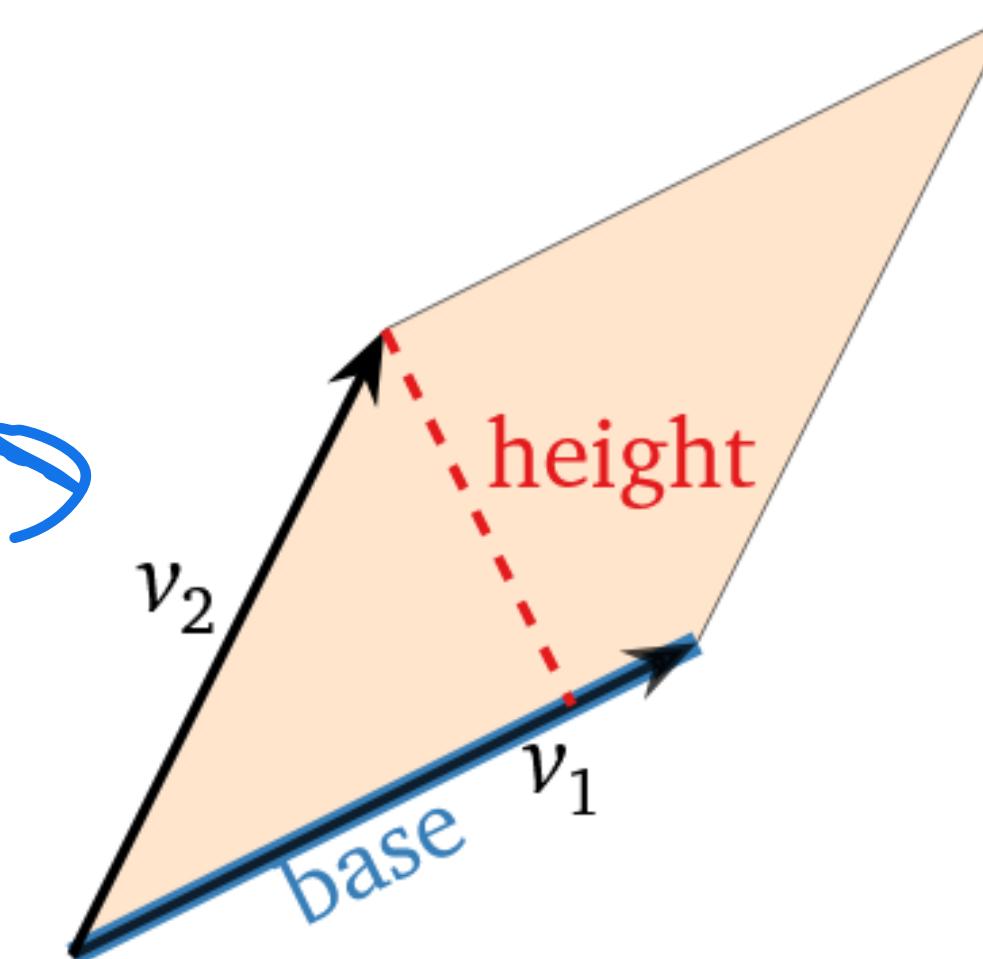
These must be all true or all false.

so  $A$  not invertible

# A Geometric Interpretation: Volume

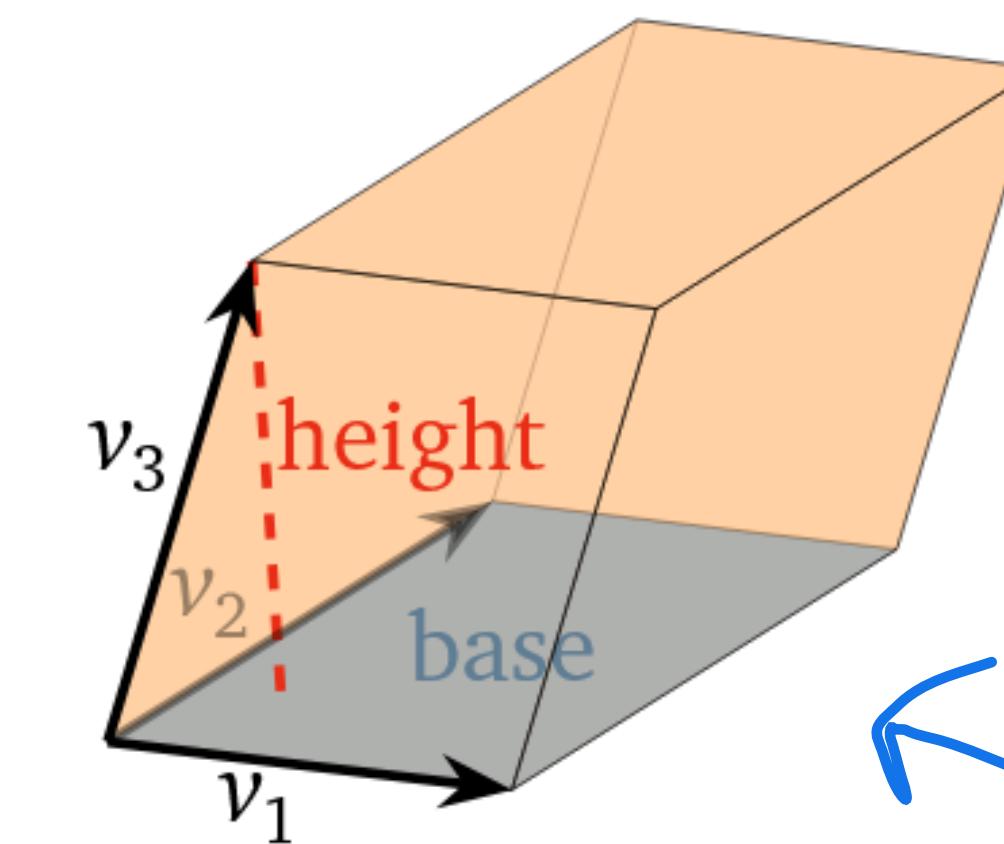
$$|\det[\vec{v}_1 \vec{v}_2]|$$

$$\text{vol}(P)$$



$$|\det[\vec{v}_1 \vec{v}_2 \vec{v}_3]|$$

$$\text{vol}(P)$$



$\wedge$  not invertible  $\Rightarrow \text{rank}(\text{col } A) < n \Rightarrow P$  has zero volume  
 $\Rightarrow \det(A) = 0$

(look up cofactor expansion also)

## Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

*echelon form*

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$$\det(A) = \frac{(-1)^s \text{ product of diagonal entries}}{c \ 0 \text{ if } A \text{ is not invertible}}$$

$U_{11} U_{22} \dots U_{nn}$

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# Example

$$S=0+1$$
$$C=1 \cdot (-\frac{1}{2})$$

$$R_2 \leftarrow R_2 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{\text{① Swap } R_2, R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & -1 \end{bmatrix} \xrightarrow{\text{② } R_2 \leftarrow -\frac{1}{2}R_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{③ } R_3 \leftarrow R_3 + 6R_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A) = \frac{(-1)^1}{-\frac{1}{2}} (1)(1)(-1) = -2$$

# Example (Again)

$$S = 0 + (+)$$

$$C = 1$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1}$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & \frac{1}{2} \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 3 & \frac{1}{2} \end{bmatrix}$$

$$\det A = \frac{(-1)^2}{1} (2)(-2)\left(\frac{1}{2}\right) = -2$$

$$\xrightarrow{R_3 + \frac{3}{2}R_2}$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The definition holds no matter  
which sequence of row  
operations you use.

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3. Determine the product of entries along the diagonal of  $U$ , call this  $P$ .
4. The determinant of  $A$  is  $\frac{(-1)^s P}{c}$ .

# Question

$$S = 1$$

$$C = 1$$

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

$$\det A = \frac{(-1)^1(1)(2)(1)}{1} = -2$$

Find the determinant of the above matrix.

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

# **Answer**

# The Shorter Version

Beyond small matrices, we'll just use a computer

With NumPy:

*numpy.linalg.det(A)*

# **Properties of Determinants**

# Properties of Determinants (1)

$$\det(AB) = \det(A) \det(B)$$

$$\det(E_1 \dots E_k A) = \det(E_1) \dots \det(E_k) \det(A)$$

It follows that  $AB$  is invertible if and only if  $A$  and  $B$  are invertible

(we won't verify this)

# Example Question

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Use the fact that  $\det(AB) = \det(A)\det(B)$  to give an expression for  $\det(A^{-1})$  in terms of  $\det(A)$ .

Hint. What is  $\det(I)$ ?  $\rightarrow 1$

$$\det(I) = \det(A \cdot A^{-1}) = \det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

## Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that  $A^T$  is invertible if and only if  $A$  is invertible.

(we also won't verify this)

# Example Question

orthogonal matrices  
If  $A^{-1} = A^T$ , then what are the possible values of  $\det(A)$ ?

$$\det(A^T) = \det(A^{-1}) = \frac{1}{\det A}$$

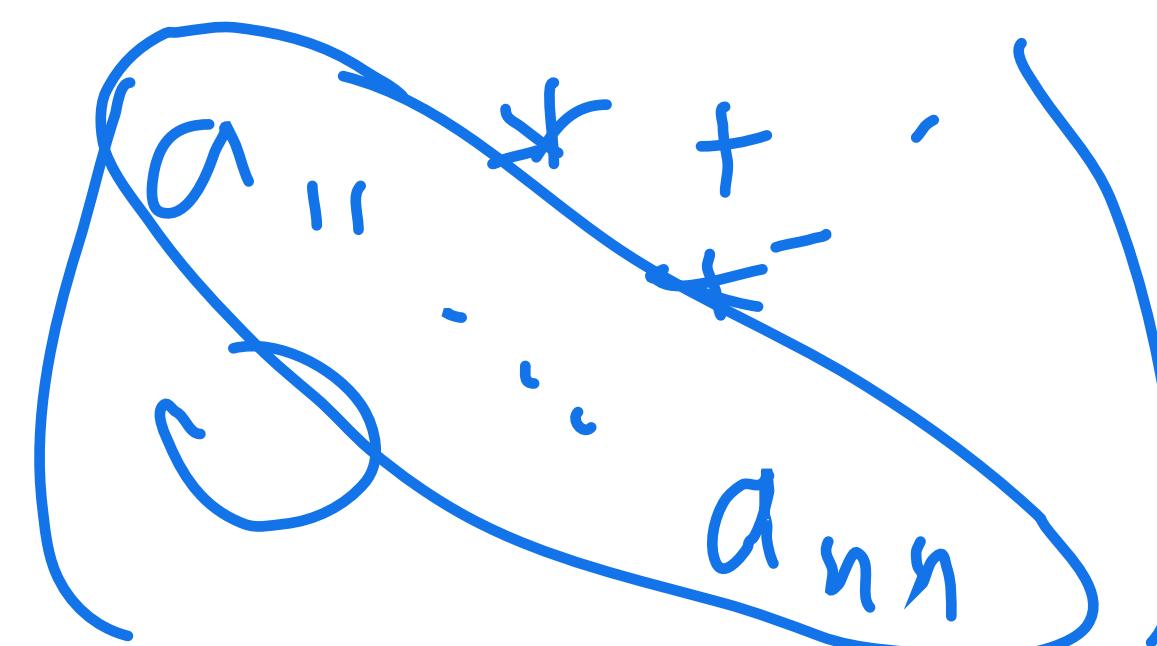
$\curvearrowleft$

$$\frac{1}{\det A} = \det A \Rightarrow (\det A)^2 = 1$$
$$\det A = \pm 1$$

# Properties of Determinants (3)

**Theorem.** If  $A$  is triangular, then  $\det(A)$  is the product of entries along the diagonal.

Verify:



$$A = \begin{pmatrix} a_{11} & & 0 \\ * & \ddots & \\ & & a_{nn} \end{pmatrix}$$

$$\det(A) = \det(A^T)$$

turns it into an upper triangular matrix

# **Answer**

# Characteristic Equation

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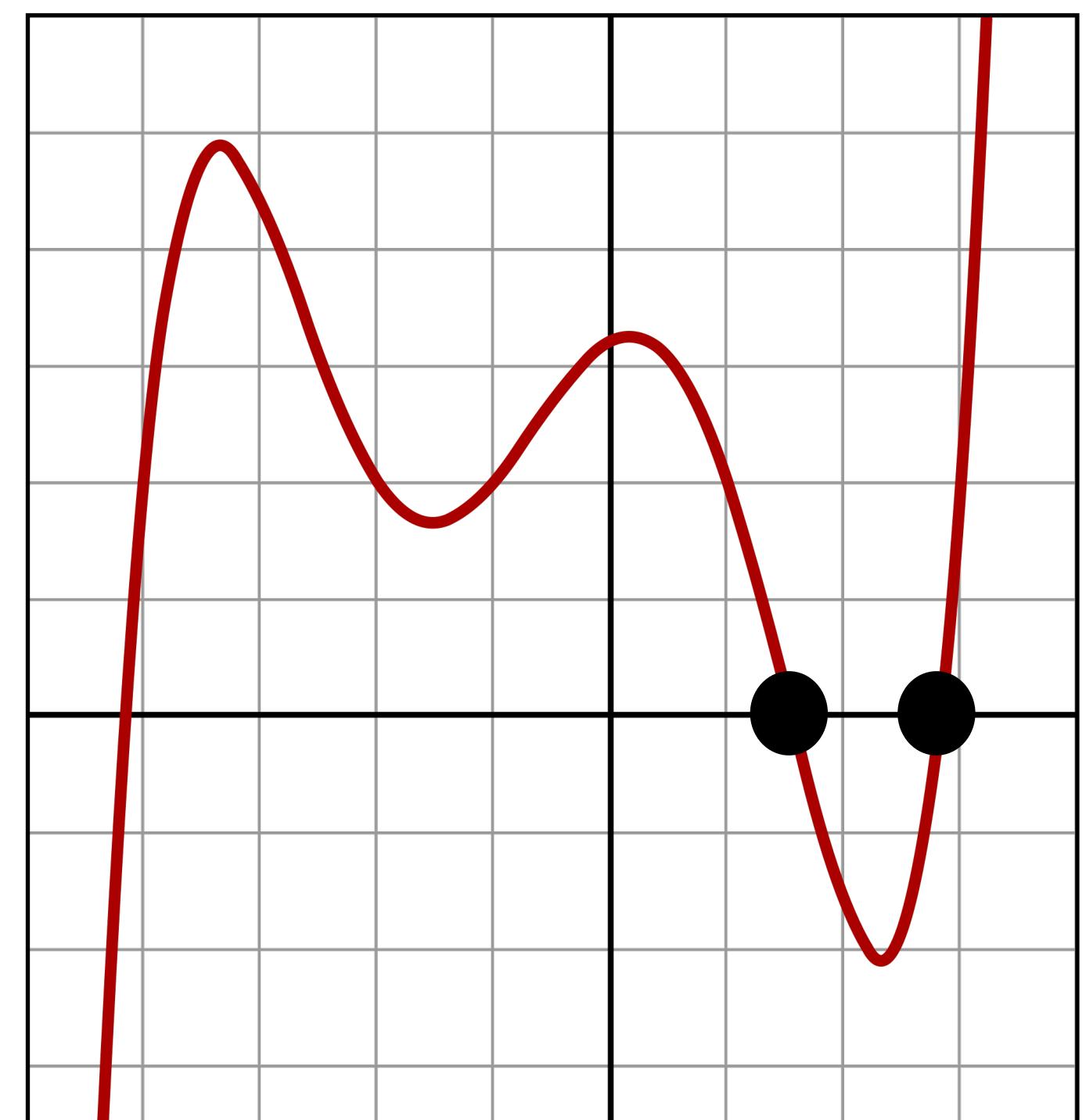
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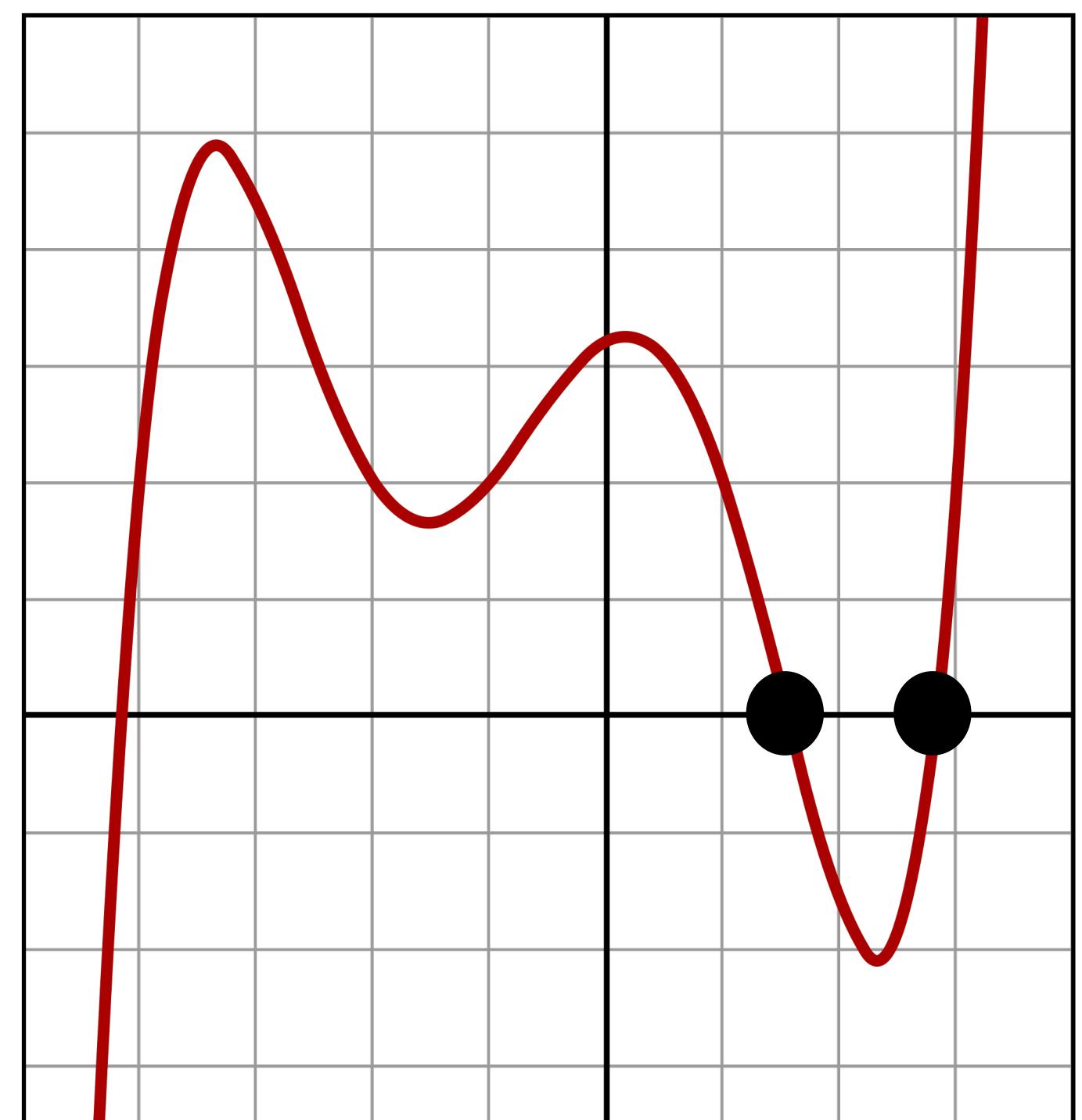
Then  $\det(A - \lambda I)$  is a **polynomial**.

# Reminder: Polynomial Roots



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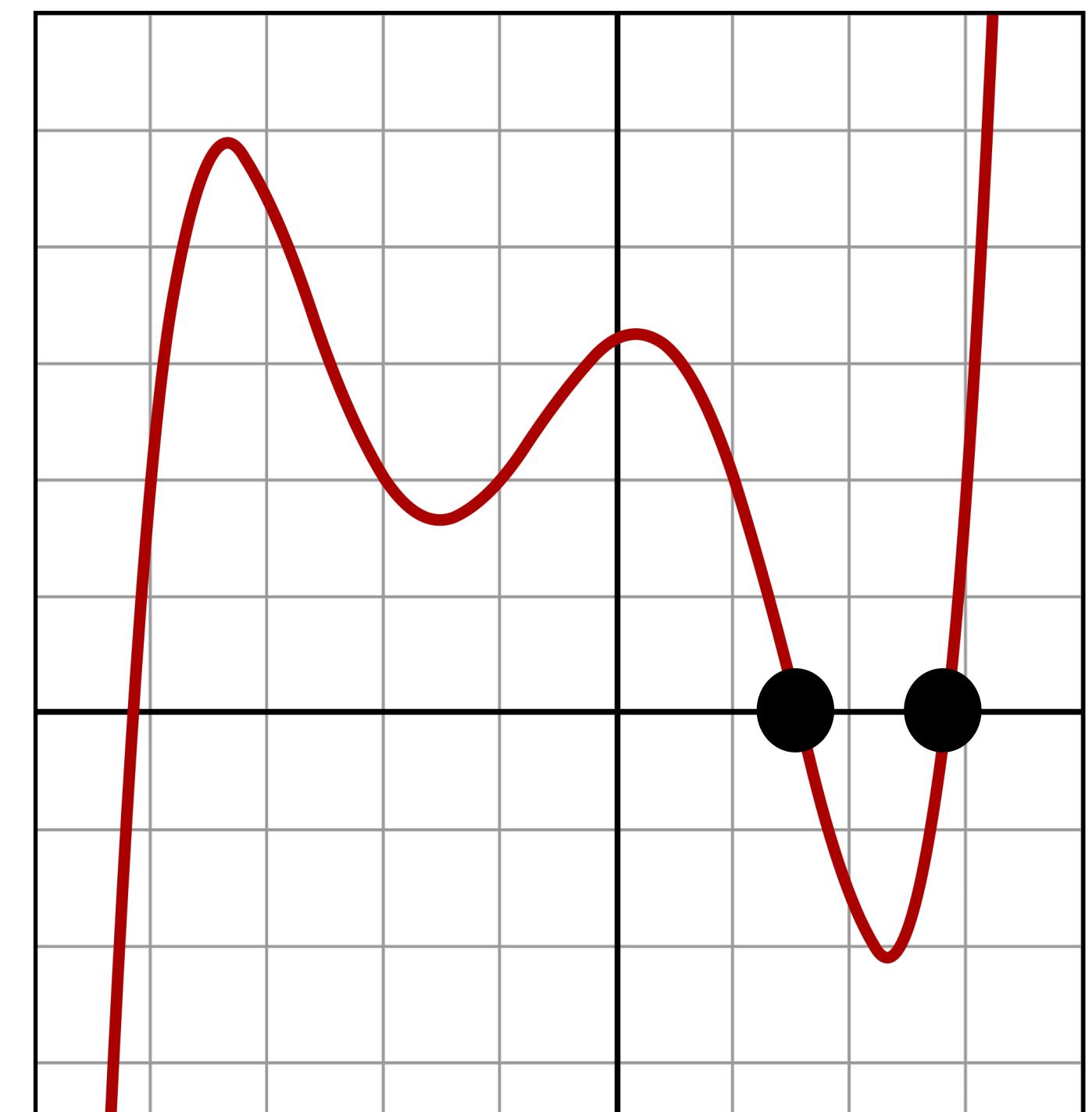
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If  $r$  is a root of  $p(x)$ , then it is possible to find a polynomial  $q(x)$  such that

$$p(x) = (x - r)q(x)$$



# **Characteristic Polynomial**

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**Definition.** The **characteristic polynomial** of a matrix  $A$  is  $\det(A - \lambda I)$  viewed as a polynomial in the variable  $\lambda$ .

This is a polynomial with the eigenvalues of  $A$  as roots.

So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

# "Deriving" the characteristic polynomial

Q: When is  $\lambda$  an eigenvalue for  $A$ ?

A: When  $(A - \lambda I)\vec{v} = 0$  has nontrivial solutions.

↓  
 $(A - \lambda I \text{ not invertible})$

$$\det(A - \lambda I) = 0$$

Hence, the characteristic polynomial

# Example: $2 \times 2$ Matrix

$$\text{rk} \left( A - \frac{1+\sqrt{5}}{2} I \right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = (1-\lambda)(-\lambda) - (1)(1)$$
$$= \boxed{\lambda^2 - \lambda - 1}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{1 \pm \sqrt{1-4(-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

# Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 0 & 6 \\ 0 & -\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = (1-\lambda)^2(-\lambda)(4-\lambda)$$
$$= (\lambda-1)^2 \lambda (\lambda-4)$$

# **How To: Finding Eigenvalues**

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**Solution.** Find the roots of the characteristic polynomial of  $A$ .

# An Observation: Multiplicity

$$\lambda^1(\lambda - 1)^2(\lambda - 4)^1 \quad \text{multiplicities}$$

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

**Is the multiplicity meaningful in this context?**

# Multiplicity and Dimension

**Theorem.** The dimension of the eigenspace of  $A$  for the eigenvalue  $\lambda$  is at most the multiplicity of  $\lambda$  in  $\det(A - \lambda I)$ .

The multiplicity is an upper bound on "how large" the eigenspace is.

dimensionality

"Eigenspace $_{\lambda}$ " =  $\text{Null}(A - \lambda I)$

not notation

# Example

$$A\vec{v} = \vec{0} \Rightarrow \vec{0} \Rightarrow \dim(\text{Nu}(A)) = 0$$

Since  $x$  not present  
as a factor

$$\Rightarrow \text{rank } A = 5$$

Let  $A$  be a  $5 \times 5$  matrix with characteristic polynomial  $(x - 1)^3(x - 3)(x + 5)$ .

» What is  $\text{rank}(A)$ ?

~~$\text{rk } A \geq 3$~~

# distinct nonzero eigenvalues

» What is the minimum possible rank of  $A - I$ ?

$$\text{rk } A - I + \text{nullity } A - I = 5$$

$$\text{rk } A - I \geq 2$$

$$\leq 3$$

$$\dim(\text{Nu}(A - I)) \leq 3$$