The Characteristic Equation

Geometric Algorithms
Lecture 19

CAS CS 132

Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Determine the dimension of the eigenspace of A for the eigenvalue 4.

(try not to do any row reductions)

Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

see A1 for answer

Objectives

1. Briefly recap eigenvalues and eigenvectors
2. Get a primer on determinants
3. Determine how to find eigenvalues (not just verify them)

Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

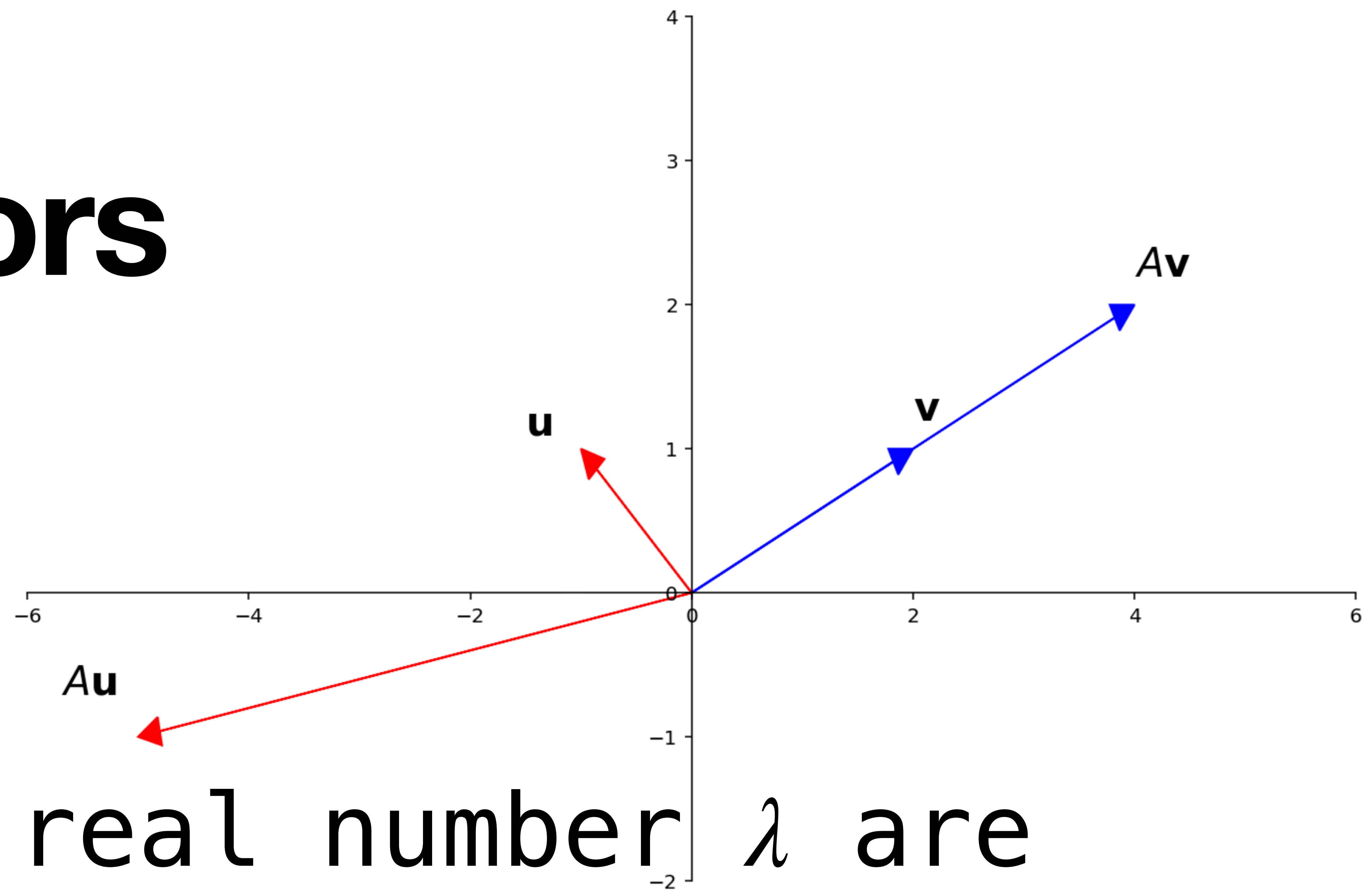
polynomial roots

triangular matrices

multiplicity

Recap

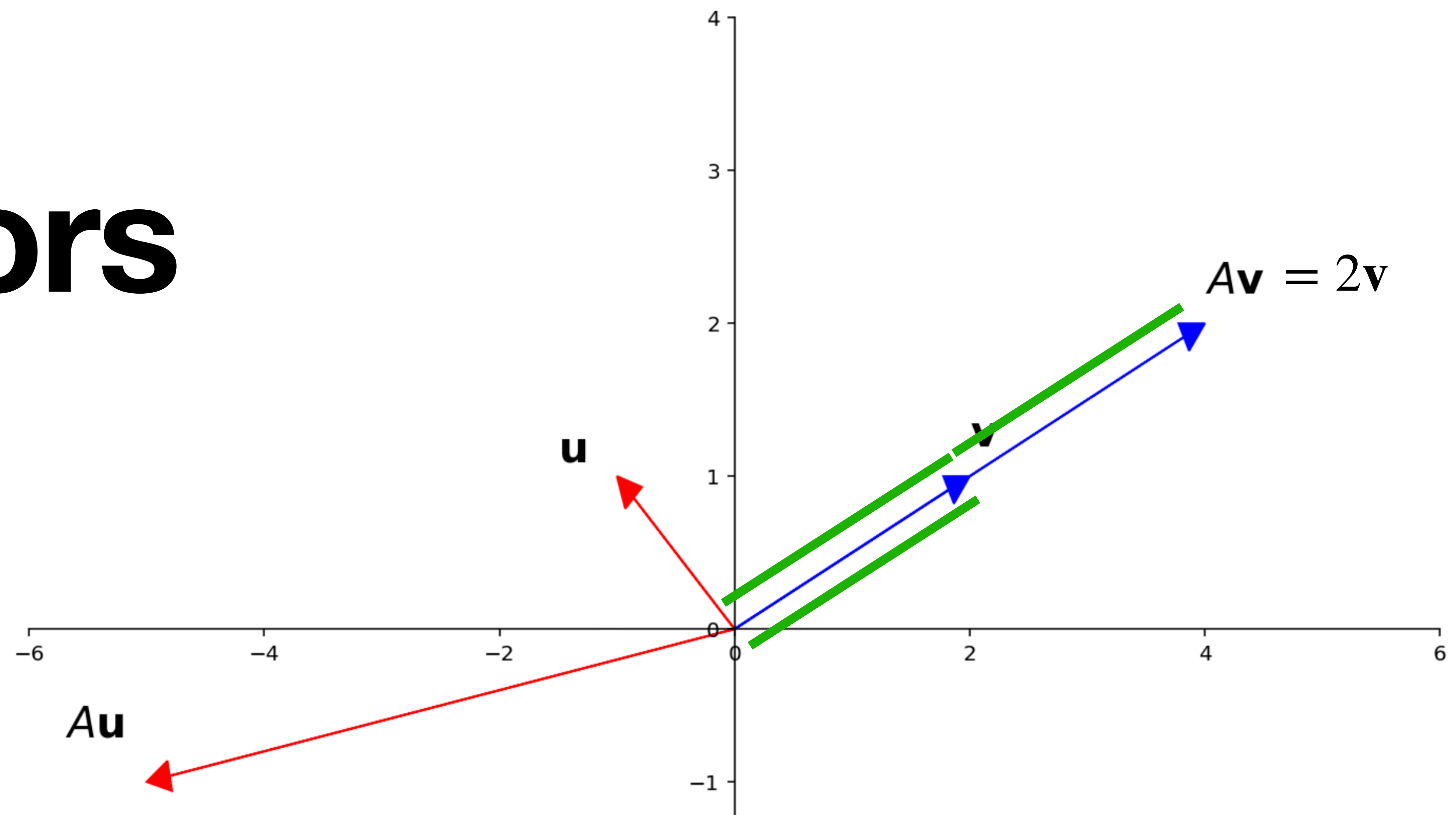
Recall: Eigenvalues/vectors



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

Recall: Eigenvalues/vectors



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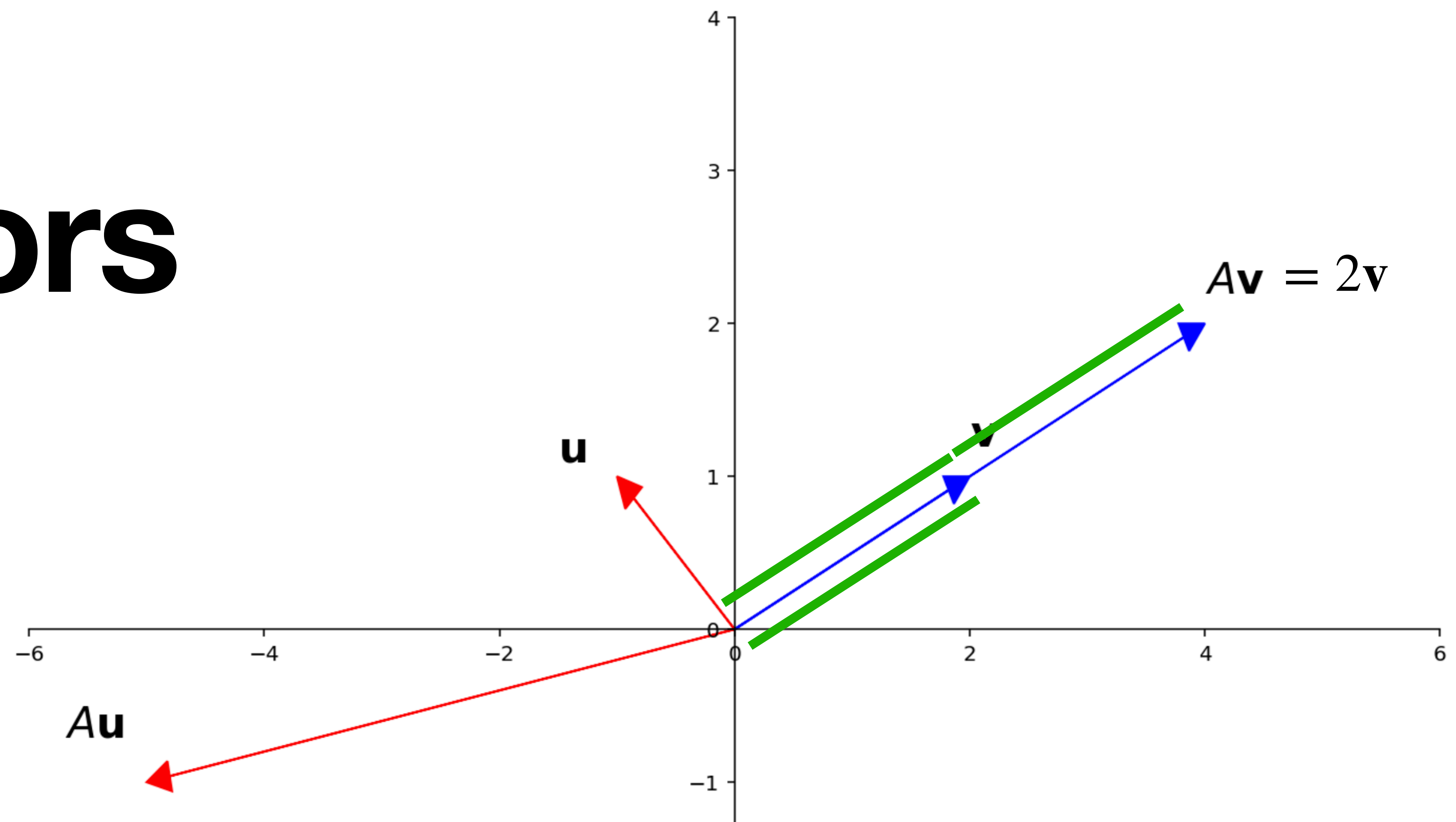

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$$A\mathbf{v} = \lambda\mathbf{v}$$

$\mathbf{v}$ **is "just scaled" by** A**, not rotated**

Recall: Verifying Eigenvectors

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Question. Determine if $\mathbf{v}$ is an eigenvector of A and determine the corresponding eigenvalues.

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Solution. Easy. Work out the matrix-vector multiplication.

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix} \times$$

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$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

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If we don't need the vector we can just show that $A - \lambda I$ is ***not*** *invertible (by IMT).*

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Solution. Find a basis for $\text{Nul}(A - \lambda I)$.

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(we did this for our recap problem)

How do eigenvectors relate to linear dynamical systems?

Recall: (Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

In other word, it does not depend on A^k and is not **recursive**

Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed for the above linear dynamical system.

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

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<u>The Key Point.</u> This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of $\mathbb{R}^2$:

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows <u>exponentially in λ_1</u>).

Verify:

Eigenbases

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We can represent vectors as ***unique*** *linear combinations of eigenvectors.*

Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

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for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

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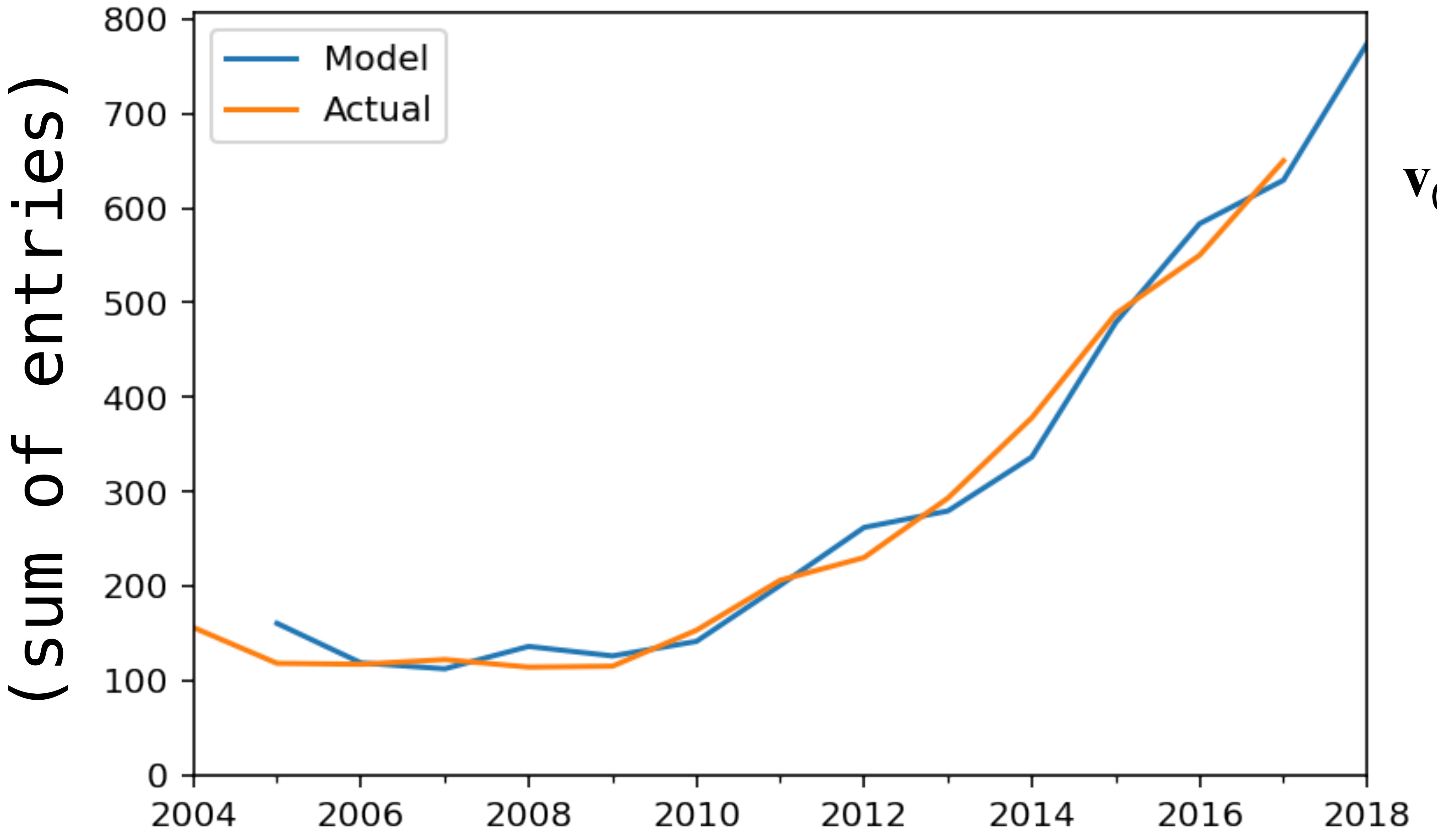
The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \texttt{\# of year 1 students enrolled in 2024} \\ \texttt{\# of year 2 students enrolled in 2024} \\ \texttt{\# of year 3 students enrolled in 2024} \\ \texttt{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the <u>largest eigenvalue</u>.

moving on...

Finding Eigenvalues

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Question. Determine the eigenvalues of A, along with their associated eigenspaces.

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Solution (Idea). Can we somehow "solve for λ" in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

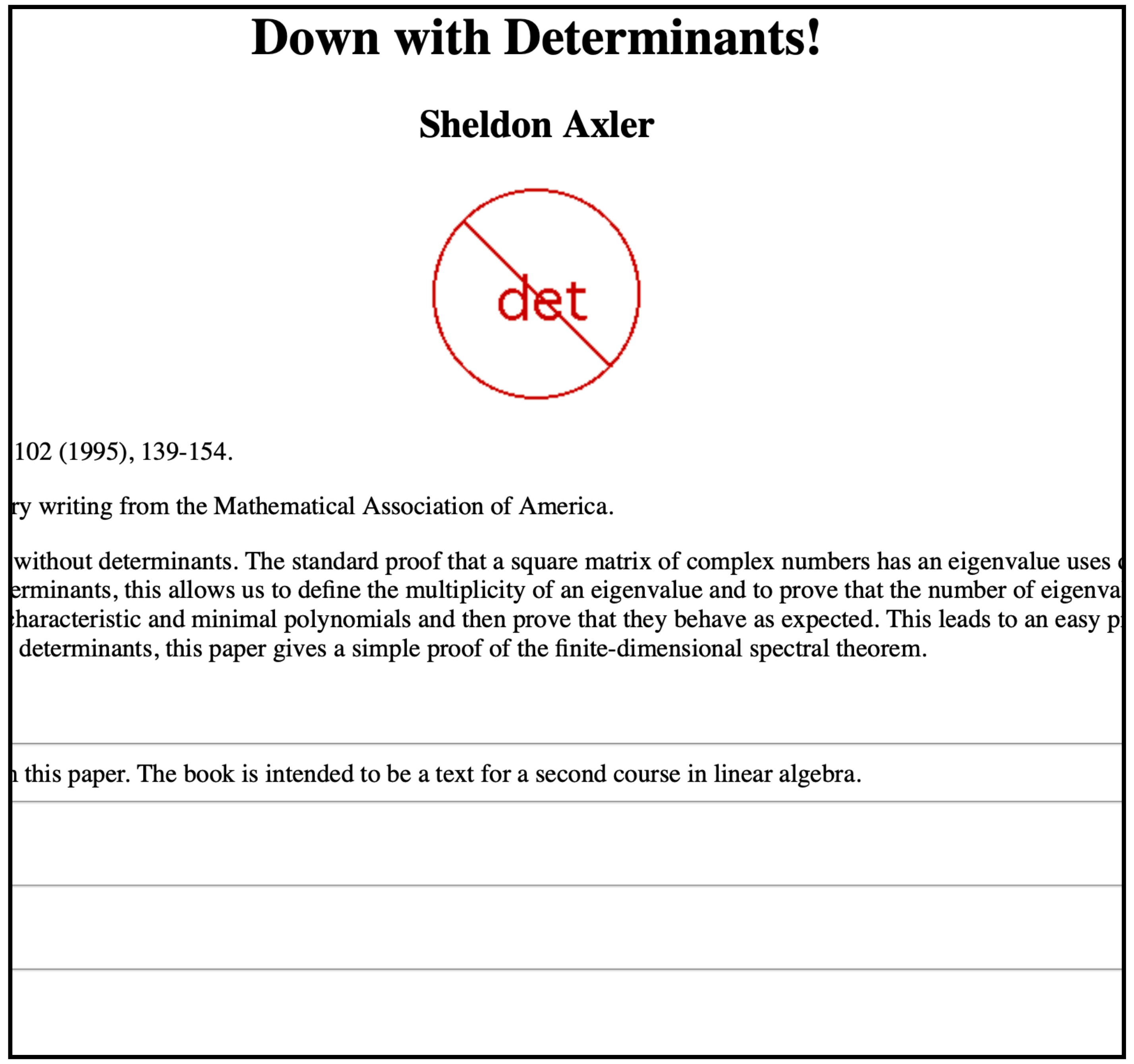
Determinants

An Aside: Determinants are Mysterious

Determinants are strangely polarizing

Some people love them, some people hate them

We'll only scratch the surface...



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Notation. We will write $\det(A)$ for the determinant of A.

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In broad strokes, it's a big sum of products of entries of A.

A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
FUNCTION det(A):
  total = 0
  FOR all matrix B we can get by swapping a bunch of rows of A:
    s = 1 IF (# of swaps necessary) is even ELSE −1
    total += s ∗ (product of the diagonal entries of B)
  RETURN total
```

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow^{0} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$(-1)^{0} ad$$

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$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow^{1} \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$(-1)^{1} cb$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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$$(-1)^0 aei$$

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^2 \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

$$(-1)^2 gbf$$

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$$(-1)^1 ahf$$

Another Perspective

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let's row reduce an arbitrary 2×2 matrix:

$$\begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim \left[\begin{array}{cc|c} a & b & 0 \\ 0 & ad-bc & 0 \end{array}\right]$$

$A\vec{x} = 0$
sol'n set

if $ad-bc = 0 \Rightarrow$ free variable

$\Rightarrow A\vec{x} = 0$ has nontrivial sol'ns

$\Rightarrow A$ not invertible

$\det(A) = ad - bc = 0 \Leftrightarrow A$ not invertible

Another Perspective

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Let's row reduce an arbitrary 3×3 matrix:

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & a(\det A) \end{bmatrix}$$

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So we can yet again extend the IMT:

» A is invertible
» $\det(A) \neq 0$
» 0 is not an eigenvalue

These must be all true or all false.

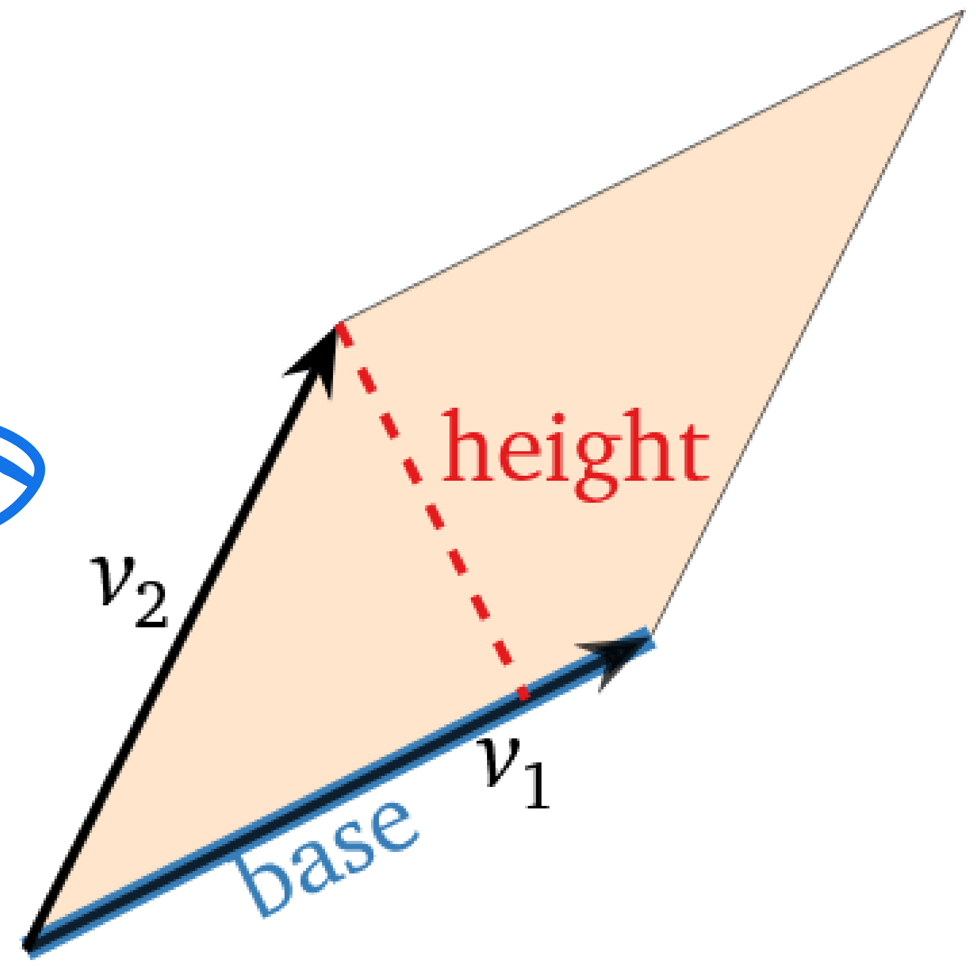
$A\vec{v} = 0\vec{v}$ has some nontrivial sol'n

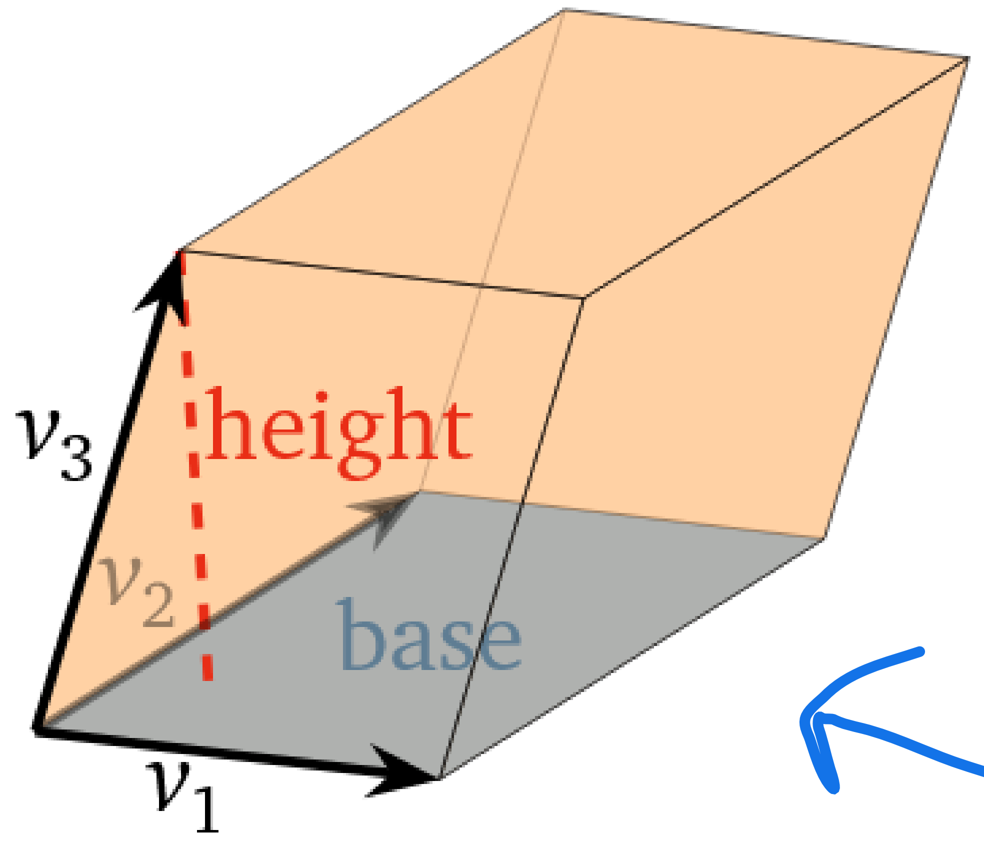
so $\mathrm{Nul}\, A \neq \{0\}$

so A not invertible

A Geometric Interpretation: Volume

$$\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\right| = \mathrm{vol}(P)$$





$$\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\right| = \mathrm{vol}(P)$$

A not invertible $\Rightarrow$ ~~span~~ $\dim(\mathrm{col}\, A) < n \Rightarrow P$ has zero volume

$\Rightarrow \det(A) = 0$

(look up cofactor expansion also)

Determinants (the definition we'll use)

echelon form

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

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product of diagonal entries

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Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} \underbrace{U_{11} U_{22} \ldots U_{nn}}$$

product of diagonal entries

0 if A is not invertible

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Example

$s = 0 + 1$

$c = 1 \cdot (-\tfrac{1}{2})$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

Let's find the determinant of this matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow[\text{② } R_2 \leftarrow -\frac{1}{2}R_2]{\text{① Swap } R_2, R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & -1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 6R_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A) = \frac{(-1)^1}{-\frac{1}{2}}(1)(1)(-1) = -2$$

Example (Again)

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$s = 0 + 1 + 1$

$c = 1$

Let's find the determinant of this matrix again but with a different sequence of row operations:

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & \frac{1}{2} \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 3 & \frac{1}{2} \end{bmatrix}$$

$\downarrow R_3 + \frac{3}{2}R_2$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\det A = \frac{(-1)^2}{1}(2)(-2)(\tfrac{1}{2}) = -2$$

The definition holds no matter which sequence of row operations you use.

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3. Determine the product of entries along the diagonal of U, call this P.

4. The determinant of A is $\dfrac{(-1)^s P}{c}$.

Question

$s = 1$

$c = 1$

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

s

$$\det A = \frac{(-1)^1}{1}(1)(2)(1)$$

$$= -2$$

Find the determinant of the above matrix.

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

The Shorter Version

`Beyond small matrices, we'll just use a computer`

`With NumPy:`

```
numpy.linalg.det(A)
```

Properties of Determinants

Properties of Determinants (1)

$$\det(AB) = \det(A)\det(B)$$

$$\det(E_k \cdots E_1 A) = \det(E_k) \cdots \det(E_1)\det(A)$$

It follows that AB is invertible if and only if A and B are invertible

(we won't verify this)

Example Question

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Use the fact that $\det(AB) = \det(A)\det(B)$ *to give an expression for* $\det(A^{-1})$ *in terms of* $\det(A)$.

Hint. What is $\det(I)$? $\to 1$

$$\det(I) = \det(A \cdot A^{-1}) = \det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that A^T is invertible if and only if A is invertible.

(we also won't verify this)

Example Question

orthogonal matrices

If $A^{-1} = A^T$, *then what are the possible values of* $\det(A)$?

$$\det(A^T) = \det(A^{-1}) = \frac{1}{\det A}$$

$$\det(A^T) = \det(A)$$

$$\frac{1}{\det A} = \det A \Rightarrow (\det A)^2 = 1$$

$$\det A = \pm 1$$

Properties of Determinants (3)

Theorem. If A is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:

$$\begin{pmatrix} a_{11} & * & * & \cdot \\ & \ddots & * & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{nn} \end{pmatrix} \qquad \det(A) = \det(A^T)$$

turns it into an upper triangular matrix

Answer

Characteristic Equation

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But a matrix may not have numbers as entries.

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We might think of the matrix $A - \lambda I$ has having *polynomials* as entries.

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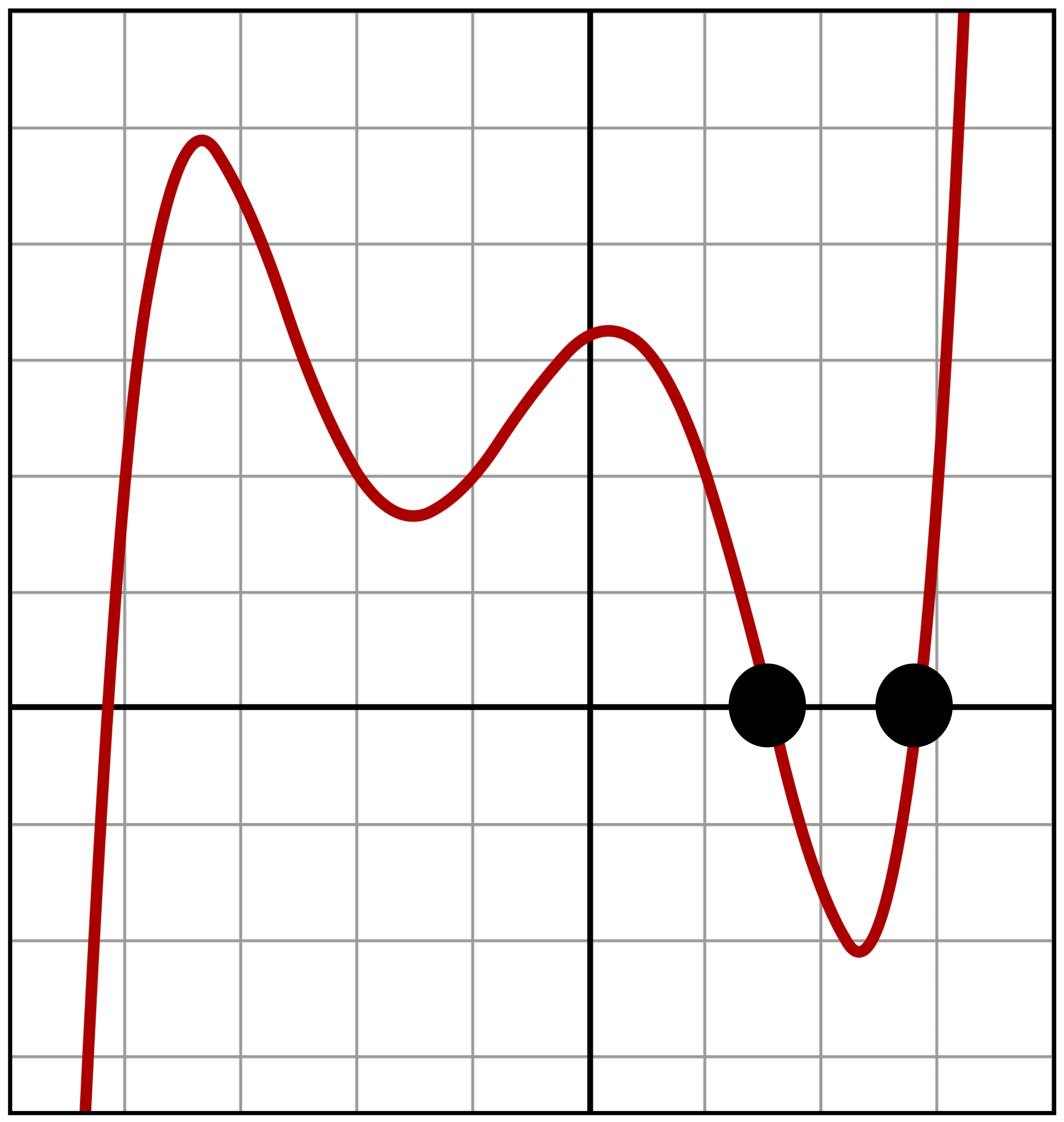
But a matrix may not have numbers as entries.

We might think of the matrix $A - \lambda I$ has having *polynomials* as entries.

Then $\det(A - \lambda I)$ is a **polynomial.**

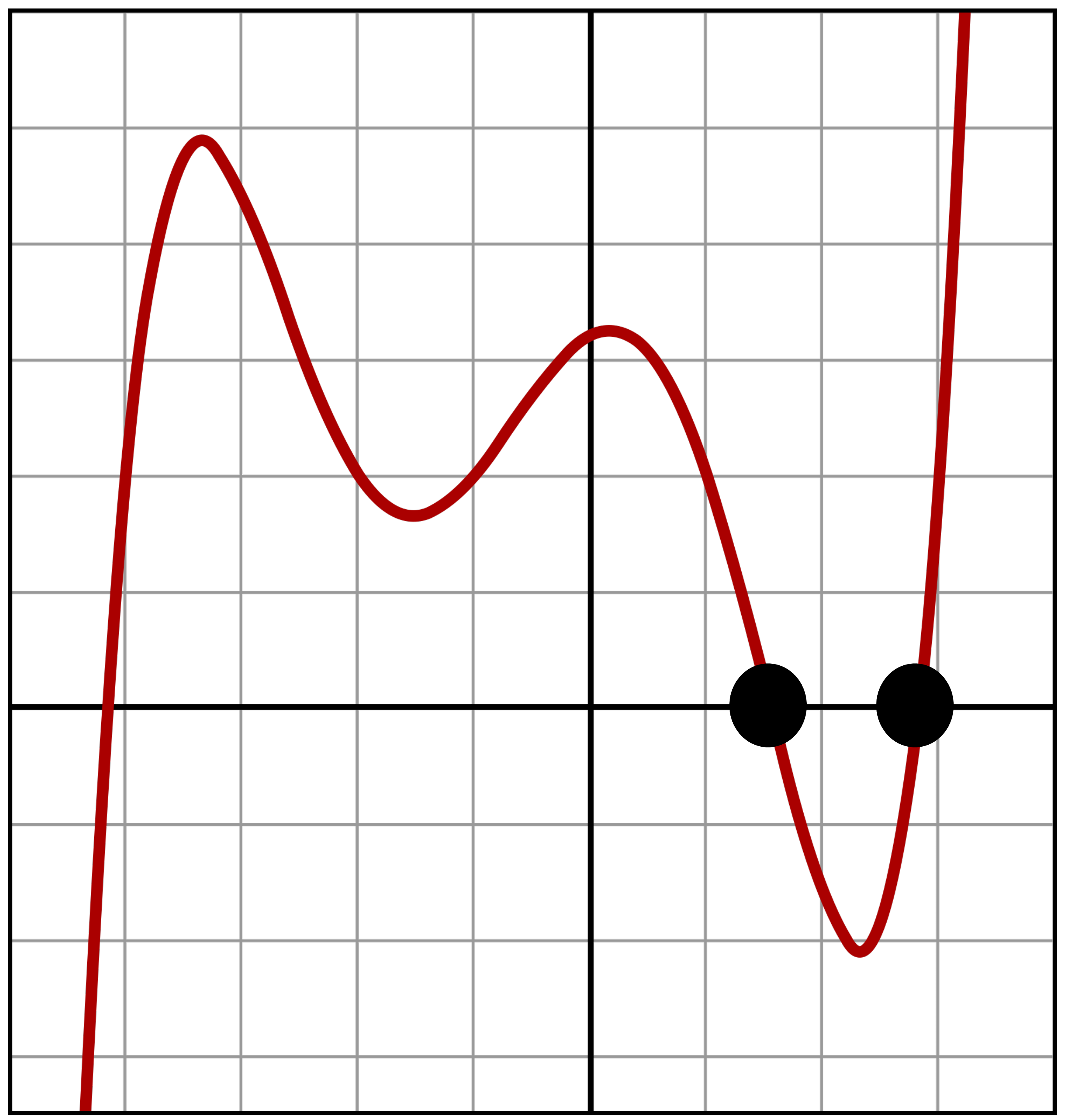
Reminder: Polynomial Roots



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.



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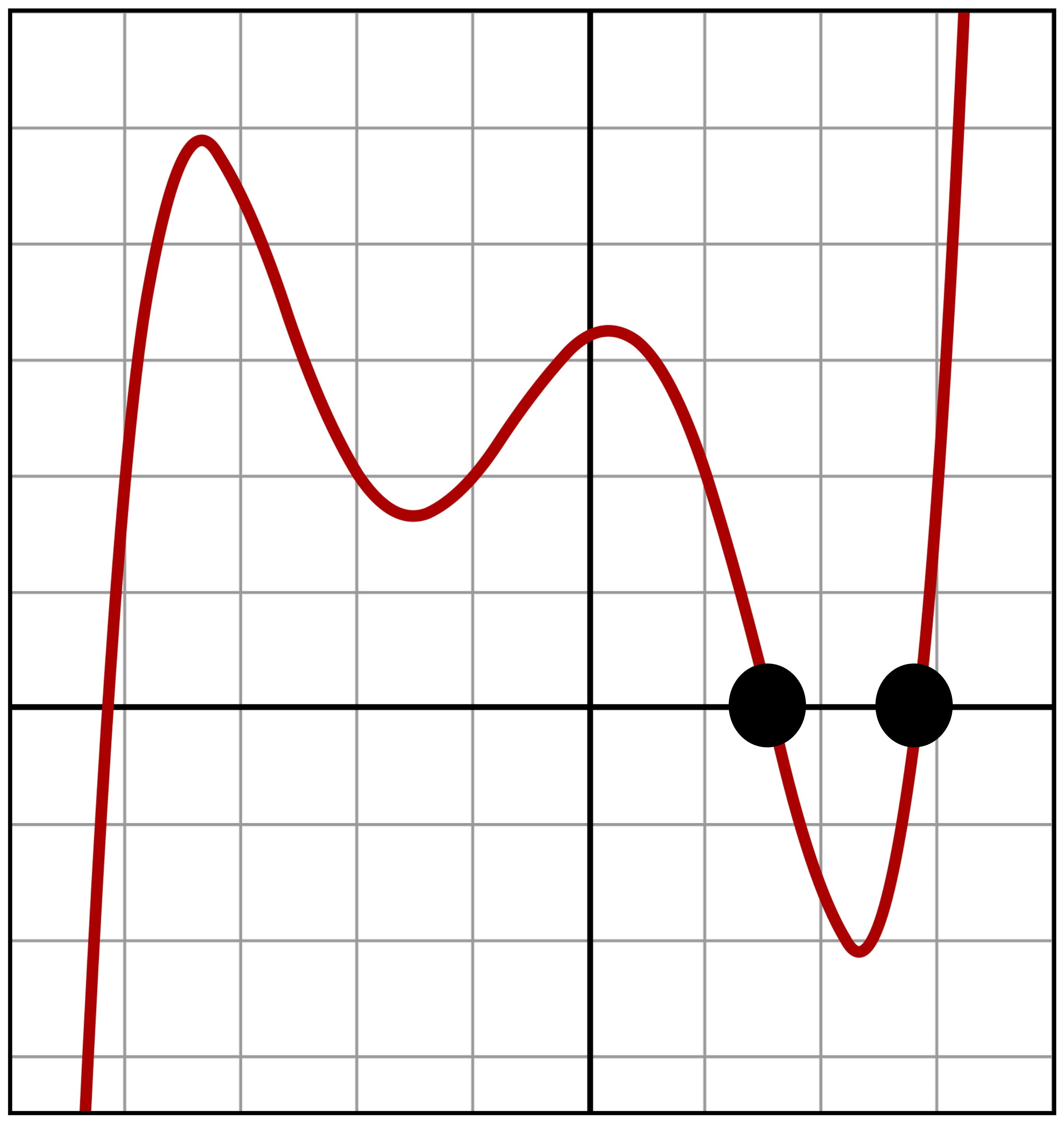
(A polynomial may have many roots)



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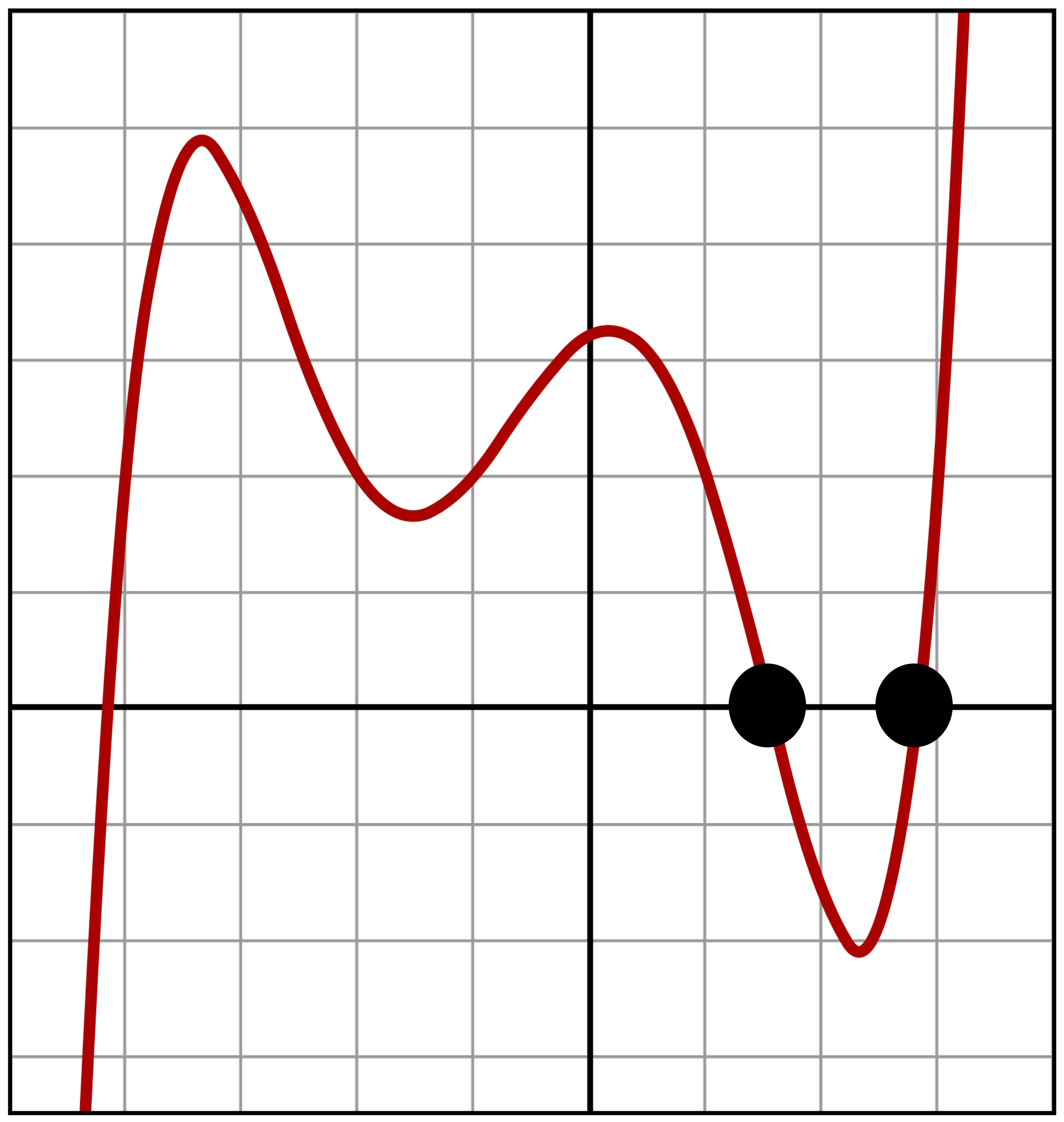
If r is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



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Characteristic Polynomial

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So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

"Deriving" the characteristic polynomial

Q: When is λ an eigenvalue for A?

A: When $(A-\lambda I)\vec{v}=0$ has nontrivial solutions.

$\Updownarrow$ ($A-\lambda I$ not invertible)

$\underline{\det(A-\lambda I)}=0$

Hence, the characteristic polynomial

Example: 2×2 Matrix

$$\mathrm{rk}\left(A - \frac{1+\sqrt{5}}{2} I\right) \quad \cancel{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \Rightarrow \det(A-\lambda I) = (1-\lambda)(-\lambda) - (1)(1)$$

$$= \boxed{\lambda^2 - \lambda - 1}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes <u>pre-factored</u>:

$$A-\lambda I = \begin{bmatrix} 1-\lambda & -3 & 0 & 6 \\ 0 & -\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} \Rightarrow \det(A-\lambda I) = (1-\lambda)^2(-\lambda)(4-\lambda)$$

$$= (\lambda-1)^2\lambda(\lambda-4)$$

How To: Finding Eigenvalues

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Question. Find all eigenvalues of the matrix A.

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Question. Find all eigenvalues of the matrix A.

Solution. Find the roots of the characteristic polynomial of A.

An Observation: Multiplicity

$$\boxed{\lambda^{1}}(\lambda - 1)^{\boxed{2}}(\lambda - 4)^{\boxed{1}}$$ multiplicities

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is <u>at most</u> the multiplicity of λ in $\det(A - \lambda I)$.

The multiplicity is an upper bound on "how large" the eigenspace is.

dimensionality

"$\text{Eigenspace}_\lambda$" $= \text{Nul}(A - \lambda I)$

not notation

Example

$A\vec{v} = 0\vec{v} = \vec{0}$

Let A be a 5×5 matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$.

» What is $\mathrm{rank}(A)$?

$\mathrm{rk}\, A \geq 3$ ← # distinct nonzero eigenvalues

» What is the minimum possible rank of $A - I$?

$\mathrm{rk}\, A - I + \underbrace{\mathrm{nullity}\, A - I}_{\leq 3} = 5$

$\boxed{\mathrm{rk}\, A - I \geq 2}$

$1 \leq \dim(\mathrm{Nul}(A-I)) \leq 3$