Linear Equations

Geometric Algorithms
Lecture 1

CAS CS 132

Outline

» Give a few motivating examples for the study of linear systems

» Formally define linear systems

» Solve some systems of linear equations

Keywords

Systems of linear equations

Solutions

Coefficient matrix

Augmented matrix

Elimination and Back-substitution

Replacement, interchange, scaling

Row Equivalence

(In)consistency

Motivation

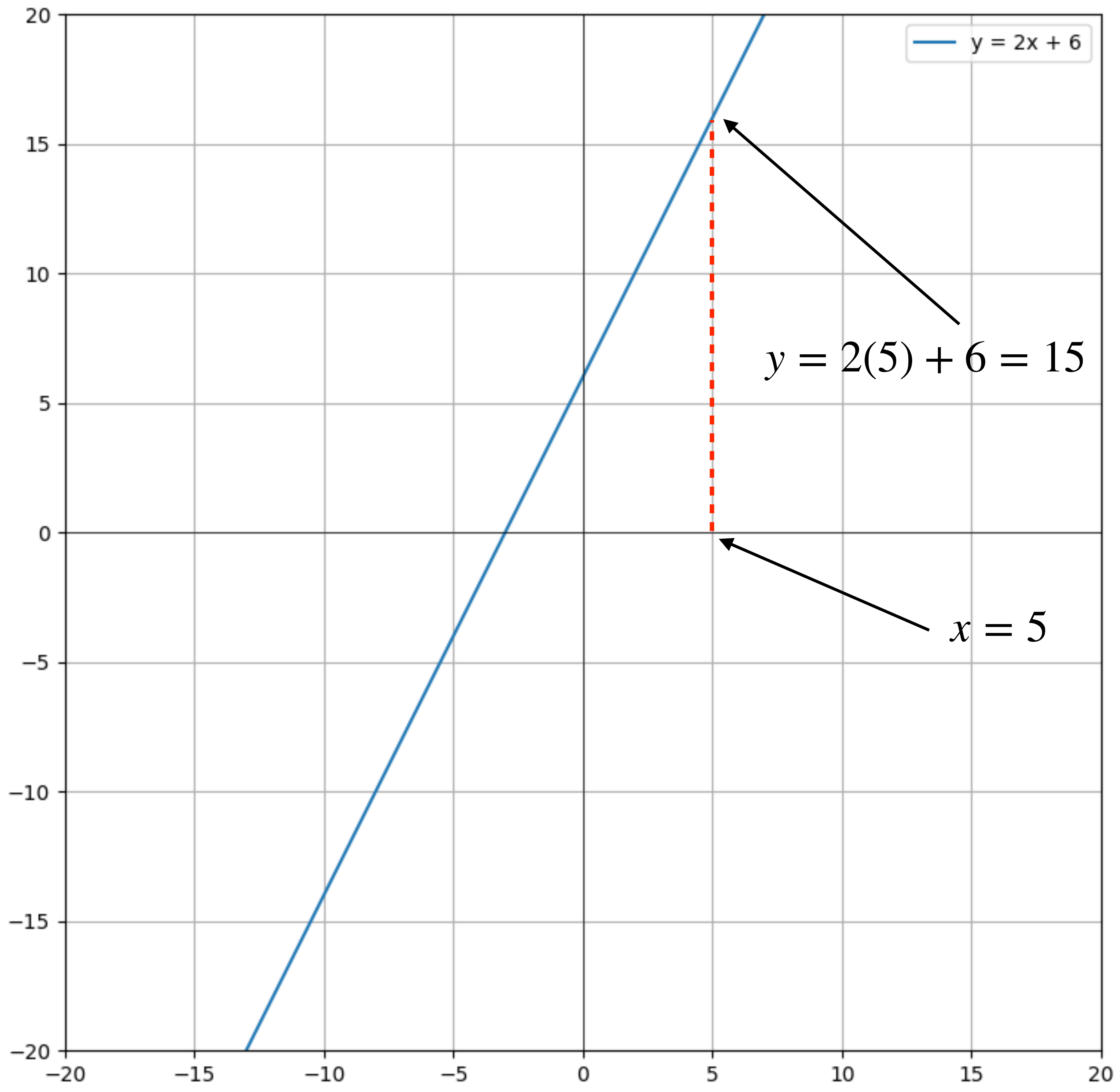
Lines (Slope-Intercept Form)

$$y = \boxed{m}x + \boxed{b}$$

slope y-intercept

Given a value of x, I can compute a value of y

Lines (Graph)



Lines (General Form)

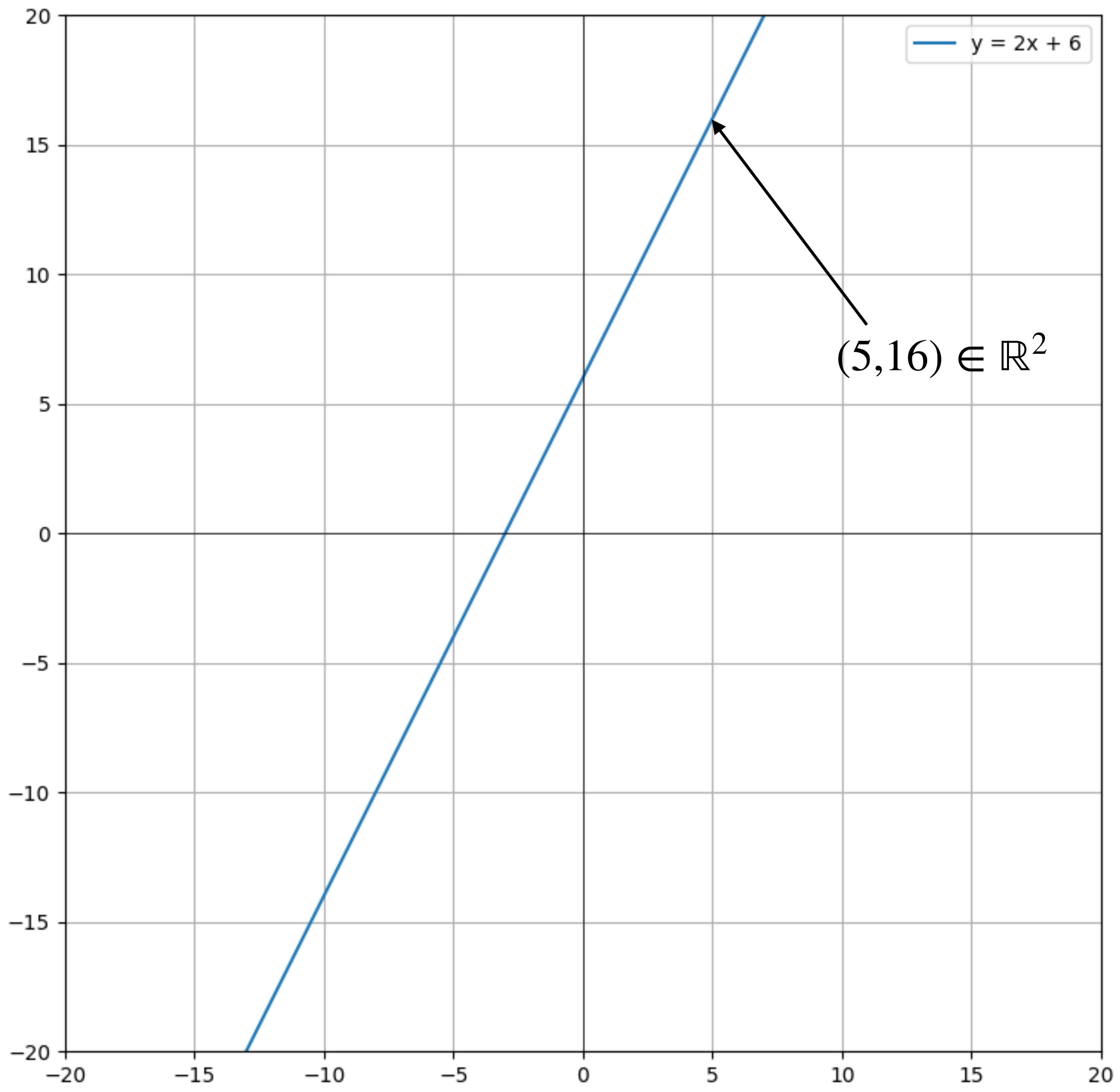
$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

What values of x and y make the equality hold?

Lines (Graph)



$$\{(x, y) : (-2)x + y = 6\}$$

Lines

slope-int → general

$$\boxed{(-m)}x + y = \boxed{b}$$

general → slope-int

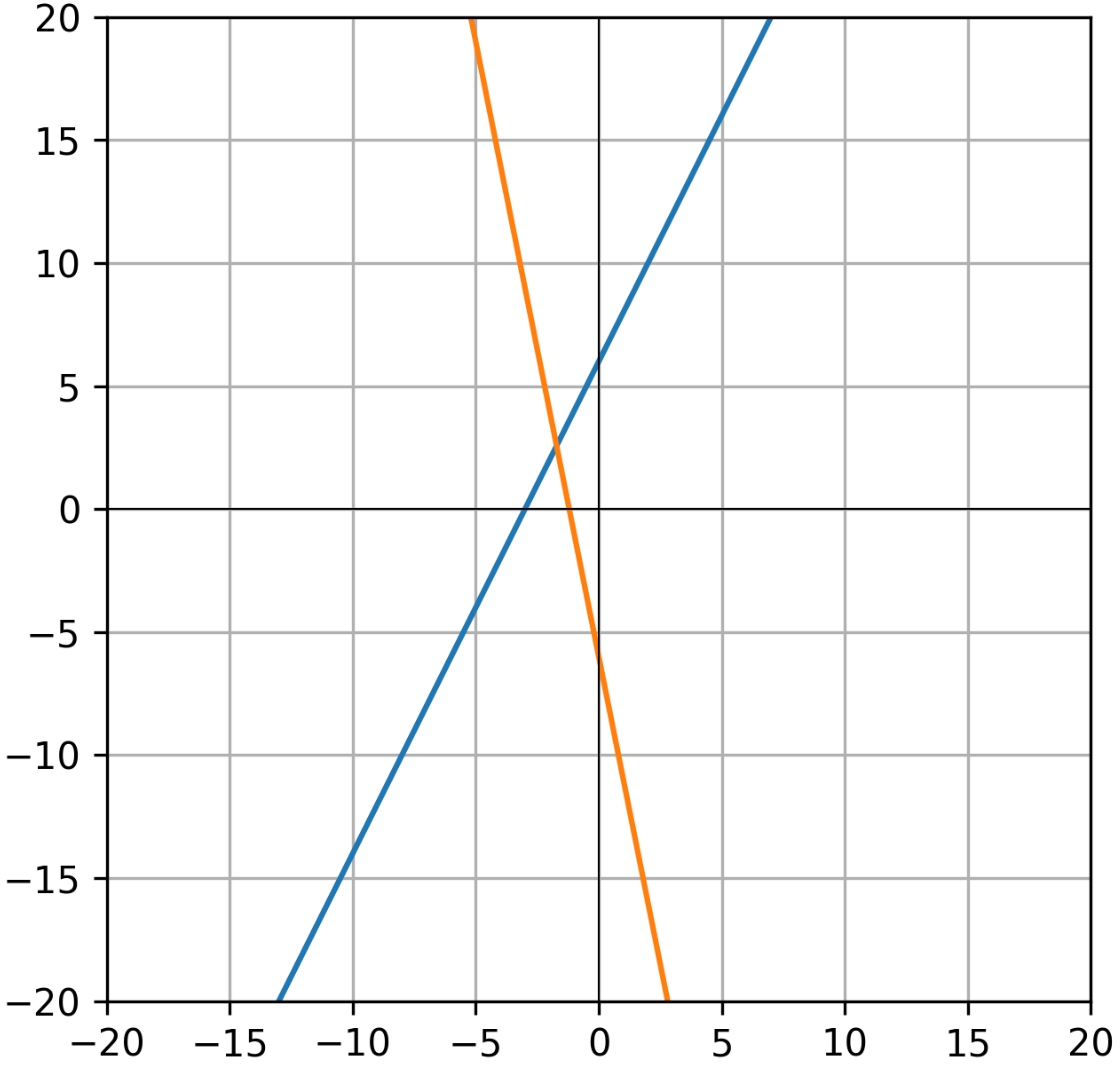
$$y = \boxed{\left(\frac{-a}{b}\right)}x + \boxed{\frac{c}{b}}$$

Line Intersection

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Question. Given two lines, where do they intersect?

Line Intersection (Graph)



Line Intersection (Alternative)

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

This is the same question

Example: Balancing Chemical Equations

$$C_6H_{12}O_6 \rightarrow C_2H_5OH + CO_2$$

Glucose Ethanol

We want to know how much ethanol is produced by fermentation (for science)

The **number of atoms** has to be *preserved* on each side of the equation

Balancing Chemical Equations

$$\alpha \mathrm{C_6H_{12}O_6} \rightarrow \beta \mathrm{C_2H_5OH} + \gamma \mathrm{CO_2}$$

Glucose Ethanol

$$6\alpha = 2\beta + \gamma \qquad \text{(C)}$$

$$12\alpha = 6\beta \qquad \text{(H)}$$

$$6\alpha = \beta + 2\gamma \qquad \text{(O)}$$

Balancing Chemical Equations

$$\alpha \mathrm{C_6H_{12}O_6} \to \beta \mathrm{C_2H_5OH} + \gamma \mathrm{CO_2}$$

`Glucose` `Ethanol`

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Formal Definitions

Linear Equations

Definition. A ***linear equation*** in variables $x_1, x_2, \ldots, x_n$ is an equation which can be written in the form

coefficients unknowns

$$\boxed{a_1}\boxed{x_1} + \boxed{a_2}\boxed{x_2} + \ldots + \boxed{a_n}\boxed{x_n} = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Examples

Linear Equations (Point sets)

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1s_1 + a_2s_2 + \ldots + a_ns_n = b\}$$

The collections of numbers such that the equation holds

Examples

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

Not a line...

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

A plane(!)

Example 1

$$0x + 0y + z = 5$$

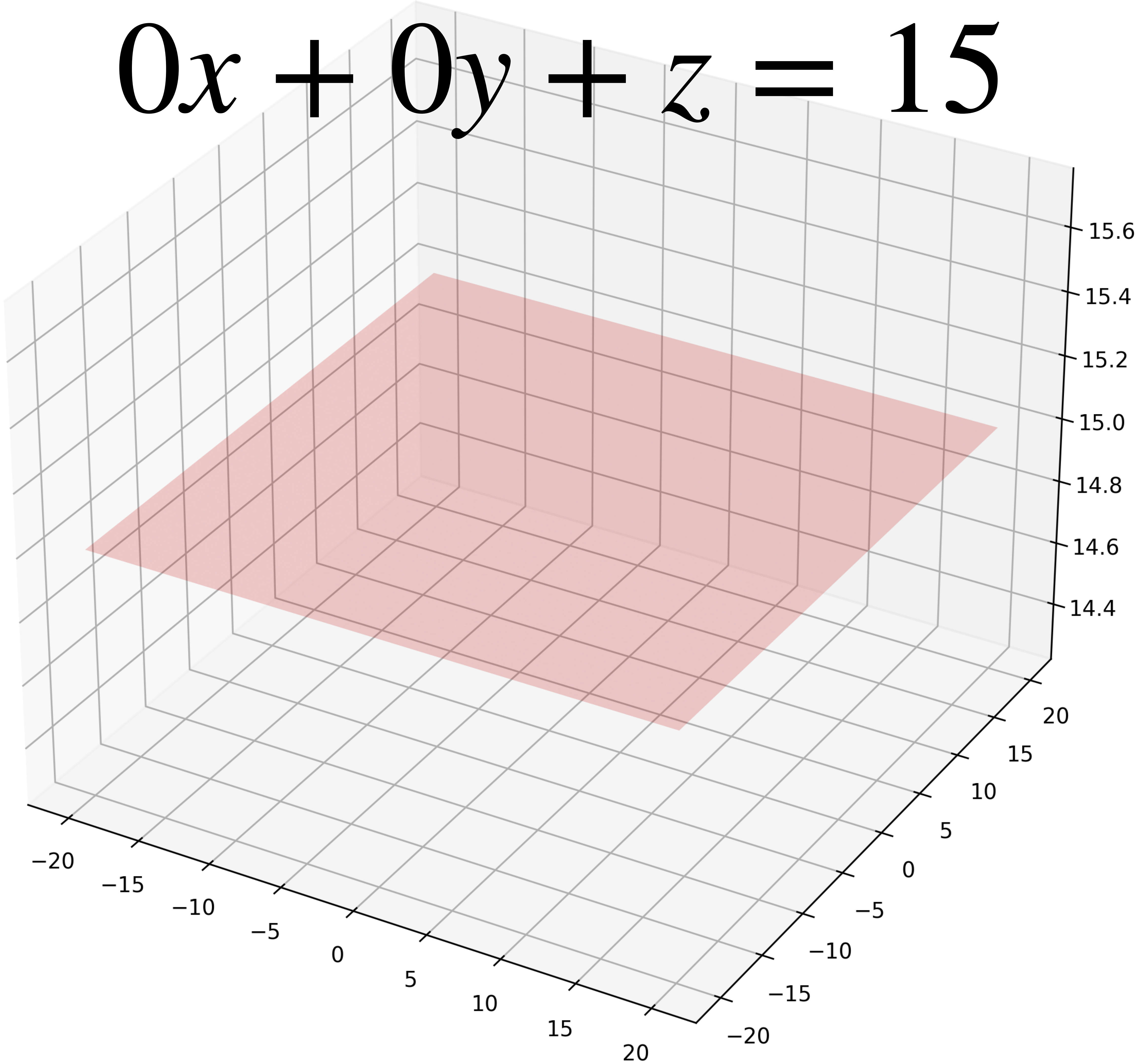
This equation describes the solution set

$$\{(x, y, z) : z = 5\}$$

so x and y can be whatever we want

Example 1

$$0x + 0y + z = 15$$



Example 2

$$-x + 0y + z = 5$$

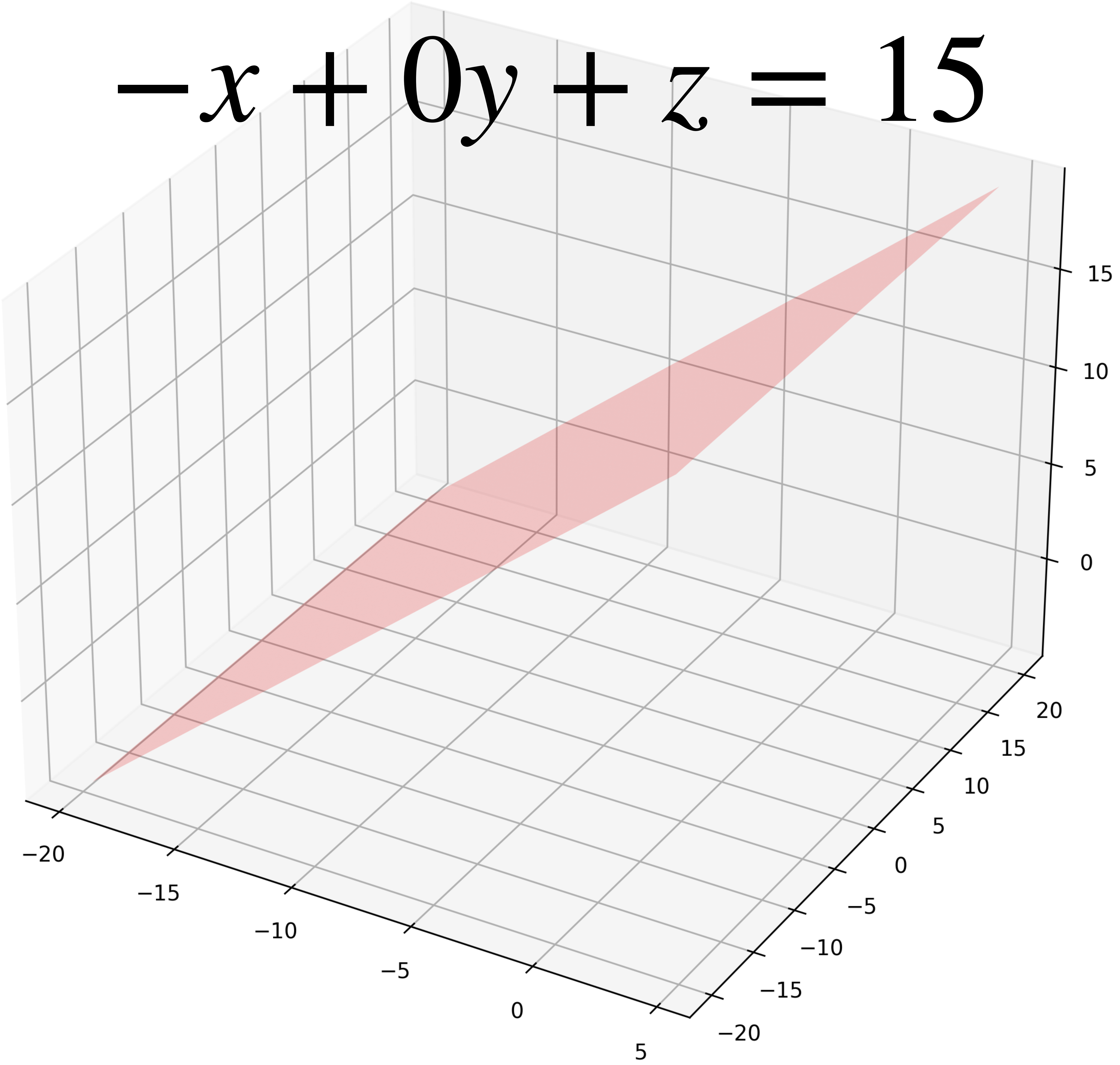
This equation describes the point set

$$\{(x, y, z) : z = x + 5\}$$

so y can be whatever we want

Example 2

$$-x + 0y + z = 15$$



Example 3

$$-x + -y + z = 5$$

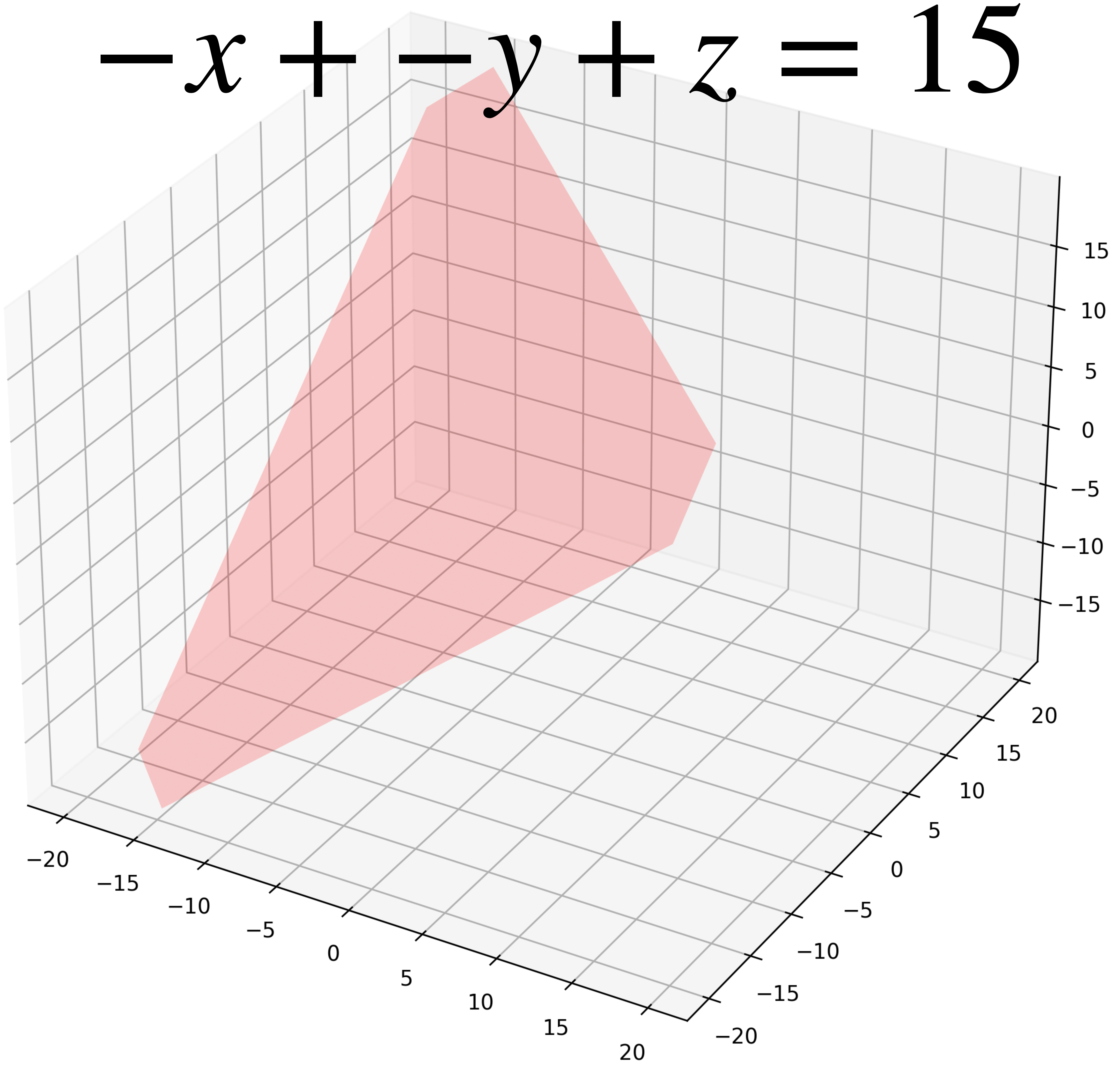
```
This equation describes the solution set
```

$$\{(x, y, z) : z = x + y + 5\}$$

```
so all variables depend on each other
```

Example 3

$$-x + -y + z = 15$$



XYZ-intercepts

$$ax + by + cz = d$$

Just like with lines, we can define

x-intercept: $\frac{d}{a}$ y-intercept: $\frac{d}{b}$ z-intercept: $\frac{d}{c}$

These three points define the plane

Question

I just lied

Give an example of a linear equation that defines a plane with an x–intercept and y–intercept but no z–intercept

Answer

Hyperplanes

After three dimensions, we can't visualize planes

The point set of a linear equation is called a **hyperplane**

<u>Theme of the course:</u> Hyperplanes "behave" like 3D planes in many respects

Systems of Linear Equations

Definition. A ***system of linear equations*** is just a collection of linear equations *over the same variables*

Definition. A ***solution*** to a system is a point that satisfies all its equations *simultaneously*

Example

linear system:

$$
\begin{aligned}
x + 2y &= 1 \\
-x - y - z &= -1 \\
2x + 6y - z &= 1
\end{aligned}
$$

solution: $(3, -1, -1)$

System of Linear equations (General-form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Does a system have a solution?

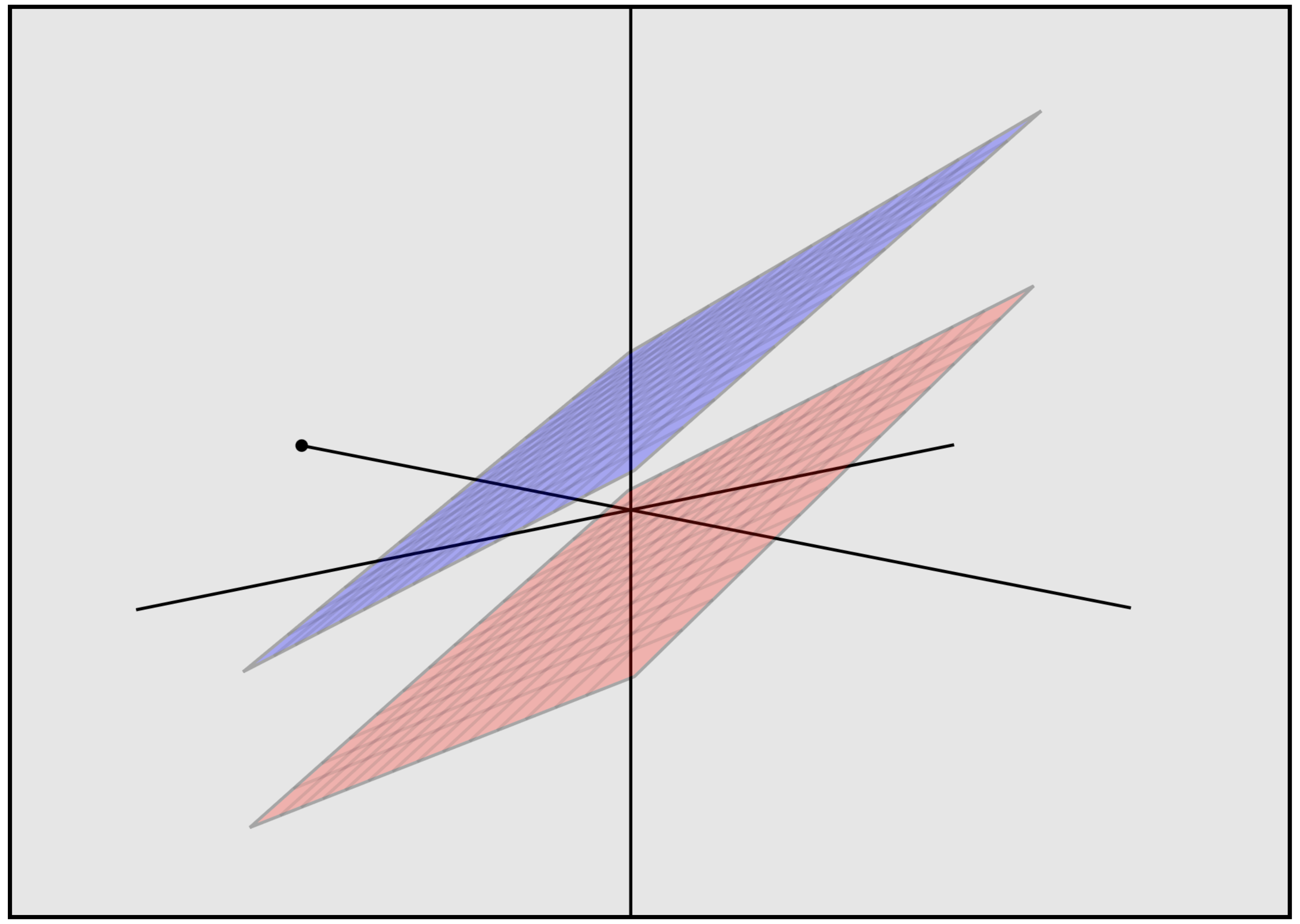
How many solutions are there?

What are its solutions?

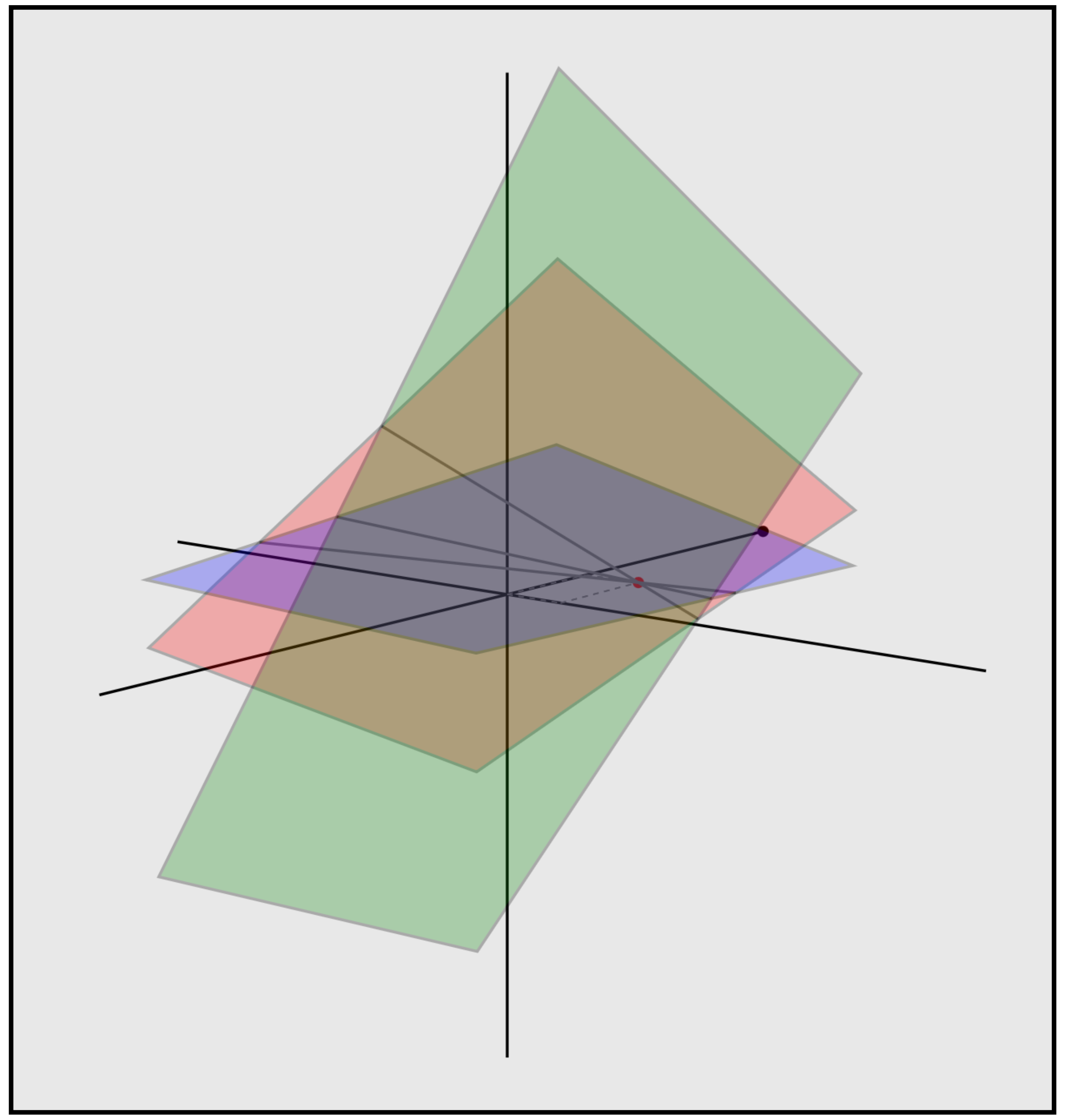
Consistency

Definition. A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has <u>no</u> solutions

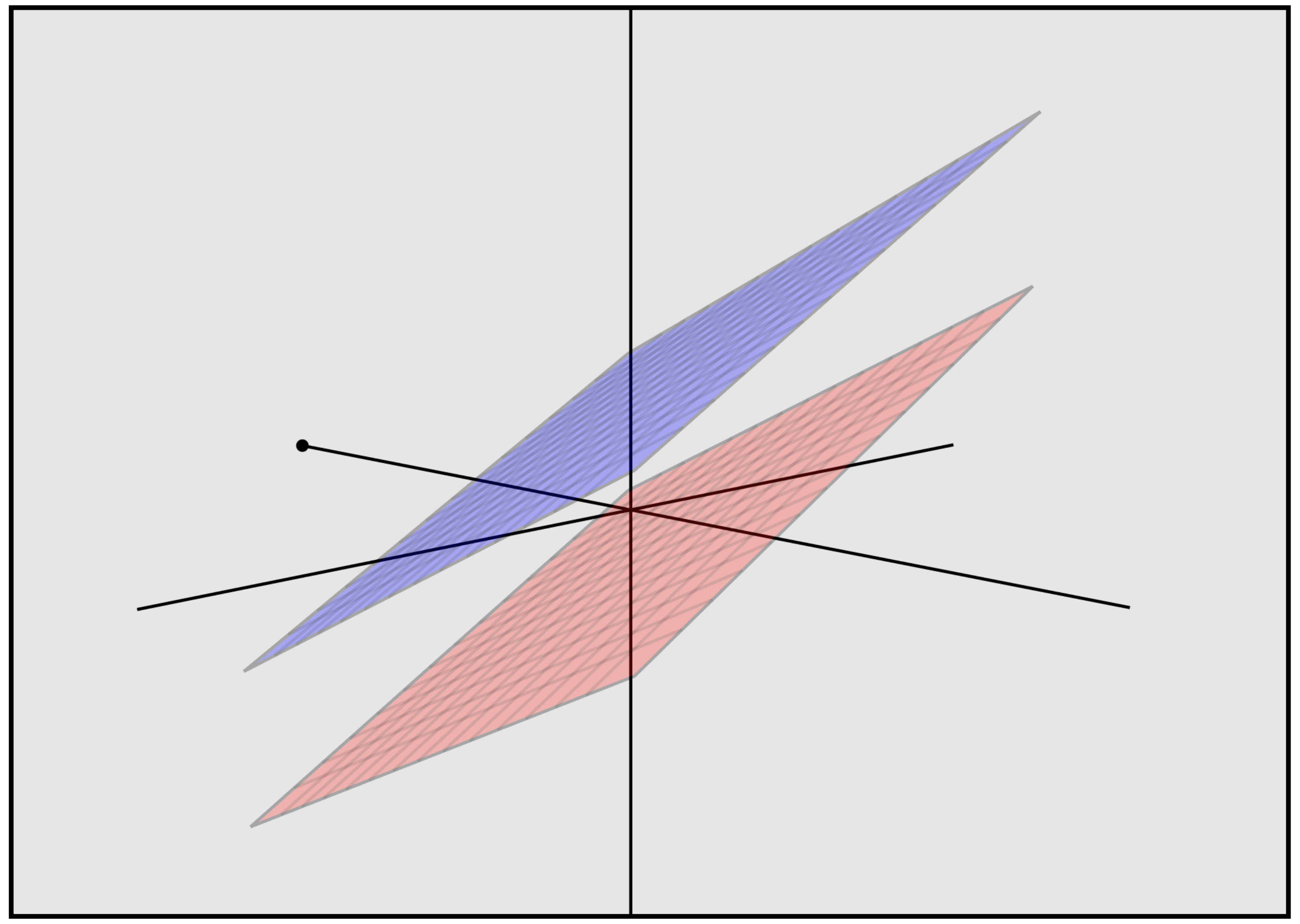


inconsistent



consistent

Example



Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

Matrix Representations

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \approx \begin{aligned} 2x + 3y &= -8 \\ y &= 2 \\ 2y &= 0 \end{aligned}$$

Writing down the unknowns is *tedious* (and more difficult to input into a computer)

We'll write down linear systems as **matrices**, which are just 2D grids of numbers with *fixed* width and height

a matrix is just a representation

Matrix Representations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coefficient matrix

Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

Solving Linear Systems

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

another perspective...

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Eliminate x *from the EQ2 and solve for* y

Eliminate y *from EQ1 and solve for* x

Let's work through it

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back–Substitution

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

The Approach

- **Eliminate x from the EQ2 and EQ3**
- Eliminate y from EQ3
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17y + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$68z - 34 - 3z = 26$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$71z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + \boxed{0} = 5$$

$$2y - 8\boxed{(0)} = -4$$

$$\boxed{z = 0}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y = 5$$
$$2y = -4$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back-Substitution

Verifying the Solution

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

Can we represent these intermediate steps as operations on matrices?

Let's look back at this...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Elementary Row Operations

scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows

These operations don't change the solutions

Scaling Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_1 \leftarrow 2R_1} \quad \begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_2 \leftarrow R_2 + R_1} \quad \begin{aligned} 2x + 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_2} \begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/(-11)} \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2/(-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1/2$$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

Question

How do we know when we're done? What is the "target" matrix?

We'll get to that next time...

Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve