

# Practice Final Exam

CAS CS 132: Geometric Algorithms

Name:

BUID:

- ▷ You will have approximately 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Please do not remove any pages from the exam.
- ▷ Please put your **final** solution in the solution box *and nothing else*. **You should do your work outside of the box!**
- ▷ You must show your work on all problems unless otherwise specified. A solution without work will be considered incorrect (and will be investigated for potential academic dishonesty).
- ▷ We will not look at any work on the pages marked “*This page is intentionally left blank.*” You should use these pages for scratch work.

## 1 Column Space and Null Space

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- A. (3 points) Determine a basis for  $\text{Nul } A$ .

*Solution.*

B.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(3 points) Determine a linear equation over the variables  $x_1, x_2, x_3, x_4$  whose solution set is  $\text{Col } A$  (that is,  $(x_1, x_2, x_3, x_4)$  satisfies the linear equation if and only if the vector  $[x_1 \ x_2 \ x_3 \ x_4]^T$  is in the column space of  $A$ ).

*Solution.*

## 2 Linear Models

$$\{(-1, 1), (0, -1), (1, 1), (2, 3), (4, 5)\}$$

(3 points) Determine the equations for finding the best-fit curve of the form

$$\beta_0 + \beta_1 x^3 + \beta_2 (2^x)$$

(where  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  are parameters) for the above data using least-squares regression. That is, determine the design matrix  $X$  and vector of observations  $\mathbf{y}$  such that the least-squares solution of

$$X \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{y}$$

determines the parameters for best-fit curve.

*Solution.*

### 3 Exponentials of Matrices

In a homework problem we saw that if  $A$  is diagonalizable (i.e., it can be expressed as  $PDP^{-1}$  for a diagonal matrix  $D$ ) then we can define  $A^{1/2}$  as  $PD^{1/2}P^{-1}$ , where

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}^{1/2} = \begin{bmatrix} d_1^{1/2} & 0 & \dots & 0 & 0 \\ 0 & d_2^{1/2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1}^{1/2} & 0 \\ 0 & 0 & \dots & 0 & d_n^{1/2} \end{bmatrix}$$

That is, we can take the square root of each entry along the diagonal of  $D$  in the diagonalization of  $A$ . It turns out that for any function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and any diagonalizable matrix  $A$  as above, we can define  $f(A)$  as  $Pf(D)P^{-1}$  where

$$f \left( \begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix} \right) = \begin{bmatrix} f(d_1) & 0 & \dots & 0 & 0 \\ 0 & f(d_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & f(d_{n-1}) & 0 \\ 0 & 0 & \dots & 0 & f(d_n) \end{bmatrix}$$

That is, we apply  $f$  to each entry along the diagonal of  $D$  in the diagonalization of  $A$ . We will use this fact to reason about exponentials of matrices. (*Problem continued on next page.*)<sup>1</sup>

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<sup>1</sup>Credit to Vishesh Jain and Abhinit Sati for suggesting a version of this problem.

A.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(4 points) Determine a diagonalization of  $A$ . Your final solution should be in the form  $PDP^{-1}$ , where  $D$  is a diagonal matrix and  $P^{-1}$  is an explicit matrix, i.e., you must compute the inverse of  $P$ .

*Solution.*

*(Problem 2A Continued)*

B.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(2 points) Determine the matrix  $2^A$ . Your final solution should be an explicit matrix, i.e, you must compute all matrix multiplications. (*Hint.* Use  $f(x) = 2^x$ .)

*Solution.*

C.

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -6 \\ 8 & -8 \end{bmatrix}$$

(4 points) Determine the characteristic polynomial of  $2^A 2^B$ . Give your solution in expanded form, i.e., not factored. (*Hint.* Don't try to compute this matrix product directly. Use properties of exponentiation).

*Solution.*

## 4 True/False Questions

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.<sup>2</sup>

- A. (1 point) For any matrix  $A$  with orthogonal columns,  $A^T A = I$ .

True  
 False

- B. (1 point) If  $A$  is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.

True  
 False

- C. (1 point) For any matrix  $A$ , if  $A$  is orthogonally diagonalizable, then so is  $A^T$ .

True  
 False

- D. (1 point) For any vector  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  in  $\mathbb{R}^n$ ,  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$ .

True  
 False

- E. (1 point) For any matrix  $A$  and vector  $\mathbf{v}$ , if  $\mathbf{v}$  is an eigenvector of the matrix  $A$  then it is also an eigenvector of the matrix  $A^2$ .

True  
 False

- F. (1 point) For any matrix  $A \in \mathbb{R}^{m \times n}$  and vector  $\mathbf{v} \in \mathbb{R}^n$ , if  $\|A\mathbf{v}\| = \mathbf{0}$  then  $\mathbf{v} \in \text{Nul } A$ .

True  
 False

- G. (1 point) For any matrix  $A$ , we have  $\text{rank}(A) = \text{rank}(A^T)$ .

True  
 False

- H. (1 point) For any vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  in  $\mathbb{R}^n$ , if  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent, then  $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ .

True  
 False

- I. (1 point) For any square matrices  $A$  and  $B$ , we have  $\det(AB^T) = \det(A)\det(B)$

True  
 False

- J. (1 point) For any quadratic form  $Q(\mathbf{x})$ , the vector  $\text{argmax}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$  is unique.

True  
 False

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<sup>2</sup>Credit to Vishesh Jain and Abhinit Sati for suggesting some parts of this question.

K. (1 point) For any matrix  $A \in \mathbb{R}^{m \times n}$  and vector  $\mathbf{b} \in \mathbb{R}^m$ , if  $A\mathbf{x} = \mathbf{b}$  has a unique least-squares solution, then  $\mathbf{x} \mapsto A\mathbf{x}$  is onto.

- True
- False

L. (1 point) If  $A$  is a square matrix with strictly positive entries, then there is diagonal matrix  $D$  such that  $AD$  is stochastic.

- True
- False

## 5 Singular Value Decomposition

A.

$$A = \begin{bmatrix} 2 & -4 & 4 \\ 2 & 2 & 1 \\ -2 & 4 & -4 \end{bmatrix} \quad AA^T = \begin{bmatrix} 36 & 0 & -36 \\ 0 & 9 & 0 \\ -36 & 0 & 36 \end{bmatrix}$$

(4 points) Determine a *reduced* singular value decomposition of  $A$ , given that  $6\sqrt{2}$  and  $3$  are its nonzero singular values and  $\text{rank}(A) = 2$ . (*Reminder.* If  $U\Sigma V^T$  is an SVD of  $A$  and  $\text{rank}(A) = r$ , then we can get a reduced SVD of  $A$  by making  $\Sigma$  an  $r \times r$  diagonal matrix and dropping columns of  $U$  and  $V$  so that the multiplication  $U\Sigma V^T$  is well-defined.)

*Solution.*

*(Problem 5A Continued)*

B.

$$\mathbf{b} = \begin{bmatrix} 10 \\ 18 \\ -14 \end{bmatrix}$$

(3 points) Let  $A$  be the matrix from the previous part. Determine the length of the *shortest* least-squares solution to the equation  $A\mathbf{x} = \mathbf{b}$ . (*Hint.* Use the pseudoinverse of  $A$ , i.e.,  $A^+ = V\Sigma^{-1}U^T$ , and keep in mind that matrices with orthonormal columns preserve lengths.)

*Solution.*

*(Problem 5B Continued)*

## 6 Dependence Relations

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$$

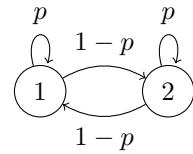
(5 points) Determine a dependence relation for the above vectors. That is, write  $\mathbf{0}$  as a linear combination of the above vectors.

*Solution.*

*(Problem 6 Continued)*

## 7 Stochastic Matrices

- A. Consider the following state diagram.



(2 points) Write the transition matrix  $T$  for the above diagram in terms of  $p$ . In the following parts,  $T$  will refer to this matrix.

*Solution.*

B. (3 points) Given  $0 < p < 1$ , determine  $\lim_{k \rightarrow \infty} T^k \mathbf{e}_1$  (where  $\mathbf{e}_1$  is the first standard basis vector). Justify your answer.

*Solution.*

*Justification.*

C. (4 points) Determine  $\lambda_2$ , the *second* largest eigenvalue of  $T$ , and a corresponding eigenvector. (*Hint.* Keep in mind that  $T$  is symmetric and, hence, orthogonally diagonalizable.)<sup>3</sup>

*Solution.*

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<sup>3</sup>The second largest eigenvalue tells us about the *rate of convergence* to the steady-state distribution. The smaller  $|\lambda_2|$ , the faster the convergence to the steady-state distribution.

*(Problem 7C Continued)*

- D. (4 points) Give a closed-form solution for  $T^k \begin{bmatrix} 1-q \\ q \end{bmatrix}$  in terms of  $p$ ,  $q$  and  $k$  (*Hint.* Again, keep in mind that  $A$  is symmetric and, hence, orthogonally diagonalizable).

*Solution.*

*(Problem 7D Continued)*

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