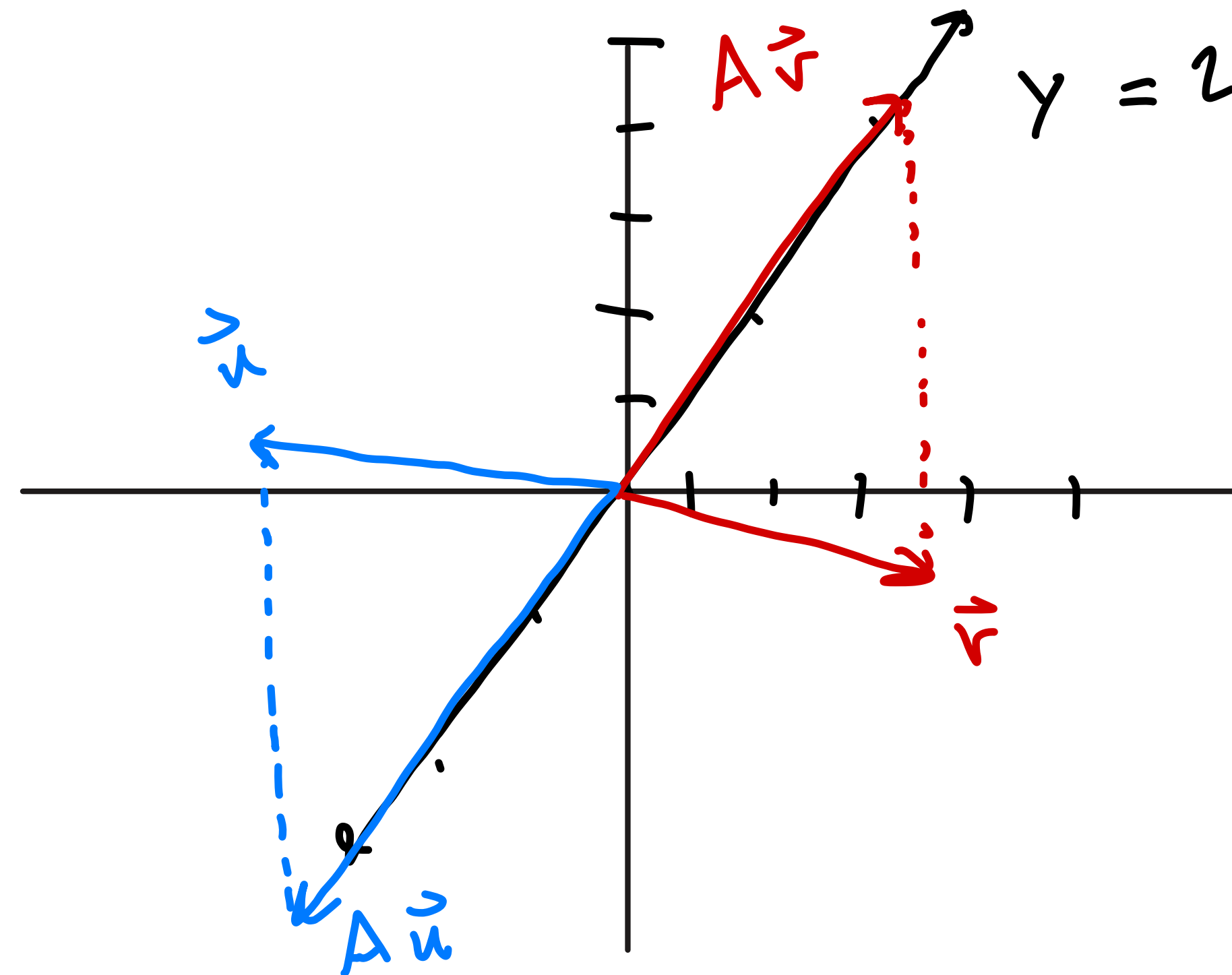


Matrix Algebra

Geometric Algorithms Lecture 9

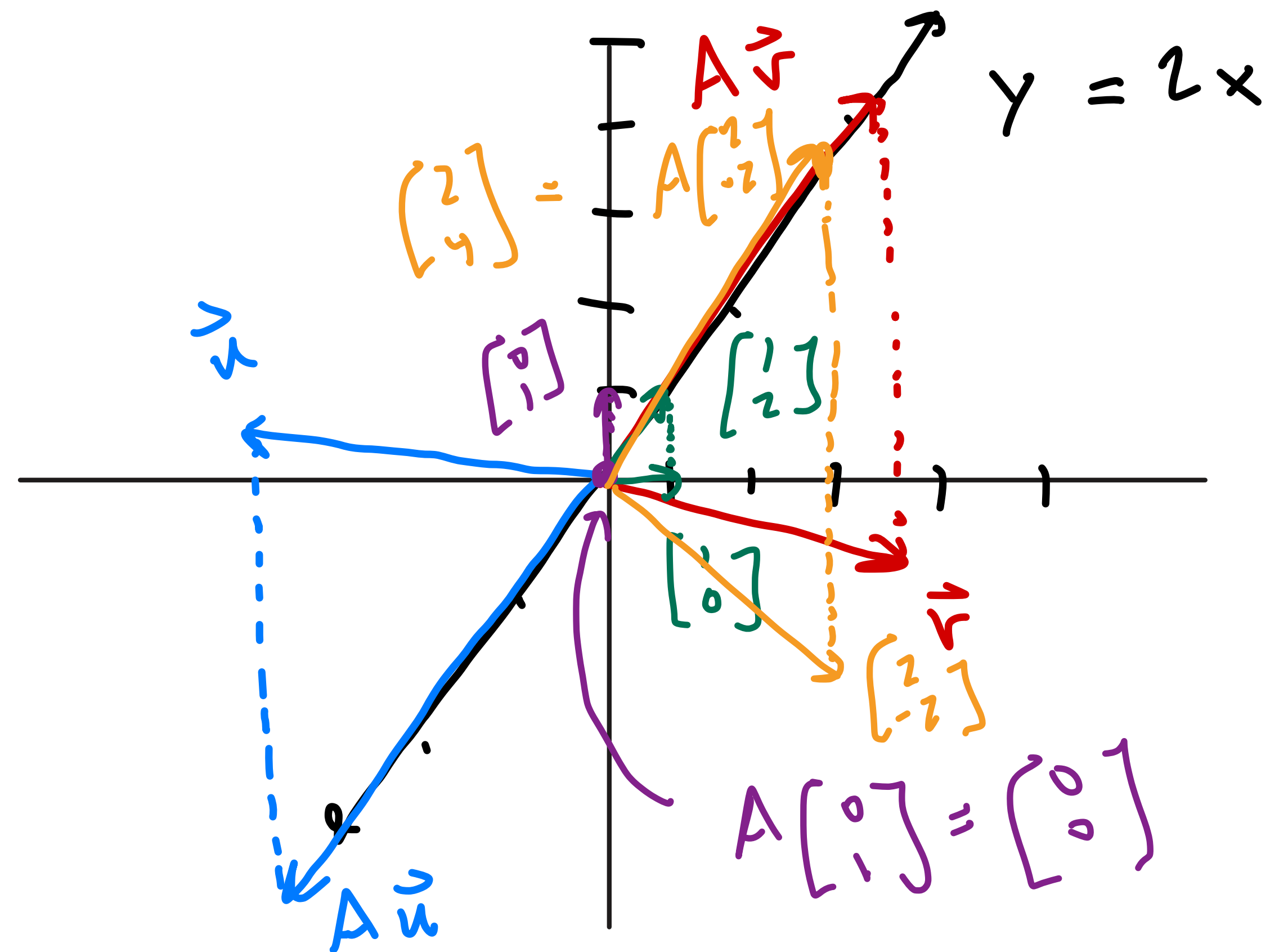
Practice Problem

Write the matrix for the transformation which projects vectors in \mathbb{R}^2 vertically onto the line $y = 2x$ ~~in \mathbb{R}^2~~ in \mathbb{R}^2



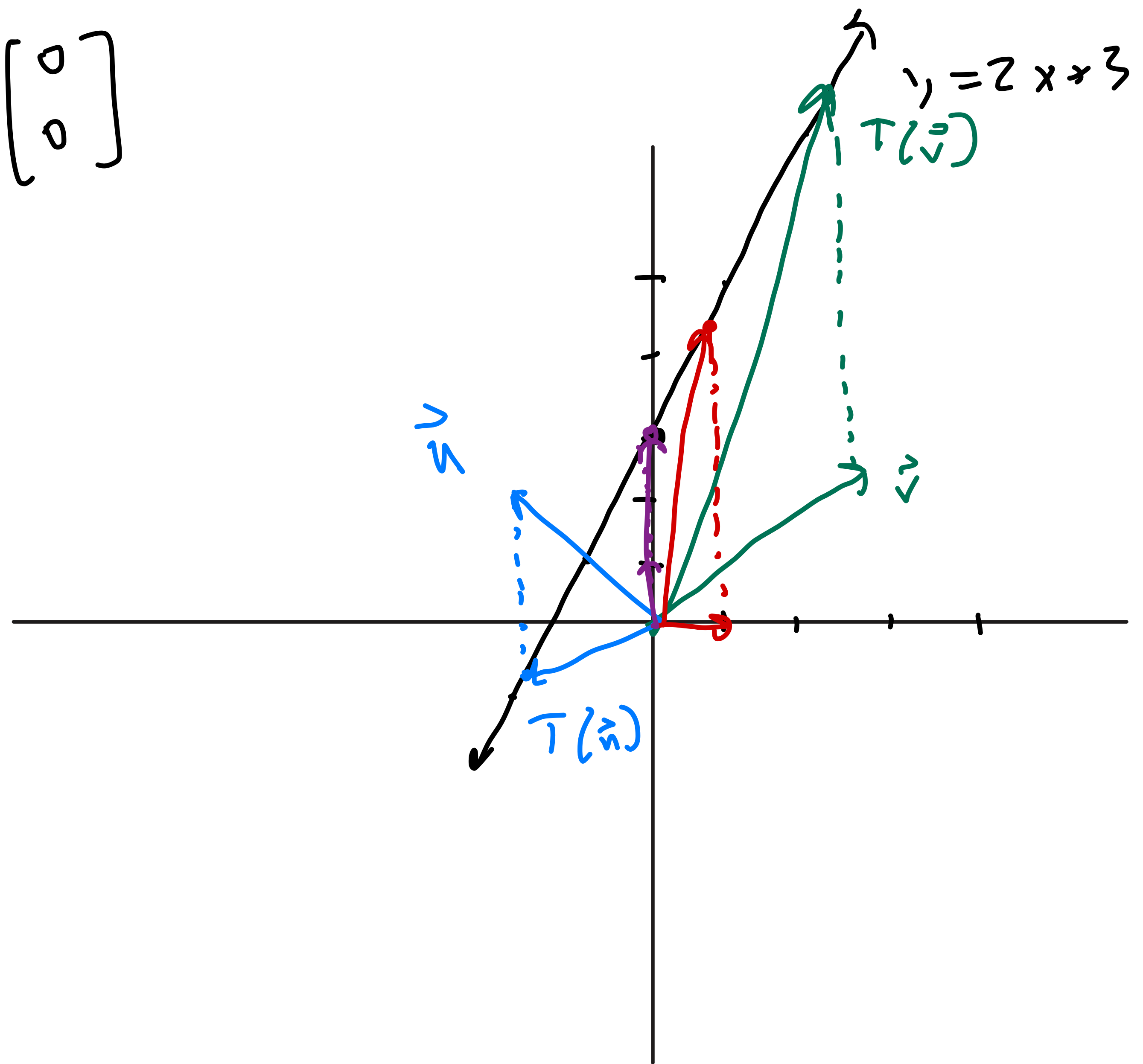
Answer

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} =$$
$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$
$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

NOT LINEAR



Objectives

- » Connect questions about **matrix equations** and **linear transformations**
- » Motivate **matrix multiplication**
- » Define matrix multiplication
- » Look at the **algebra** of matrix multiplication

Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

Recap: Geometry of Linear Transformations

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors

The transformed vector lies in the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

$$T(\vec{v} + \vec{v}) = T(\vec{v}) + T(\vec{v})$$

$$T(c\vec{v}) = cT(\vec{v})$$

map a vector \mathbf{x} to the vector $A\mathbf{x}$

Recall: Motivating Questions

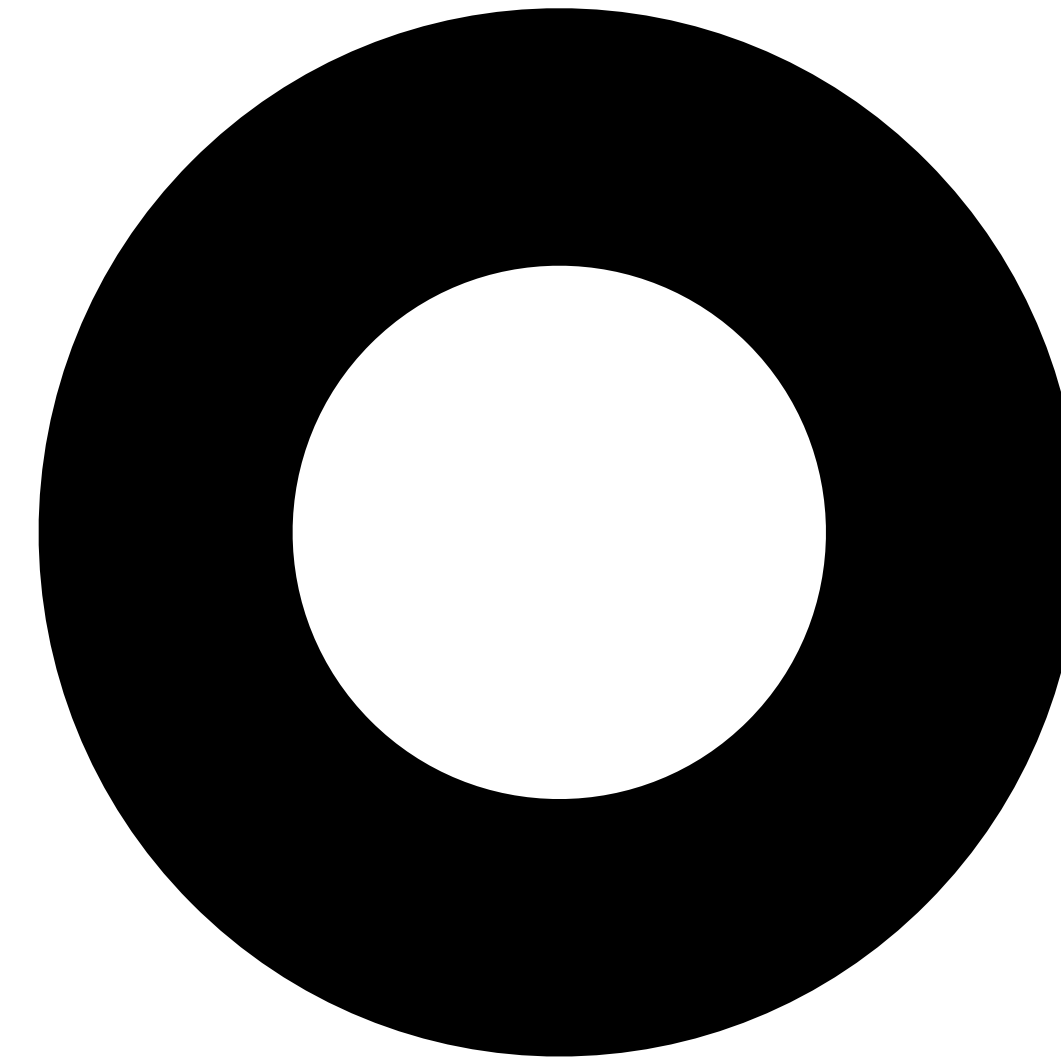
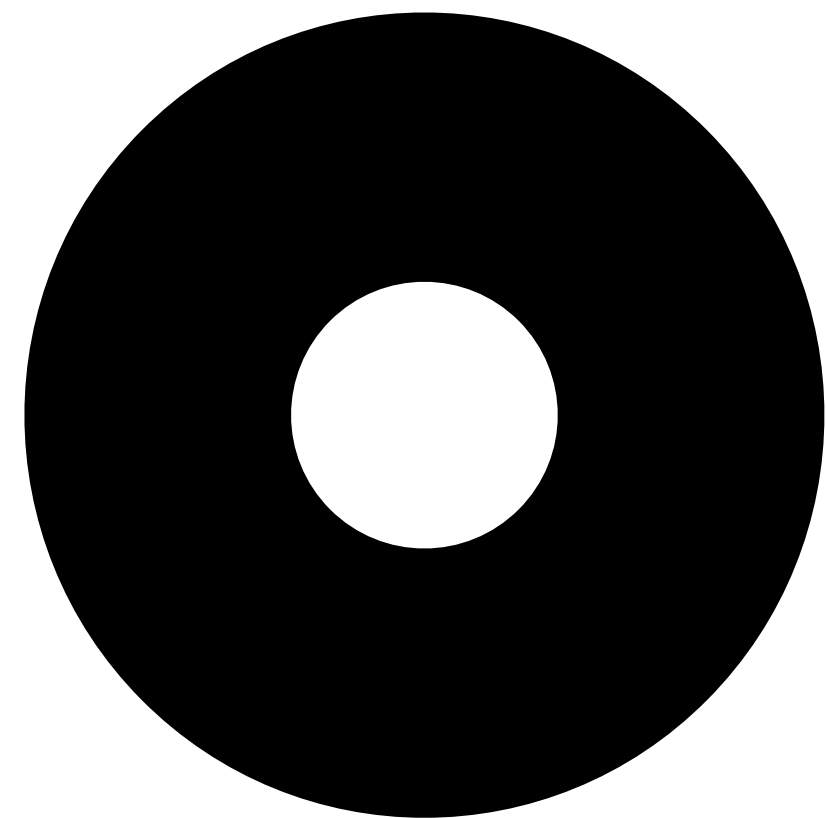
What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

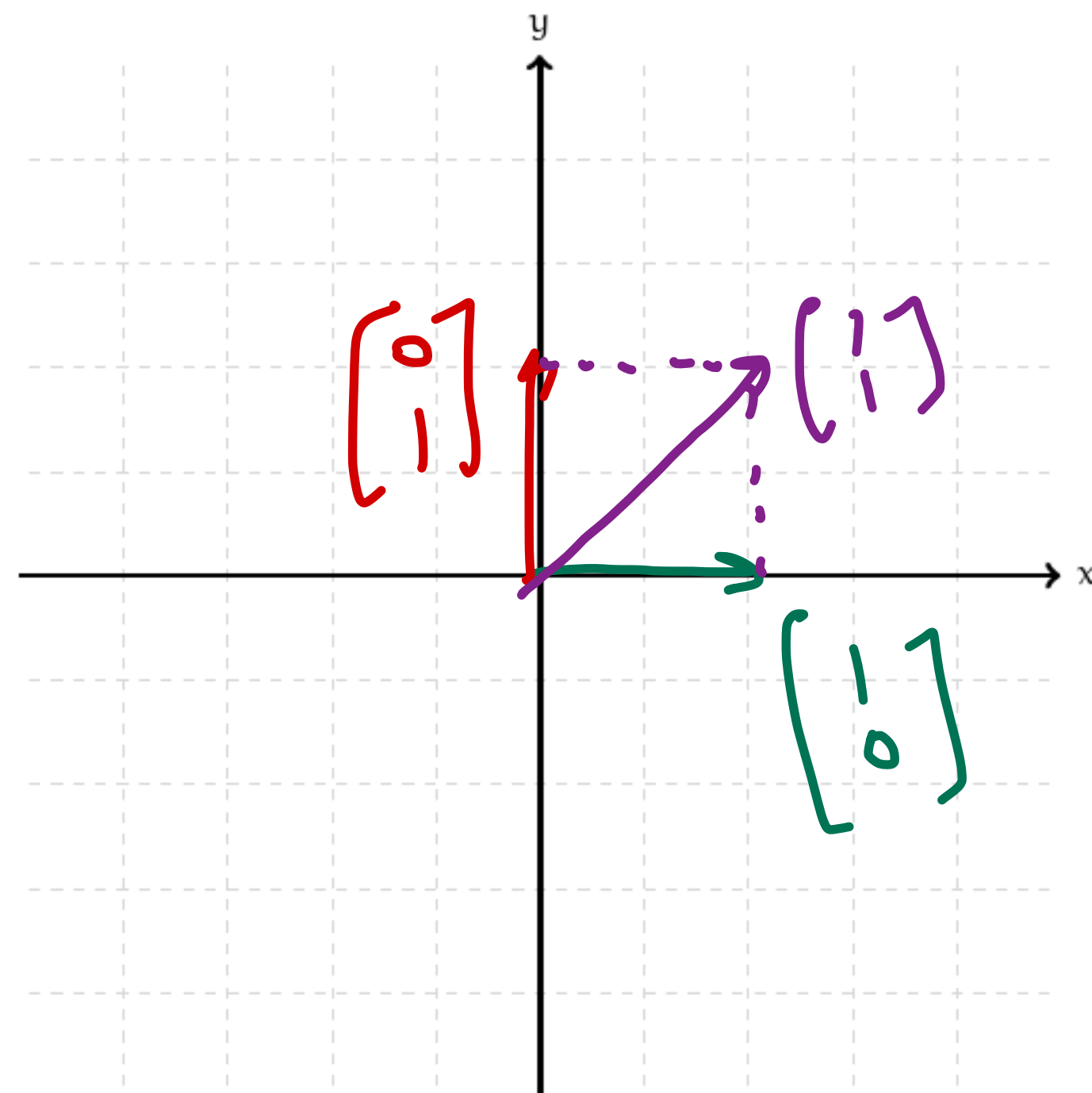
Matrix transformations change the
"shape" of a set of set of
vectors (points)

Example: Dilation



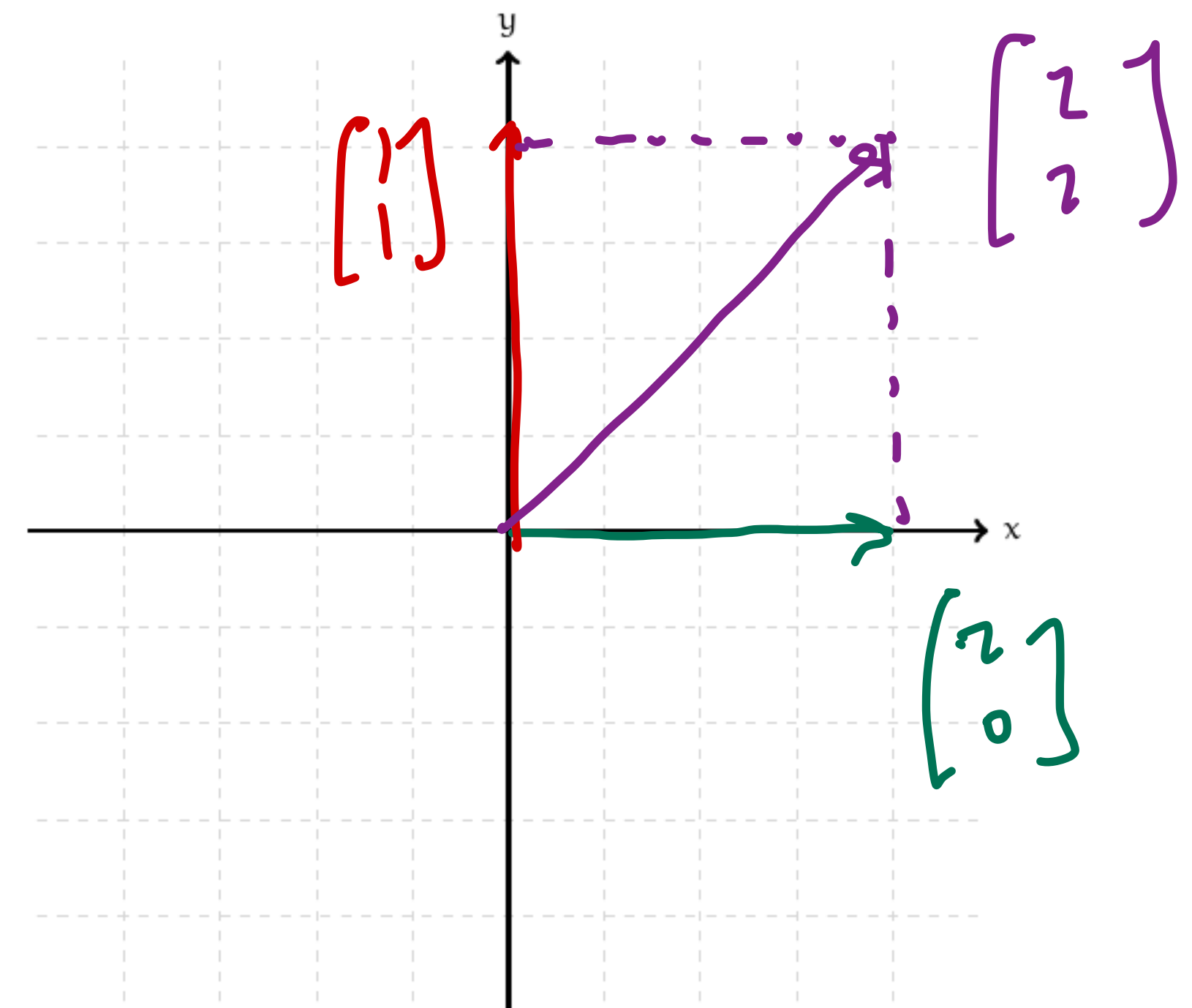
Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



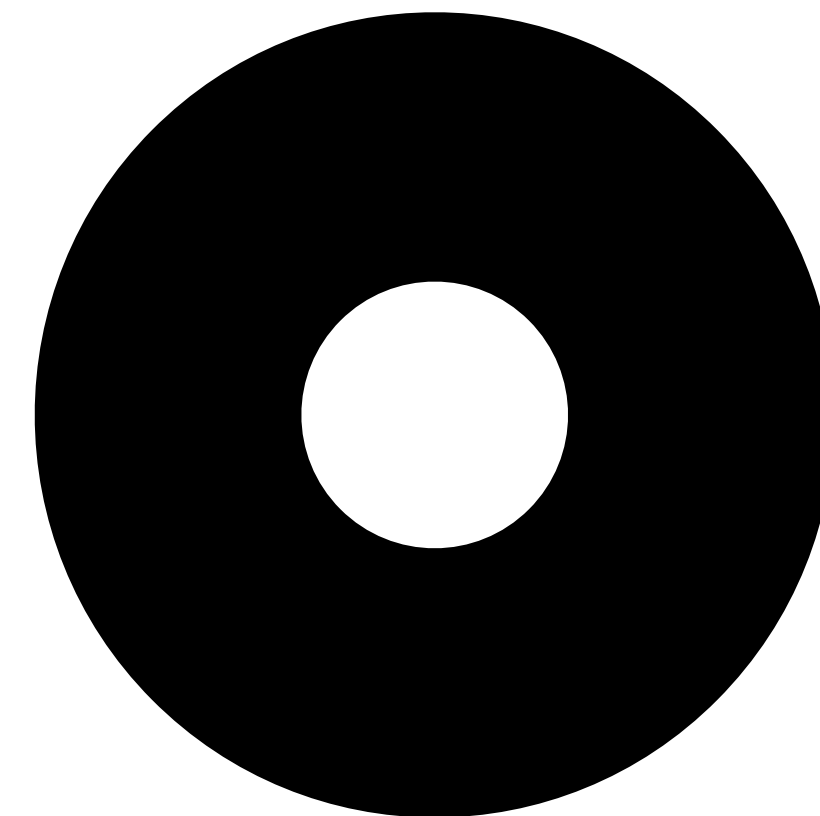
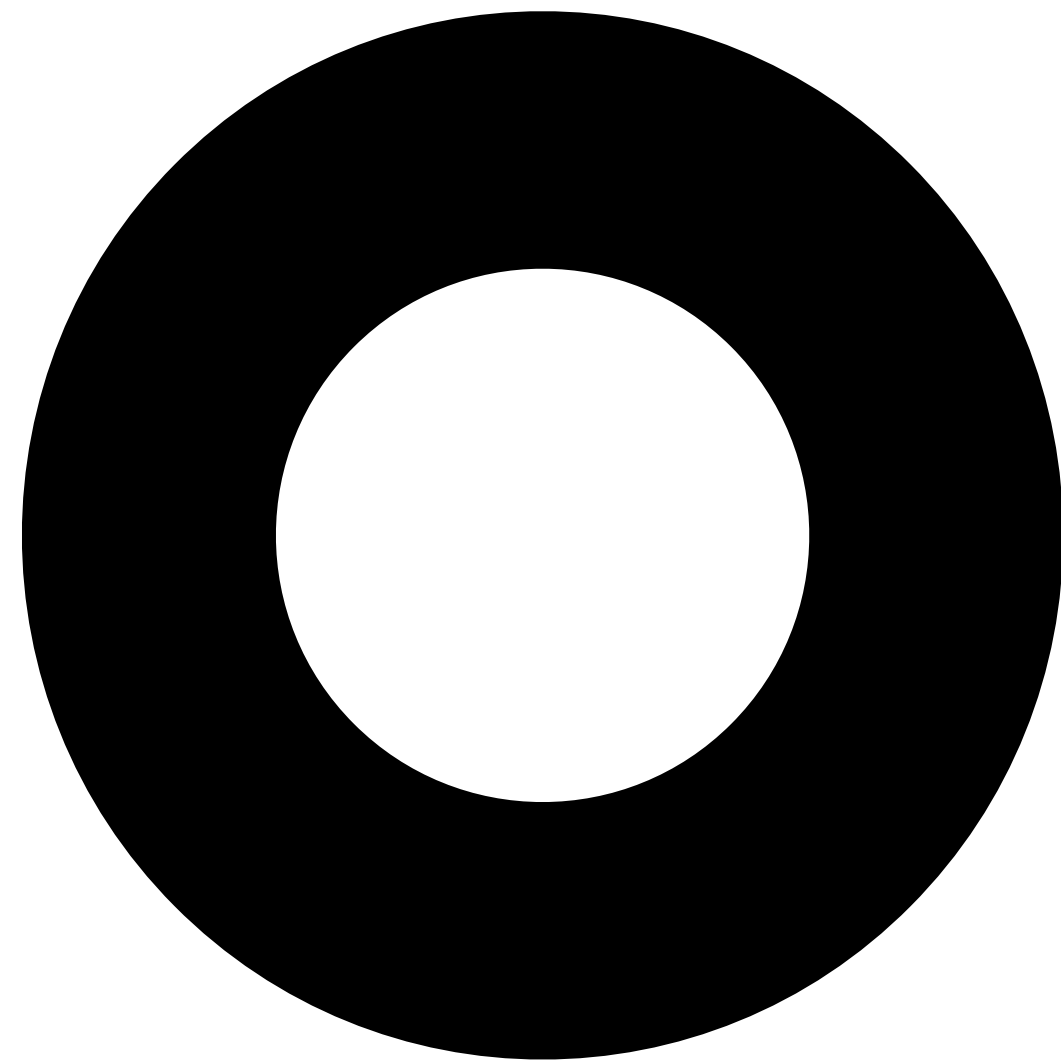
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

A large black arrow points from the first graph to the second graph, indicating the transformation.



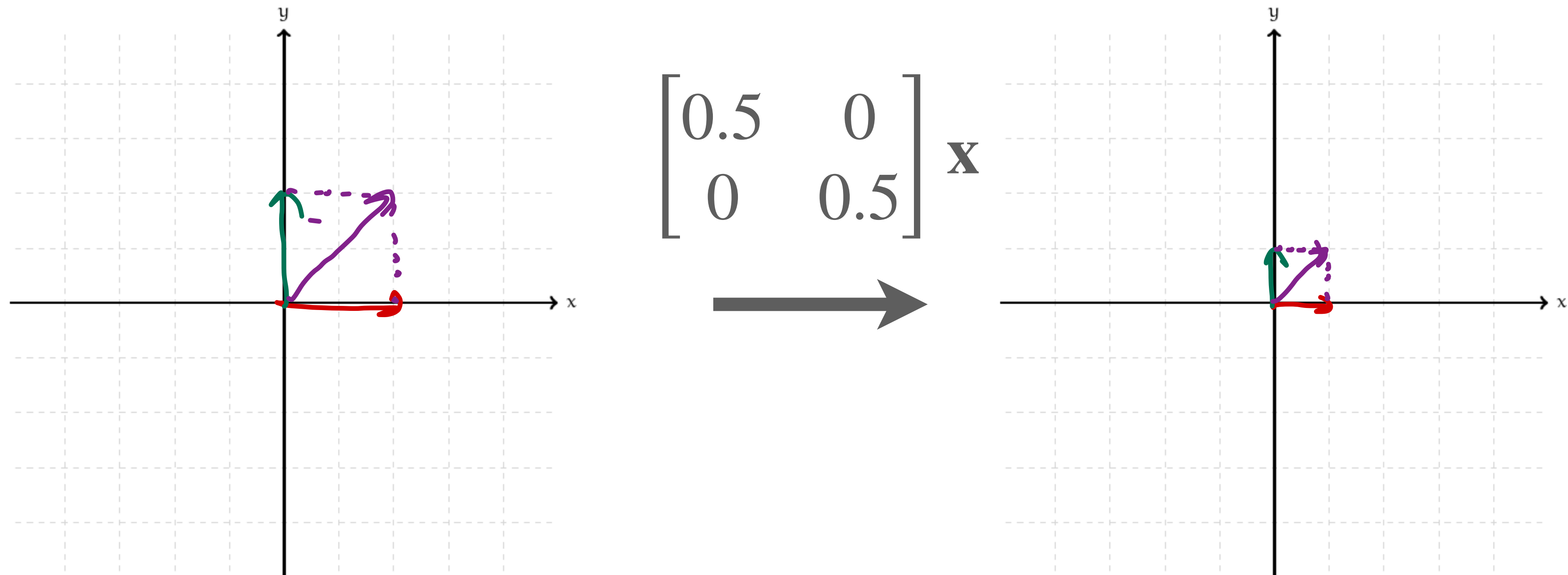
if $r > 1$, then the transformation pushes points away from the origin

Example: Contraction



Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



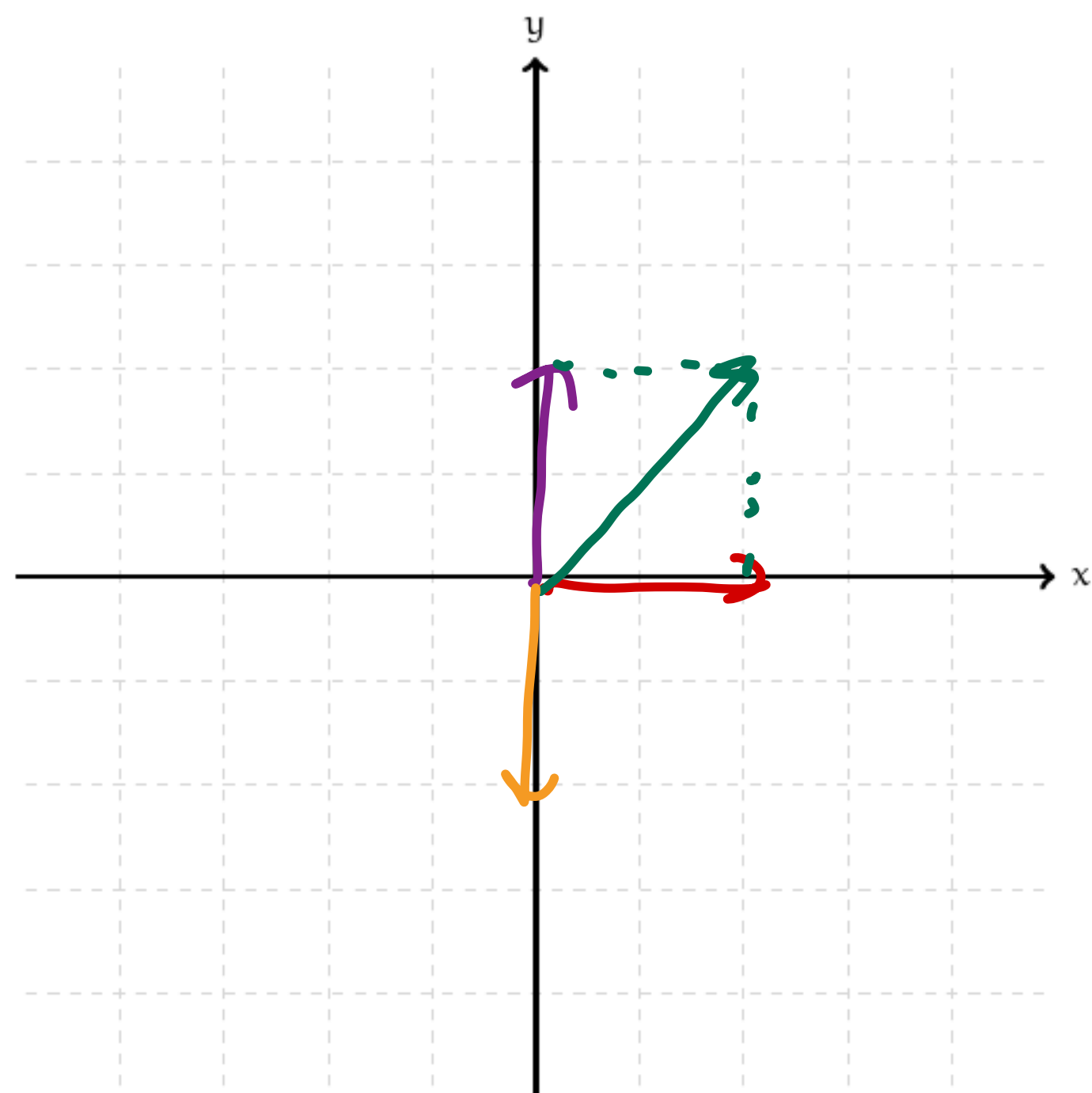
if $0 \leq r \leq 1$, then the transformation
pulls points towards the origin


Example: Shearing

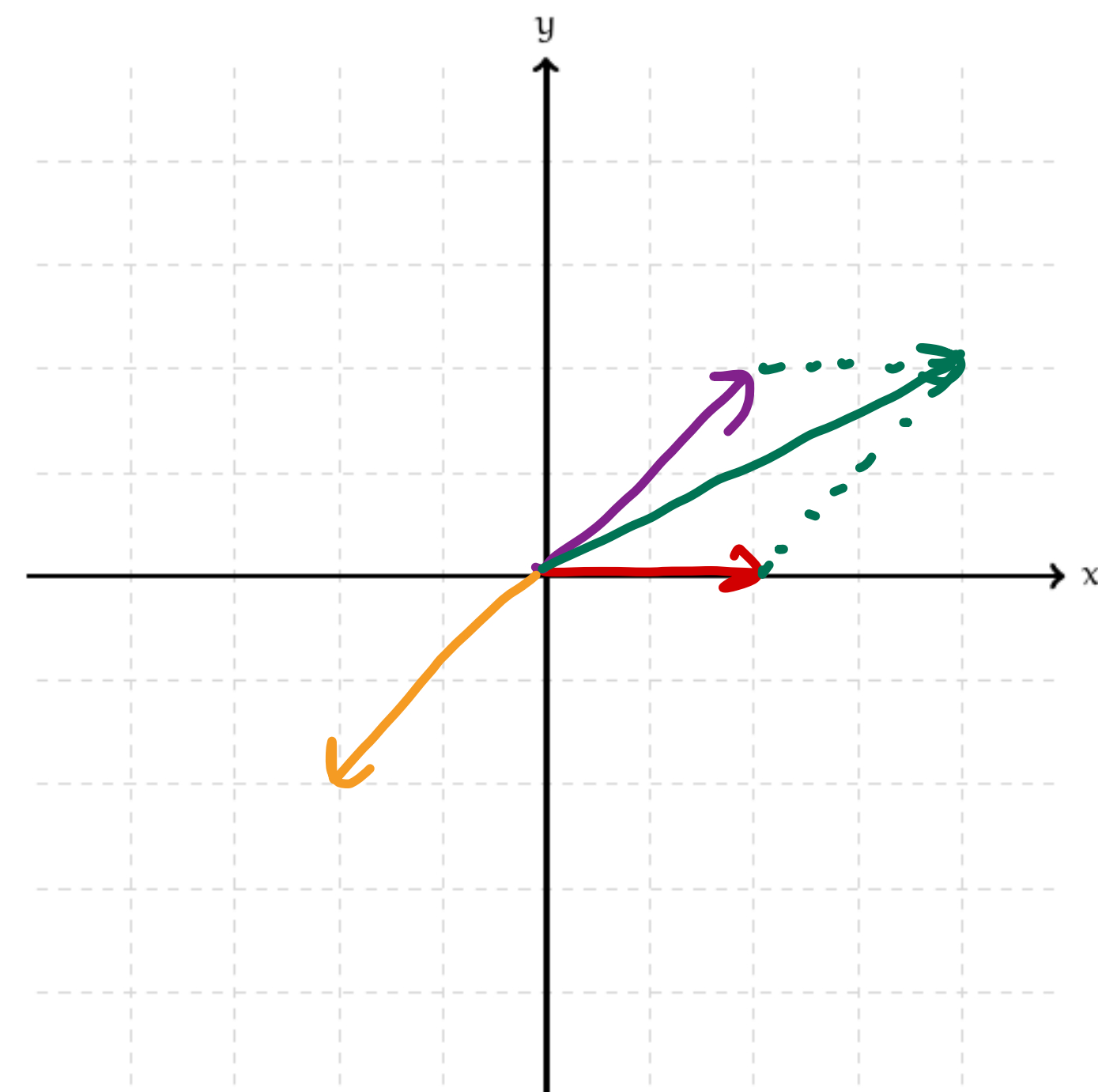


Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

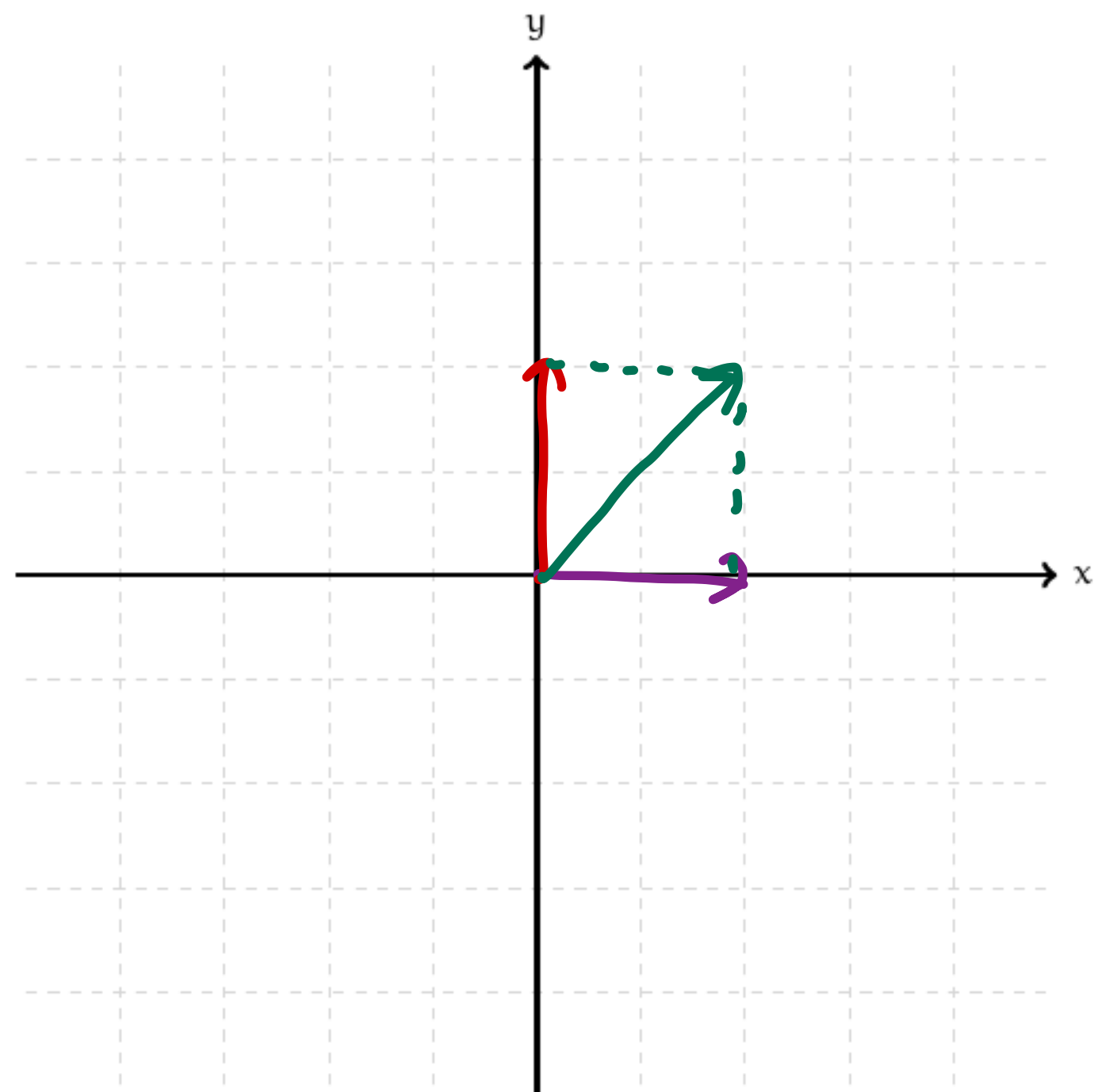


$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x}$$


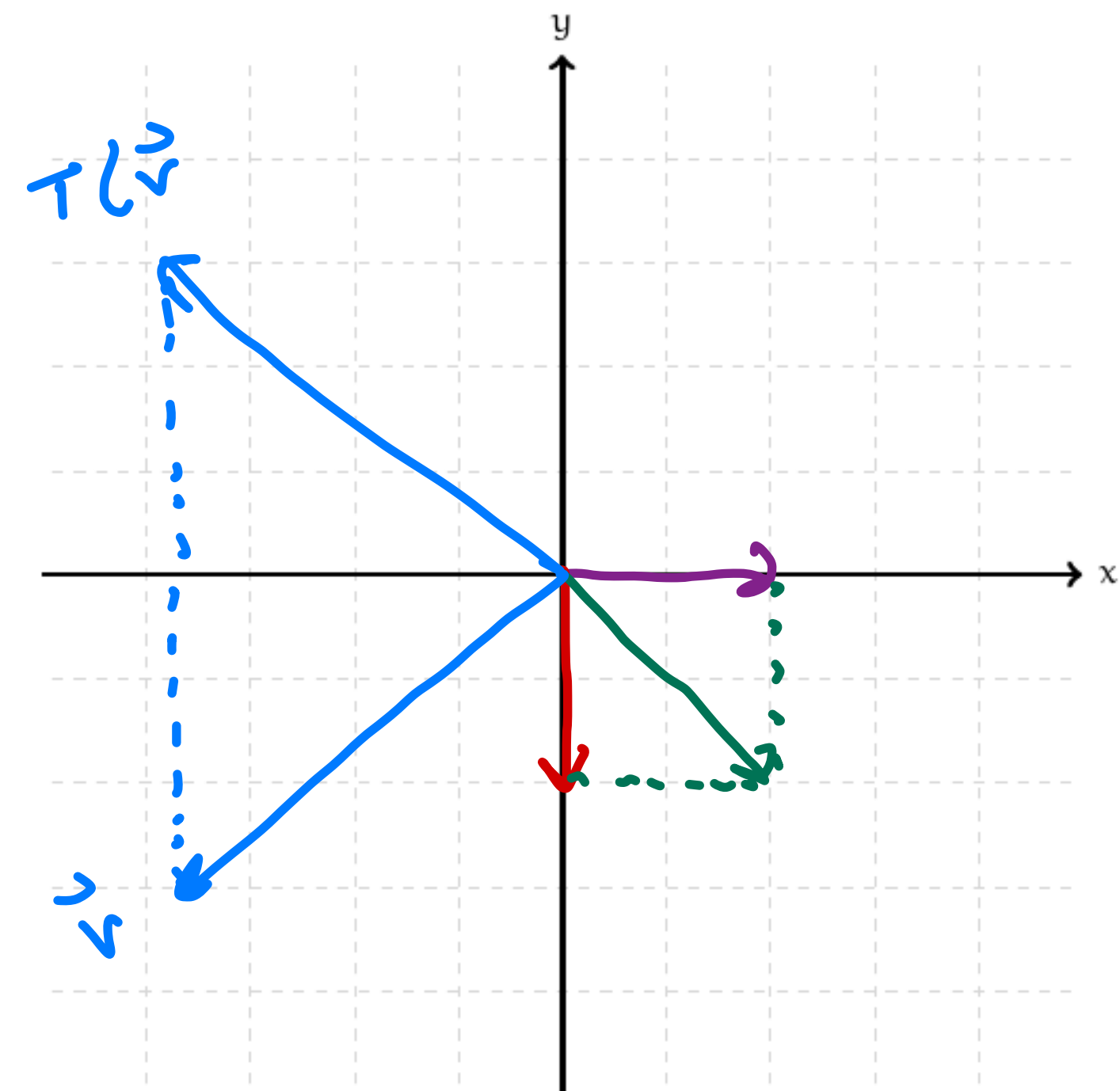


Imagine shearing like with rocks or metal

Example: Reflection

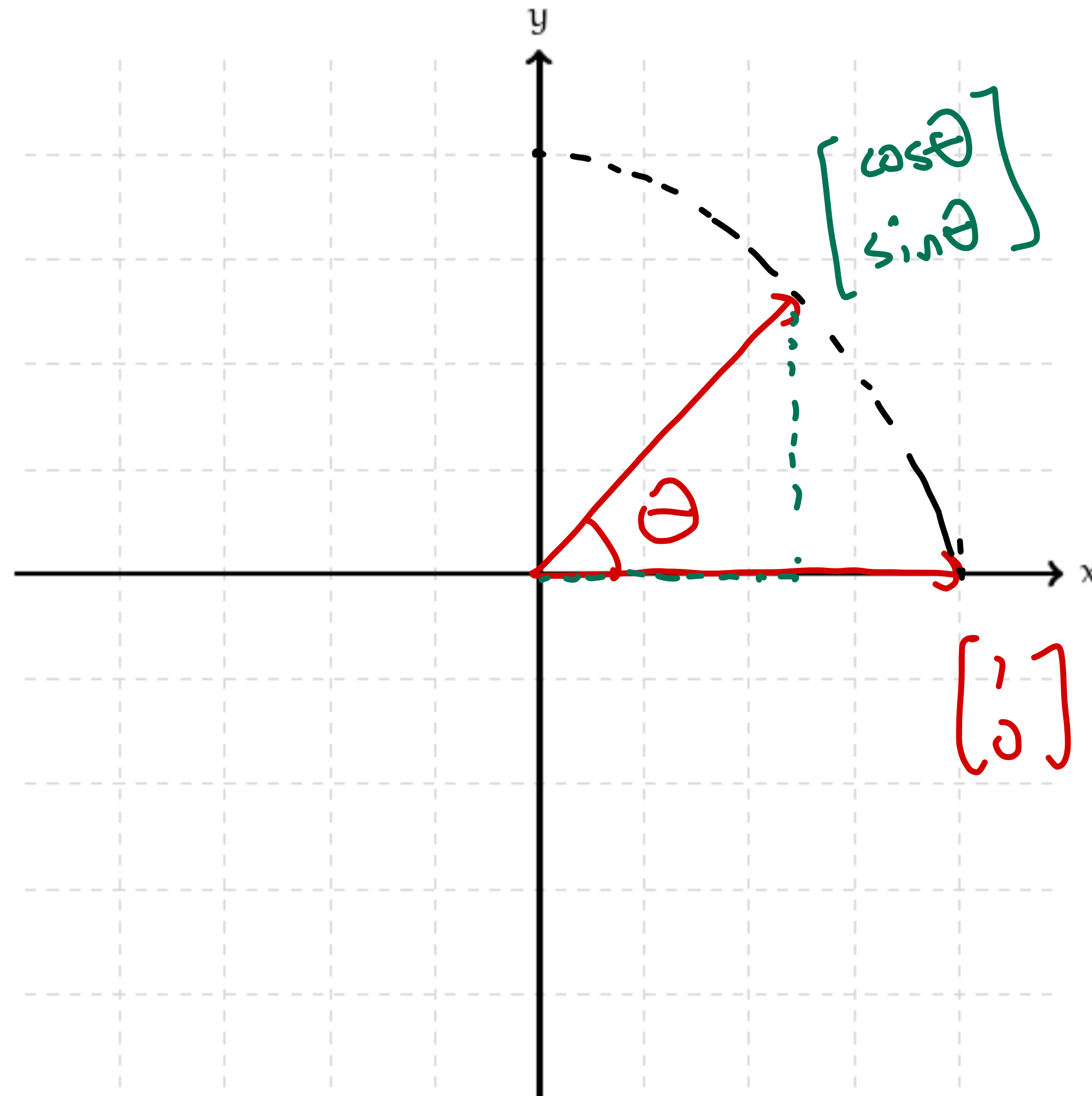


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



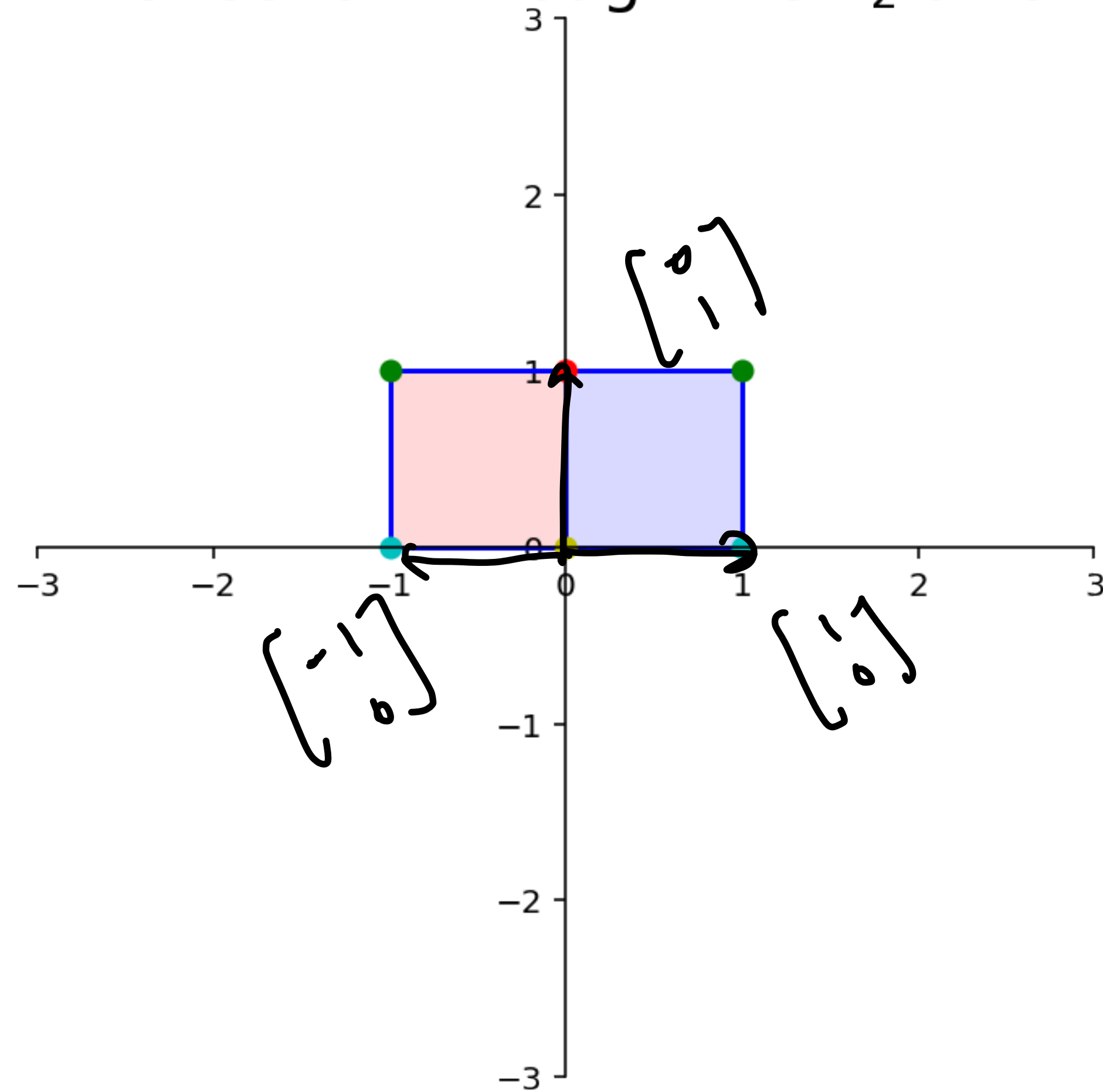
General Rotation

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



Example: Reflection through the x_2 -axis

Reflection through the x_2 axis

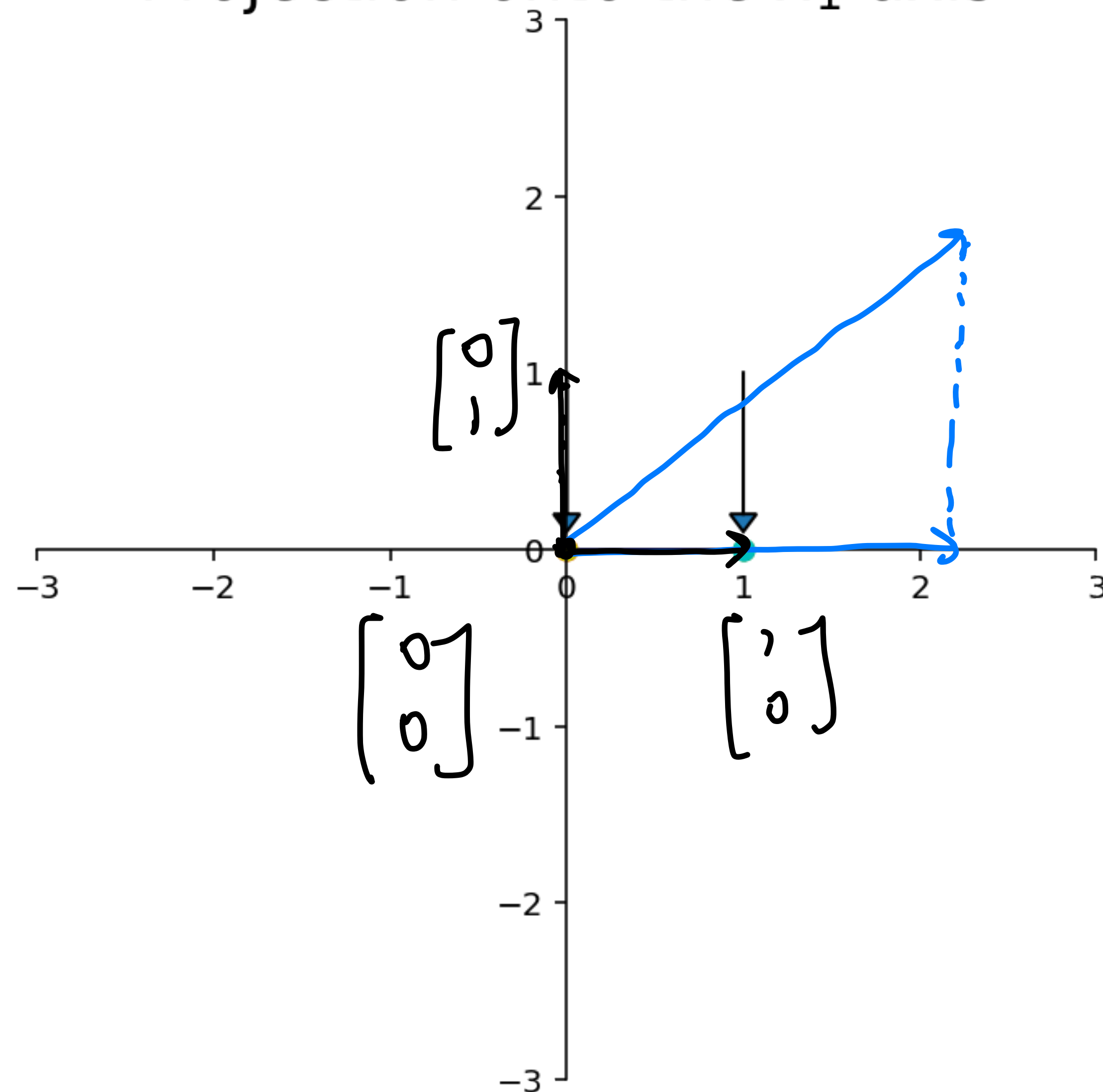


$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

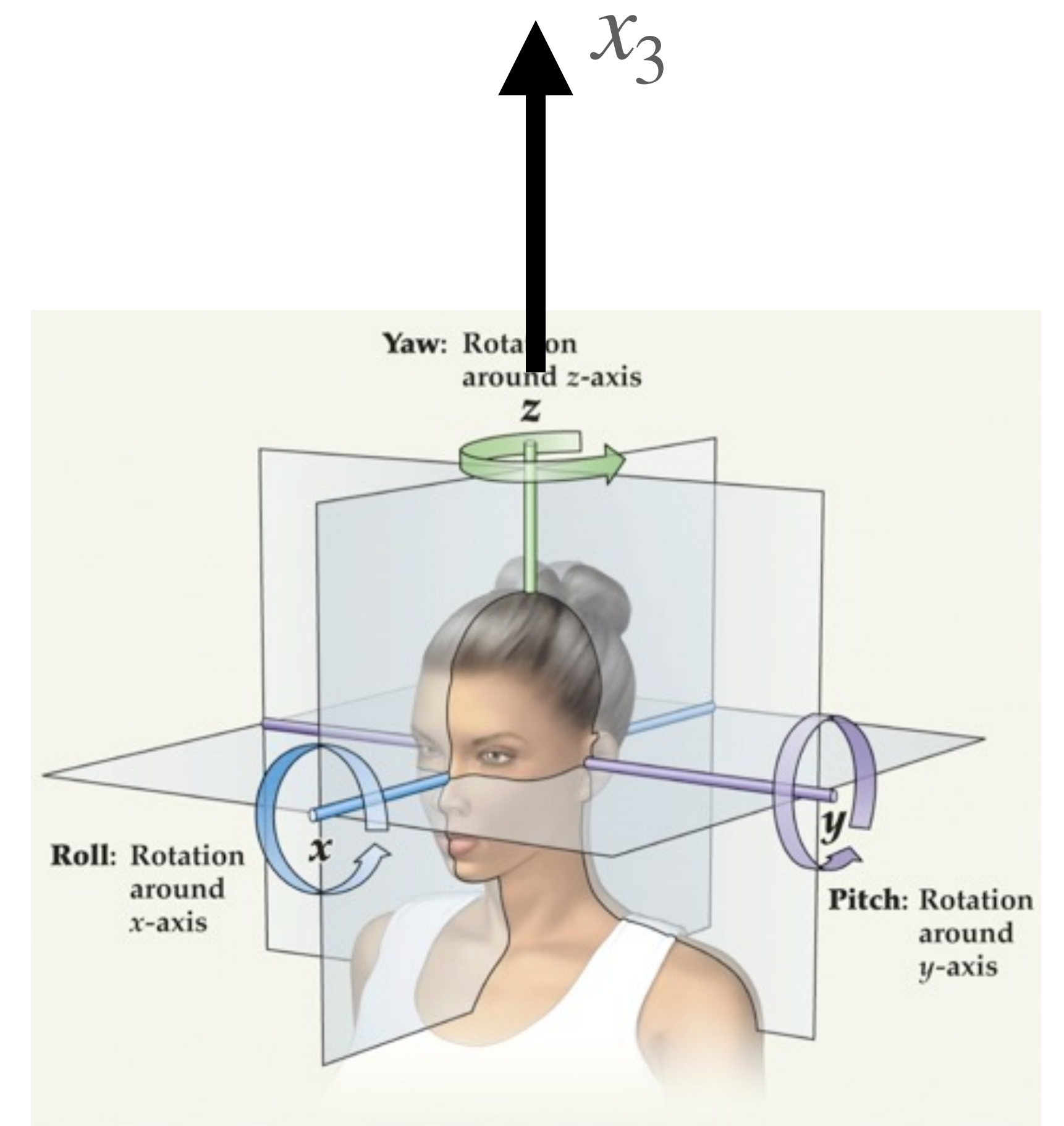
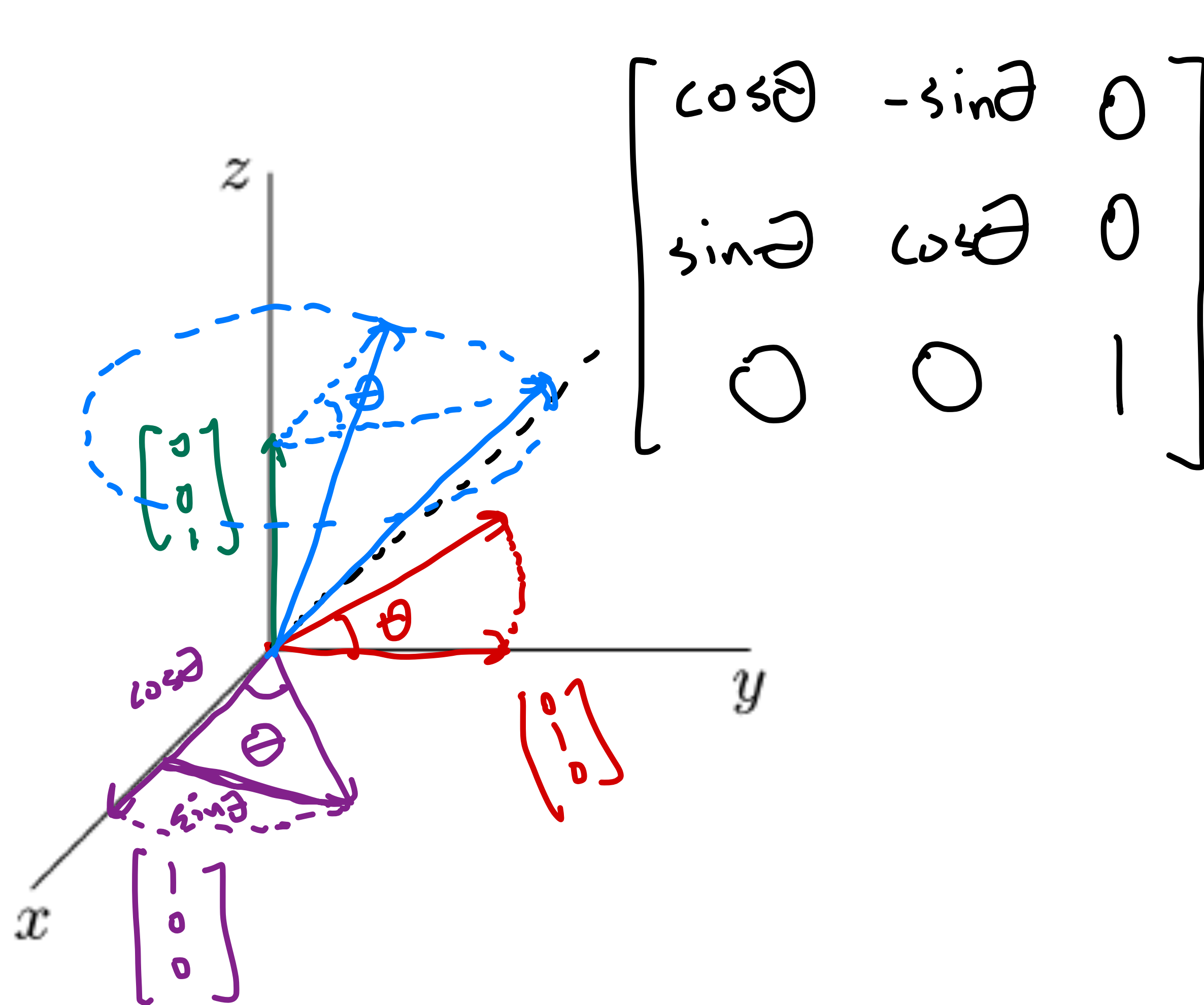
Example: Projections

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Projection onto the x_1 axis

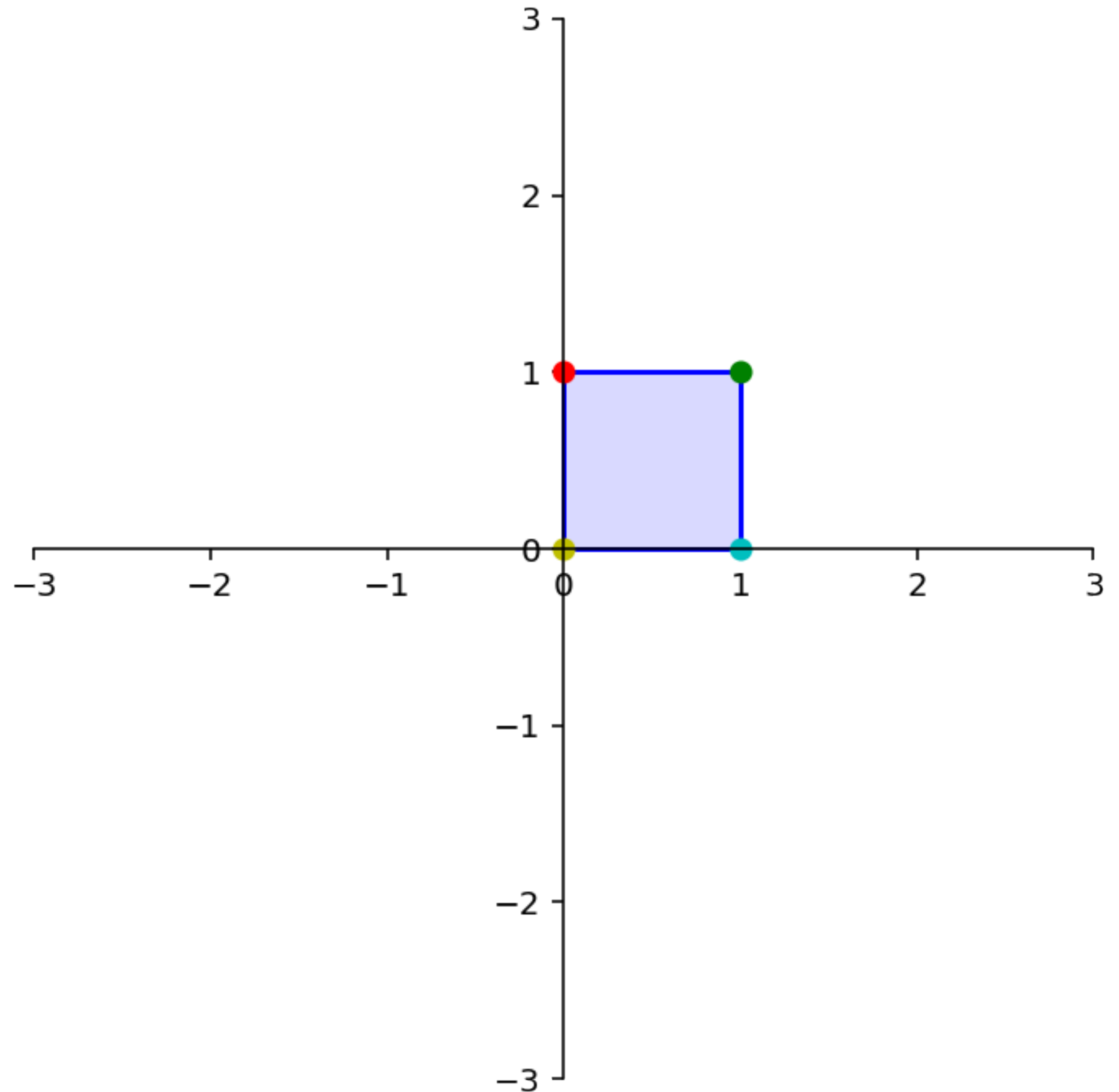


3D Example: Rotation about the x_3 -Axis (z -Axis)



The Unit Square

The *unit square* is the enclosed by the points



How To: The Unit Square and Matrices

How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

How To: The Unit Square and Matrices

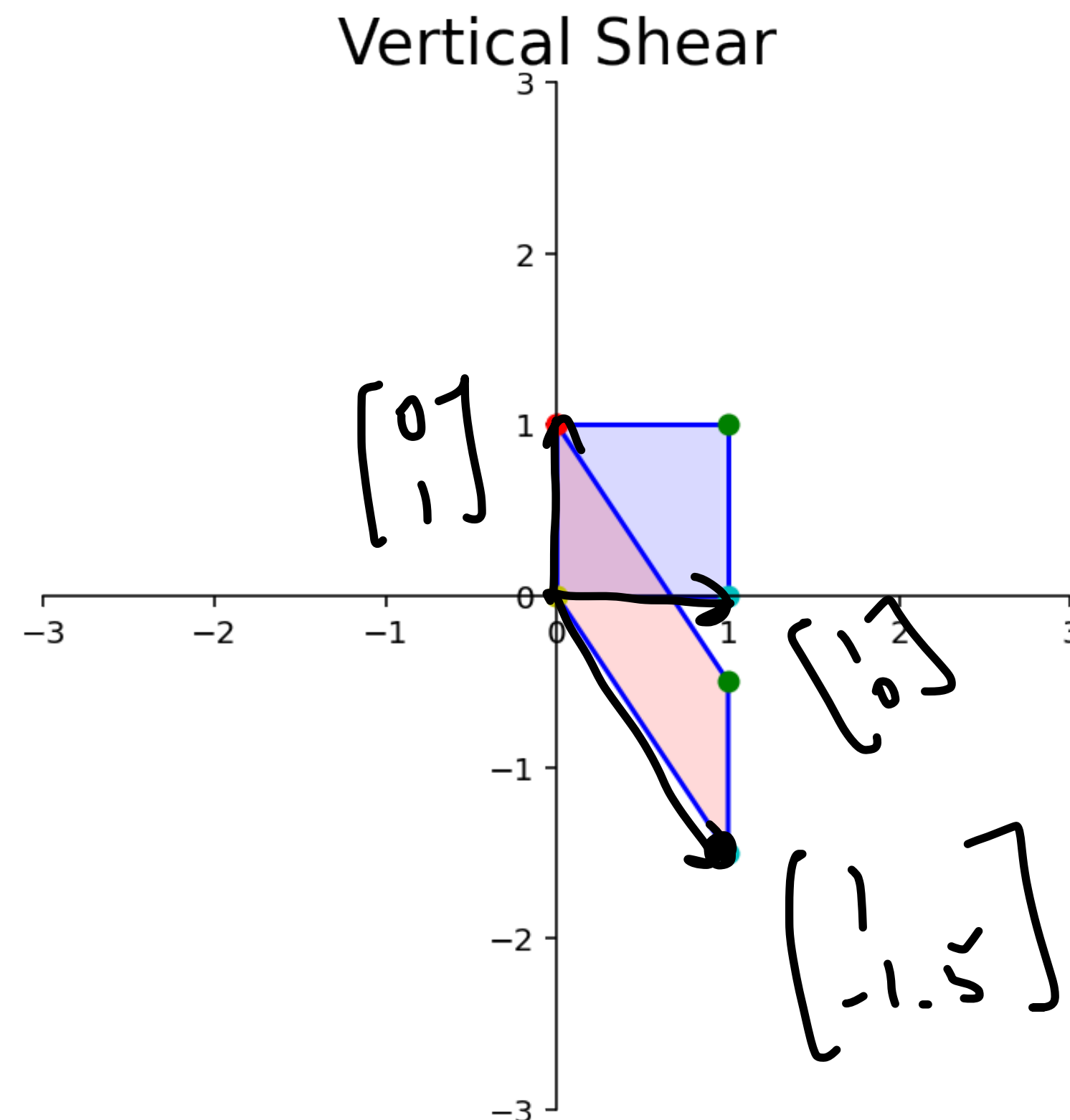
Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

Solution. Find where the standard basis vectors go

Example

Write down the matrix for the following shearing operation using this method

$$\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}$$



You need to know these matrices, but you don't need to
memorize them

Remember: What does this matrix do to the unit square?
Then build the matrix from there

List of Important 2D Linear Transformations

- » dilation, contraction
- » reflections
- » projections
- » horizontal/vertical contractions
- » horizontal/vertical shearing

Look through the notes for a comprehensive
collection of pictures or... (demo)

One-to-One and Onto

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A
transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A
transforms into \mathbf{b}

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? \equiv is there a vector which A
transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A
transforms into \mathbf{b}

What about other questions?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Wait

$A\mathbf{x} = \mathbf{0}$ has a
unique solution

\equiv $A\mathbf{x} = \mathbf{b}$ has **at most one
solution**

why?:

$$A\vec{r} = \vec{0}$$

$$\vec{r} \neq \vec{0}$$

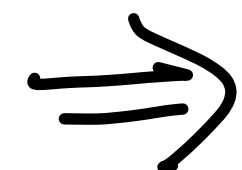


$A\vec{x} = \vec{0}$ has more than 1
solution.

nontrivial solution to $A\vec{x} = \vec{0}$

$$A\vec{r} = \vec{b} \quad \vec{u} \neq \vec{r}$$

$$A\vec{u} = \vec{b}$$



$$A(\underbrace{\vec{u} - \vec{r}}_{\vec{0} \neq}) = A\vec{u} - A\vec{r} = \vec{b} - \vec{b} = \vec{0}$$

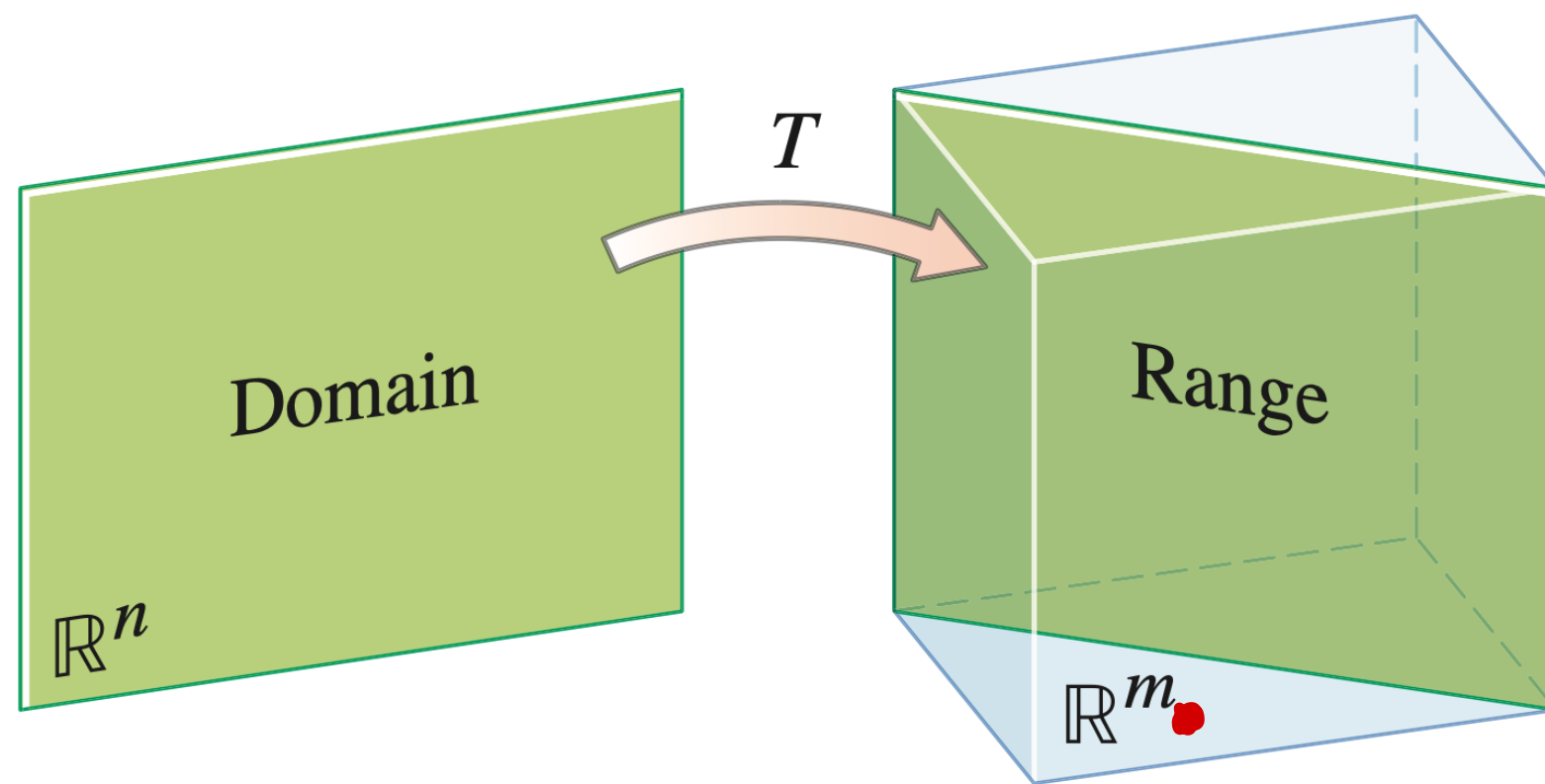
Onto Transformations

Onto Transformations

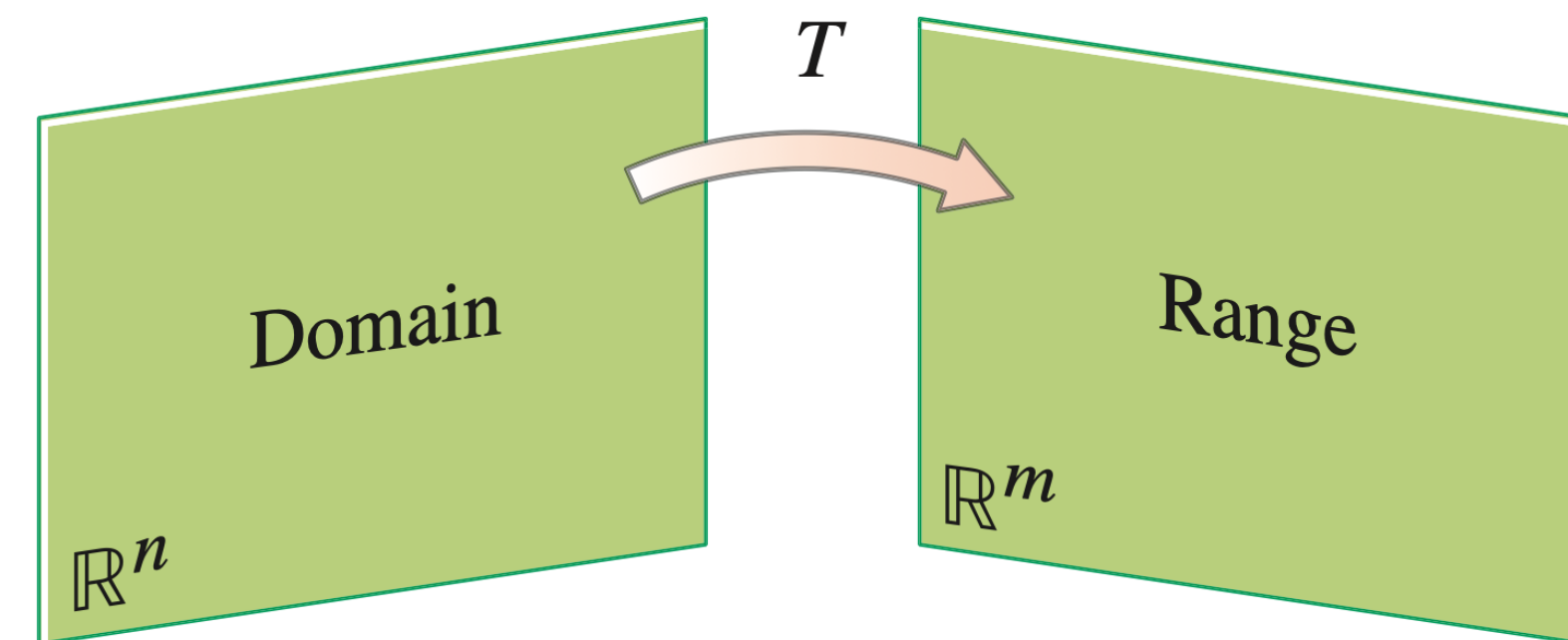
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

Onto Transformations

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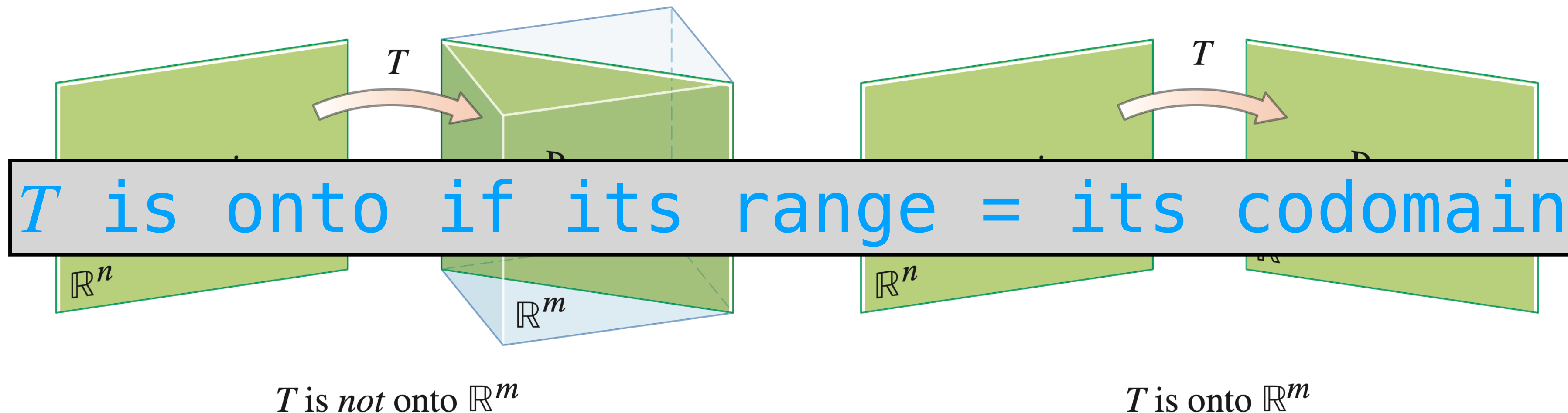
T is *not* onto \mathbb{R}^m



T is onto \mathbb{R}^m

Onto Transformations

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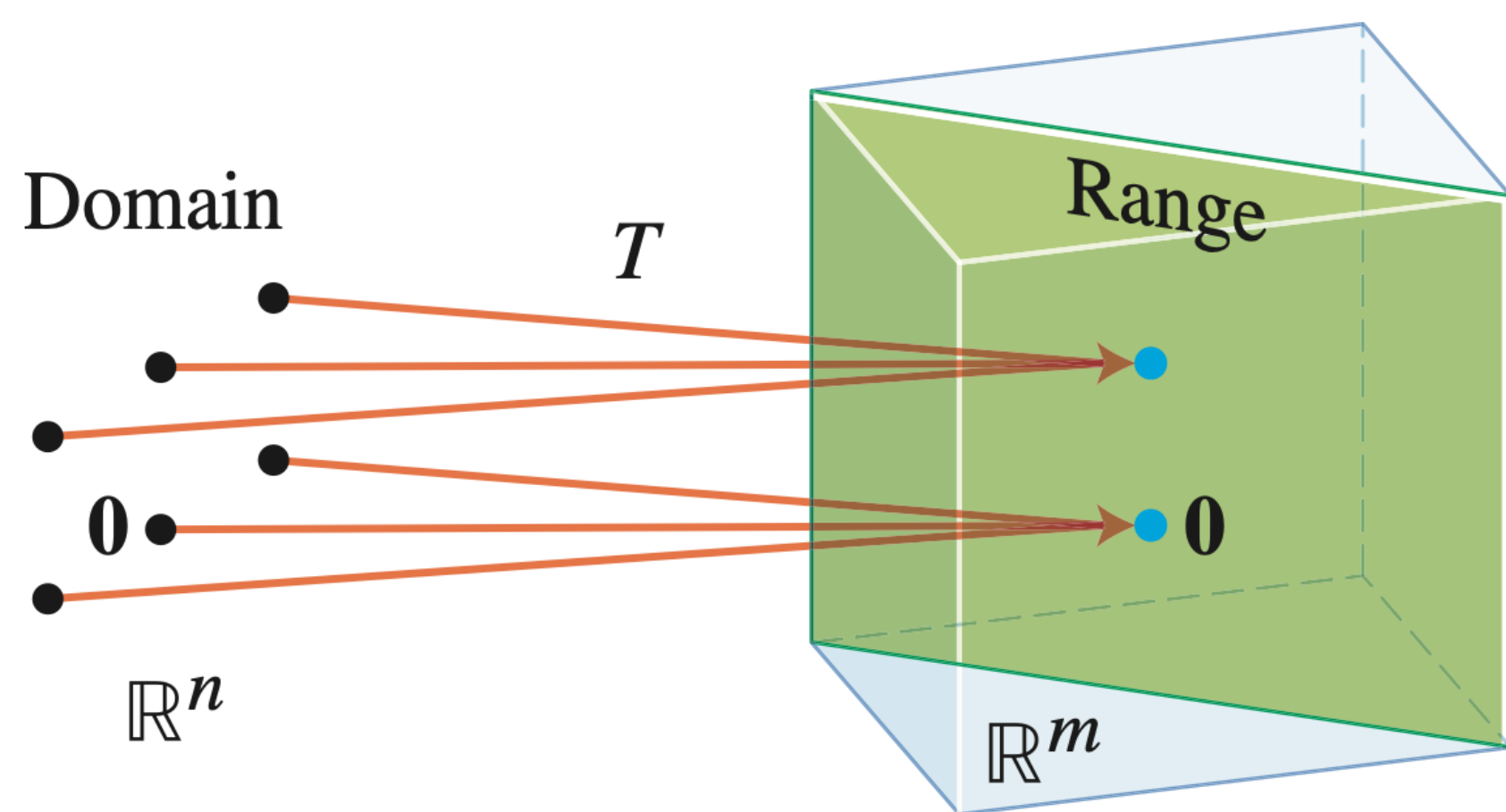
One-to-one Transformations

One-to-one Transformations

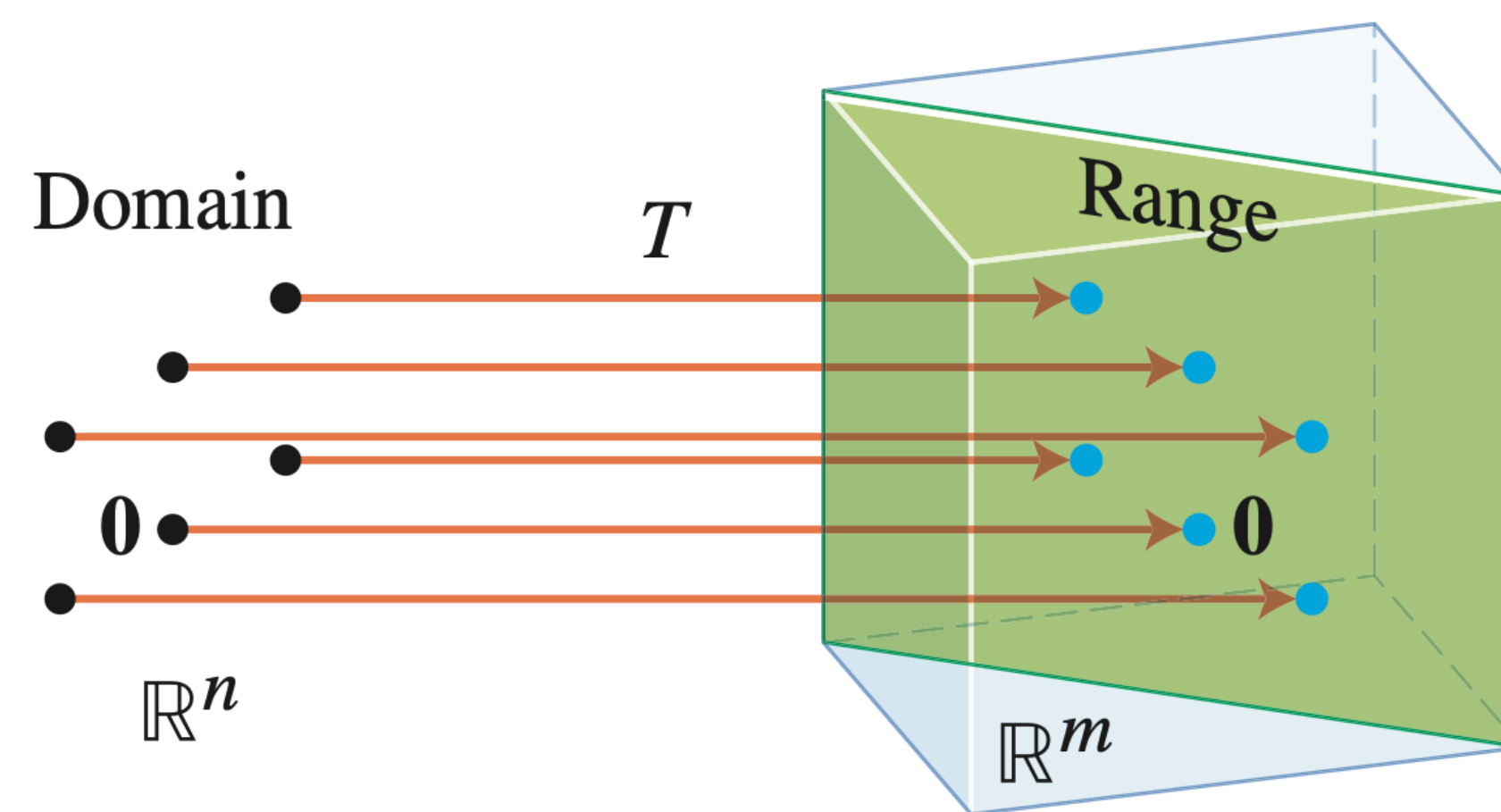
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

One-to-one Transformations

Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

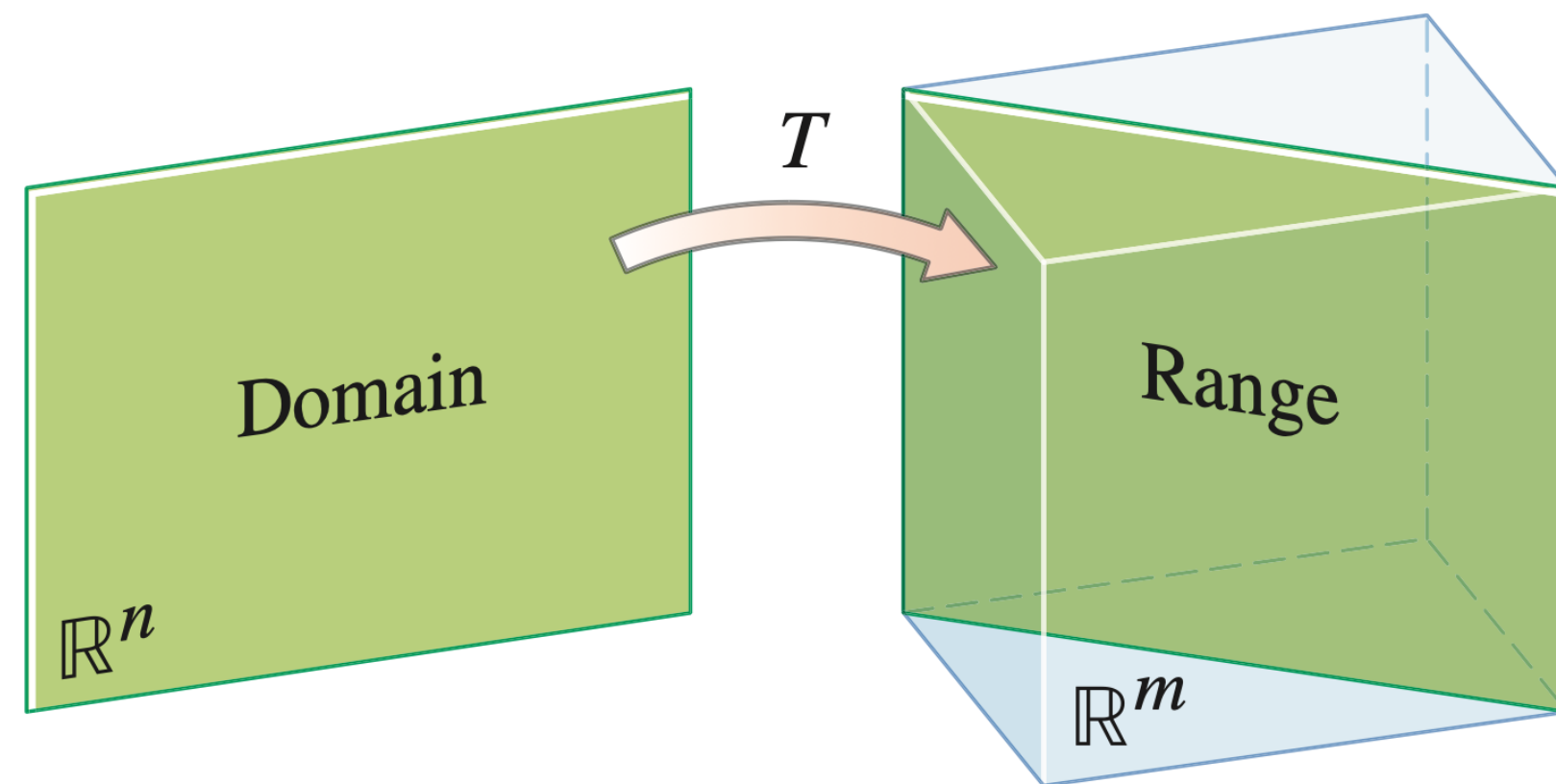


T is *not* one-to-one

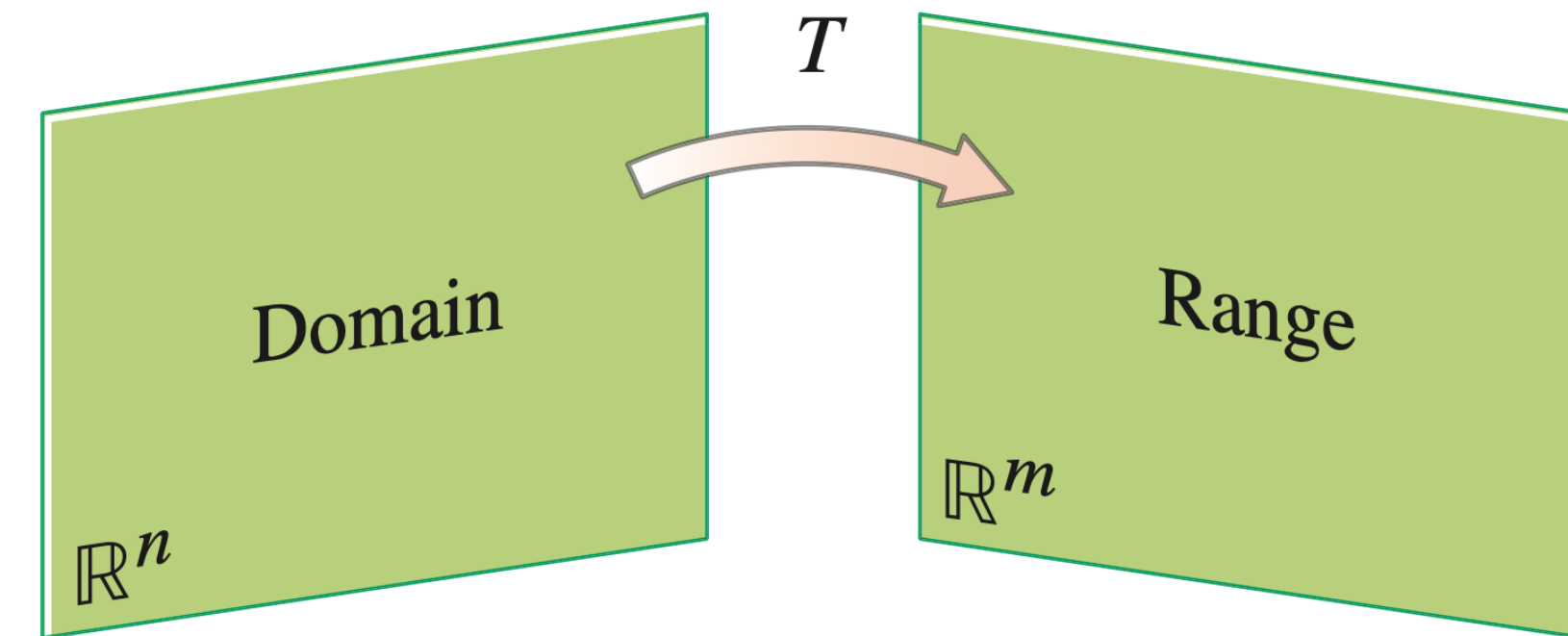


T is one-to-one

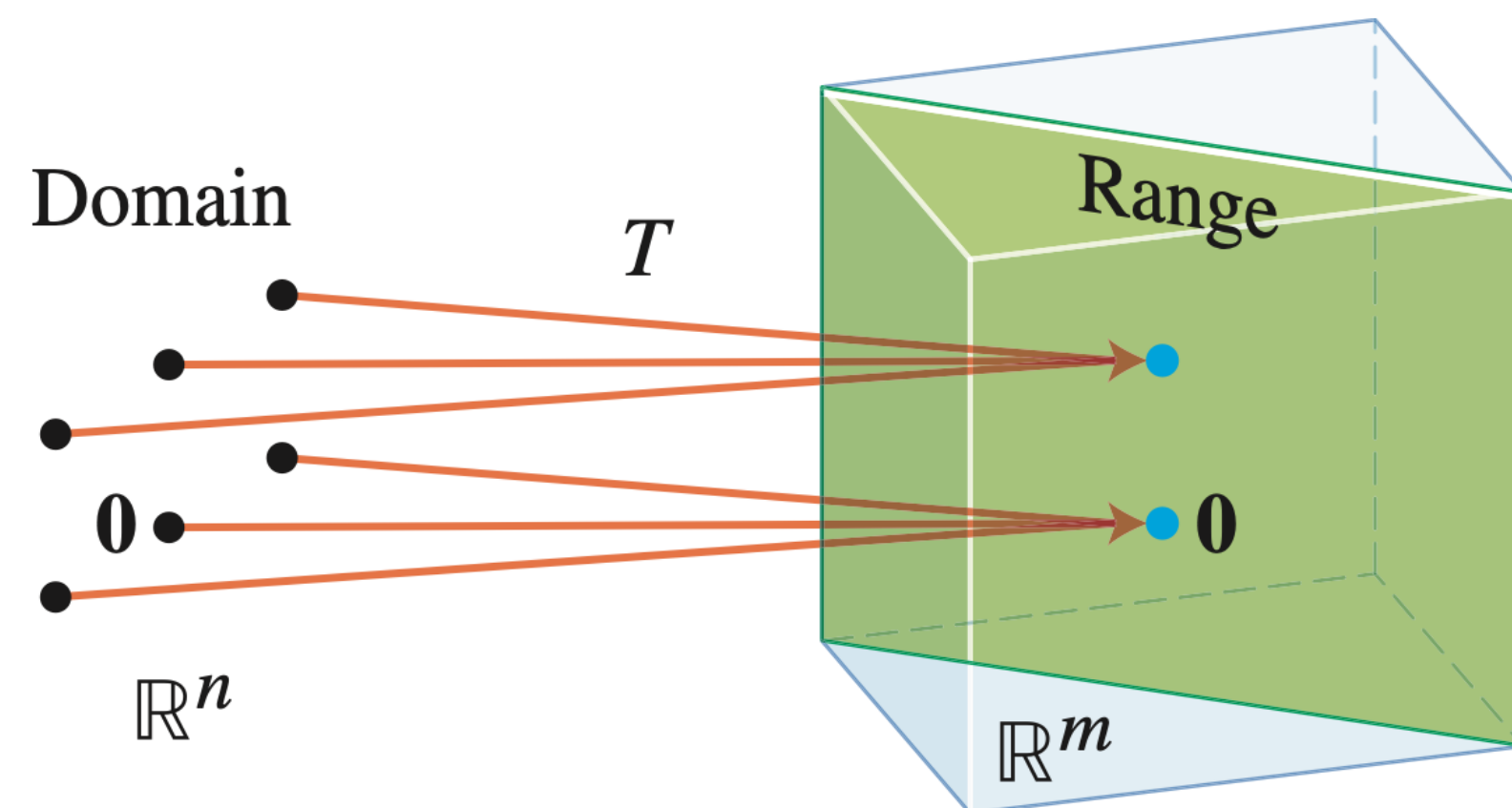
Comparing Pictures



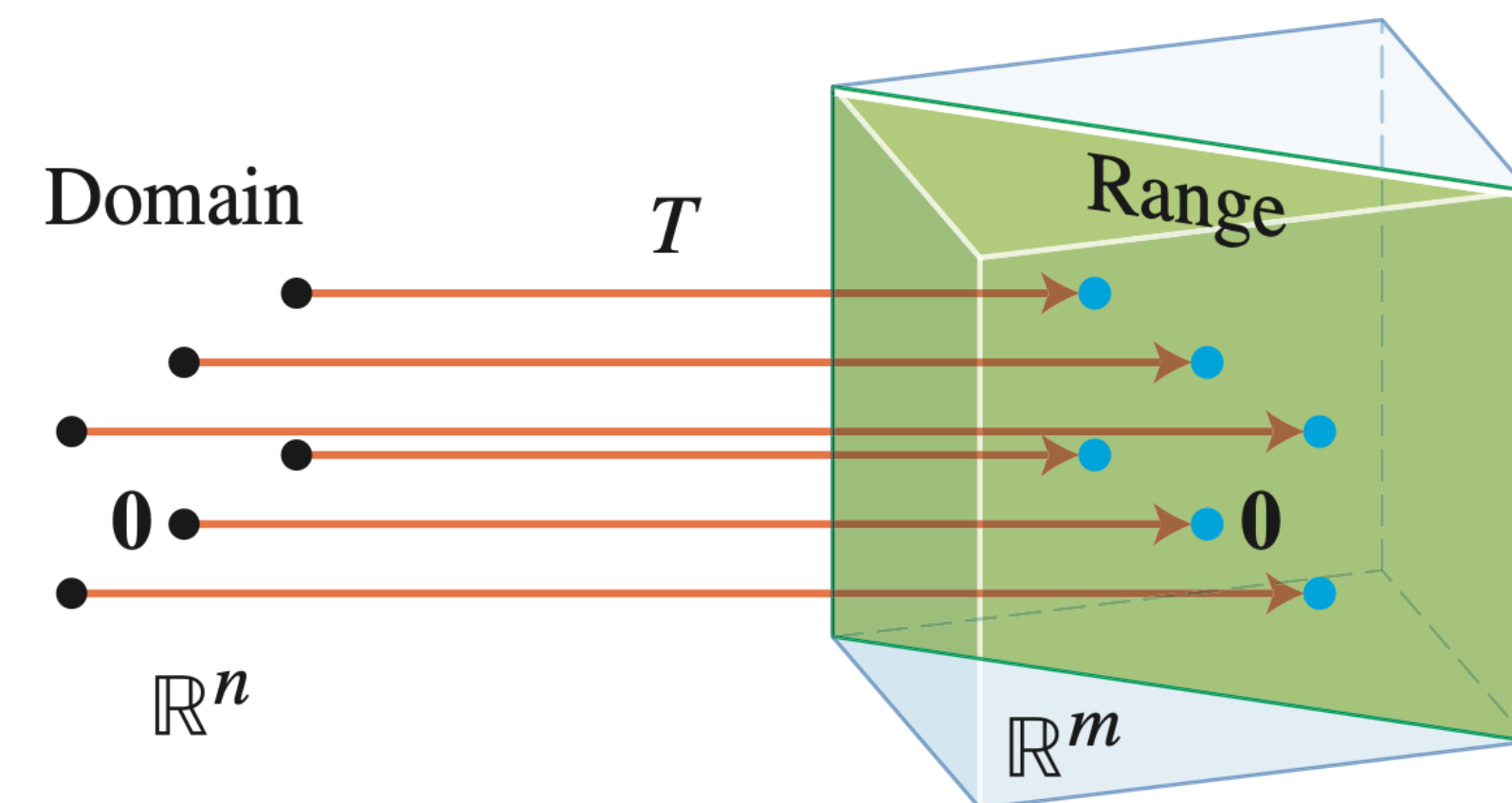
T is not onto \mathbb{R}^m



T is onto \mathbb{R}^m



T is not one-to-one



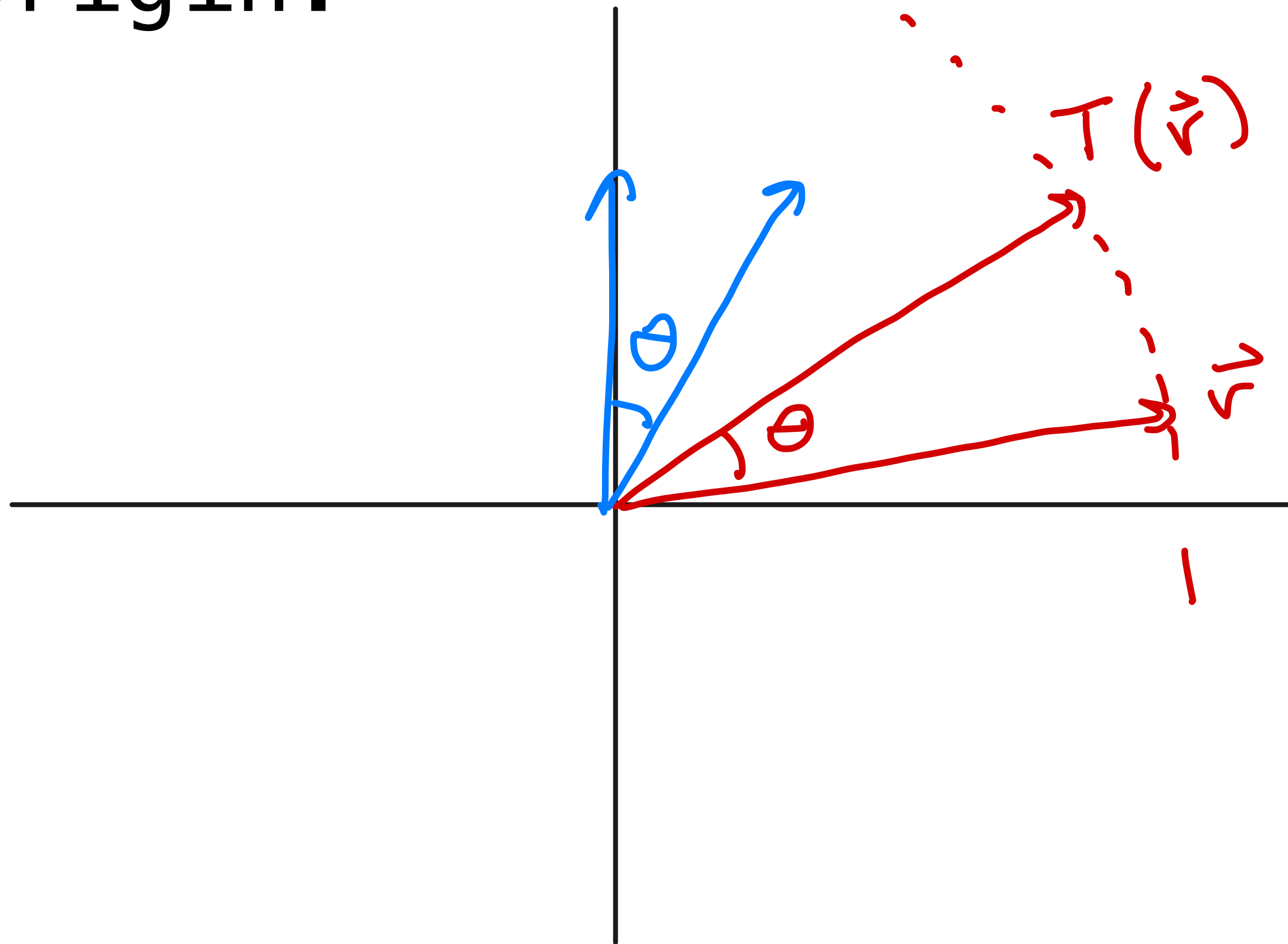
T is one-to-one

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :



Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

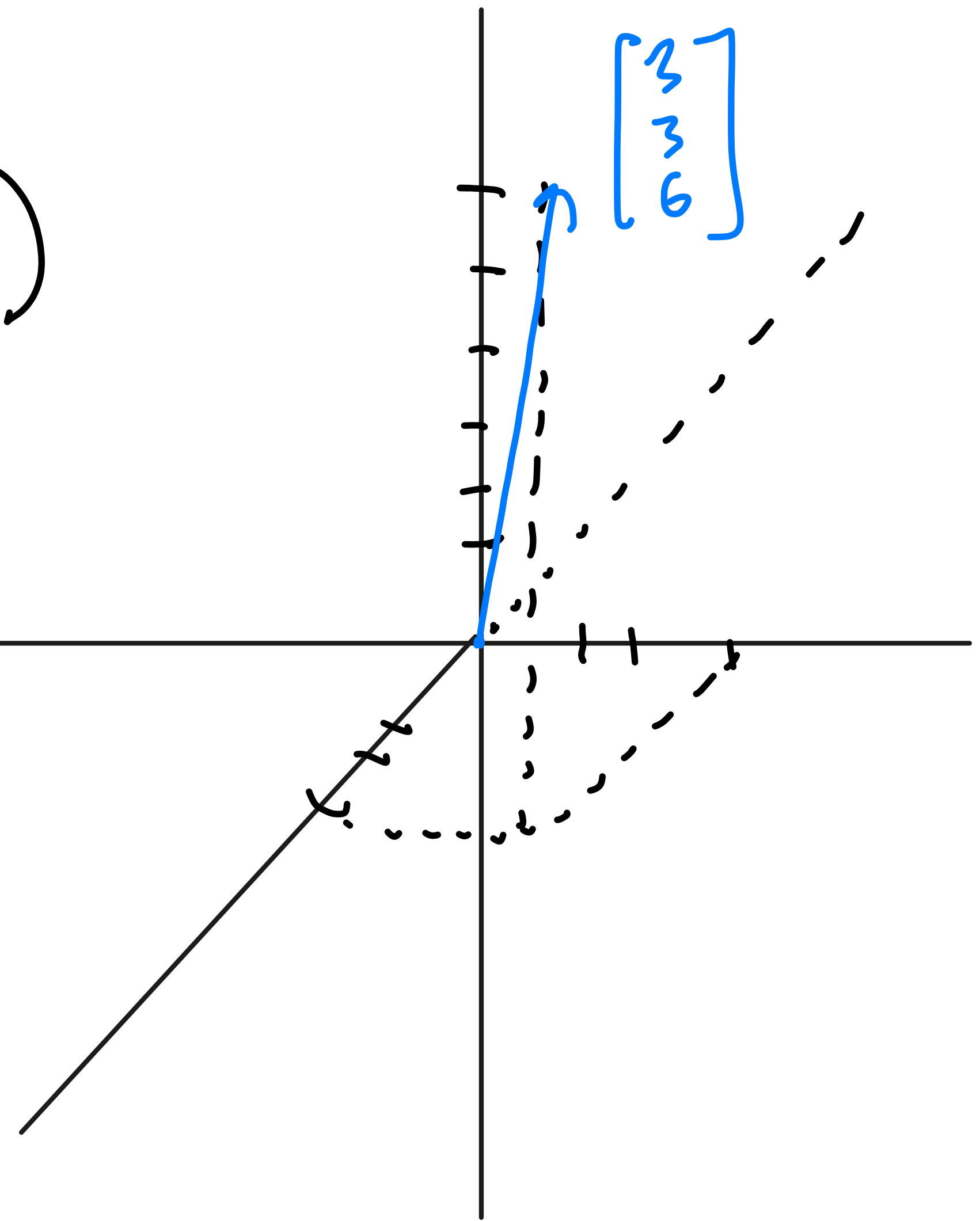
why? :

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \notin \text{ran}(T)$$

not onto

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

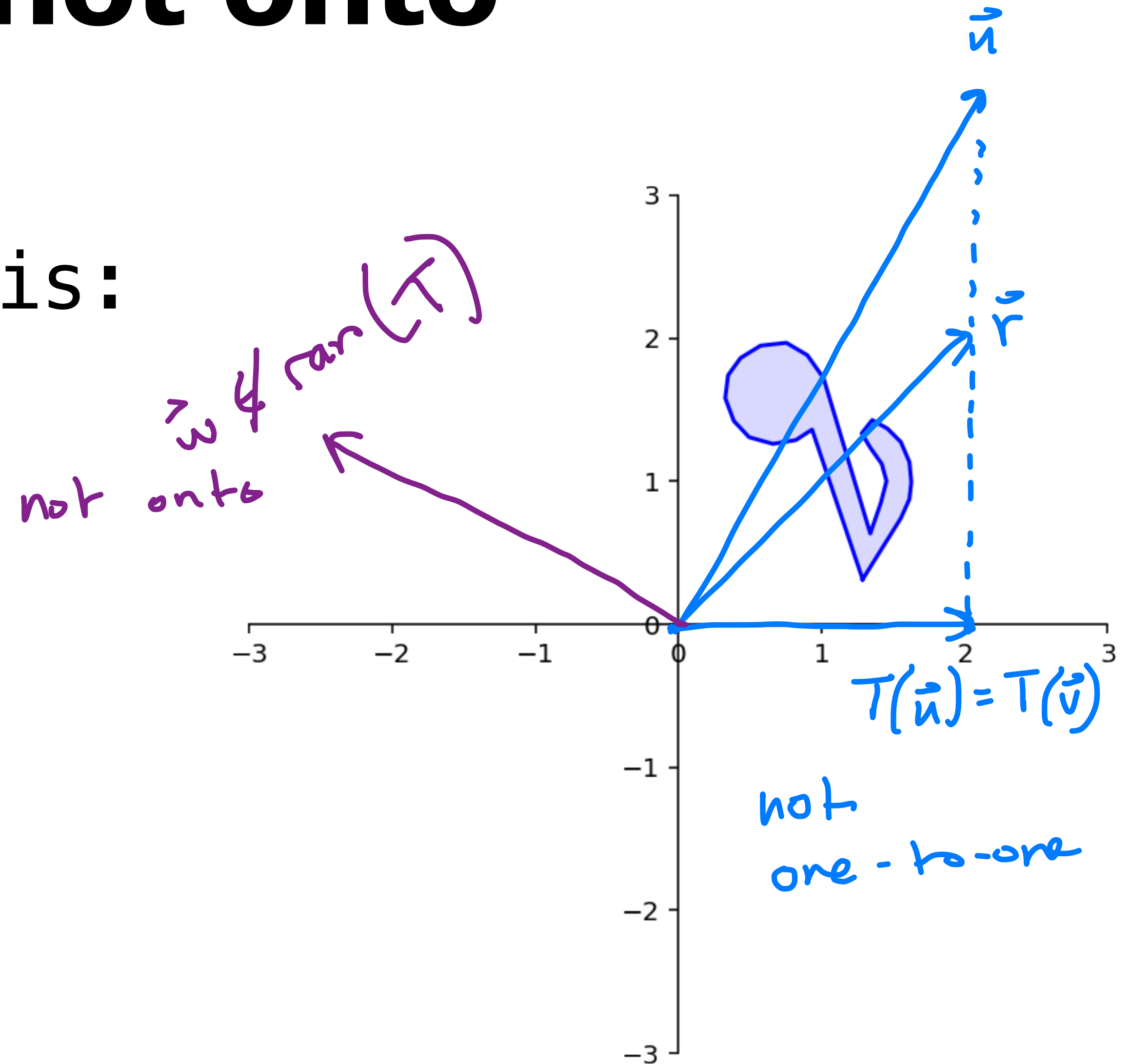


Example: not 1-1, not onto

Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :



Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

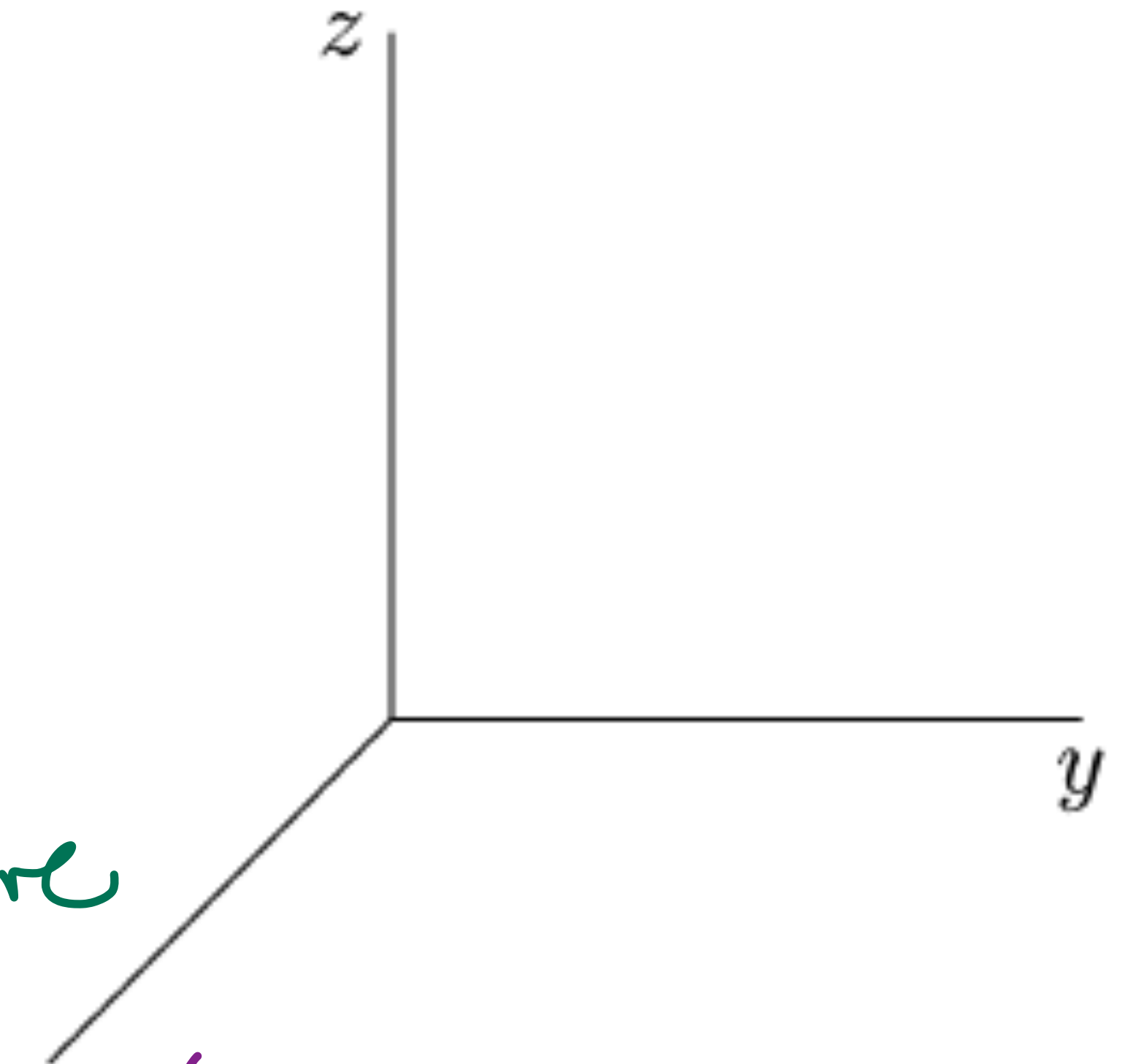
why? :

$$\left. \begin{array}{l} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \right\}$$

not one-to-one

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

onto ✓



Taking Stock: Onto

Taking Stock: Onto

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

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Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

- » T is onto
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- » $\text{range}(T) = \text{codomain}(T)$

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- » the columns of A span \mathbb{R}^m
- » A has a pivot position in every row

Taking Stock: One-to-One

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- » T is one-to-one
- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}

Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

- » T is one-to-one
- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
- » $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

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- » $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- » The columns of A are linearly independent
- » A has a pivot position in every column

How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \sim \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\sin^2 \theta \end{bmatrix}$$

(Note: The handwritten matrix above is likely a typo for the standard rotation matrix inverse.)

why? :

$$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\sin^2 \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta + \frac{\sin^2 \theta}{\cos \theta} \end{bmatrix} = \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta} = \frac{1}{\cos \theta}$$

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

why? :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \cancel{1} & 0 & 1 \\ 0 & \cancel{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: not 1-1, not o

Projection onto the x_1 axis:

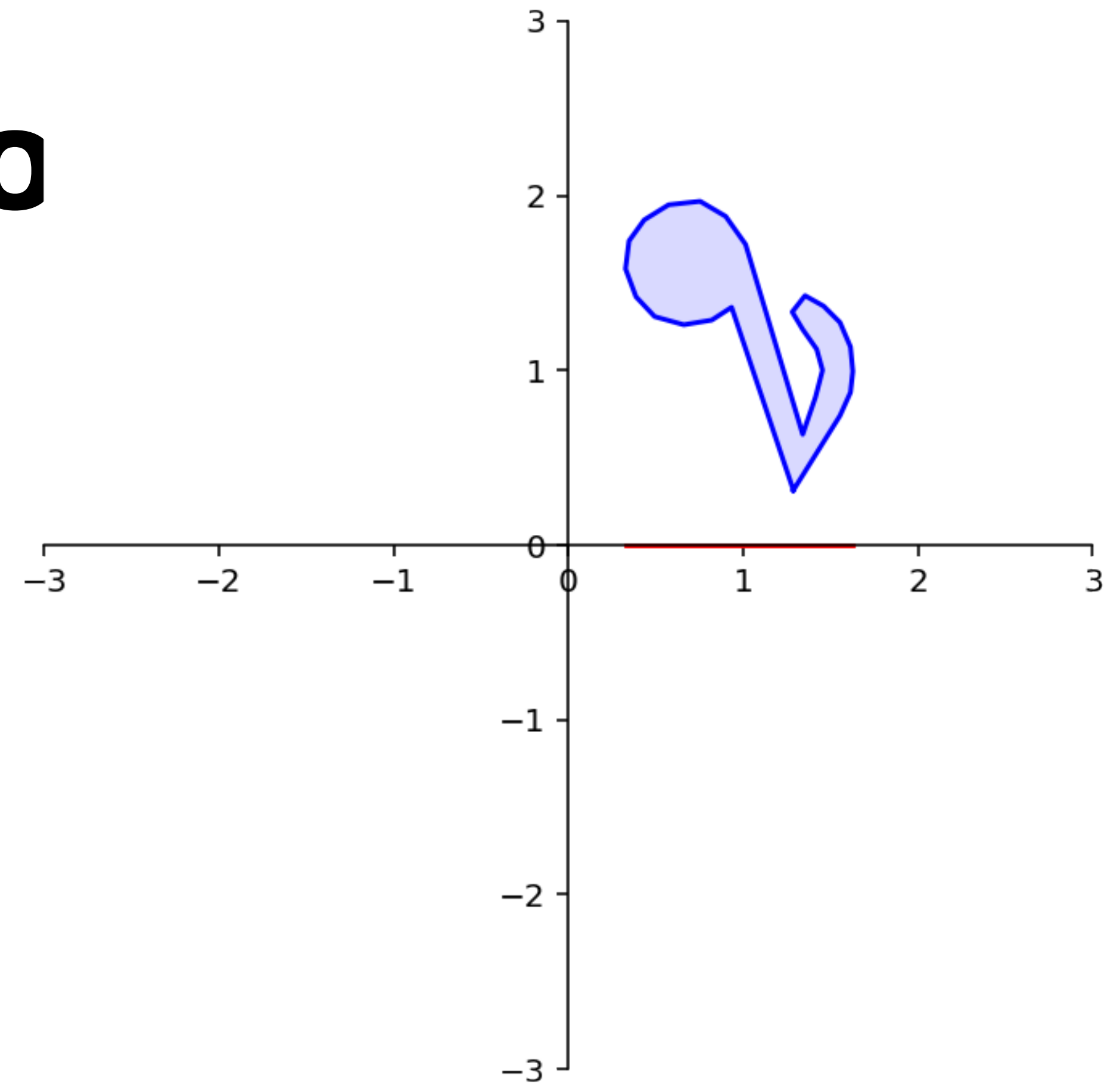
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

1 p.r.

2 col.

2 rows



Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

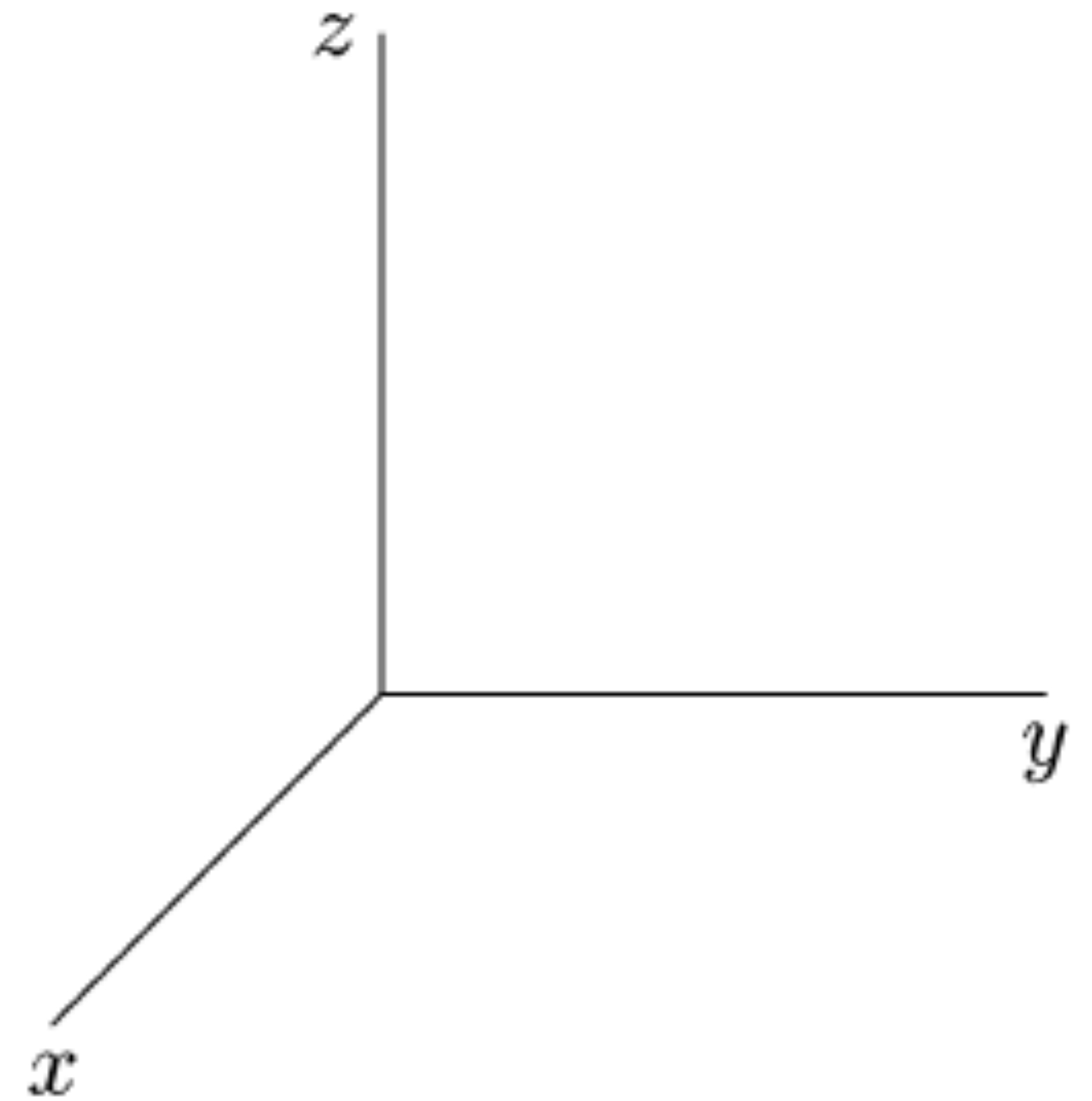
why? :

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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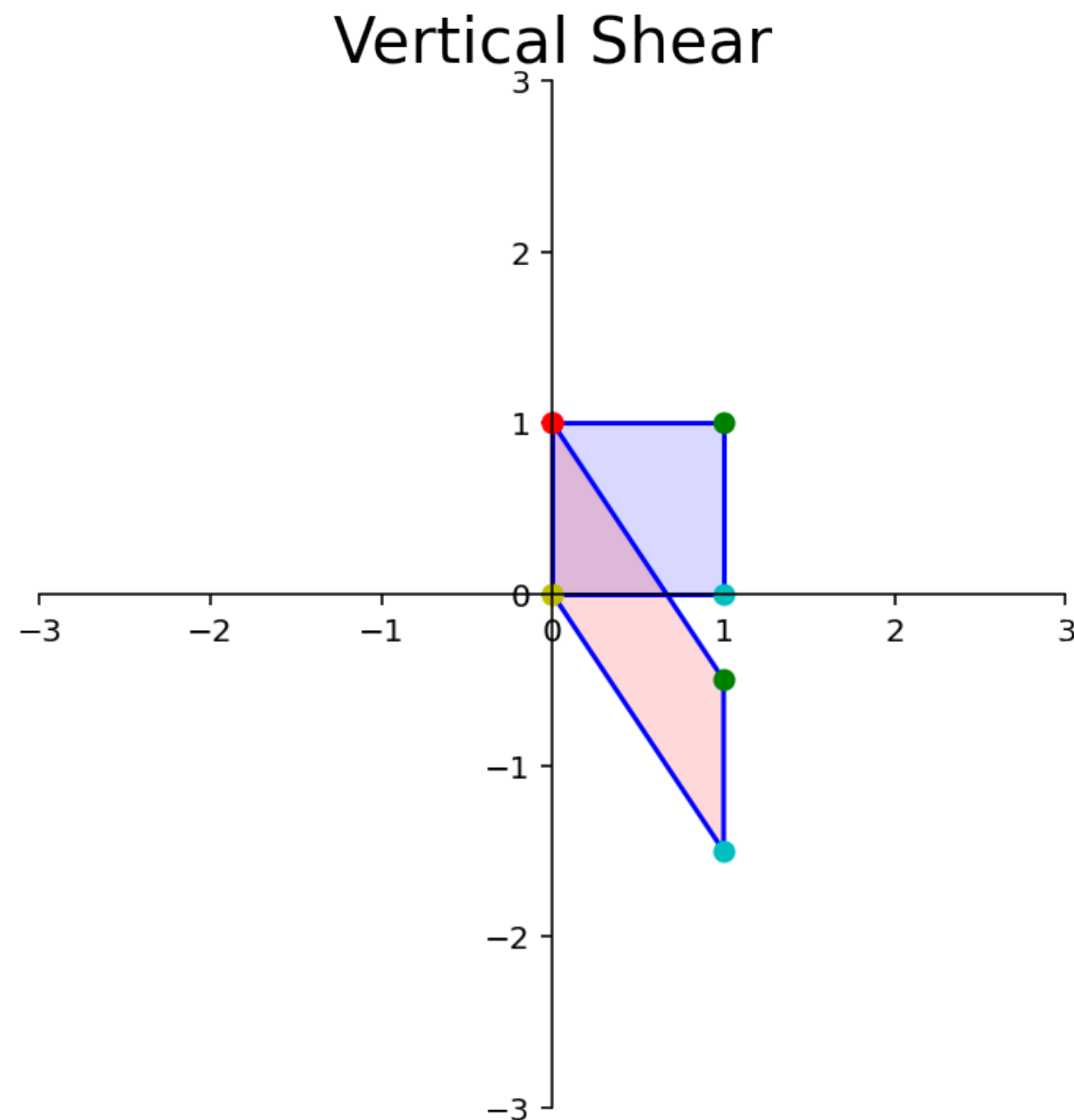
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



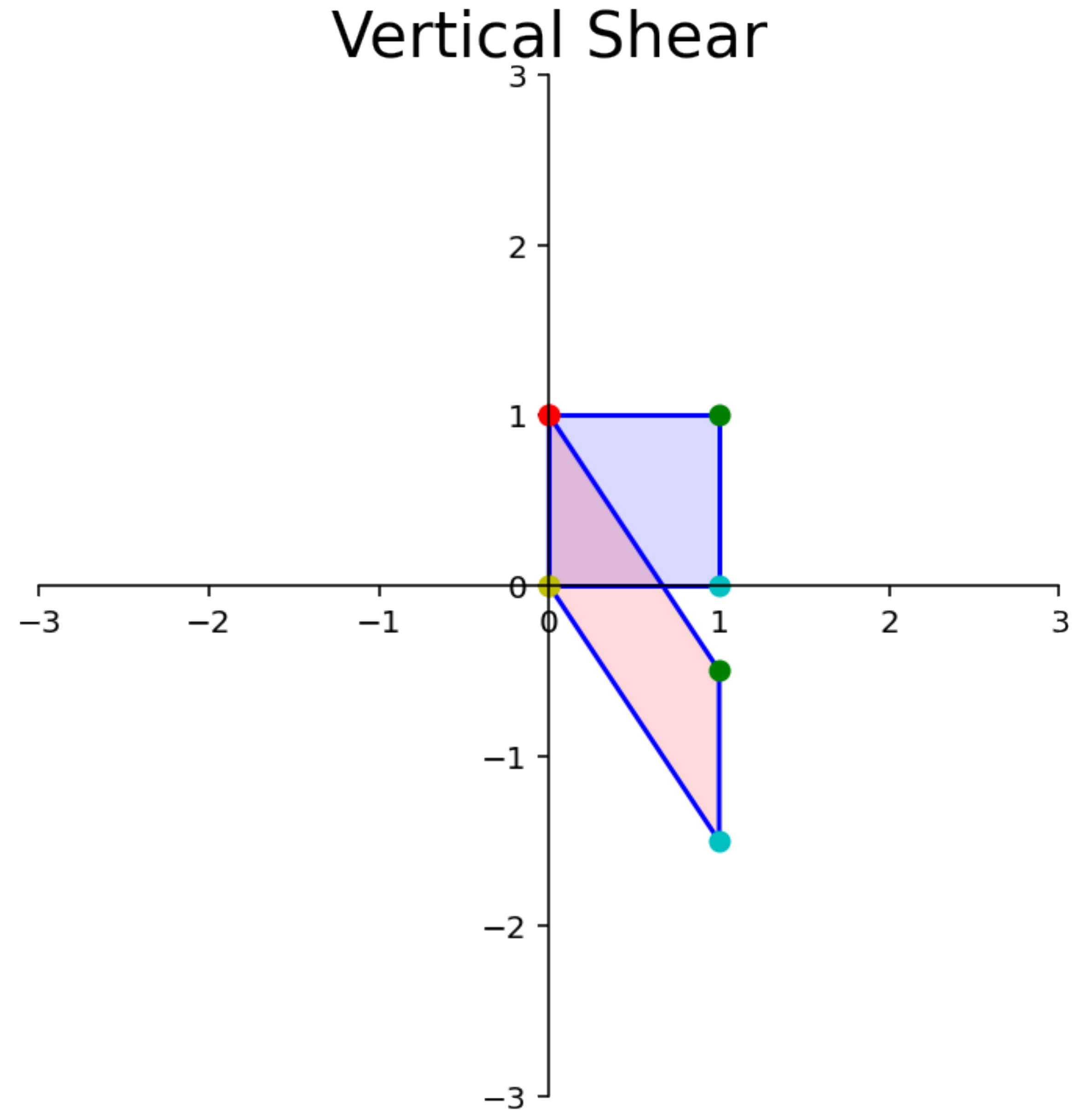
Question

Is vertical shearing a 1-1 transformation?
Justify your answer

Exercise :

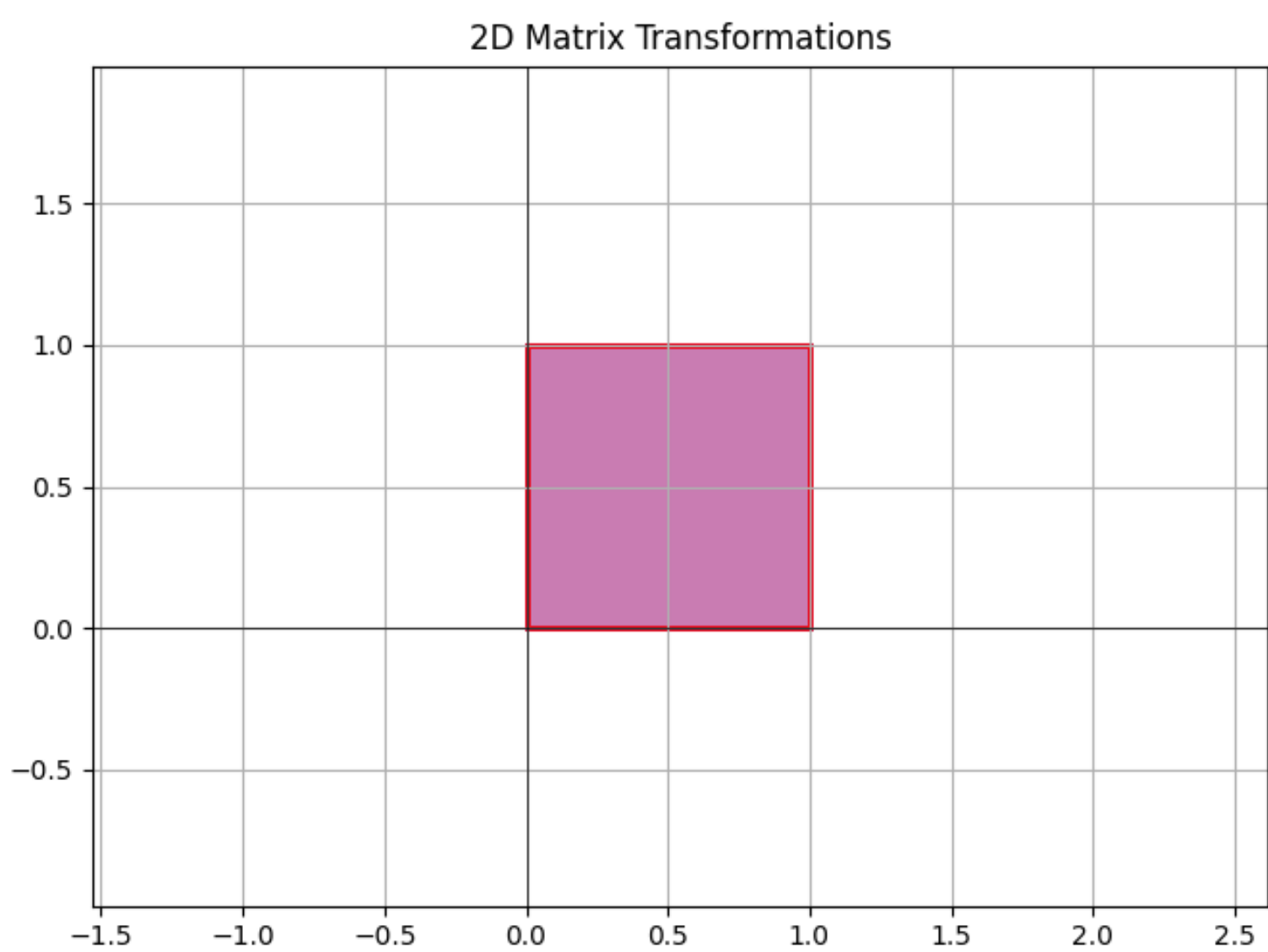


Answer: Yes

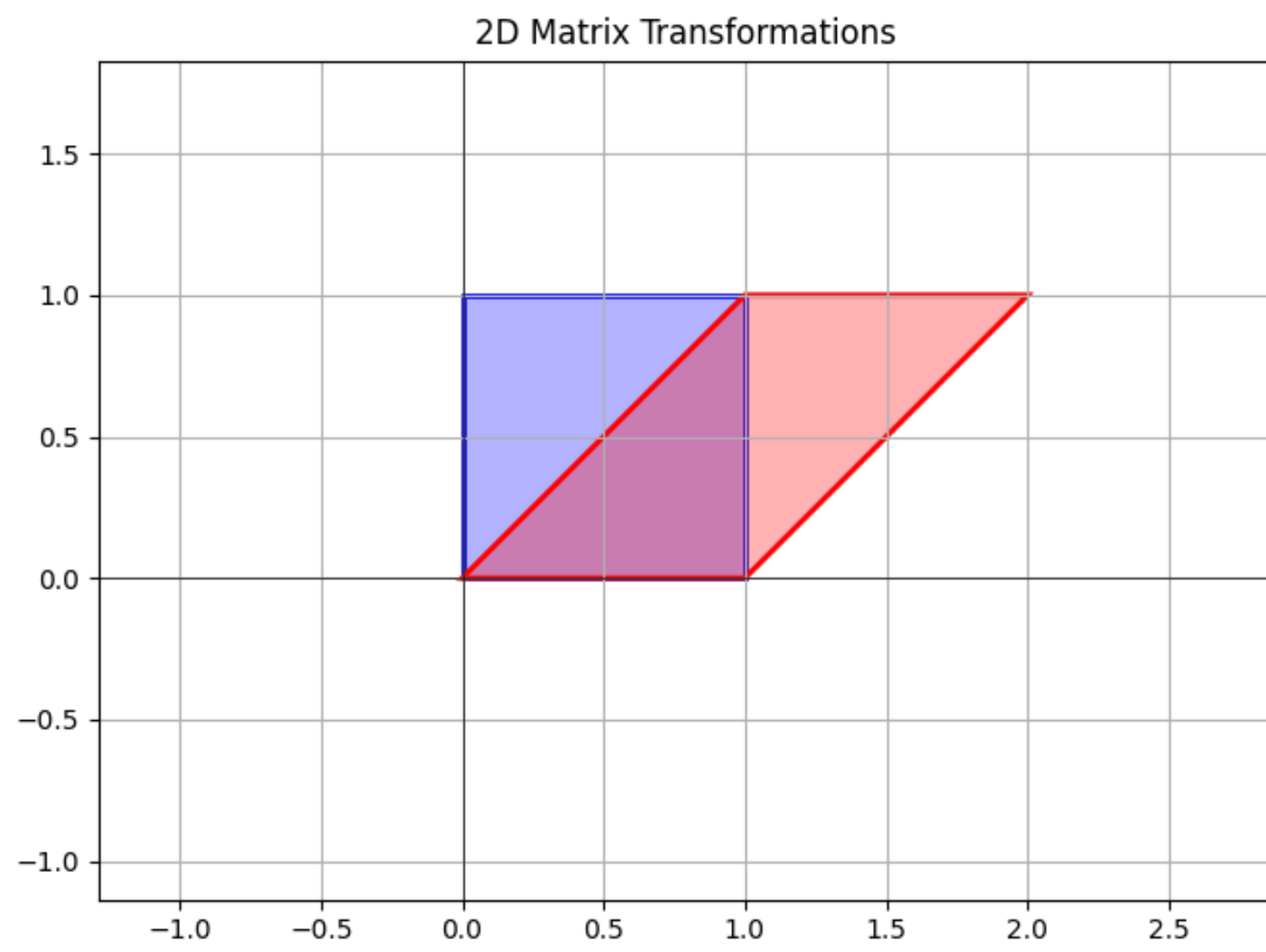


Composing Linear Transformations

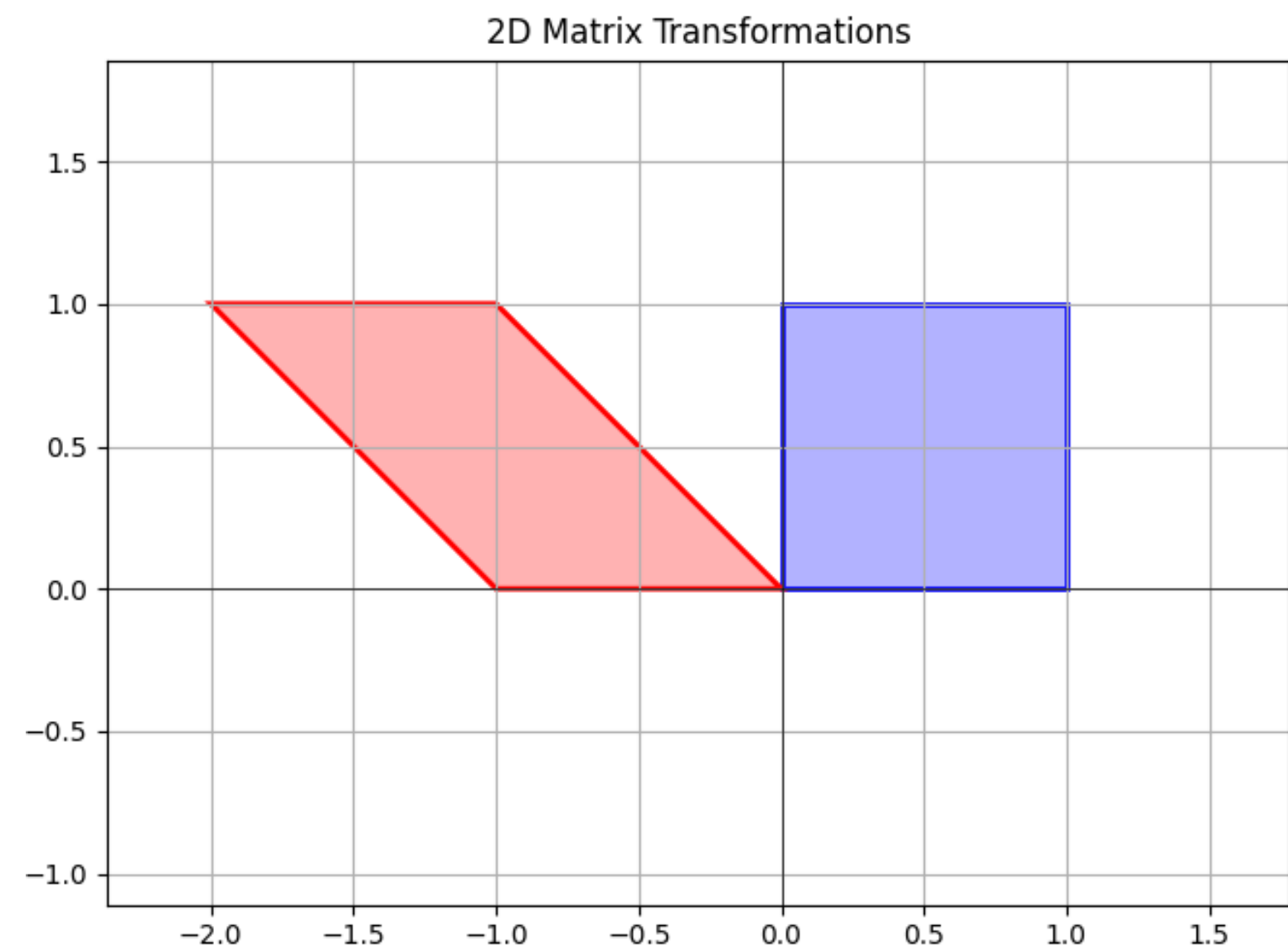
Shearing and Reflecting (Geometrically)



shear



reflect

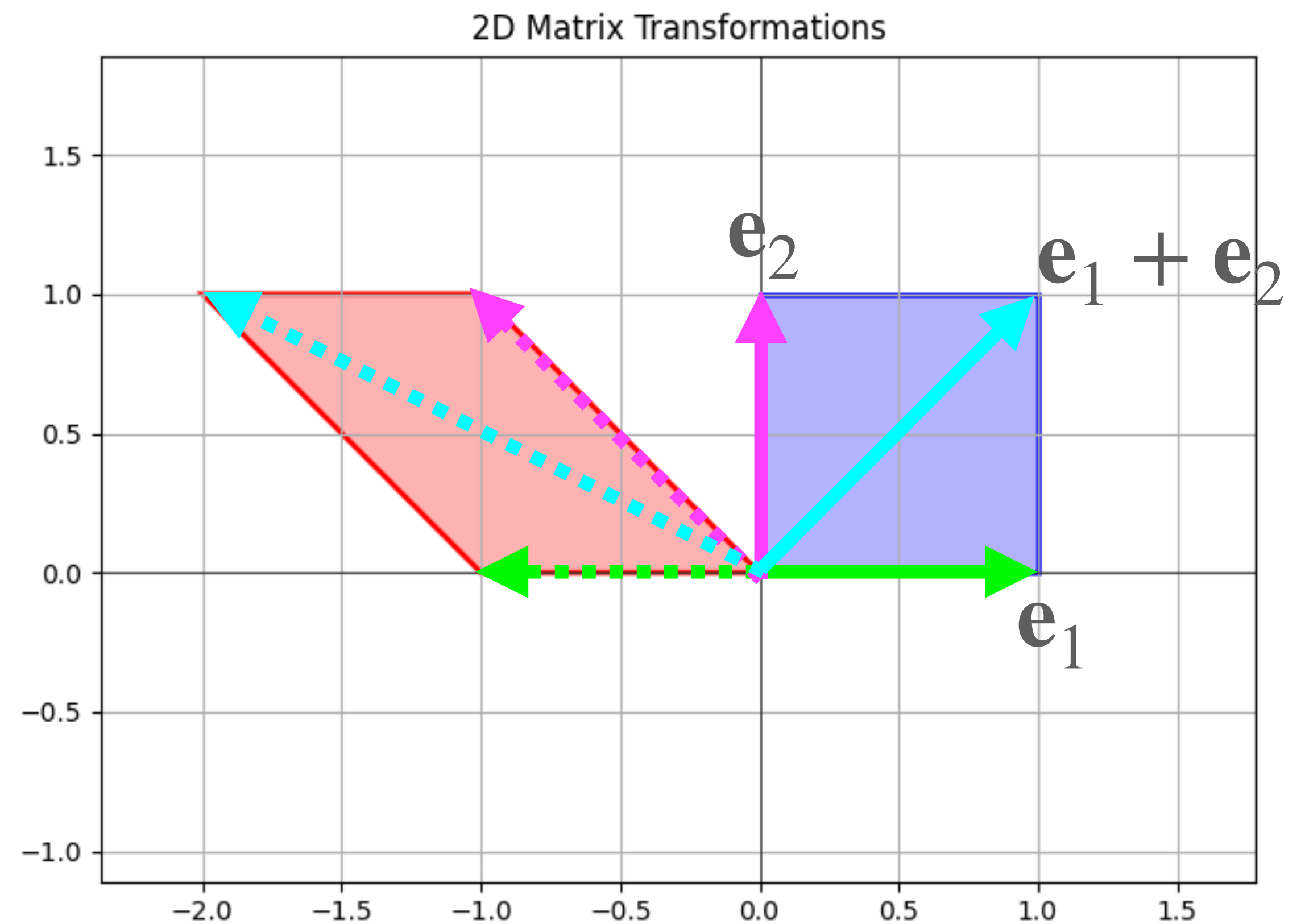


Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply
by reflection matrix

Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply
by reflection matrix

This gives us the same transformation

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

The Key Fact

Fact. The composition of two linear transformations is a linear transformation

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Verify:

The Key Fact

Fact. The composition of two linear transformations is a linear transformation

Verify:

This means the composition of two matrix transformations can be represented as a *single* matrix

The Key Question

*Given two linear transformations,
how to we compute the matrix which
implements their composition?*

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Matrix Multiplication

Matrix Multiplication

Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column

Tracking Dimensions

This only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates matrix multiplication with dimension tracking. It shows three matrices arranged in a sequence separated by an equals sign. The first matrix is a 5x3 matrix, represented by a blue vertical line on the left labeled m and a red horizontal line on top labeled n . The second matrix is a 3x4 matrix, represented by a red vertical line on the left labeled n and a purple horizontal line on top labeled k . The third matrix is a 5x4 matrix, represented by a blue vertical line on the left labeled m and a purple horizontal line on top labeled k . Each matrix contains asterisks representing elements. Below each matrix, its dimensions are written in colored boxes: $(m \times n)$ for the first, $(n \times k)$ for the second, and $(m \times k)$ for the third.

$$\begin{matrix} m \\ \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{array} \right] \end{matrix} \begin{matrix} n \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix} = \begin{matrix} m \\ \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \end{matrix}$$

$(m \times n)$ $(n \times k)$ $(m \times k)$

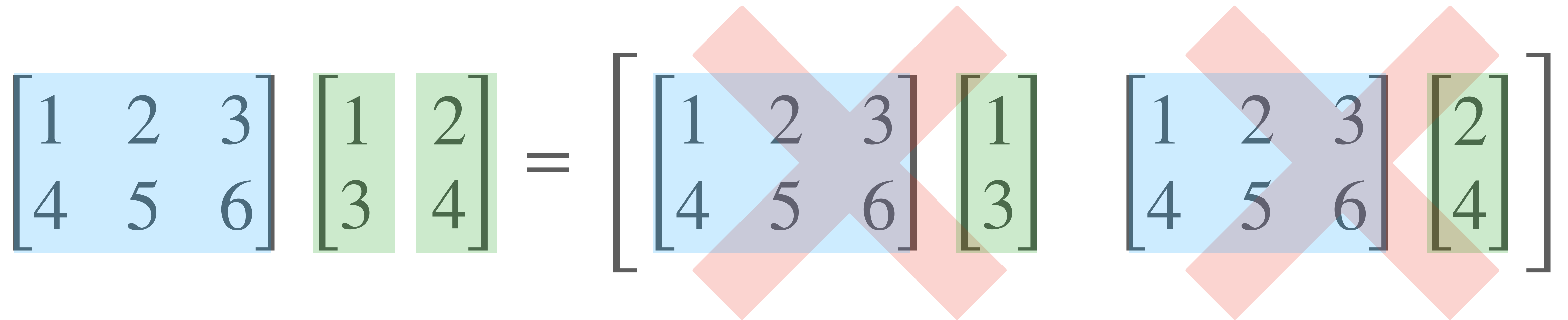
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$


These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices: a 5x3 matrix A on the left, a 3x4 matrix B in the middle, and a 5x4 matrix C on the right. The first row of A is highlighted in light blue, the first column of B is highlighted in light red, and the first row of C is highlighted in light purple. The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A (left) with its first row highlighted in light blue.
- A 3x4 matrix B (middle) with its second column highlighted in light red.
- The resulting 5x4 matrix AB (right) with its first row and second column highlighted in light purple.

The matrices are separated by an equals sign, indicating the multiplication operation.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its third column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the first row and third column is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the third column of the second matrix.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with its first row highlighted in light blue. The second matrix is a 3x4 matrix with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element in the first row and fourth column is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the fourth column of the second matrix.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks. The third matrix is a 5x4 matrix, also with all elements represented by asterisks. An equals sign (=) is placed between the second and third matrices. The first matrix has its second row highlighted in light blue. The second matrix has its first column highlighted in light red. The third matrix has its first row highlighted in light purple. This highlights the specific row and column used in the calculation of the element at the intersection of the second row and first column of the product matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements represented by asterisks (*). The third matrix is a 5x4 matrix, also with all elements represented by asterisks (*). The second matrix is positioned between the first and third matrices, with an equals sign (=) to its right. The first matrix has its second row highlighted in light blue. The second matrix has its second column highlighted in light red. The third matrix has its second column highlighted in light purple. This visualizes the calculation of the element in the second row and second column of the product matrix, which is the dot product of the second row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all elements represented by asterisks (*). The second matrix is a 3x4 matrix, also with all elements as asterisks. The third matrix is a 5x4 matrix, also with all elements as asterisks. An equals sign (=) is placed between the second and third matrices. In the first matrix, the second row is highlighted with a light blue background. In the second matrix, the third column is highlighted with a light red background. In the third matrix, the element in the second row and third column is highlighted with a light purple background, representing the result of the dot product of the second row of the first matrix and the third column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with all cells containing an asterisk (*). The second matrix is a 3x4 matrix, also with all cells containing an asterisk (*). The third matrix is a 5x4 matrix, also with all cells containing an asterisk (*). The second matrix is highlighted with a light red background, and the third matrix is highlighted with a light purple background. The first matrix is highlighted with a light blue background. The second and third matrices are separated by an equals sign (=).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 6x3 matrix with asterisks in each cell; the third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the first column is highlighted in light red. An equals sign follows. The third matrix is a 6x4 matrix with asterisks; the element in the third row and first column is highlighted in light purple, representing the dot product of the third row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the third row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the second column is highlighted in light red. An equals sign follows, and then a 5x4 matrix where the element in the third row and second column is highlighted in light purple, representing the result of the dot product of the third row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with a light blue horizontal band highlighting its third row. The second matrix is a 3x4 matrix with a light red vertical band highlighting its third column. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the third row and third column is highlighted with a light purple square, representing the result of the dot product of the highlighted row and column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its third row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the third row and fourth column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the fourth row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the first column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the fourth row and first column is highlighted in light purple, representing the dot product of the fourth row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices arranged horizontally, separated by an equals sign. The first matrix is a 5x3 matrix with asterisks in each cell; its fourth row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; its second column is highlighted in light red. The third matrix is a 5x4 matrix with asterisks; its fourth row, second column element is highlighted in light purple. This visualizes the calculation of the element at the intersection of the fourth row and second column of the product matrix, which is the dot product of the fourth row of the first matrix and the second column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with a light blue horizontal band highlighting its fourth row. The second matrix is a 3x4 matrix with a light red vertical band highlighting its third column. An equals sign follows, and then a 5x4 matrix where the element at the intersection of the fourth row and third column is highlighted with a light purple square, representing the result of the dot product of the highlighted row and column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its fourth row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the fourth row and fourth column is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication using three matrices of asterisks. The first matrix is a 5x3 grid with its bottom row highlighted in light blue. The second matrix is a 3x4 grid with its first column highlighted in light red. An equals sign follows, and then a 5x4 grid where the element at the intersection of the first row and the first column (the bottom-left element) is highlighted in light purple, representing the result of the dot product of the first row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in each cell; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks; the second column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks; the element in the bottom row and second column is highlighted in light purple, representing the result of the dot product of the first row and second column.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C .

Matrix A is represented as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Matrix B is represented as:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

The third column of matrix B is highlighted in red. An equals sign follows matrix B .

Matrix C is represented as:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

The element in the fifth row and third column of matrix C is highlighted in purple, representing the result of the dot product of the fifth row of A and the third column of B .

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in all cells; the bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in all cells; the fourth column is highlighted in light red. An equals sign follows. The third matrix is a 5x4 matrix with asterisks in all cells; the bottom-right element (row 5, column 4) is highlighted in light purple.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

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This is exactly the same as vector addition, but for matrices

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices

Algebraic Properties (Addition and Scaling)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

We need to know/memorize these

Algebraic Properties (Addition and Scaling)

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

We need to know/memorize these

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1

(also find a pair where they are the same)

Answer: Rotation and Reflection

Computational Aspects of Matrix Multiplication

Matrix Operations in Numpy

Let `a` and `b` be 2D numpy arrays and let `c` be a floating point number

» `a @ b` (matrix multiplication)

» `a + b` (matrix addition)

» `c * a` (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more

Analyzing Linear Algebra Algorithms

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We will not use $O(\cdot)$ notation!

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For numerics, we care about number of **F**loating-oint
Operations (FLOPs):

- >> addition
- >> subtraction
- >> multiplication
- >> division
- >> square root

Analyzing Linear Algebra Algorithms

We will not use $O(\cdot)$ notation!

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Operations (FLOPs):

- >> addition
- >> subtraction
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- >> division
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$2n$ vs. n is very different
when $n \sim 10^{20}$

Dominant Terms

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that said, we don't care about *exact* bounds

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A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \rightarrow \infty} \frac{f(i)}{g(i)} = 1$$

Dominant Terms

that said, we don't care about *exact* bounds

A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \rightarrow \infty} \frac{f(i)}{g(i)} = 1$$

for polynomials, they are equivalent to their dominant term

Dominant Terms

the dominant term of a polynomial is the monomial with the highest degree

$$\lim_{i \rightarrow \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$ dominates the function even though the coefficient for x^2 is so large

A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Suppose A and B are $n \times n$ matrices

This operations takes n multiplications and n divisions ($2n$ FLOPS total)

Repeating for each entry gives $\sim 2n^3$ FLOPS

A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

The main part of this procedure is highly parallelizable

A Note on Parallelization

```
a = np.array(...)  
b = np.array(...)  
prod = np.zeros([a.shape[0], b.shape[1]])  
for i in range(a.shape[0]):  
    for j in range(b.shape[1]):  
        prod[i, j] = np.dot(a[i], b[:,j])
```

The main part of this procedure is highly parallelizable

One processor per entry gets you to $\sim 2n$ FLOPS

A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers

Best leave it to experts (or do research in the area)

LAPACK is the state of the art library for matrix operations

numpy uses LAPACK

Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations

Matrix multiplication coincides with composition of linear transformations

There is an algebra of matrices which is consistent with the algebra of vectors