

Singular Value Decomposition

**Geometric Algorithms
Lecture 26**

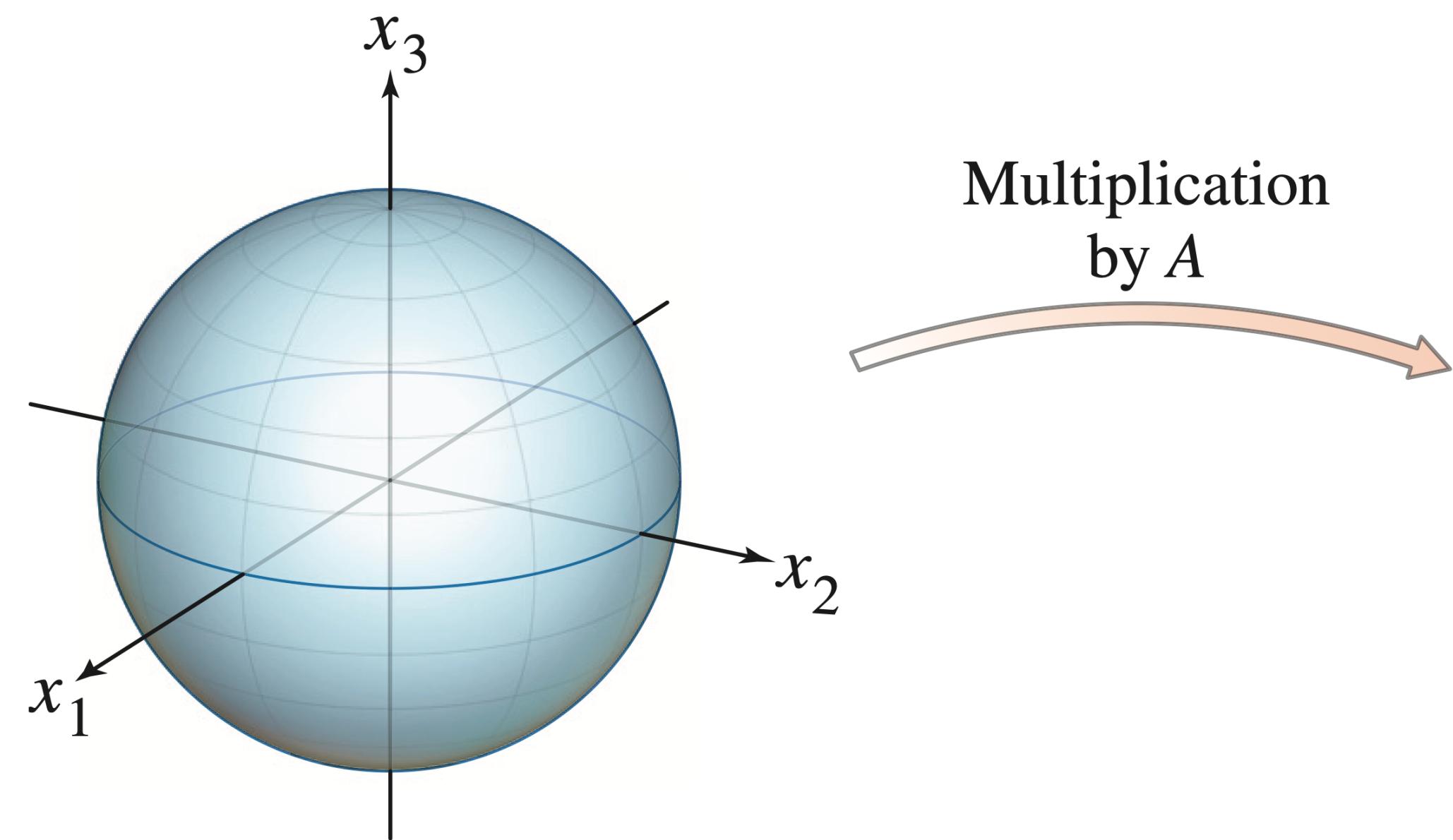
Objectives

1. Introduce the **singular value decomposition** (probably the most important matrix decomposition for computer science)
2. Talk very briefly about what to do after this course if you want (or have to) to see more linear algebra
3. ~~Fill out course evals(!)~~ Next time

Motivation

Question

What shape is a the unit sphere after a linear transformation?

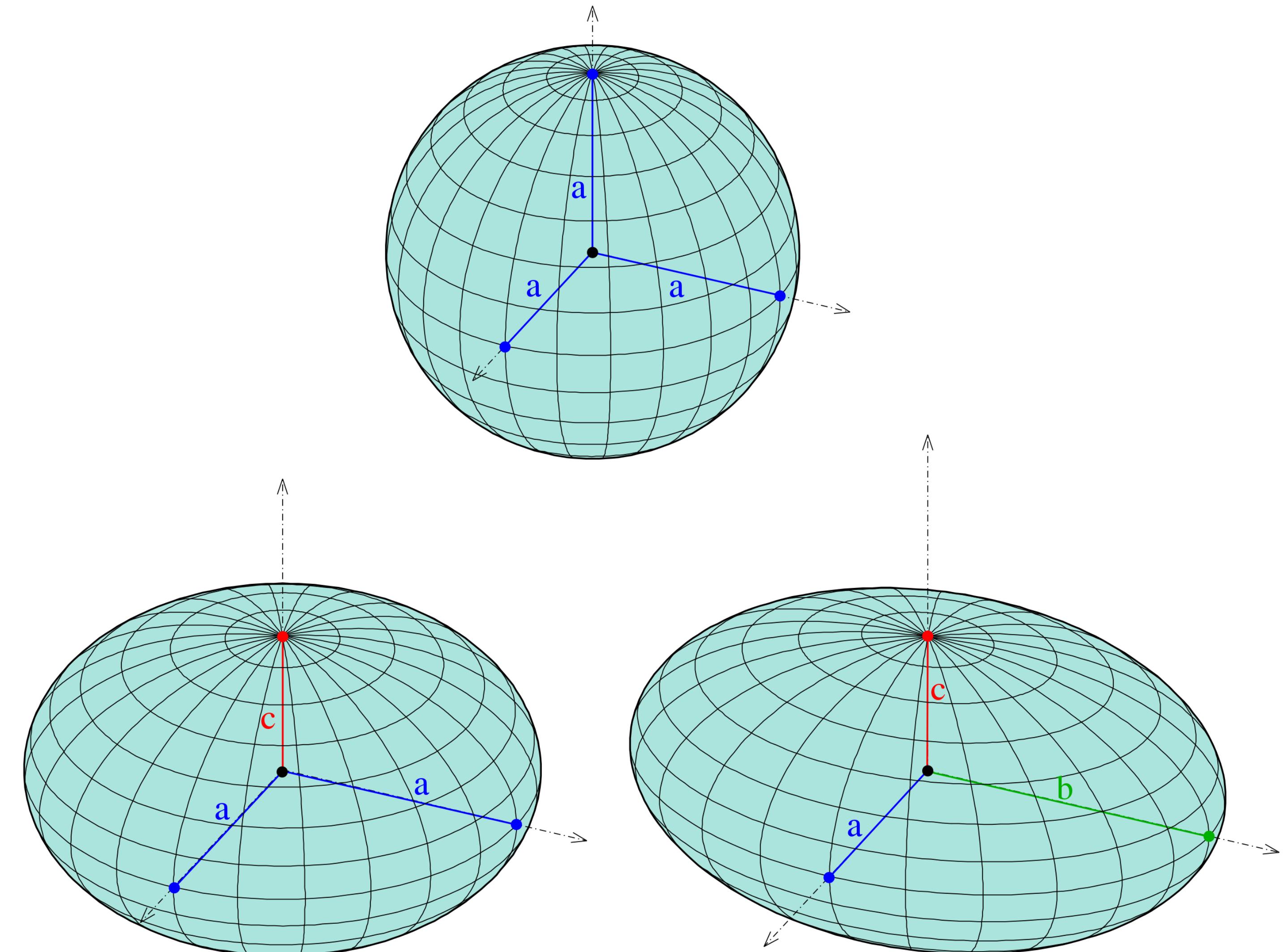


???

Ellipsoids

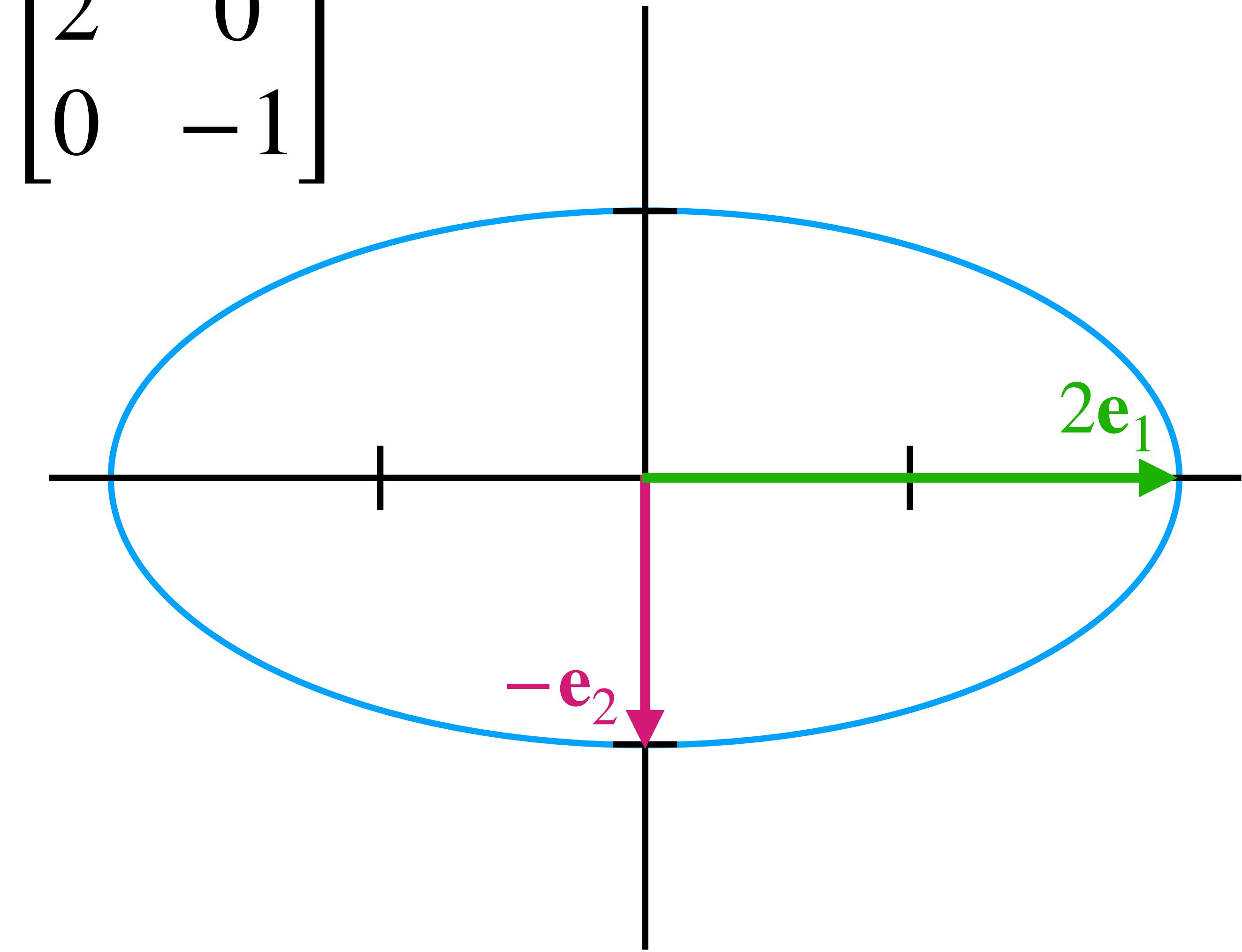
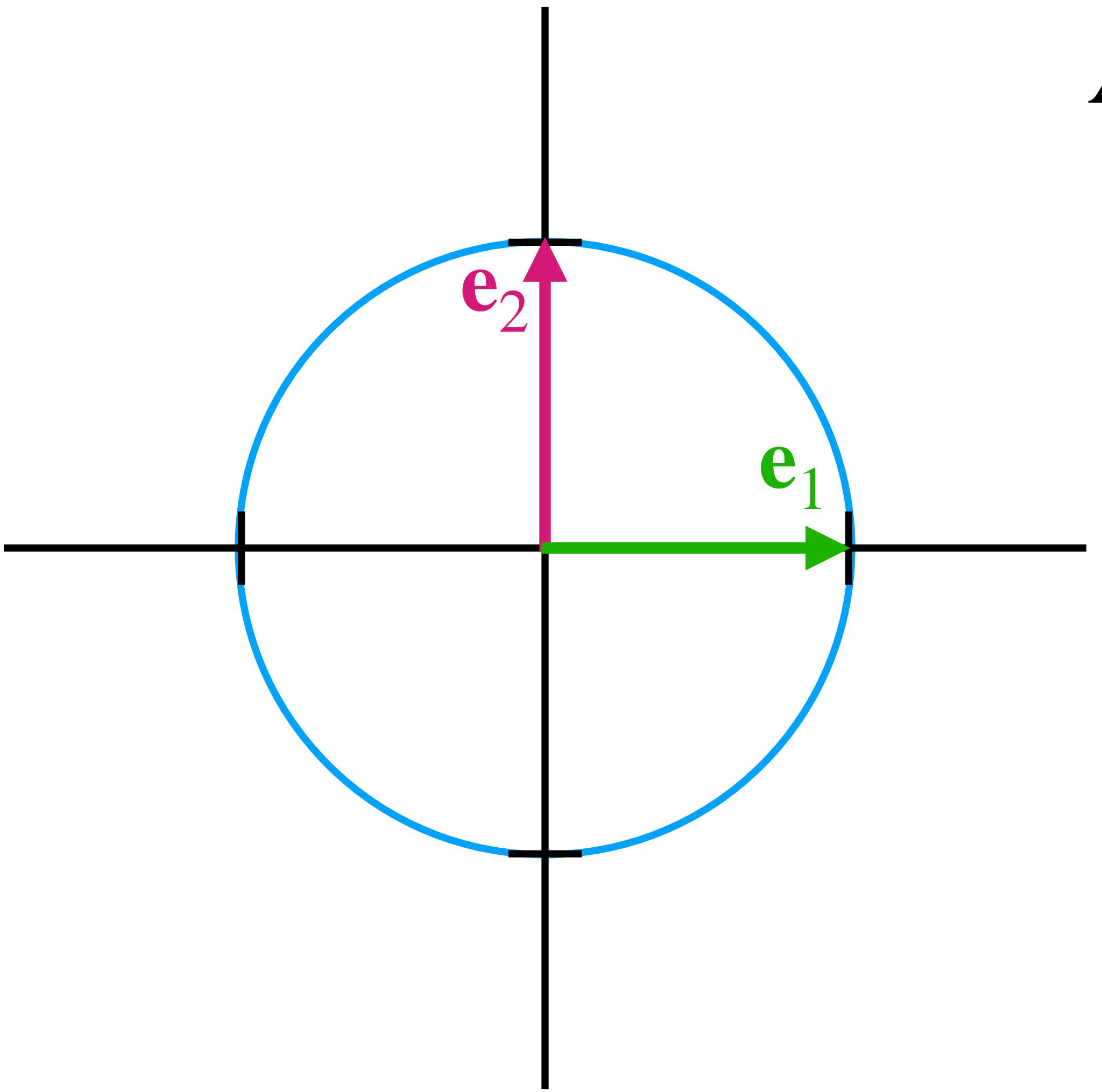
Ellipsoids are spheres "stretched" in orthogonal directions called the **axes of symmetry** or the **principle axes**.

Linear transformations maps spheres to ellipsoids.

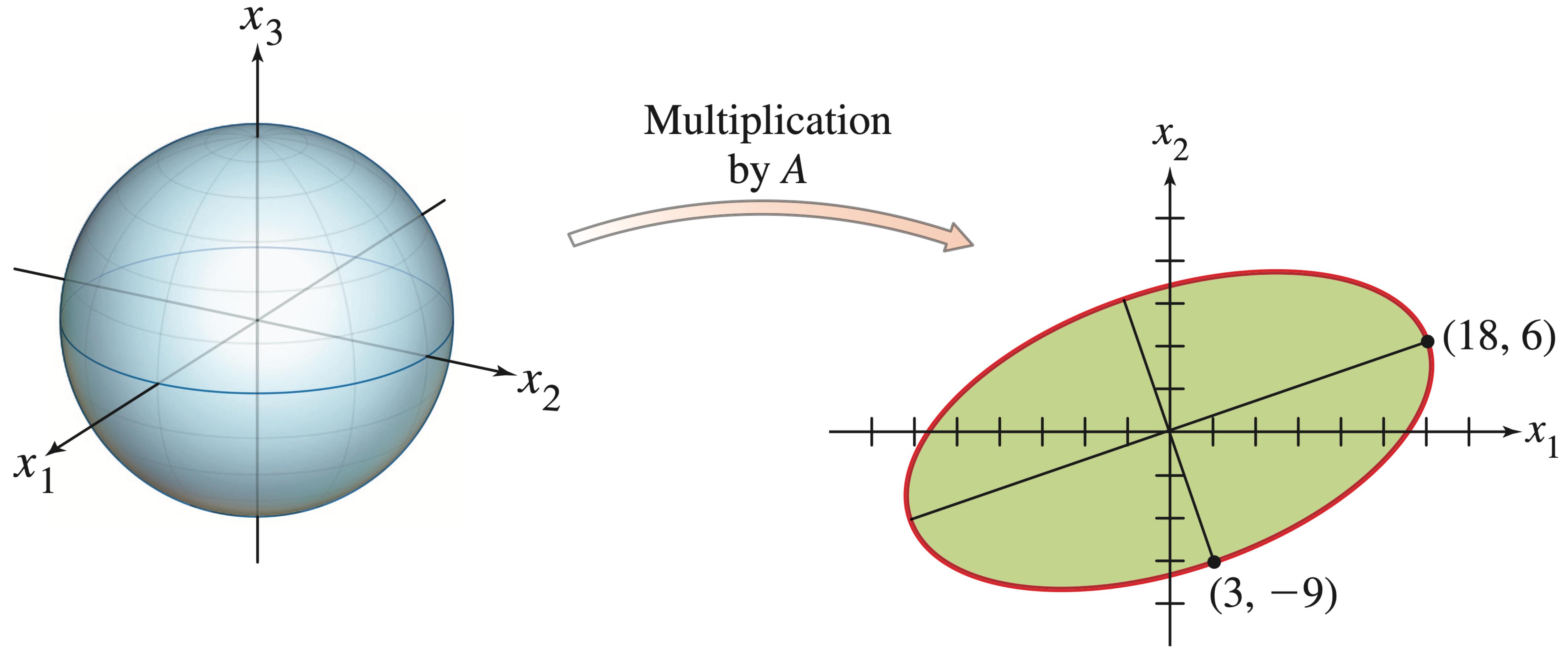


Simple Example : Scaling Matrices

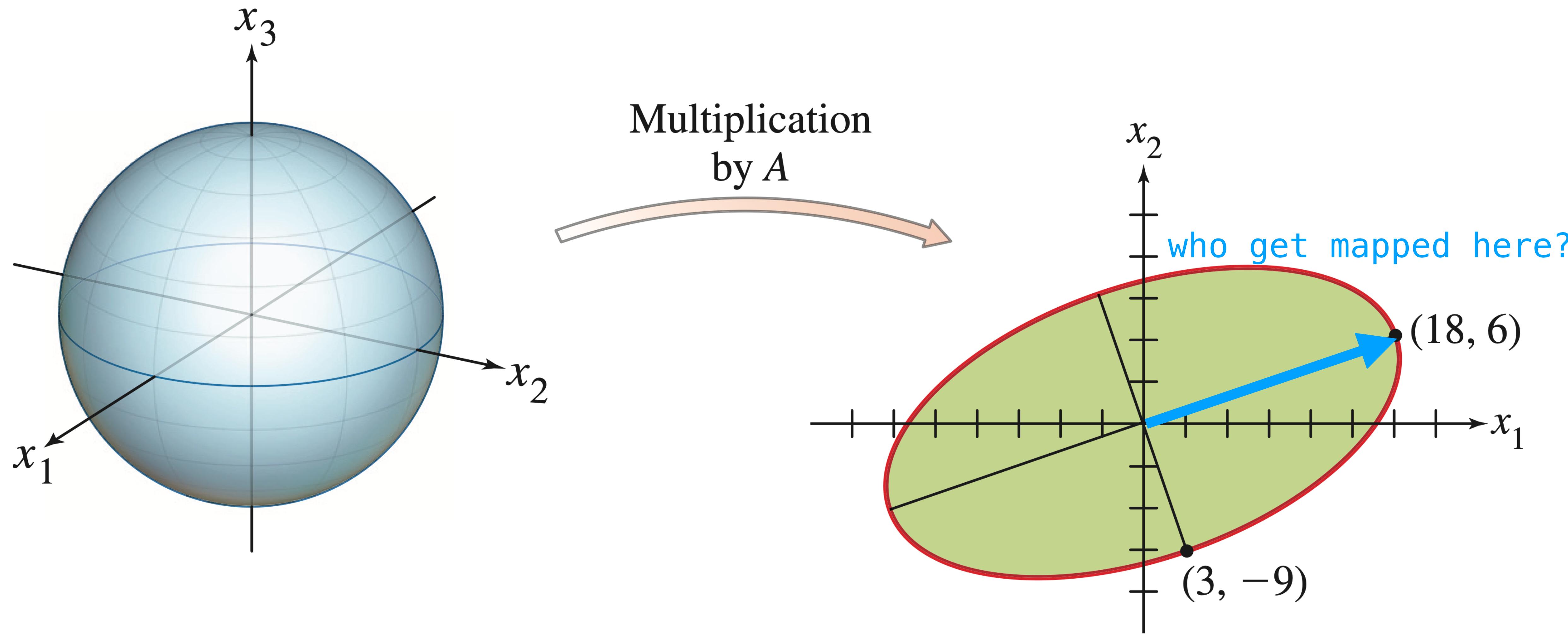
$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$



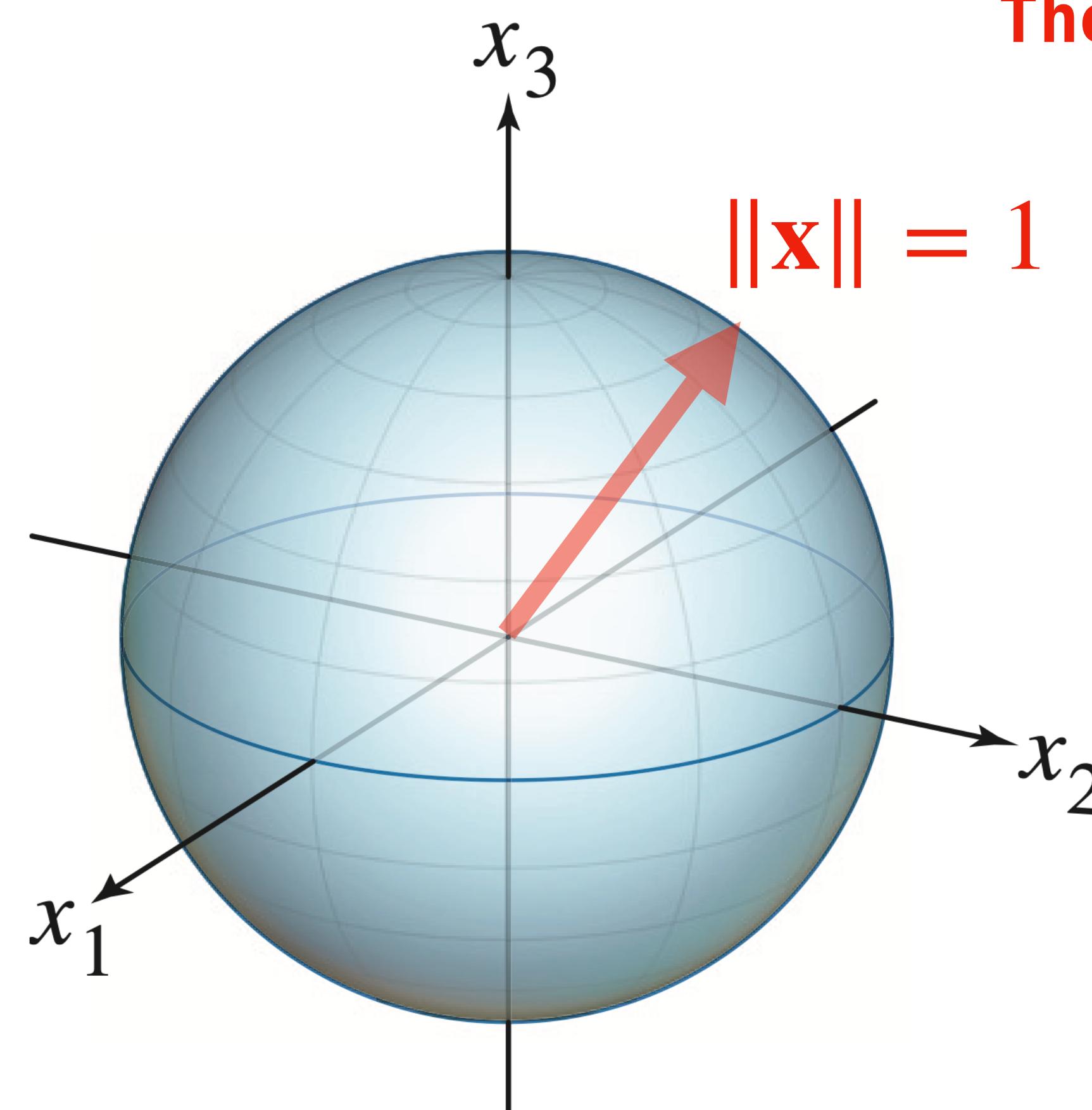
The Picture



The Picture

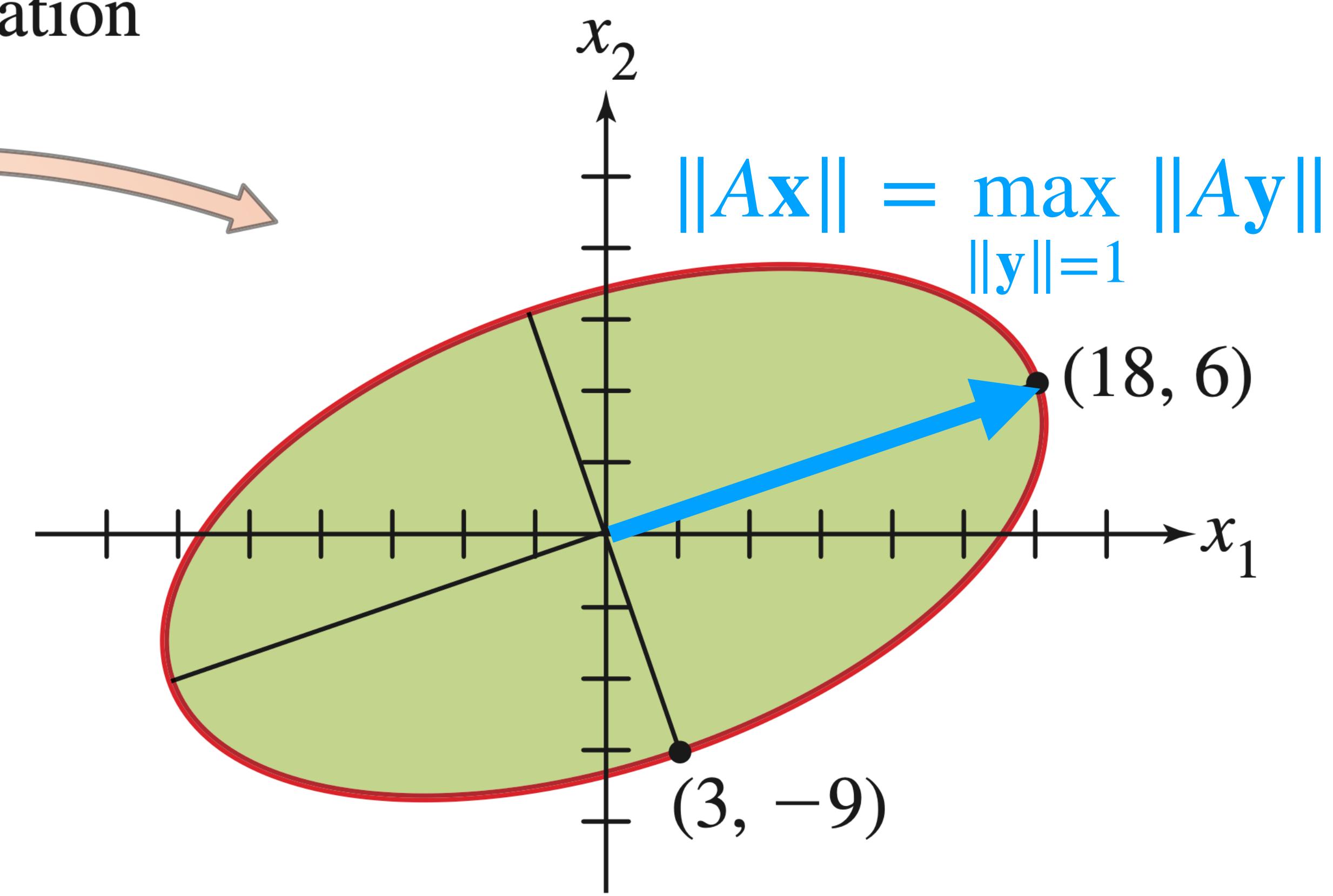


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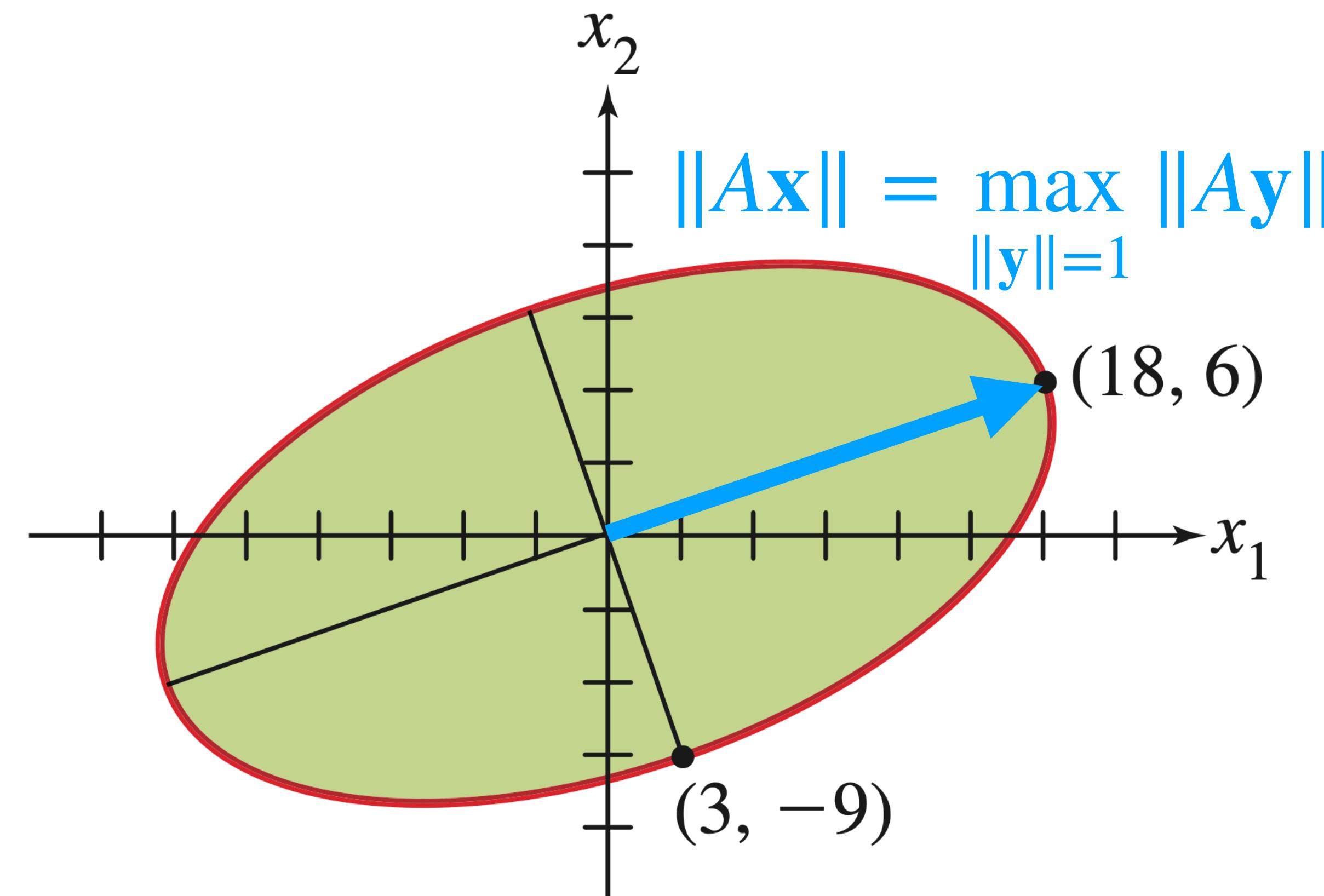


The longest end of the ellipse is the solution to a constrained optimization problem

Multiplication
by A

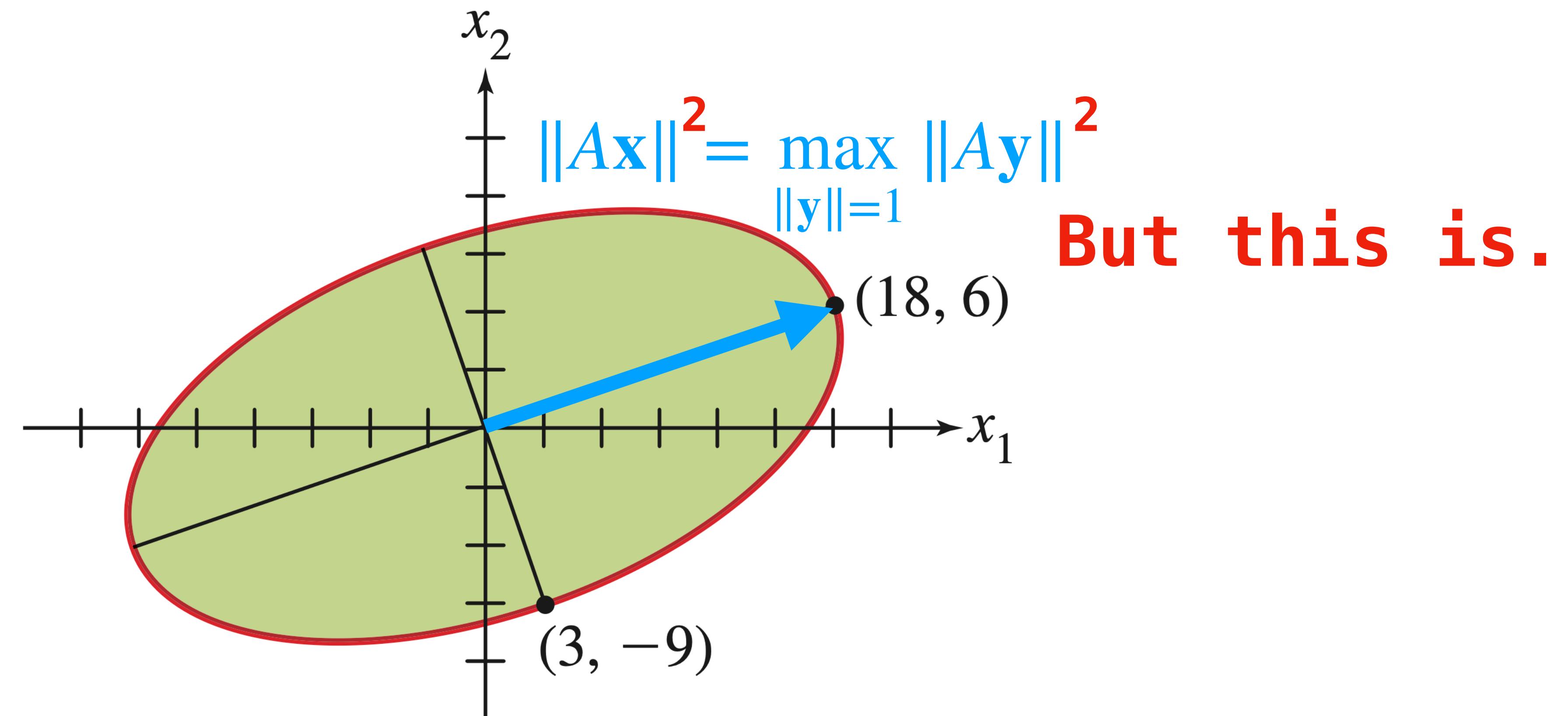


The Picture



This is not a quadratic form...

The Picture



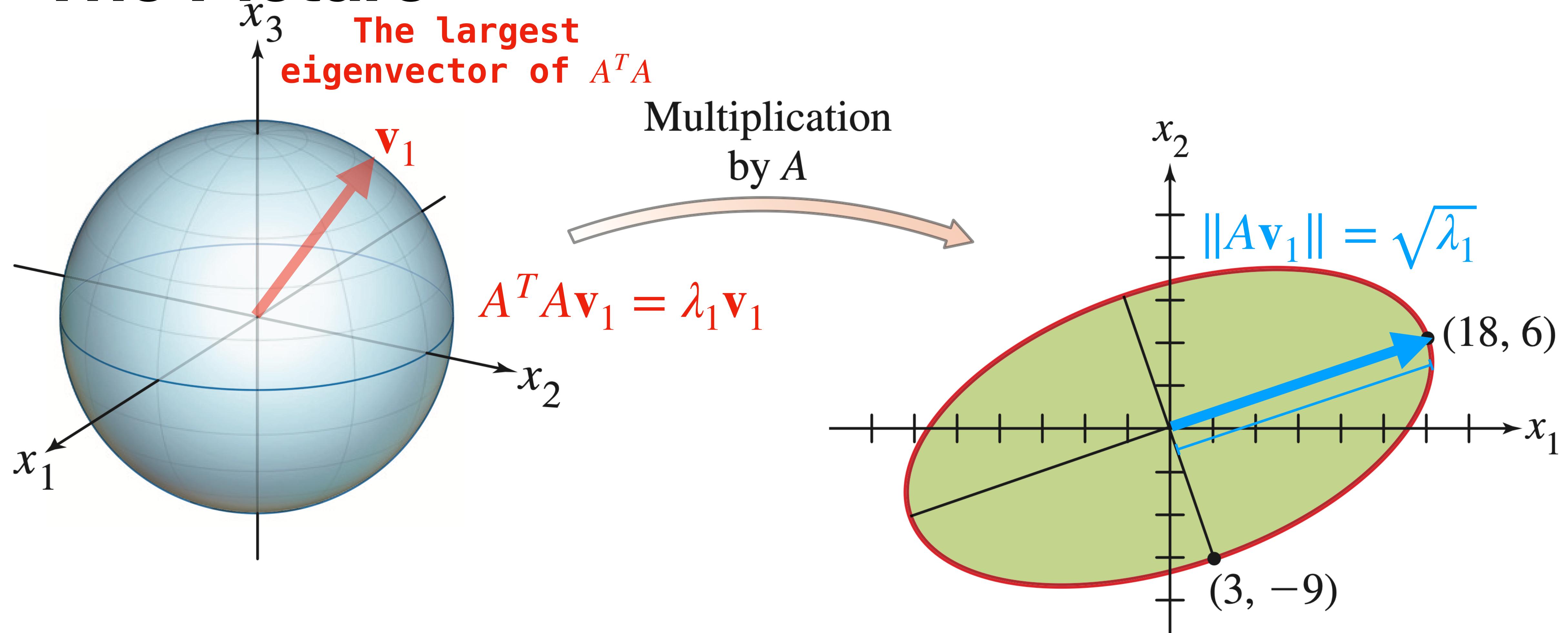
This is not a quadratic form...

A Quadratic Form

What does $\|Ax\|^2$ look like?:

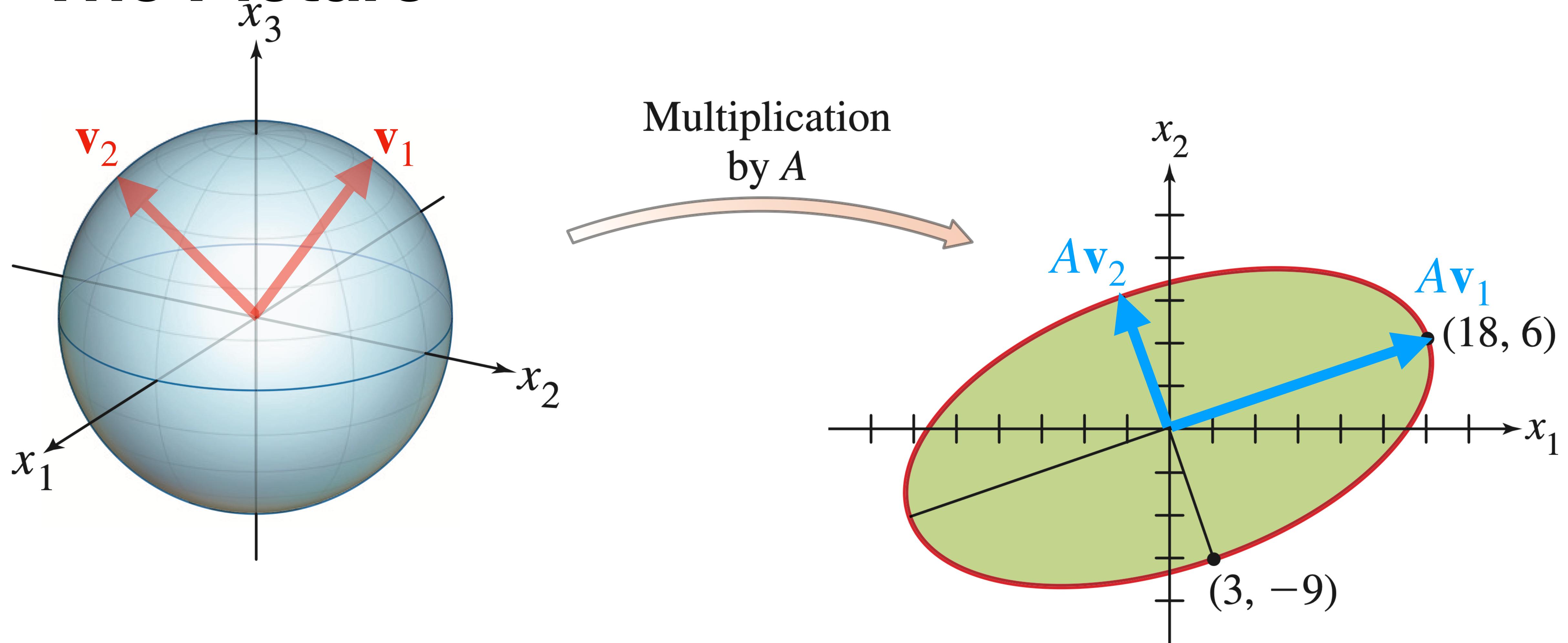
$$(Ax)^T(Ax) = \underbrace{\vec{x}^T A^T A \vec{x}}_{\text{symmetric}}$$

The Picture



\mathbf{v}_1 solves the constrained optimization problem.

The Picture



The second eigenvector is sent to the *minimum* principle axis

Properties of $A^T A$

Properties of $(A^T A)^T = A^T A$

» It's symmetric

Properties of $A^T A$

- » It's symmetric
- » So its orthogonally diagonalizable

Properties of $A^T A$

- » It's symmetric
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- » There is an orthogonal basis of eigenvectors

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Properties of $A^T A$

- » It's symmetric
- » So its orthogonally diagonalizable
- » There is an orthogonal basis of eigenvectors
- » Its eigenvalues are nonnegative
- » It's positive semidefinite

if $\vec{x} \neq 0$, $\vec{x}^T A^T A \vec{x} = ||A\vec{x}||^2 \geq 0$

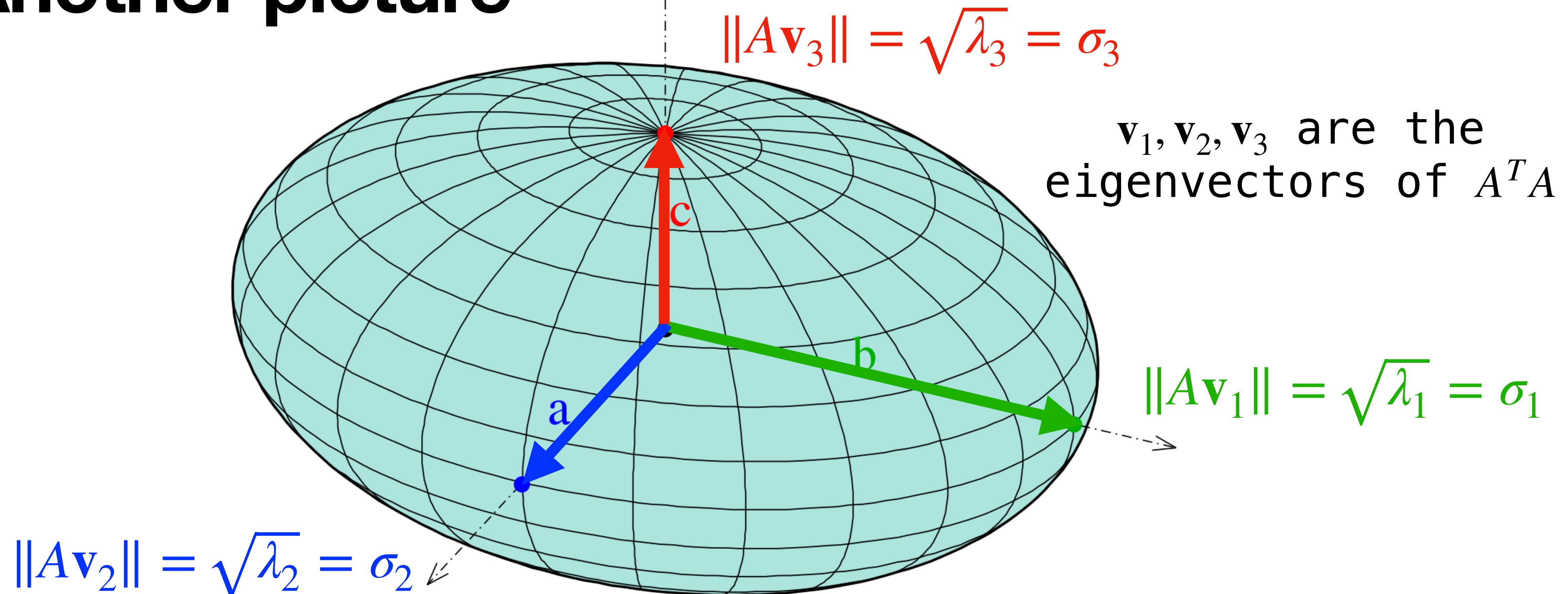
Singular Values

Definition. For an $m \times n$ matrix A , the **singular values** of A are the n values

$$\sigma_1 \geq \sigma_2 \dots \geq \sigma_n \geq 0$$

where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is an eigenvalue of $A^T A$.

Another picture



The **singular values** are the lengths of the *axes of symmetry* of the ellipsoid after transforming the unit sphere.

Every $m \times n$ matrix transforms the
unit \cancel{n} -sphere into an \cancel{m} -ellipsoid

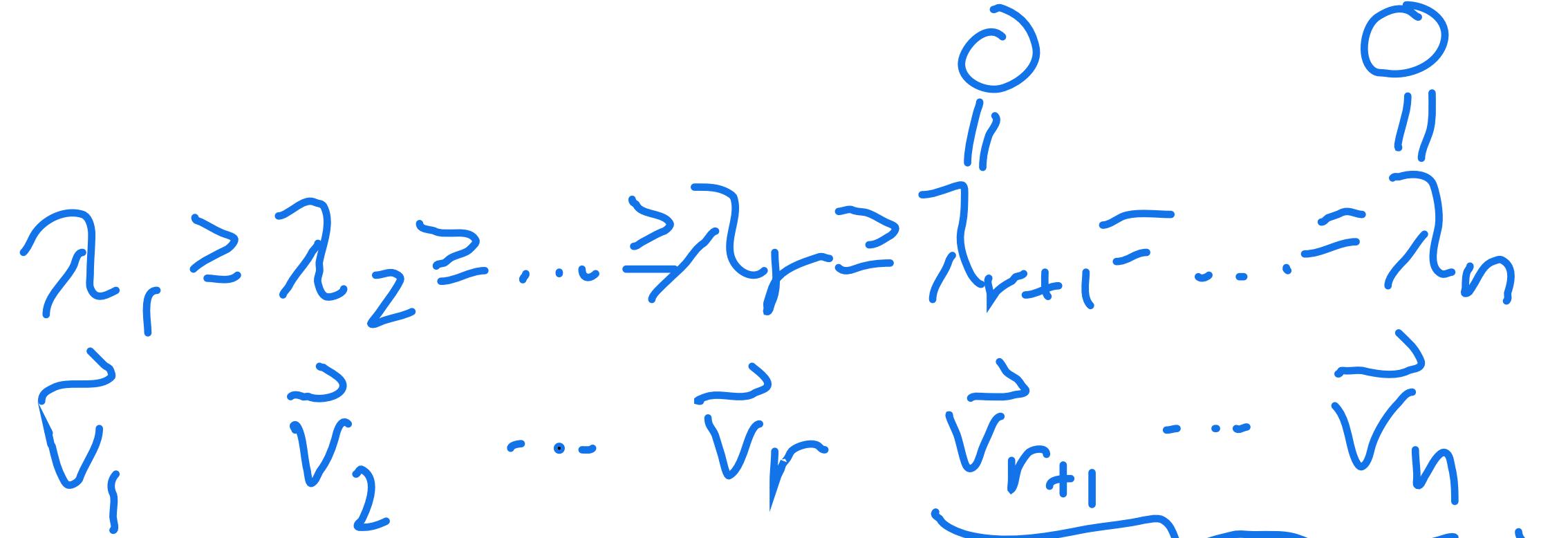
So every $m \times n$ matrix has
 $\textcolor{blue}{m}$ singular values

What else can we say?

Let v_1, \dots, v_n be an **orthogonal** eigenbasis of \mathbb{R}^n for $A^T A$ and suppose A has r nonzero singular values

Theorem. Av_1, \dots, Av_r is an orthogonal basis of $\text{Col}(A)$

What else can we say?

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = \dots = \lambda_n$$
$$\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_r \quad \vec{v}_{r+1} \quad \dots \quad \vec{v}_n$$


Let $\underline{v_1, \dots, v_n}$ be an **orthogonal** eigenbasis of \mathbb{R}^n for $A^T A$ and suppose A has r nonzero singular values

Theorem. $A\underline{v_1, \dots, v_r}$ is an orthogonal basis of $\text{Col}(A)$

This is the most important theorem for SVD

Verifying it

$$\lambda_i \neq \lambda_j$$

Let's show Av_1, \dots, Av_r are orthogonal (and linearly independent):

$$\vec{v}_j^T A^T A \vec{v}_i = \vec{v}_j^T \lambda_i \vec{v}_i = \lambda_i \vec{v}_j^T \vec{v}_i = 0$$

~~$$(A^T A) \vec{v}_i = \lambda_i \vec{v}_i$$~~

~~$$(\lambda_i - \lambda_j) \vec{v}_i^T \vec{v}_j = 0$$~~

Recall $\vec{v}_1, \dots, \vec{v}_n$ orthonormal eigenbasis

Verifying it

& $\vec{v}_{r+1}, \dots, \vec{v}_n$ basis for $Nu / A^T A$

Let's show $A\vec{v}_1, \dots, A\vec{v}_r$ span $\text{Col}(A)$:

For any $A\vec{x}$ want

$$A\vec{x} = c_1 A\vec{v}_1 + \dots + c_r A\vec{v}_r$$

for some constants c_1, \dots, c_r

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$\Rightarrow A\vec{x} = c_1 A\vec{v}_1 + \dots + c_n A\vec{v}_n$$

Q: Can we lop off the last
 $n-r$ terms?

Yes!

$$A\vec{x} = c_1 A\vec{v}_1 + \dots + c_r A\vec{v}_r$$

DONE

Fact: $Nu / (A) = Nul(A^T A)$

PF: $\vec{x} \in Nul(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow A^T A\vec{x} = \vec{0}$

$$\Rightarrow \vec{x} \in Nul(A^T A)$$

$$\text{so } Nul(A) \subseteq Nul(A^T A)$$

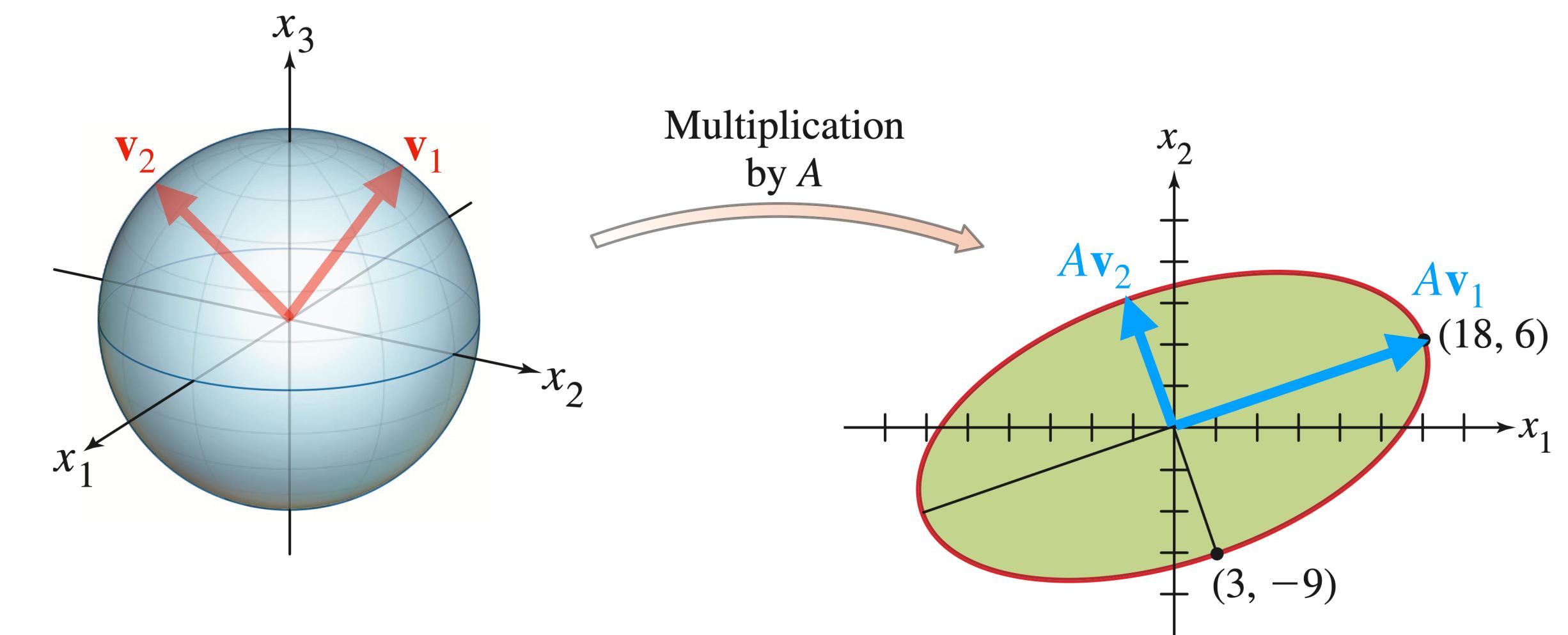
$$\vec{x} \in Nul(A^T A) \Rightarrow A^T A\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x}^T A^T A\vec{x} = 0 \Rightarrow \|A\vec{x}\|^2 = 0$$

$$\Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} \in Nul(A)$$

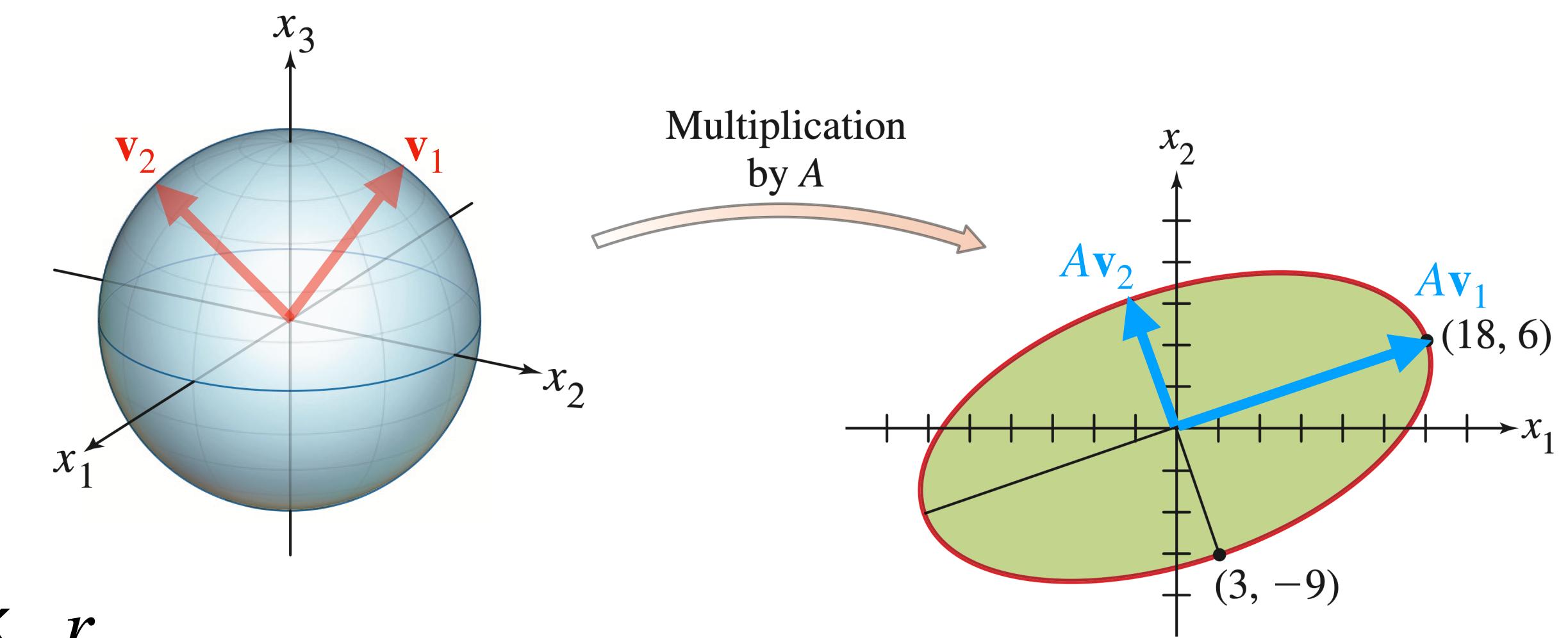
$$\text{so } Nul(A^T A) \subseteq Nul(A)$$

Putting it all together



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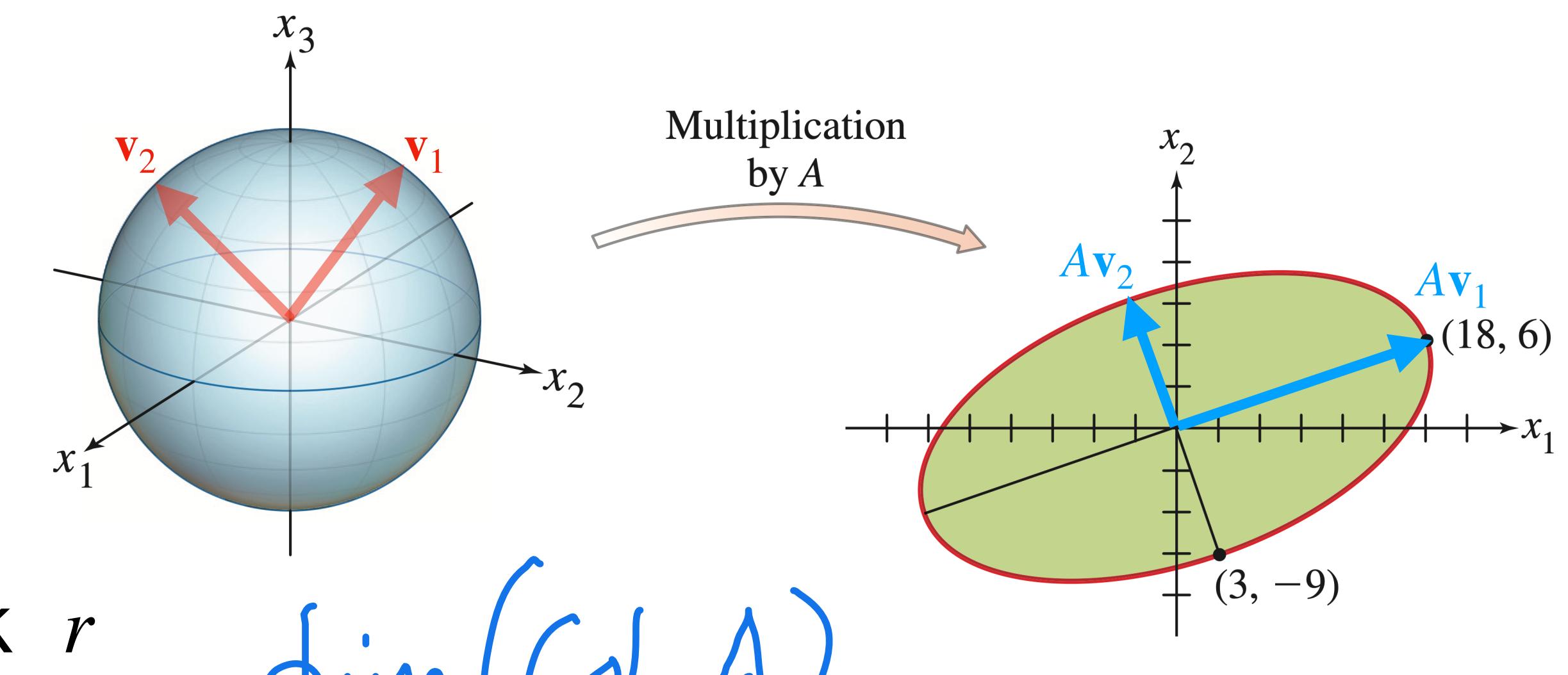
Let A be an $m \times n$ matrix of rank r



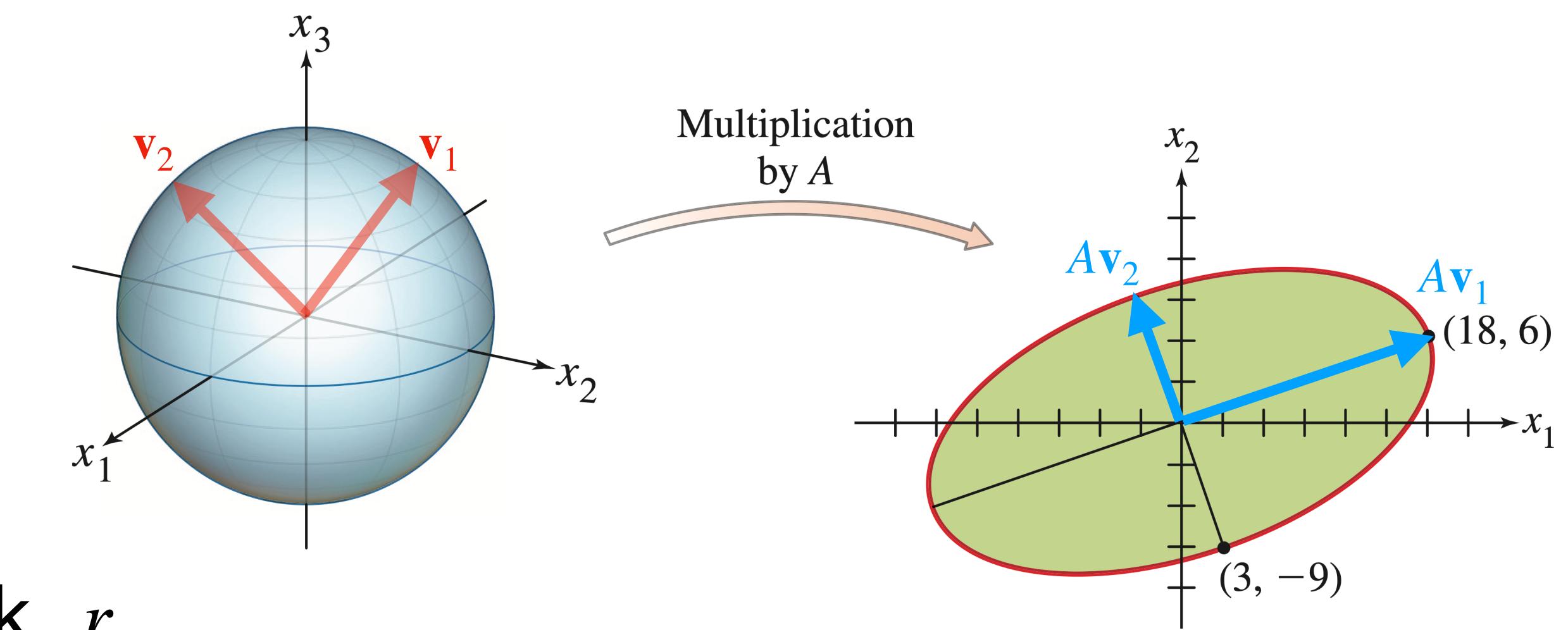
Putting it all together

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What we know:



Putting it all together

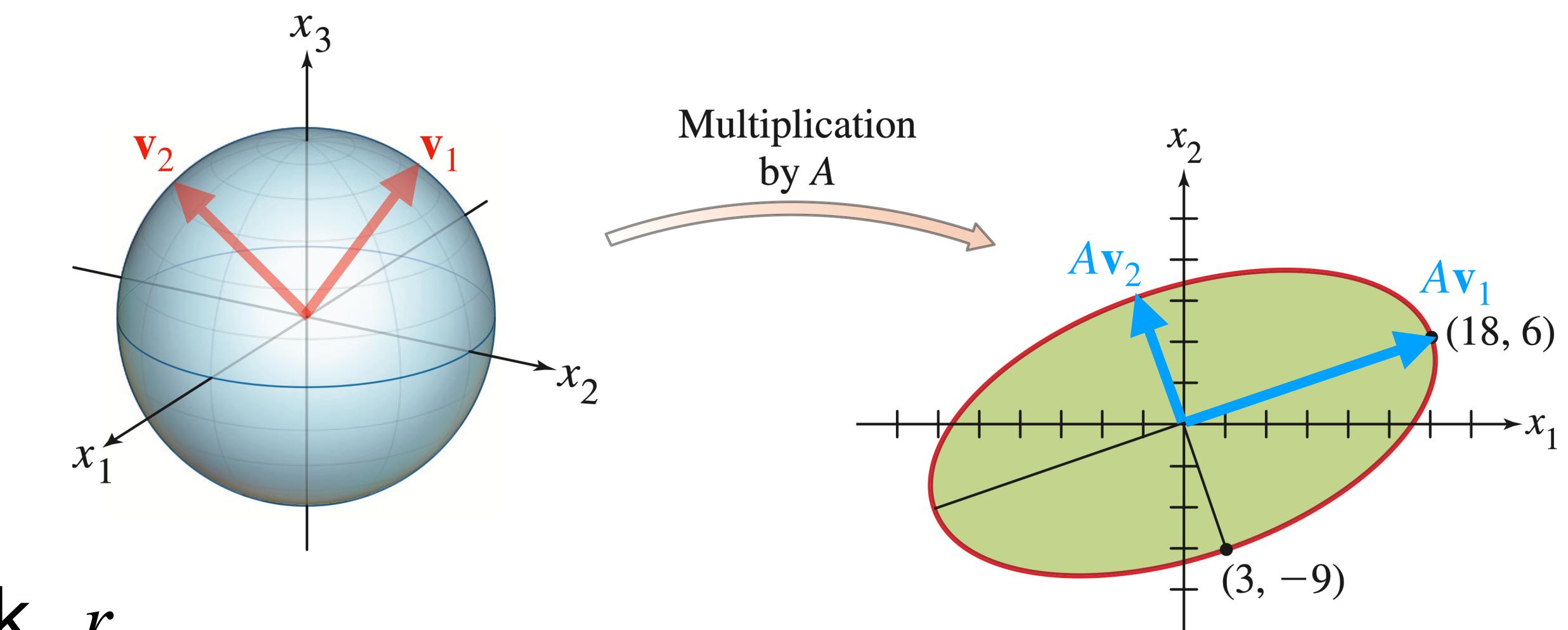


Let A be an $m \times n$ matrix of rank r

What we know:

- » We can find orthonormal vectors v_1, \dots, v_n in \mathbb{R}^n such that Av_1, \dots, Av_r in \mathbb{R}^m form an orthogonal basis for $\text{Col}(A)$

Putting it all together

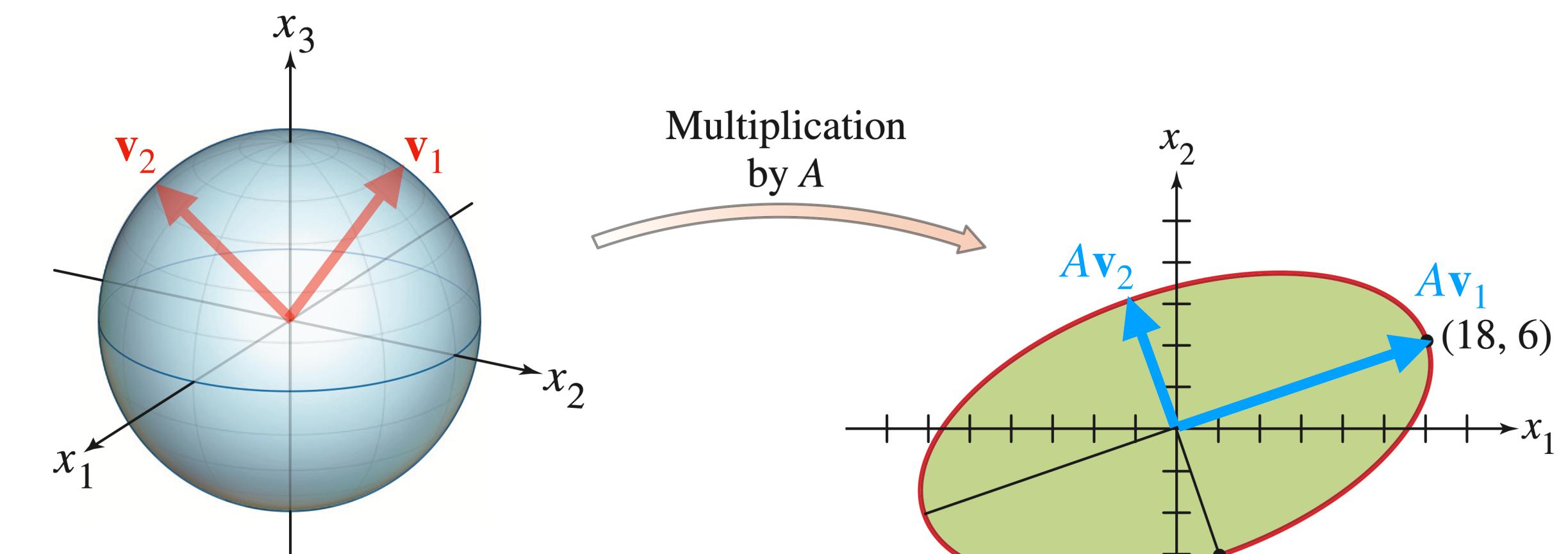


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Putting it all together

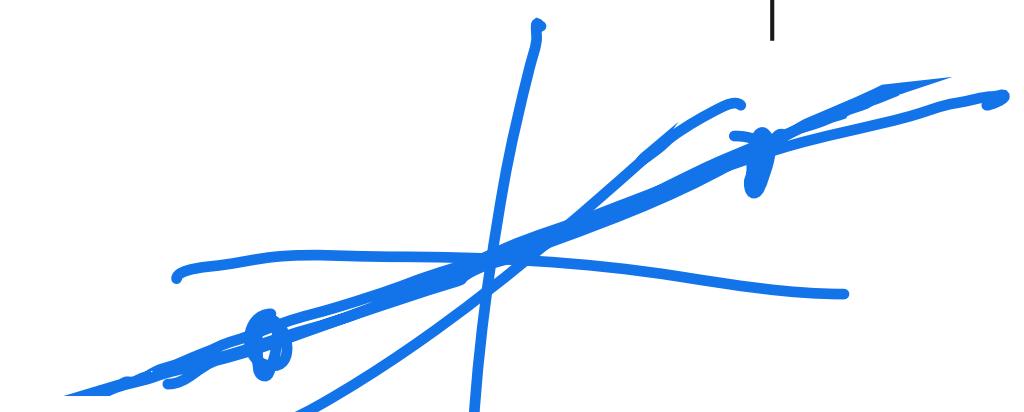
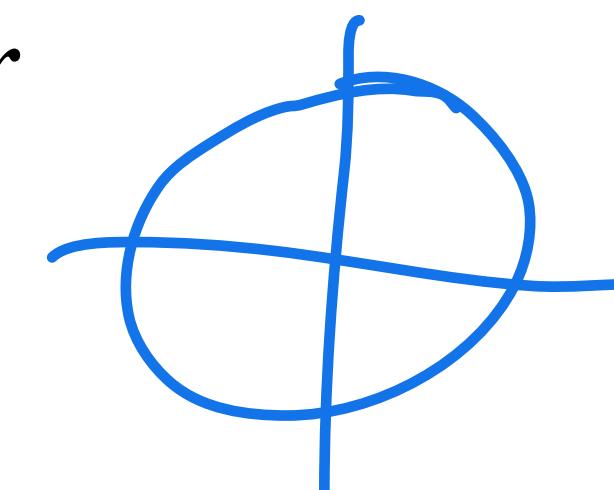


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What we know:

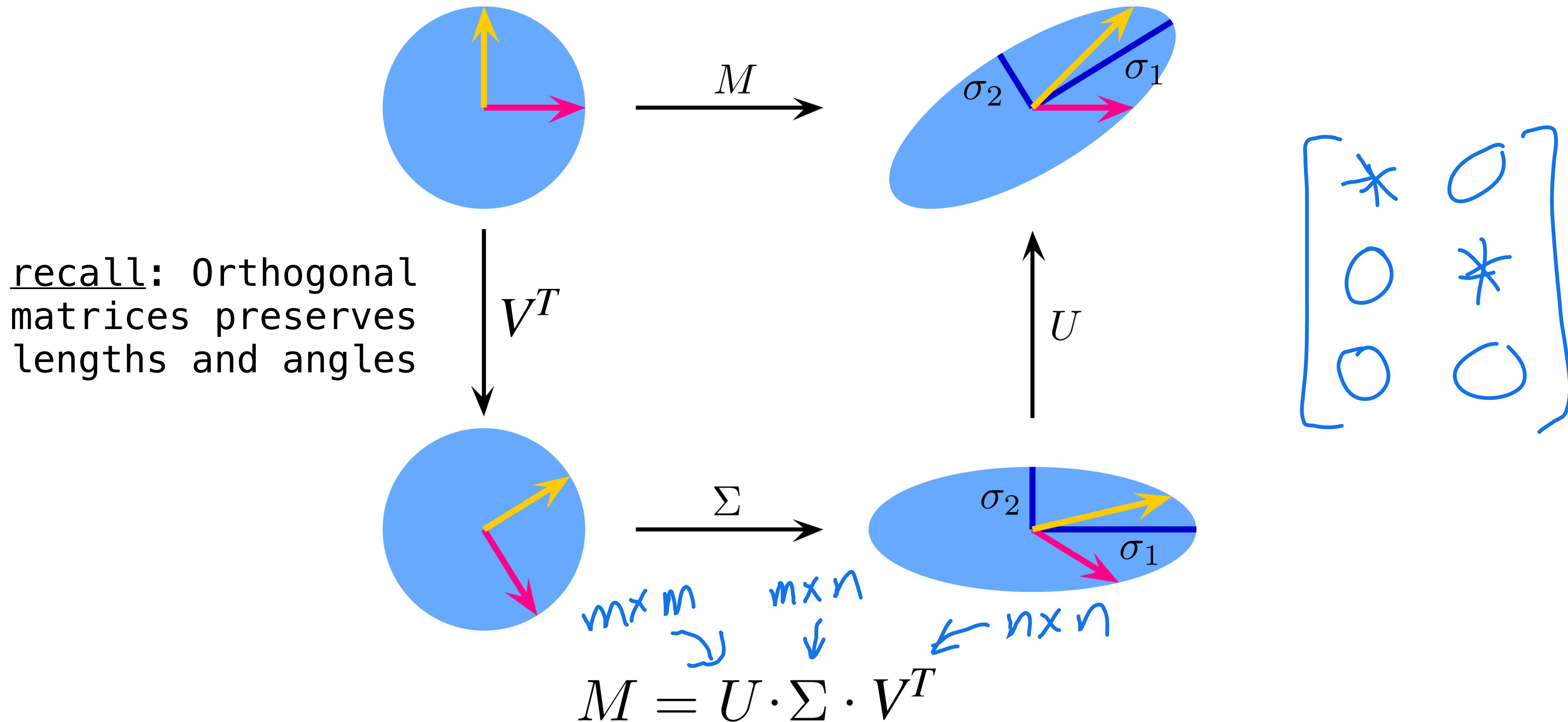
- » We can find orthonormal vectors v_1, \dots, v_n in \mathbb{R}^n such that Av_1, \dots, Av_r in \mathbb{R}^m form an orthogonal basis for $\text{Col}(A)$
- » So if we take $u_i = \frac{Av_i}{\|Av_i\|}$, we get an **orthonormal** basis of $\text{Col}(A)$
- » And we can extend this to u_1, \dots, u_m an orthonormal basis of \mathbb{R}^m (via Gram-Schmidt).

(didn't cover technically)



Singular Value Decomposition

High Level View of the Decomposition



The Important Equality

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$$

$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

The Important Equality

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$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value,
which is the length $\|A\mathbf{v}_i\|$

The Important Equality

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Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value,
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What happens when we write this in matrix form?

The Important Equality

$$A[\mathbf{v}_1 \dots \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \dots \sigma_n \mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

The Important Equality

$$A[\mathbf{v}_1 \dots \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \dots \sigma_n \mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

The Important Equality

$$A[\mathbf{v}_1 \dots \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \dots \sigma_n \mathbf{u}_n]$$

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Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

$m > n$

$m < n$

$m = n$

The Important Equality

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Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

remember: U is orthonormal

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The Important Equality

$$AV = U\Sigma$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

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$m > n$ $m < n$ $m = n$

The Important Equality

$$\begin{matrix} m \times n \\ n \times n \end{matrix} A V = \begin{matrix} m \times m \\ m \times n \end{matrix} U \Sigma$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

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$m > n$ $m < n$ $m = n$

The Important Equality

$$AVV^T = U\Sigma V^T$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value,
which is the length $\|A\mathbf{v}_i\|$. ↳ Id

Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

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$$\text{or } \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix} \quad m < n$$

$$\text{or } \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix} \quad m = n$$

The Important Equality

$$A = U\Sigma V^T$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \dots \mathbf{u}_m]$ and

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The Important Equality

singular value decomposition

$$A = U\Sigma V^T$$

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Singular Value Decomposition

Theorem. For a $m \times n$ matrix A , there are *orthogonal* matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = \underset{m \times n}{U} \underset{m \times m}{\Sigma} \underset{n \times n}{V^T}$$

where diagonal entries* of Σ are $\sigma_1, \dots, \sigma_n$ the singular values of A .

* these are diagonal entries in a non-square matrix.

Singular Value Decomposition

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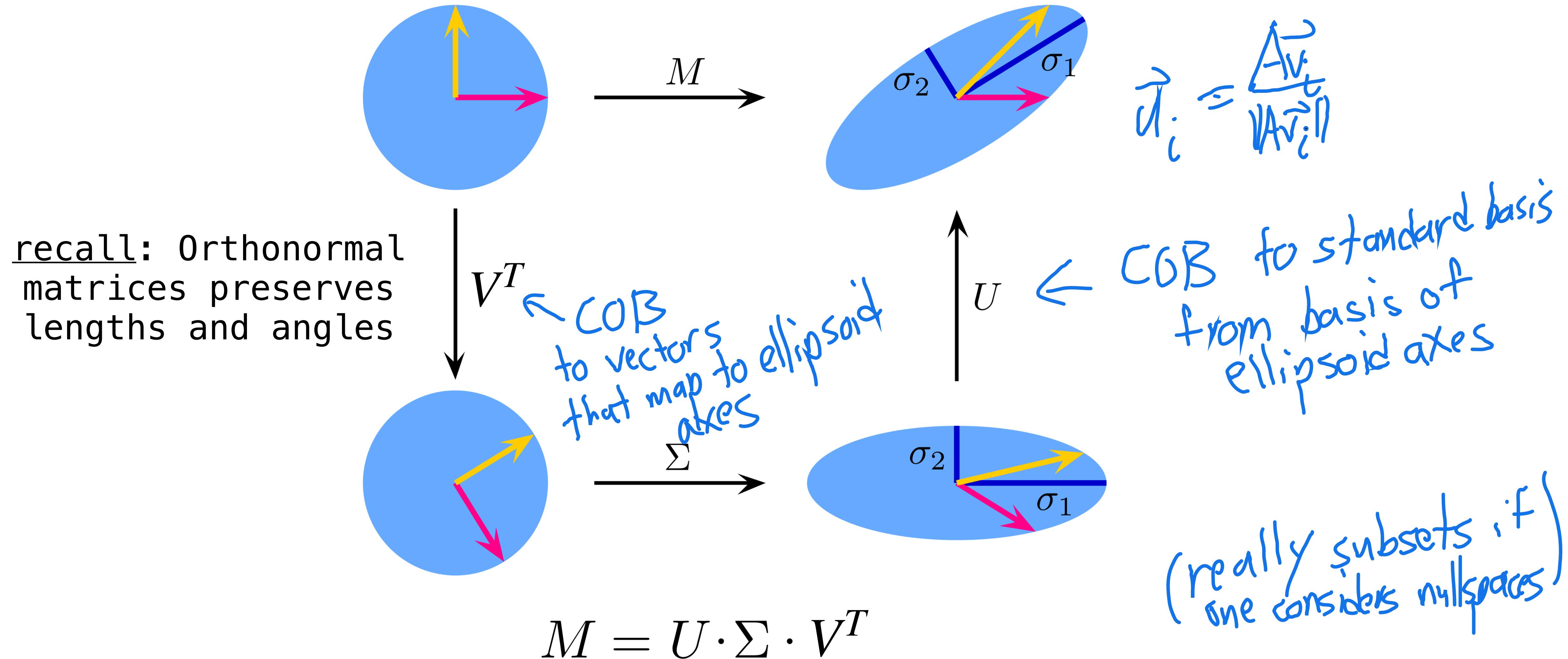
left singular vectors **right singular vectors**

$$A = U \underset{m \times n}{\Sigma} V^T$$

where diagonal entries* of Σ are $\sigma_1, \dots, \sigma_n$ the singular values of A .

* these are diagonal entries in a non-square matrix.

The Picture (Again)



How To: Finding a SVD

Step 1: Set up Σ

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

The **singular values** are the square roots of the eigenvalues of $A^T A$ (or AA^T):

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (9-\lambda)^2 - 81$$

$$\begin{aligned} &= \lambda^2 - 18\lambda + 81 - 81 = \lambda^2 - 18\lambda = \lambda(\lambda - 18) \\ &\lambda_1 = 18 \geq \lambda_2 = 0 \\ &\sigma_1 = \sqrt{18} \\ &\sigma_2 = 0 \end{aligned}$$

Step 2: Set up V

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

Find an orthonormal eigenbasis for $A^T A$:

$$\lambda_1 = 18$$

$$A^T A - 18I = \begin{bmatrix} -9 & -9 & 0 \\ -9 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0 \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 3: Set up U (Part 1)

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

If $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an eigenbasis of \mathbb{R}^n (in decreasing order of eigenvalue), then $A\mathbf{v}_1, \dots, A\mathbf{v}_r$ is an eigenbasis of $\text{Col}(A)$ (where r is the rank of A). These vectors can be normalized and made the first r columns of U :

$$r=1$$

$$A\vec{v}_1 = A \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \alpha \cancel{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -4 \end{bmatrix} \alpha \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\tilde{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

Step 4: Set up U (Part 2)

$$m=3 \quad U_{n \times m} \quad \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

If $m > r$, then extend $\mathbf{u}_1, \dots, \mathbf{u}_r$ until it has m orthonormal vectors:

$$\begin{aligned} \mathbf{v}_1 &> \\ \mathbf{v}_1 \cdot \mathbf{v}_2 &= 0 \\ \mathbf{x} \cdot \mathbf{v}_1 &= 0 \\ \mathbf{x} \cdot \mathbf{v}_2 &= 0 \end{aligned}$$

$$\begin{aligned} &\left[\begin{array}{c} -1/\sqrt{3} \\ 2/\sqrt{3} \\ -2/\sqrt{3} \end{array} \right], \left[\begin{array}{c} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{array} \right], \left[\begin{array}{c} 4/\sqrt{18} \\ 1/\sqrt{18} \\ -1/\sqrt{18} \end{array} \right] \\ &\left[\begin{array}{c} -1 \\ 2 \\ -2 \end{array} \right] \times \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 4 \\ -1 \\ -1 \end{array} \right] \end{aligned}$$

$$\begin{aligned} &\checkmark n \times n \\ &\checkmark 2 \times 2 \\ &\left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 1 \end{array} \right] = \left[\begin{array}{ccc} 2(1) - (-2)(1) \\ -(-1) \\ -1 \end{array} \right] \end{aligned}$$

Step 5: Put everything together

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$A = U \sum V^T$$

$$= \begin{bmatrix} -\frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

↑

SVD in NumPy

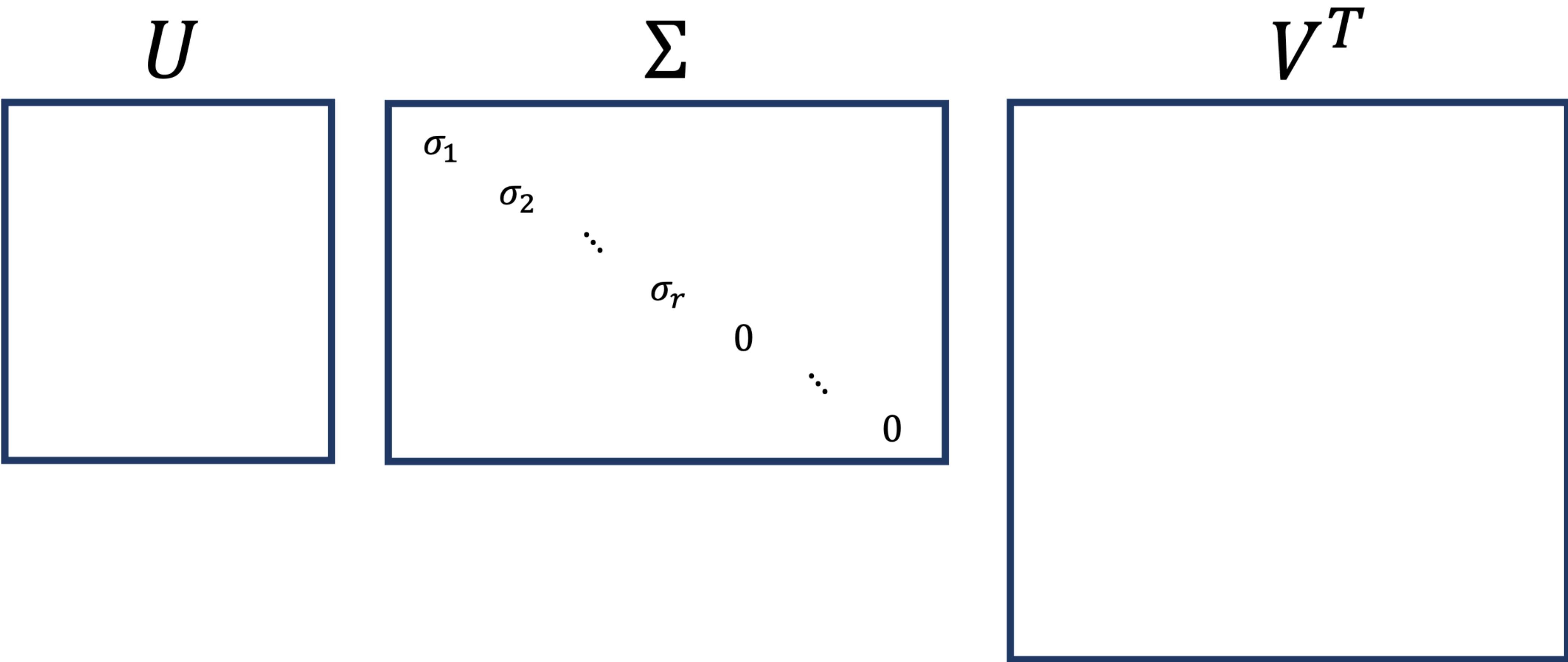
In reality, we will almost never build SVDs by hand. We can use:

numpy.linalg.svd



Pseudoinverses

SVD (The Picture)



Reduced SVD (The Picture)

$$A = U \Sigma V^T$$

The diagram illustrates the Reduced Singular Value Decomposition (SVD) of a matrix A . It consists of three main components arranged horizontally:

- U :** A matrix represented by a blue-bordered box containing columns u_1, u_2, \dots, u_r . The rows are indicated by vertical ellipses (dots).
- Σ :** A diagonal matrix represented by a blue-bordered box. The diagonal elements are labeled $\sigma_1, \sigma_2, \dots, \sigma_r$, followed by zeros. Ellipses indicate the continuation of the zero entries.
- V^T :** A matrix represented by a blue-bordered box containing rows $v_1^T, v_2^T, \dots, v_r^T$. The columns are indicated by horizontal ellipses (dots).

If we just want a decomposition of A , we don't need all the 0 singular values in Σ

The Reduced SVD

Theorem. For every matrix A of rank r , there is an orthonormal matrix $U \in \mathbb{R}^{m \times r}$, a diagonal matrix $\Sigma \in \mathbb{R}^{r \times r}$ with **positive** entries on the diagonal, and an orthonormal matrix $V \in \mathbb{R}^{n \times r}$ such that

$$A = U\Sigma V^T$$

The Pseudoinverse

Definition. Given a reduced SVD $A = U\Sigma V^T$, the **pseudoinverse** of A is $A^+ = V\Sigma^{-1}U^T$

Theorem. $A^+\mathbf{b}$ is the *minimum length least squares solution* of $A\mathbf{x} = \mathbf{b}$

(in Python we have `numpy.linalg.pinv`)

Recall: Least Squares in NumPy

numpy.linalg.lstsq

`linalg.lstsq(a, b, rcond='warn')`

[\[source\]](#)

Return the least-squares solution to a linear matrix equation.

Computes the vector x that approximately solves the equation $a @ x = b$. The equation may be under-, well-, or over-determined (i.e., the number of linearly independent rows of a can be less than, equal to, or greater than its number of linearly independent columns). If a is square and of full rank, then x (but for round-off error) is the “exact” solution of the equation. Else, x minimizes the Euclidean 2-norm $\|b - ax\|$. If there are multiple minimizing solutions, the one with the smallest 2-norm $\|x\|$ is returned.

Parameters: `a` : *(M, N) array_like*

“Coefficient” matrix.

`b` : *{(M,), (M, K)} array_like*

Ordinate or “dependent variable” values. If b is two-dimensional, the least-squares solution is calculated for each of the K columns of b .

`rcond` : *float. optional*

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because they use SVD!

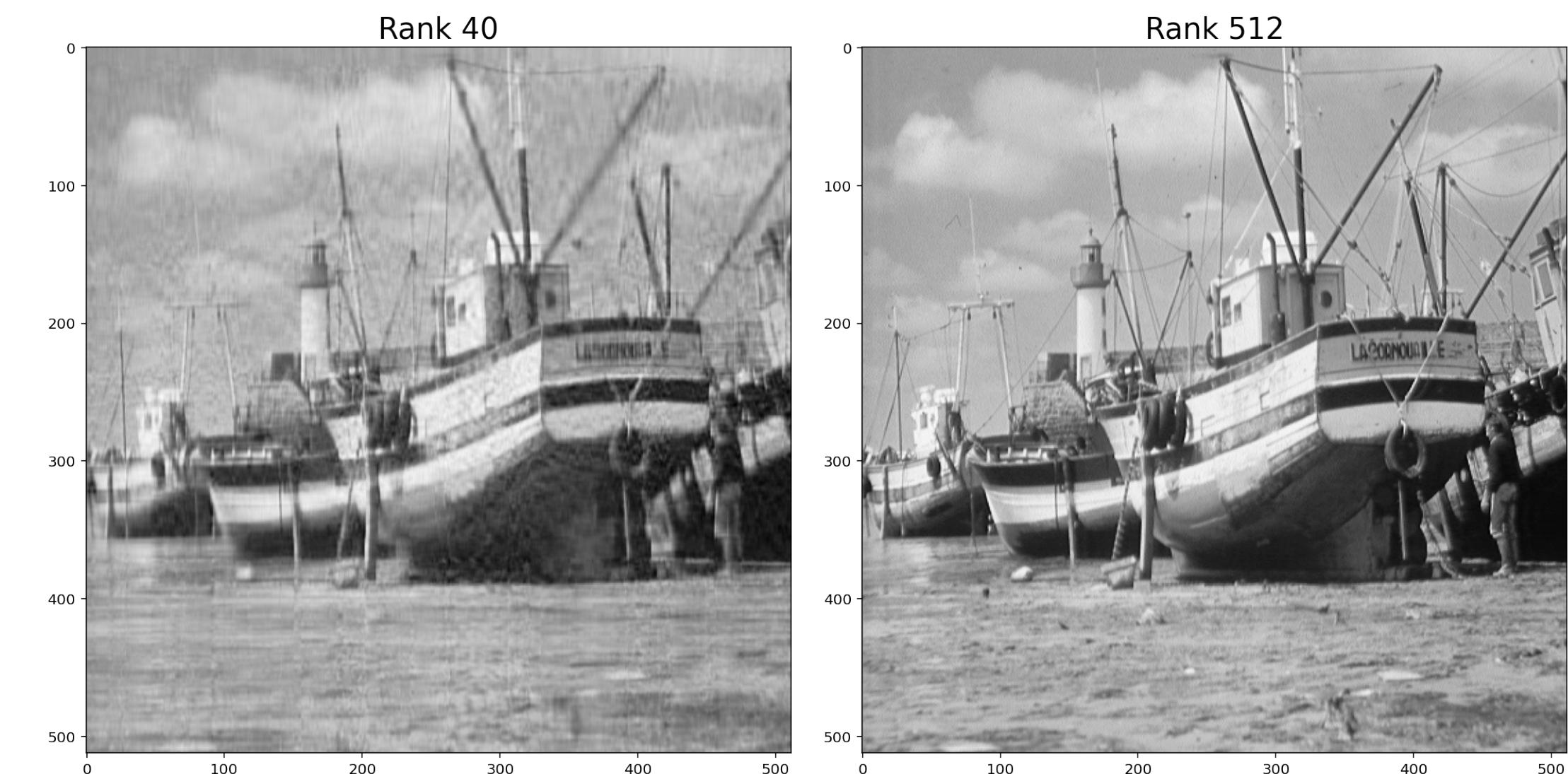
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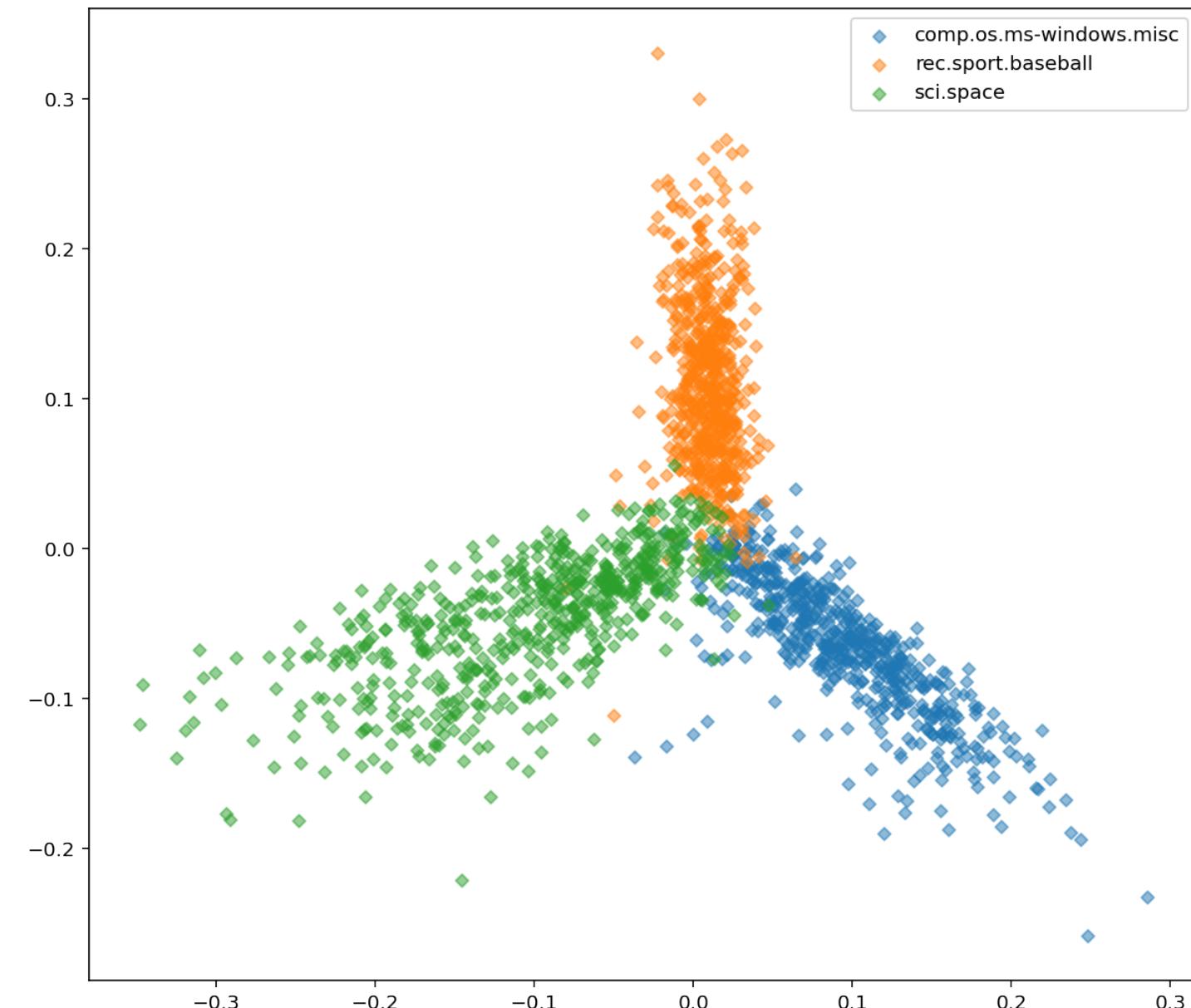
**What's next?
A couple final thoughts**

Applications of SVD

image compression



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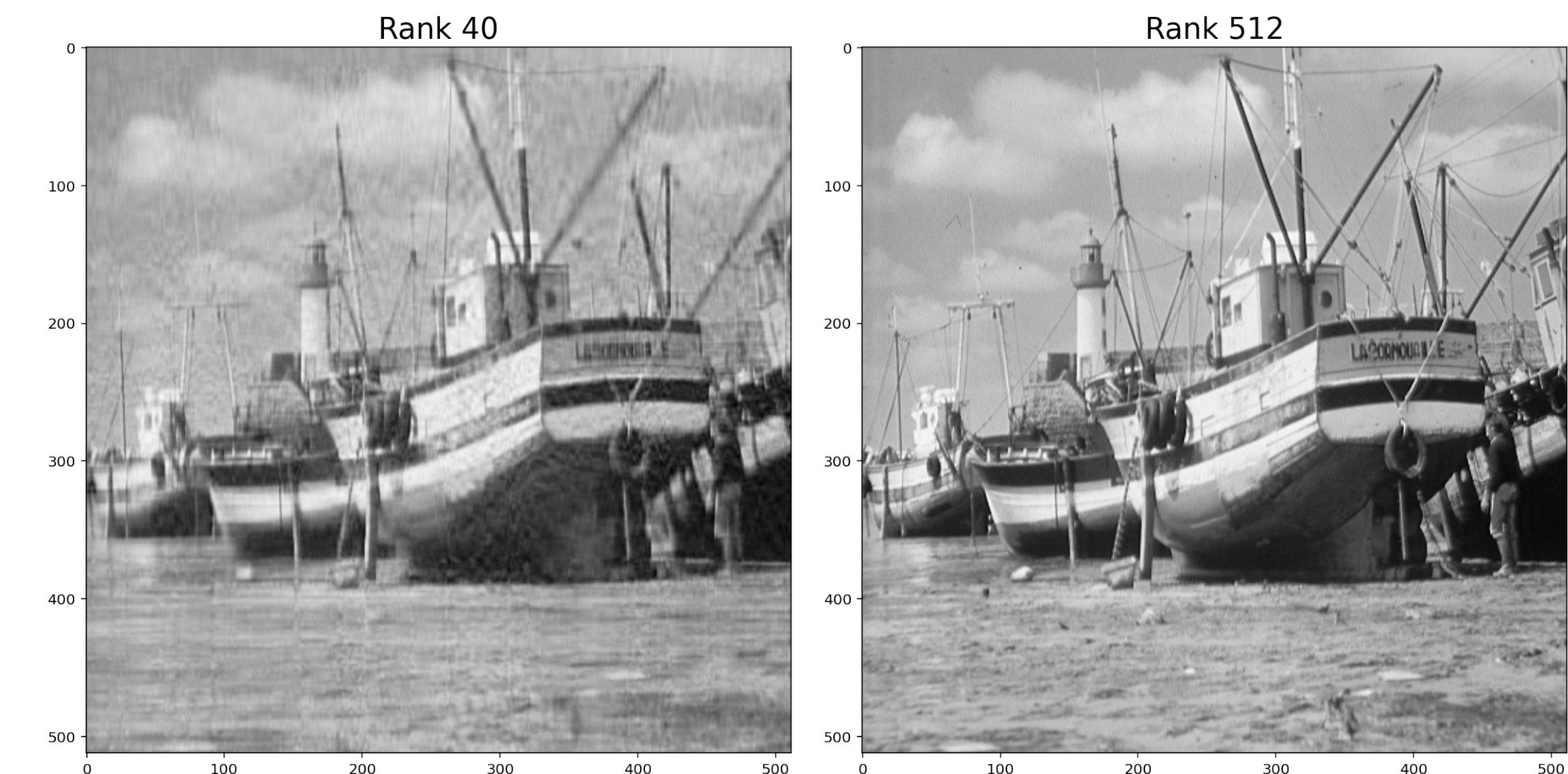


document
classification

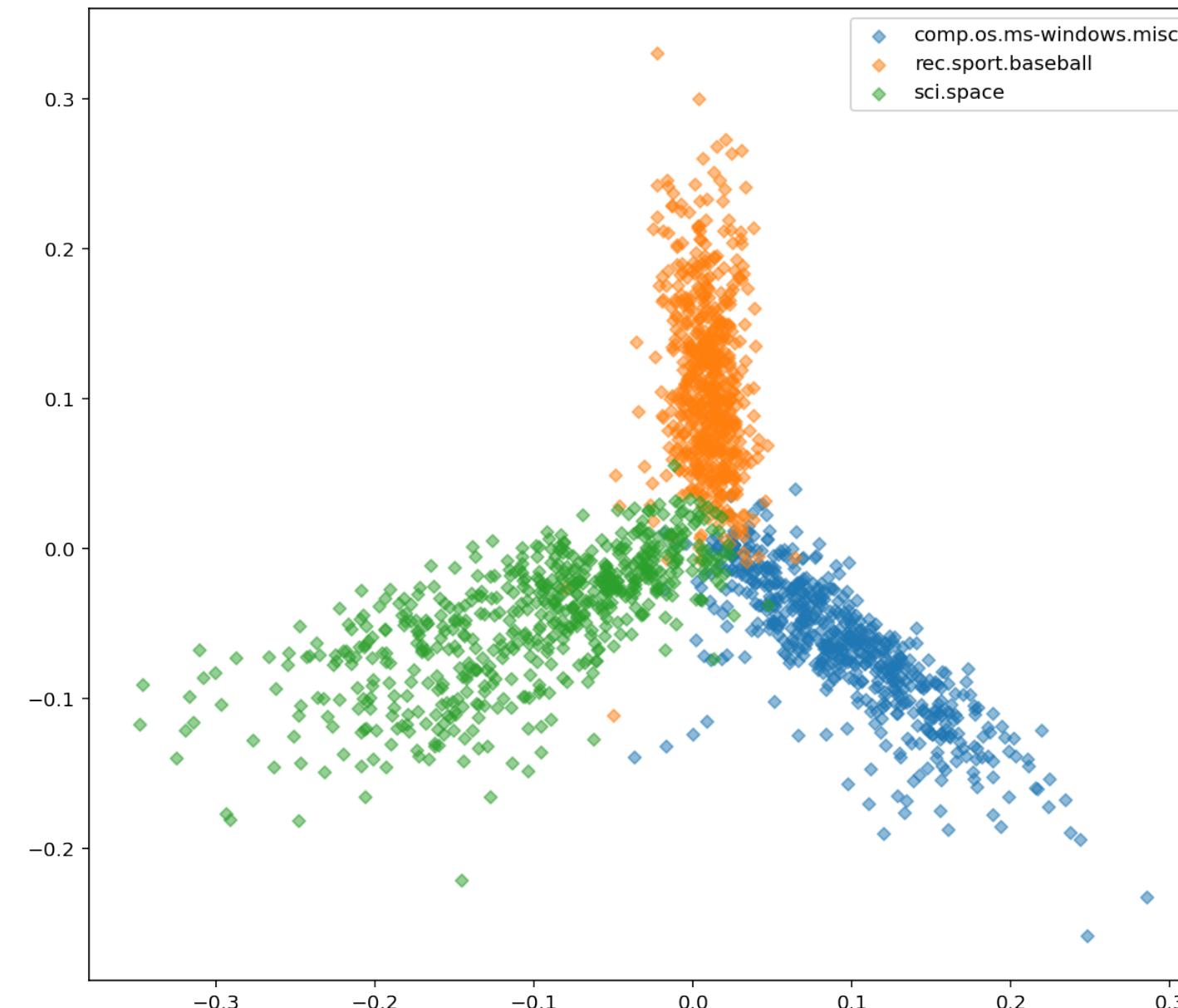
Applications of SVD

- Reduced SVD, pseudoinverses and least squares

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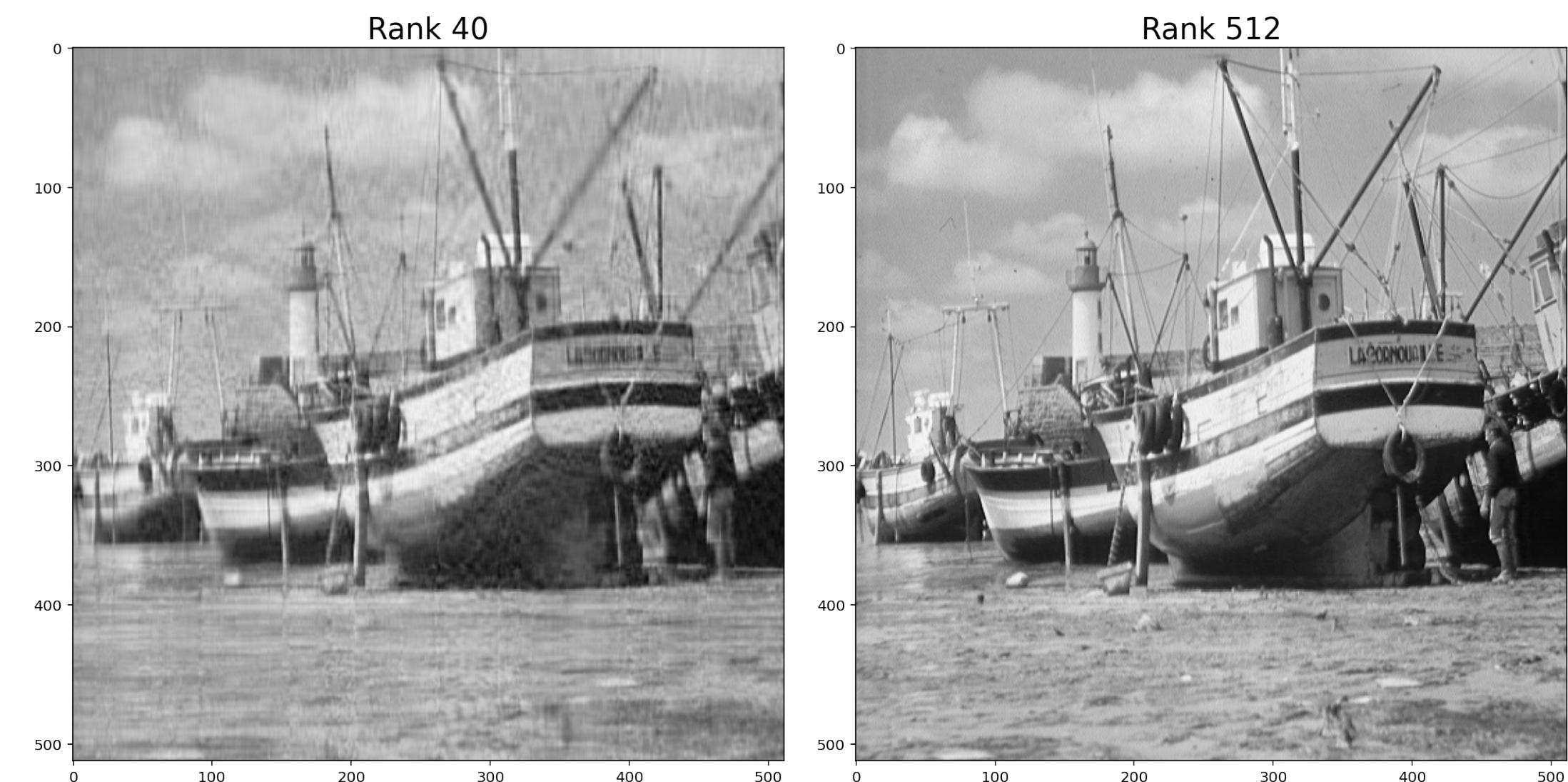


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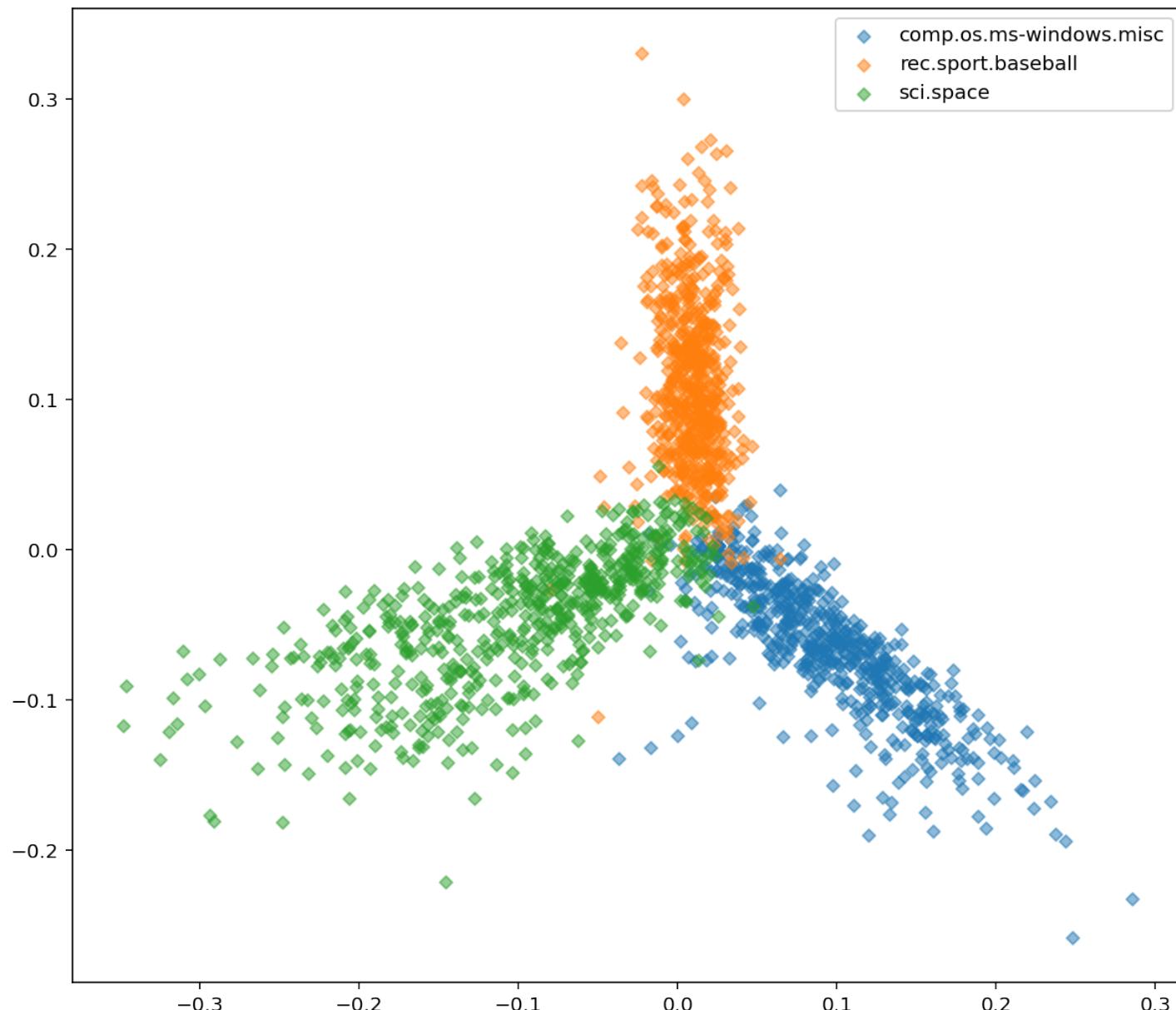
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 - If $A^+ = V\Sigma^{-1}U^T$, then $A^+\mathbf{b}$ is a least squares solution of minimum length

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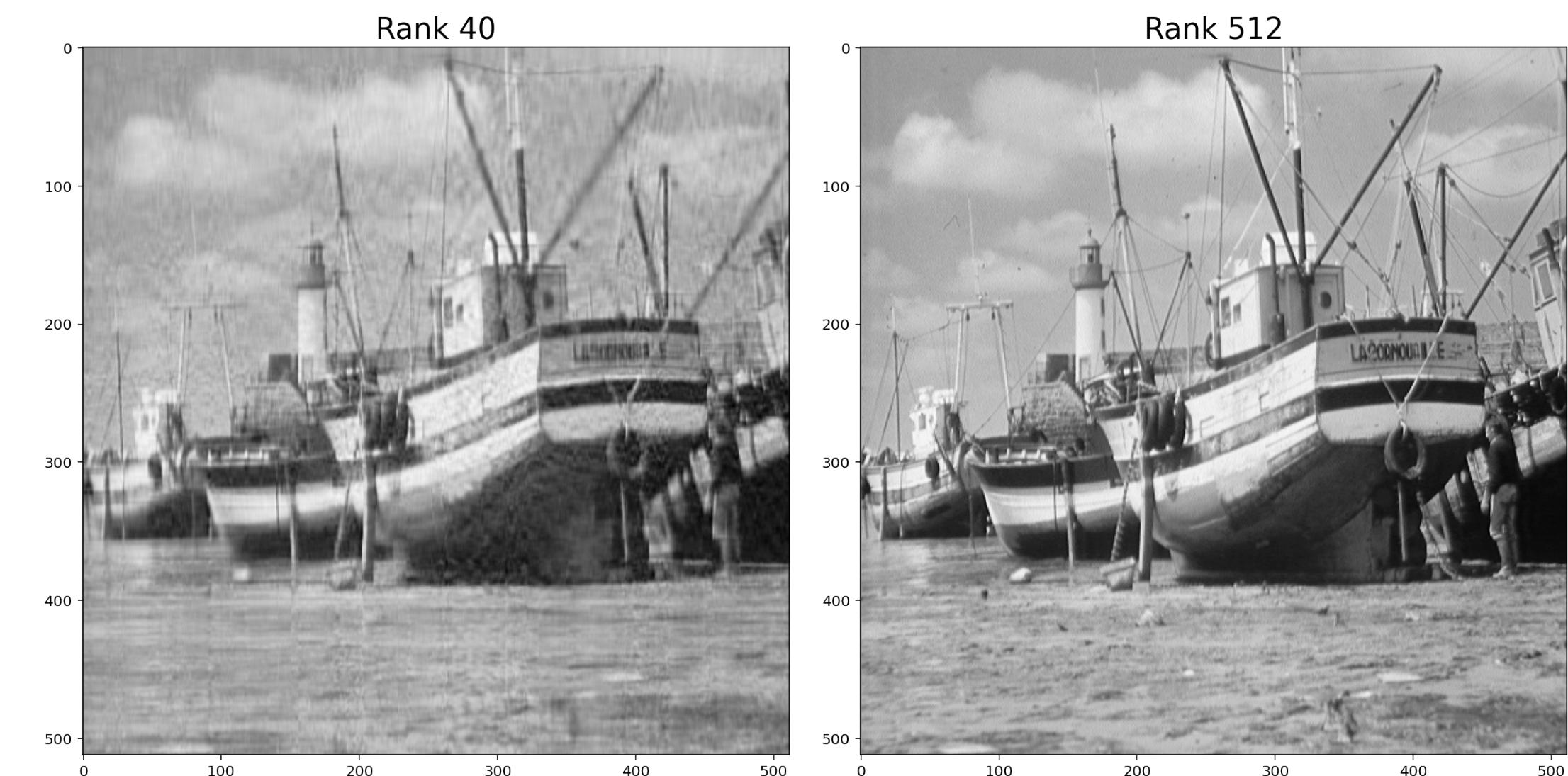


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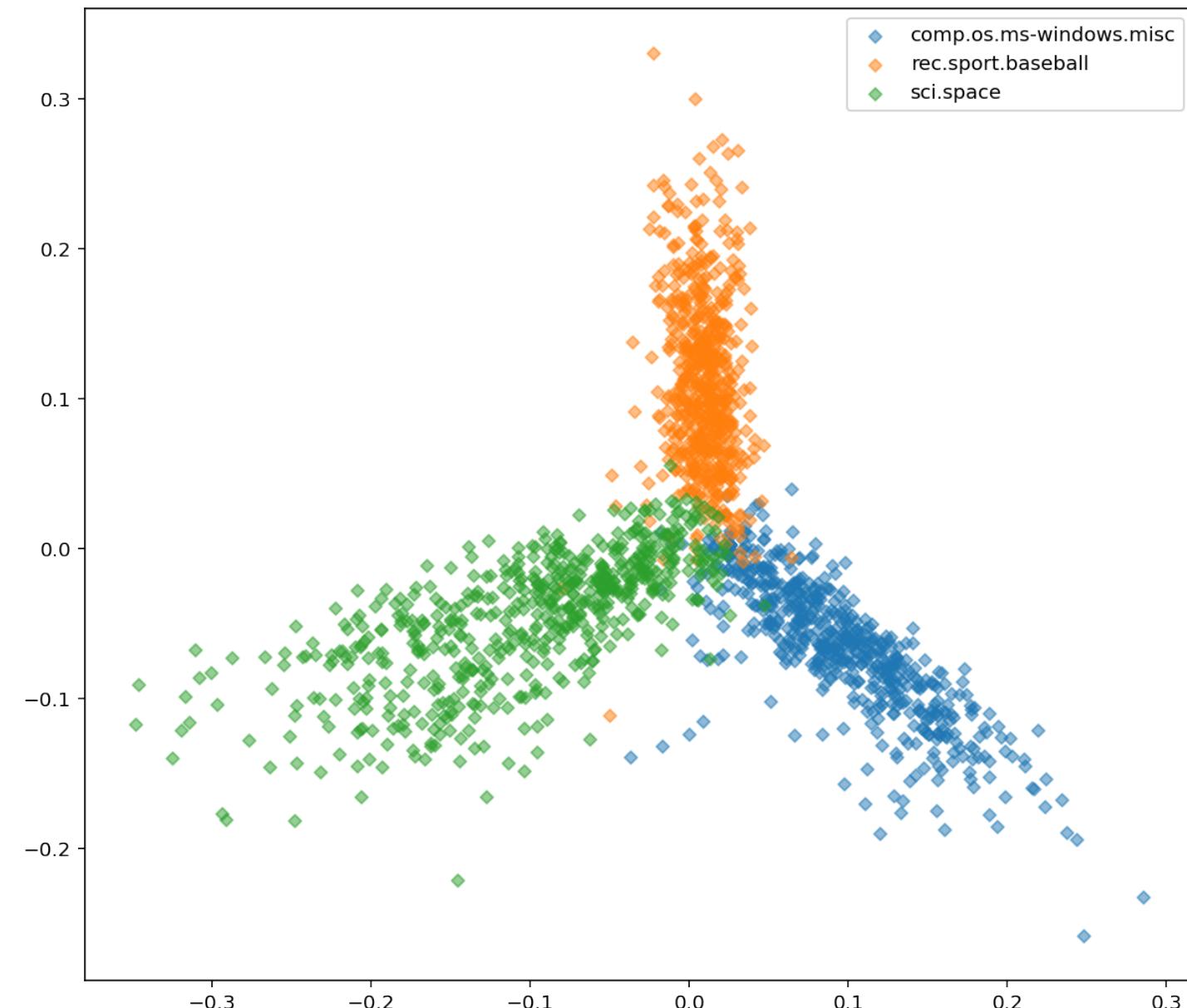
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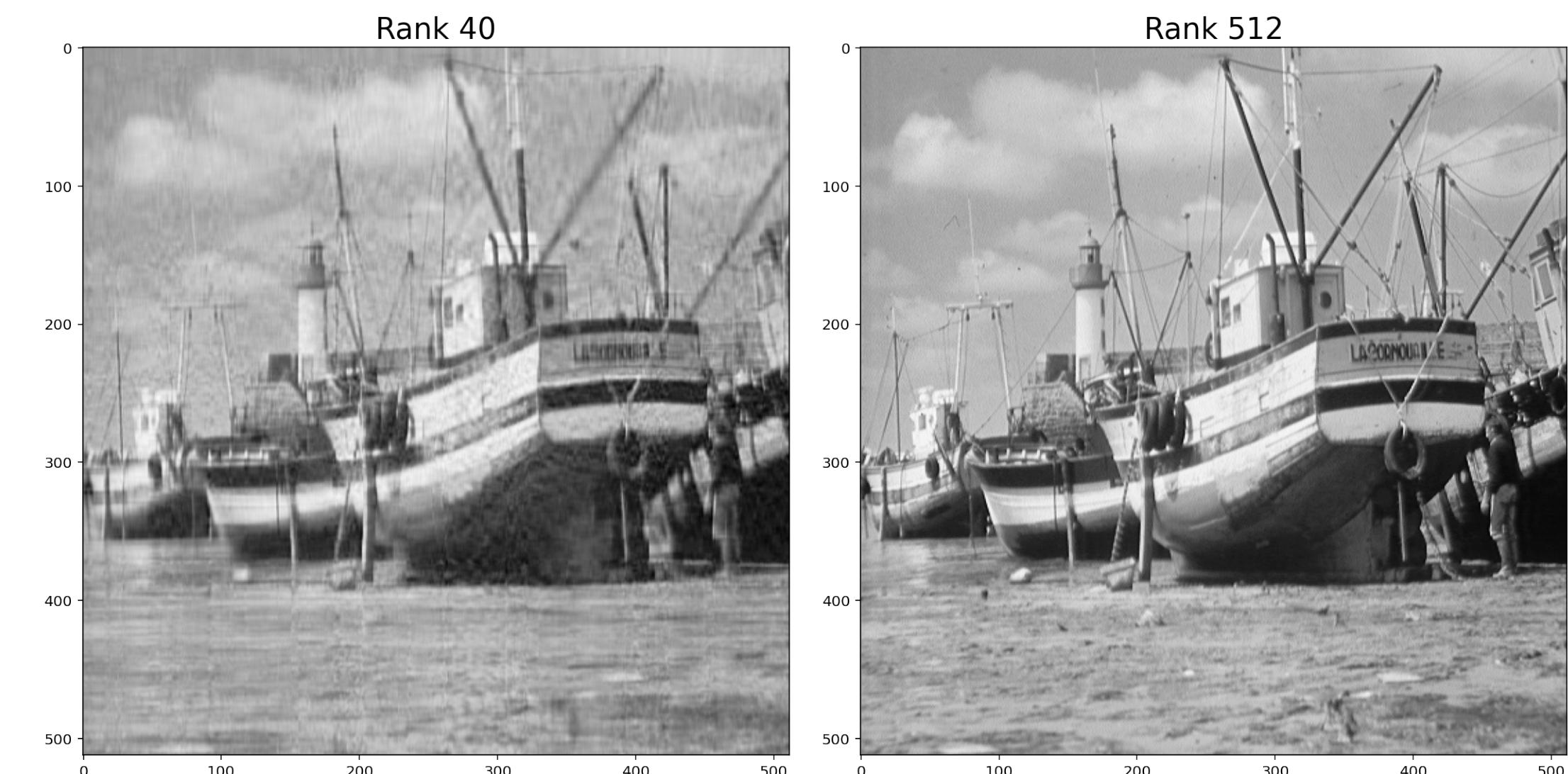


document
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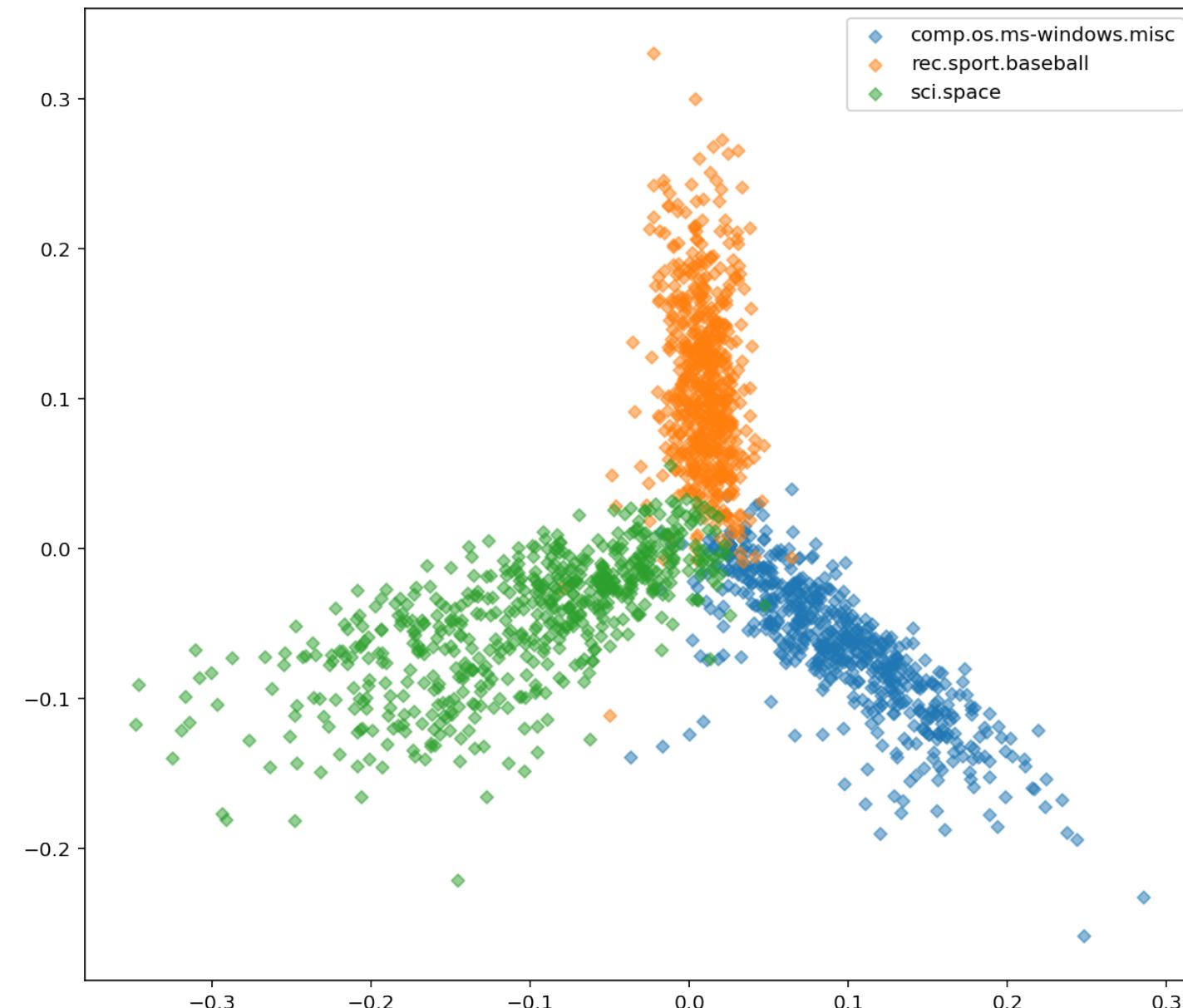
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image compression



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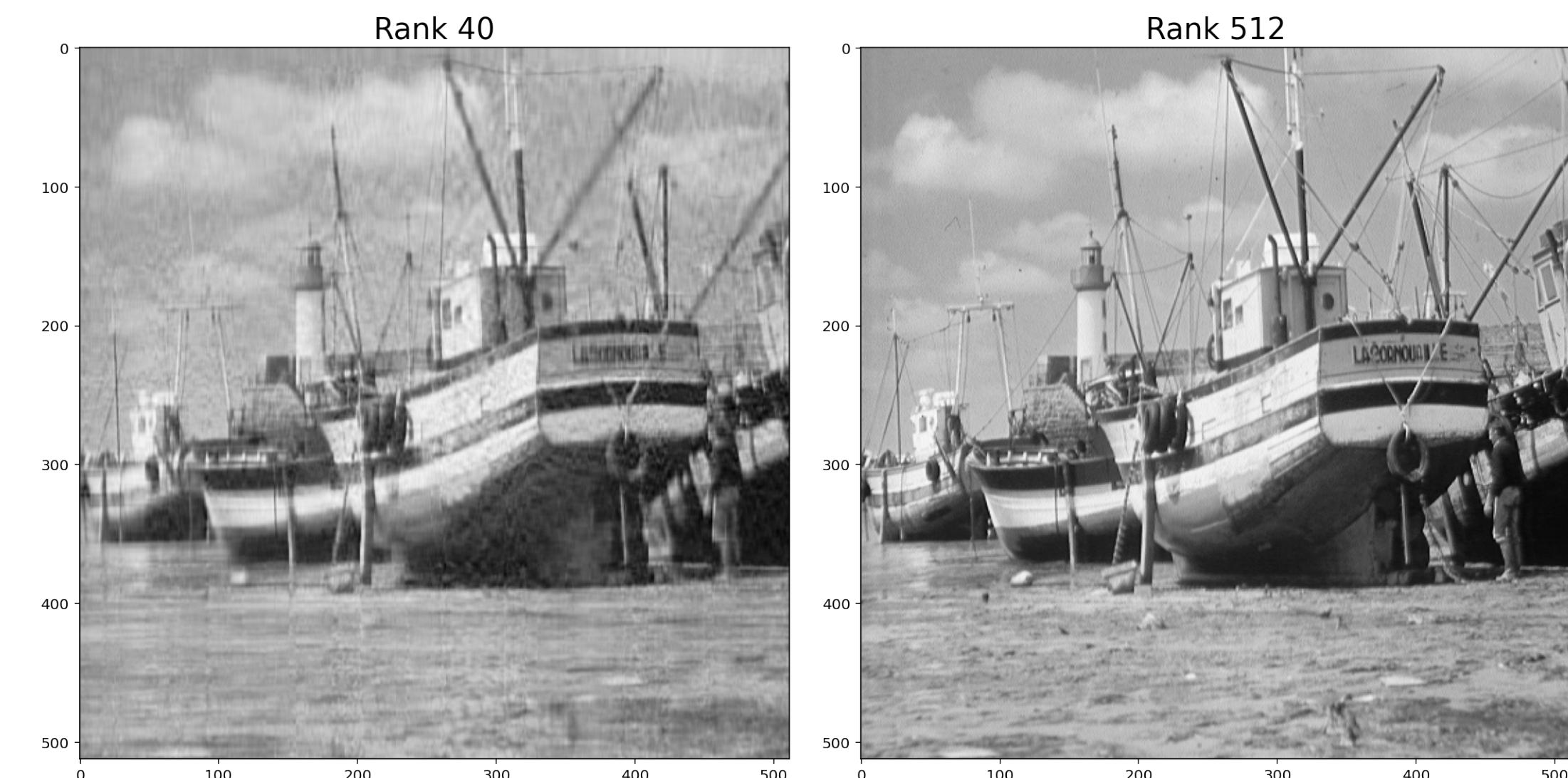


document classification

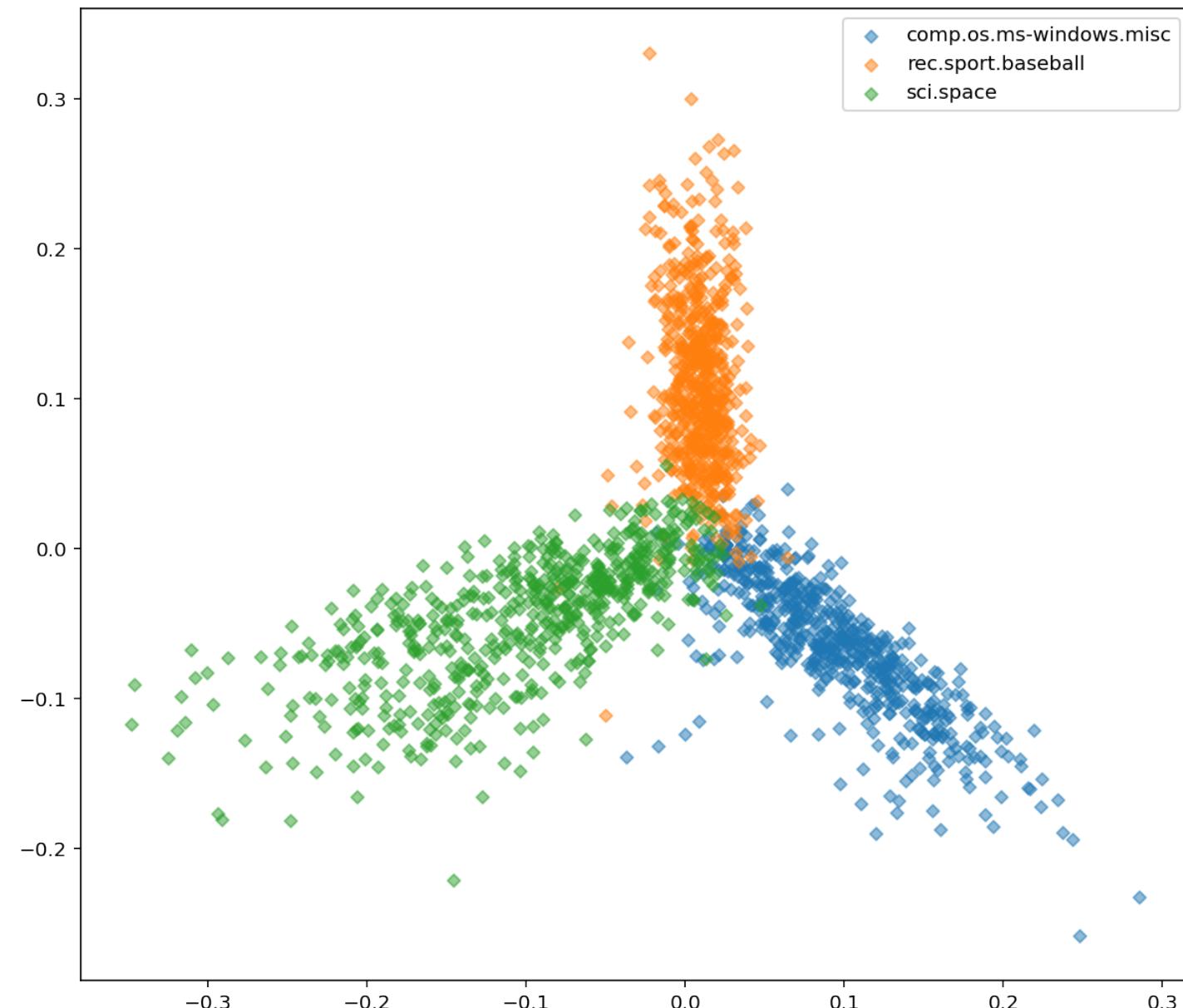
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 - This is used for image compression

image compression



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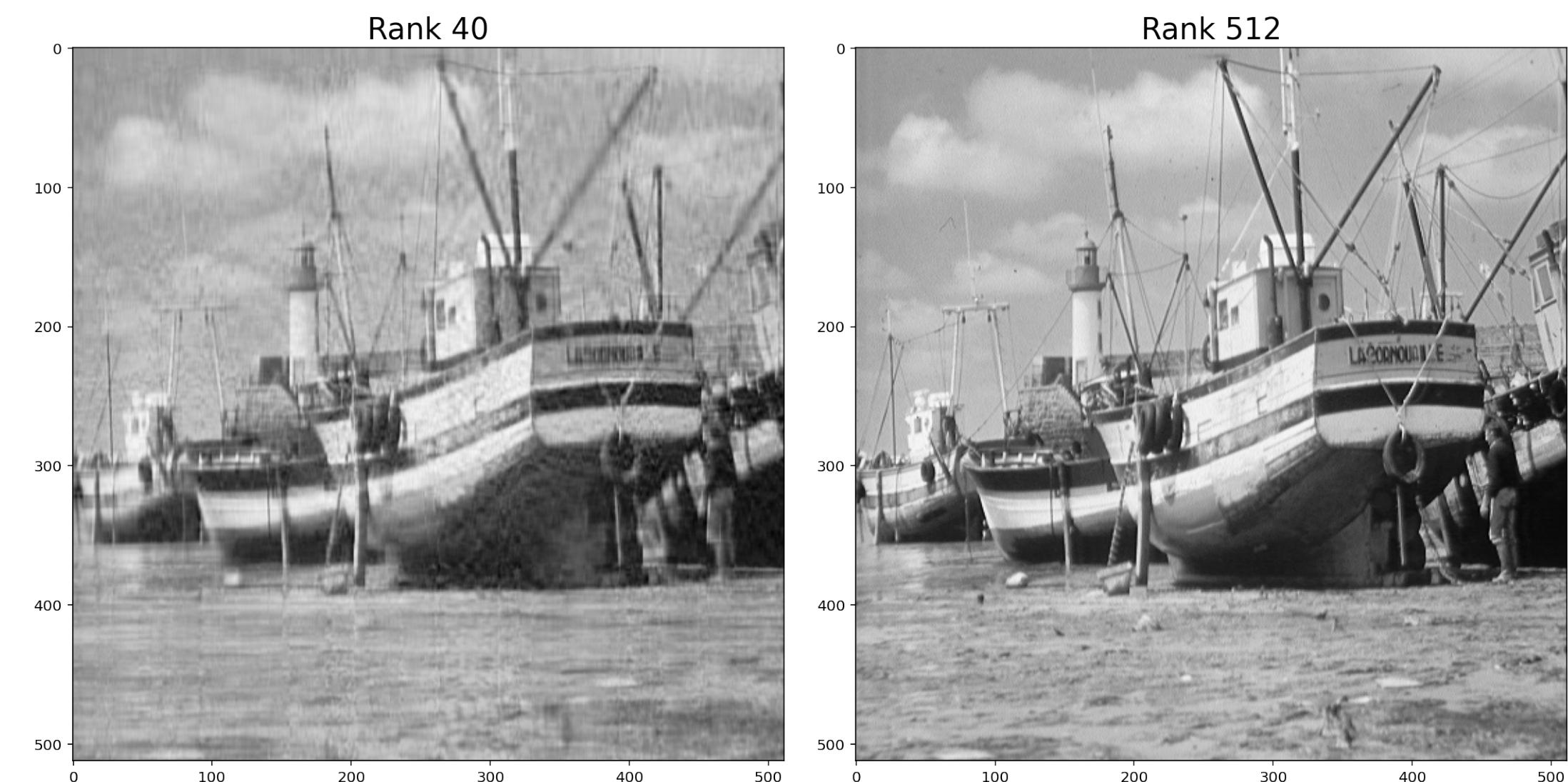


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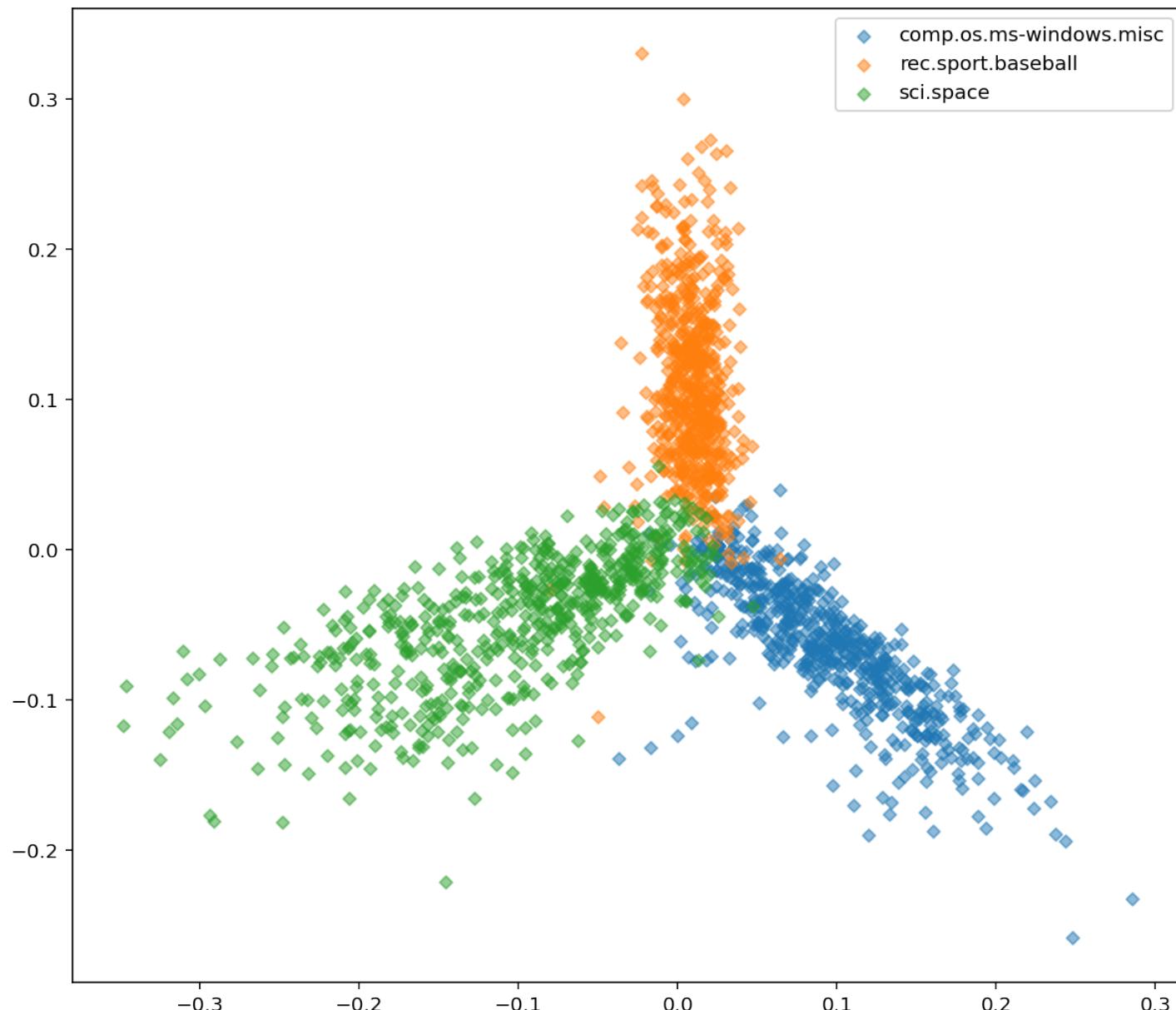
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- Principle Component Analysis

image compression



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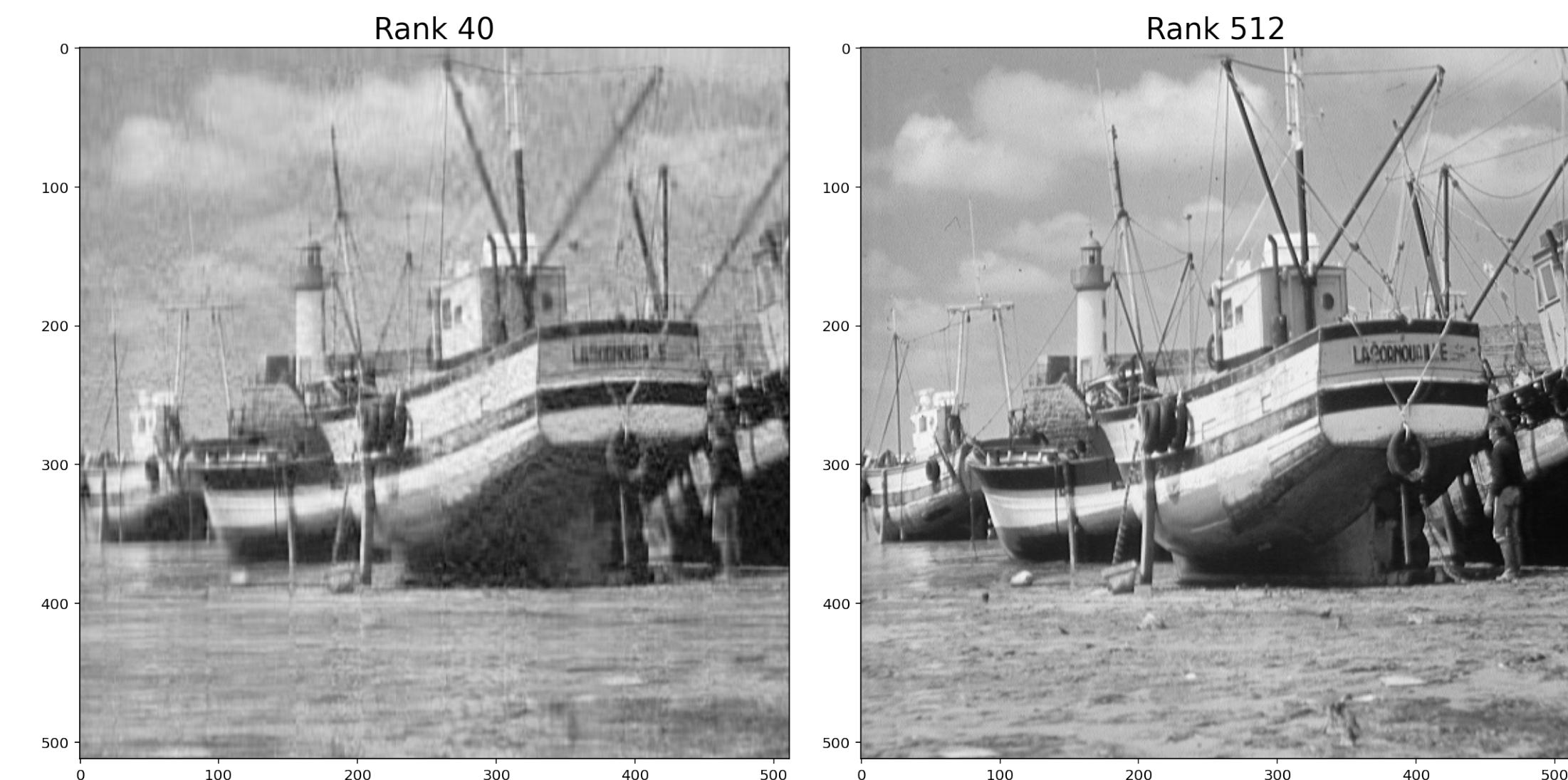


document classification

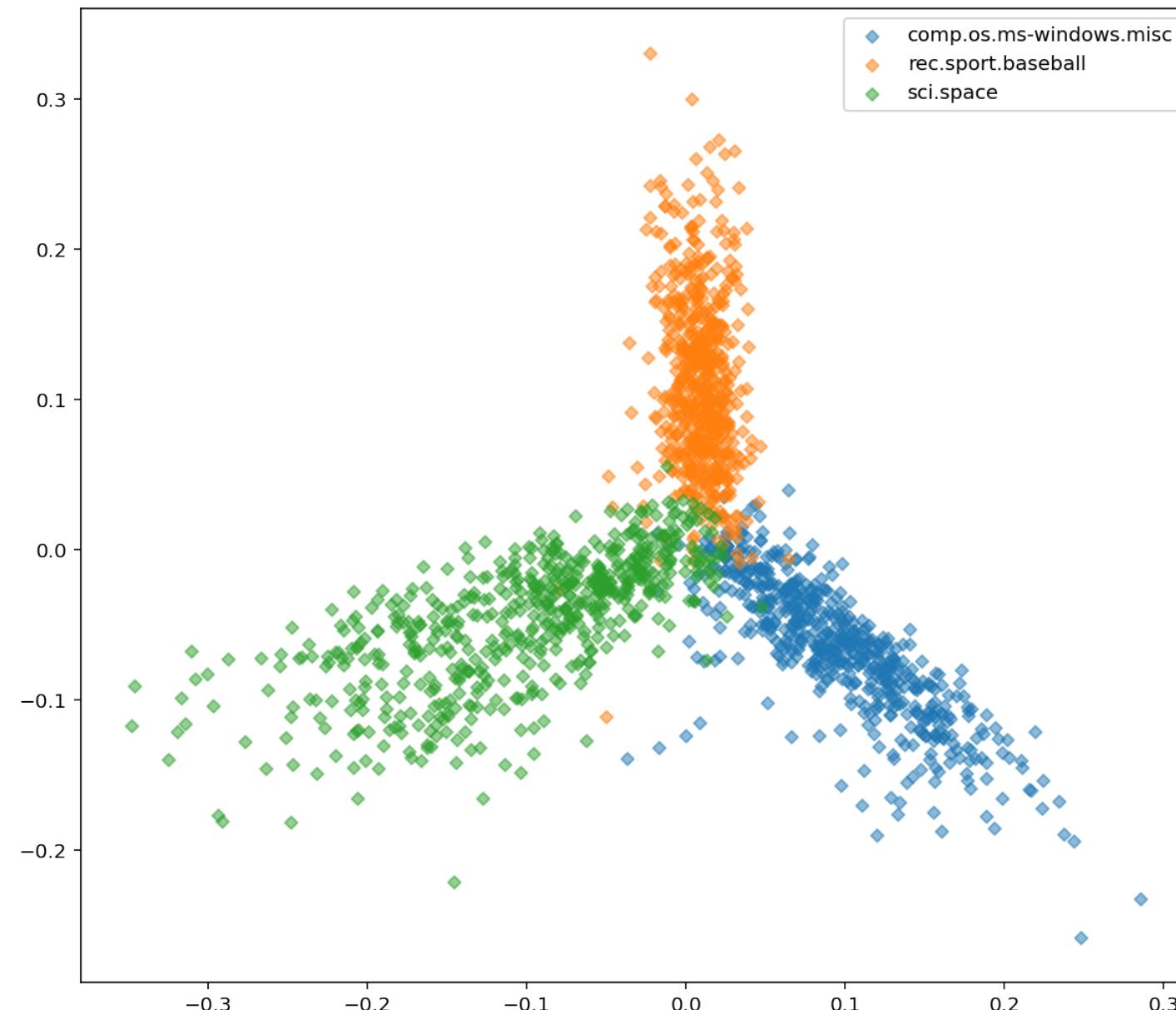
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 - Large singular vectors are "most affected."

image compression



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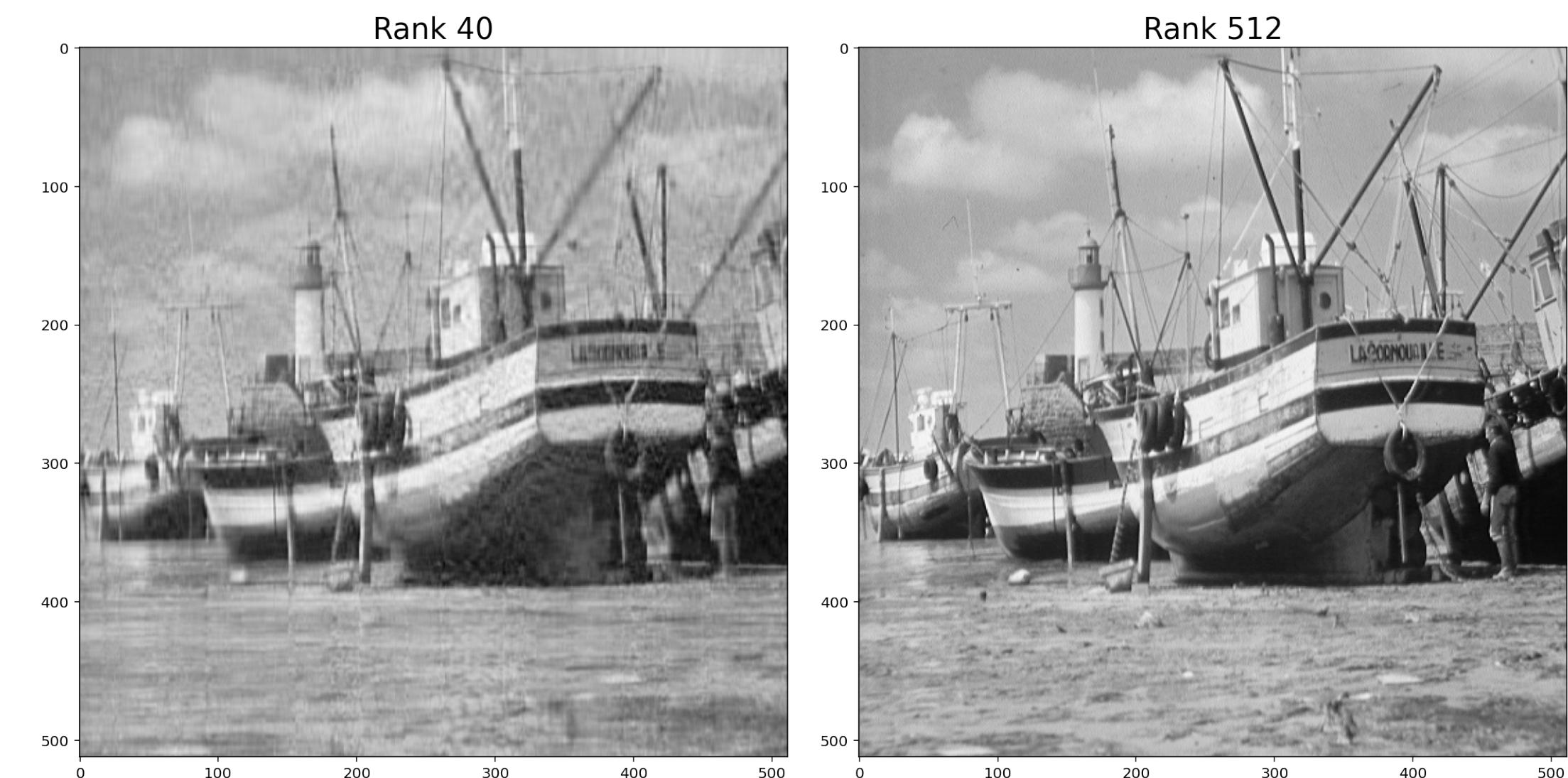


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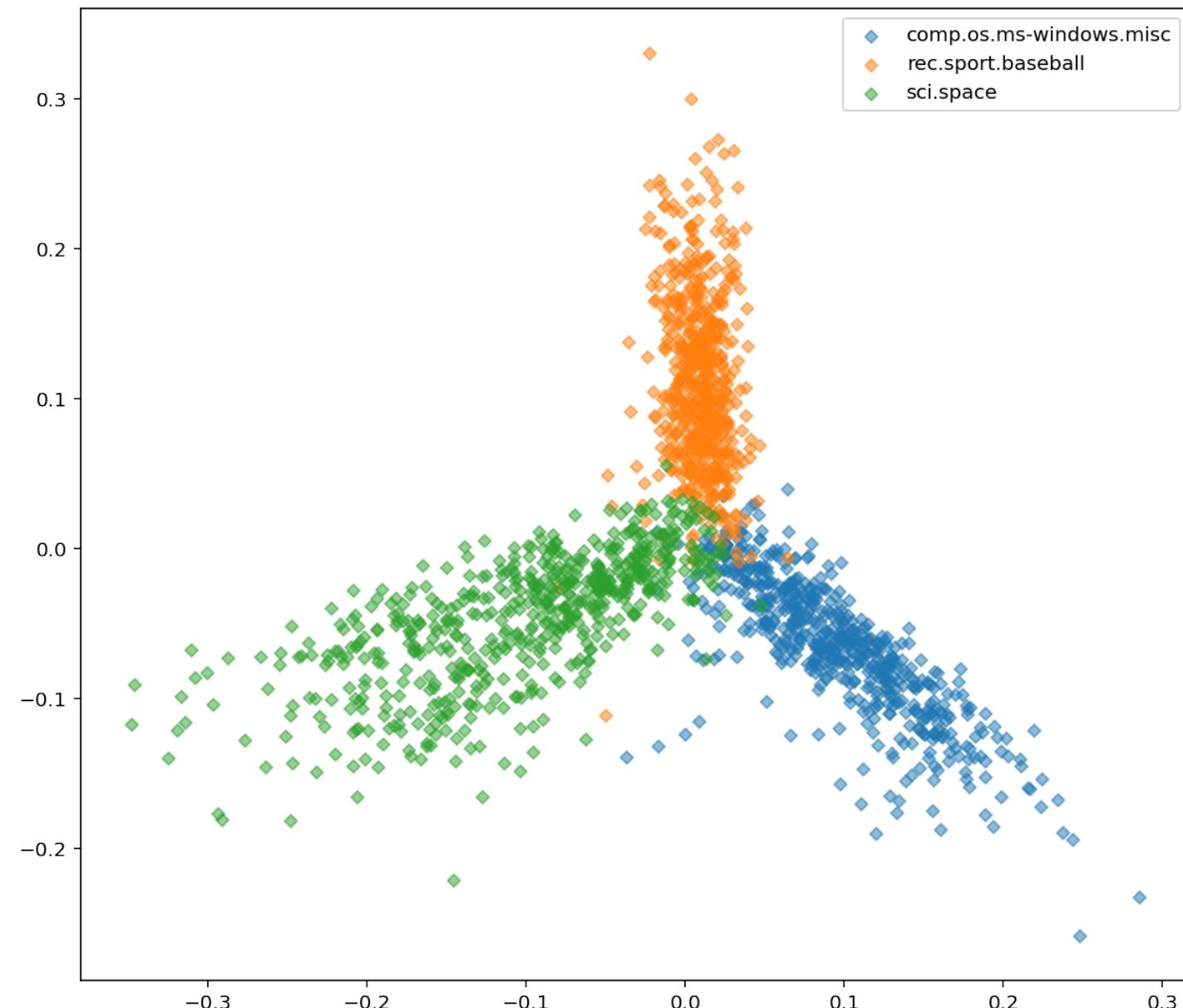
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 - This is used for image compression
- Principle Component Analysis
 - Large singular vectors are "most affected."
 - These are good vectors to look at for classifying data

image compression

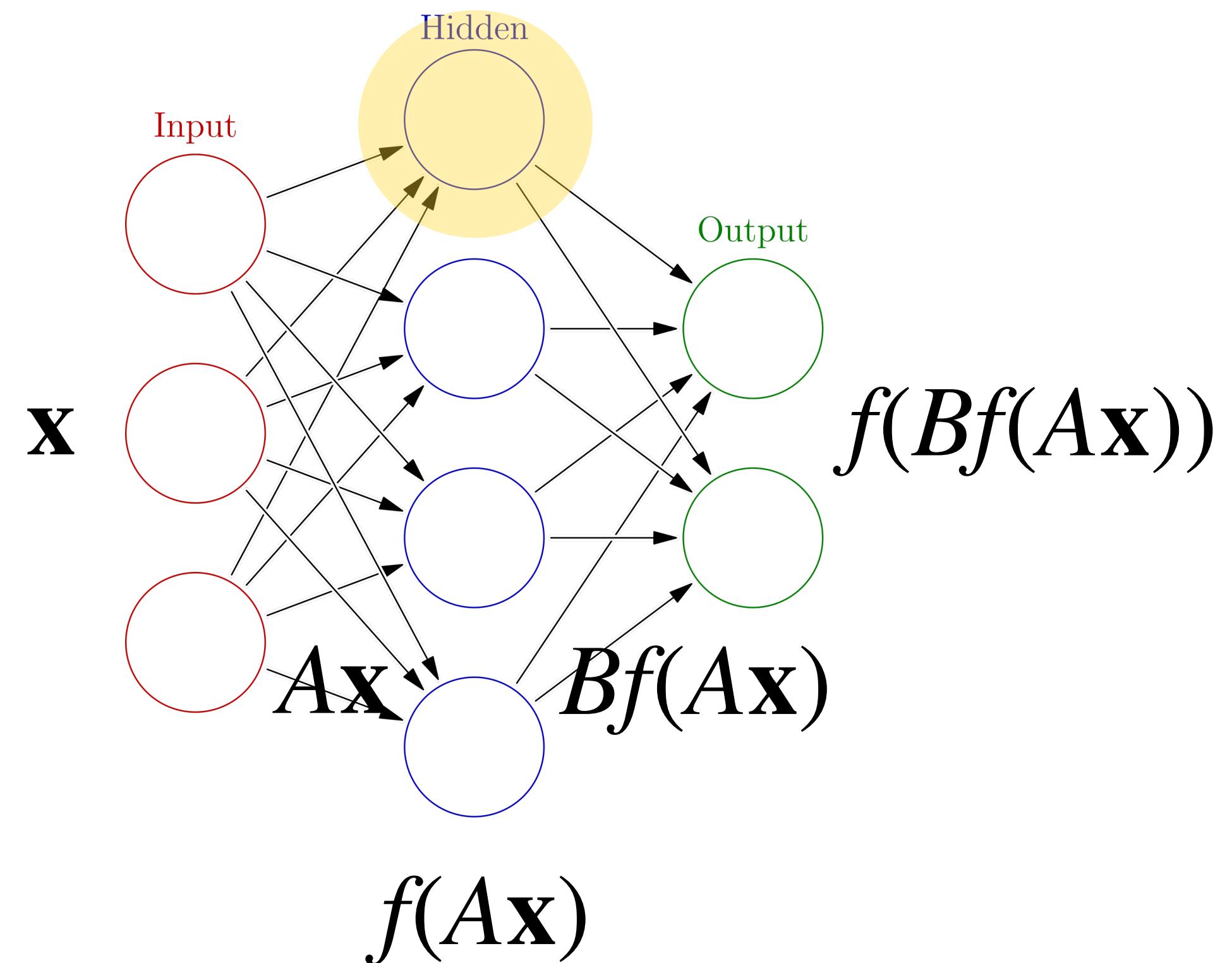
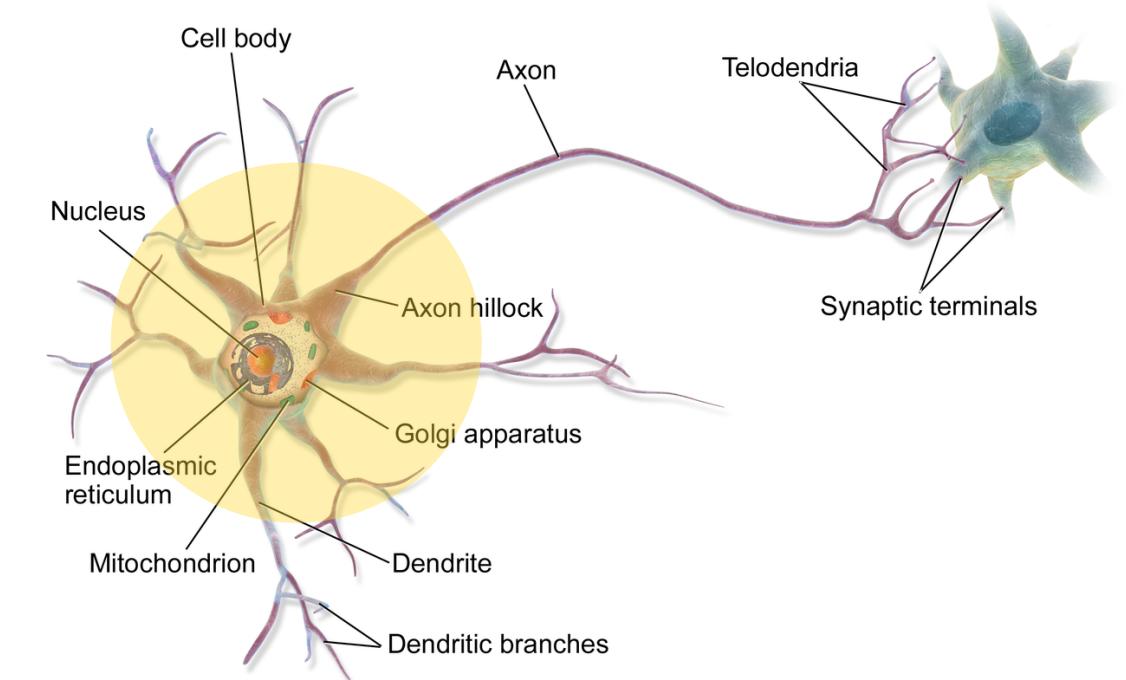


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document
classification

Neural Networks (Non-Linearity)

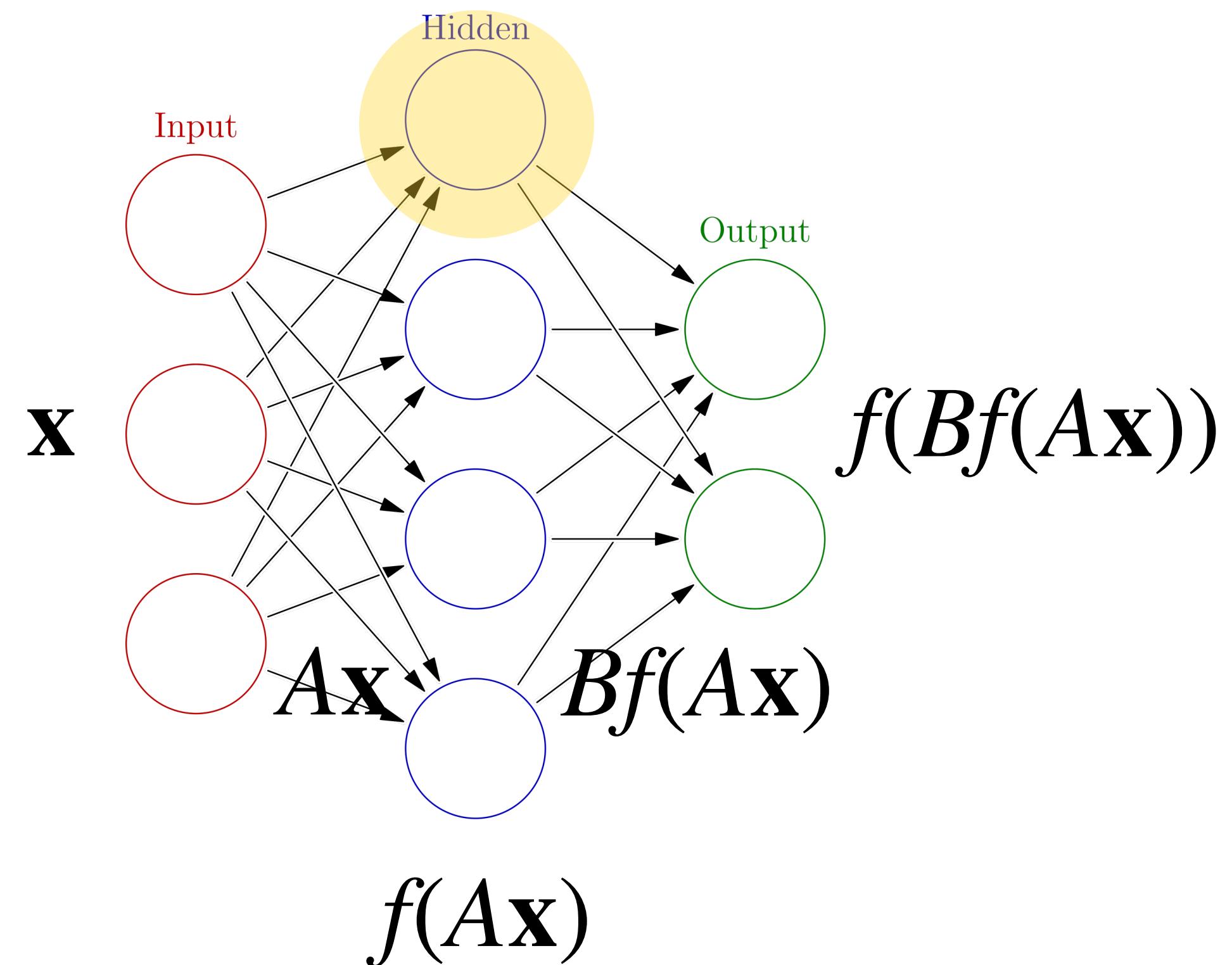
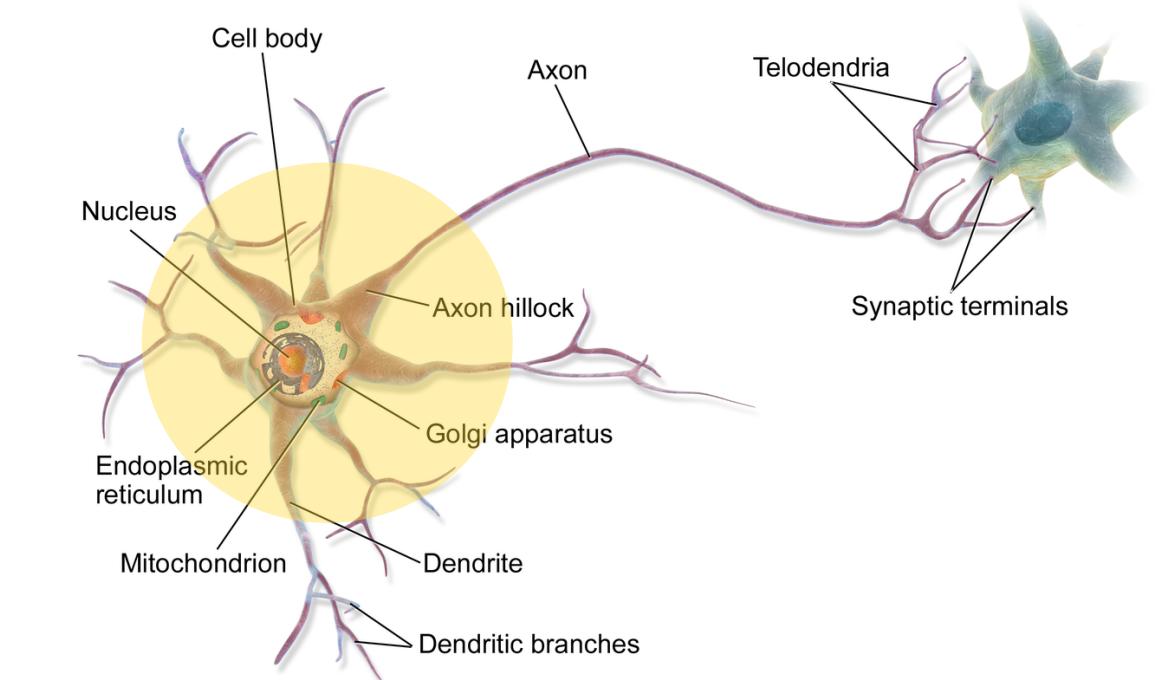


https://commons.wikimedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png

https://commons.wikimedia.org/wiki/File:Colored_neural_network.svg

Neural Networks (Non-Linearity)

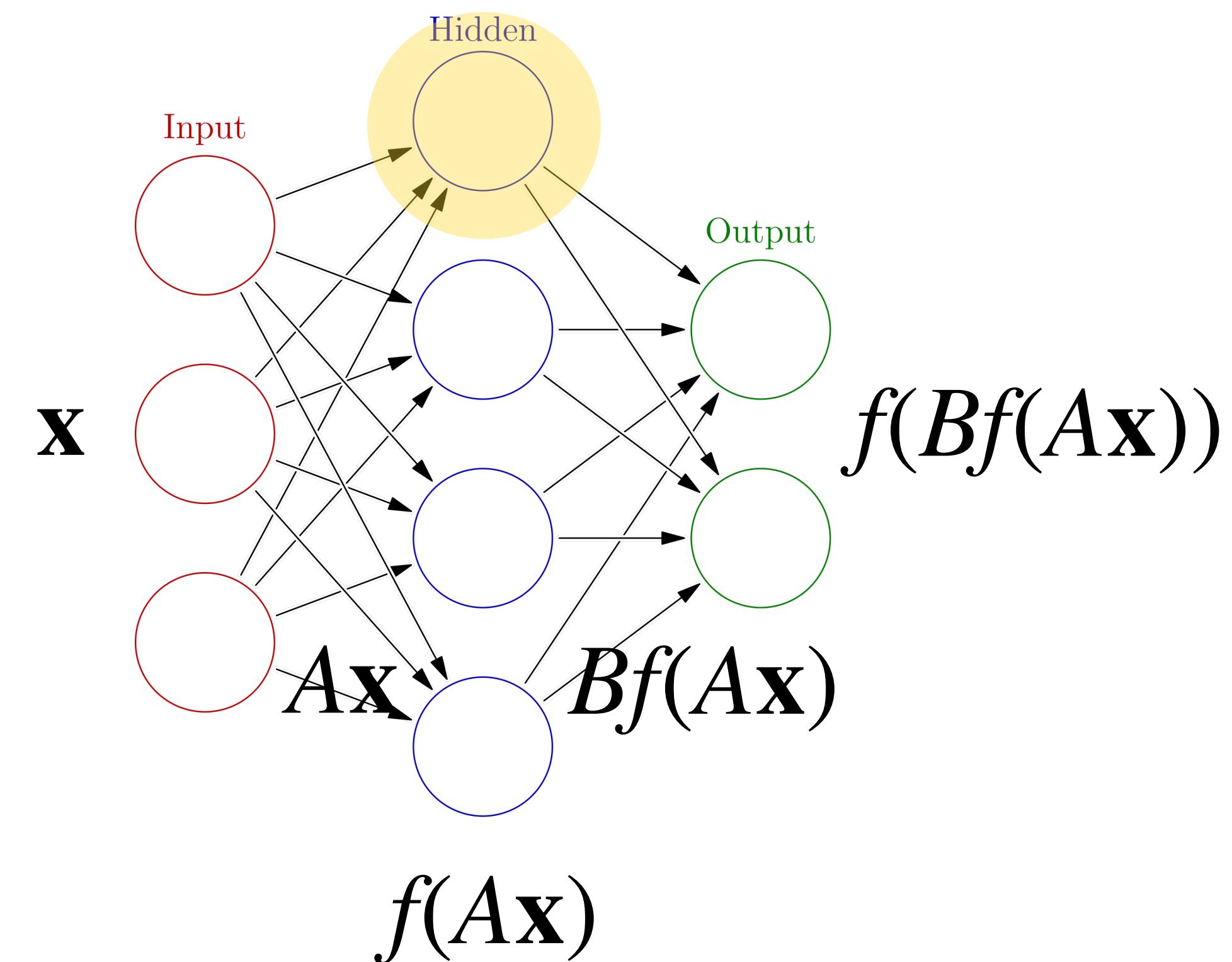
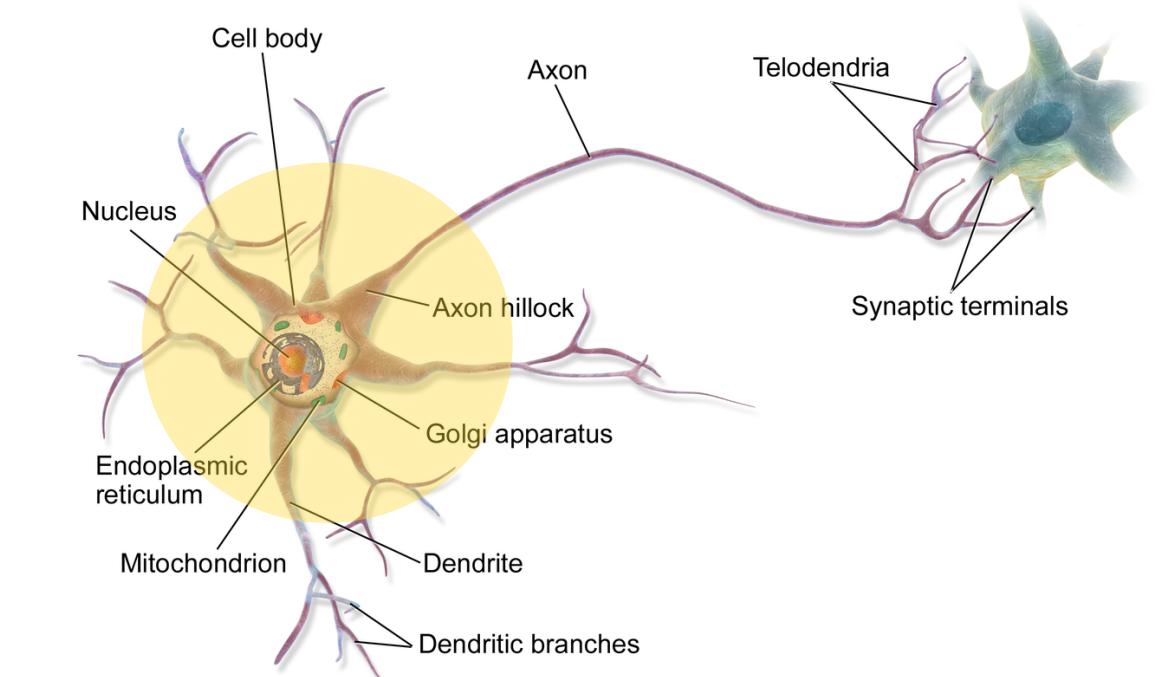
Neural networks are models of artificial neurons bundles.



Neural Networks (Non-Linearity)

Neural networks are models of artificial neurons bundles.

Given an input vector \mathbf{x} , it is transformed into a *hidden* vector $A\mathbf{x}$ by a linear transformation, and then an *activation function* f is applied to the result.

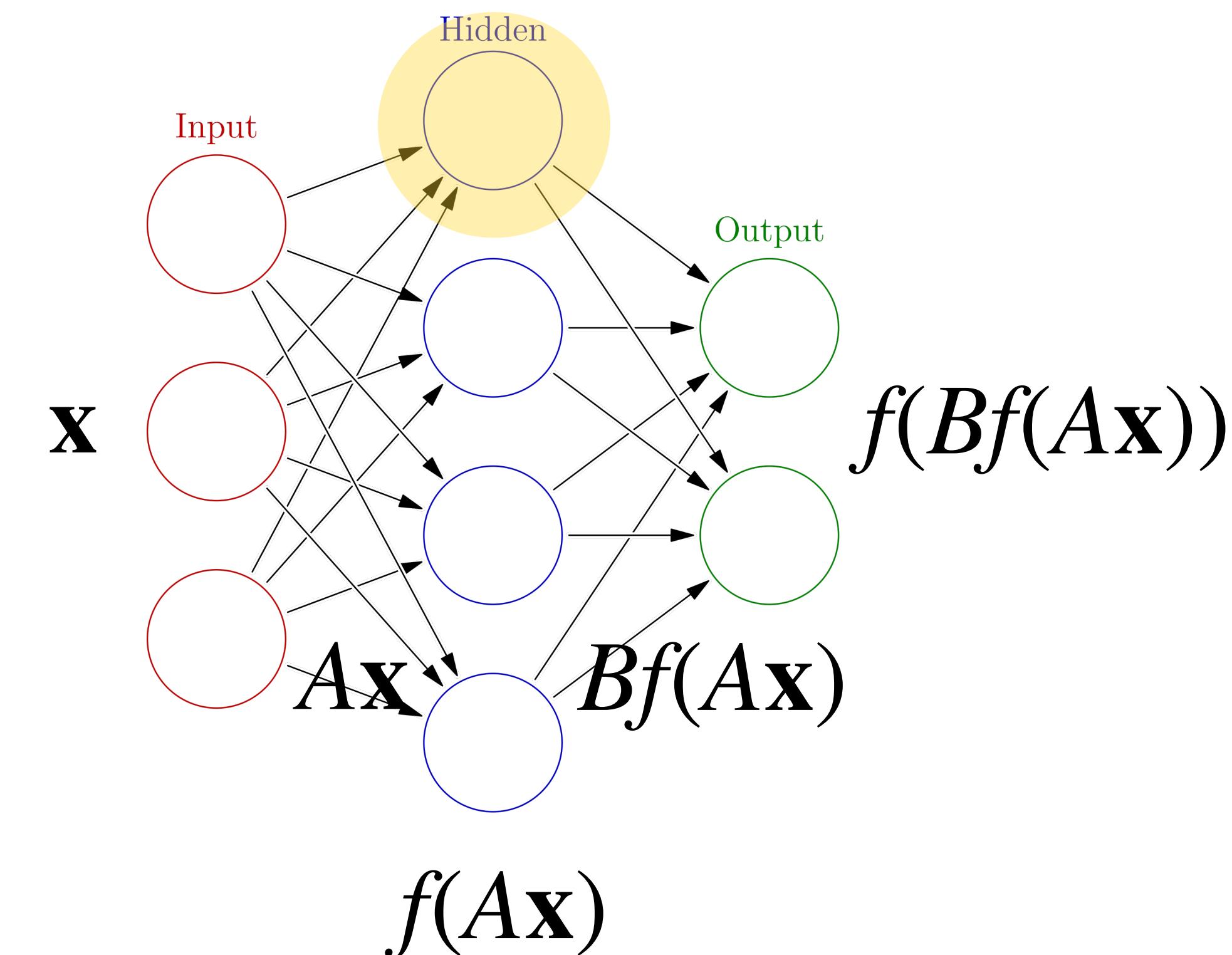
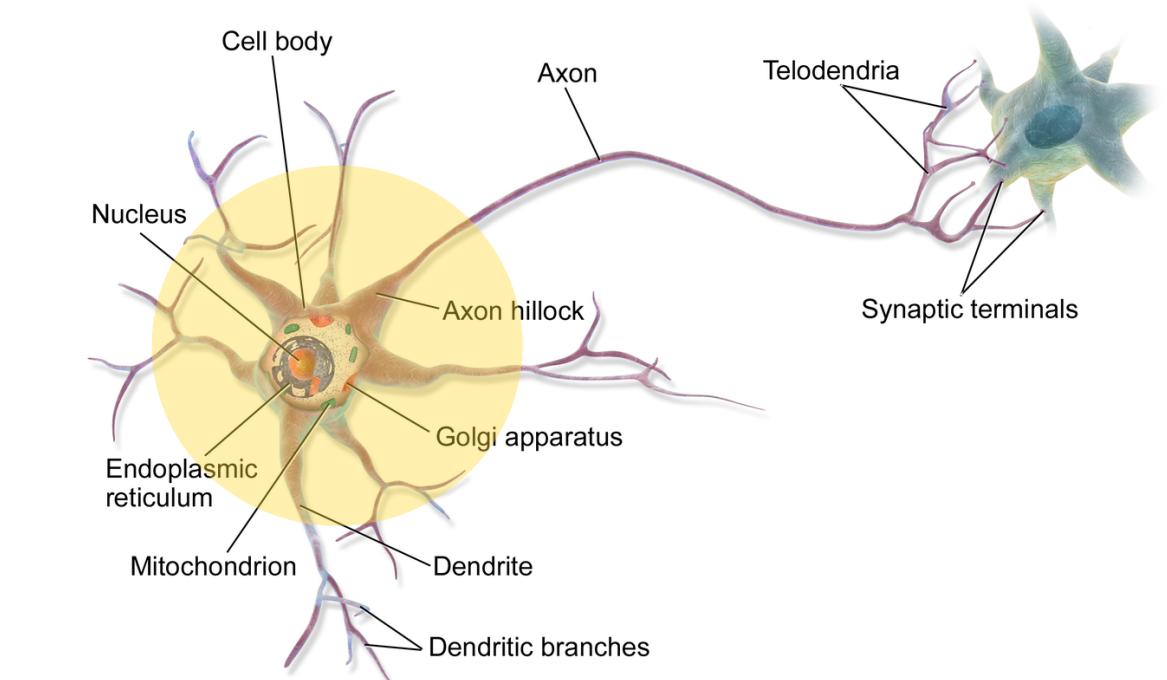


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Given an input vector x , it is transformed into a *hidden* vector Ax by a linear transformation, and then an *activation function* f is applied to the result.

Neural networks are just matrix multiplications with intermediate calls to a nonlinear function f .



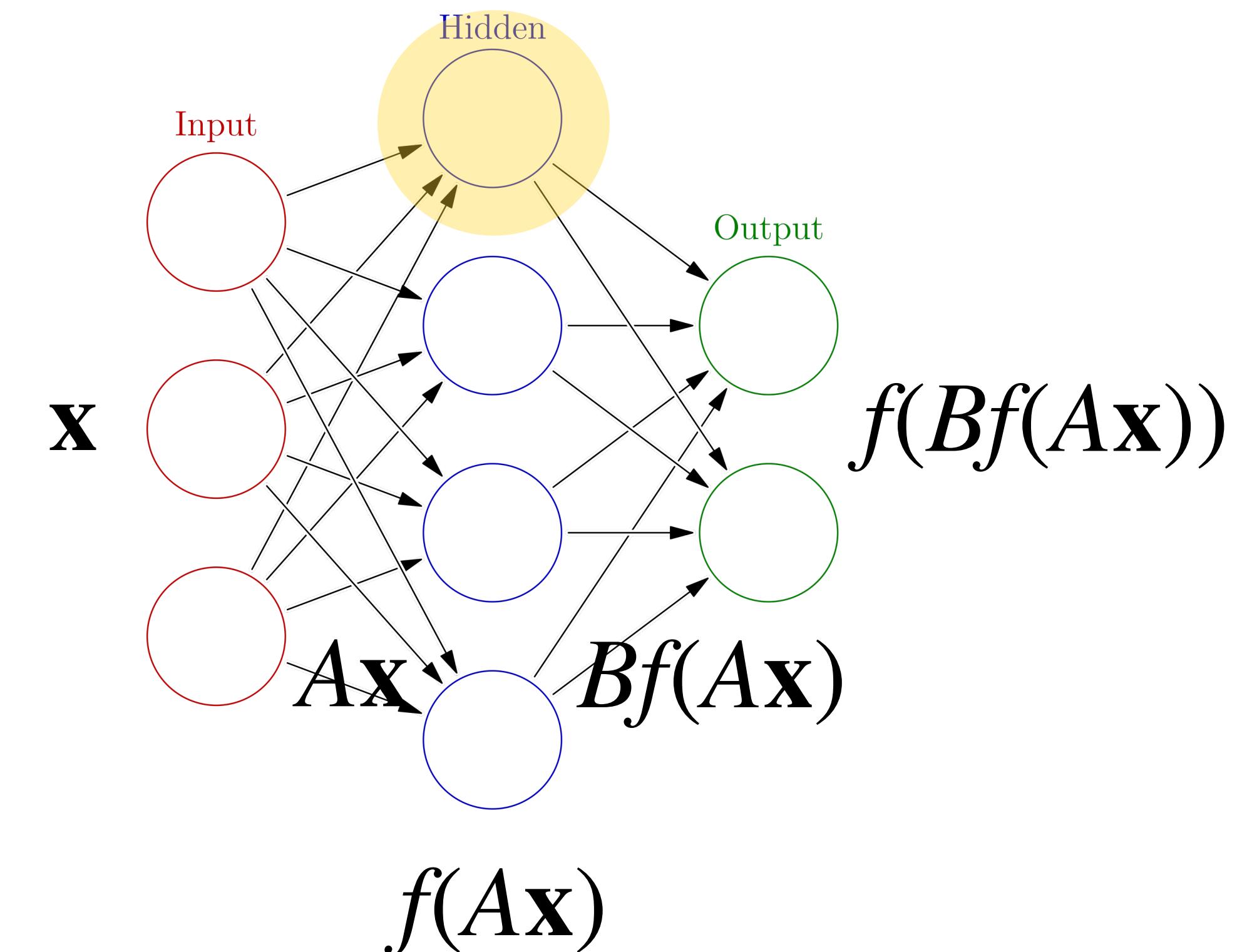
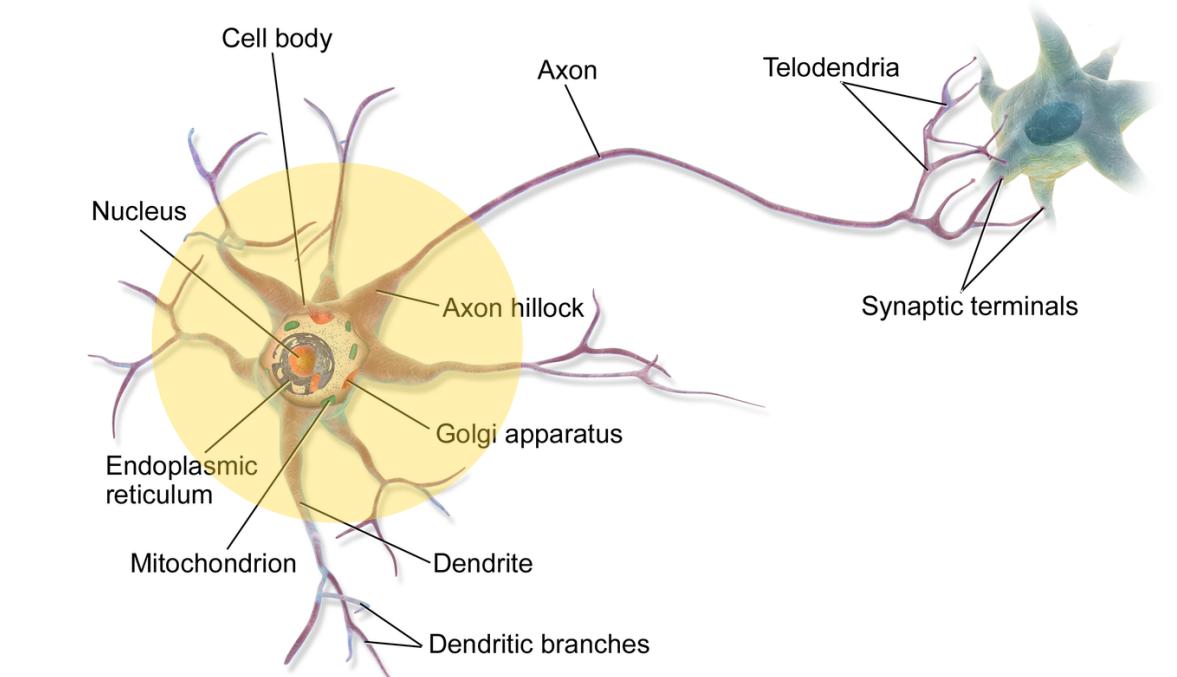
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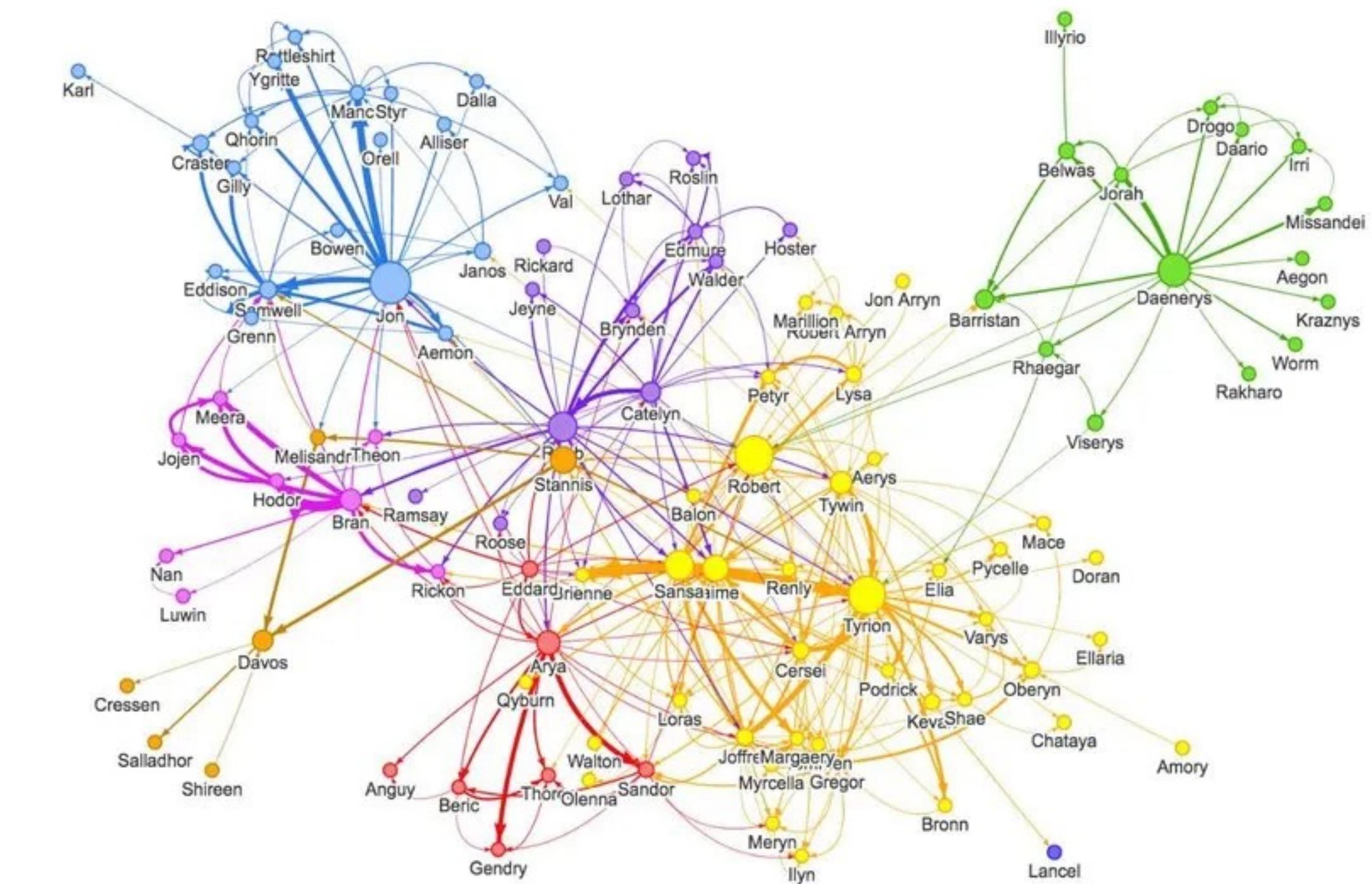
$$\text{NN}(x) = f(A_k(f(A_{k-1} \dots f(A_1 x)))$$



Spectral/Algebraic Graph Theory

Graphs can be viewed as matrices.

The finding eigenvalues in graphs can gives user better clustering and cutting algorithms.



Abstract Algebra

$$\frac{U}{\text{Nul}(f)} \cong \text{Range}(f)$$

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \searrow \\ & & U/\text{Nul}(f) \end{array}$$

There's a lot of beautiful structure in the algebra we've done in this course.

And there are lots of directions to go from here (infinite dimensional spaces, less restrictive settings like groups and modules, ...)

Course List

- CS 365 Foundations of Data Science
- CS 440 Intro to Artificial Intelligence
- CS 480 Intro to Computer Graphics
- CS 505 Intro to Natural Language Processing
- CS 506 Tools for Data Science
- CS 507 Intro to Optimization in ML
- CS 523 Deep Learning
- CS 530 Advanced Algorithms
- CS 531 Advanced Optimization Algorithms
- CS 542 Machine Learning
- CS 565 Algorithmic Data Mining
- CS 581 Computational Fabrication
- CS 583 Audio Computation

•CS 582 Geometry Processing

Some of these may not exist anymore...

Appreciations

The Course Staff

I'd like to thank:

~~Abhinit Sati, Vishesh Jain, Ieva Sagaitis, Kevin Wrenn, Jin Zhang, Sohan Atluri, Fynn Buesnel, Aseef Imran, Eugene Jung, Chris Min, Wyatt Napier, Kyle Yung~~

If you see them around you should thank them as well

Rahul Mitra, Gor Matcakian, Helen Zhou, Ian Sun

The CS Department Staff

If you're ever in the CS Department office, be kind to the people who work there. They work very hard to keep all our courses running

The Students of CS132

Thanks for sticking with it

Thanks for giving feedback

Thanks for participating

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