Eigenvalues and Eigenvectors

Geometric Algorithms
Lecture 18

CAS CS 132

Practice Problem

Suppose A *is a* 234×300 *matrix. What is the smallest possible value for* $\dim(\mathrm{Nul}(A))$*? What is the largest possible value?*

What is the smallest possible value for $\mathrm{rank}(A)$*? What is the largest possible value?*

Answer

$$A \text{ is } \overset{m}{234} \overset{\times}{\times} \overset{n}{300}$$

$$\underbrace{\dim(\text{Col } A)}_{\text{"rank"}} + \underbrace{\dim(\text{Nul } A)}_{\text{"nullity"}} = n$$

$$66 \leq \dim(\text{Nul } A) \leq 300$$

$$0 \leq \dim(\text{Col } A) \leq 234$$

if $\dim(\text{Nul } A) = 300$

&

$\dim(\text{Col } A) = 0$

A is 0 matrix

Objectives

1. <u>Motivate</u> and introduce the fundamental notion of eigenvalues and eigenvectors
2. Determine how to <u>verify</u> eigenvalues and eigenvectors
3. Look at the <u>subspace</u> generated by eigenvectors
4. Apply the study of eigenvectors to <u>dynamical linear systems</u>

Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

Motivation

demo

How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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$\Big\}$ All matrices do some combination of these things

* square matrices

How can matrices transform vectors?*

In 2D and 3D we've seen:

- rotations
- projections
- shearing
- reflection
- scaling/stretching
- ... **Today's focus**

All matrices do some combination of these things

* square matrices

What's special about scaling?

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`We don't need a whole matrix to do scaling`

$$\mathbf{x} \mapsto c\mathbf{x}$$

What's special about scaling?

We don't need a whole matrix to do scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

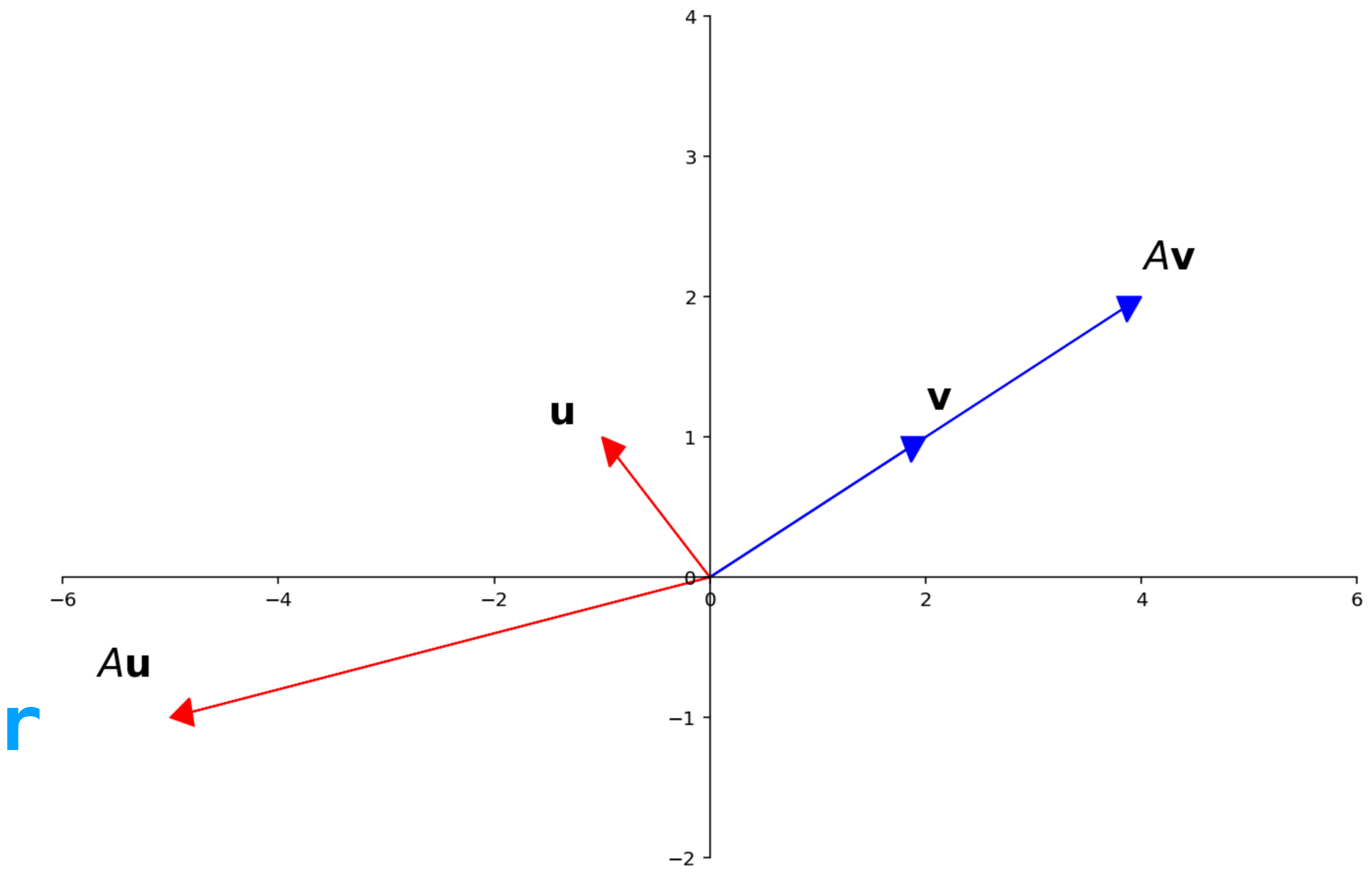
So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what A does to $\mathbf{v}$.

Eigenvectors (Informal)

eigenvalue

$$A\mathbf{v} = \lambda \mathbf{v}$$

eigenvector



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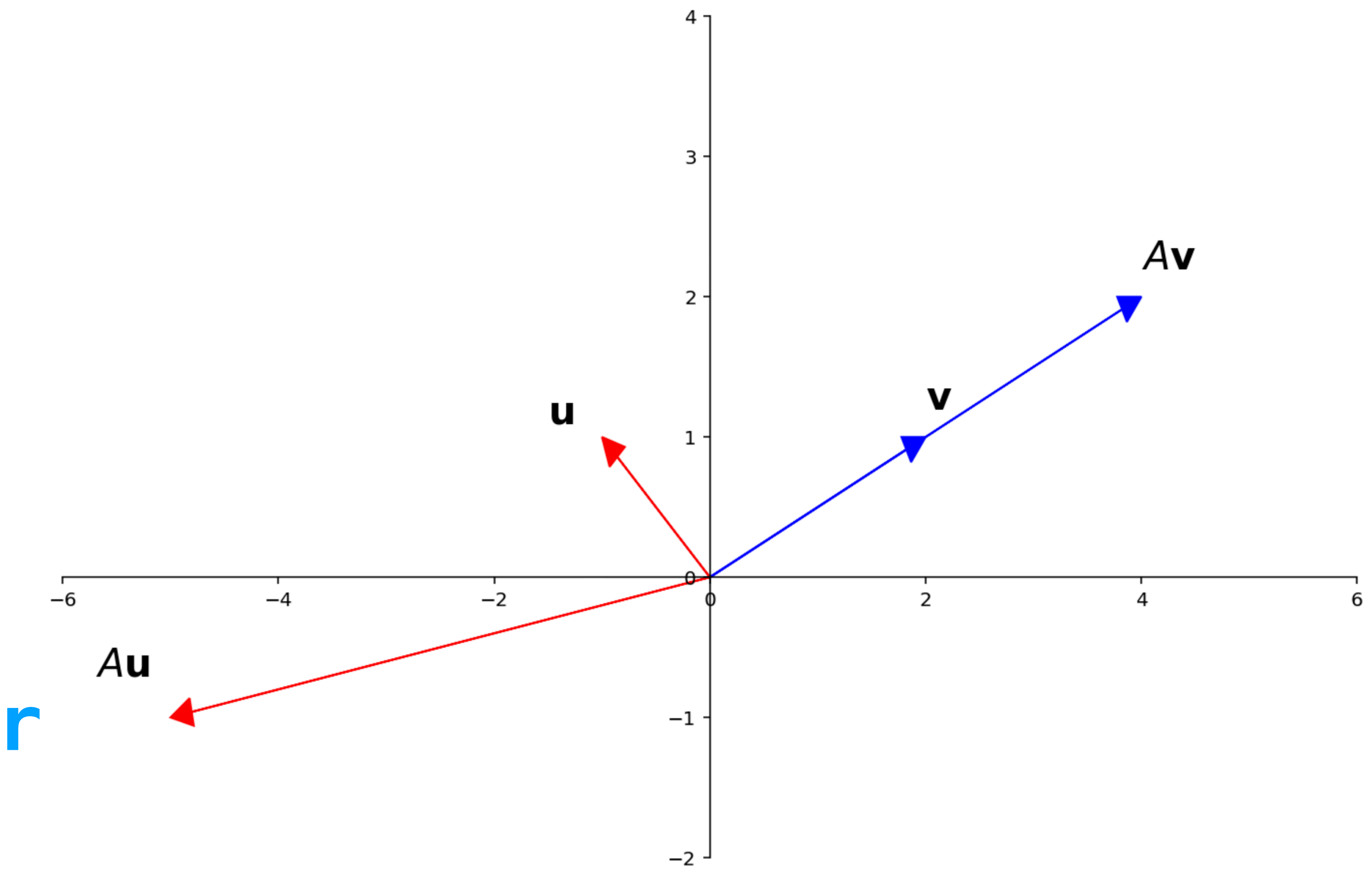
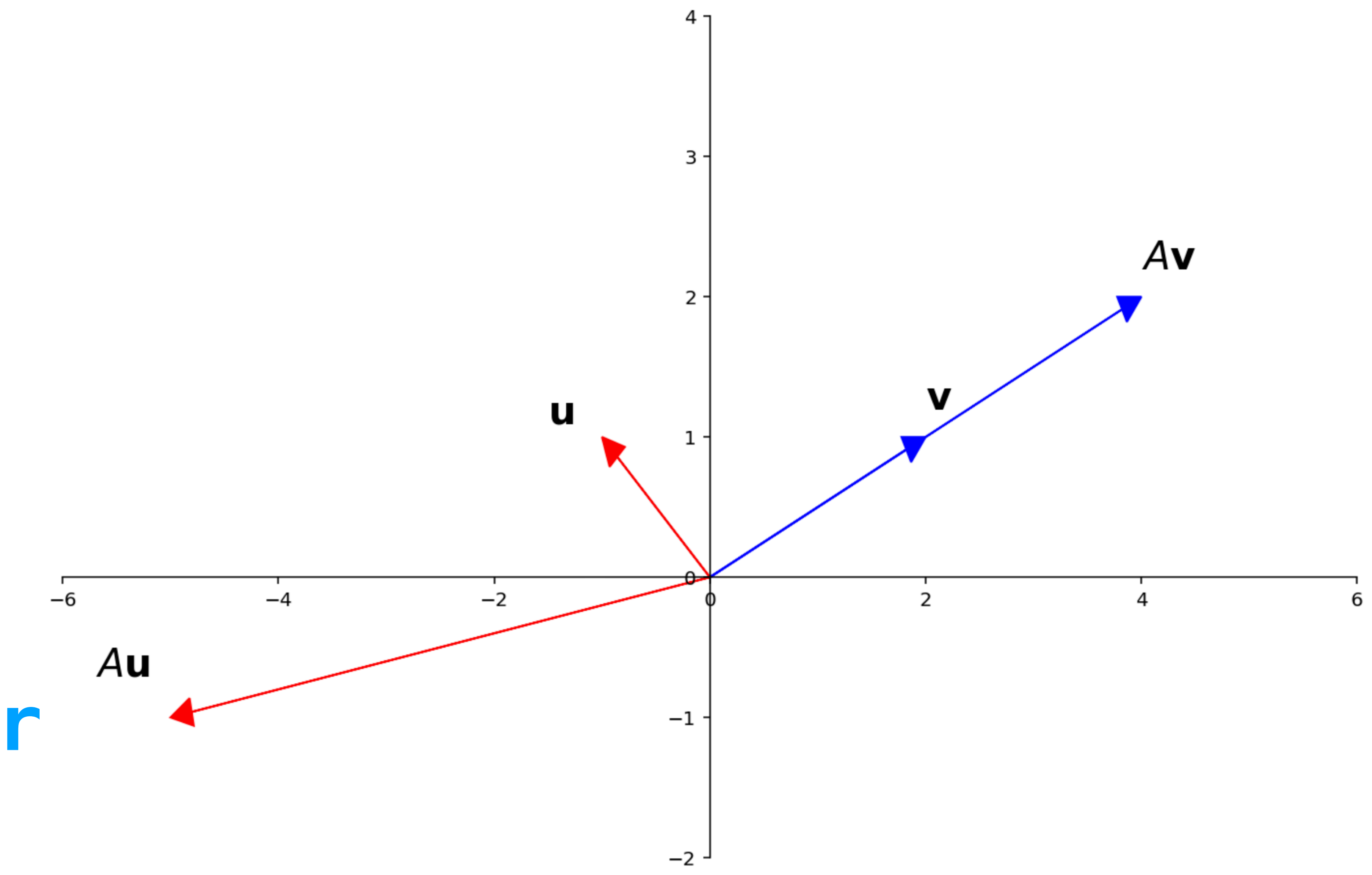


Eigenvectors of A are stretched by A without changing their direction.

Eigenvectors (Informal)

$$A\mathbf{v} = \overset{\text{eigenvalue}}{\lambda}\underset{\text{eigenvector}}{\mathbf{v}}$$



Eigenvectors of A are stretched by A without changing their direction.

The amount they are stretched is called the **eigenvalue.**

Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

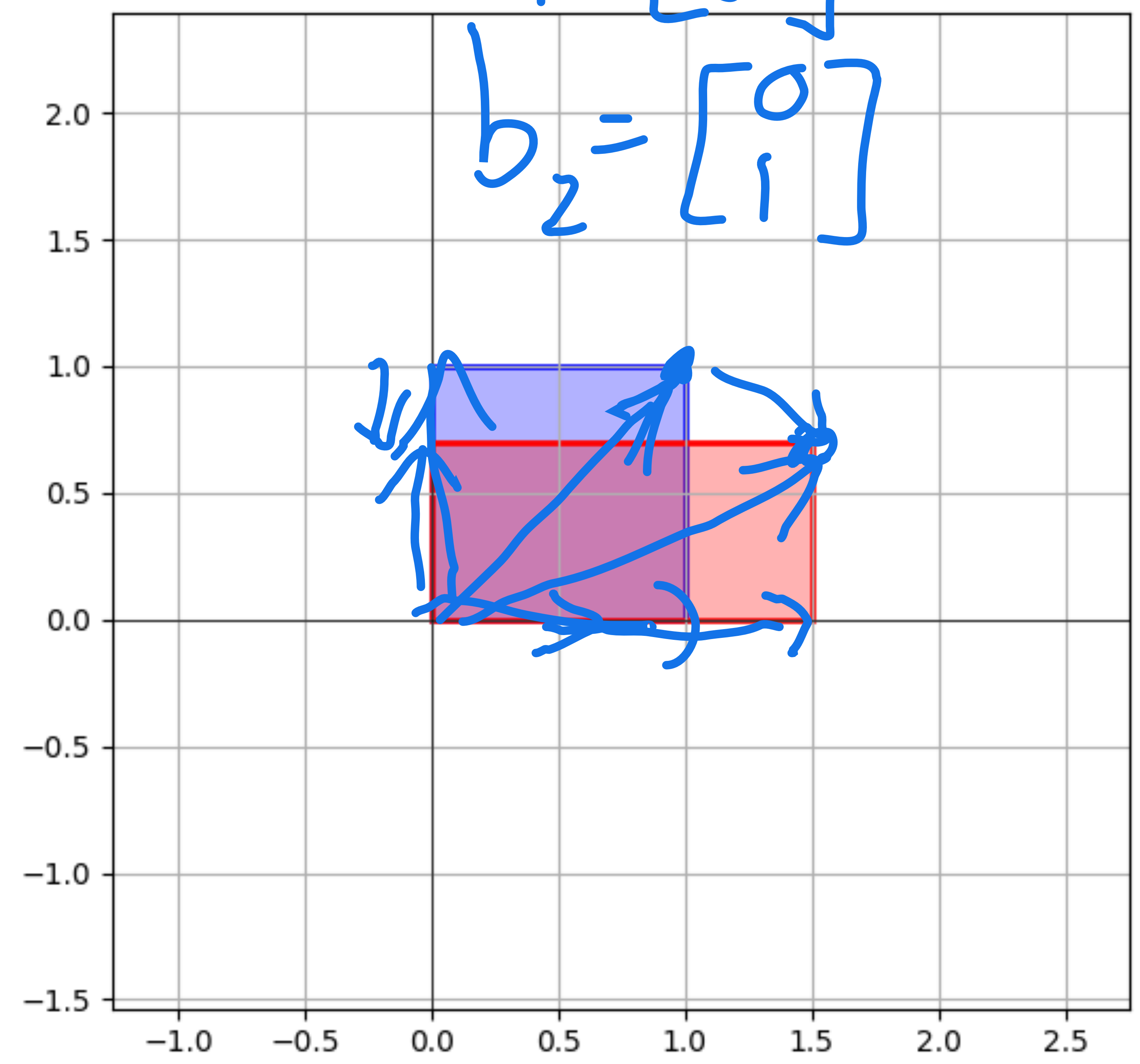
It transforms each entry individually and then combines them.

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = (1.5)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \text{''} & \text{''} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix} = (0.7)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Eigenbases (Informal)

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of* A. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

Eigenbases (Informal)

$\lambda_1, \lambda_2, \lambda_3$

Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of A*. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

It's "easy to describe" how A transforms $\mathbf{v}$.

It *transforms each "component" individually and then combines them.*

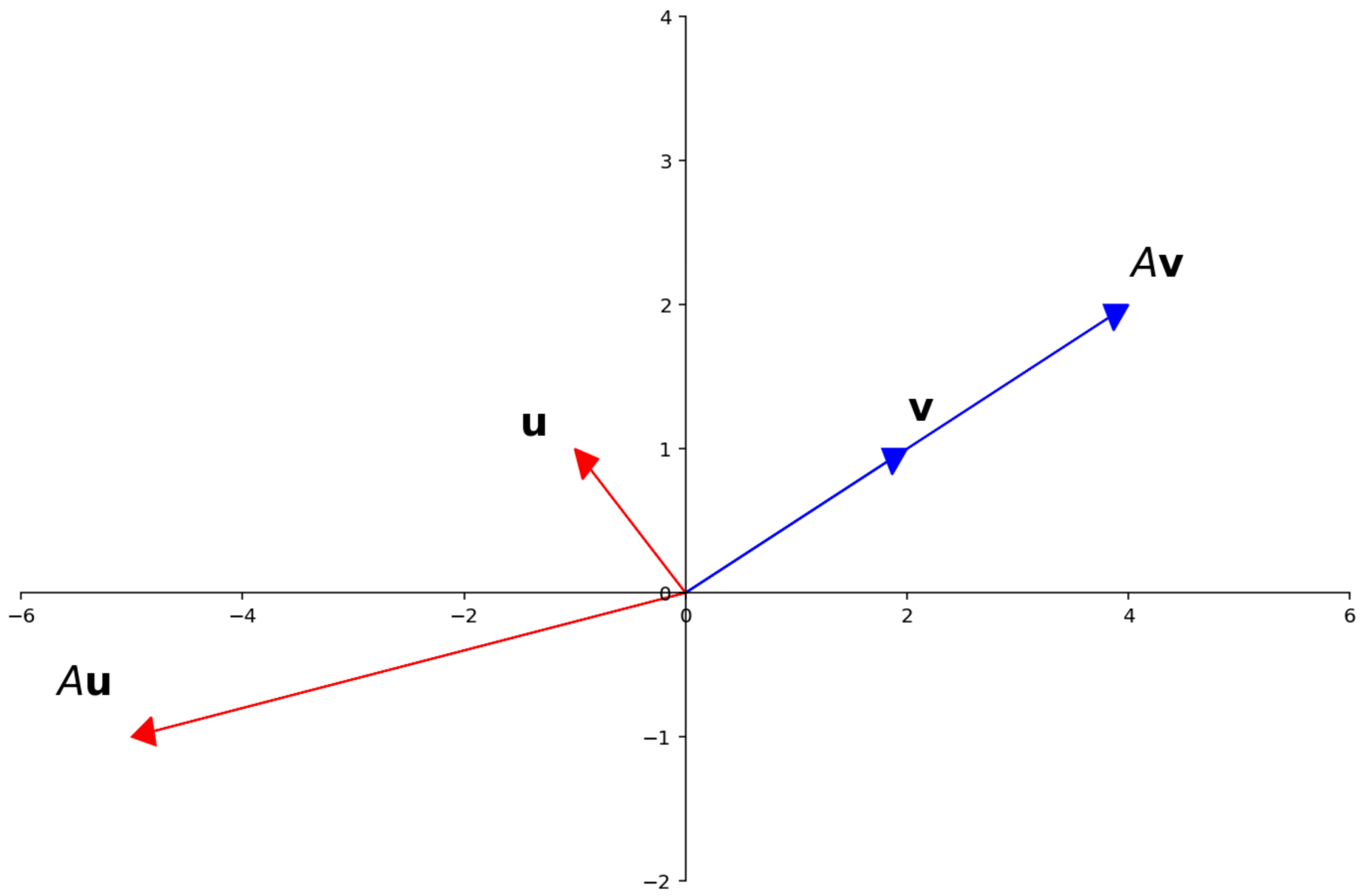
Verify: $A\vec{v} = A(2\vec{b}_1 - \vec{b}_2 - 5\vec{b}_3) = A(2\vec{b}_1) - A\vec{b}_2 - A(5\vec{b}_3)$

$= 2A\vec{b}_1 - A\vec{b}_2 - 5A\vec{b}_3$

$= 2\lambda_1\vec{b}_1 - \lambda_2\vec{b}_2 - 5\lambda_3\vec{b}_3$

Eigenvalues and Eigenvectors

Formal Definition



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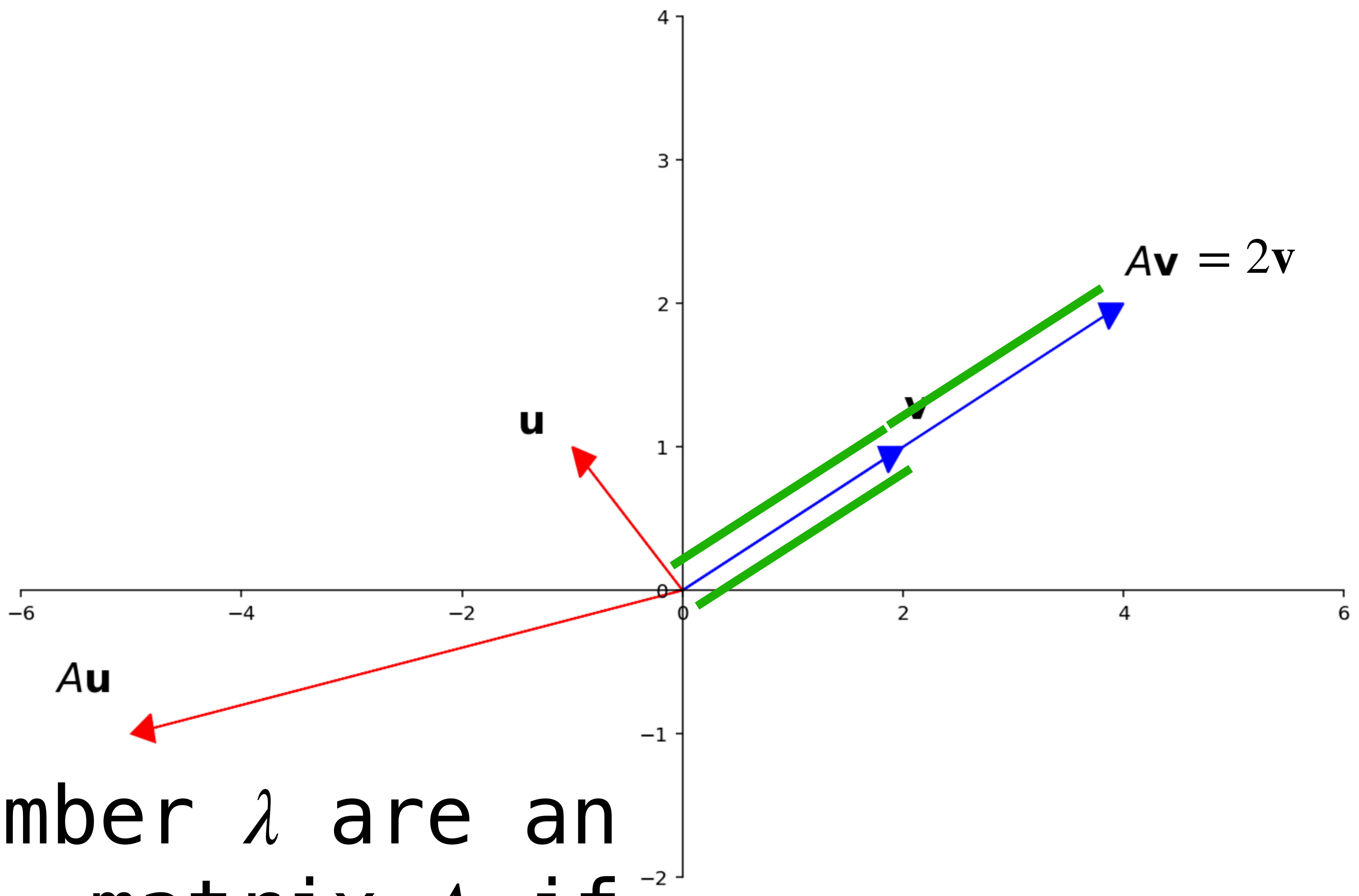

A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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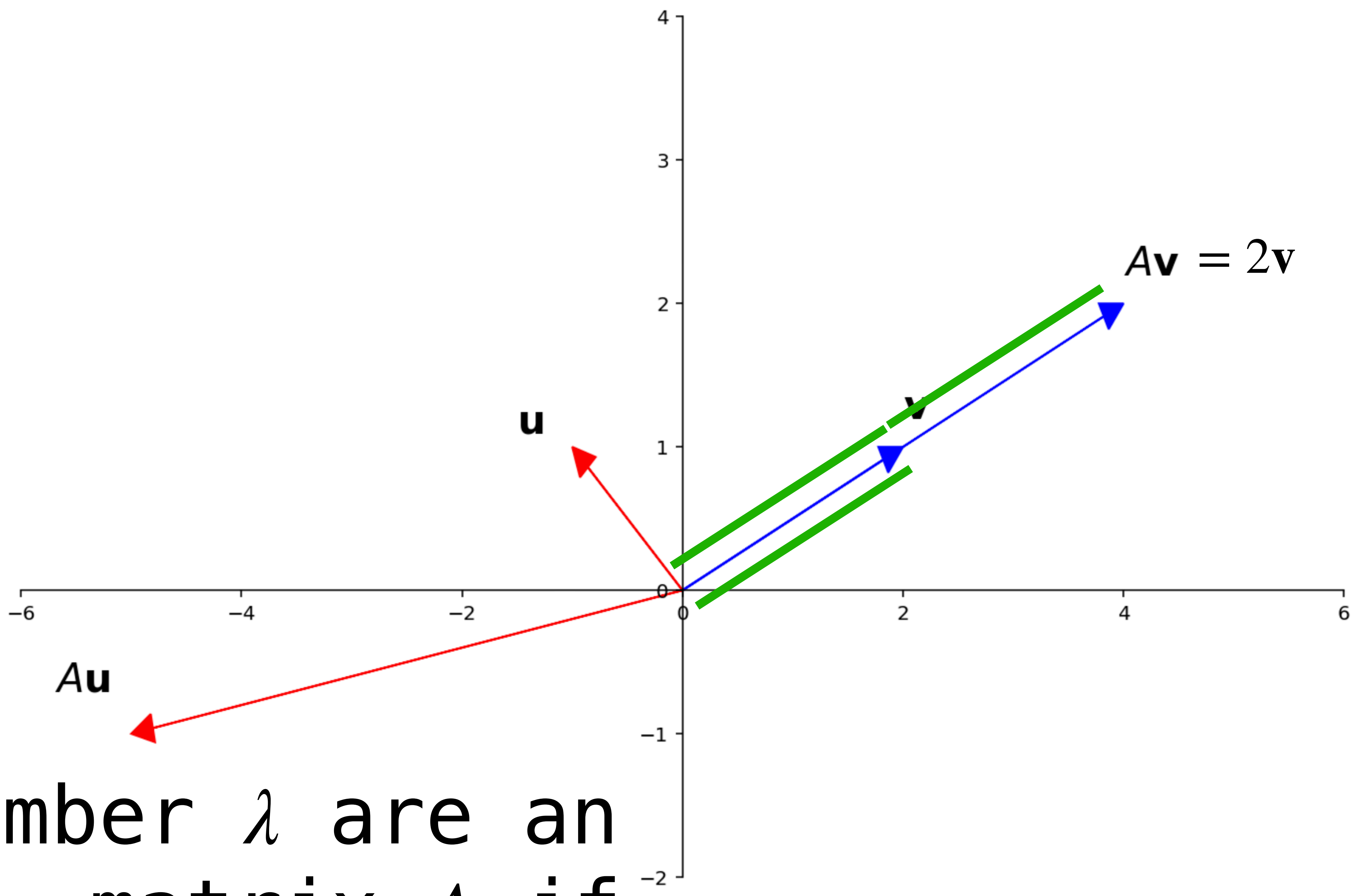

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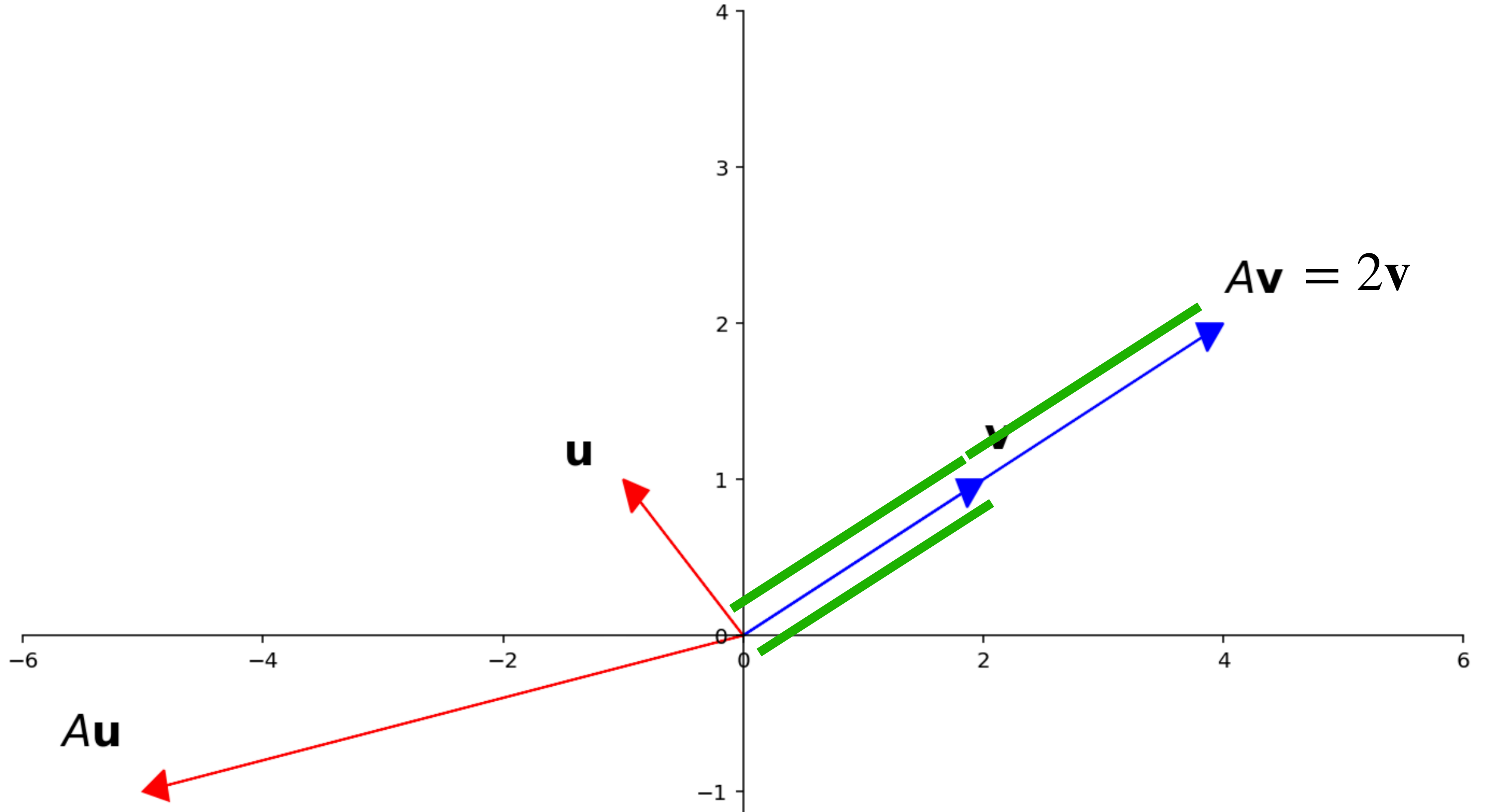
We will say that $\mathbf{v}$ is an eigenvector <u>of/for</u> the eigenvalue λ, and that λ is the eigenvalue <u>of/corresponding to</u> $\mathbf{v}$.

Formal Definition

$A\vec{0} = 0 \cdot \vec{0}$



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

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Note. Eigenvectors <u>must</u> be nonzero, but it is possible for 0 to be an eigenvalue.

What if 0 is an eigenvalue?

What if 0 is an eigenvalue?

If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

$$\{ \mathbf{v} \mid \underline{A\mathbf{v} = 0\mathbf{v} = \mathbf{0}} \} = \text{Null}(A)$$

What if 0 is an eigenvalue?

If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$$

In other words,

- $\mathbf{v} \in \mathsf{Nul}(A)$
- $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

Extending the IMT (Again)

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Theorem. A $n \times n$ matrix is invertible if and only if it does not have 0 as an eigenvalue.

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To reiterate. An eigenvalue 0 is equivalent to

Extending the IMT (Again)

Theorem. A $n \times n$ matrix is invertible if and only if it does not have 0 as an eigenvalue.

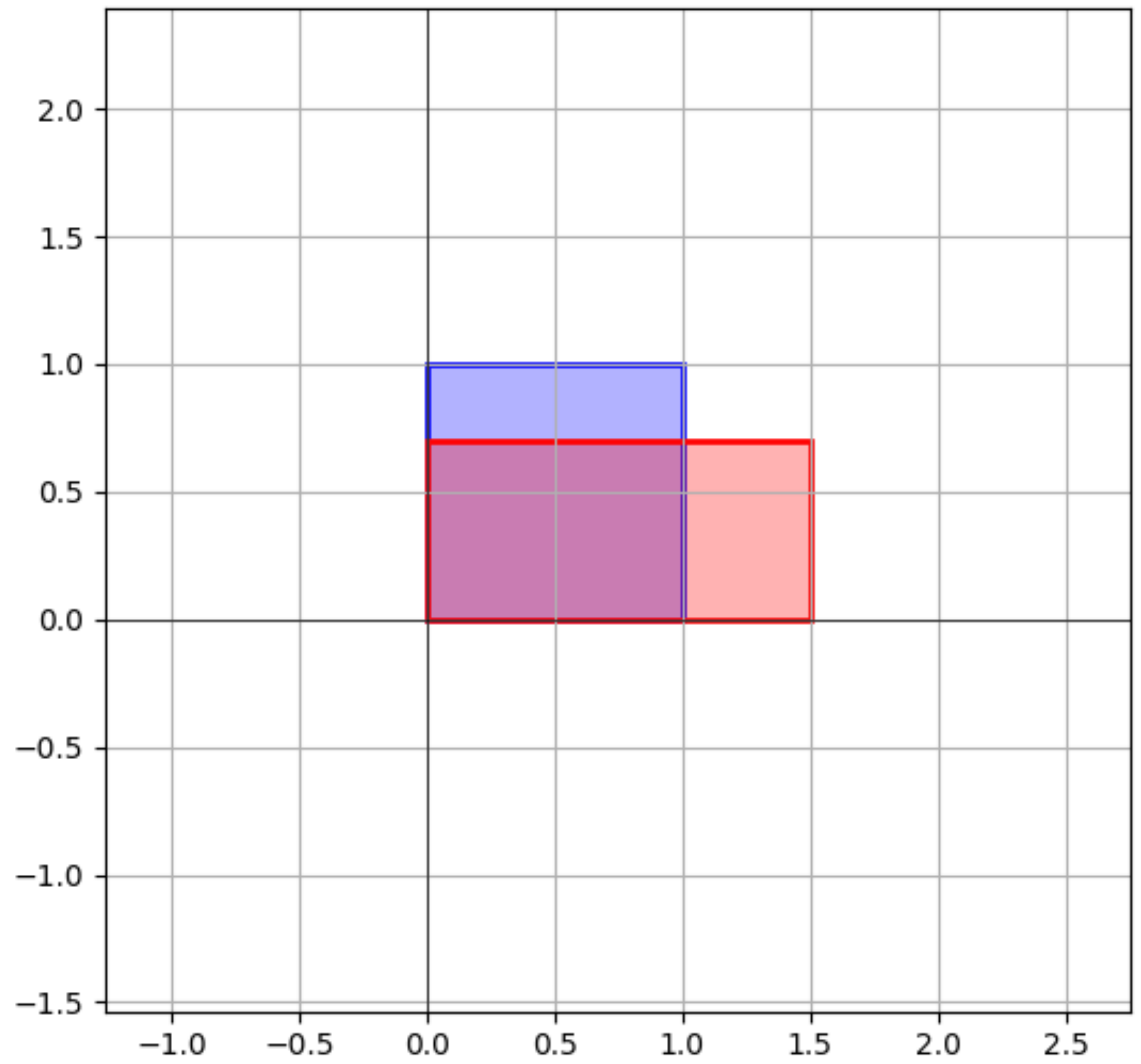
$\mathrm{Nul}\,A = \{0\}$

To reiterate. An eigenvalue 0 is equivalent to

- $A\mathbf{x} = \mathbf{0}$ has ~~no~~ nontrivial solutions
- the columns of A are linearly dependent
- $\mathrm{Col}(A) \neq \mathbb{R}^n$
- ...

Example: Unequal Scaling

Let's determine it's eigenvalues and eigenvectors:



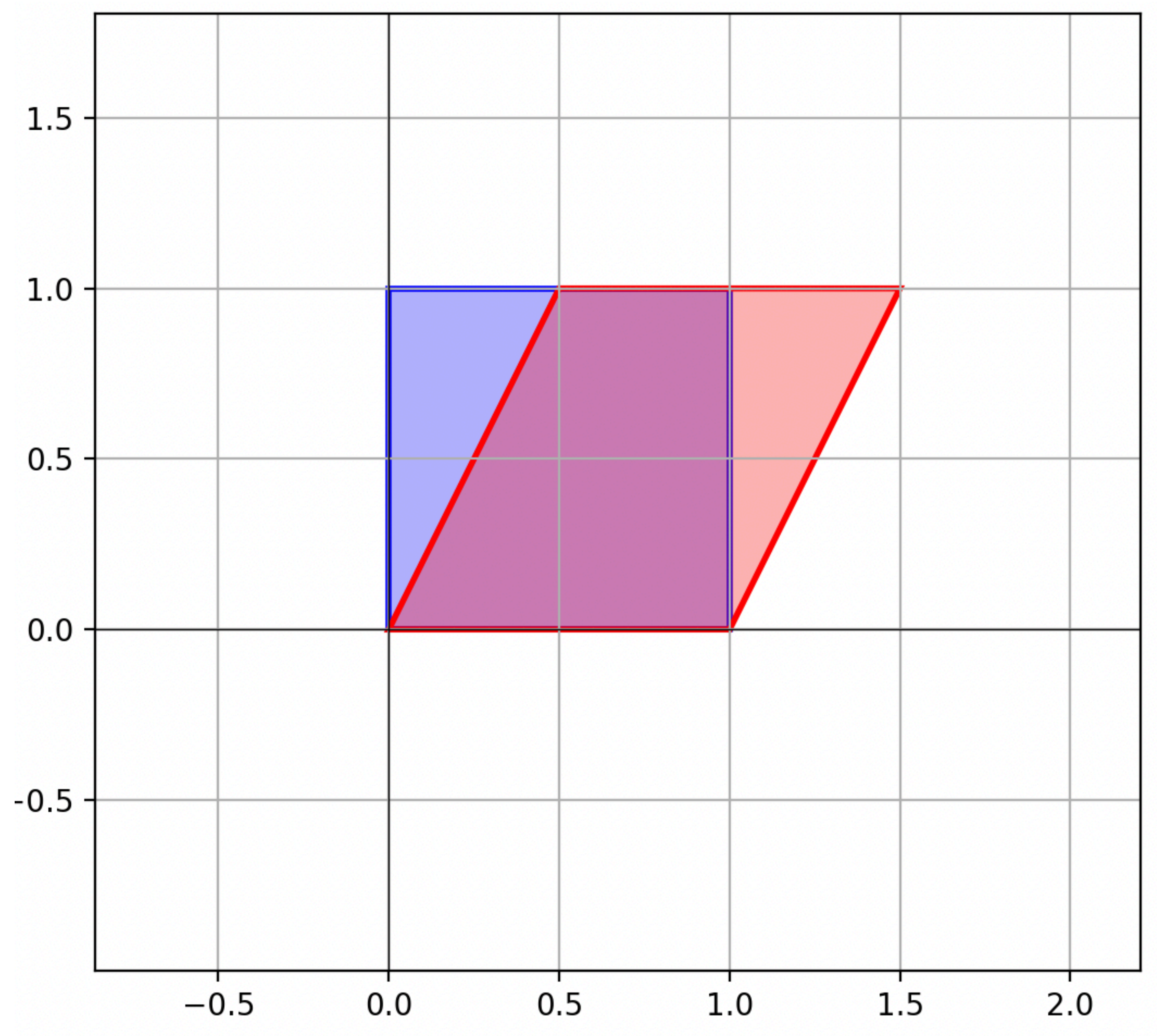
$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

1.5 0.7

Example: Shearing

Let's determine it's eigenvalues and eigenvectors:

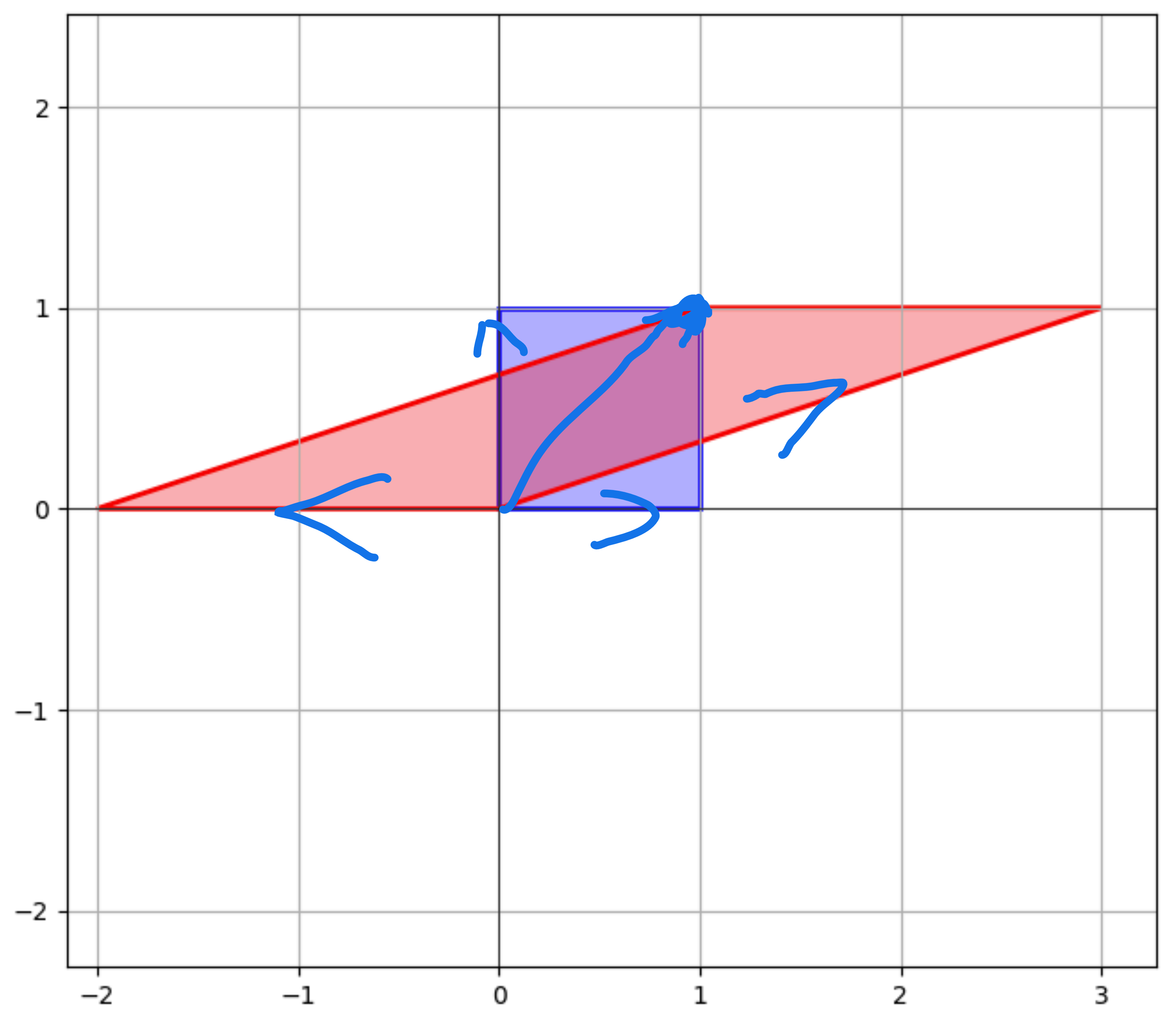
$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{(1)}_{\lambda} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$



$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \overset{\lambda_u}{(1)}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underset{\lambda_v}{2}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

How do we verify eigenvalues and eigenvectors?

Verifying Eigenvectors

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.* ask

Solution. Easy. Work out the matrix-vector multiplication.

Verifying Eigenvectors

$\vec{v}_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = \underset{\lambda_1}{(-4)}\begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$ $\Rightarrow \vec{v}_2$ not an eigenvector

Verifying Eigenvalues

Verifying Eigenvalues

`This is harder...`

Verifying Eigenvalues

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Question. Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

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What vector do we check???

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Before we go over how to do this...

Verifying Eigenvalues (Warm Up)

Question. Verify that 1 is an eigenvalue of

$$A = \begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

$(A - I)\vec{x} = 0$

Hint. Recall our discussion of Markov Chains.

Solution: A is regular $\Rightarrow$ there is a unique steady state

$$A\vec{x} = \vec{x} = (1)\vec{x}$$

$$\Leftrightarrow A\vec{x} - \vec{x} = 0 \Leftrightarrow (A - I)\vec{x} = 0$$

Steady-States and Eigenvectors

Steady-state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue 1.

How did we find steady-state vectors?:

look for (nontrivial) sol'ns to $(A - I)\vec{x} = 0$

Steady-States and Eigenvectors

$$\mathbf{v} \text{ is a steady-state vector}^{*} \equiv \quad \mathbf{v} \in \mathsf{Nul}(A - I)$$

$*$It must also be a probability vector

Verifying Eigenvalues

This is harder...

Question. Show that λ is an eigenvalue of A.

(there exists $\vec{v} \neq 0$)

Solution:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

$$\vec{v} \in \mathrm{Nul}(A - \lambda I)$$

Verifying Eigenvalues

$\mathbf{v}$ `is an eigenvector for` $\lambda \quad \equiv \quad \mathbf{v} \in \mathrm{Nul}(A - \lambda I)$

(and $\vec{v} \neq 0$)

if just $\{0\}$

λ not an eigenvalue

Verifying Eigenvalues

$(A - 7I)\vec{x} = 0$

This is harder...

Question. Show that $\overset{\lambda}{\overset{\shortparallel}{7}}$ is an eigenvalue of $\overset{A}{\overset{\shortparallel}{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}}}$.

Solution: $A - 7I = \begin{bmatrix} -6 & 6 \\ 6 & -5 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = x_2$

x_2 free

Yes!

Nonzero sol'ns

$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem

Ask

$(A-2I)\vec{x}=0$

$A\vec{x}=2\vec{x}$

Verify that $\underset{\lambda}{2}$ is an eigenvalue of $\underset{A}{\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}}$

$$A-2I=\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2x_1 - x_2 + 6x_3 = 0$$

$$2x_1 = x_2 - 6x_3$$

$$x_1 = \left(\tfrac{1}{2}\right)x_2 - 3x_3$$

x_2 free

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x} = x_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

How many eigenvectors can a matrix have?

Linear Independence of Eigenvectors

Theorem.* If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

So an $n \times n$ matrix can have at most n eigenvalues.

Why?: more than n eigenvalues ⇒ more than n lin. ind. eigenvectors
not possible

*We won't prove this.

Eigenspace

Fact. The set of eigenvectors for a eigenvalue λ of $A \in \mathbb{R}^{n\times n}$ form a subspace of $\mathbb{R}^n$.

Verify: $\mathrm{Nul}(A-\lambda I)$

closure under add'n: $\vec{v}, \vec{w}$ eigenvectors

$$A(\vec{v}+\vec{w}) = A\vec{v}+A\vec{w} = \lambda\vec{v}+\lambda\vec{w}$$
$$= \lambda(\vec{v}+\vec{w})$$

closure under scaling: $\vec{v}$ eigenvector

$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$

Eigenspace

Definition. The set of eigenvectors for a eigenvalue λ of A is called the **eigenspace** of A corresponding to λ.

It is the same as $\mathrm{Nul}(A - \lambda I)$.

How To: Basis of an Eigenspace

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\mathrm{Nul}(A - \lambda I)$.

We know how to do this.

Example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -1 \\ -4 & 0 & 5 \end{bmatrix}$$

$\vec{x} \in \mathrm{Nul}(A - I)$

$(A - I)\vec{x} = 0$

λ

Determine a basis for the eigenspace corresponding to the eigenvalue 1*:*

$$A - I = \begin{bmatrix} -3 & 0 & 3 \\ 1 & 0 & -1 \\ -4 & 0 & 4 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_3$

x_2 free $\Leftrightarrow x_2 = x_2$

x_3 free $\Leftrightarrow x_3 = x_3$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

How do we find eigenvalues?

How do we find eigenvalues?

We'll cover this next time...

Eigenvalues of Triangular Matrices

Theorem. The eigenvalues of a triangular matrix are its entries along the diagonal.

Verify:

(Space)

Example

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Determine the eigenvectors and values of the above matrix:

Linear Dynamical Systems

Recall: Linear Dynamical Systems

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Definition. A **(discrete time) linear dynamical system** is described by a $n \times n$ matrix A. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after i time steps:

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A tells us how our system evolves over time.

Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after i time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

Recall: State Vectors

$$
\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}
$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number k time steps

This is also called a *recurrence relation* or a *linear difference function*

Recall: State Vectors

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0)$$

$$\boxed{\mathbf{v}_k = A^k\mathbf{v}_0}$$

$$\mathbf{v}_5 = A\mathbf{v}_4 = A(AAAA\mathbf{v}_0)$$

$$\vdots$$

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It's also difficult computationally because matrix multiplication is expensive

(Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

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In other word, it does not depend on A^k and is not **recursive**

Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed for the above linear dynamical system.

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

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No dependence on A^k or $\mathbf{v}_{k-1}$

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Solutions with Eigenvectors as Initial States

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No dependence on A^k or $\mathbf{v}_{k-1}$

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

<u>The Key Point.</u> This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of $\mathbb{R}^2$:

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows <u>exponentially in λ_1</u>).

Verify:

Eigenbases

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Definition. An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix A is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of A.

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*We can represent vectors as **unique** linear combinations of eigenvectors.*

Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

Another Example: Golden Ratio

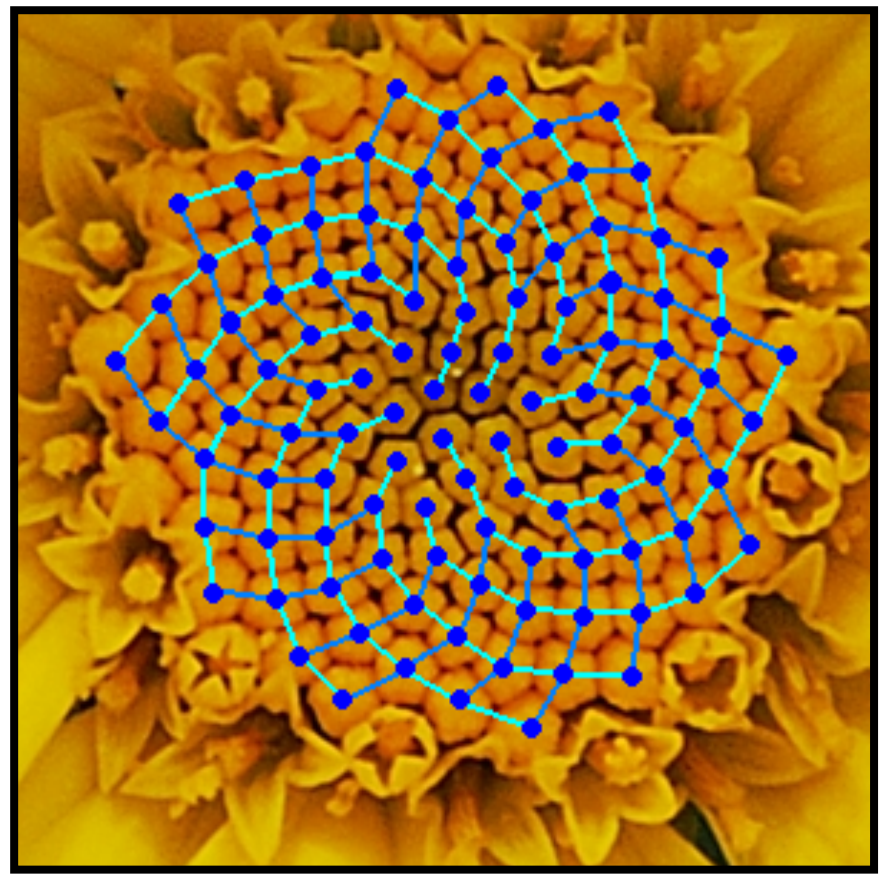
A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the system given by the above matrix.

What does this matrix represent?:

Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature.

https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG

Golden Ratio

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

`The "long term behavior" is the Fibonacci sequence is defined by the golden ratio.`

`This is the largest eigenvalue of` $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**