

# Vector Equations

**Geometric Algorithms**  
**Lecture 4**

# Practice Problem

*Suppose that  $A$  is a  $322 \times 245$  augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that  $A$  can have?*

# Answer

2 4 5

2 4 4

1 2 2

2 4 3

$A \in \mathbb{R}^{10 \times 5}$   
 sy. matrix with  
 inf. many sol.

3

$A \in \mathbb{R}^{10 \times 5}$  and  
 A is aug. matrix  
 consistent system

$$\left[ \begin{array}{ccc|c} 2 & 4 & 5 & 2 \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$0 = 1$

$A \in \mathbb{R}^{10 \times 5}$

rows cols.

4, 9, 10

5

$\min(m, n)$

4

$$\left[ \begin{array}{c} \boxed{1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

# Outline

- » Formally define vectors
- » Discuss vector operations and vector algebra
- » Draw the connection between vectors and systems of linear equations

# Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

# **Motivation (An Aside)**

# Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all  $n$

# Changing Perspective

$$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & - & 1 \\ & 9 & 9 & 9 & 9 & & \end{array} =$$

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100\dots000 - 000\dots001 = 011\dots111$$

show that this holds for all  $n$



# Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

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show that this holds for all  $n$

much easier in binary

# Motivation?

vectors will be one of the most important  
shifts of perspective in this course

the insight is simple yet elegant

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maybe I'm reaching...

# Big Data (More Practical Motivation)

A piece of data is a bunch of distinct values  
(numbers)

How can we tell if two piece of data are similar?

Maybe if they are **close together** in a geometric  
sense

# A Note on Algebra

$$\mathbf{v} = \mathbf{w}$$

$$\mathbf{v} + \mathbf{w}$$

$$a\mathbf{v}$$

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In programming an *interface* is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

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$$a\mathbf{v}$$

We're defining a new thing called a *column vector*

We need to define what **equality** and **adding** and **multiplying by a number** means for column vectors

# Vectors

# What is a vector (in $\mathbb{R}^n$ )?

- A. an  $n$ -tuple of real numbers
- B. a point in  $\mathbb{R}^n$
- C. a 1-column matrix with real values
- D. all of the above
- E. none of the above?

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E. none of the above?

it's common to conflate points and vectors

# Column Vectors

**Definition.** a *column vector* is a matrix with a single column, e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathbb{R}^n$$

# A Note on Matrix Size

an  $(m \times n)$  matrix is a matrix with  $m$  rows and  $n$  columns

$$\begin{array}{c} m \end{array} \left[ \begin{array}{ccccc} * & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \\ * & * & \dots & * & * \end{array} \right]$$

A diagram of a general matrix with  $m$  rows and  $n$  columns. A blue vertical line to the left of the matrix is labeled  $m$ . A red horizontal line above the matrix is labeled  $n$ . The matrix elements are represented by asterisks and ellipses.

$$4 \left[ \begin{array}{c} 2 \\ 3 \\ 0.1 \\ -2 \end{array} \right]$$

A diagram of a specific matrix with 4 rows and 1 column. A blue vertical line to the left of the matrix is labeled 4. A red horizontal line above the matrix is labeled 1. The matrix elements are 2, 3, 0.1, and -2.



# A Note on Matrix Size

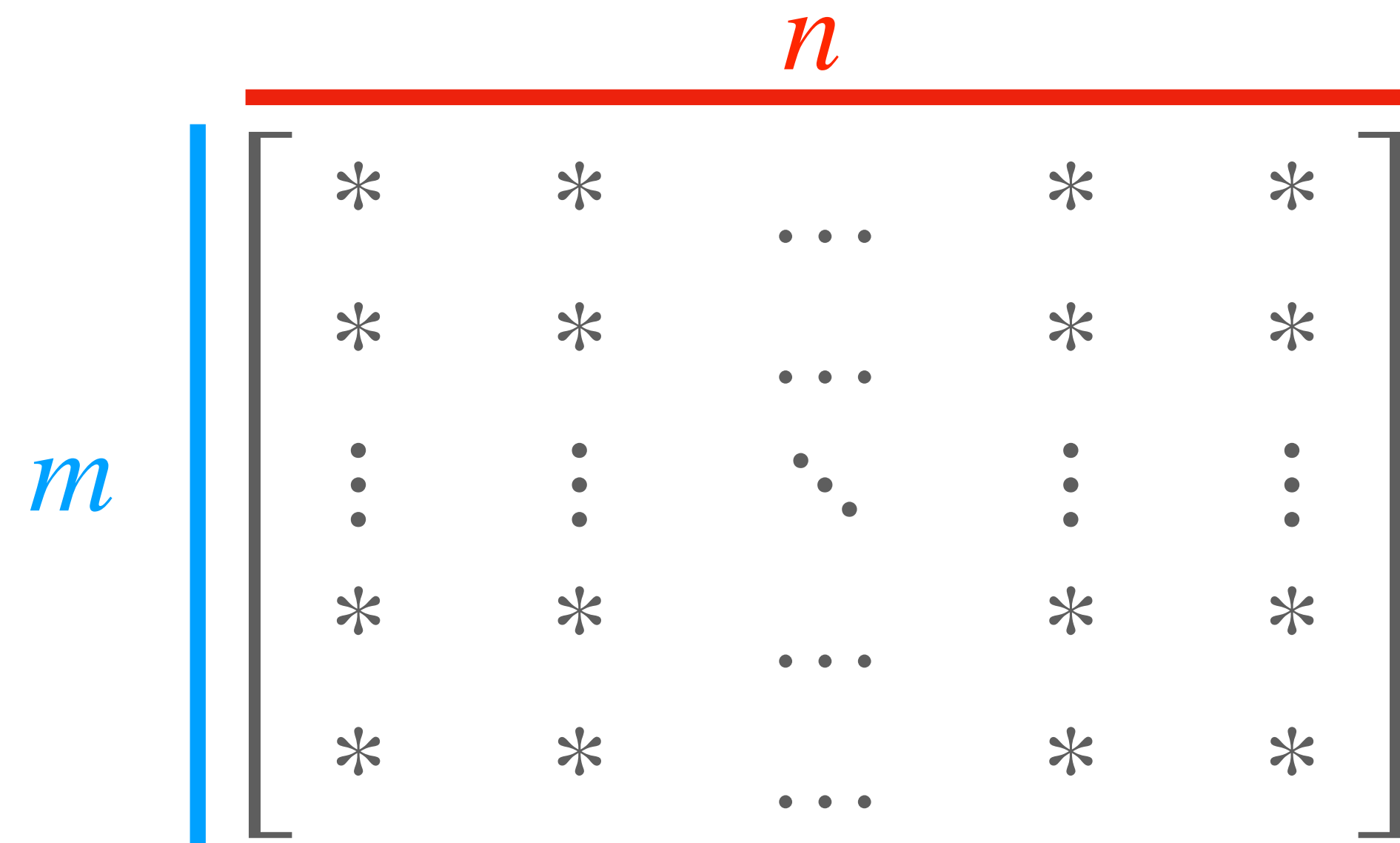
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$$\begin{array}{c} m \\ \left[ \begin{array}{ccccc} * & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \\ * & * & \dots & * & * \end{array} \right] \end{array} \overset{n}{\in \mathbb{R}^{m \times n}}$$
$$\begin{array}{c} 4 \\ \left[ \begin{array}{c} 2 \\ 3 \\ 0.1 \\ -2 \end{array} \right] \end{array} \overset{1}{\in \mathbb{R}^{4 \times 1} = \mathbb{R}^4}$$

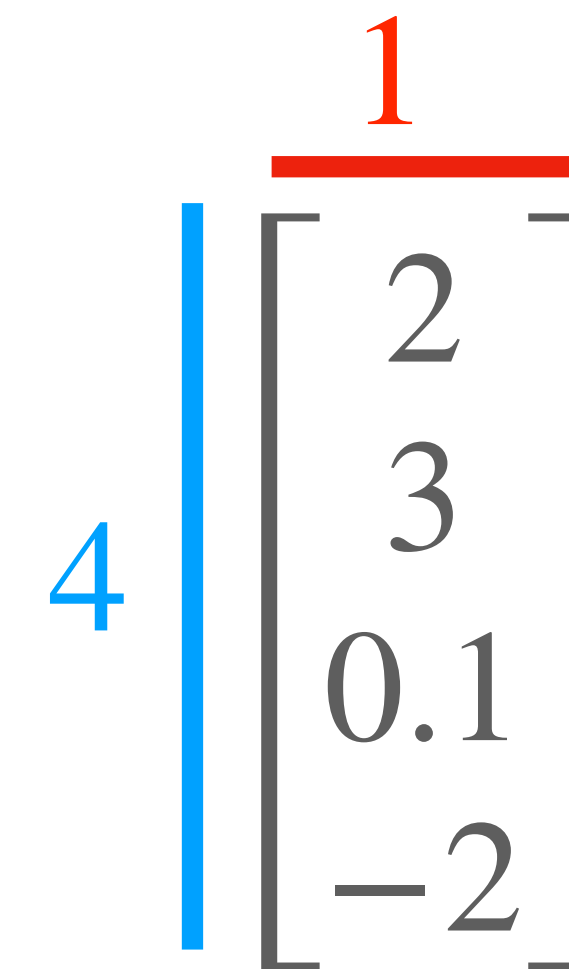
$\mathbb{R}^{m \times n}$  is set of matrices with  $\mathbb{R}$  entries

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A diagram of a general matrix with  $m$  rows and  $n$  columns. A blue vertical line to the left of the matrix is labeled  $m$ . A red horizontal line above the matrix is labeled  $n$ . The matrix is enclosed in large square brackets and contains asterisks (\*) and ellipses (...) representing its entries.



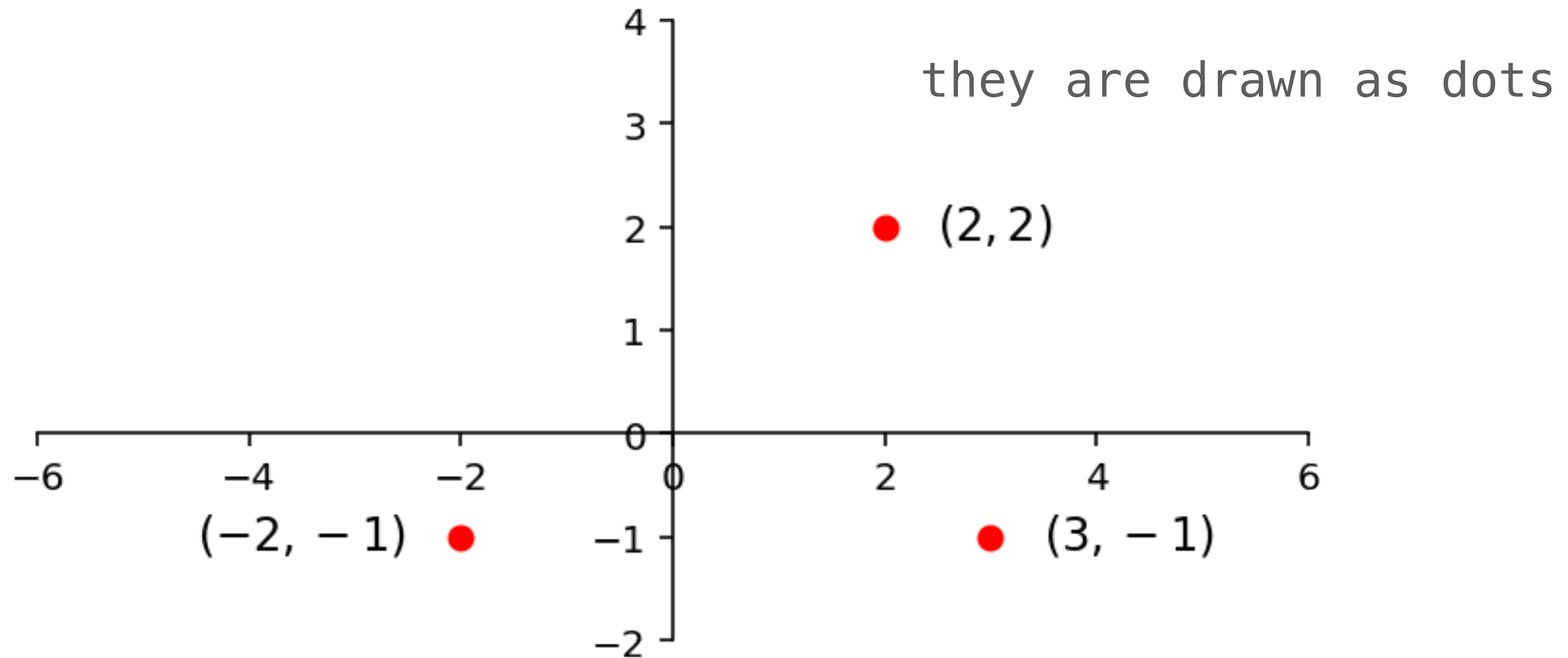
A diagram of a vector with 4 rows and 1 column. A blue vertical line to the left of the vector is labeled 4. A red horizontal line above the vector is labeled 1. The vector is enclosed in large square brackets and contains the values 2, 3, 0.1, and -2.

the number of rows  
of a vectors is  
called its **dimension**

$\mathbb{R}^{m \times n}$  is set of matrices with  $\mathbb{R}$  entries

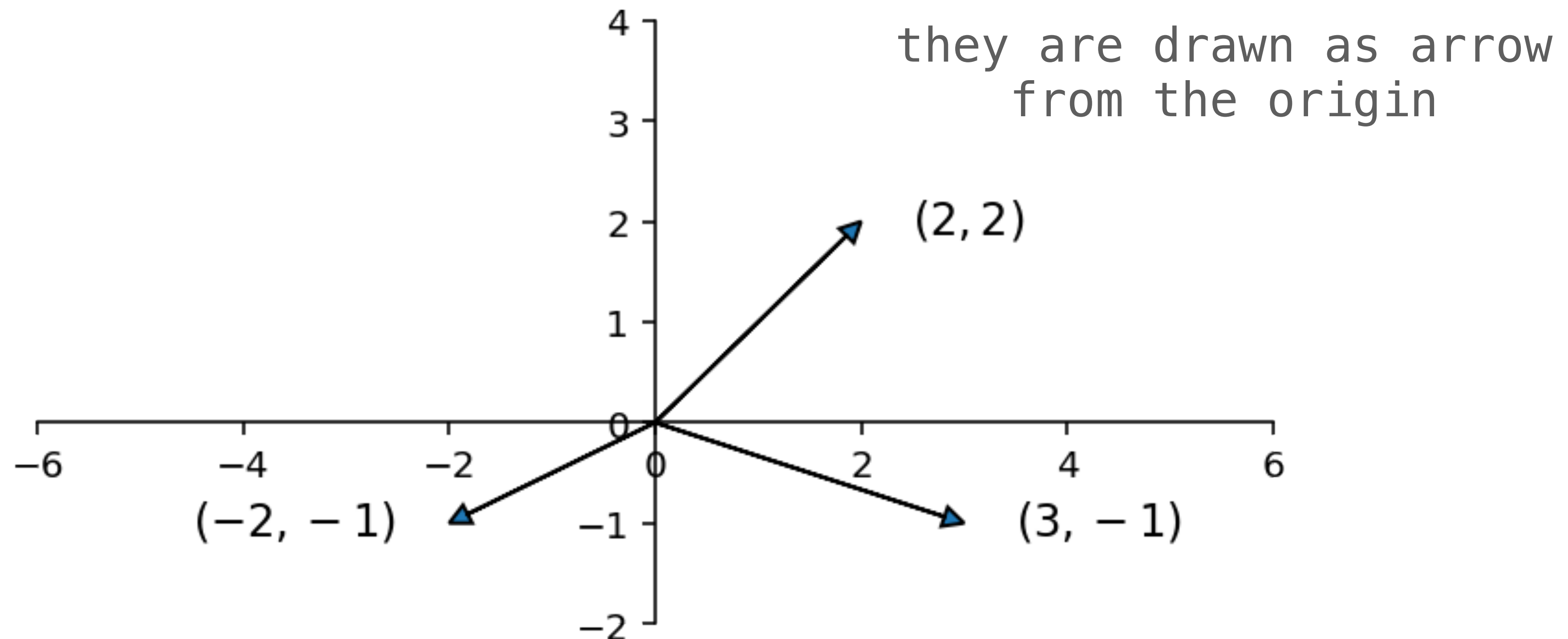
# Examples

# Notation (Points)



points in  $\mathbb{R}^2$  are notated as  $(a, b)$

# Notation (Vectors)



vectors in  $\mathbb{R}^2$  are notated as  $\begin{bmatrix} a \\ b \end{bmatrix}$

# Notation (Looking ahead)

we will often write  $[a_1 \ a_2 \ \dots \ a_n]^T$  for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$[a_1 \ a_2 \ \dots \ a_n] \in \mathbb{R}^{1 \times n}$$

"row vector"

**!!IMPORTANT!!**

$(a_1, a_2, \dots, a_n)$  is not the same as  $[a_1 \ a_2 \ \dots \ a_n]$

# Vector Operations

# Vector "Interface"



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What properties do they need to satisfy?

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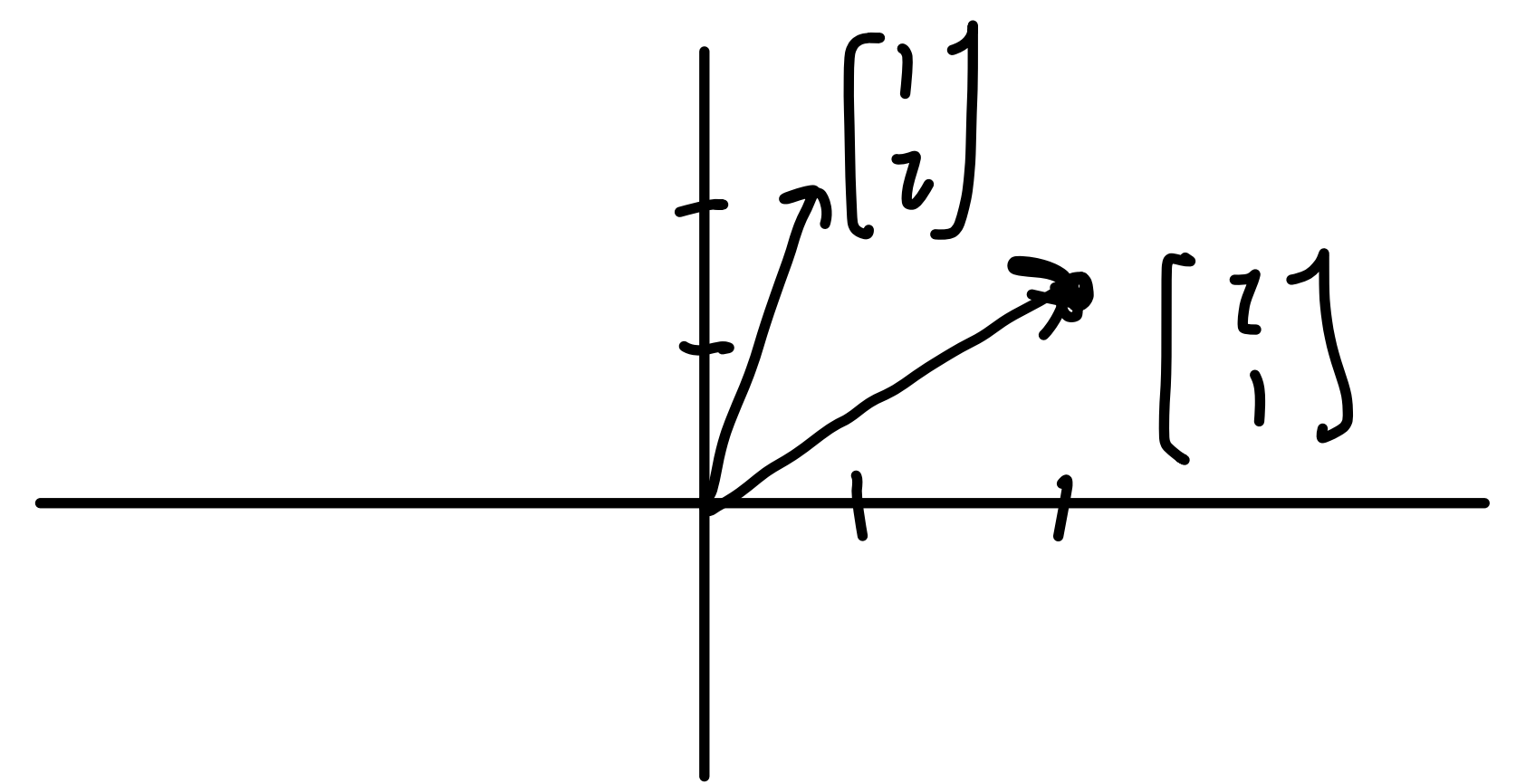
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# Vector Equality



two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

**!!IMPORTANT!!**  
**ORDER MATTERS**

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

# Vector Equality

$$\in \mathbb{R}^{n \times 1}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is the same as

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ &\vdots \\ a_n &= b_n \end{aligned}$$

$$[a_1 \ \dots] \in \mathbb{R}^{1 \times n}$$

~~$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$~~   
not a question

# Examples

$$\begin{bmatrix} 1 \\ - \\ 2 \\ - \end{bmatrix} = \begin{bmatrix} 1 \\ - \\ 2 \\ - \end{bmatrix}$$



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# Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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**!! IMPORTANT !!**

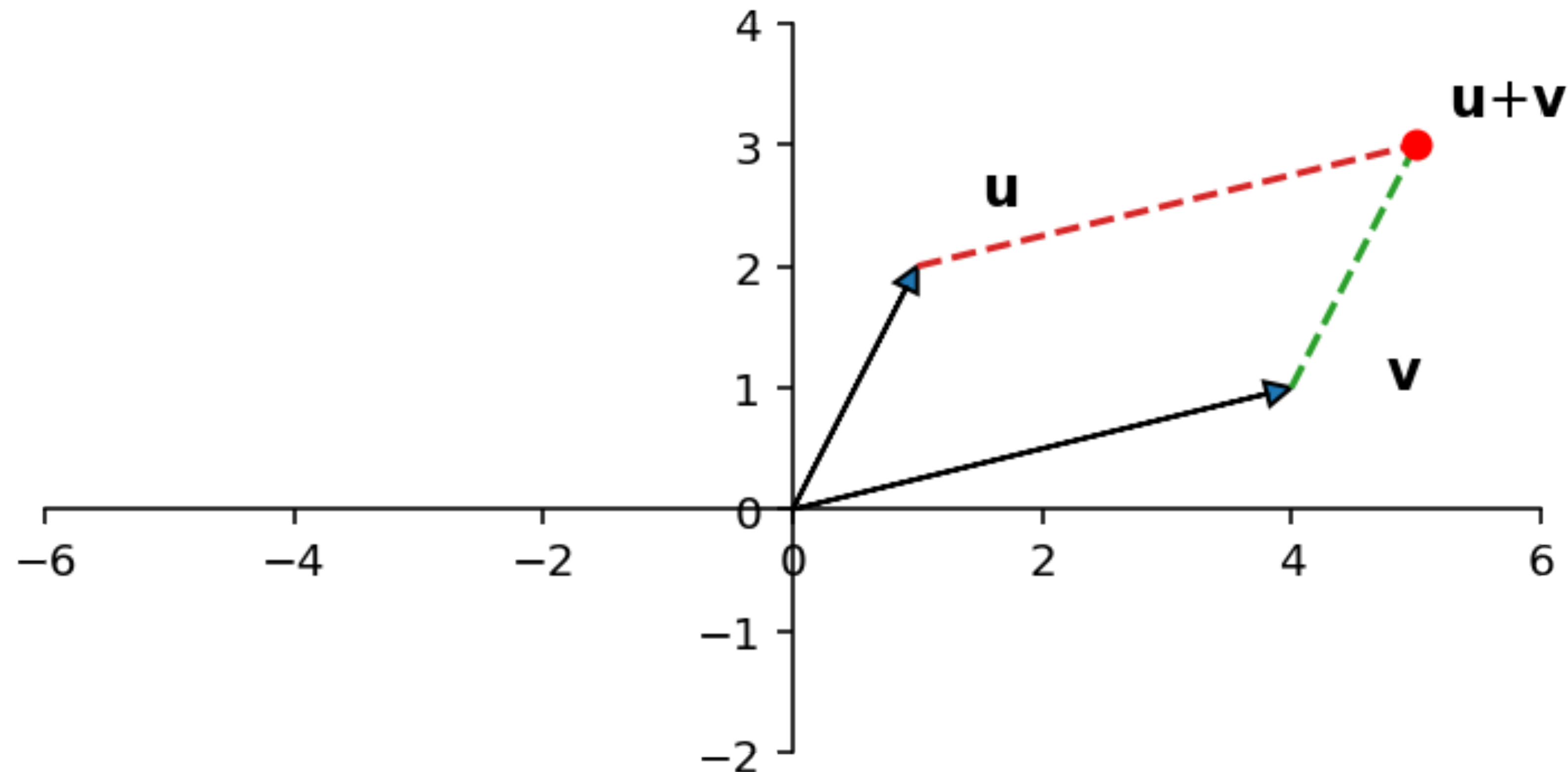
**WE CAN ONLY ADD VECTORS OF THE SAME SIZE**

# Examples

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

# Vector Addition (Geometrically)

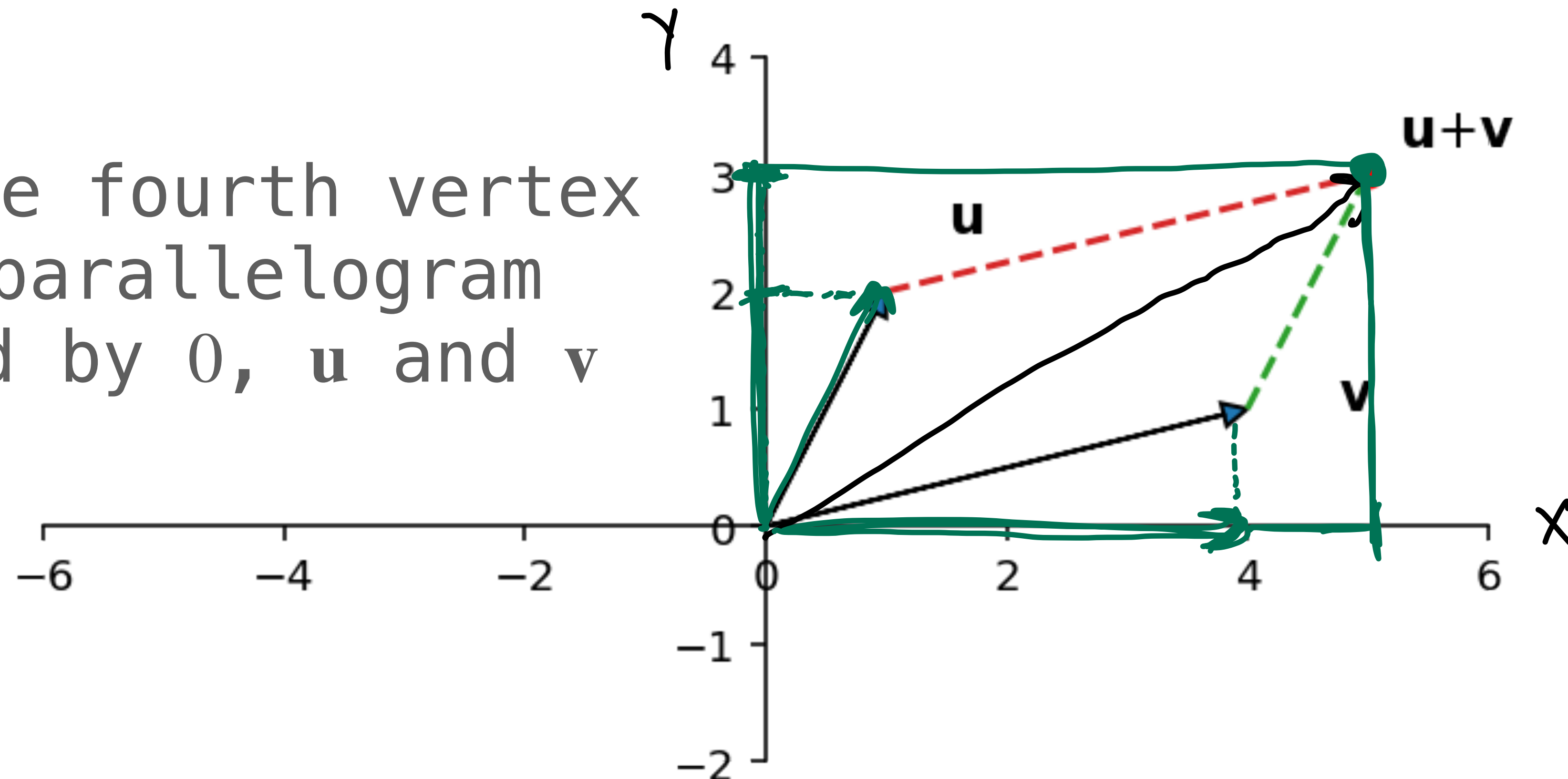
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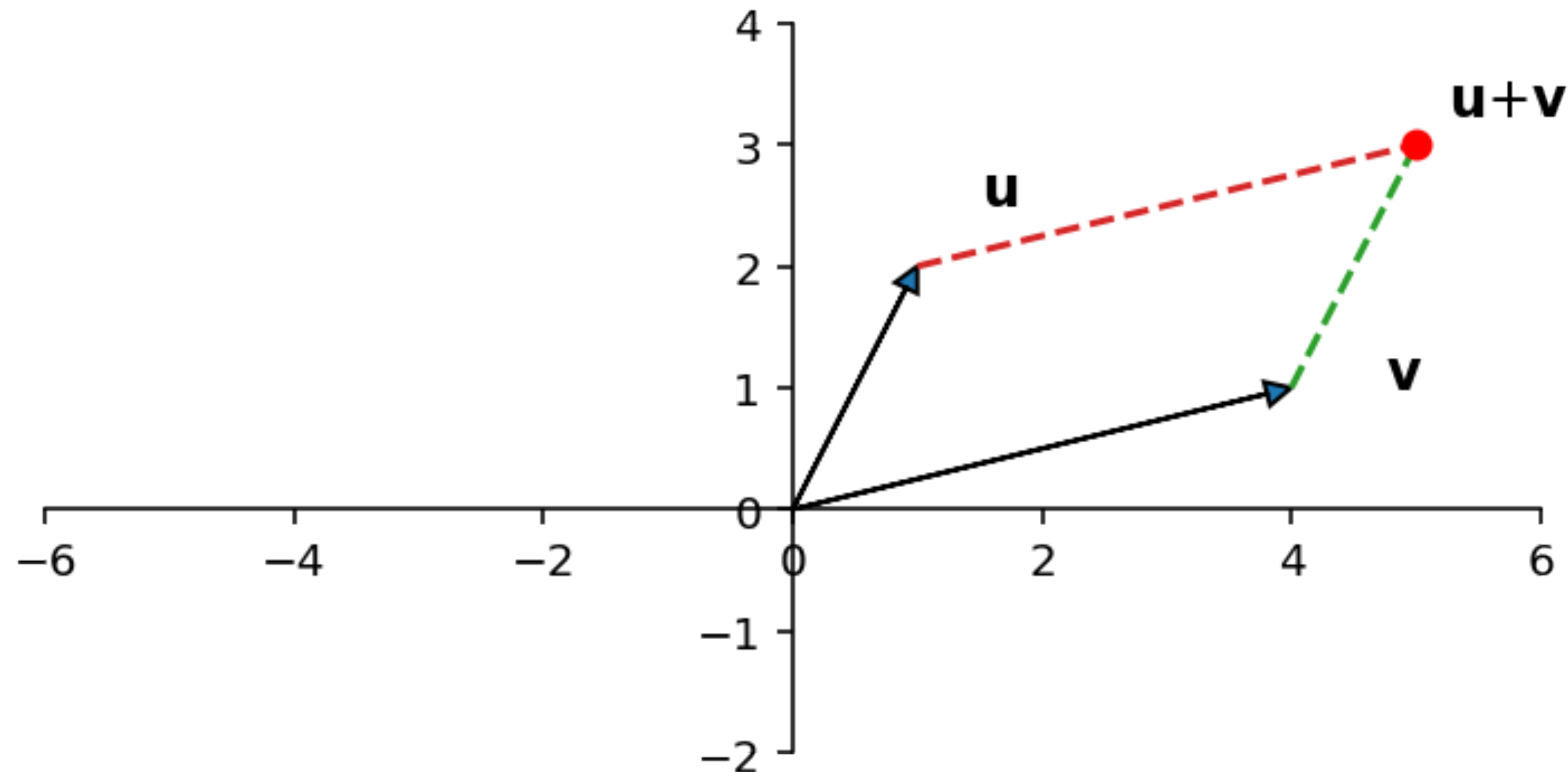
in  $\mathbb{R}^2$  it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$  is the fourth vertex  
of the parallelogram  
generated by  $\mathbf{0}$ ,  $\mathbf{u}$  and  $\mathbf{v}$



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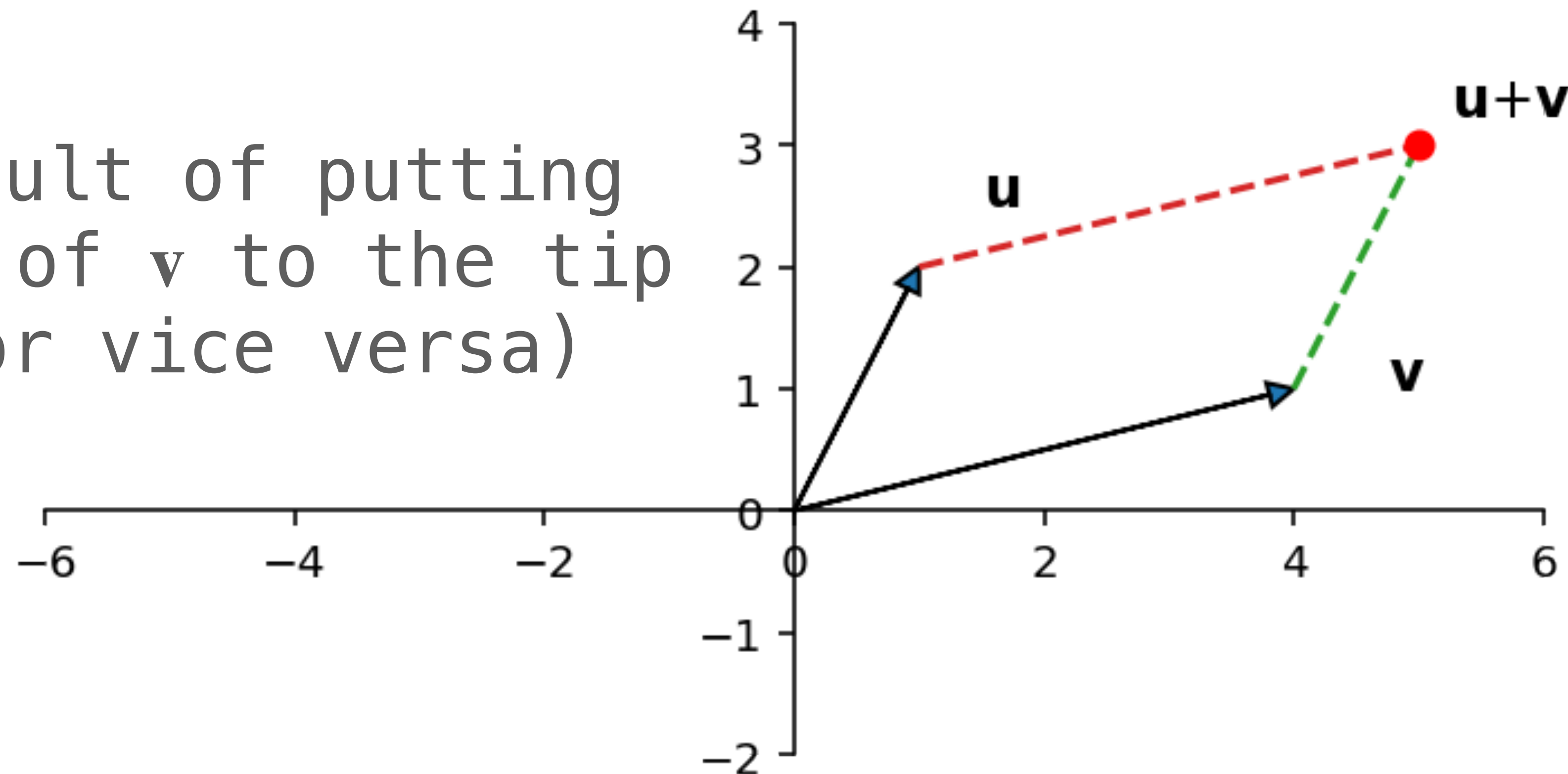
or the *tip-to-tail rule*



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$\mathbf{u} + \mathbf{v}$  result of putting  
the tail of  $\mathbf{v}$  to the tip  
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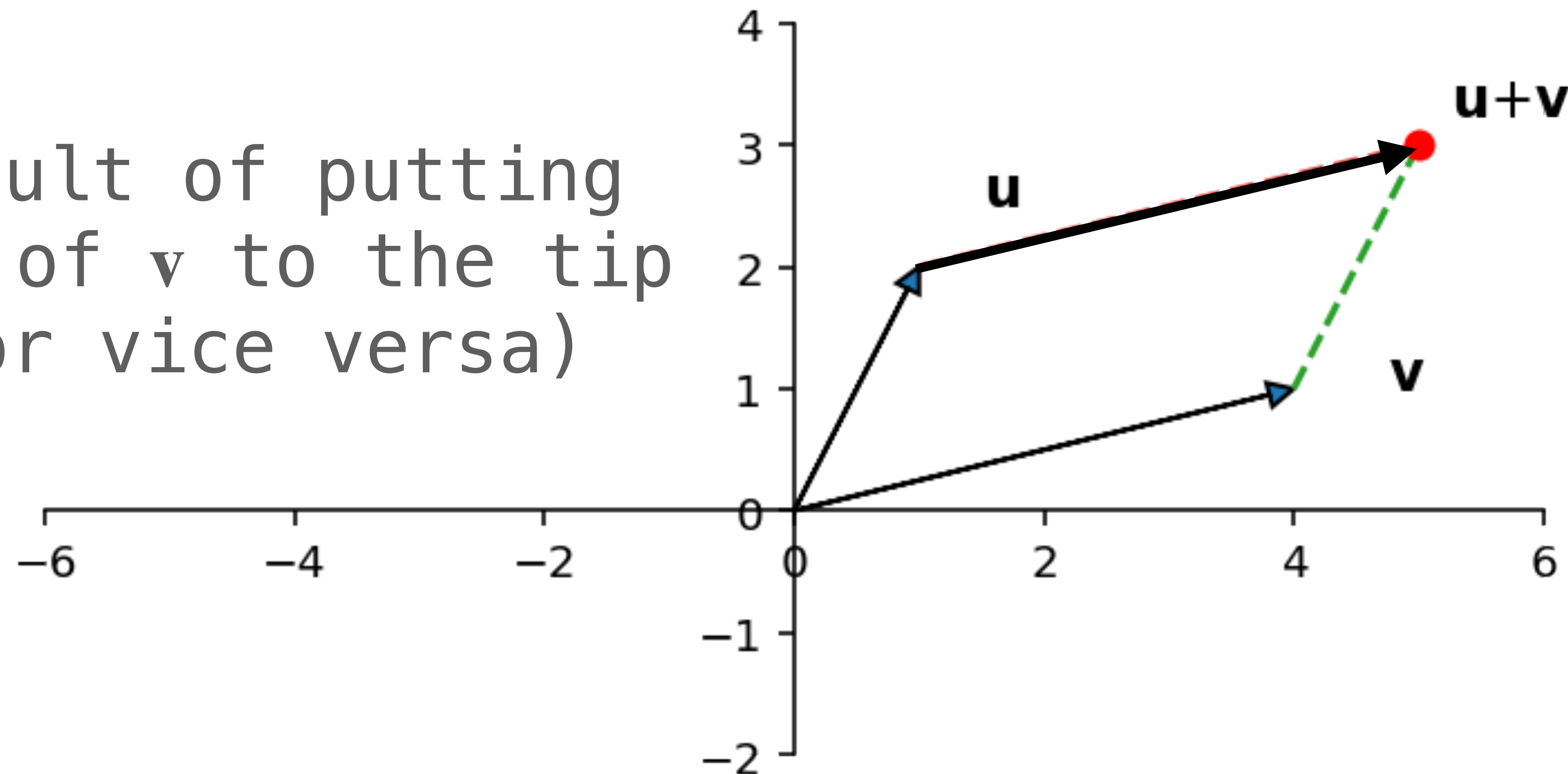




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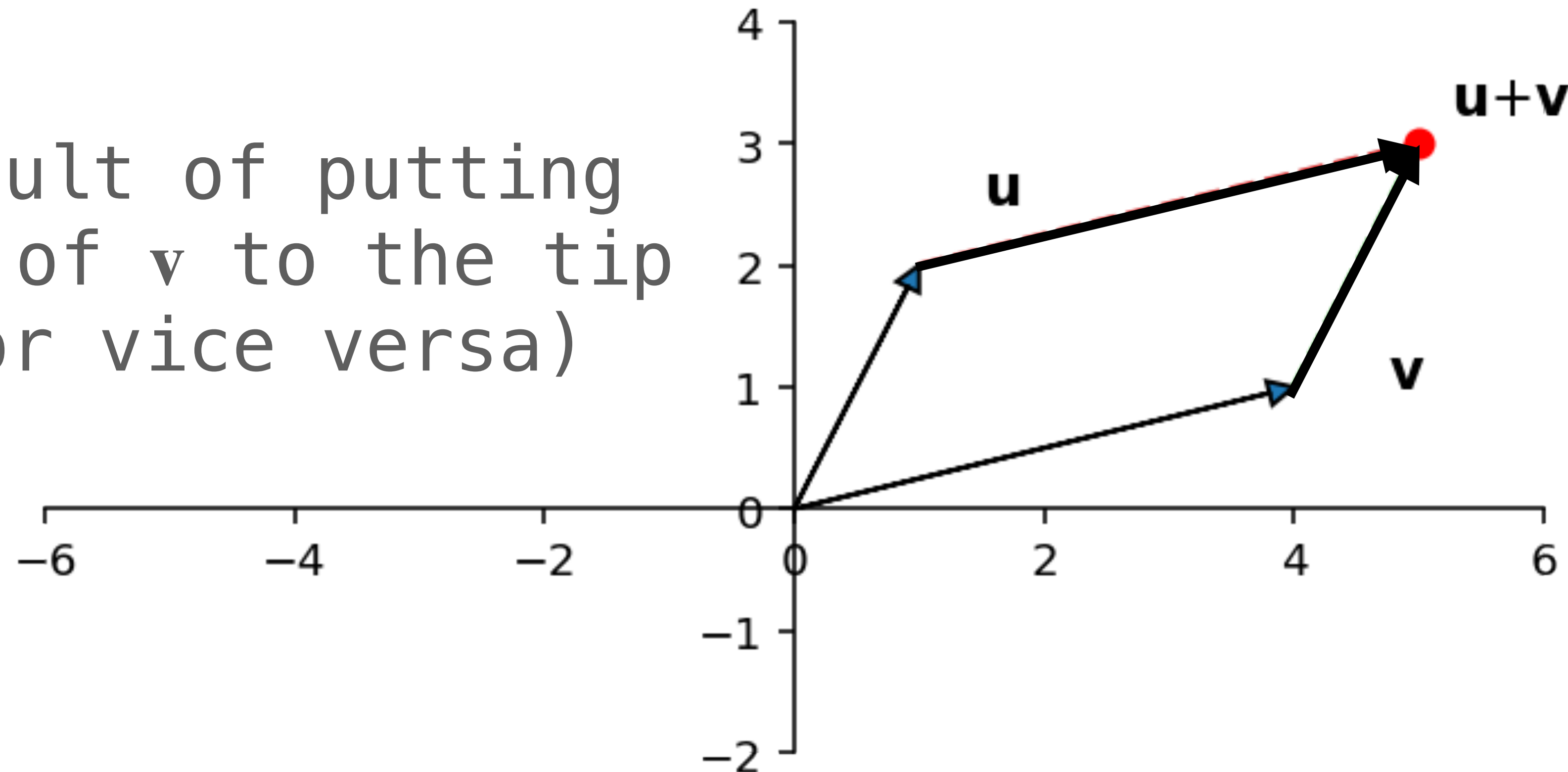
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demo  
(from ILA)

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# Vector Scaling/Multiplication

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

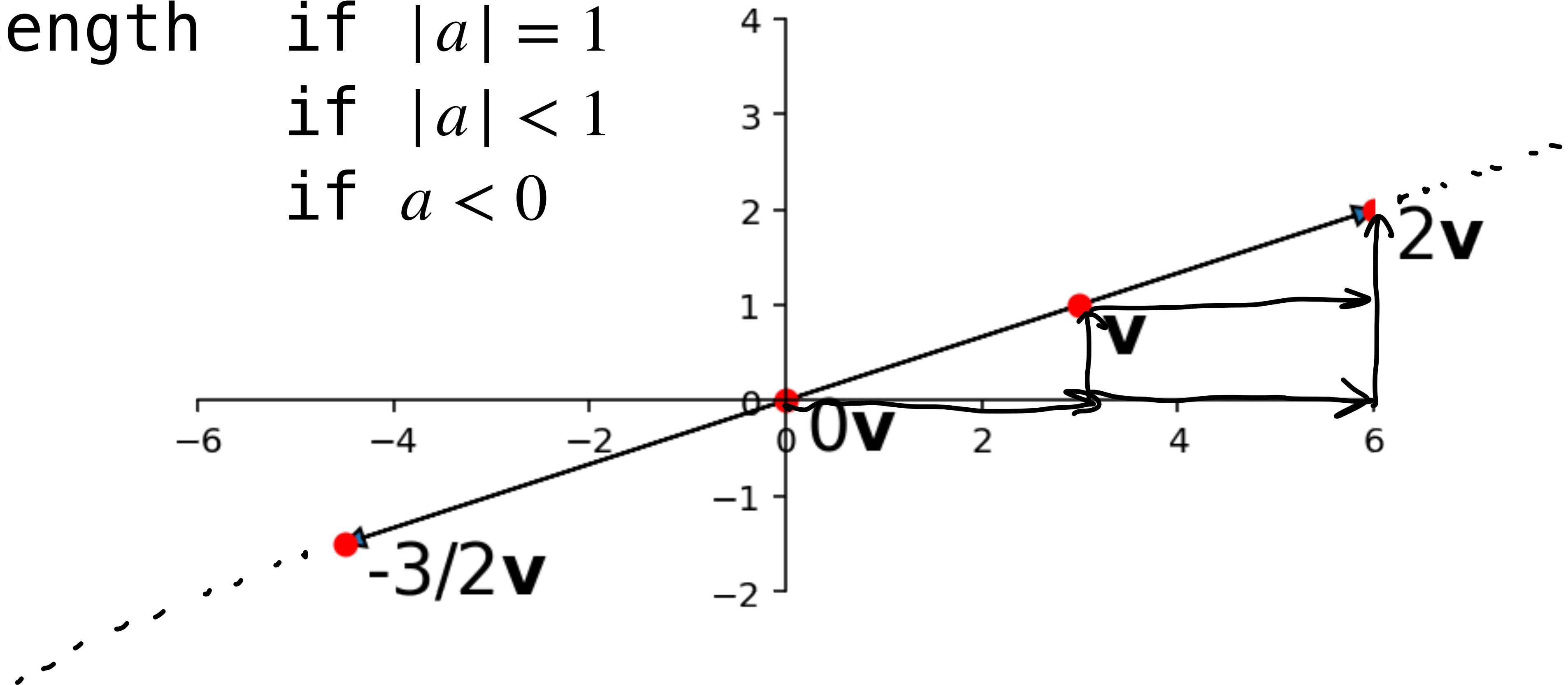
# Vector Scaling/Multiplication (Example)

Scaling/multiplying a vector by a number means multiplying each of it's elements

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 10.5 \\ 12 \end{bmatrix}$$

# Vector Scaling (Geometrically)

longer	if $ a  > 1$
the same length	if $ a  = 1$
shorter	if $ a  < 1$
reversed	if $a < 0$



demo  
(from ILA)



# Algebraic Properties

For any vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  and any real numbers  $c, d$ :

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

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these are requirements for any **vector space**  
they matter more for *bizarre* vector spaces

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

# Example "Proof"

$$\begin{aligned} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \end{aligned}$$

# Question (Practice)

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

*Compute the value of the above vector*

**Answer**

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

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What vectors can we make in this way?

# Linear Combinations

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**Definition.** a *linear combination* of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$$

where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are in  $\mathbb{R}$

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Looks suspiciously like  
a linear equation...

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weights

# Linear Combinations (Example)

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demo  
(from ILA)



# The Fundamental Concern

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Can  $\mathbf{u}$  be written as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ ?

That is, are there weights  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that  $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}$ ?

# **Why is this fundamental?**

*I'm going to ask that you suspend your disbelief...*

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For now, how do we solve this problem?

# Vector Equations and Linear Systems

# The Fundamental Connection

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We don't know the weights, that's what we want to find

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What if we write them as *unknowns*?

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$



# Some Symbol Pushing...

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# Some Symbol Pushing...

$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

# Some Symbol Pushing...

$$x_1 + 2x_2 = 7$$

$$(-2)x_1 + 5x_2 = 4$$

$$-5x_1 + 6x_2 = -3$$

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we get a system  
of linear  
equations we  
know how to  
solve

# General Vector Equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \dots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

by vector scaling



# General Vector Equations

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

by vector addition

# General Vector Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

by vector equality

# The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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this is our big  
shift in  
perspective

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**Solution.** Solve the system of linear equations with the augmented matrix

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \quad \mathbf{b}]$$

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$$\left[ \begin{array}{c|c|c|c|c} \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{array} \right]$$

← this is notation for building a matrix out of column vectors

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# Question

Can  $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$  be written as a linear combination of  $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ ?

Exercise.

**Answer**

$$3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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↑  
such that

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**$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  exactly when  $\mathbf{u}$  can be expressed as a linear combination of those vectors**

# Spans

$$\text{span}\{v_1, v_2\} = \left\{ \alpha \vec{v}_1 + \beta \vec{v}_2, \begin{matrix} 1v_1 + 0v_2 \\ 2v_1 + 0v_2 \\ \vdots \end{matrix} \right\}$$

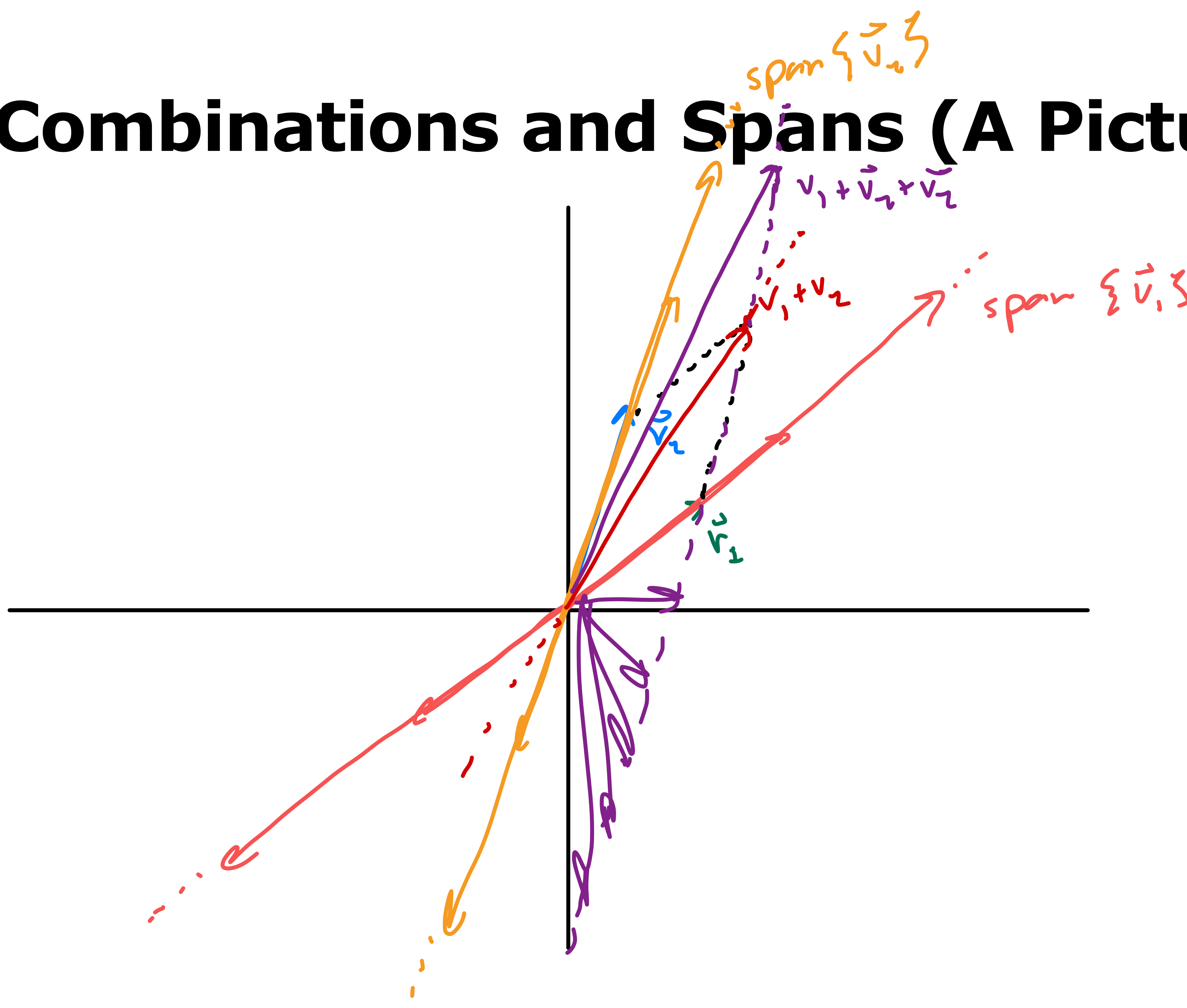
**Definition.** the *span* of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{v_1, v_2, \dots, v_n\} = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_1, \alpha_2, \dots, \alpha_n \text{ are in } \mathbb{R}\}$$

read:  $u$  is an element of  $\text{span}\{v_1, v_2, \dots, v_n\}$

$u \in \text{span}\{v_1, v_2, \dots, v_n\}$  exactly when  $u$  can be expressed as a linear combination of those vectors

# Linear Combinations and Spans (A Picture)



# Spans (Geometrically)

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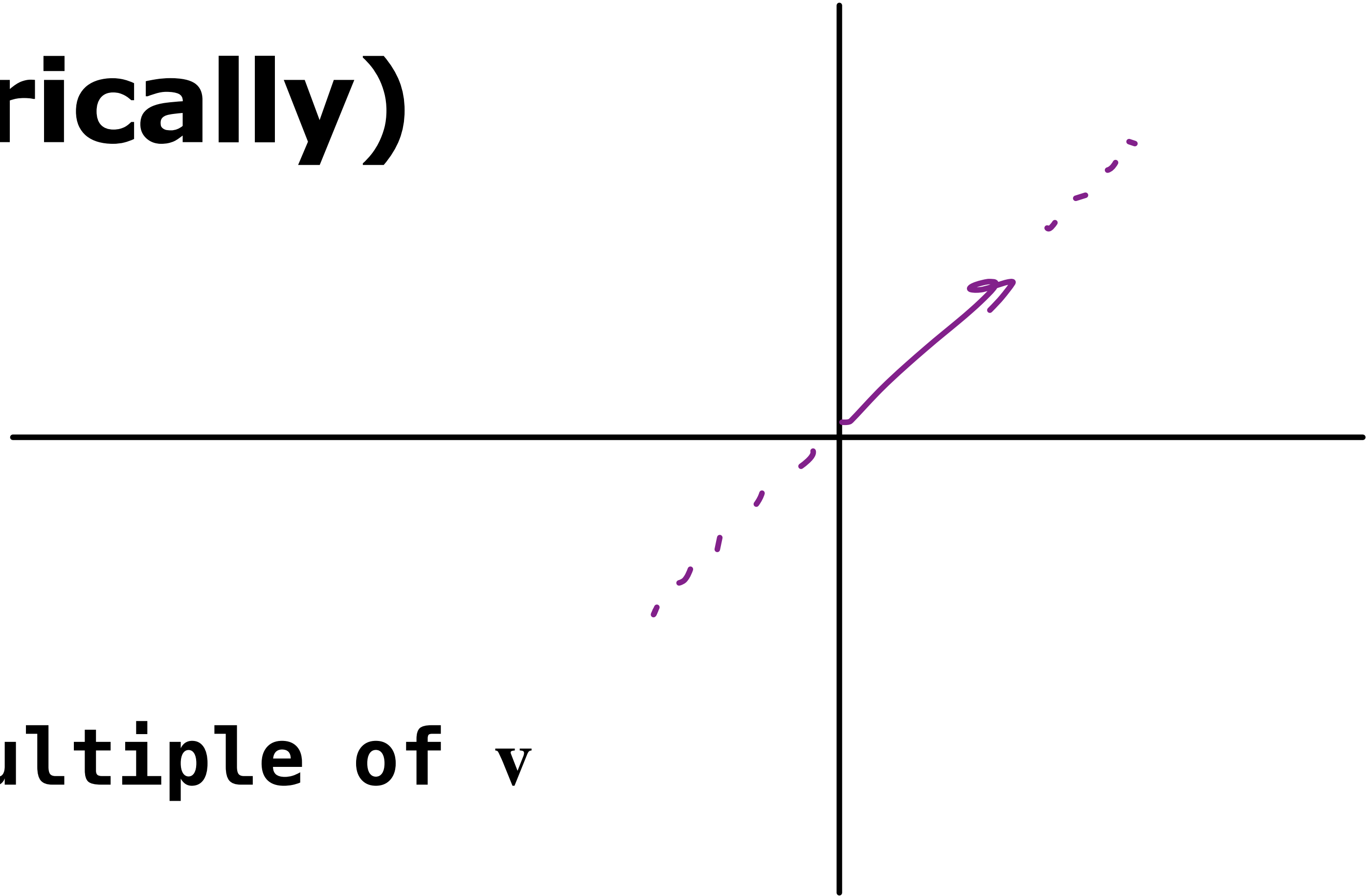
# Spans (Geometrically)

for one vector

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$

this is **all** scalar multiple of  $\mathbf{v}$

the span of one vector is a **line**



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**!!IMPORTANT!!**

**In all cases they pass through the origin**

demo  
(from ILA)

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you know how to do this now

# Example

$$\text{Is } \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \text{ in span } \left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\} ?$$

# Question (Conceptual)

*What does it mean geometrically if  $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ ?*

demo  
(from ILA)

# How To: Inconsistency and Spans

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**Question.** find a vector  $\mathbf{b}$  which *does not* appear  
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There is **no way** to write  $\mathbf{b}$  as a linear combination

# Example

*Find a vector **not** in*  $\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$

# Summary

**Vectors** are fundamental objects

We can think of them as the **columns** of a linear system

We can **scale** them and **add** them together

They can **span** spaces which represent **hyperplanes**