Administrivia

+ HW3 due Thursday

+ Project comments on slack

+ Updated calender

Intuitionistic Propositional Logic

Lecture 18

Objectives

+ connect IPL with STLC
+ recall soundness + completeness
+ computational interpretations of CPL

Recall STLC:

$$\frac{}{\vdash \bot : \mathrm{Type}}$$

$$\frac{\vdash A : \mathrm{Type} \qquad \vdash B : \mathrm{Type}}{\vdash A \to B : \mathrm{Type}}$$

$$\frac{\Gamma \vdash \star \qquad \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash x : A} \ (x \notin \Gamma)$$

$$\frac{\Gamma \vdash m : A \qquad \vdash B : \mathrm{Type}}{\Gamma, x : B \vdash m : A} \ (x \notin \Gamma)$$

$$(\text{abs}) \ \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \to B}$$

$$(\text{app}) \ \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M N : B}$$

Minimal Logic

$$\underset{\text{assumptions}}{\Gamma} \vdash \underset{\text{consequent}}{A}$$

$$\frac{}{\vdash \bot : \mathrm{Prop}}$$

$$\frac{\vdash A : \mathrm{Prop} \qquad \vdash B : \mathrm{Prop}}{\vdash A \to B : \mathrm{Prop}}$$

$$\frac{\Gamma \vdash * \qquad \vdash A : \mathrm{Prop}}{\Gamma, A \vdash A}$$

$$\frac{\Gamma \vdash P \qquad \vdash B : \mathrm{Prop}}{\Gamma, B \vdash P}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

(modus ponens)

Example

$$
\dfrac{\dfrac{\dfrac{x:\bot\to\bot,\ y:\bot \vdash x:\bot\to\bot \qquad x:\bot\to\bot,\ y:\bot \vdash y:\bot}{x:\bot\to\bot,\ y:\bot \vdash xy:\bot}}{x:\bot\to\bot \vdash \lambda y.\,xy : \bot\to\bot}}{\vdash \lambda x.\,\lambda y.\,xy : (\bot\to\bot)\to(\bot\to\bot)}
$$

STLC with Type Variables

$$\dfrac{}{\Gamma \vdash \bot : \mathrm{Type}}$$

$$\dfrac{\Gamma \vdash A : \mathrm{Type} \qquad \Gamma \vdash B : \mathrm{Type}}{\Gamma \vdash A \to B : \mathrm{Type}}$$

$\Gamma \vdash M : A$

$$\dfrac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, X : \mathrm{Type} \vdash X : \mathrm{Type}} \quad (\cancel{X} \notin \Gamma)$$

$\Gamma \vdash M : A$

$$\dfrac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, \cancel{X} : \mathrm{Type} \vdash A : \mathrm{Type}} \quad (\cancel{X} \notin \Gamma)$$

$\Gamma, B : \mathrm{Type} \vdash M : A$

$$\dfrac{\Gamma \vdash * \qquad \Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash x : A}$$

$$\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \mathrm{Type}}{\Gamma, x : B \vdash M : A}$$

$$\dfrac{\Gamma \vdash m : A \to B \qquad \Gamma \vdash n : A}{\Gamma \vdash mn : B}$$

$$\dfrac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda x^A M : A \to B}$$

Meta Theoretic Lemma

$$\Gamma \vdash M : A \implies$$

$$\boxed{\Delta}, \boxed{\Psi} \vdash M : A \quad \text{and} \quad \Gamma = \pi(\Delta, \Psi)$$

for some permutation π.

type variables

term variables

Minimal Log with Prop. Variables

$$\boxed{\Delta} \mid \boxed{\Gamma} \vdash \boxed{A}$$

Prop. var. — assumptions — consequent.

$$\frac{}{\emptyset \mid \emptyset \vdash \bot : Prop}$$

$$\frac{\Delta \mid \Gamma \vdash A : Prop \quad \Delta \mid \Gamma \vdash B : Prop}{\Delta \mid \Gamma \vdash A \to B : Prop}$$

$$\frac{\Delta \mid \Gamma \vdash *}{\Delta, B \mid \Gamma \vdash *} \quad B \notin \Delta$$

$$\frac{\Delta \mid \Gamma \vdash P \quad \Delta \mid \Gamma \vdash A : Prop}{\Delta \mid \Gamma, A \vdash P}$$

$$\frac{\Delta \mid \Gamma \vdash * \quad \Delta \mid \Gamma \vdash A : Prop}{\Delta \mid \Gamma, A \vdash A}$$

$$\frac{\Delta \mid \Gamma, A \vdash B}{\Delta \mid \Gamma \vdash A \to B}$$

$$\frac{\Delta \mid \Gamma \vdash A \to B \quad \Delta \mid \Gamma \vdash A}{\Delta \mid \Gamma \vdash B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A, B \mid \emptyset \vdash A : \mathrm{Prop}}{A, B \mid A \vdash A} \qquad A, B \mid A \vdash B : \mathrm{Prop}}{A, B \mid A, B \vdash A}}{A, B \mid A \vdash B \to A}}{A, B \mid \emptyset \vdash A \to B \to A}$$

Intuitionistic Propositional Logic

$$\Delta \mid \Gamma \vdash P \iff$$

there is an m s.t.

$\boxed{\Psi}, \boxed{\Gamma} \vdash m : P$ is STLC

type var.

term var.

Questions

What is the relationship between CPL and IPL?

- what do we gain?
- what do we lose?

Law of Excluded Middle

$$A \mid \emptyset \vdash_{CPL} A \vee \neg A$$

$$A \mid \emptyset \nvdash_{IPL} A \vee \neg A$$

(we will prove this on thursday)

$$\dfrac{\dfrac{A \mid A \vdash_c A}{A \mid A \vdash_c A \vee \neg A} \qquad \dfrac{A \mid \neg A \vdash_c \neg A}{A \mid \neg A \vdash_c A \vee \neg A}}{A \mid \emptyset \vdash_c A \vee \neg A}$$

Other Classical Propositions

Double Negation elimination

$$A \mid \not{\phi} \vdash_{CPL} \neg\neg A \to A$$

$$A, B \mid \not{\phi} \vdash_{CPL} ((A \to B) \to A) \to A$$

$$A, B \mid A \to B \vdash_{CPL} \neg A \vee B$$

Disjunction Property.

Thm: $\Delta \mid \Gamma \vdash_{IPL} A \vee B \implies$

$\Delta \mid \Gamma \vdash_{IPL} A$ or $\Delta \mid \Gamma \vdash_{IPL} B$

Not in CPC

$A \mid \emptyset \vdash_{CPL} A \vee \neg A$