

The Lambda Calculus: Meta-Theory

Type Theory and Mechanized Reasoning
Lecture 11

Introduction

Administrivia

Homework 4 is due on *Thursday by 11:59PM*.

Homework 5 will be released on Friday (it will be short).

Objectives

Finish our discussion on the `operational semantics` of the lambda calculus.

Introduce `semantic notions` of the lambda calculus.

Demonstrate how to `encode` data.

If we have time, use `De Bruijn indices` to avoid issues of α -equivalence.

Recap

Recall: Lambda Terms

(Fix a set of variables.)

Recall: Lambda Terms

(Fix a set of variables.)

Definition. The collection of **lambda terms** is defined inductively.

Recall: Lambda Terms

(Fix a set of variables.)

Definition. The collection of **lambda terms** is defined inductively.

- Every variable x is a lambda term.

variables

Recall: Lambda Terms

(Fix a set of variables.)

Definition. The collection of **lambda terms** is defined inductively.

- Every variable x is a lambda term.
- If M and N are lambda terms, then so is (MN)

variables

application

Recall: Lambda Terms

(Fix a set of variables.)

Definition. The collection of **lambda terms** is defined inductively.

- Every variable x is a lambda term.
- If M and N are lambda terms, then so is (MN)
- If M is a lambda term, then so is $(\lambda x.M)$ for any variable x

variables

application

abstraction

Examples (Again)

x, y

$$I \triangleq \lambda x . x$$

$$K \triangleq \lambda x . \lambda y . x$$

$$A \triangleq \lambda x . \lambda y . xy$$

$$\omega \triangleq \lambda x . xx$$

$$\Omega \triangleq \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

Evaluation (High Level)

$(\lambda x. \text{hat-on}(x))\text{cat}$

should evaluate to

$\text{hat-on}(\text{cat})$

Evaluation (High Level)

$(\lambda x. \text{hat-on}(x))\text{cat}$

should evaluate to

$\text{hat-on}(\text{cat})$

We need to be able to **replace** the variable x in $\text{hat-on}(x)$ with the argument to the function.

Evaluation (High Level)

$(\lambda x. \text{hat-on}(x))\text{cat}$
should evaluate to
 $\text{hat-on}(\text{cat})$

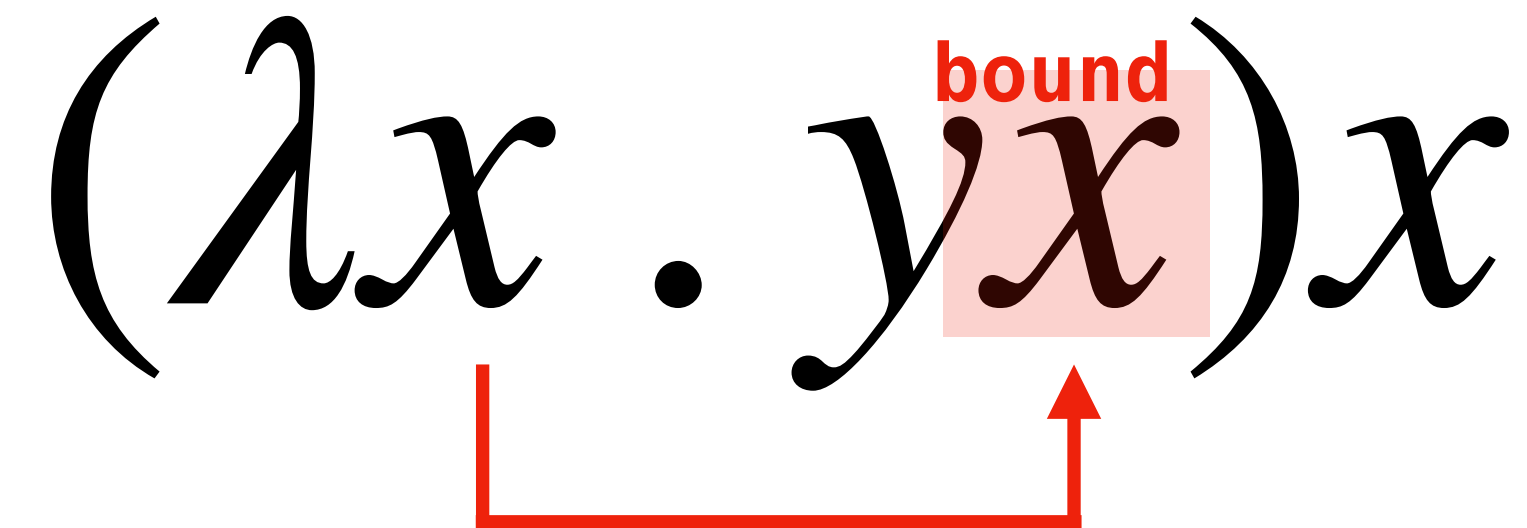
We need to be able to **replace** the variable x in $\text{hat-on}(x)$ with the argument to the function.

The variable x is able to be replaced in $\text{hat-on}(x)$ because it is not **bound** by anything.

Recall: Free and Bound Variables

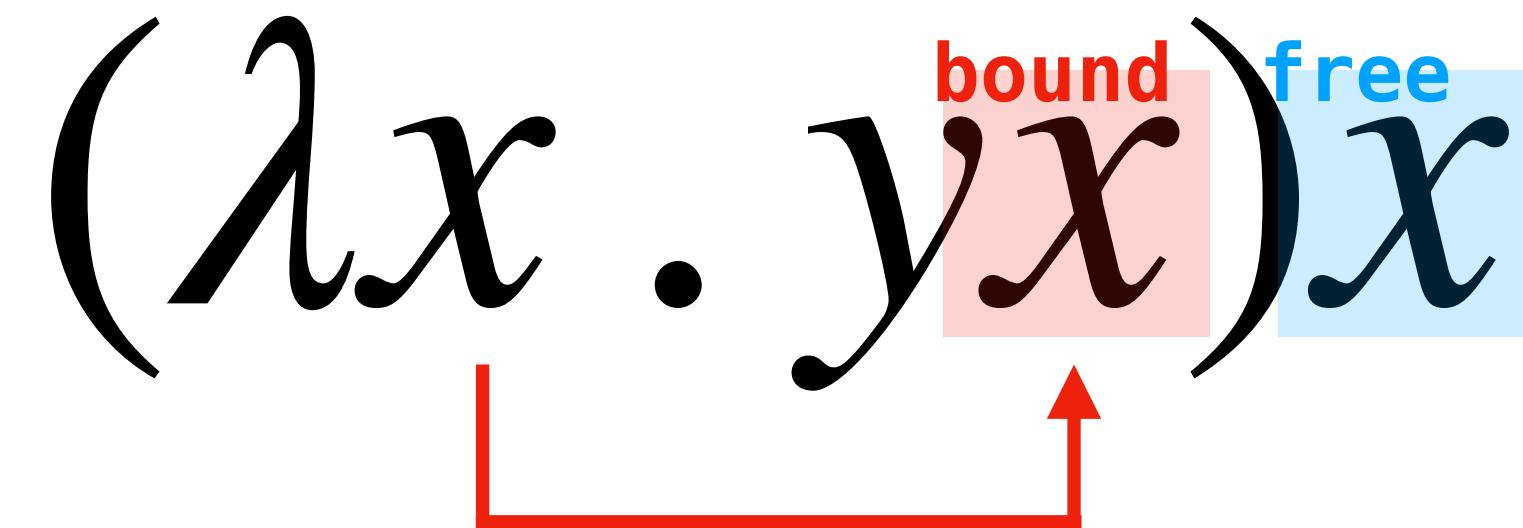
$$(\lambda x. yx)x$$

Recall: Free and Bound Variables

$$(\lambda x. y \mathbf{x})x$$


A variable x is **bound** if it appears in the body of an abstraction over x .

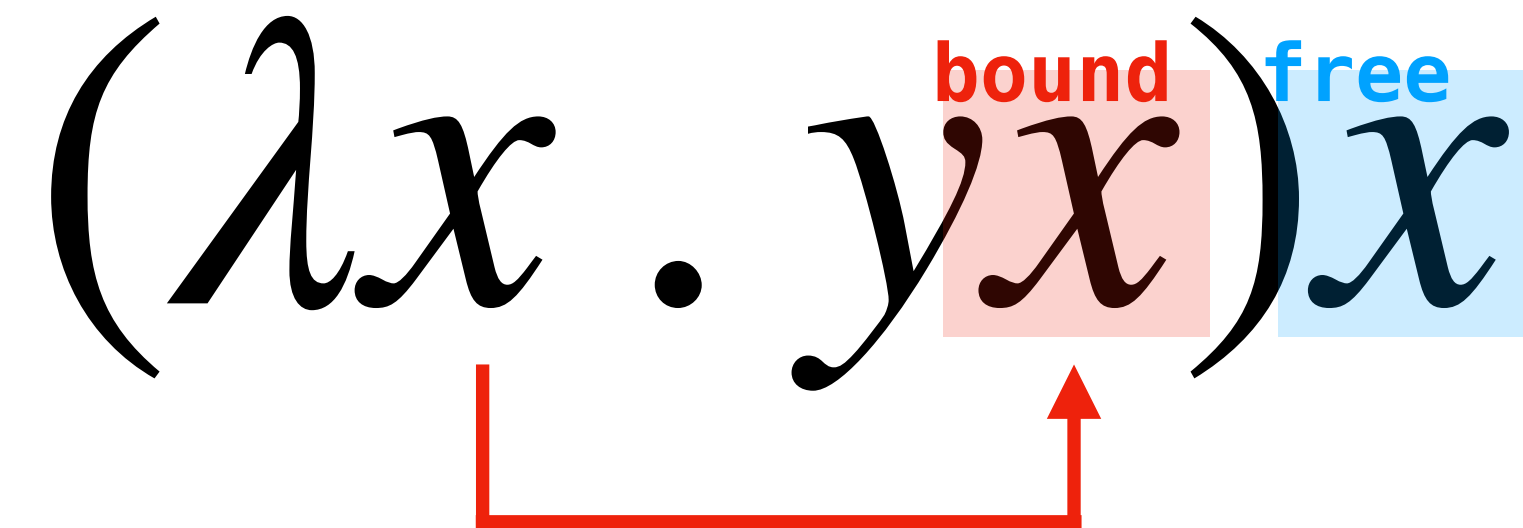
Recall: Free and Bound Variables



A variable x is **bound** if it appears in the body of an abstraction over x .

Otherwise it is **free**.

Recall: Free and Bound Variables



A variable x is **bound** if it appears in the body of an abstraction over x .

Otherwise it is **free**.

Definition. A term is **closed** if it has no free variables. Such a term is called a **combinator**.

Recall: Substitution

Recall: Substitution

Definition. Substitution of N for a free variable x in M , written $M[N/x]$ is defined recursively on M .

Recall: Substitution

Definition. Substitution of N for a free variable x in M , written $M[N/x]$ is defined recursively on M .

- $y[N/x] = \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases}$

Recall: Substitution

Definition. Substitution of N for a free variable x in M , written $M[N/x]$ is defined recursively on M .

- $y[N/x] = \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases}$
- $(M_1M_2)[N/x] = (M_1[N/x])(M_2[N/x])$

Recall: Substitution

Definition. Substitution of N for a free variable x in M , written $M[N/x]$ is defined recursively on M .

- $y[N/x] = \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases}$
- $(M_1 M_2)[N/x] = (M_1[N/x])(M_2[N/x])$
- $(\lambda y . M)[N/x] = \begin{cases} \lambda y . M & y = x \\ \lambda y . M[N/x] & \text{otherwise} \end{cases}$

Recall: Substitution

Definition. Substitution of N for a free variable x in M , written $M[N/x]$ is defined recursively on M .

- $y[N/x] = \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases}$
- $(M_1 M_2)[N/x] = (M_1[N/x])(M_2[N/x])$
- $(\lambda y . M)[N/x] = \begin{cases} \lambda y . M & y = x \\ \lambda y . M[N/x] & \text{otherwise} \end{cases}$
this is not quite right.

Recall: Alpha-Equivalence

$$\lambda x . xz =_{\alpha} \lambda y . yz \neq_{\alpha} \lambda z . zz$$

Recall: Alpha-Equivalence

$$\lambda x . xz =_{\alpha} \lambda y . yz \neq_{\alpha} \lambda z . zz$$

Definition (Informal). Lambda terms M and N are α -equivalent if they are the same up to *valid* renaming bound variables.

Recall: Alpha-Equivalence

$$\lambda x . xz =_{\alpha} \lambda y . yz \neq_{\alpha} \lambda z . zz$$

Definition (Informal). Lambda terms M and N are α -equivalent if they are the same up to *valid* renaming bound variables.

We cannot rename bound variables to names of existing free variables.

Recall: Alpha-Equivalence

$$\lambda x . xz =_{\alpha} \lambda y . yz \neq_{\alpha} \lambda z . zz$$

Definition (Informal). Lambda terms M and N are α -equivalent if they are the same up to *valid* renaming bound variables.

We cannot rename bound variables to names of existing free variables.

We will always consider terms up to α -equivalence.

moving on...

Captured Variables

$$M =_{\alpha} M'$$

should imply

$$M[N/x] =_{\alpha} M'[N/x]$$

Captured Variables

$$M =_{\alpha} M'$$

should imply

$$M[N/x] =_{\alpha} M'[N/x]$$

If we consider terms up to $=_{\alpha}$, then substitution should preserve $=_{\alpha}$.

Captured Variables

$$M =_{\alpha} M'$$

should imply

$$M[N/x] =_{\alpha} M'[N/x]$$

If we consider terms up to $=_{\alpha}$, then substitution should preserve $=_{\alpha}$.

Our current definition doesn't do this.

Captured Variables

$$\lambda x . y =_{\alpha} \lambda z . y$$

but

$$(\lambda x . y)[x/y] \neq_{\alpha} (\lambda z . y)[x/y]$$

Captured Variables

$$\lambda x . y =_{\alpha} \lambda z . y$$

but

$$(\lambda x . y)[x/y] \neq_{\alpha} (\lambda z . y)[x/y]$$

Since x appears free in the value being substituted in, it will be **captured**.

Substitution (Again)

Definition. Substitution of N for x in M , written $M[N/x]$ is defined recursively on M .

- $y[N/x] = \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases}$
- $(M_1 M_2)[N/x] = (M_1[N/x])(M_2[N/x])$
- $(\lambda y . M)[N/x] = \begin{cases} \lambda y . M & y = x \\ (\lambda z . M[z/y])[N/x] & \text{otherwise} \end{cases}$
where z does not appear free in M or N

Fresh Variables

Fresh Variables

Definition. A variable x is **fresh** with respect to a term M if x does not appear free in M .

Fresh Variables

Definition. A variable x is **fresh** with respect to a term M if x does not appear free in M .

A couple ways to deal with it:

Fresh Variables

Definition. A variable x is **fresh** with respect to a term M if x does not appear free in M .

A couple ways to deal with it:

- » Keep track of a counter (this requires state)
- » If variables are indexed, take the max of all indices + 1
- » Use De Bruijn indices to avoid this all together

Fresh Variables

Definition. A variable x is **fresh** with respect to a term M if x does not appear free in M .

A couple ways to deal with it:

- » Keep track of a counter (this requires state)
- » If variables are indexed, take the max of all indices + 1
- » Use De Bruijn indices to avoid this all together

(Finding fresh variables is more difficult in the functional setting.)

Reduction

Beta Reduction

Beta Reduction

Definition. We define the relation $M \rightarrow_{\beta} N$ as follows.

Beta Reduction

Definition. We define the relation $M \rightarrow_{\beta} N$ as follows.

- $(\lambda x. M)N \rightarrow_{\beta} M[N/x]$

Beta Reduction

Definition. We define the relation $M \rightarrow_{\beta} N$ as follows.

- $(\lambda x . M)N \rightarrow_{\beta} M[N/x]$
- $M \rightarrow_{\beta} M'$ implies $MN \rightarrow_{\beta} M'N$ and $NM \rightarrow_{\beta} NM'$ and $\lambda x . M \rightarrow_{\beta} \lambda x . M'$

Beta Reduction

Definition. We define the relation $M \rightarrow_{\beta} N$ as follows.

- $(\lambda x . M)N \rightarrow_{\beta} M[N/x]$
- $M \rightarrow_{\beta} M'$ implies $MN \rightarrow_{\beta} M'N$ and $NM \rightarrow_{\beta} NM'$ and $\lambda x . M \rightarrow_{\beta} \lambda x . M'$

This is a **relation** not a function.

Reduction

Reduction

Definition. \rightarrow_{β} for the reflexive transitive closure of \rightarrow_{β} .

Reduction

Definition. \rightarrow_{β} for the reflexive transitive closure of \rightarrow_{β} .

That is, $M \rightarrow_{\beta} N$ if there is a (possibly empty) sequence of reductions

Reduction

Definition. \rightarrow_{β} for the reflexive transitive closure of \rightarrow_{β} .

That is, $M \rightarrow_{\beta} N$ if there is a (possibly empty) sequence of reductions

$$M = P_1 \rightarrow_{\beta} P_2 \dots \rightarrow_{\beta} P_k = N$$

Reduction

Definition. \rightarrow_{β} for the reflexive transitive closure of \rightarrow_{β} .

That is, $M \rightarrow_{\beta} N$ if there is a (possibly empty) sequence of reductions

$$M = P_1 \rightarrow_{\beta} P_2 \dots \rightarrow_{\beta} P_k = N$$

This captures what happens when we "compute" a lambda term.

Redex

$$\dots((\lambda x . M)N)\dots \rightarrow_{\beta} \dots(M[N/x])\dots$$

Redex

$$\dots((\lambda x . M)N)\dots \rightarrow_{\beta} \dots(M[N/x])\dots$$

Definition. A subterm is a **redex** if it is of the form.

Redex

$$\dots((\lambda x . M)N)\dots \rightarrow_{\beta} \dots(M[N/x])\dots$$

Definition. A subterm is a **redex** if it is of the form.

A term may have many redexes, which means there may be **multiple ways to β -reduce a term.**

Normal Forms

Normal Forms

Definition. A β -normal form is a term M such that there is no N where $M \rightarrow_{\beta} N$.

Normal Forms

Definition. A β -normal form is a term M such that there is no N where $M \rightarrow_{\beta} N$.

Normal forms cannot be further reduced, they are like the **values** of a computation.

Normal Forms

Definition. A β -normal form is a term M such that there is no N where $M \rightarrow_{\beta} N$.

Normal forms cannot be further reduced, they are like the **values** of a computation.

Examples. $\lambda x.x$, $\lambda x.\lambda y.x$, $\lambda x.xx$, are normal forms whereas $(\lambda x.x)(\lambda x.x)$ is not.

Meta-Theory

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Omega Combinator

$$\Omega = \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

Omega Combinator

$$\Omega = \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

This is the `infinite-loop` combinator:

Omega Combinator

$$\Omega = \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

This is the `infinite-loop` combinator:

$$(\lambda x . xx)(\lambda x . xx) \rightarrow_{\beta} (xx)[\lambda x . xx/x] = (\lambda x . xx)(\lambda x . xx)$$

Omega Combinator

$$\Omega = \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

This is the `infinite-loop` combinator:

$$(\lambda x . xx)(\lambda x . xx) \rightarrow_{\beta} (xx)[\lambda x . xx/x] = (\lambda x . xx)(\lambda x . xx)$$

So this term does not have a normal form.

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

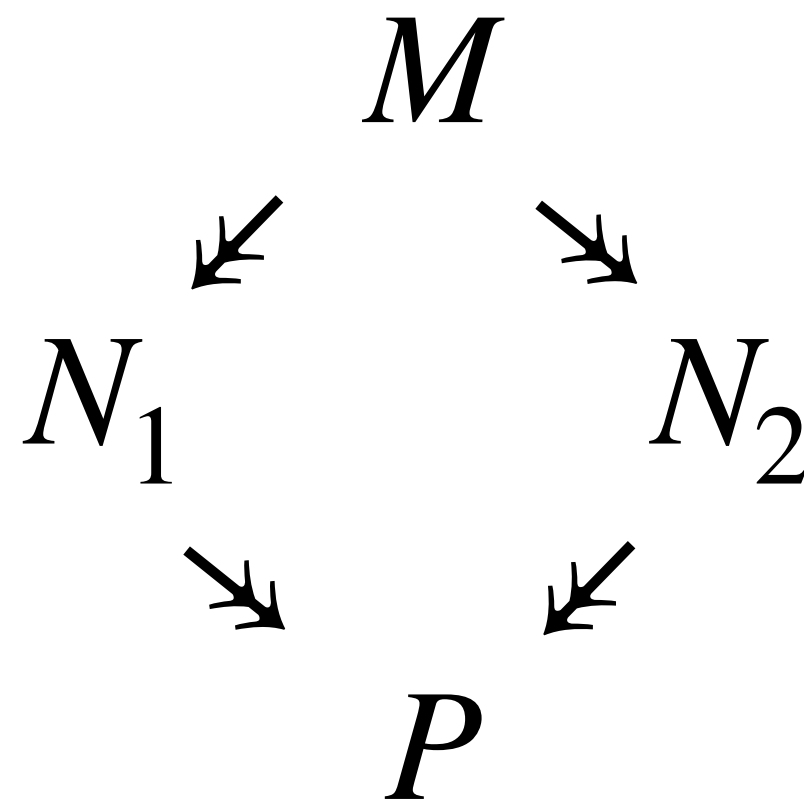
Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Confluence (Church-Rosser)

Theorem. If $M \rightarrow_{\beta} N_1$ and $M \rightarrow_{\beta} N_2$ then there is a term P such that $N_1 \rightarrow_{\beta} P$ and $N_2 \rightarrow_{\beta} P$.

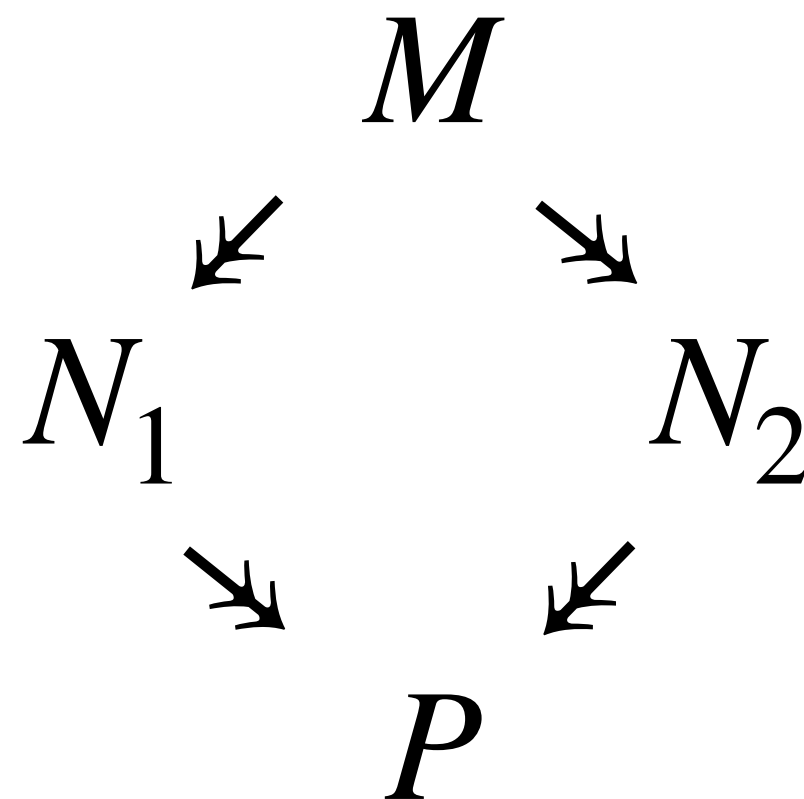
We can't "completely diverge" after reducing.



Confluence (Church-Rosser)

Theorem. If $M \rightarrow_{\beta} N_1$ and $M \rightarrow_{\beta} N_2$ then there is a term P such that $N_1 \rightarrow_{\beta} P$ and $N_2 \rightarrow_{\beta} P$.

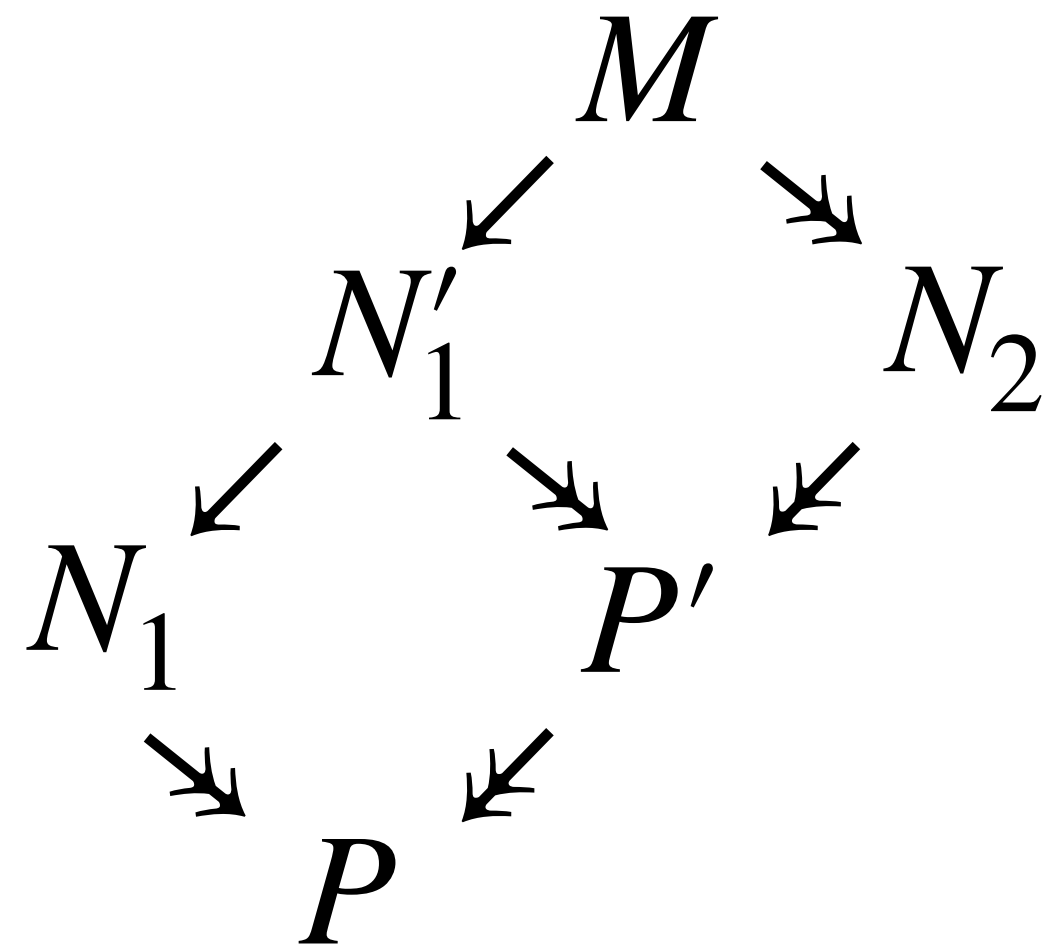
We can't "completely diverge" after reducing.



Confluence: Very Rough Proof Sketch

Theorem. If $M \twoheadrightarrow_{\beta} N_1$ and $M \twoheadrightarrow_{\beta} N_2$ then there is a term P such that $N_1 \twoheadrightarrow_{\beta} P$ and $N_2 \twoheadrightarrow P$.

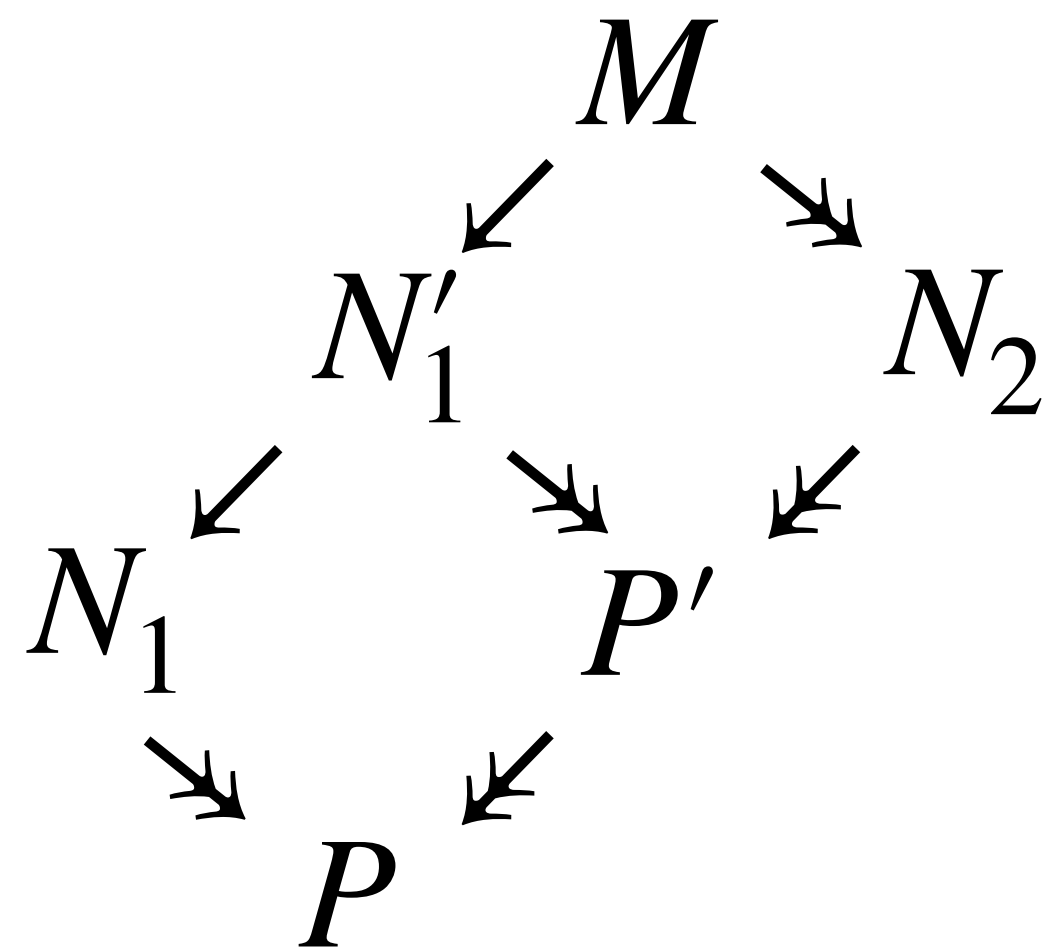
We "unravel" each step on the paths and fill in the parallelogram.



Confluence: Very Rough Proof Sketch

Theorem. If $M \rightarrow_{\beta}^* N_1$ and $M \rightarrow_{\beta}^* N_2$ then there is a term P such that $N_1 \rightarrow_{\beta}^* P$ and $N_2 \rightarrow_{\beta}^* P$.

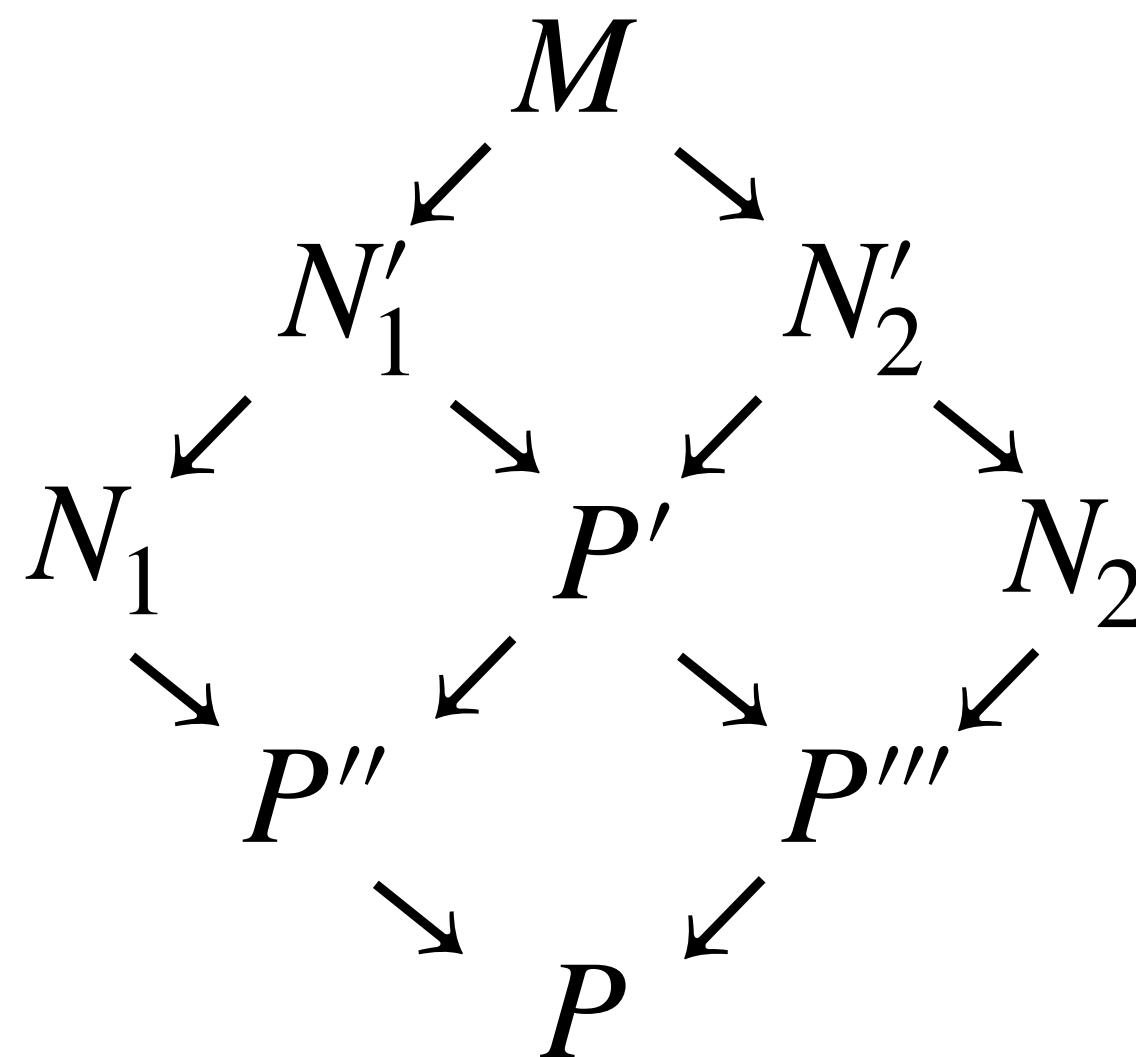
We "unravel" each step on the paths and fill in the parallelogram.



Confluence: Very Rough Proof Sketch

Theorem. If $M \twoheadrightarrow_{\beta} N_1$ and $M \twoheadrightarrow_{\beta} N_2$ then there is a term P such that $N_1 \twoheadrightarrow_{\beta} P$ and $N_2 \twoheadrightarrow_{\beta} P$.

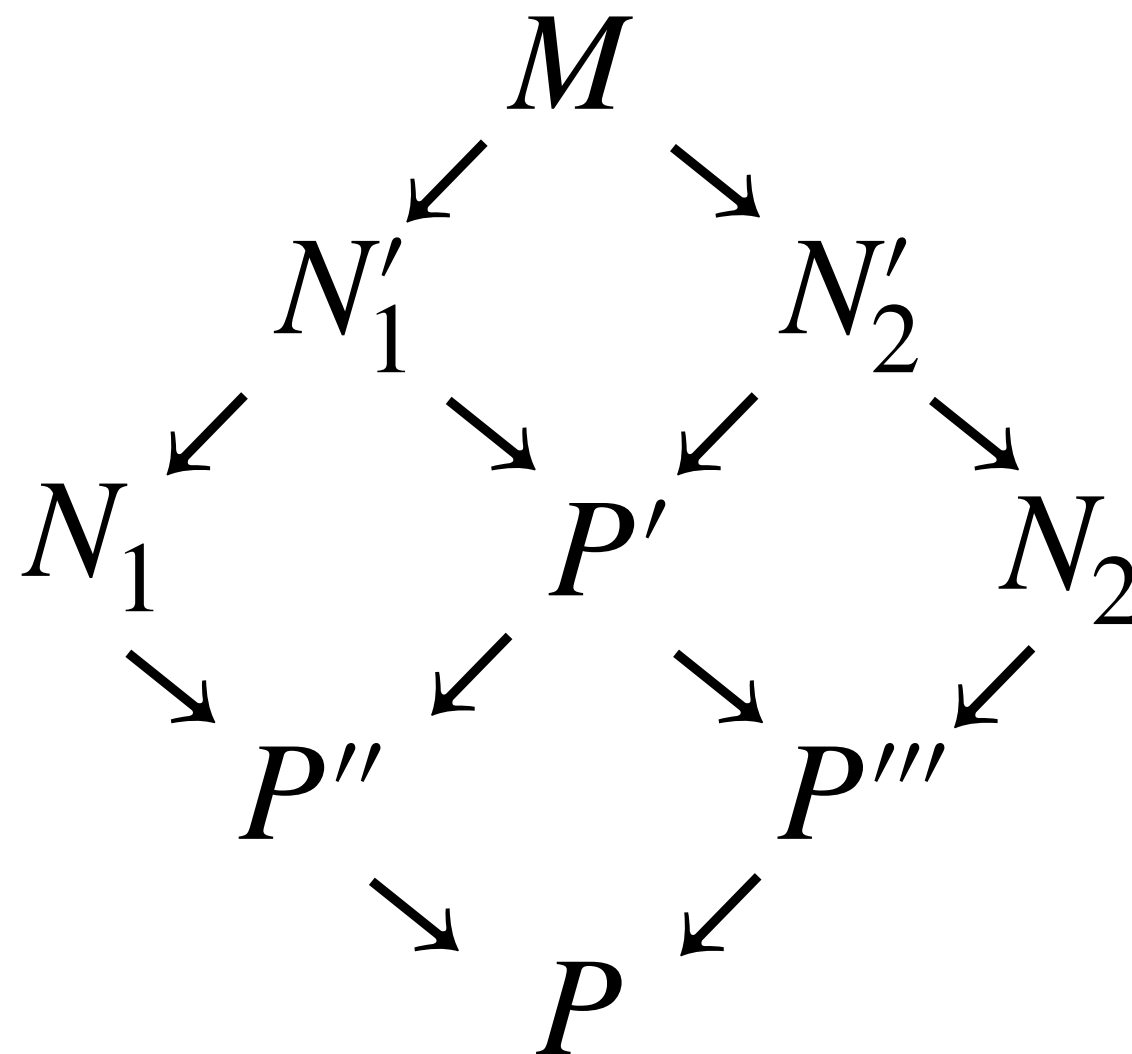
We "unravel" each step on the paths and fill in the parallelogram.



Confluence: Very Rough Proof Sketch

Theorem. If $M \twoheadrightarrow_{\beta} N_1$ and $M \twoheadrightarrow_{\beta} N_2$ then there is a term P such that $N_1 \twoheadrightarrow_{\beta} P$ and $N_2 \twoheadrightarrow_{\beta} P$.

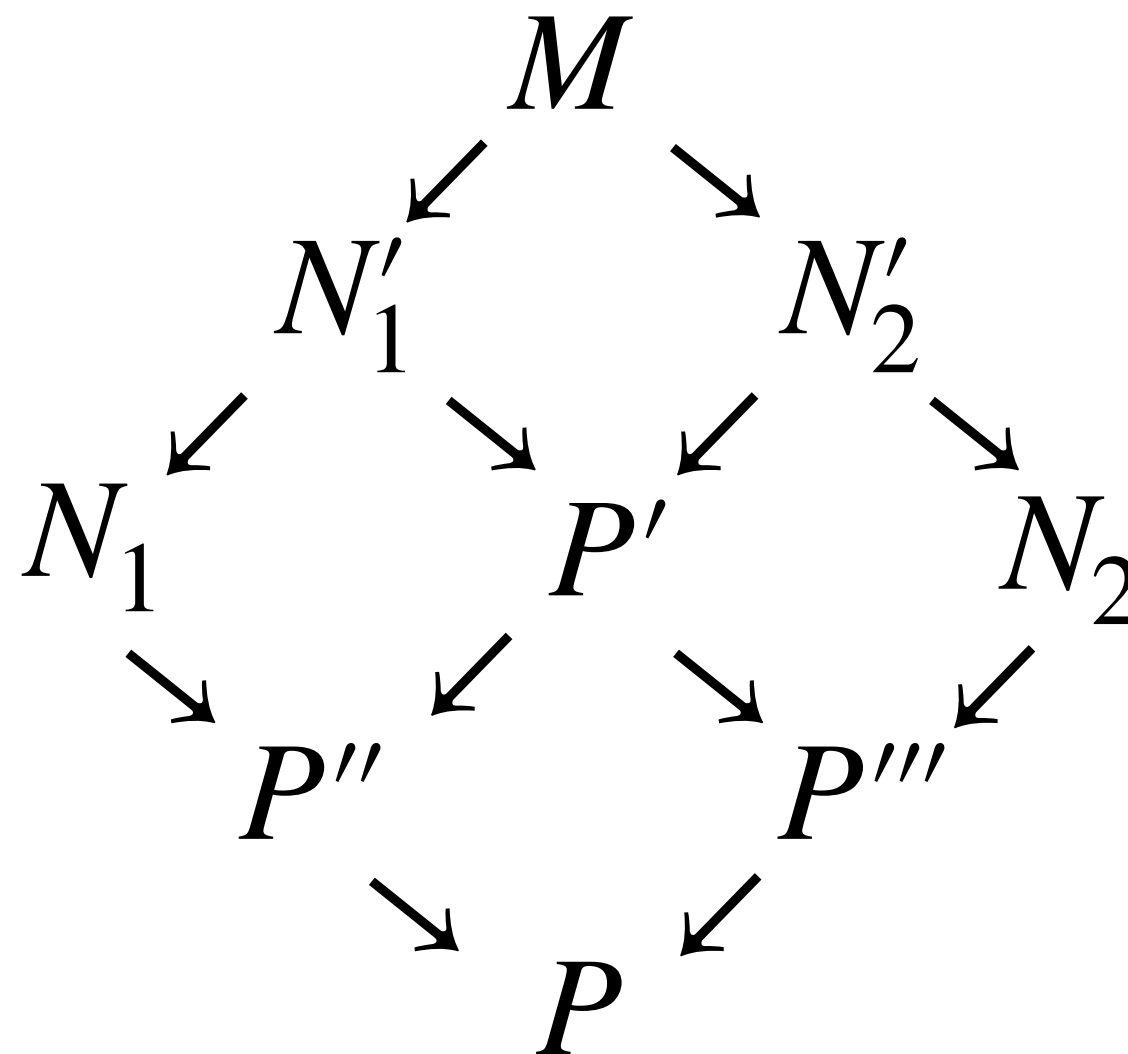
We "unravel" each step on the paths and fill in the parallelogram.



Confluence: Very Rough Proof Sketch

Theorem. If $M \rightarrow_{\beta}^* N_1$ and $M \rightarrow_{\beta}^* N_2$ then there is a term P such that $N_1 \rightarrow_{\beta}^* P$ and $N_2 \rightarrow_{\beta}^* P$.

We "unravel" each step on the paths and fill in the parallelogram.



It's a bit more complicated than this in reality.

Unique Normal Forms

Theorem. If $M \rightarrow_{\beta}^* N$ and $M \rightarrow_{\beta}^* P$ and N and P are normal forms, then $N = P$.

Proof. There is a term Z such that $N \rightarrow_{\beta}^* Z$ and $P \rightarrow_{\beta}^* Z$. Since N and P are normal forms, it must be that

$$N = Z = P$$

Unique Normal Forms

Theorem. If $M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$ and N and P are normal forms, then $N = P$.

Proof. There is a term Z such that $N \rightarrow_{\beta} Z$ and $P \rightarrow_{\beta} Z$. Since N and P are normal forms, it must be that

$$N = Z = P$$

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

Meta-Theoretic Questions

- Do all terms have normal forms?
- If a term has a normal form, is it unique?
- If a term has a normal form, is there always a way to find it?

This is a subtle question...

Evaluation

Evaluation

Evaluation is the process of **reducing** a term to a **normal form**, if possible. But we have to worry about the order in which we choose redexes.

Evaluation

Evaluation is the process of **reducing** a term to a **normal form**, if possible. But we have to worry about the order in which we choose redexes.

Definition (Informal). A (one-step) **evaluation strategy** is a way of determining which redexes to reduce.

Evaluation

Evaluation is the process of **reducing** a term to a **normal form**, if possible. But we have to worry about the order in which we choose redexes.

Definition (Informal). A (one-step) **evaluation strategy** is a way of determining which redexes to reduce.

We apply our strategy over and over until we reach a normal form (or run forever).

Two Common Evaluation Strategies

Two Common Evaluation Strategies

Leftmost outermost (Normal). Reduce the leftmost redex that does not appear in any other redexes.

Two Common Evaluation Strategies

Leftmost outermost (Normal). Reduce the leftmost redex that does not appear in any other redexes.

$$(\lambda x . xx)((\lambda x . x)y) \rightarrow_{\beta} ((\lambda x . x)y)((\lambda x . x)y)$$

Two Common Evaluation Strategies

Leftmost outermost (Normal). Reduce the leftmost redex that does not appear in any other redexes.

$$(\lambda x . xx)((\lambda x . x)y) \rightarrow_{\beta} ((\lambda x . x)y)((\lambda x . x)y)$$

Leftmost innermost (Applicative). Reduce the leftmost redex which does not contain any other redexes.

Two Common Evaluation Strategies

Leftmost outermost (Normal). Reduce the leftmost redex that does not appear in any other redexes.

$$(\lambda x . xx)((\lambda x . x)y) \rightarrow_{\beta} ((\lambda x . x)y)((\lambda x . x)y)$$

Leftmost innermost (Applicative). Reduce the leftmost redex which does not contain any other redexes.

$$(\lambda x . xx)((\lambda x . x)y) \rightarrow_{\beta} (\lambda x . xx)y$$

Normal Order

Normal Order

A function is *immediately applied* to its argument.

Normal Order

A function is **immediately applied** to its argument.

This roughly corresponds to "Call-by-Name" evaluation (though not exactly).

Normal Order

A function is **immediately applied** to its argument.

This roughly corresponds to "Call-by-Name" evaluation (though not exactly).

Theorem. If $M \rightarrow_{\beta} N$ and N is a normal form, then the normal order reduction strategy reduces M to N .

Normal Order

A function is **immediately applied** to its argument.

This roughly corresponds to "Call-by-Name" evaluation (though not exactly).

Theorem. If $M \rightarrow_{\beta} N$ and N is a normal form, then the normal order reduction strategy reduces M to N .

Example. $(\lambda f. \lambda x. f(fx))((\lambda x. \lambda y. x)z)$ (on the board)

Applicative Order

Applicative Order

The argument is fully *evaluated before* the function is called.

Applicative Order

The argument is fully *evaluated before* the function is called.

This roughly corresponds to "Call-by-value".

Applicative Order

The argument is fully **evaluated before** the function is called.

This roughly corresponds to "Call-by-value".

This strategy may not terminate, even if there is a normal form:

Applicative Order

The argument is fully *evaluated before* the function is called.

This roughly corresponds to "Call-by-value".

This strategy may not terminate, even if there is a normal form:

$$KI\Omega = (\lambda x . \lambda y . x)(\lambda x . x)\Omega$$

Weak and Strong Normalization

Weak and Strong Normalization

Definition. A term is **weakly normalizing** if it has a normal form.

Weak and Strong Normalization

Definition. A term is **weakly normalizing** if it has a normal form.

Definition. A term is **strongly normalizing** if it has no infinite reduction sequences.

Weak and Strong Normalization

Definition. A term is **weakly normalizing** if it has a normal form.

Definition. A term is **strongly normalizing** if it has no infinite reduction sequences.

Strong normalization means we (barring complexity concerns) we **don't need to think about** the reduction strategy.

Weak and Strong Normalization

Definition. A term is **weakly normalizing** if it has a normal form.

Definition. A term is **strongly normalizing** if it has no infinite reduction sequences.

Strong normalization means we (barring complexity concerns) we **don't need to think about** the reduction strategy.

Examples. $(\lambda x.x)(\lambda x.x)$ is SN. $KI\Omega$ is WN but not SN. Ω is neither.

Encoding

Church Booleans

$$\text{tru} = \lambda x . \lambda y . x$$

$$\text{fls} = \lambda x . \lambda y . y$$

Booleans are represented as *computations* which, given two values, chooses one based on the Boolean value we're representing.

Question. Can we implement ***if-then-else***?

Church Numerals

$$\text{zero} = \lambda f. \lambda x. x$$

$$\text{one} = \lambda f. \lambda x. fx$$

$$\text{two} = \lambda f. \lambda x. f(fx)$$

$$\text{suc} = \lambda n. \lambda f. \lambda x. nf(fx)$$

Numbers can be represented as "folds" or "recursors". Given a function f and a base value k , n is represented by the computation that applies f to k a total of n times.

Question. Can we implement ***add***?

Computability and the Lambda Calculus

Theorem (Informal). The lambda calculus is Turing-complete.

Any partial function on numbers which can be written as a Turing Machine (or a Python program) can be written as a lambda term on Church numerals.

Computability and the Lambda Calculus

Theorem (Informal). The lambda calculus is Turing-complete.

Any partial function on numbers which can be written as a Turing Machine (or a Python program) can be written as a lambda term on Church numerals.

De Bruijn Indices

Motivation

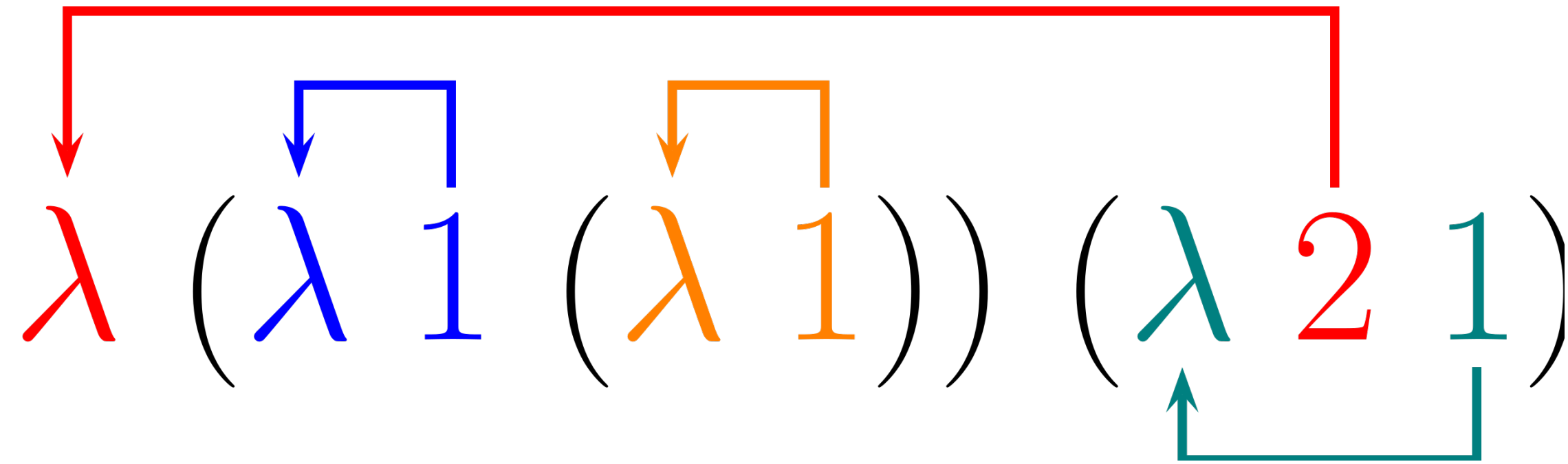
$$\lambda x . xz =_{\alpha} \lambda y . yz \neq_{\alpha} \lambda \boxed{\bullet} . \boxed{\bullet} z$$

We always consider terms up to $=_{\alpha}$.

What we *really* want is to be able to replace the binding variable with a **pointer**.

In math speak, we want to give a "canonical element" for the α -equivalence class.

De Bruijn Indices



The idea. Bound variables are represented as numbers, the depth away from the *binding site*.

$$M ::= \mathbb{N} \mid \lambda M \mid MM$$

This gives an incredibly simple grammar.

What about free variables?

We can use numbers larger than the depth of the formula.

$$\lambda x . xz \implies \lambda(1\ 2)$$

Or we can use the "locally nameless representation":

$$\lambda x . xz \implies \lambda(1\ z)$$

We keep free variables as they are, and use De Bruijn indices for bound variables.

Pros and Cons

- We no longer need to consider $=_{\alpha}$, equality is structural equality
- β -reduction is a bit harder, now we need to update De-Bruijn indices at each step.
- They make terms harder to read.