$\boxed{\text{Kripke Semantics}}$

Lecture 19

Valuations

Def. A <u>valuation</u> is a function

$$v : V \to \{0, 1\}$$

FALSE TRUE

Given valuation v, we defined

$$\bar{v} : P \to \{0, 1\}$$

e.g.

$$\bar{v}(P \vee Q) = \begin{cases} 1 & \bar{v}(P) = 1 \text{ or } \bar{v}(Q) = 1 \\ 0 & \text{o.w.} \end{cases}$$

Semantics of Classical Prop. Logic (CPL)

- $v \models \varphi$ if $\bar{v}(\varphi) = 1$
- $\models \varphi$ if $v \models \varphi$ for any v
- $\Gamma \models \varphi$ if $v \models \bigwedge \Gamma$ implies $v \models \varphi$

ψ_1, ψ_2, ψ_3 $\qquad\qquad$ $\psi_1 \wedge \psi_2 \wedge \psi_3$

Last Time:

Defined the proof system IPL and CPL

Notation: $\Delta \mid \Gamma \vdash_c \phi$ if there is a derivation in this system

Soundness

Def. A proof system $(\vdash)$ is **sound** w.r.t a semantics $(\models)$ if $\Gamma \vdash \phi$ implies $\Gamma \models \phi$

"if we can prove it then it is true."

CPL is sound w.r.t. valuations.

Thm: For any Δ, Γ, ϕ, if $\Delta \mid \Gamma \vdash_c \phi$ then $\Gamma \vDash \phi$ (for any valuation only over the variables in Δ)

Example: $A, B \mid A \vee B \vdash_c \neg(\neg A \wedge \neg B)$

$$\Downarrow$$

For any valuation $v : \{A, B\} \to \{0, 1\}$, if $\bar{v}(A \vee B) = 1$ then $\bar{v}(\neg(\neg A \wedge \neg B)) = 1$

Proof of Soundness for CPL

Induction on derivations.

e.g.

$$\dfrac{\overset{\Gamma \models A}{\Delta \mid \Gamma \vdash A} \qquad \overset{\Gamma \models A}{\Delta \mid \Gamma \vdash B}}{\Delta \mid \Gamma \vdash A \wedge B}$$

Suppose $\bar{v}(\Gamma)$ then $\bar{v}(A)$ and $\bar{v}(B)$

so by def. $\bar{v}(A \wedge B)$

Completeness

Def. A $(\vdash)$ is is <u>complete</u> w.r.t $(\models)$

if $\Gamma \models \phi$ implies $\Gamma \vdash \phi$

"Anything true, we can prove"

CPL is complete w.r.t valuations

Thm. For any Δ, Γ, φ, if

$$\Gamma \vDash_\Delta \varphi \quad \text{then} \quad \Delta \mid \Gamma \vdash_c \varphi$$

Lemma: If $\vDash_\Delta \varphi$ then $\Delta \mid \emptyset \vdash_c \varphi$

e.g. $\vDash_{\{A\}} \neg\neg A \to A$ so $A \mid \emptyset \vdash_c \neg\neg A \to A$

IPL is <u>not</u> completeness w.r.t valuations

(1) $\vDash_{\{A\}} A \vee \neg A$ ✓ since if $v : \{A\} \to \{0, 1\}$

then $\bar{v}(A \vee \neg A) = \begin{cases} 1 & \text{if } \bar{v}(A) = 1 \text{ or } \bar{v}(\neg A) = 1 \\ 0 & \text{o.w.} \end{cases}$

$= \begin{cases} 1 & v(A) = 1 \text{ or } v(A) = 0 \\ 0 & \text{o.w.} \end{cases}$

$= 1$

(2) $A \mid \emptyset \nvdash_I A \vee \neg A$ (today)

Law of Excluded Middle in CPL

$$\dfrac{\dfrac{A \mid A \vdash A}{A \mid A \vdash A \vee \neg A}\,(\vee\text{-}I_l) \qquad \dfrac{A \mid \neg A \vdash \neg A}{A \mid \neg A \vdash A \vee \neg A}\,(\vee\text{-}I_r)}{A \mid \emptyset \vdash_c A \vee \neg A}\,(\mathrm{SNE})$$

Other principle:

Double Negation Elim: $\neg\neg A \to A$

$\to$ to $\vee$: $(A \to B) \to (\neg A \vee B)$

Peirce's Law : $((A \to B) \to A) \to A$

IPL is <u>sound</u> w.r.t valuations

$$\mathrm{IPL} \subset \mathrm{CPL}$$

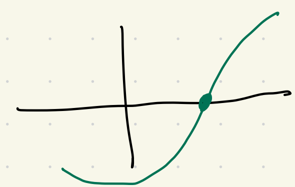
Kipke Model

Intuitionism and Proof

There is no notion truth except what is provable.

- this does not allow for non-constructive existence proof.

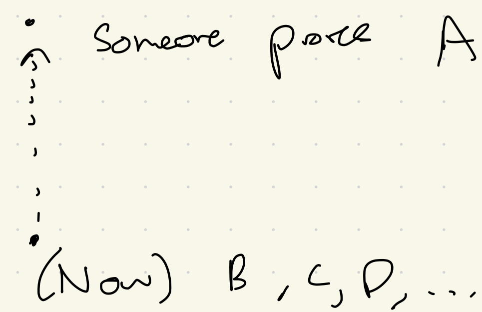
  e.g. Intermediate Value Theorem.



- We have computational interpretation of logic and proof.

Intuitionism and Time

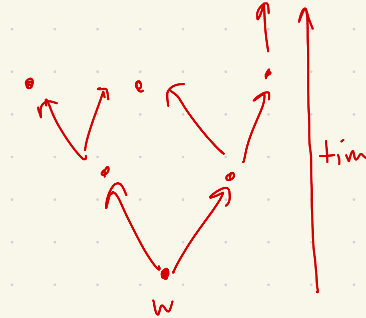
- We have to wait for a proof of A to say is true



- $\neg A$ holds only if no one can eventually prove A.

Kripke Models

$\mathcal{M}$



Def. A model is made up of

- $W_{\mathcal{M}}$: set of possible worlds
- $\alpha_{\mathcal{M}} : W \to 2^{V}$ (a set prop. variables
- $\leq_{\mathcal{M}} \subseteq W \times W$ which reflexive and transitive

$$w \leq_{\mathcal{M}} w \quad \text{(reflexive)}$$

$$w \leq_{\mathcal{M}} u, \; u \leq v \text{ then } w \leq v \quad \text{(transitive)}$$

- $\leq_{\mathcal{M}}$ is montonic w.r.t $\alpha_{\mathcal{M}}$, i.e.

$$w \leq_{\mathcal{M}} u \text{ then } \alpha(w) \subseteq \alpha(u)$$

Know only increases with time

Kripke models as semantics

Given $\mathcal{M} = (W, \alpha, \leq)$ and $w \in W$ we define

① $\mathcal{M}, w \models A$ for $A \in V$ if $A \in \alpha(w)$

② $\mathcal{M}, w \models P \vee Q$ if $\mathcal{M}, w \models P$ or $\mathcal{M}, w \models Q$

③ $\mathcal{M}, w \models P \wedge Q$ if $\mathcal{M}, w \models P$ and $\mathcal{M}, w \models \phi$

④ $\mathcal{M}, w \not\models \bot$

⑤ $\mathcal{M}, w \models A \to B$ if $w \leq v$ and $\mathcal{M}, v \models A$ then $\mathcal{M}, v \models B$

⑥ $\mathcal{M}, w \models \neg A \equiv A \to \bot$ if $w \leq v$ $\mathcal{M}, v \not\models A$.

Semantic Notions

$M \models_K \phi$ if $M, w \models_K \phi$ for any $w \in W_M$

$\models_K \phi$ if $M \models_K \phi$ for any model.

$\Gamma \models_K \phi$ if $\models_K \bigwedge \Gamma \to \phi$.

A new semantics based of Kripke model.

Main Theorem

Theorem. IPL is sound and complete w.r.t. Kripke models.

$$\Delta \mid \Gamma \vdash_I \varphi \iff \Gamma \vDash_K \varphi$$

Lemma: If $M, w \nvDash \varphi$ then

$$\nvdash_I \varphi$$

Kripke Counterexamples

Let's consider $A \mid \emptyset \nvdash_I A \vee \neg A$.

1. $W = \{w, r\}$

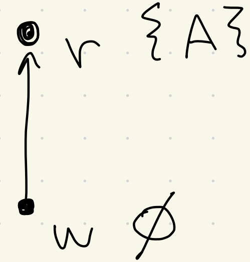
2. $\alpha(w) = \emptyset$

   $\alpha(r) = \{A\}$

3. $w \leq w \quad w \leq r \quad r \leq r$

4. monotonicity ?

   $\alpha(w) \subseteq \alpha(r)$ ✓



$M, w \nvDash A \vee \neg A$

$\Uparrow$

$M, w \nvDash A$ and $M, w \nvDash \neg A$

$\Uparrow \qquad\qquad \Uparrow$

$A \notin \alpha(w) \qquad M, r \vDash A$ and $w \leq r$

$\Uparrow$

$A \in \alpha(r)$

Computational interpetation of CPL

$$A : \mathrm{Type} \vdash \boxed{M} : A \vee \neg A$$

what could M be ?

try :

{ code

catch e :

{ do other thing

$$\frac{\Gamma, x : \neg A \vdash M : B \qquad \Gamma, x : A \vdash N : B}{\Gamma \vdash \mathrm{try}\ x.\ M \quad N}$$

$$\mathrm{try}\ x.\ (\ldots \mathrm{explode}\ x\ P \ldots)\ N$$

$$\downarrow_\beta$$

$$N[P/x]$$

$$\mathrm{try}\ x.\ M \quad N \to M$$

$x \notin M$ free