

Administrivia

Homework 7 is due by Thursday at 11:59PM.

The BUGWU is on strike.

Case Study: STLC in Agda

Type Theory and Mechanized Reasoning
Lecture 16

Outline

See how to represent the simply typed lambda calculus *in Agda*.

Prove meta-theoretic lemmas about STLC, leading to a proof of *type preservation*.

Recap

Recall: Lambda Terms

(Fix a set of variables.)

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variables

application

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Definition. The collection of **lambda terms** is defined inductively.

- Every variable x is a lambda term.
- If M and N are lambda terms, then so is (MN)
- If M is a lambda term, then so is $(\lambda x.M)$ for any variable x

variables

application

abstraction

Recall: Examples

x, y

$$I \triangleq \lambda x . x$$

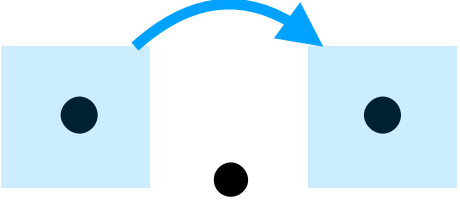
$$K \triangleq \lambda x . \lambda y . x$$

$$A \triangleq \lambda x . \lambda y . xy$$

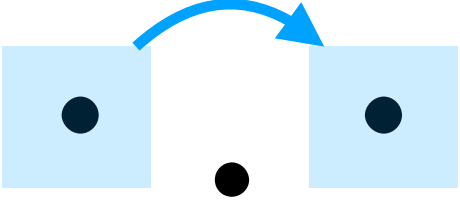
$$\omega \triangleq \lambda x . xx$$

$$\Omega \triangleq \omega\omega = (\lambda x . xx)(\lambda x . xx)$$

Recall: Motivating De Bruijn Indices

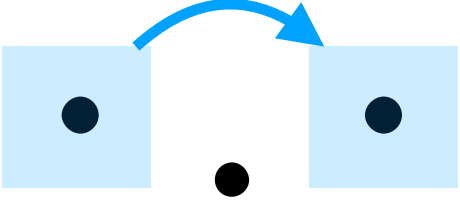
$$\lambda x . xz =_{\alpha} \lambda y . yz =_{\alpha} \lambda \boxed{\bullet} . \boxed{\bullet} z$$


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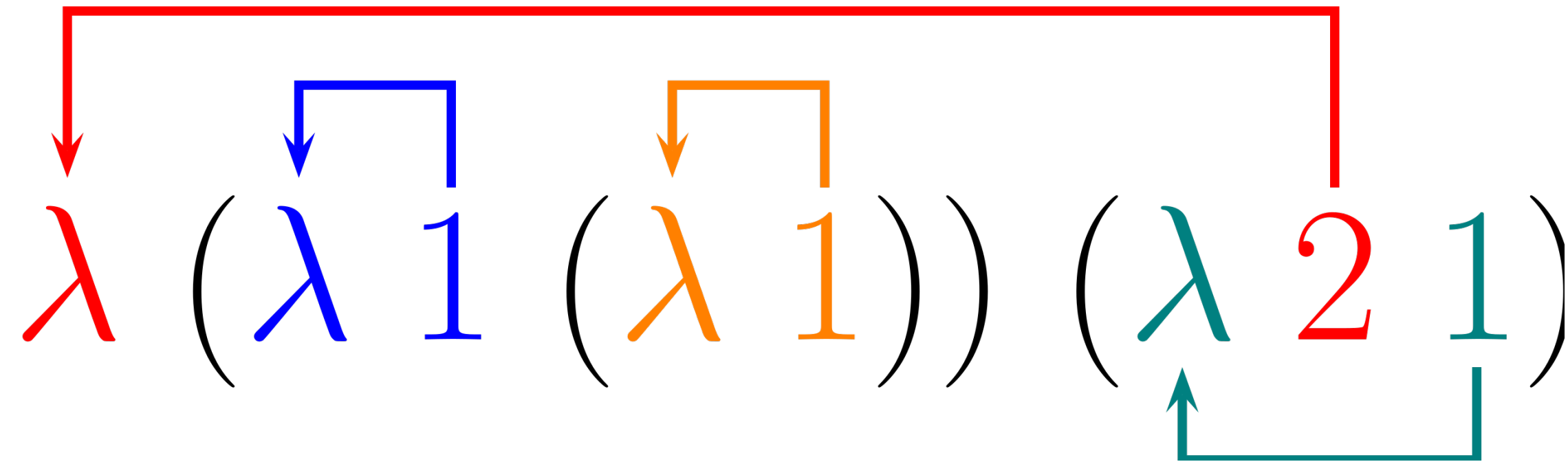
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What we *really* want is to be able to replace the binding variable with a **pointer**.

In math speak, we want to give a **canonical element** for the α -equivalence class.

Recall: De Bruijn Indices



The idea. Bound variables are represented as numbers, the depth away from the *binding site*.

$$M ::= \mathbb{N} \mid \lambda M \mid MM$$

This gives an incredibly simple grammar.

Recall: Free Variables and De Bruijn Indices

$$\lambda x . x(yz) \longrightarrow \lambda . 1(23)$$

Today, we will be using numbers ***larger than the depth*** of the term to represent free variables.

(This will make contexts easier to represent.)

(There is also a very nice trick for representing De Bruijn indices using dependent types.)

demo

(let's define these in Agda)

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$$f : \perp \rightarrow \perp \vdash \lambda x . fx : \perp \rightarrow \perp$$

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term *simple type*
typing statement

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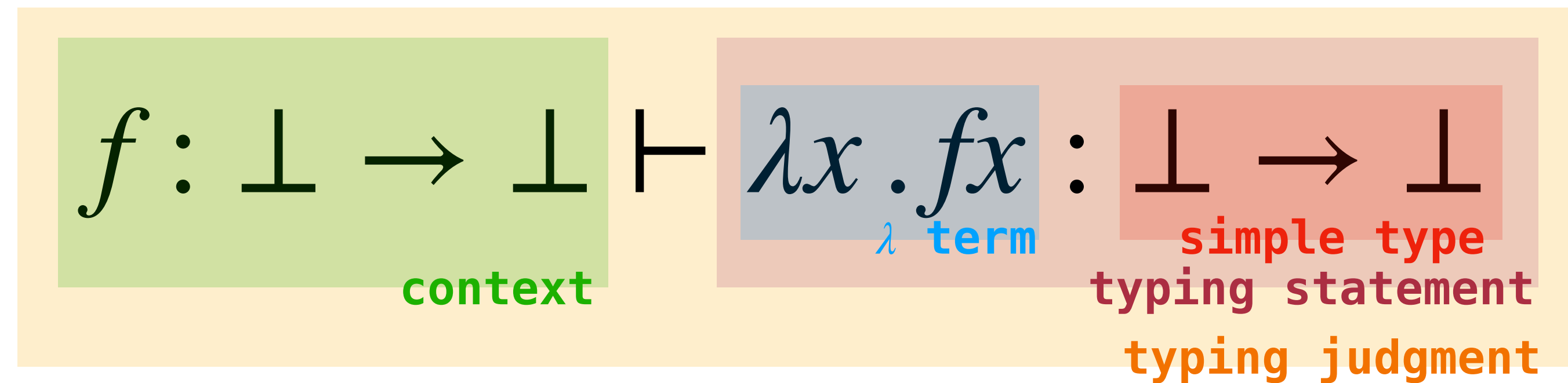
Recall: The Simply Typed Lambda Calculus

$f : \perp \rightarrow \perp$ **context** \vdash $\lambda x. fx$ **term** $:$ $\perp \rightarrow \perp$ **simple type** **typing statement**

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Examples. $\perp \rightarrow \perp$, $(\perp \rightarrow \perp) \rightarrow (\perp \rightarrow (\perp \rightarrow \perp))$

Simply Typed Lambda Calculus (Types)

$$\frac{}{\emptyset \vdash \perp : \text{Type}}$$

$$\frac{\emptyset \vdash A : \text{Type} \quad \emptyset \vdash B : \text{Type}}{\emptyset \vdash A \rightarrow B : \text{Type}}$$

Type formation rules are used to build types within and for judgments.

(These are the same as our inductive rules, but written as typing judgments)

Simply Typed Lambda Calculus (Terms)

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma, x : A \vdash x : A} \quad (x \notin \Gamma)$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : \text{Type}}{\Gamma, x : B \vdash M : A} \quad (x \notin \Gamma)$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Term formation rules are used to generate typeable terms in the simply typed lambda calculus.

Simply Typed Lambda Calculus (Terms)

start

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma, x : A \vdash x : A} \quad (x \notin \Gamma)$$

abstraction

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B}$$

weakening

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : \text{Type}}{\Gamma, x : B \vdash M : A} \quad (x \notin \Gamma)$$

application

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Term formation rules are used to generate typeable terms in the simply typed lambda calculus.

demo

(let's define these in Agda)

Variable Shifting

$$\lambda x . x(\lambda y . y(x(zw))) \longrightarrow \lambda . 0(\lambda . 0(1(23))) \longrightarrow \lambda . 0(\lambda . 0(1(56)))$$

One of the trickier aspects of working with De Bruijn indices is that we often have to ***shift around the values of free variables.***

We will write $\text{shift}_{m,p}(M)$ for the function which increases all free variables at least value m by p .

Example: Weakening

$$x : A \vdash \lambda y . x : C \rightarrow A$$
$$A \vdash \lambda . 1 : C \rightarrow A$$


weakening

$$x : A, z : B \vdash \lambda y . x : C \rightarrow A$$
$$A, B \vdash \lambda . 2 : C \rightarrow A$$

When we represent the variables in a context, they are in increasing order from right to left.

So weakening requires *changing* the typed term.

demo

(let's define these in Agda)

Recall: Induction on Derivations

$$\frac{\vdots}{\Gamma \vdash M : A}$$

If we want to prove that P holds of all *typeable* terms, we have to show that it holds of all terms M ***for any choice of the last inference rule in a derivation of M .***

Thinning Lemma

Theorem. If $\Gamma, \Delta \vdash M : A$ and x does not appear in Δ , then $\Gamma, x : B, \Delta \vdash M : A$.

Using De Bruijn indices:

If $\Gamma, \Delta \vdash M : A$ and $|\Gamma| = m$, then $\Gamma, B, \Delta \vdash \text{shift}_{|\Gamma|, 1}(M) : A$

Proof. By induction on the structure of derivations.

demo

(let's define these in Agda)

Simultaneous Substitution

Let M be a term with free variables $\vec{x} = x_1, \dots, x_k$.
We define $M[\vec{N}/\vec{x}]$ inductively as follows.

$$\gg x_i[N_1/x_1] \dots [N_k/x_k] = N_i$$

lookup

$$\gg (MP)[\vec{N}/\vec{x}] = (M[\vec{N}/\vec{x}]) (P[\vec{N}/\vec{x}])$$

recurse

$$\gg (\lambda M)[\vec{N}/\vec{x}] = \lambda(M[\vec{N}'/\vec{x}]) \text{ where } N'_i = \text{shift}_{0,1}(N_i)$$

recurse and shift

Recall: Simultaneous Substitution

Theorem. If $y_1 : A_1, \dots, y_k : A_k \vdash M : B$ and

$$\Gamma \vdash N_1 : A_1 \text{ and } \dots \text{ and } \Gamma \vdash N_k : A_k$$

then $\Gamma \vdash M[N_1/y_1][N_2/y_2]\dots[N_k/y_k] : B$

Proof. By induction on the structure of derivations.

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This is a **relation** not a function.

Type Preservation

Theorem. If $\Gamma \vdash M : A$ and $M \rightarrow_{\beta} N$ then $\Gamma \vdash N : A$.

Beta reduction doesn't change typability, or the type.

Proof. By induction on the β -reduction relation...(!)

demo

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