Gaussian Elimination (and Numerics)

Geometric Algorithms
Lecture 3

CAS CS 132

Practice Problem

$$\begin{aligned} x + hy &= 3 \\ 2x - 5y &= k \end{aligned}$$

For what values of h and k is the above system inconsistent?

Solution

$$x + hy = 3$$
$$2x - 5y = k$$

Outline

» Look at pseudocode of Gaussian Elimination

» Discuss number representations, and look at the consequences of floating point representations

» *If there's time:* Analyze the running time of Gaussian Elimination

Keywords

forward elimination

back substitution

floating point numbers

IEEE-754

relative error

numpy.isclose

ill-conditioned problems

Recap

Recap: Echelon Form

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ = nonzero, * = anything

Recap: Echelon Form

next leading entry
to the right

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all-zero rows at
the bottom

$\blacksquare$ = nonzero, $*$ = anything

Recap: Reduced Echelon Form

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recap: Reduced Echelon Form

leading entries are 1

$$\begin{bmatrix} 0 & \boxed{1} & * & 0 & 0 & 0 & * & * & \boxed{0} & * \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & * & * & \boxed{0} & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & \boxed{0} & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & \boxed{0} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

other column entries are 0

Recap: The Fundamental Points

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Point 1. we can "read off" the solutions of a system of linear equations from its RREF

Recap: The Fundamental Points

Point 1. we can "read off" the solutions of a system of linear equations from its RREF

Point 2. *every* matrix is row equivalent to a <u>unique</u> matrix in reduced echelon form

Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 `is free`

Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

1. For each pivot position (i,j), isolate x_j in the equation in row i

Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

1. For each pivot position (i, j), isolate x_j in the equation in row i

2. If x_i is not in a pivot column then write

x_i **is free**

Gaussian Elimination

At a High Level

At a High Level

`eliminations + back–substitution`

At a High Level

eliminations + back–substitution

we've already done this

At a High Level

eliminations + back–substitution

we've already done this

but now we'll give the algorithm as <u>pseudocode</u>

A Word of Warning

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The details of Gaussian elimination are tricky

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The goal is not to understand it entirely

A Word of Warning

The details of Gaussian elimination are tricky

The goal is not to understand it entirely

You should understand Gaussian elimination well enough to use it when solving a system by hand

demo

Gaussian Elimination (Specification)

```
FUNCTION GE(A):
  # INPUT: m × n matrix A
  # OUTPUT: row equivalent m × n RREF matrix
  ...
```

Gaussian Elimination (High Level)

```
FUNCTION fwd_elim(A):
  # INPUT: m × n matrix A
  # OUTPUT: row equivalent m × n echelon form matrix
  ...

FUNCTION back_sub(A):
  # INPUT: m × n echelon form matrix A
  # OUTPUT: row equivalent m × n RREF matrix
  ...

FUNCTION GE(A):
  RETURN back_sub(fwd_elim(A))
```

Elimination Stage

Elimination Stage (High Level)

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Input: matrix A of size $m \times n$

Output: echelon form of A

Elimination Stage (High Level)

Input: matrix A of size $m \times n$

Output: echelon form of A

Basic idea: starting top left and moving down, find a leading entry and eliminate it from latter equations

Edge cases

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What if the first equation doesn't have the variable x_1?

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What if the first equation doesn't have the variable x_1?

Swap rows with an equation that does.

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Swap rows with an equation that does.

What if *none* of the equations have the variable x_1?

Edge cases

What if the first equation doesn't have the variable x_1?

Swap rows with an equation that does.

What if *none* of the equations have the variable x_1?

Find the *leftmost* variable which appears in *any* of the remaining equations.

Elimination Stage (Example)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
nonzero
entry

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \boxed{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
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$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \boxed{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Swap R_1 and R_3

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$
\begin{array}{r}
\\
\texttt{next entry} \\
\texttt{to zero} \\
\\
\end{array}
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
\boxed{3} & -7 & 8 & -5 & 8 & 9 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
$$

Elimination Stage (Example)

$$\begin{array}{c} \texttt{next entry} \\ \texttt{to zero} \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ \boxed{3} & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1$$

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{array}{c} \texttt{leftmost} \\ \texttt{nonzero} \\ \texttt{entry} \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

swap R_2 with R_2

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{array}{r} \\ \\ \texttt{next entry to zero} \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

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$$R_3 \leftarrow R_3 - \frac{3R_2}{2}$$

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Elimination Stage (Example)

$$
\begin{array}{c}
\texttt{leftmost} \\ \texttt{nonzero} \\ \texttt{entry}
\end{array}
\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}
$$

Elimination Stage (Example)

leftmost nonzero entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

swap R_3 with R_3

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

done with elimination stage
going to back substitution stage

Elimination Stage (Pseudocode)

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```
FUNCTION fwd_elim(A):
```

Elimination Stage (Pseudocode)

```
FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
```

Elimination Stage (Pseudocode)

```
FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [rows i...m are all-zeros]: # if remaining rows are zero
```

Elimination Stage (Pseudocode)

```
FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [rows i...m are all-zeros]: # if remaining rows are zero
      RETURN A
```

Elimination Stage (Pseudocode)

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FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [rows i...m are all-zeros]: # if remaining rows are zero
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      (j, k) ← [position of leftmost entry in the rows i...m]
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      (j, k) ← [position of leftmost entry in the rows i...m]
      [swap row i and row j]
```

Elimination Stage (Pseudocode)

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FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [rows i...m are all-zeros]: # if remaining rows are zero
      RETURN A
    ELSE:
      (j, k) ← [position of leftmost entry in the rows i...m]
      [swap row i and row j]
      FOR [l from i + 1 to m]: # for all remaining rows
```

Elimination Stage (Pseudocode)

```
FUNCTION fwd_elim(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [rows i...m are all-zeros]: # if remaining rows are zero
      RETURN A
    ELSE:
      (j, k) ← [position of leftmost entry in the rows i...m]
      [swap row i and row j]
      FOR [l from i + 1 to m]: # for all remaining rows
        [zero out A[l, k] using a replacement operation]
```

Elimination Stage (Pseudocode)

```
FUNCTION fwd_elim(A):
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      [swap row i and row j]
      FOR [l from i + 1 to m]: # for all remaining rows
        [zero out A[l, k] using a replacement operation]
  RETURN A
```

Elimination Stage (Example)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Elimination Stage (Example)

$$
\begin{array}{c}
\\
\\
\texttt{leftmost} \\
\texttt{nonzero} \\
\texttt{entry}
\end{array}
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
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\boxed{3} & -9 & 12 & -9 & 6 & 15
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$$

Elimination Stage (Example)

$$\begin{array}{c} \\ \\ \texttt{leftmost nonzero entry} \end{array} \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \boxed{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Swap R_1 and R_3

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{array}{r l} & \\ \texttt{next entry to zero} & \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ \boxed{3} & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Elimination Stage (Example)

$$\begin{array}{r} \\ \texttt{next entry} \\ \texttt{to zero} \\ \\ \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1$$

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
nonzero
entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
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entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

swap R_2 with R_2

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{array}{r} \\ \\ \texttt{next entry to zero} \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{array}{r} \\ \\ \texttt{next entry to zero} \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & \boxed{3} & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - \frac{3R_2}{2}$$

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Elimination Stage (Example)

leftmost
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entry

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Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

leftmost nonzero entry

swap R_3 with R_3

Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Elimination Stage (Example)

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

done with elimination stage
going to back substitution stage

Back Substitution Stage

Back Substitution Stage (High Level)

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Input: matrix A of size $m \times n$ in echelon form

Output: reduced echelon form of A

Back Substitution Stage (High Level)

Input: matrix A of size $m \times n$ in echelon form

Output: reduced echelon form of A

Basic idea: scale pivot positions and eliminate the variable for that column from the other equations

Gaussian Elimination (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{array}{c} \text{pivot} \\ \text{position} \end{array} \quad \begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

pivot position

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 \,/\, 3$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{array}{cc} & \\ \texttt{pivot position} & \left[\begin{array}{cccccc} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

Gaussian Elimination (Example)

$$\begin{array}{r} \\ \texttt{pivot position} \\ \\ \end{array} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 \,/\, 2$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry to zero

$$\begin{bmatrix} 1 & \boxed{-3} & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & \boxed{-3} & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{array}{c} \\ \\ \text{pivot position} \end{array} \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{array}{cc} & \\ & \\ \texttt{pivot position} & \end{array}\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

$$R_3 \leftarrow R_3 \,/\, 1$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_3$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & \boxed{5} & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & \boxed{5} & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 5R_3$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

done with back substitution phase

Back Substitution (Psuedocode)

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
      j ← index of leading entry of row i
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
      j ← index of leading entry of row i
      R_i(A) ← R_i(A) / A[i, j] # divide by leading entry
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
      j ← index of leading entry of row i
      R_i(A) ← R_i(A) / A[i, j] # divide by leading entry
      FOR [k from 1 to i - 1]: # for the rows above the current one
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
      j ← index of leading entry of row i
      R_i(A) ← R_i(A) / A[i, j] # divide by leading entry
      FOR [k from 1 to i − 1]: # for the rows above the current one
        R_k(A) ← R_k(A) − R[k, j] ∗ R_i(A)
        # zero out R[k, j] above the leading entry
```

Back Substitution (Psuedocode)

```
FUNCTION back_sub(A):
  FOR [i from 1 to m]: # for each row from top to bottom
    IF [row i has a leading entry]:
      j ← index of leading entry of row i
      R_i(A) ← R_i(A) / A[i, j] # divide by leading entry
      FOR [k from 1 to i − 1]: # for the rows above the current one
        R_k(A) ← R_k(A) − R[k, j] ∗ R_i(A)
        # zero out R[k, j] above the leading entry
  RETURN A
```

Gaussian Elimination (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

pivot position

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

pivot position

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \leftarrow R_1 / 3$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$
\begin{array}{c} \text{pivot} \\ \text{position} \end{array}
\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}
$$

Gaussian Elimination (Example)

$$\begin{array}{r} \\ \text{pivot position} \\ \\ \end{array} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 \,/\, 2$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & \boxed{-3} & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry to zero

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{array}{c} \\ \\ \text{pivot position} \end{array} \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$
\begin{array}{r c}
 & \\
 & \\
\texttt{pivot position} &
\end{array}
\begin{bmatrix}
1 & 0 & -2 & 3 & 5 & -4 \\
0 & 1 & -2 & 2 & 1 & -3 \\
0 & 0 & 0 & 0 & \boxed{1} & 4
\end{bmatrix}
$$

$$R_3 \leftarrow R_3 \;/\; 1$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_3$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & \boxed{5} & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

next entry to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & \boxed{5} & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 5R_3$$

Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination (Example)

$$\begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

done with back substitution phase

Gaussian Elimination (Example)

$$x_1 = (-24) + 2x_3 - 3x_4$$

$$x_2 = (-7) + 2x_3 - 2x_4$$

x_3 `is free`

x_4 `is free`

$$x_5 = 4$$

Recap: Solving a System of Linear Equations

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1. Write your system as an augmented matrix

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1. Write your system as an augmented matrix

2. Find the RREF of that matrix
Gaussian elimination

3. Read off the solution from the RREF

Numerics

demo

Significant Figures (Sig Figs)

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Do you remember sig figs from science class?

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Rulers only work up to $\pm 1\mathrm{mm}$

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so we can't talk about nanometer differences

Significant Figures (Sig Figs)

Do you remember sig figs from science class?

Rulers only work up to ± 1mm

so we can't talk about nanometer differences

We run into a similar problem with decimal numbers and computers

Number Representations

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Your computer is a collection of fixed size registers

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Each register holds a sequence of bits

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The Goal: represent numbers so they fit in those registers

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Each register holds a sequence of bits

The Goal: represent numbers so they fit in those registers

this is, of course, ~~a lie~~ an abstraction

Number Representations

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Number Representations

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Question. Which bits represent what?

Number Representations

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Question. Which bits represent what?

<u>things to consider:</u>
» simple idea (easy to understand)
» maximize coverage (not too redundant)
» simple numeric operations (easy to use)

Unsigned Integers

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

`value`

`binary value (we should know this by now)`

`e.g. 10001010 represents`

$$1(2^7) + 0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$$

Signed Integers

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

sign value

sign bit + binary value

e.g. 10001010 represents

$$-1 \times \left(0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)\right)$$

Floating-Point Numbers (Figures)

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`floats in python use` *`64 bits`*

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floats in python use *64 bits*

That's 1.8×10^{19} possible values

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floats in python use *64 bits*

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We can't represent everything. We'll have to choose and then round

Question. Which ones should we represent?

Floating-Point Numbers (Idea)

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Integers work because they are **discrete and evenly spaced**

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What if we evenly discretize a range of values?

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Integers work because they are **discrete and evenly spaced**

What if we evenly discretize a range of values?

i.e., represent

..., −0.001, 0, 0.0001, 0.002, 0.003, 0.004,...

Discuss the advantages and disadvantages of this approach

Floating-Point Numbers (IEEE-754)

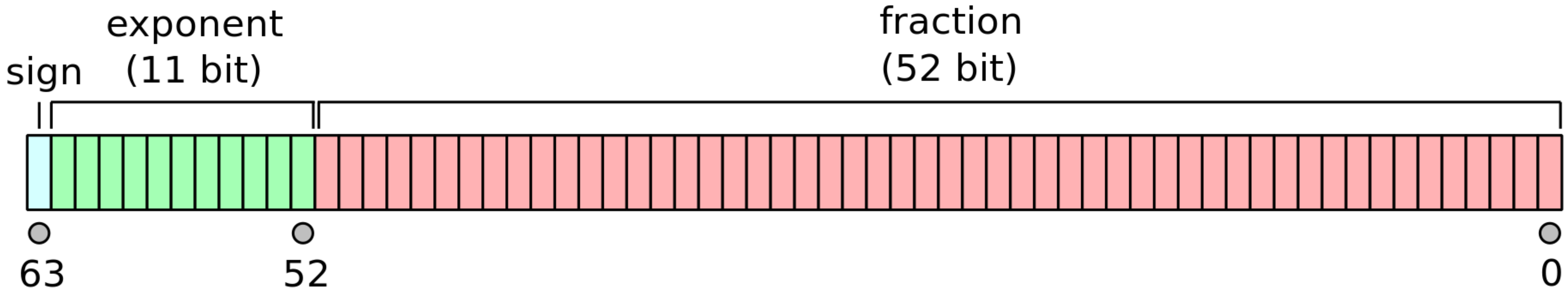
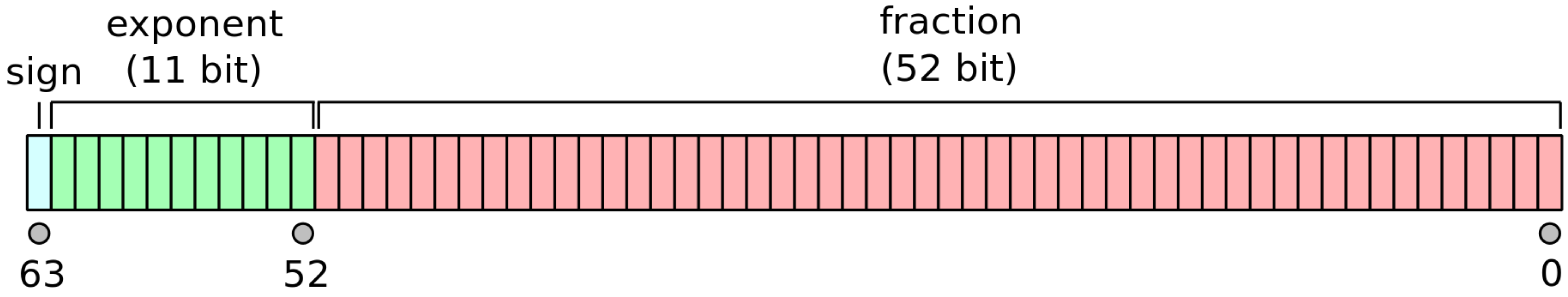


image source

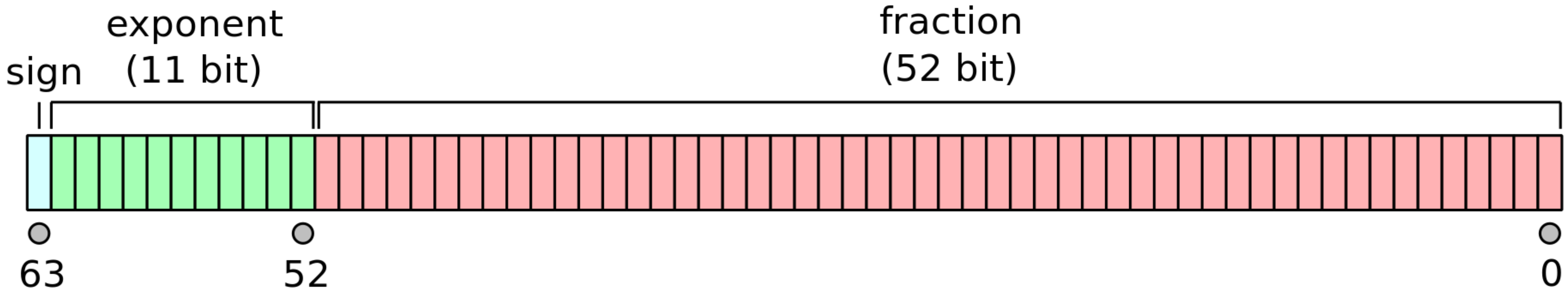
Floating-Point Numbers (IEEE-754)



This is like scientific notation, but binary:

image source

Floating-Point Numbers (IEEE-754)



This is like scientific notation, but binary:

$$(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent} - (2^{10} - 1)}$$

It's an accepted standard, not perfect, but it works well

image source

Question

$$(-1)^{\mathsf{sign}} \times \left(1 + \frac{\mathsf{fraction}}{2^{52}}\right) \times 2^{\mathsf{exponent}-(2^{10}-1)}$$

Any ideas why this is better/worse?

And why not have a sign bit for the exponent?

Step Size

$$(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

image source

$$(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

Step Size

⟵ step size increases with magnitude ⟶

Definition. *step size* is the space between two floating-point representations

<u>image source</u>

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Definition. *step size* is the space between two floating-point representations

for fixed exponent n two numbers are at least

$$0.00\ldots001 \times 2^n = 2^{-52} \times 2^n$$

away (why?)

image source

Step Size

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← step size increases with magnitude →

Definition. *step size* is the space between two floating-point representations

for fixed exponent n two numbers are at least

$$0.00\ldots001 \times 2^n = 2^{-52} \times 2^n$$

away (why?)

Step size <u>doubles</u> for each exponent

image source

Things to Keep in Mind

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`IEEE-754 defines a subset of decimal numbers`

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operations on floating point numbers attempt to give you the <u>closest</u> to the actual value, though there will be errors

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operations on floating point numbers attempt to give you the <u>closest</u> to the actual value, though there will be errors

we can assume when we write down a number like '0.3' we get the closest IEEE-754 value

Relative Error

Observation. ± 0.001 is *tiny* error for 10^{20} but *massive* for 10^{-20}

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Relative Error.

$$\text{err}_{\text{rel}} = \frac{\text{err}}{\text{val}}$$

IEEE-754 keeps relative error <u>small</u>

Relative Error (Calculation)

(fix an exponent n)

$$(-1)^{\mathsf{sign}} \times \left(1 + \frac{\mathsf{fraction}}{2^{52}}\right) \times 2^{\mathsf{exponent} - (2^{10} - 1)}$$

Relative Error (Calculation)

(fix an exponent n)

$$(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

error is determined by step-size

$$\text{err} \leq 2^{-52} \times 2^{n}$$

Relative Error (Calculation)

(fix an exponent n)

$$(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

the smallest number we can represent is 1.0×2^n

$$\text{val} \geq 1.0 \times 2^n$$

(why do we care about a lower bound on val?)

Relative Error (Calculation)

(fix an exponent n*)*

$$(-1)^{\mathsf{sign}} \times \left(1 + \frac{\mathsf{fraction}}{2^{52}}\right) \times 2^{\mathsf{exponent}-(2^{10}-1)}$$

Relative Error (Calculation)

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the relative error is *small*

$$\mathsf{val} \geq 1.0 \times 2^n$$

$$\mathsf{err} \leq 2^{-52} \times 2^n$$

Relative Error (Calculation)

(fix an exponent n)

$$(-1)^{\mathsf{sign}} \times \left(1 + \frac{\mathsf{fraction}}{2^{52}}\right) \times 2^{\mathsf{exponent}-(2^{10}-1)}$$

`the relative error is` *`small`*

$$\mathsf{val} \geq 1.0 \times 2^n$$

$$\mathsf{err} \leq 2^{-52} \times 2^n$$

$$\mathsf{err}_{\mathsf{rel}} = \frac{\mathsf{err}}{\mathsf{val}} \leq \frac{2^{-52} \times 2^n}{1.0 \times 2^n} = 2^{-52} \approx 10^{-16}$$

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(fix an exponent n)

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$\approx$16 digits of accuracy

Not bad, but also not great

demo

(example from the notes)

The Takeaways

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`operations on floating-points are not exact`

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What can we do about it?

Principle 1: Closeness

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When doing floating-point calculations in a program, define an error margin and use that for equality checking

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When doing floating-point calculations in a program, define an error margin and use that for equality checking

In Practice.

```
Replace    x == y
with       numpy.isclose(x, y)
```

demo

Principle 2: Small Differences

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Make sure you understand your error tolerance when looking that the small differences of large numbers.

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In Practice. Don't expect $a - b$ to be small when a and b are "close" but very large.

demo

Principle 3: Ill-Conditioned Problems

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Make sure your problem is not sensitive to small errors.

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Make sure your problem is not sensitive to small errors.

In Practice. for example, don't divide by numbers much smaller than your error tolerance

demo

One Last Note: Special Numbers

`0`	(we can't already represent 0?)
`nan`	stands for not a number, .e.g, sqrt(-2)
`inf`	symbolic infinity, behaves as expected

Extra Topic:
Analyzing the Algorithm

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We will not use $O(\cdot)$ notation!

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For numerics, we care about number of **FL**oating–oint **OP**erations (FLOPs):

>> addition
>> subtraction
>> multiplication
>> division
>> square root

Analyzing the Algorithm

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For numerics, we care about number of **FL**oating-oint **OP**erations (FLOPs):

>> addition
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$2n$ vs. n is very different when $n \sim 10^{20}$

Dominant Terms

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that said, we don't care about *exact* bounds

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A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \to \infty} \frac{f(i)}{g(i)} = 1$$

Dominant Terms

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A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \to \infty} \frac{f(i)}{g(i)} = 1$$

for polynomials, they are equivalent to their dominant term

Dominant Terms

The **dominant term** of a polynomial is the monomial with the highest degree

$$\lim_{i \to \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$ dominates the function even though the coefficient for x^2 is so large

Parameters

n : number of variables

m : number of equations (we will assume $m = n$)

$n+1$: number of rows in the augmented matrix

The Cost of a Row Operation

$$R_i \leftarrow R_i + aR_j$$

$n+1$ multiplications for the scaling

$n+1$ additions for the row additions

Tally: $2(n+1)$ FLOPS

Cost of First Iteration of Elimination

$$\begin{aligned} R_2 &\leftarrow R_2 + a_2 R_1 \\ R_3 &\leftarrow R_3 + a_3 R_1 \\ &\vdots \\ R_n &\leftarrow R_n + a_n R_1 \end{aligned}$$

Repeated row operations for each row except the first

Tally: $\approx 2n(n+1)$ FLOPS

Rough Cost of Elimination

repeating this last process at most n times gives us a dominant term $2n^3$

we can give a better estimation...

Tally: $\approx 2n^2(n+1)$ FLOPS

Cost of Elimination

$$
\begin{bmatrix}
0 & \blacksquare & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

At iteration i, we're only interested in rows after i

And to the right of column i

Cost of Elimination

```
Iteration 1:
```
$2n(n+1)$

```
Iteraiton 2:
```
$2(n-1)n$

```
Iteration 3:
```
$2(n-2)(n-1)$

$\vdots$ $\qquad +$

$$\sum_{k=1}^{n} 2k(k+1) \approx \frac{2n(n+1)(2n+1)}{6} \sim (2/3)n^3$$

Tally: $\sim (2/3)n^3$ FLOPS

Cost of Back Substitution

Assume no free variables, for each pivot, we only need to:

» zero out a position in 1 row (0 FLOPS)

» add a value to the last row (1 FLOP)

at most 1 FLOP per row per pivot $\sim n^2$

Tally: $\sim (2/3)n^3$ FLOPS

Cost of Gaussian Elimination

Tally: $\sim (2/3)n^3$ FLOPS

(dominated by elimination)

Summary

Gaussian elimination is a codification of forward elimination and back-substitution

Decimal numbers are represented in your computer as **floating points**, which can result in error when doing computations

The running time of Gaussian elimination is $\sim (2/3)n^3$ FLOPS