Linear Equations

Geometric Algorithms
Lecture 1

CAS CS 132

Outline

- Motivating examples of linear systems
- Formally define linear systems
- Solve systems of linear equations

Keywords

linear equations

coefficient

unknowns

point sets

hyperplanes

systems of linear equations

consistency

augmented matrix

coefficient matrix

forward-elimination

back-substitution

row operations

row equivalence

Motivation

Lines (Slope-Intercept Form)

$$y = mx + b$$

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$$y = \boxed{m}x + \boxed{b}$$

slope y-intercept

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$$y = \boxed{m}x + \boxed{b}$$

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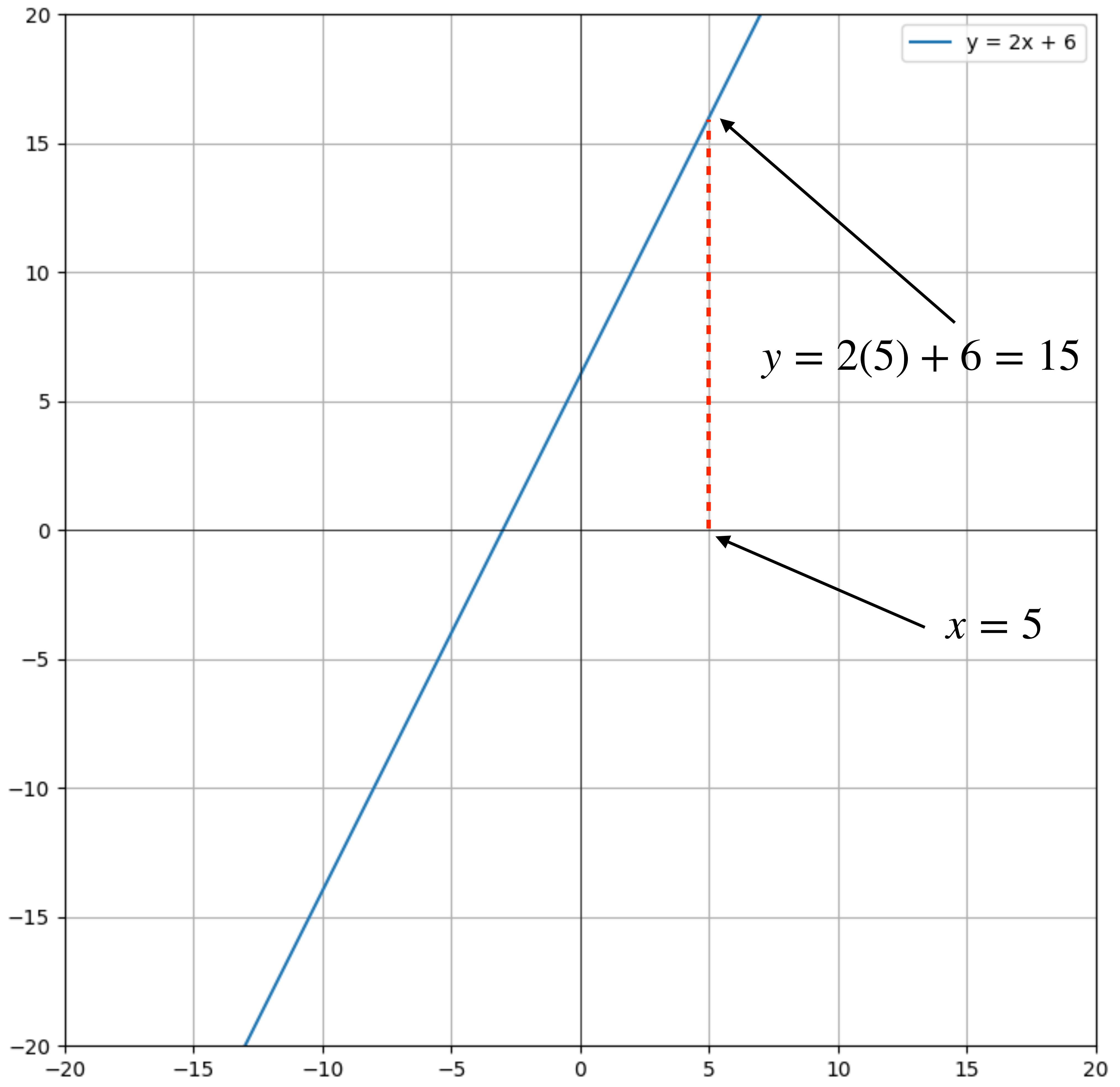
Given a value of x, compute a value of y

Lines (Graph)



Lines (General Form)

$$ax + by = c$$

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x-intercept: $\frac{c}{a}$

Lines (General Form)

$$\boxed{b}\,y \text{ with } ax + by = \boxed{c}$$

x-intercept: $\frac{c}{a}$

y-intercept: $\boxed{\frac{c}{b}}$

Lines (General Form)

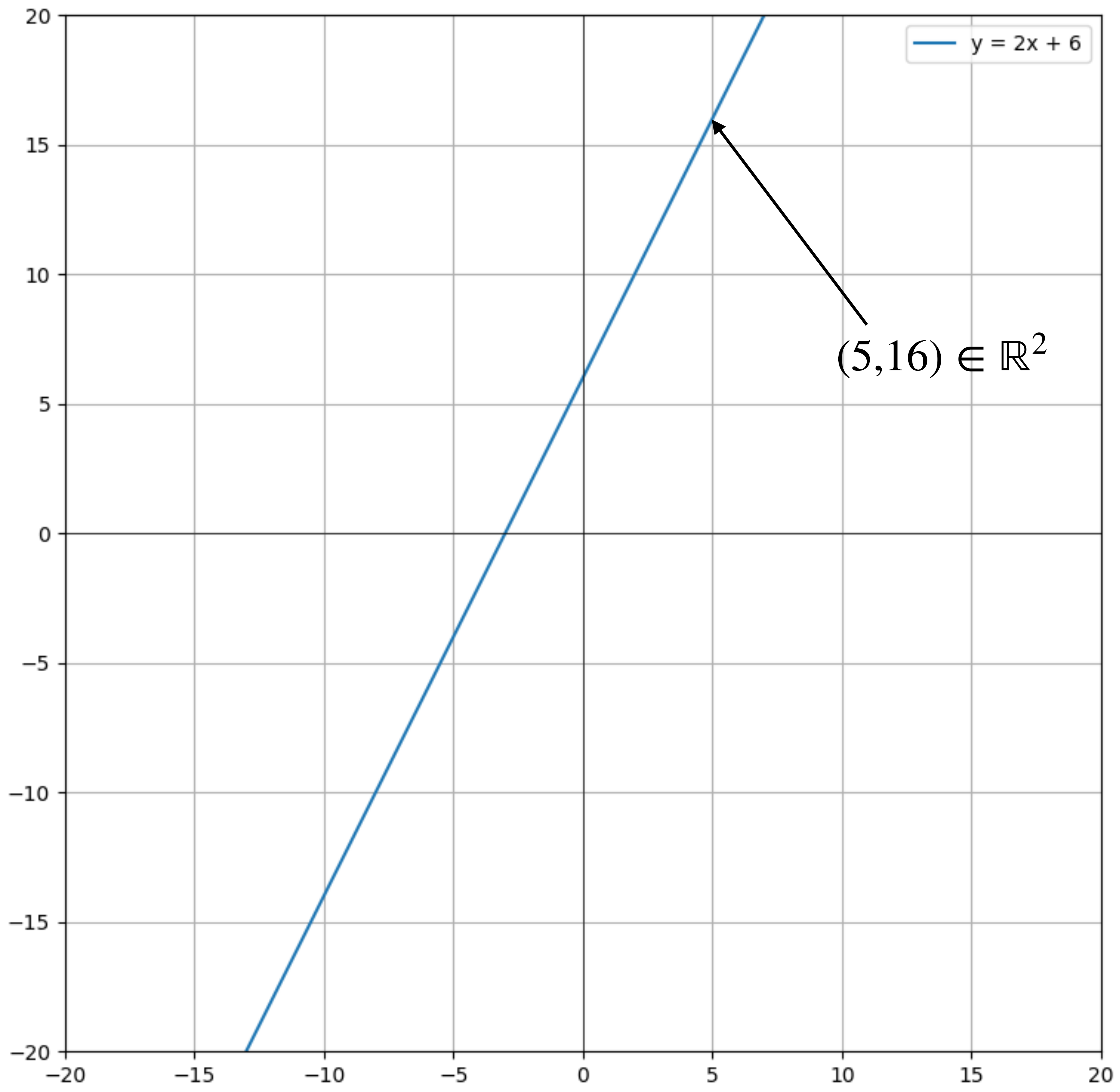
$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

What values of x and y make the equality hold?

Lines (Graph)



$$\{(x, y) : (-2)x + y = 6\}$$

Lines

slope-int → general

$$(-m)x + y = b$$

general → slope-int

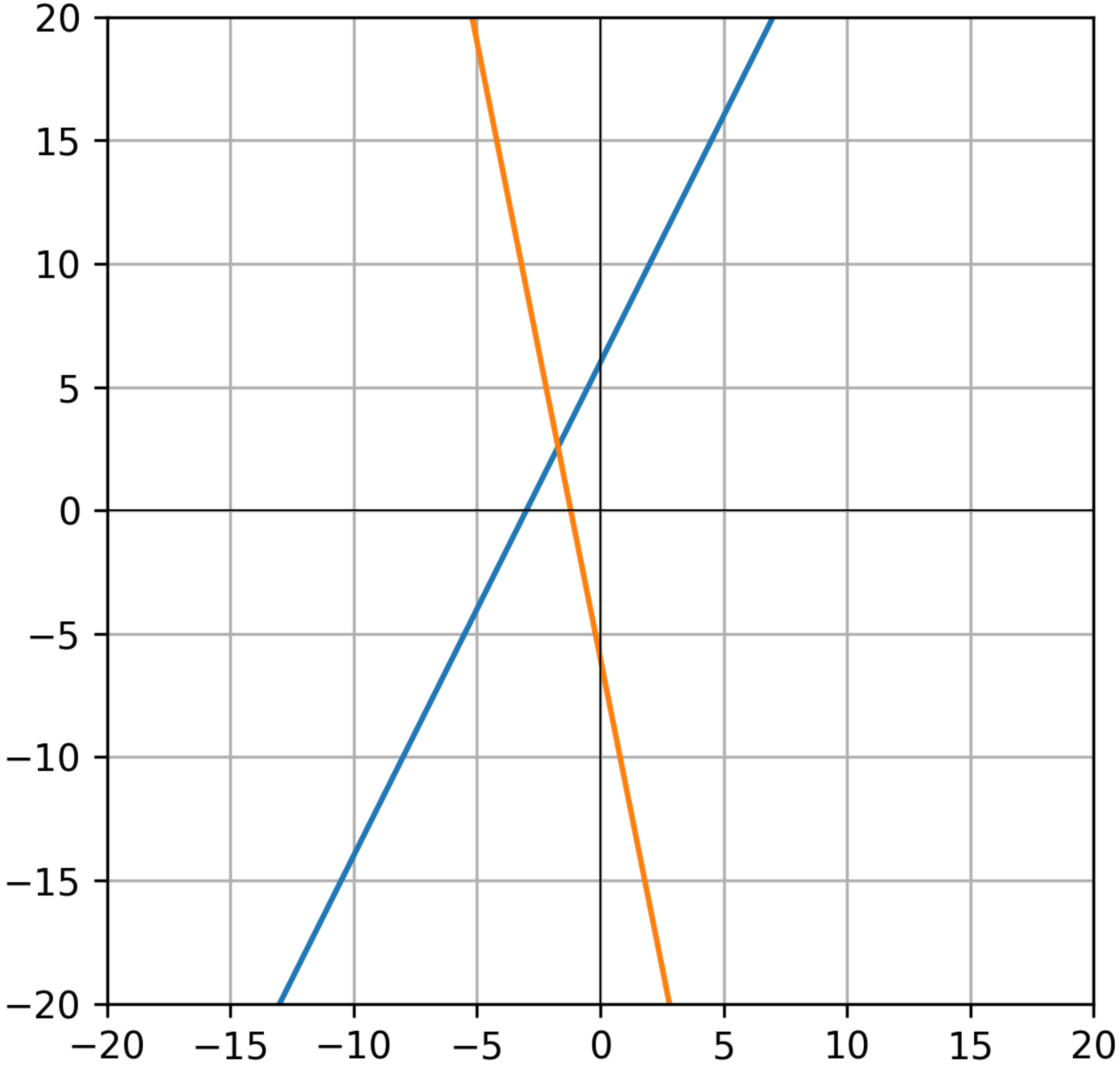
$$y = \left(\frac{-a}{b}\right)x + \frac{c}{b}$$

Line Intersection

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Question. Given two lines, where do they intersect?

Line Intersection (Graph)



Line Intersection (Alternative)

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

Line Intersection (Alternative)

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

This is the same question

Example: Balancing Chemical Equations

$$C_6H_{12}O_6 \rightarrow C_2H_5OH + CO_2$$

Glucose Ethanol

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(for science)

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Glucose   Ethanol

How much ethanol is produced by fermentation?
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The **number of atoms** has to be *preserved* on each side of the equation

Balancing Chemical Equations

$$\alpha \mathrm{C_6H_{12}O_6} \rightarrow \beta \mathrm{C_2H_5OH} + \gamma \mathrm{CO_2}$$

Glucose Ethanol

Balancing Chemical Equations

$$\underset{\texttt{Glucose}}{\alpha C_6H_{12}O_6} \rightarrow \underset{\texttt{Ethanol}}{\beta C_2H_5OH} + \gamma CO_2$$

$$6\alpha = 2\beta + \gamma \qquad \text{(C)}$$

$$12\alpha = 6\beta \qquad \text{(H)}$$

$$6\alpha = \beta + 2\gamma \qquad \text{(O)}$$

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose Ethanol

$$6\alpha - 2\beta - \gamma = 0 \qquad (C)$$

$$12\alpha - 6\beta = 0 \qquad (H)$$

$$6\alpha - \beta - 2\gamma = 0 \qquad (O)$$

Formal Definitions

Linear Equations

Definition. A ***linear equation*** in variables $x_1, x_2, \ldots, x_n$ is an equation which can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Linear Equations

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coefficients

$$\boxed{a_1} x_1 + \boxed{a_2} x_2 + \ldots + \boxed{a_n} x_n = b$$

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Linear Equations

Definition. A ***linear equation*** in variables $x_1, x_2, \ldots, x_n$ is an equation which can be written in the form

coefficients unknowns

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where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Examples

Point sets

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

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The points such that the equation holds

Examples

Linear Equations, Geometrically

If a 2D linear equation represents a ***line*** in the plane, then a 3D linear equation is...

Linear Equations, Geometrically

If a 2D linear equation represents a ***line*** in the plane, then a 3D linear equation is...

Not a line...

Demo

$$0x + 0y + z = 5$$

What does the point set of this linear equation look like?

Demo

$$-x + 0y + z = 5$$

How about this one?

Demo

$$-x + -y + z = 5$$

How about this one?

Hyperplanes

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The point set of a general linear equation is called a **hyperplane**

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After three dimensions, we can no longer visualize point sets

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After three dimensions, we can no longer visualize point sets

Theme of the course: Hyperplanes "behave" like lines and planes in many respects

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Linear Equations, Geometrically

If a 2D linear equation represents a ***line*** in the plane, then a 3D linear equation is...

A plane!

Systems of Linear Equations

Definition. A ***system of linear equations*** is just a collection of linear equations *over the same variables*

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Definition. A ***system of linear equations*** is just a collection of linear equations *over the same variables*

Definition. A ***solution*** to a system is a point that satisfies all its equations *simultaneously*

Example

$$x + 2y = 1$$
$$-x - y - z = -1$$
$$2x + 6y - z = 1$$

Show that $(3, -1, -1)$ *is solution to the above linear system*

System of Linear equations (General-form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

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$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Consistency

Definition. A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has <u>no</u> solutions

demo

(consistency/inconsistency in $\mathbb{R}^3$)

Number of Solutions

```
zero the system is inconsistent

one  the system has a unique solution

many the system has infinity solutions
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Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

Matrix Representations

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \approx \begin{aligned} 2x + 3y &= -8 \\ y &= 2 \\ 2y &= 0 \end{aligned}$$

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`Writing down the unknowns is` *`tedious`* `(and more difficult to input into a computer)`

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We'll write down linear systems as **matrices**, i.e., 2D grids of numbers with *fixed* width and height

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We'll write down linear systems as **matrices**, i.e., 2D grids of numbers with *fixed* width and height

a matrix is just a representation

Matrix Representations

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Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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augmented matrix

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coefficient matrix

Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

Solving Linear Systems

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

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The Approach

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Solve for x in terms of y in EQ1

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The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Question

This works, but *why might we want a different approach?*

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Another Approach

Elimination

Eliminate x from the EQ2 and solve for y

Eliminate y from EQ1 and solve for x

Back-Substitution

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

Can we represent these intermediate steps as operations on matrices?

Let's look back at this...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Elementary Row Operations

scaling	multiply a row by a **NONZERO** number
replacement	add a multiple of one row to another
interchange	switch two rows

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scaling	multiply a row by a **NONZERO** number
replacement	add a multiple of one row to another
interchange	switch two rows

These operations don't change the solutions

Scaling Example

$$2x + 3y = -6$$
$$4x - 5y = 10$$

$$\xrightarrow{R_1 \leftarrow 2R_1}$$

$$4x + 6y = -12$$
$$4x - 5y = 10$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \qquad \xrightarrow{R_2 \leftarrow R_2 + R_1} \qquad \begin{aligned} 2x + 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

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$$\xrightarrow{R_2 \leftarrow R_2/(-11)} \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

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$$\xrightarrow{R_2 \leftarrow R_2/(-11)} \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

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$$\xrightarrow{R_1 \leftarrow R_1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

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\begin{aligned}
R_2 &\leftarrow R_2 - 2R_1 \\
R_2 &\leftarrow R_2/(-11) \\
R_1 &\leftarrow R_1 - 3R_2 \\
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\end{aligned}
$$

$$
\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}
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Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2/(-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1/2$$

back-substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

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We can compute solutions by sequence of row operations

Question

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When the final matrix "looks like" a solution

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When the final matrix "looks like" a solution

We'll get more into that next time...

Example

$$\begin{aligned} x - 2y + z &= 5 \\ 2y - 8z &= -4 \\ 6x + 5y + 9z &= -4 \end{aligned}$$

Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve