

Linear Equations

Geometric Algorithms
Lecture 1

Outline

- » Motivating examples of linear systems
- » Formally define linear systems
- » Solve systems of linear equations

Keywords

linear equations

coefficient

unknowns

point sets

hyperplanes

systems of linear equations

consistency

augmented matrix

coefficient matrix

forward-elimination

back-substitution

row operations

row equivalence

Motivation

Lines (Slope-Intercept Form)

$$y = mx + b$$

Lines (Slope-Intercept Form)

$$y = \textcolor{red}{mx} + \textcolor{blue}{b}$$

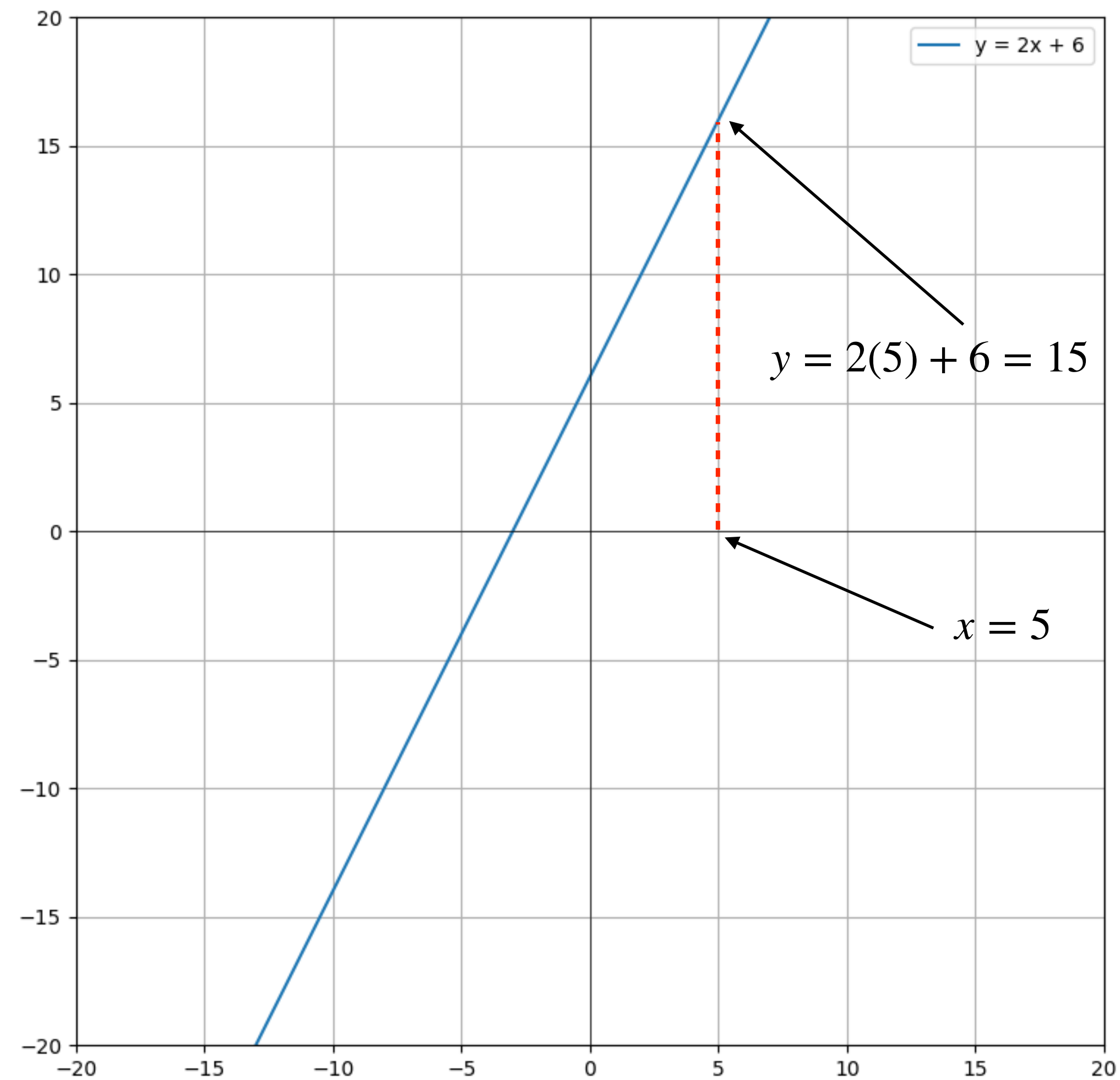
slope y-intercept

Lines (Slope-Intercept Form)

$$y = \underbrace{mx}_{\text{slope}} + \underbrace{b}_{\text{y-intercept}}$$

Given a value of x , compute a value of y

Lines (Graph)



Lines (General Form)

$$ax + by = c$$

Lines (General Form)

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x-intercept: $-\frac{c}{a}$

Lines (General Form)

$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

Lines (General Form)

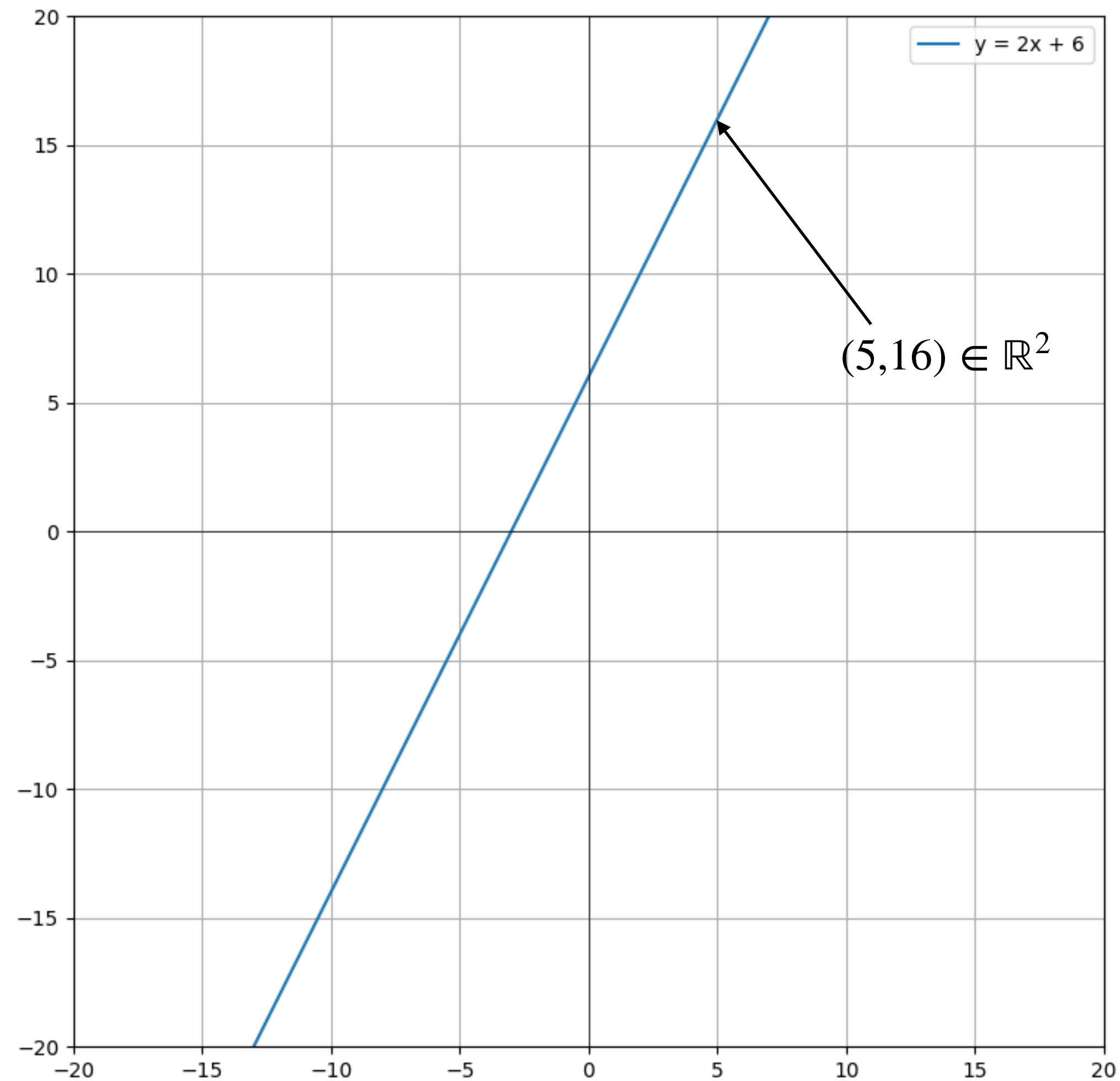
$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

What values of x and y make the equality hold?

Lines (Graph)



$$\begin{matrix} 5 & 16 \\ \{(x, y) : (-2)x + y = 6\} \end{matrix}$$

Lines

slope-int \rightarrow general

$$(-m)x + y = b$$

general \rightarrow slope-int

$$y = \left(\frac{-a}{b} \right) x + \frac{c}{b}$$

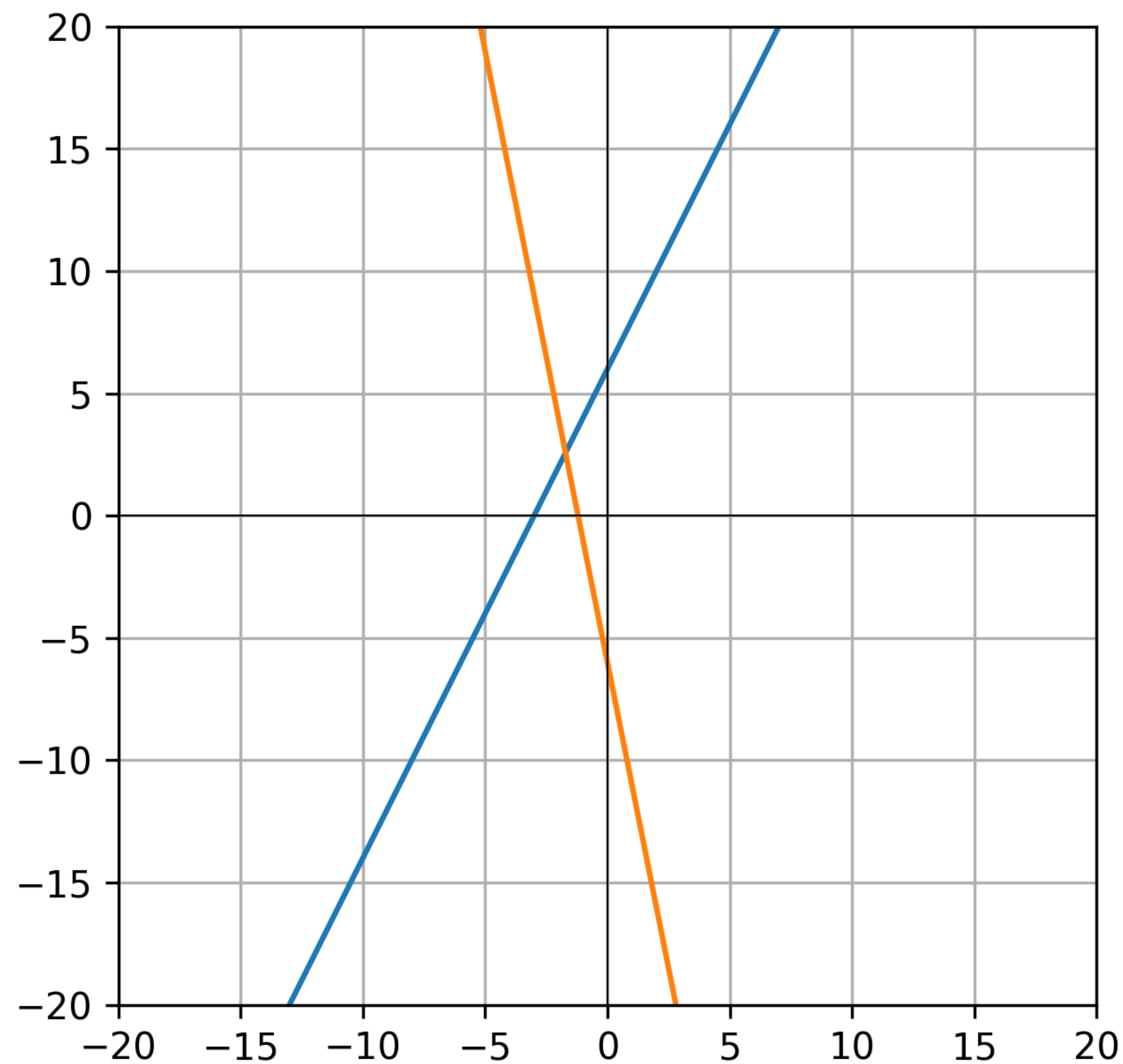
Line Intersection

$$y = m_1x + b_1$$

$$y = m_2x + b_2$$

Question. Given two lines, where do they intersect?

Line Intersection (Graph)



Line Intersection (Alternative)

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

Line Intersection (Alternative)

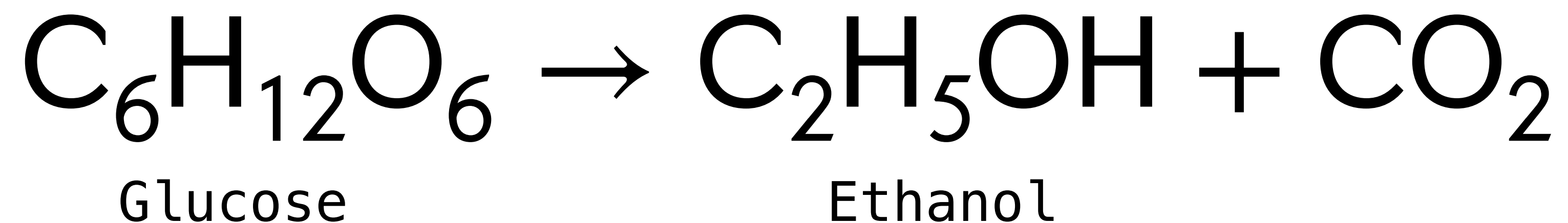
$$a_1x + b_1y = c_1$$

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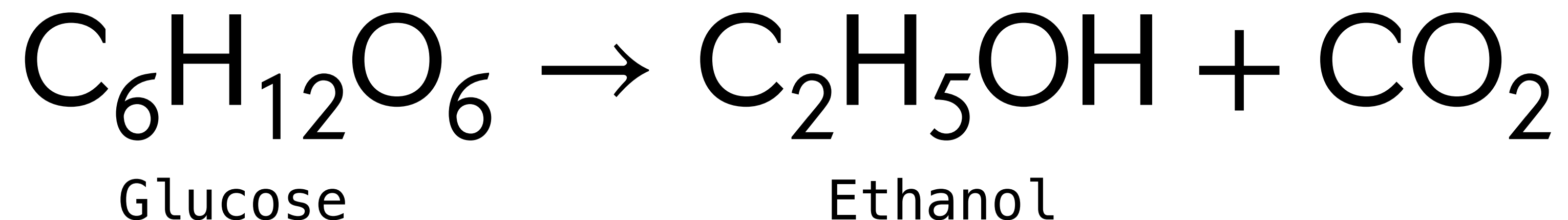
Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

This is the same question

Example: Balancing Chemical Equations

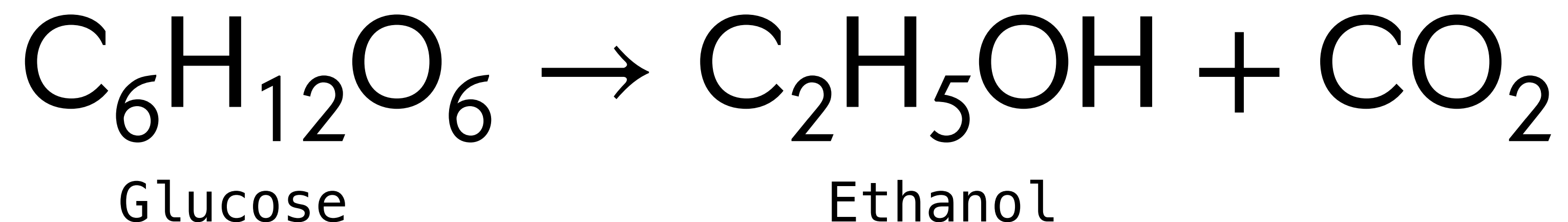


Example: Balancing Chemical Equations



How much ethanol is produced by fermentation?
(for science)

Example: Balancing Chemical Equations



How much ethanol is produced by fermentation?
(for science)

The **number of atoms** has to be *preserved* on each side of the equation

Balancing Chemical Equations



Balancing Chemical Equations



$$6\alpha = 2\beta + \gamma \quad (\text{C})$$

$$12\alpha = 6\beta \quad (\text{H})$$

$$6\alpha = \beta + 2\gamma \quad (\text{O})$$

Balancing Chemical Equations



$$6\alpha - 2\beta - \gamma = 0 \quad (\text{C})$$

$$12\alpha - 6\beta = 0 \quad (\text{H})$$

$$6\alpha - \beta - 2\gamma = 0 \quad (\text{O})$$

Formal Definitions

Linear Equations

Definition. A *linear equation* in variables x_1, x_2, \dots, x_n is an equation which can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where a_1, a_2, \dots, a_n, b are real numbers (\mathbb{R})

Linear Equations

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coefficients

$$\boxed{a_1}x_1 + \boxed{a_2}x_2 + \dots + \boxed{a_n}x_n = b$$

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Linear Equations

Definition. A *linear equation* in variables x_1, x_2, \dots, x_n is an equation which can be written in the form

$$\begin{array}{c} \text{coefficients} \qquad \qquad \qquad \text{unknowns} \\ a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \end{array}$$

where a_1, a_2, \dots, a_n, b are real numbers (\mathbb{R})

Examples

$$2x + 3y - 4z = 5$$

$$2x_1 + 3x_2 - 4x_3 = 5$$

$$\pi x + e y = 4$$

$$2x = 3y$$

\Downarrow

$$2x - 3y = 0$$

Point sets

Linear equations describe *point sets*:

$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \dots + a_n s_n = b\}$$

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$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1s_1 + a_2s_2 + \dots + a_ns_n = b\}$$

The points such that the equation holds

Examples

$$2x - 3y + 4z = 5$$

$$(1, -1, 0)$$

$$2(1) - 3(-1) + 4(0) = 2 + 3 = 5 \quad \checkmark$$

$$(2, 1, 1)$$

$$2(2) - 3(1) + 4(1) = 4 - 3 + 4 = 5 \quad \checkmark$$

Linear Equations, Geometrically

If a 2D linear equation represents a *line* in the plane, then a 3D linear equation is...

Linear Equations, Geometrically

If a 2D linear equation represents a *line* in the plane, then a 3D linear equation is...

Not a line...

Demo

$$z = 5$$

$$0x + 0y + z = 5$$

What does the point set of this linear equation look like?

$$\{ (x, y, z) \in \mathbb{R}^3 : z = 5 \}$$

$$(0, 0, 5)$$

$$(94, 2, 5)$$

$$(\pi, e, 5)$$

Demo

$$-x + 0y + z = 5$$

How about this one?

$$\{ (x, y, z) : z = x + 5 \}$$

$$(-5, 0, 0)$$

$$(1, 1, 6)$$

$$(-5, \pi, 0)$$

$$(1, \pi, 6)$$

Demo

$$-x + -y + z = 5$$

How about this one?

$$\{ (x, y, z) ; z = x + y + 5 \}$$

$$(1, 1, 7)$$

$$(2, 0, 7)$$

$$(3, 3, 11)$$

Hyperplanes

Hyperplanes

The point set of a general linear equation
is called a **hyperplane**

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After three dimensions, we can no longer visualize point sets

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After three dimensions, we can no longer visualize point sets

Theme of the course: Hyperplanes "behave" like lines and planes in many respects

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Linear Equations, Geometrically

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A plane!

Systems of Linear Equations

Definition. A *system of linear equations* is just a collection of linear equations over the same variables

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Definition. A *solution* to a system is a point that satisfies all its equations *simultaneously*

Example

$$x + 2y + 0z = 1$$

$$-x - y - z = -1$$

$$2x + 6y - z = 1$$

Show that $(3, -1, -1)$ is solution to the above linear system

$$3 + 2(-1) + 0(-1) = 3 + (-2) = 1 \quad \checkmark$$

$$-3 - (-1) - (-1) = -3 + 1 + 1 = -1 \quad \checkmark$$

$$2(3) + 6(-1) - (-1) = 6 - 6 + 1 = 1 \quad \checkmark$$

System of Linear equations (General-form)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

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Does a system have a solution?

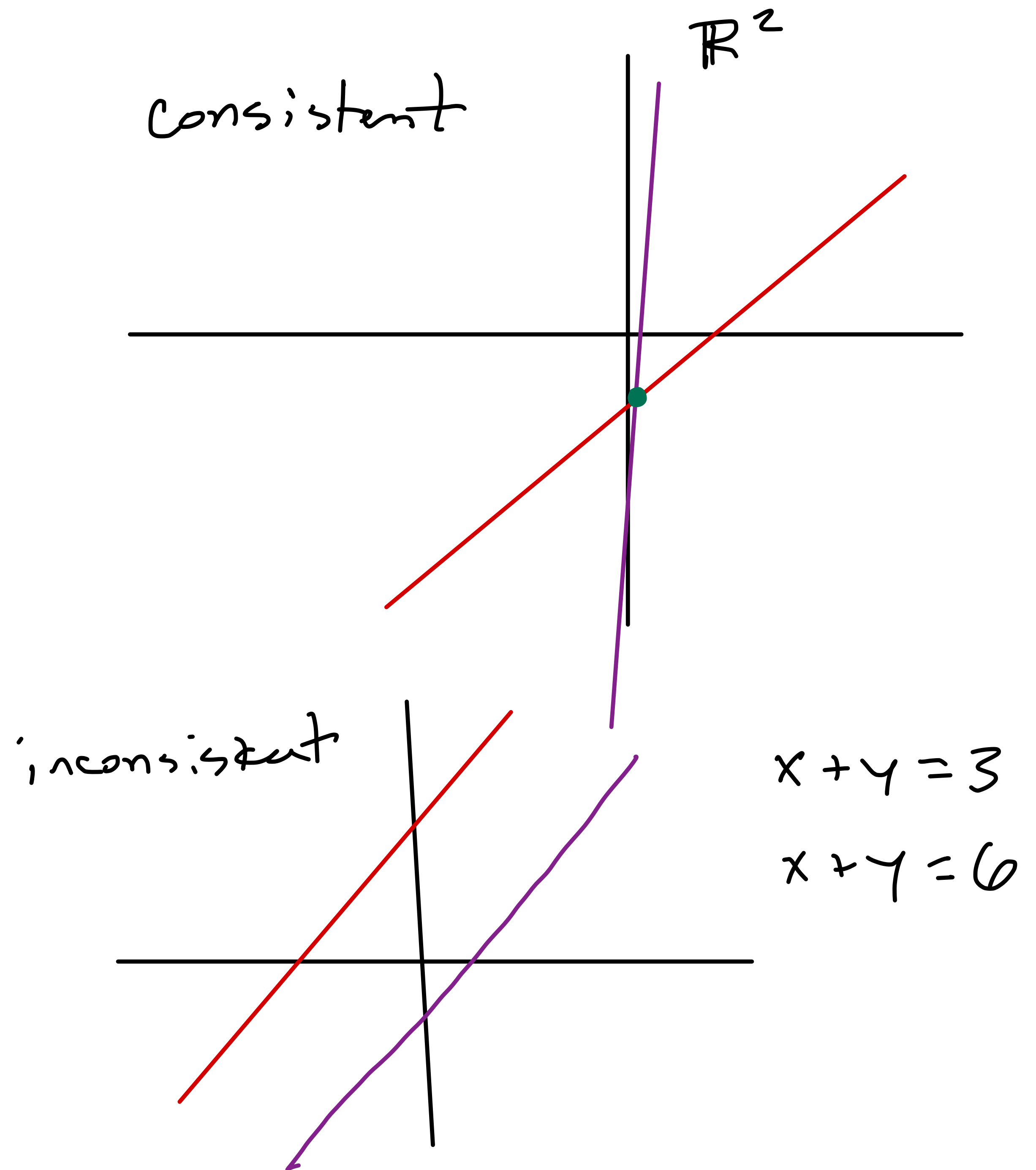
How many solutions are there?

What are its solutions?

Consistency

Definition. A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has no solutions



demo

(consistency/inconsistency in \mathbb{R}^3)

Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

Matrix Representations

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \approx \begin{array}{l} 2x + 3y = -8 \\ y = 2 \\ 2y = 0 \end{array}$$

Matrix Representations

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We'll write down linear systems as **matrices**, i.e., 2D grids of numbers with *fixed* width and height

a matrix is just a representation

Matrix Representations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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augmented matrix

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coefficient matrix

Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \quad (\text{C})$$

$$12\alpha - 6\beta \overset{+0\gamma}{=} 0 \quad (\text{H})$$

$$6\alpha - \beta - 2\gamma = 0 \quad (\text{O})$$

Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

Solving Linear Systems

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solving Systems with Two Variables

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The Approach

Solve for x in terms of y in EQ1

Solving Systems with Two Variables

$$2x + 3y = -6$$

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The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Question

This works, but *why might we want a different approach?*

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Another Approach

Elimination

Eliminate x from the EQ2 and solve for y

Eliminate y from EQ1 and solve for x

Back-Substitution

Solving Systems with Two Variables

$$\begin{array}{rcl} 2x + 3y = -6 & \xrightarrow{2*} & 4x - 5y = 10 \\ 4x - 5y = 10 & & -4x + 6y = -12 \\ \hline & & -11y = 22 \end{array}$$

$\begin{array}{r} 2x + 3y = -6 \\ 4x - 5y = 10 \\ \hline 6x - 2y = 4 \end{array}$

\Downarrow

$$\begin{array}{l} 2x + 3y = -6 \\ -11y = 22 \Rightarrow \end{array}$$

$$\begin{array}{l} 2x + 3y = -6 \\ y = -2 \end{array}$$



Exercise

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Can we represent these intermediate steps as operations on matrices?

Let's look back at this...

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Elementary Row Operations

scaling	multiply a row by a NONZERO number
replacement	add a multiple of one row to another
interchange	switch two rows


Elementary Row Operations

scaling	multiply a row by a NONZERO number
replacement	add a multiple of one row to another
interchange	switch two rows

These operations don't change the solutions

Scaling Example

$$\begin{array}{l} 2x + 3y = -6 \\ 4x - 5y = 10 \end{array}$$

$$R_1 \leftarrow 2R_1$$


$$\begin{array}{l} 4x + 6y = -12 \\ 4x - 5y = 10 \end{array}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$\begin{array}{l} 2x + 3y = -6 \\ 4x - 5y = 10 \end{array}$$

$$R_2 \leftarrow R_2 + R_1$$



$$\begin{array}{l} 2x + 3y = -6 \\ 6x - 2y = 4 \end{array}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$2x + 3y = -6$$

$$4x - 5y = 10$$

$$R_1 \leftrightarrow R_2$$



$$4x - 5y = 10$$

$$2x + 3y = -6$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

Example: Row Reductions

$$\begin{array}{ccc} & R_2 \leftarrow R_2 - 2R_1 & \\ \begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix} \\ & R_2 \leftarrow R_2 / (-11) & \\ & \longrightarrow & \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix} \end{array}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$R_2 \leftarrow R_2 / (-11)$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 / 2$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / (-11)$$

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$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / (-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_1 \leftarrow R_1 / 2$$

back-substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

Question

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How do we know when we're done? What is the "target" matrix?

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When the final matrix "looks like" a solution

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How do we know when we're done? What is the "target" matrix?

When the final matrix "looks like" a solution

We'll get more into that next time...

Example

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve