Echelon Forms

Geometric Algorithms
Lecture 2

CAS CS 132

Practice Problem

None today...

Outline

» Introduce echelon forms as matrices which "look like" solutions

» Learn to "read off" a solution from an echelon form

Keywords

leading entries

echelon form

(row-)reduced echelon form (RREF)

pivot positions

pivot columns

free variables

basic variables

general form solutions

forward elimination

back substitution

Recap

Recall: Linear Systems (General-form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Recall: Linear Systems (General-form)

$$
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&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Recall: Matrix Representations

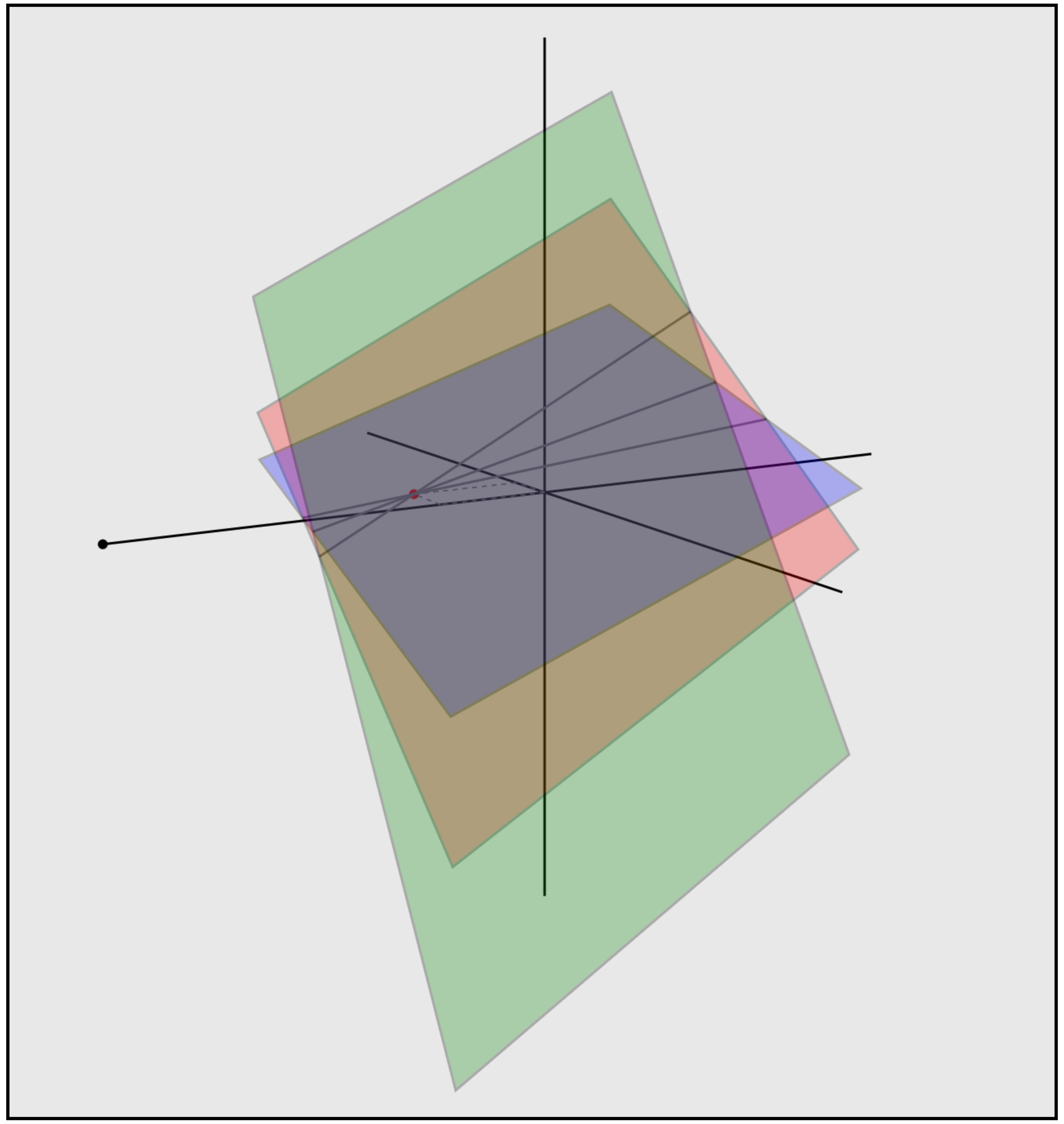
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Recall: Linear Systems, Geometrically



Recall: Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

Recall: Number of Solutions

`zero the system is inconsistent`

`one  the system has a unique solution`

`many the system has infinity solutions`

These are the **only** options

where we left off...

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Another Approach

Elimination

Eliminate x from the EQ2 and solve for y

Eliminate y from EQ1 and solve for x

Back-Substitution

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$\boxed{4x} - 5y = 10$$

$$2x = -3y - 6$$
$$x = \boxed{\tfrac{1}{2}(-3y - 6)}$$

$$\begin{array}{r} 4x - 5y = 10 \\ -\quad 4x + 6y = -12 \\ \hline 0x - 11y = 22 \end{array}$$

$$2x + 3y = -6$$
$$-11y = 22$$
$$\Downarrow$$
$$2x + 3y = -6$$
$$y = -2$$

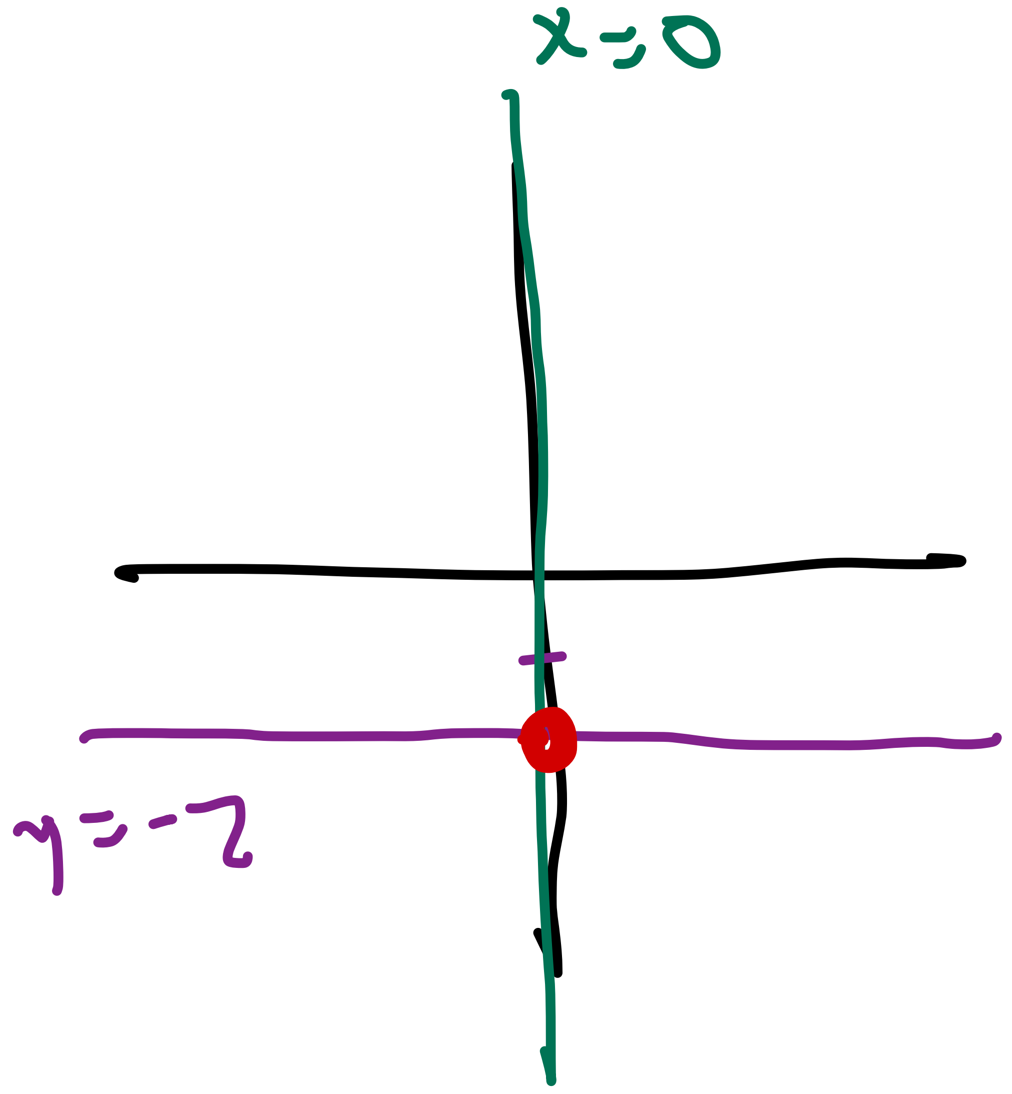
$$\Longrightarrow$$

$$\begin{array}{r} 2x + 3y = -6 \\ -\qquad 3y = -6 \\ \hline 2x + 0y = 0 \end{array}$$

$$2x = 0$$
$$y = -2$$

$$\Longrightarrow$$

$$x = 0$$
$$y = -2$$



Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

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Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

Can we represent these intermediate steps as operations on matrices?

Let's look back at this...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4^{-4} & -5^{-6} & 10^{+12} \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1}$$

$$4x - 5 = 10$$
$$-2(2x - 3y = -6)$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 / (-11)}$$

$2x + 3y = -6$

$-11y = 22$

$$\begin{bmatrix} 2 & 3^{-3} & -6^{+6} \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_1 \leftarrow R_1 / 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$x = 0$

$y = -2$

Elementary Row Operations

scaling	multiply a row by a **NONZERO** number
replacement	add a multiple of one row to another
interchange	switch two rows

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These operations don't change the solutions

Scaling Example

$x = 5$

$2x = 10$

$3x = 15$

$\pi x = 5\pi$

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_1 \leftarrow 2R_1} \quad \begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$x = 5$

$y = 10$

$\Downarrow$

$x = 5$

$x + y = 15$

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_2 \leftarrow R_2 + R_1} \quad \begin{aligned} 2x + 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & ? \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

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Example: Row Reductions

$$\begin{aligned} R_2 &\leftarrow R_2 - 2R_1 \\ R_2 &\leftarrow R_2/(-11) \\ R_1 &\leftarrow R_1 - 3R_2 \\ R_1 &\leftarrow R_1/2 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2/(-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1/2$$

back-substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

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We can compute solutions by sequence of row operations

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elimination

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We can compute solutions by sequence of row operations

!!!IMPORTANT!!!

Row equivalent augmented matrices represent linear system with the same solution set

Example

$$\begin{aligned} x - 2y + z &= 5 \\ 2y - 8z &= -4 \\ 6x + 5y + 9z &= -4 \end{aligned}$$

How do we know when we're done?
What's the "target" matrix?

Answer: when we are able to "read off" a solution

Motivating Questions

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Motivating Questions

echelon forms

What matrices "represent solutions"? (which have solutions that are easy to "read off"?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

Unique Solution Case

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$
$$y = 2$$
$$z = 3$$

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$
$$y = 2$$
$$z = 3$$

Like all the examples we've seen so far

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1s along the diagonal

0s elsewhere

Unique Solution Case

$$\begin{array}{c} \text{coefficient matrix} \\ \left[\begin{array}{ccc|c} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}\right] \end{array}$$

a system of linear equations whose **coefficient matrix** is the identity matrix represents a unique solution

No Solution Case

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No Solution Case

$R_3 \leftrightarrow R_1$

$R_2 \leftrightarrow R_1$

$R_3 \leftarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 0 = 1$$

two parallel planes

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

two parallel planes

row representing $0 = 1$

No Solution Case

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row representing $0 = 1$

a system with no solutions can be reduced to a matrix with the row

$$0\ 0\ \ldots\ 0\ 1$$

Infinite Solution Case

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$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$(0, 1, 2)$

$$x_1 + x_3 = 2$$

$(1, 1, 1)$

$$x_2 = 1$$

$(2, 1, 0)$

a system with infinity solutions can be reduced to a system which leaves a variable <u>unrestricted</u>

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 2$$
$$x_2 = 1$$
$$x_3 = 0$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 1.5$$
$$x_2 = 1$$
$$x_3 = 0.5$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 20$$
$$x_2 = 1$$
$$x_3 = -18$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

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general form

In Sum

none	reduces to a system with the equation $0 = 1$
one	reduces to a system whose coefficient matrix is the identity matrix
infinity	reduces to a system which leaves a variable unrestricted

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none	reduces to a system with the equation $0 = 1$
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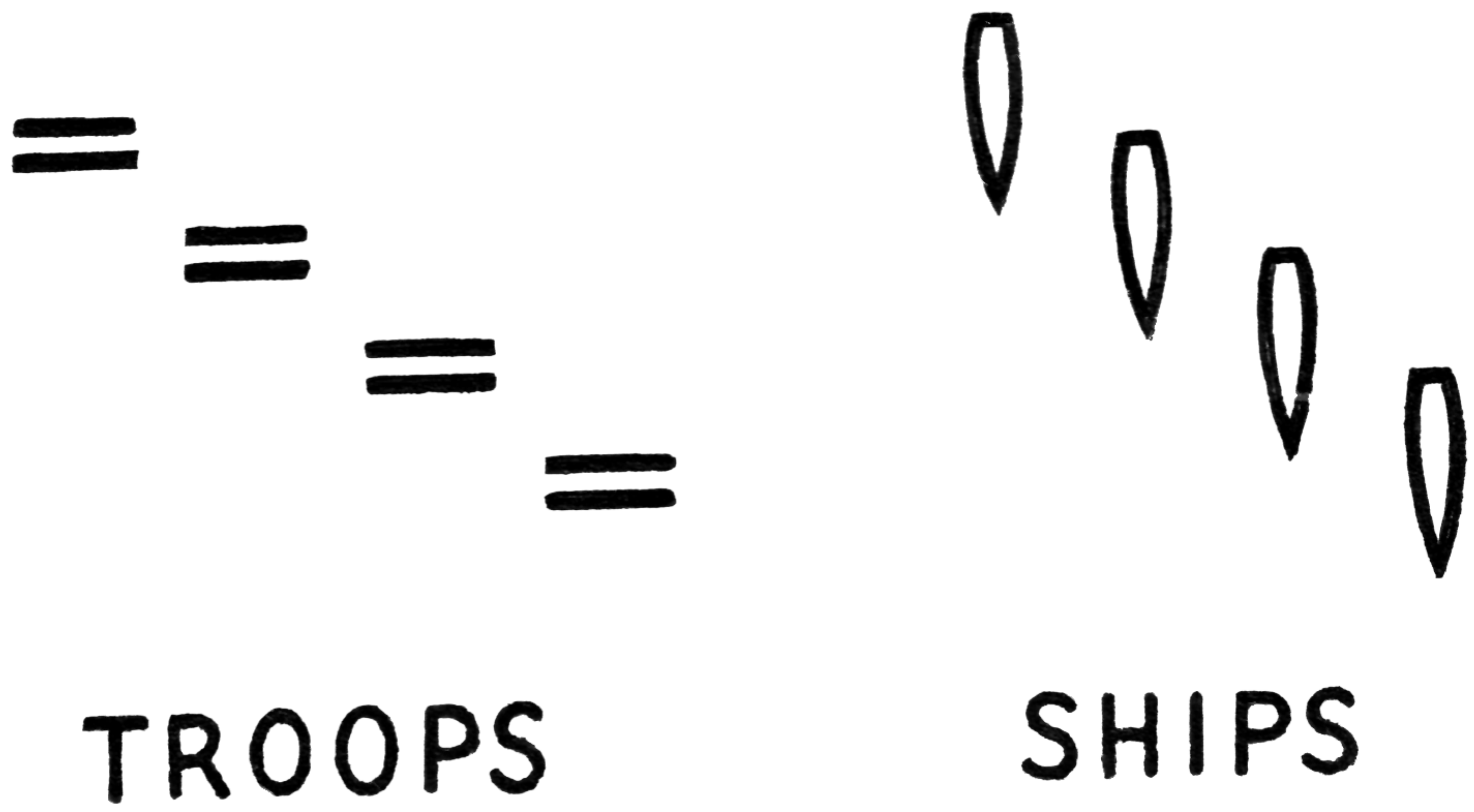
infinity	reduces to a system which leaves a variable unrestricted

Ideally, we want one *form* that handles all three cases

Echelon Form

The Picture (and a bit of history)



https://commons.wikimedia.org/wiki/File:Echelon_1_(PSF).png

Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ = nonzero, * = anything

Leading Entries

Definition. the ***leading entry*** of a row is the first nonzero value

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-3} & 3 \\ 0 & 0 & 0 \\ \boxed{1} & -1 & 10 \end{bmatrix} \longleftarrow \text{no leading entry}$$

Echelon Form

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1. The leading entry of each row appears to the right of the leading entry above it

Echelon Form

Definition. A matrix is in ***echelon form*** if

1. The leading entry of each row appears to the right of the leading entry above it

2. Every all-zeros row appears below any non-zero rows

Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ = nonzero, * = anything

Echelon Form (Pictorially)

next leading entry
to the right

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all-zero rows at
the bottom

$\blacksquare$ = nonzero, * = anything

Question

Is the identity matrix in echelon form?

Answer: Yes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the leading entries of each row appears to the right of the leading entry above it

it has no all-zero rows

Question

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Is this matrix in echelon form?

Answer: No

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

The leading entry of the least row is not to the right of the leading entry of the second row

What's special about Echelon forms?

Theorem. Let A be the augmented matrix of an <u>inconsistent</u> linear system. If $A \sim B$ and B is in echelon form then B has the row

$$[0\ 0\ \ \dots\ \ 0\ 0\ \blacksquare]$$

What's special about Echelon forms?

Theorem. Let A be the augmented matrix of an <u>inconsistent</u> linear system. If $A \sim B$ and B is in echelon form then B has the row

$$[0\ 0\ \ldots\ 0\ 0\ \blacksquare]$$

If all we care about is consistency then we just need to find an echelon form

Example

$$x - 2z = 4$$
$$-x + y + 5z = -3$$
$$x + 2y + 4z = 7$$

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ -1^{+1} & 1 & 5^{-2} & -3^{+4} \\ 1^{-1} & 2 & 4^{+2} & 7^{-4} \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 + R_1}$$

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 2^{-2} & 6^{-6} & 3^{-2} \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$0 = 1$

The Problem with Echelon Forms

If our system *is* consistent, we can't "read off" a solution from an echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

$$x + 2y = 3$$

$$y = 4$$

Reduced Echelon Form

Row-Reduced Echelon Form (RREF)

Definition. A matrix is in ***(row-)reduced echelon form*** if

1. The leading entry of each row appears to the right of the leading entry above it

2. Every all-zeros row appears below any non-zero rows

3. The leading entries of non-zero rows are 1

4. the leading entries are the only non-zero entries of their columns

Reduced Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Echelon Form (Pictorially)

leading entries are 1

$$\begin{bmatrix} 0 & \boxed{1} & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

other column
entries are 0

Basic Example

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

Basic Example

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

The Fundamental Points

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Point 1. we can "read off" the solutions of a system of linear equations from its RREF

The Fundamental Points

Point 1. we can "read off" the solutions of a system of linear equations from its RREF

Point 2. *every* matrix is row equivalent to a <u>unique</u> matrix in reduced echelon form

How-To: Solving a System of Linear Equations

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Our next topic

What's special about RREF?

Every leading variable can be written in terms of only non–leading variables

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2 - x_3$

$x_2 = 1$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

So, why we care about RREFs?

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reduced echelon forms describe solutions to linear equations

General-Form Solutions

Motivating Question

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how do we characterize all solutions in the infinite solution case?

Basic and Free Variables

Basic and Free Variables

pivots pivots

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

x_1 is basic

x_2 is basic

x_3 is free

Solutions of Reduced Echelon Forms

the row i of a pivot position describes the value of x_i in a solution to the system, in terms of the free variables

basic

free

$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & \\
\end{array}
$$

$$
\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}
$$

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

How-To: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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How-To: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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1. For each pivot position (i,j), isolate x_j in the equation in row i

How-To: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

x_3 is free

1. For each pivot position (i, j), isolate x_j in the equation in row i

2. If x_i is not in a pivot column then write

x_i **is free**

Example

$$x_1 + 2x_2 - 2x_4 = 4$$

$$x_3 + 3x_4 = 5$$

$$(4, 0, 5, 0)$$

$$(2, 1, 0, 0)$$

$x_1 = 4 - 2x_2 + 2x_4$

x_2 is free

$x_3 = 5 - 3x_4$

x_4 is free

$$\begin{array}{cccc} & & & \\ \left[\begin{array}{ccccc} 1 & 2 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \end{array}$$

basic

free

Question

basic free

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Circle the pivot positions, highlight the pivot rows.

Which variables are free? Which are basic?

Write down a solution in general form for this reduced echelon form matrix.

*Write down a **particular** solution given the general form.*

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 1 - 3x_4$

x_2 is free

$x_3 = 4 - 2x_4$

x_4 is free

Summary

Echelon form matrices "represent solutions"

General form solutions can be used to describe the infinite solution sets