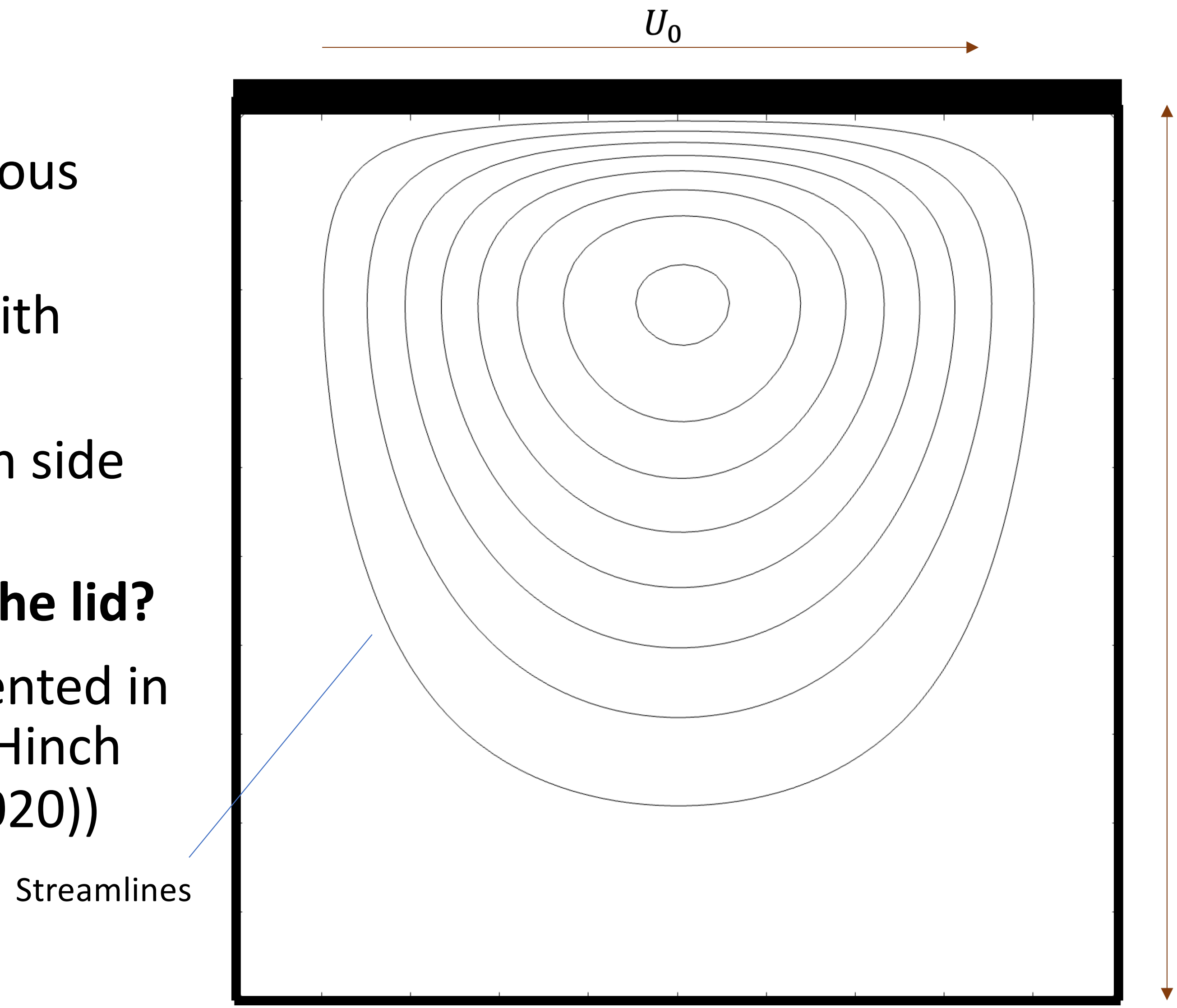
B1 Engineering Computation: Force on a driven lid over a cavity

Wouter Mostert

MT 2023

Problem statement

- Consider a box filled with a viscous liquid, density ρ and viscosity μ
- The lid can slide over the box with velocity U_0
- The box is square (and 2D), with side length L_0
- **What is the resisting force on the lid?**
- (Based on the problem as presented in *Think before you compute*, E.J. Hinch (Cambridge University Press, 2020))



Lid-driven cavity

- The force on the lid is

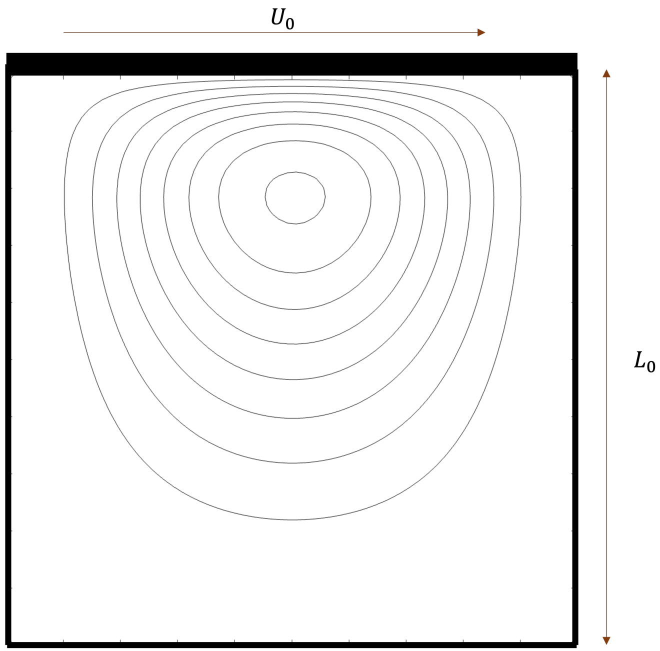
$$F = \mu \int_0^{L_0} \frac{\partial u_x}{\partial y}\Big|_{y=L_0} dx$$

- Hard to estimate analytically because u_x isn't known at the top of the box
- So use a numerical approach
- This is a fundamental CFD problem
- Requires solution of [incompressible] Navier-Stokes:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \boldsymbol{u}$$

(without gravity)

- …but this is too hard for you to attempt in a few weeks



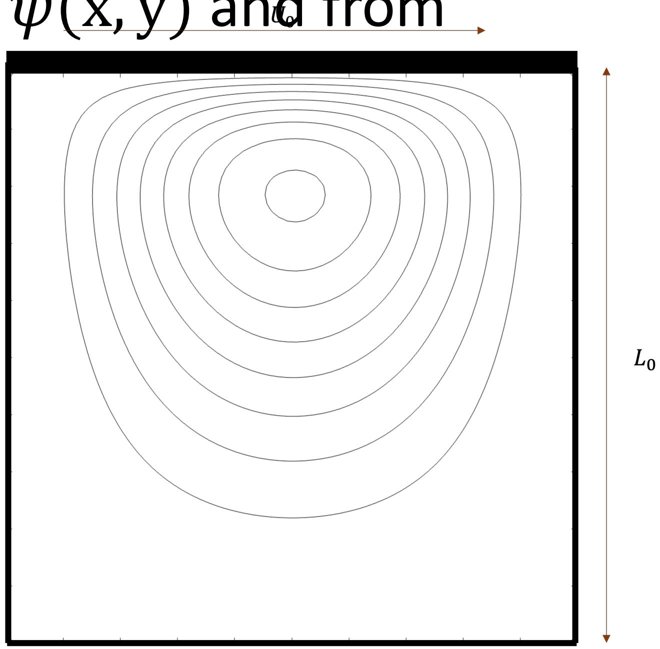
Lid-driven cavity [sane]

- Instead you will solve the **Poisson equation**:

$$\nabla^2\psi = \omega$$

where $\psi(x, y)$ is a streamfunction, $\omega(x, y)$ is a vorticity field, and with some boundary conditions

- I will give you the data for $\omega(x, y)$, you have to solve for $\psi(\mathrm{x}, \mathrm{y})$ and from there get the force on the lid



Why does that solve the problem?

- Because the streamfunction is defined,

$$\frac{\partial \psi}{\partial x} = -u_y, \qquad \frac{\partial \psi}{\partial y} = u_x$$

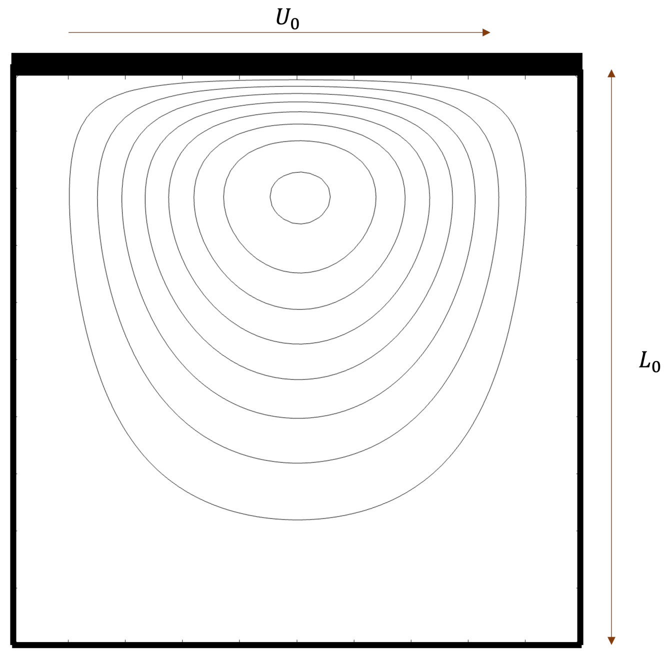
- And vorticity is defined,

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

Plug one into the other to get the Poisson equation

- Normally you would also solve a coupled PDE to get ω
- This is easier than solving Navier-Stokes
- But still hard
- So I'm just giving you ω (at some fixed time)

UNIVERSITY OF OXFORD



How would I solve this problem?

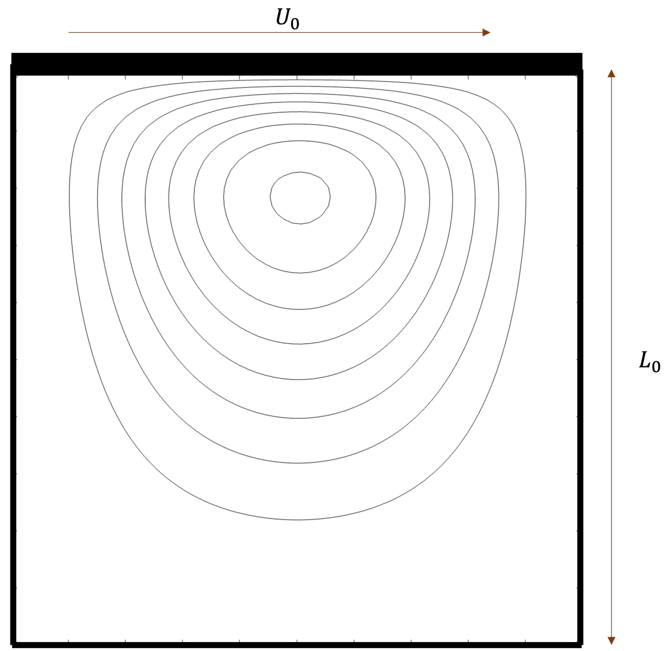
- You will use a finite difference method (FDM)
- You will cover these in the lectures
- It involves *discretising* the Poisson equation over space into $N \times N$ interior points,

$$\nabla^2 \psi = \omega$$

- So that it transforms into a linear system of equations

$$A\Psi = \boldsymbol{b}$$

- You will then solve this system to get an estimate for ψ
- You will use an iterative solver by the successive over-relaxation (SOR) method (see A1 notes)



How do I know my answer is right?

- You can check your solver works by testing against a known analytical result
- E.g. it is known that

$$\nabla^2\psi = 2\pi^2 \sin \pi x \sin \pi y$$

has an exact solution

$$\psi(x, y) = \sin \pi x \sin \pi y$$

- Your code should be able to get the exact solution for this test problem
- If it works for this then it will (likely) work for the real one

UNIVERSITY OF OXFORD

How do I know I'm solving it efficiently & accurately?

- The discretisation and SOR method depend on two numerical parameters,
  - The discretising resolution N
  - The over-relaxation parameter r
- These have no physical meaning but (especially N) may influence your interpretation
- **Do you want your prediction of a physical phenomenon to depend on a numerical parameter?**
- If you choose r poorly, the SOR method will converge too slowly
- If you choose N poorly, either your approximation will be bad, or your simulation will be take too long or eat too much memory
- **How do you choose these values?**

UNIVERSITY OF OXFORD

What is the force on the lid?

- You can then apply your code to the problem.
- I'll give you three different datasets of ω, each at a different resolution
- You must estimate to the best of your ability the force on the lid
- **This will not necessarily be what you get directly out of your solver at the finest resolution!**

- (There are a few more subtleties but these are covered in the handout)

UNIVERSITY OF OXFORD

How do I convince anybody that I've thought about this?

- Write a report!
- Describe the physical, mathematical & numerical problem briefly but enough to show you know what's going on
- Describe your numerical solver, how you constructed it, and its performance on the test problem
- Describe your solver applied to the actual problem
- Answer the basic engineering question
- Discuss the limitations of the model
- Does the existence of limitations make the answer unreliable? Why or why not?
- **Think critically and use your judgment**

UNIVERSITY OF OXFORD