B1 Optimization Examples

Victor Adrian Prisacariu, Michaelmas Term 2023

1. The Rosenbrock function is
$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2$$

   (a) Compute the gradient and Hessian of $f(x, y)$.

   (b) Show that that $f(x, y)$ has zero gradient at the point $(1, 1)$.

   (c) By considering the Hessian matrix at $(x, y) = (1, 1)$, show that this point is a minimum.

2. In Newton type minimization schemes the update step is of the form

$$\delta\mathbf{x} = -\mathtt{H}^{-1}\mathbf{g}$$

   where $\mathbf{g} = \nabla f$. By considering $\mathbf{g}.\delta\mathbf{x}$ compare the convergence of:

   (a) Newton, to

   (b) Gauss Newton

   for a general function $f(\mathbf{x})$ (i.e. where $\mathtt{H}$ may not be positive definite).

3. Explain how you could use the Gauss Newton method to solve a set of simultaneous non-linear equations.

4. Sketch the feasible regions defined by the following inequalities and comment on the possible optimal values.

   (a)

$$
\begin{aligned}
-x_1 + x_2 &\geq 2 \\
x_1 + x_2 &\leq 1 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

   (b)

$$
\begin{aligned}
2x_1 - x_2 &\leq 2 \\
x_1 &\leq 4 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

5. More on linear programming.

   (a) Show that the optimization

$$\min_{\mathbf{x}} \sum_i |\mathbf{a}_i^\top \mathbf{x} - b_i|$$

   where the vectors $\mathbf{a}_i$ and scalars b_i are given, can be formulated as a linear programming problem.

(b) Solve the following linear programming problem using Matlab:

$$\max_{x_1,x_2} \quad 40x_1 + 88x_2$$

$$\text{subject to}$$

$$
\begin{aligned}
2x_1 + 8x_2 &\leq 60 \\
5x_1 + 2x_2 &\leq 60 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

6. Interior point method using a barrier function. Show that the following 1D problem

$$\text{minimize} \quad f(\mathbf{x}) = x^2, x \in \mathbb{R}$$
$$\text{subject to} \quad x - 1 \geq 0$$

can be reformulated using a logarithmic barrier method as

$$\text{minimize } x^2 - r\log(x - 1)$$

Determine the solution (as a function of r), and show that the global optimum is obtained as $r \to 0$.

7. Determine in each case if the following functions are convex:

   (a) The sum of quadratic functions $f(x) = a_1(x - b_1)^2 + a_2(x - b_2)^2$, for $a_i > 0$
   (b) The piecewise linear function $f(x) = \max_{i=1,\dots,m}(\mathbf{a}_i^\top \mathbf{x} + b_i)$
   (c) $f(x) = \max\{x, 1/x\}$ for $x > 0$
   (d) $f(\mathbf{x}) = ||\mathbf{Ax} - \mathbf{b}||^2$