Department of Engineering Science

UNIVERSITY OF OXFORD

B1 The Finite Element Method
Lecture 4: Numerical integration

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Reminder: element discretisation

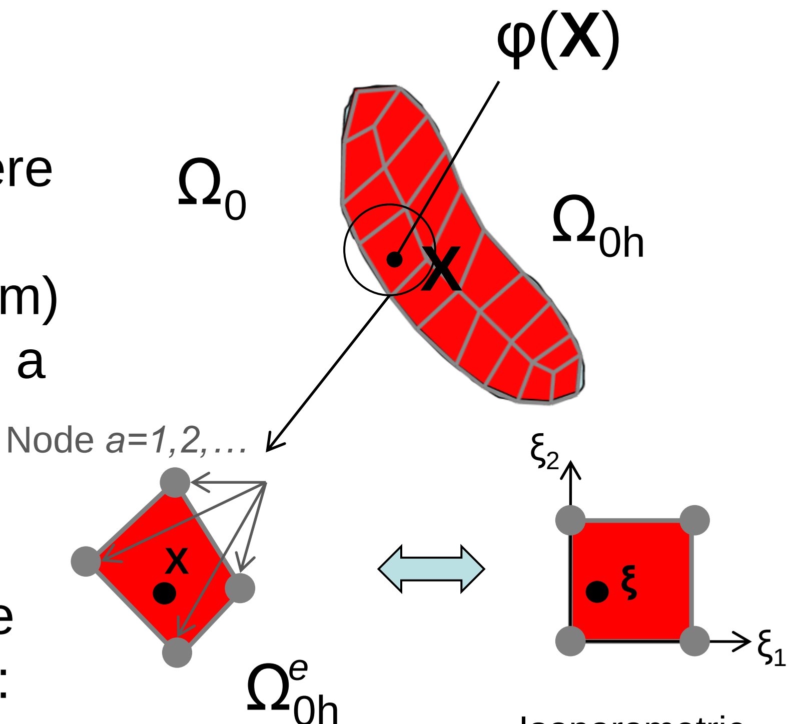
- Unknown field $\boldsymbol{\varphi}: (\Omega_0, \mathbb{R}^+) \rightarrow \Omega$

$$(\boldsymbol{X}, t) \mapsto \boldsymbol{x}$$

- For a given **X** inside an element *e*, there exists a unique **ξ** in the isoparametric coordinate system (by homeomorphism)
- We define the shape function $N_a(\boldsymbol{\xi})$ as a polynomial of order *k* such that

$$N_a(\boldsymbol{\xi}_b) = \delta_{ab}$$

for nodes *a* and *b* of element *e*
- By knowing the displacement of all the nodes *a*, we can approximate φ(**X**) by:



Isoparametric coordinate system of **all** elements *e*

with $\varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = \sum_a N_a(\xi)\boldsymbol{x}^a$

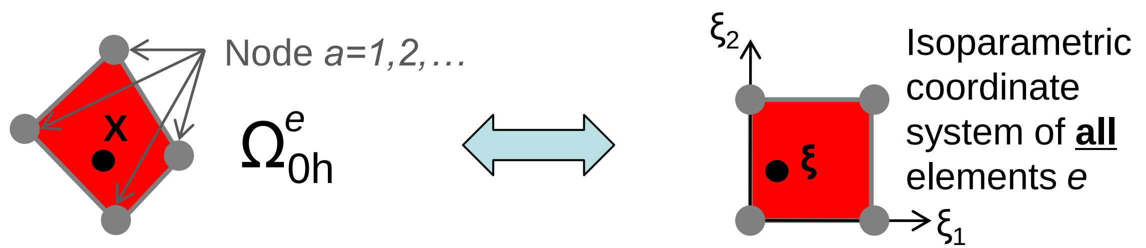
$$\varphi_h \in \left\{ \varphi_h \in C^0(\Omega_{0h}) \middle| \varphi_h^e = \varphi_h|_{\Omega_{0h}^e} \in P^k(\Omega_{0h}^e) \forall \Omega_{0h}^e \in \Omega_{0h} \right\}$$

Shape function derivation

- Before element assembly, stiffness matrix contribution of a given element is: $\iiint_{\Omega^e} C^e_{ijkl} N^e_{a,j} N^e_{b,l} dV^e$
- By derivation, we have: $\frac{\partial N^e_a}{\partial \xi_\alpha} = \frac{\partial N^e_a}{\partial X_i} \frac{\partial X_i}{\partial \xi_\alpha}$

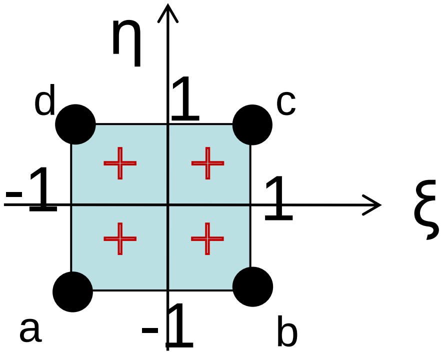
  where $\frac{\partial X_i}{\partial \xi_\alpha}$ is the Jacobian matrix $\boldsymbol{J}^e$ between the reference

  configuration and the isoparametric coordinate system



$$\Rightarrow \quad \frac{\partial N^e_a}{\partial \boldsymbol{X}} = \frac{\partial N^e_a}{\partial \boldsymbol{\xi}} \cdot \boldsymbol{J}^{e-1} \quad \text{and} \quad J^e_{\alpha i} = \sum_a \frac{\partial N^e_a}{\partial \xi_\alpha} X^a_i$$

2D linear example: square element

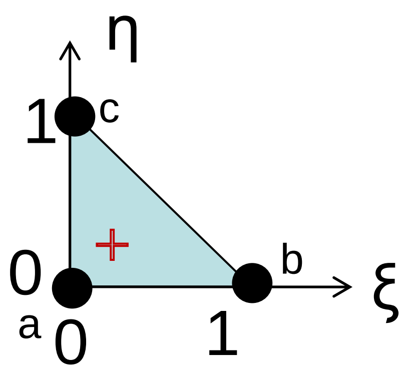


$$N_a(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)\qquad N_c(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$

$$N_b(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)\qquad N_d(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

$$\boldsymbol{J}^e=\begin{bmatrix}-\frac{1}{4}(1-\eta) & \frac{1}{4}(1-\eta) & \frac{1}{4}(1+\eta) & -\frac{1}{4}(1+\eta)\\ -\frac{1}{4}(1-\xi) & -\frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) & \frac{1}{4}(1-\xi)\end{bmatrix}\cdot\begin{bmatrix}X_a & Y_a\\ X_b & Y_b\\ X_c & Y_c\\ X_d & Y_d\end{bmatrix}$$

2D linear example: triangle element



$$N_a(\xi,\eta) = 1 - \xi - \eta$$

$$N_b(\xi,\eta) = \xi$$

$$N_c(\xi,\eta) = \eta$$

$$J^e = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_a & Y_a \\ X_b & Y_b \\ X_c & Y_c \end{bmatrix}$$

Quadrature points

- Let's consider the integration of a function *f*:

$$I = \iiint_{\Omega^e} f(\boldsymbol{X}) dV^e = \iiint_{\Omega^{iso}} f(\boldsymbol{\xi}) \det(\boldsymbol{J}^e) dV^{iso}$$

where the Jacobian $J=\det(\boldsymbol{J}^e)$ depends on $\boldsymbol{\xi}$

- Let's pick a set of Q "quadrature points" $\boldsymbol{\xi}_q$ with weight w_q inside the element e such that:

$$I \approx \sum_q w_q\, f\left(\boldsymbol{\xi}_q\right) J(\boldsymbol{\xi}_q)$$

- In the following, for simplification we assume $J=1$

Gauss quadrature (1/3)

- Choose $\boldsymbol{\xi}_q$ and w_q such that the sum is equal to the integral for the highest ranking polynomial form of f
- Such quadrature points are called "Gauss points"
- For Q=1 in a 1D problem, if we have $f(\xi)=a_0+a_1\xi$,

$$I = \iiint_{\Omega^{iso}} f(\boldsymbol{\xi})dV^{iso} = \int_{-1}^{1}(a_0 + a_1\xi)d\xi = 2a_0$$

$$I \approx \sum_q w_q\, f(\boldsymbol{\xi}_q) = w_1\, f(\xi_1) = w_1\, a_0 + w_1 a_1 \xi_1$$

$\Rightarrow$ ξ_1 =0 and w_1=2

Gauss quadrature (2/3)

- For Q=1 in a 1D problem, if we have $f(\xi)=a_0+a_1\xi+a_2\xi^2$,

$$I = 2a_0 + \frac{2a_2}{3}$$

$$I \approx w_1\, a_0 + w_1 a_1 \xi_1 + w_1 a_2 \xi_1^2$$

$\Rightarrow$ ξ_1 =0 and w_1=2 (and a_2=0)

- For Q=1, the maximum polynomial is of order 1

Gauss quadrature (3/3)

- For Q=2 in a 1D problem, if we have $f(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3+a_4\xi^4$,

$$I = 2a_0 + \frac{2a_2}{3} + \frac{2a_4}{5}$$

$$I \approx (w_1 + w_2)a_0 + (w_1\xi_1 + w_2\xi_2)a_1$$

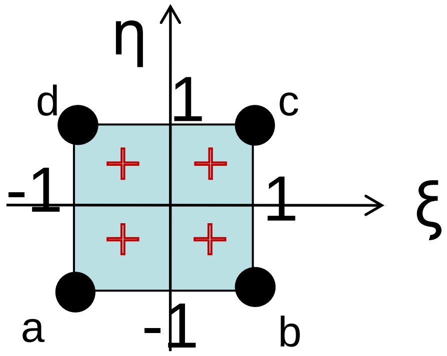
$$+ (w_1\xi_1^2 + w_2\xi_2^2)a_2 + (w_1\xi_1^3 + w_2\xi_2^3)a_3 + (w_1\xi_1^4 + w_2\xi_2^4)a_4$$

⇒ $\xi_1=1/\sqrt{3}$, $\xi_2=-1/\sqrt{3}$ and $w_1=w_2=1$ (and $a_4=0$)

- For Q=2, the maximum polynomial is of order 3

In 1D it can be proven that Q Gauss points can solve <u>exactly</u> a polynomial with a rank up to 2Q-1

2D linear example: square element



$$I = \iiint_{\Omega^{iso}} f(\boldsymbol{\xi}) dV^{iso} = \int_{-1}^{1}\int_{-1}^{1} (a_0 + a_1\xi + a_2\eta + a_3\xi\eta + a_4\xi^2 + a_5\eta^2) d\xi d\eta$$

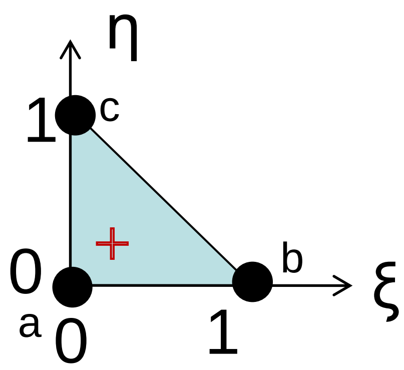
$$= 4a_0 + \frac{4}{3}(a_4 + a_5)$$

$$I \approx \sum_q w_q\, f(\boldsymbol{\xi}_q) = \left(\sum_i w_i\right) a_0 + \left(\sum_i w_i\xi_i\right) a_1 + \left(\sum_i w_i\eta_i\right) a_2$$

$$+ \left(\sum_i w_i\xi_i\eta_i\right) a_3 + \left(\sum_i w_i{\xi_i}^2\right) a_4 + \left(\sum_i w_i{\eta_i}^2\right) a_5$$

$\Rightarrow$ $\xi_i = \pm 1/\sqrt{3}$, $\eta_i = \pm 1/\sqrt{3}$ and $w_1 = w_2 = w_3 = w_4 = 1$

2D linear example: triangle element



$$I = \iiint_{\Omega^{iso}} f(\boldsymbol{\xi})dV^{iso} = \int_0^1 \int_0^{1-\eta} (a_0 + a_1\xi + a_2\eta)d\xi d\eta$$

$$= \frac{1}{2}a_0 + \frac{1}{6}a_1 + \frac{1}{6}a_2$$

$$I \approx \sum_q w_q f(\boldsymbol{\xi}_q) = w_1 a_0 + w_1 \xi_1 a_1 + w_1 \eta_1 a_2$$

$\Rightarrow$ ξ_1=1/3, η_1=1/3 and w_1=*1/2*

Element assembly

$$K_{iakb}\, u_k^b = f_{ia}^{ext}$$

$$\left(\sum_e K_{iakb}^e\right) u_k^b = \sum_e f_{ia}^{e\ ext}$$

$$\iiint_{\Omega^e} C_{ijkl}^e N_{a,j}^e N_{b,l}^e dV^e$$

$$\sum_q w_q C_{ijkl}^e N_{a,j}^e(\boldsymbol{\xi}_q) N_{b,l}^e\,(\boldsymbol{\xi}_q) J(\boldsymbol{\xi}_q)$$