Department of Engineering Science

UNIVERSITY OF OXFORD

B1 The Finite Element Method
Lecture 2: Continuum mechanics

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The Poisson problem (heat equation)

Strong formulation:

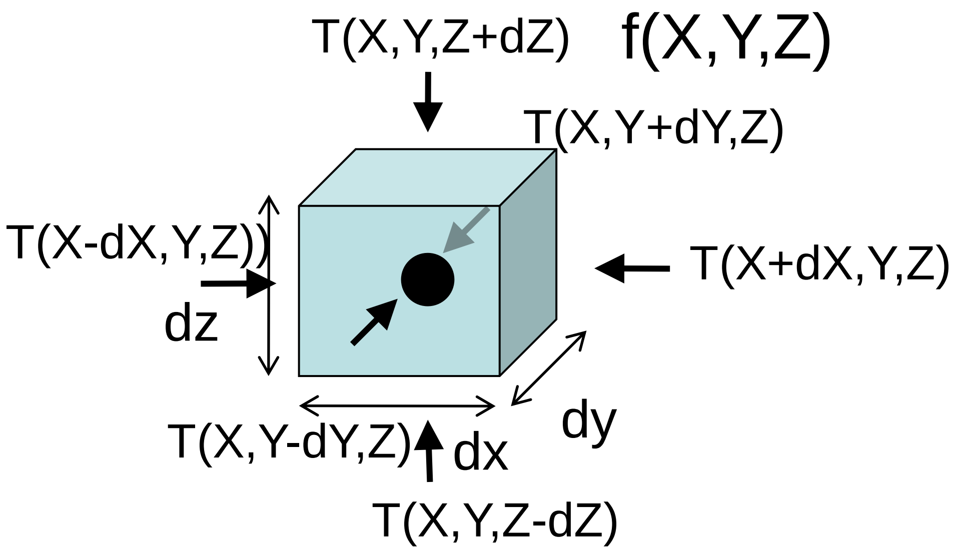
The following conditions are valid for all $\boldsymbol{X}(t)$ of $\Omega_0(t)$ for all t

- Conservation of internal energy
- Initial conditions
- Boundary conditions

Conservation of internal energy (1/2)

Volumetric heat source: f
Temperature: T

(Fourier's law: $\mathbf{q} = -k \nabla T$)



Along x | Along y | Along z

- k(T(X+dX)-T(X)) dY dZ/dX - k(T(Y+dY)-T(Y)) dX dZ/dY - k(T(Z+dZ)-T(Z)) dY dX/dZ

- k(T(X-dX)-T(X)) dY dZ/dX - k(T(Y-dY)-T(Y)) dX dZ/dY - k(T(Z-dZ)-T(Z)) dY dX/dZ

+ f dX dY dZ = 0

Conservation of internal energy (2/2)

Static:

$$-k\,\nabla^2 T = f$$

Dynamic:

$$\frac{\partial T}{\partial t} = k\,\nabla^2 T + f$$

(from now on, assume k=1)

Initial/Boundary Conditions

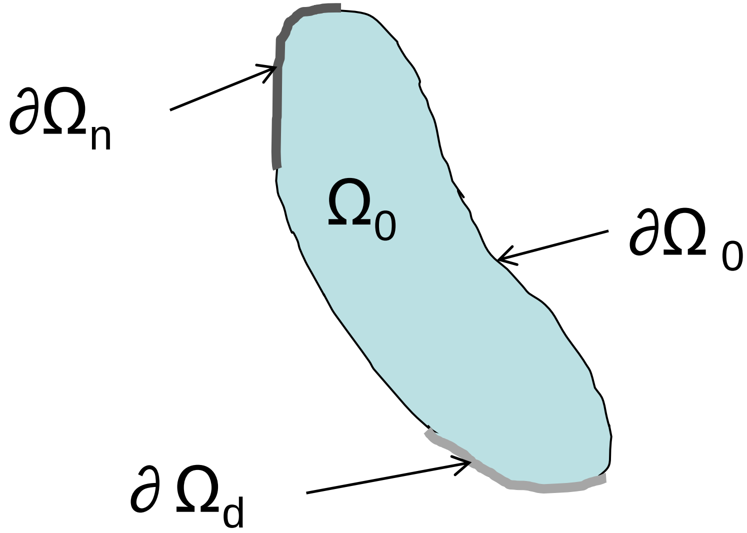
- Initial conditions:

$$\boldsymbol{\varphi}\colon (\Omega_0, \mathbb{R}^+) \to \mathbb{R}^+$$
$$(\boldsymbol{X}, t) \mapsto T$$
$$\boldsymbol{\varphi}(\boldsymbol{X}, 0) = T_0(\boldsymbol{X}) \quad \forall \boldsymbol{X} \in \Omega_0$$

- Boundary conditions:

$$\boldsymbol{q} = \bar{\boldsymbol{q}} \quad \forall \boldsymbol{X} \in \partial\Omega_n$$

$$T = \bar{T} \quad \forall \boldsymbol{X} \in \partial\Omega_d$$



Heat equation functional

- The functional $J(T)$ is defined by:

$$
\begin{aligned}
F(\boldsymbol{\nabla} T) &= \frac{1}{2}(\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T) - Tf \\
\Phi(\boldsymbol{u}) &= T\, n_i\, \bar{q}_i \qquad \text{on } \partial\Omega_n \\
&= n_i\, q_i\, (T - \bar{T}) \qquad \text{on } \partial\Omega_d
\end{aligned}
$$

- Applying Euler-Lagrange (wrt T, $\boldsymbol{\nabla} T$) we obtain:

$$
\nabla^2 T + f = 0 \quad \text{in } \Omega_0 \qquad \left\{ \begin{array}{ll} q_i = \bar{q}_i & \text{on } \partial\Omega_n \\ 0 = 0 & \text{on } \partial\Omega_d \end{array} \right.
$$

$$
J(\boldsymbol{u}) = \iiint_{\Omega_0} F(\boldsymbol{X}, T, \boldsymbol{\nabla} T, \dots)\, dV - \iint_{\partial\Omega_n} \Phi(\boldsymbol{X}, T, \boldsymbol{\nabla} T, \dots)\, dS
$$

$$
\boxed{J(\boldsymbol{u}) = \iiint_{\Omega_0} \left(\frac{1}{2}(\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T) - Tf\right) dV - \iint_{\partial\Omega_n} \mathrm{T}\ \bar{\mathbf{q}} \cdot \boldsymbol{n}\, dS}
$$

All for what?!

Weak form!

T minimises J

$$\Leftrightarrow$$

$$\iiint_{\Omega_0} \dot{T}\,\eta\, dV + \iiint_{\Omega_0} \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T\, dV = \iint_{\partial\Omega_\mathrm{n}} \bar{q}_i n_i \eta dS + \iiint_{\Omega_0} f\eta\, dV, \forall\eta \in \mathbb{R}^+$$

$$\left(\iiint_{\Omega_0} (-\nabla^2 T = f)\,\eta\, dV = 0, \forall\eta \in \mathbb{R}^+ \right)$$

Strong form

Continuum mechanics

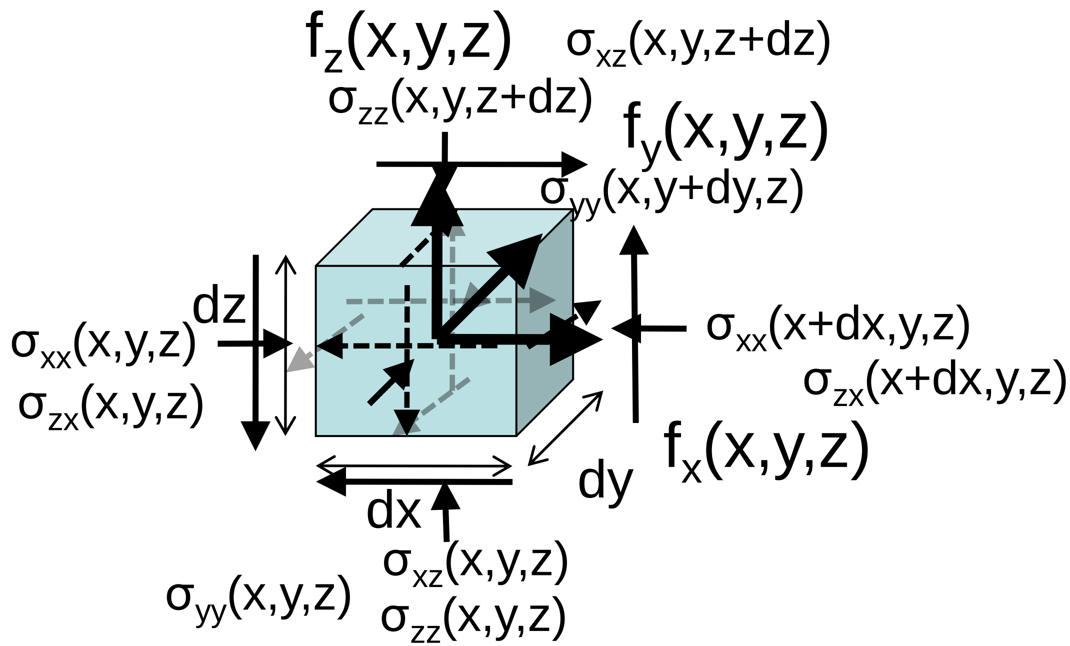
Strong formulation:

The following conditions are valid for all $\boldsymbol{x}(t)$ of $\Omega(t)$ for all t

- Conservation of linear momentum
- Conservation of angular momentum
- Initial conditions
- Boundary conditions (Neumann & Dirichlet)

Conservation of linear momentum (1/2)

Body forces: **f**
Stress tensor: **σ**



Along x	Along y	Along z
$(\sigma_{xx}(x+dx)-\sigma_{xx}(x))\ dy\ dz$	$(\sigma_{yy}(y+dy)-\sigma_{yy}(y))\ dx\ dz$	$(\sigma_{zz}(z+dz)-\sigma_{zz}(z))\ dy\ dx$
$+\ (\sigma_{xy}(y+dy)-\sigma_{xy}(y))\ dx\ dz$	$+(\sigma_{yz}(z+dz)-\sigma_{yz}(z))\ dx\ dy$	$+(\sigma_{zx}(x+dx)-\sigma_{zx}(x))\ dy\ dz$
$+\ (\sigma_{xz}(z+dz)-\sigma_{xz}(z))\ dx\ dy$	$+(\sigma_{yx}(x+dx)-\sigma_{yx}(x))\ dy\ dz$	$+(\sigma_{zy}(y+dy)-\sigma_{zy}(y))\ dx\ dz$
$+\ \ f_x\ dx\ dy\ dz \quad = 0$	$+\ \ f_y\ dx\ dy\ dz \quad = 0$	$+\ f_z\ dx\ dy\ dz \quad = 0$

Conservation of linear momentum (2/2)

<u>Static:</u>

$$f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

$$\boldsymbol{f} + \boldsymbol{div}(\boldsymbol{\sigma}) = 0$$

<u>Dynamic:</u>

$$\boldsymbol{f} + \boldsymbol{div}(\boldsymbol{\sigma}) = \rho \ddot{\boldsymbol{x}}$$

Conservation of angular momentum

- Applying the same approach, the conservation of angular momentum yields:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- This is often applied directly through the constitutive law and doesn't need to be "reinforced"

Initial/Boundary Conditions

- Initial conditions:

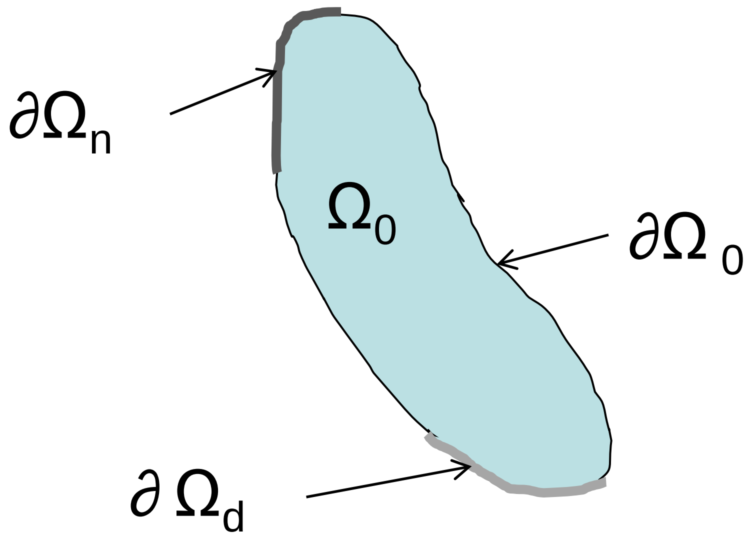
$$\boldsymbol{\varphi}: (\Omega_0, \mathbb{R}^+) \to \Omega$$
$$(\boldsymbol{X}, t) \mapsto \boldsymbol{x}$$

$$\boldsymbol{\varphi}(\boldsymbol{X}, 0) = \boldsymbol{\varphi}_0(\boldsymbol{X}), \dot{\boldsymbol{\varphi}}(\boldsymbol{X}, 0) = \dot{\boldsymbol{\varphi}}_0(\boldsymbol{X}) \quad \forall \boldsymbol{X} \in \Omega_0$$

- Boundary conditions:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}, \quad \forall \boldsymbol{X} \in \partial\Omega_n$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}}, \quad \forall \boldsymbol{X} \in \partial\Omega_d$$



Hu-Washizu functional

- Applied to small deformation of materials, defined with Cauchy stress tensor $\boldsymbol{\sigma}$, strain tensor $\boldsymbol{\varepsilon}$ and Helmholtz free energy $W(\boldsymbol{\varepsilon})$, the Hu-Washizu functional $J(\boldsymbol{u})$ is defined by:

$$F(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = W(\underline{\boldsymbol{\varepsilon}}) - f_i u_i + \sigma_{ij}\,(\,v(i,j) - \varepsilon_{ij}\,)$$

$$\begin{aligned} \Phi(\boldsymbol{u}) &= u_i\,\bar{t}_i && \text{on } \partial\Omega_n \\ &= n_j\,\sigma_{ij}\,(u_i - \bar{u}_i) && \text{on } \partial\Omega_d \end{aligned}$$

- Applying Euler-Lagrange (wrt $\mathbf{u}$,$\boldsymbol{\varepsilon}$ *and* $\boldsymbol{\sigma}$) we obtain:

$$\left\{\begin{aligned} & f_i + \sigma_{ij,j} = 0 \\ & \frac{\partial W}{\partial \varepsilon_{ij}} - \sigma_{ij} = 0 \qquad \text{in } \Omega \\ & v(i,j) - \varepsilon_{ij} = 0 \end{aligned}\right. \qquad \left\{\begin{aligned} & \sigma_{ij} n_j = \bar{t}_i \quad \text{on } \partial\Omega_n \\ & 0 = 0 \quad \text{on } \partial\Omega_d \end{aligned}\right.$$

$$J(\boldsymbol{u}) = \iiint_{\Omega} F(\boldsymbol{X},\boldsymbol{u},\boldsymbol{\nabla u},\dots)\,dV - \iint_{\partial\Omega} \Phi(\boldsymbol{X},\boldsymbol{u},\boldsymbol{\nabla u},\dots)\,dS$$

$$\boxed{J(\boldsymbol{u}) = \iiint_{\Omega} (W(\boldsymbol{\varepsilon}) - f_i u_i)\,dV - \iint_{\partial\Omega_n} \bar{t}_i u_i\,dS}$$

All for what?!

Weak form!

u minimises J

$$\Leftrightarrow$$

$$\iiint_{\Omega} \rho \ddot{x}_i \eta_i dV + \iiint_{\Omega} \sigma_{ij} \eta_{i,j}\, dV = \iint_{\partial\Omega_{\mathrm{n}}} \bar{t}_i \eta_i dS + \iiint_{\Omega} f_i \eta_i\, dV, \forall \boldsymbol{\eta} \in \Omega$$

$$\iiint_{\Omega} \left(f_i + \sigma_{ij,j} = \rho \ddot{x}_i\right) \eta_i dV = 0, \forall \boldsymbol{\eta} \in \Omega$$

Strong form