Department of Engineering Science

UNIVERSITY OF OXFORD

B1 The Finite Element Method
Lecture 3: Spatial discretisation

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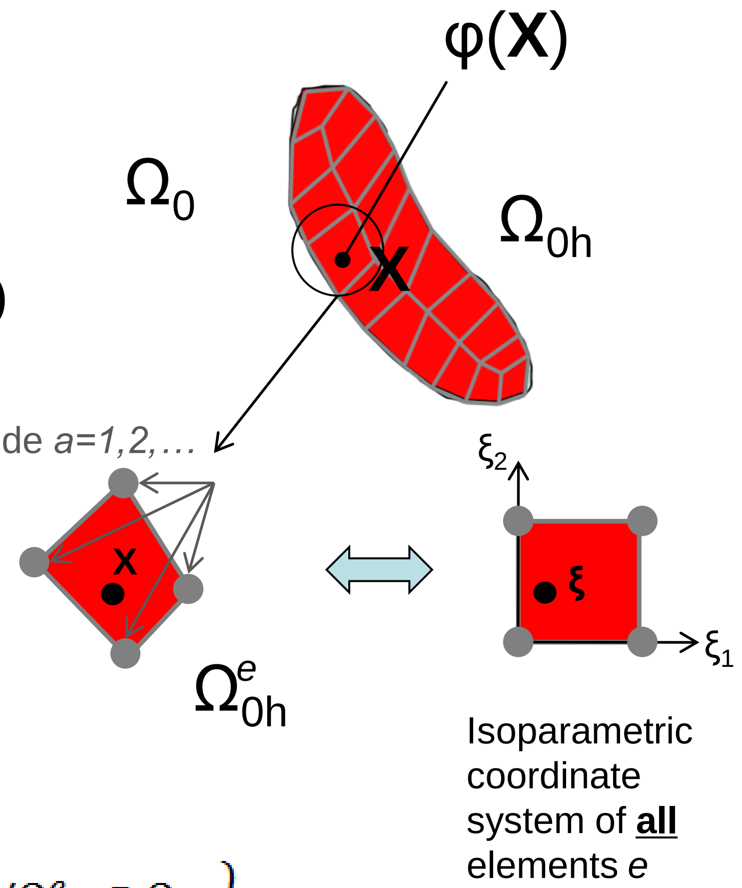
Element discretisation

- Unknown field $\boldsymbol{\varphi}: (\Omega_0, \mathbb{R}^+) \to \Omega$
  $$(\boldsymbol{X}, t) \mapsto \boldsymbol{x}$$

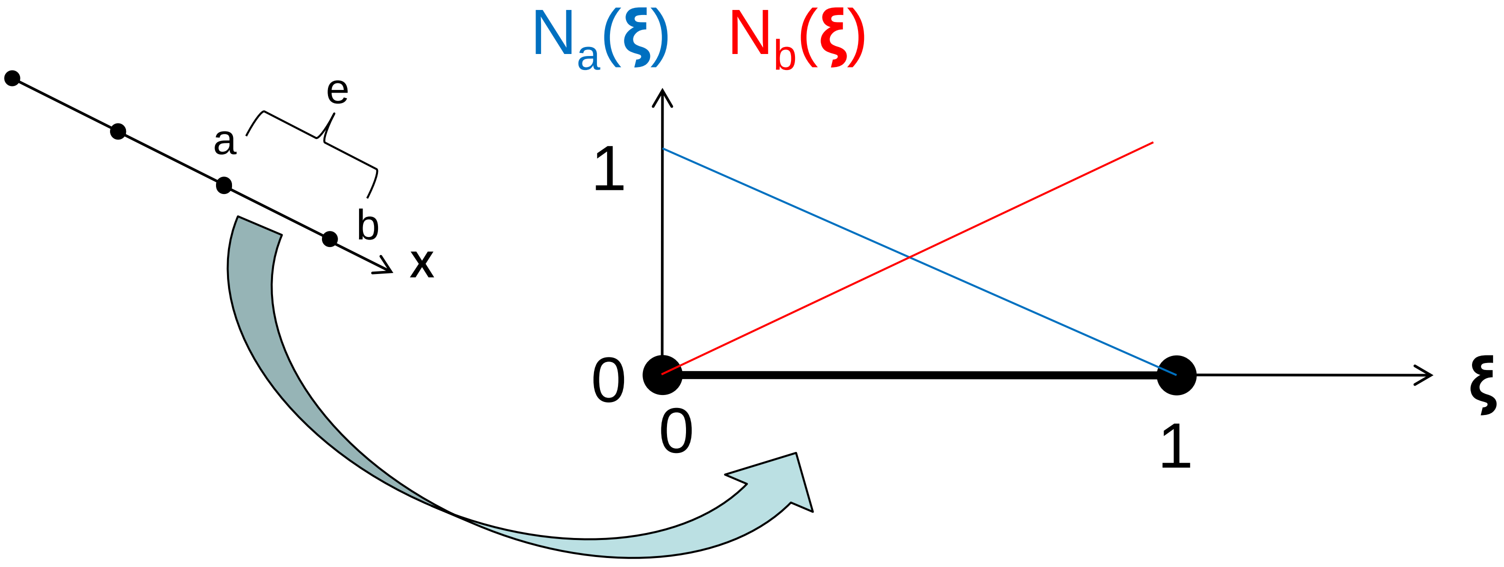
- For a given **X** inside an element *e*, there exists a unique **ξ** in the isoparametric coordinate system (by homeomorphism)
- We define the shape function $N_a(\boldsymbol{\xi})$ as a polynomial of order *k* such that
  $$N_a(\boldsymbol{\xi}_b) = \delta_{ab}$$
  for nodes *a* and *b* of element *e*
- By knowing the displacement of all the nodes *a*, we can approximate φ(**X**) by:



with $$\varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = \sum_a N_a(\xi)\boldsymbol{x}^a$$

$$\varphi_h \in \left\{\varphi_h \in C^0(\Omega_{0h}) | \varphi_h^e = \varphi_h|_{\Omega^e_{0h}} \in P^k(\Omega^e_{0h}) \forall \Omega^e_{0h} \in \Omega_{0h}\right\}$$

1D linear example

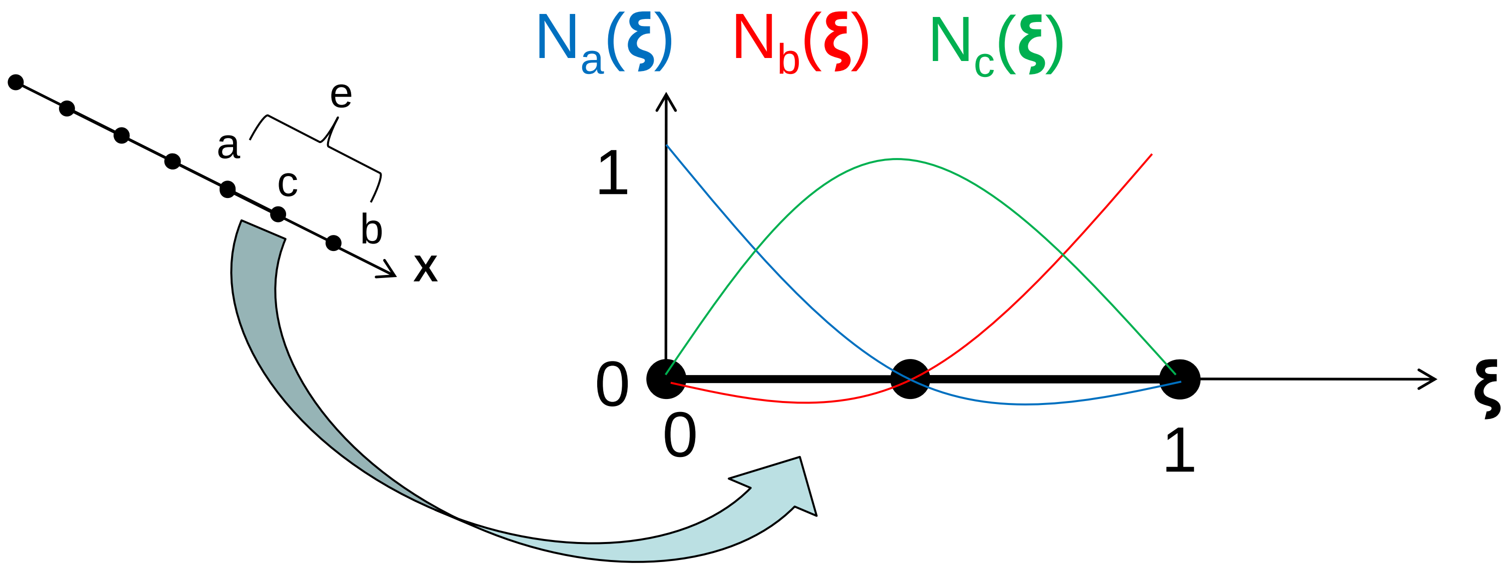


$N_a(\xi)=1-\xi$ $\quad$ $N_b(\xi)=\xi$

$$\varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = \sum_a N_a(\xi)\boldsymbol{x}^a$$

$$\Rightarrow \varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = (1-\xi)\boldsymbol{x}^a + \xi\boldsymbol{x}^b$$

1D quadratic example

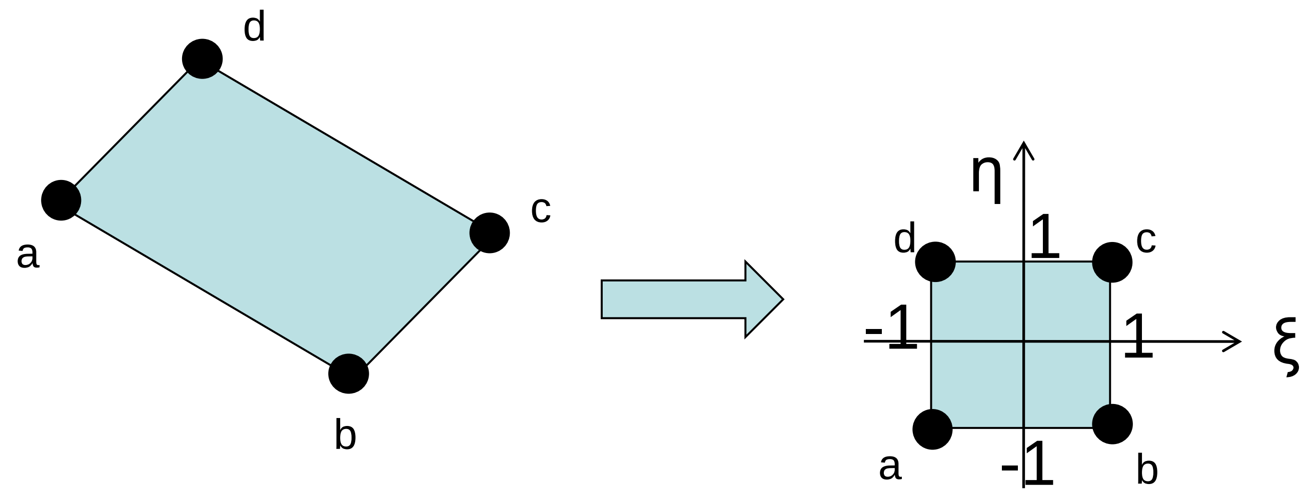


$N_a(\xi)=(1-\xi)(1-2\xi)$ $N_b(\xi)=\xi(2\xi-1)$ $N_c(\xi)=4\xi(1-\xi)$

$$\varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = \sum_a N_a(\boldsymbol{\xi})\boldsymbol{x}^a$$

$$\varphi_h(\boldsymbol{X}) = \varphi_h^e(\boldsymbol{X}) = (1-\xi)(1-2\xi)\boldsymbol{x}^a + \xi(2\xi-1)\boldsymbol{x}^b + 4\xi(1-\xi)\boldsymbol{x}^c$$

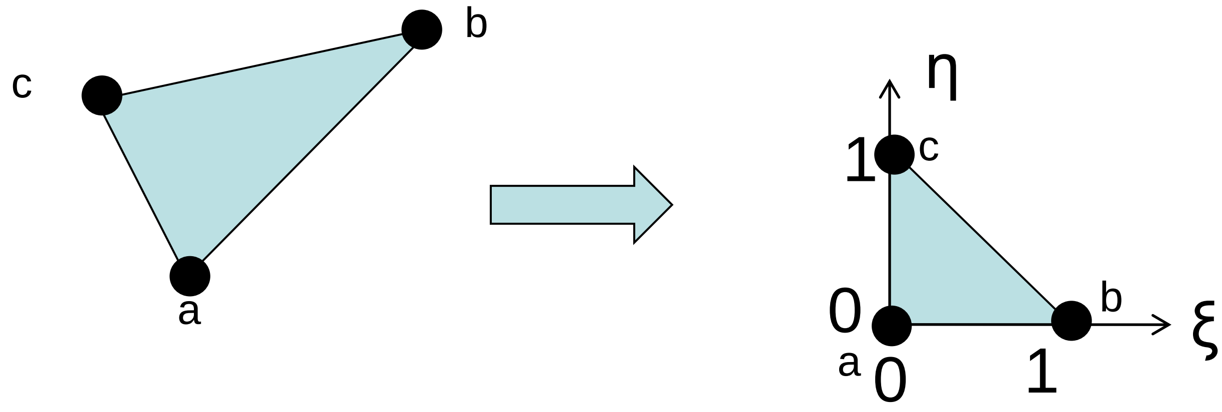
2D linear example: square element



$$N_a(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_c(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_b(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) \qquad N_d(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

2D linear example: triangle element



$$N_a(\xi,\eta) = 1 - \xi - \eta$$

$$N_b(\xi,\eta) = \xi$$

$$N_c(\xi,\eta) = \eta$$

Reminder: minimisation of *J*

$$J(\boldsymbol{u}) = \iiint_{\Omega} \left(W(\boldsymbol{\varepsilon}) - f_i u_i\right) dV - \iint_{\partial\Omega_n} \bar{t}_i u_i \, dS$$

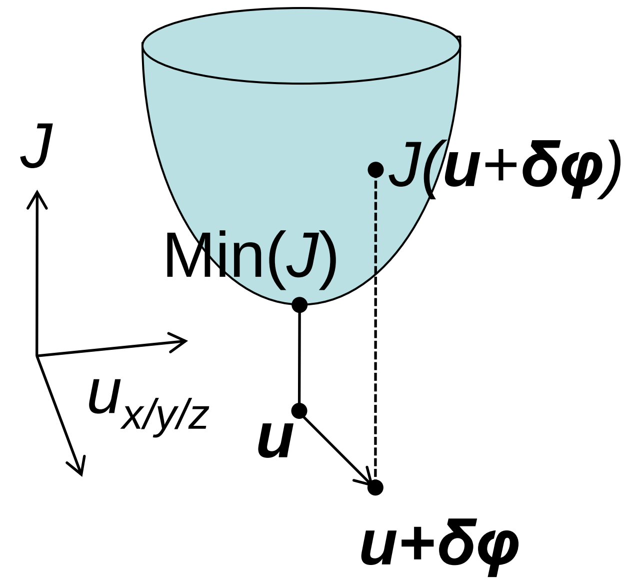
u minimises *J*

$$\Leftrightarrow$$

$$\left.\frac{dJ(\boldsymbol{u} + \varepsilon\boldsymbol{\eta})}{d\varepsilon}\right|_{\varepsilon=0} = 0, \forall \boldsymbol{\eta} \in \Omega$$

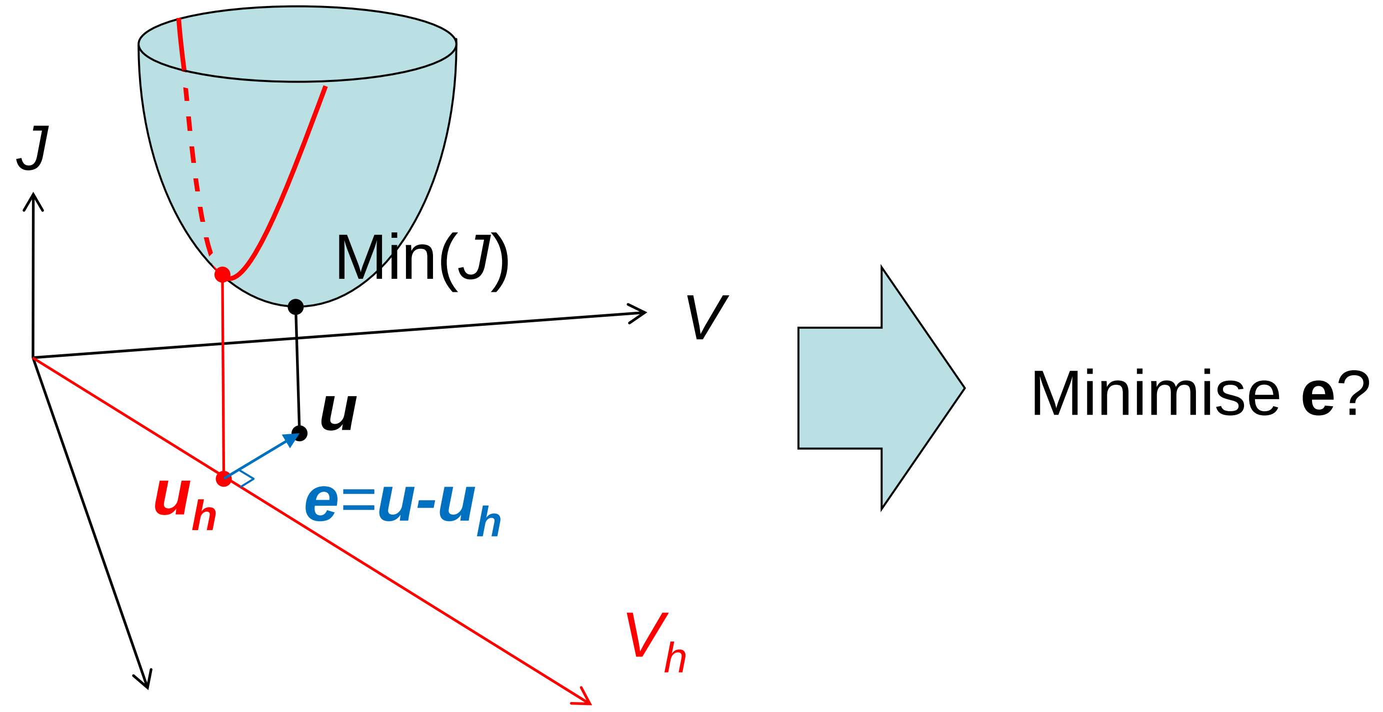
$$\Leftrightarrow$$

$$\iiint_{\Omega} \sigma_{ij}\eta_{i,j} \, dV = \iint_{\partial\Omega_{\mathrm{n}}} \bar{t}_i \eta_i dS + \iiint_{\Omega} f_i \eta_i \, dV, \forall \boldsymbol{\eta} \in \Omega$$



Finding u_h…

- Let $\boldsymbol{u_h}=\boldsymbol{\varphi_h}(\boldsymbol{X})-\boldsymbol{X}$ be the approximated displacement field minimising J
- Let V and V_h be the admissible spaces for u and u_h respectively



The special case of linear elasticity

- Linear elasticity
  - Strain: $\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i})$
  - Cauchy stress: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$
  - Strain energy: $W = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2}\,C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}$
- Solving for $\boldsymbol{u_h}$ can be done by:
  - Minimising $J(\boldsymbol{u_h})$ (Rayleigh-Ritz method)
  - Solving weak form using $\boldsymbol{u_h}$ and $\boldsymbol{\eta_h}$ (Galerkin method)

$$\Leftrightarrow$$

  - Minimising $e=||\boldsymbol{u} - \boldsymbol{u_h}||$
  - Finding $\boldsymbol{u_h}$ such that:

$$\langle \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\eta}_h \rangle = 0, \forall \boldsymbol{\eta}_h \in V_h$$

Math again!

- In finite dimension, all norms are equivalent
  let's use another norm (or inner product)
- Let's define the symmetric bilinear form such that:

$$a(\boldsymbol{u},\boldsymbol{v}) = \iiint_{\Omega} C_{ijkl}\, u_{k,l} v_{i,j}\, dV \,, \forall(\boldsymbol{u},\boldsymbol{v}) \in \Omega^2$$

- Here, *a* is also definite positive *i.e.* $||\boldsymbol{u}||=\sqrt{a(\boldsymbol{u},\boldsymbol{u})}$ is a norm, ($a(\mathbf{u},\mathbf{u})\geq 0$ and $a(\mathbf{u},\mathbf{u})=0$ iff $\boldsymbol{u}=\boldsymbol{0}$) and *a* is an inner product.

$$\begin{pmatrix} a(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}) = a(\boldsymbol{u},\boldsymbol{w}) + a(\boldsymbol{v},\boldsymbol{w}) \\ a(\boldsymbol{u},\boldsymbol{v}+\boldsymbol{w}) = a(\boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{w}) \\ a(\boldsymbol{u},\lambda\boldsymbol{v}) = a(\lambda\boldsymbol{u},\boldsymbol{v}) = \lambda a(\boldsymbol{u},\boldsymbol{v}) \end{pmatrix}$$

We can use this norm/inner product to solve the minimisation problem

Weak form equivalent to Min(**e**)? (1/2)

- Weak form in linear elasticity:

$$\iiint_{\Omega} \sigma_{ij}\eta_{i,j}\, dV = \iint_{\partial\Omega_{\mathrm{n}}} \bar{t}_i \eta_i dS + \iiint_{\Omega} f_i \eta_i\, dV, \forall \boldsymbol{\eta} \in \mathrm{V}$$

with

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

we have:

$$\iiint_{\Omega} C_{ijkl} u_{k,l} \eta_{i,j} dV = \iint_{\partial\Omega_{\mathrm{n}}} \bar{t}_i \eta_i dS + \iiint_{\Omega} f_i \eta_i\, dV, \forall \boldsymbol{\eta} \in \mathrm{V}$$

$$\Leftrightarrow$$

$$a(\boldsymbol{u}, \boldsymbol{\eta}) = l(\boldsymbol{\eta}), \forall \boldsymbol{\eta} \in \mathrm{V}$$

$$\Rightarrow \quad \boxed{a(\boldsymbol{u}, \boldsymbol{\eta}_{\boldsymbol{h}}) = l(\boldsymbol{\eta}_{\boldsymbol{h}})\ \forall \boldsymbol{\eta}_h \in \mathrm{V}_h}$$

Weak form equivalent to Min(**e**)? (2/2)

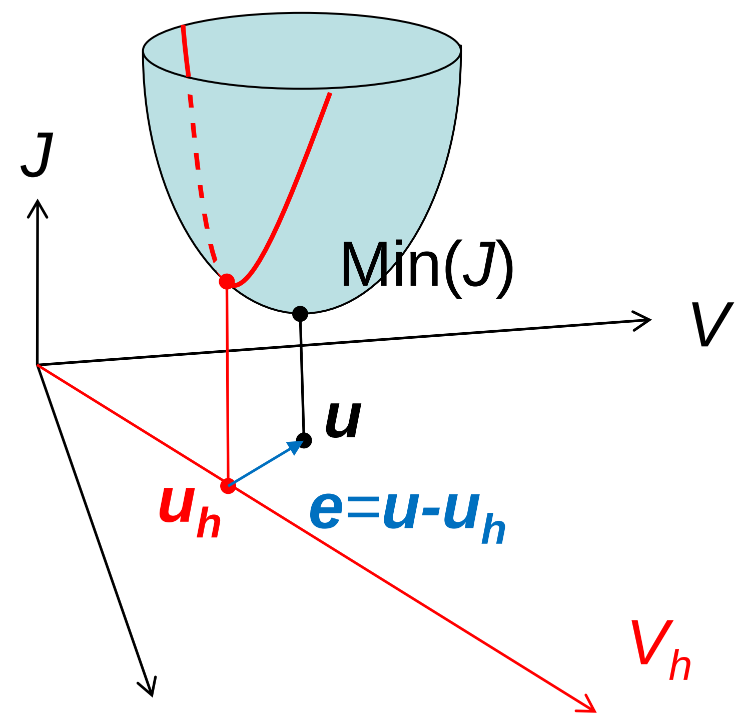
$$a(\boldsymbol{u}, \boldsymbol{\eta_h}) = l(\boldsymbol{\eta_h})\ \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

- But $\boldsymbol{u}_h$ also minimises J in V_h so

$$a(\boldsymbol{u_h}, \boldsymbol{\eta_h}) = l(\boldsymbol{\eta_h})\ \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

$$\boxed{\langle \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\eta}_h \rangle = 0, \forall \boldsymbol{\eta}_h \in V_h}$$

Methods equivalent!



Finding u_h… the Galerkin way (1/2)

- Replacing $\boldsymbol{u}$ and $\boldsymbol{\eta}$ (in V) by $\boldsymbol{u_h}$ and $\boldsymbol{\eta_h}$ (in V_h):

we have: $$\boldsymbol{u}_h = \sum_a N_a(\boldsymbol{\xi})\boldsymbol{u}^a\ , \qquad \boldsymbol{\eta}_h = \sum_a N_a(\boldsymbol{\xi})\boldsymbol{\eta}^a$$

$$a(\boldsymbol{u_h}, \boldsymbol{\eta_h}) = l(\boldsymbol{\eta_h})\ \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

$$\Leftrightarrow$$

$$\iiint_\Omega C_{ijkl}\left(\sum_a N_{a,l}\, u_k^a\right)\left(\sum_b N_{b,j}\eta_i^b\right) dV = \iint_{\partial\Omega_\mathrm{n}} \bar{t}_i\left(\sum_b N_b\eta_i^b\right) dS$$
$$+ \iiint_\Omega f_i\left(\sum_b N_b\eta_i^b\right) dV, \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

Finding u_h… the Galerkin way (2/2)

$$\iiint_\Omega C_{ijkl}\left(\sum_a N_{a,l}\,u_k^a\right)\left(\sum_b N_{b,j}\eta_i^b\right)dV = \iint_{\partial\Omega_\mathrm{n}} \bar{t}_i\left(\sum_b N_b\eta_i^b\right)dS$$

$$+\iiint_\Omega f_i\left(\sum_b N_b\eta_i^b\right)dV, \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

$$\Leftrightarrow$$

$$\sum_b \eta_i^b\left[\sum_a\left(\iiint_\Omega C_{ijkl}N_{a,l}N_{b,j}\,dV\right)u_k^a - \iint_{\partial\Omega_\mathrm{n}} \bar{t}_i N_b\,dS - \iiint_\Omega f_i N_b\,dV\right], \forall \boldsymbol{\eta}_h \in \mathrm{V}_h$$

$$K_{ibka} \qquad\qquad f_{ib}^{ext}$$

$$\Leftrightarrow$$

$$K_{iakb}\,u_k^b = f_{ia}^{ext}$$

Element assembly

$$K_{iakb}\, u_k^b = f_{ia}^{ext}$$

$$\left(\sum_e K_{iakb}^e\right) u_k^b = \sum_e f_{ia}^{e\ \ ext}$$

$$\iiint_{\Omega^e} C_{ijkl}^e N_{a,j}^e N_{b,l}^e dV^e \qquad \iint_{\partial\Omega_\mathrm{n}^\mathrm{e}} \bar{t}_i N_b^e dS^e + \iiint_{\Omega^e} f_i N_b^e \, dV^e$$

Lax-Milgram Theorem

If there exists *k* and *K* such that for all *u* and *v* in Hilbert space *V*:

$$\begin{cases} ||a(\boldsymbol{u},\boldsymbol{v})|| \leq K||\boldsymbol{u}||\,||\boldsymbol{v}|| \\ ||a(\boldsymbol{u},\boldsymbol{u})|| \geq k||\boldsymbol{u}||^2 \end{cases}$$

Then there is a unique solution ***u*** verifying

$$a(\boldsymbol{u},\boldsymbol{\eta}) = l(\boldsymbol{\eta}), \forall \boldsymbol{\eta} \in \mathrm{V}$$

And same theorem for u_h in V_h!