Department of Engineering Science

B1 The Finite Element Method
Lecture 1: Energy minimisation

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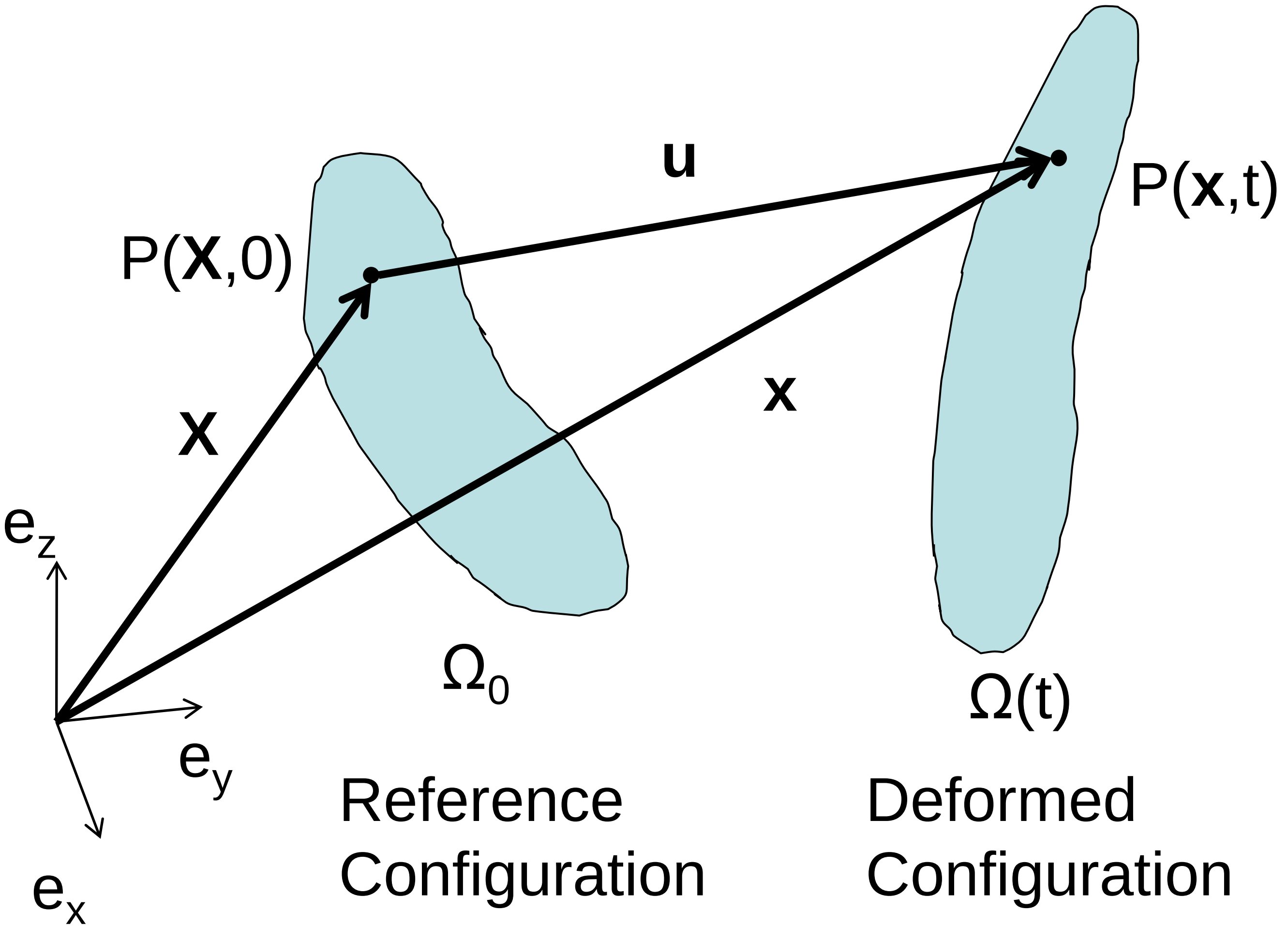
10/2/2023

Continuum configurations (1/3)

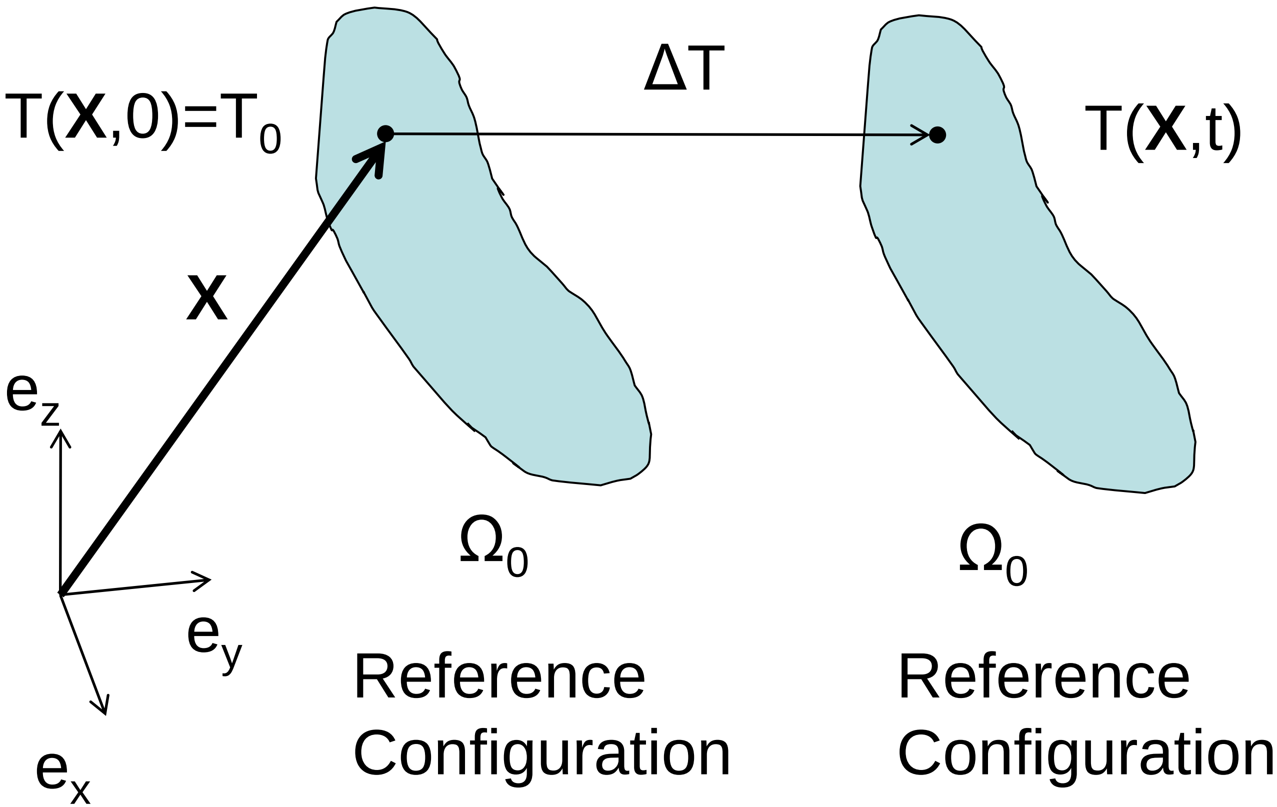


Reference Configuration

Deformed Configuration

$$\mathbf{X} + \mathbf{u} = \mathbf{x}$$

Continuum configurations (2/3)

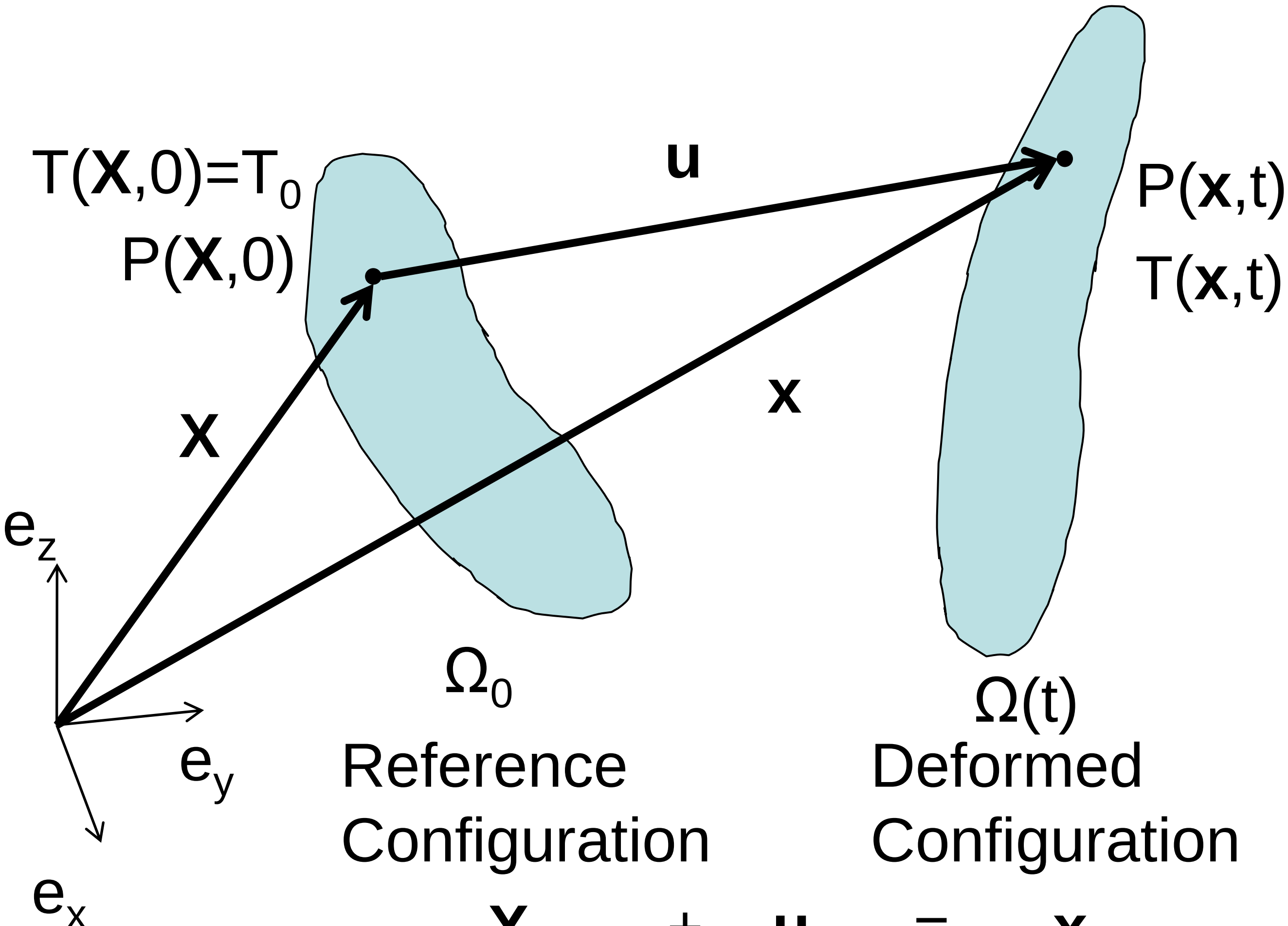


Reference Configuration

Reference Configuration

$$T_0 \quad + \quad \Delta T \quad = \quad T$$

Continuum configurations (3/3)



$$\mathbf{X} \quad + \quad \mathbf{u} \quad = \quad \mathbf{x}$$

$$T_0 \quad + \quad \Delta T \quad = \quad T$$

A bit of Math: variational calculus

- Solving a continuum mechanics boundary value problem = finding the state of unknown (deformation, temperature, etc.) minimising the energy of the system
- This can be history dependent!

- Let J(**u**) be a functional of **u**:

$$J(\boldsymbol{u}) = \iiint_{\Omega} F(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{u}, \dots)\, dV - \iint_{\partial\Omega} \Phi(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{u}, \dots)\, dS$$

- Goal: Find **u** minimising J(**u**)

Minimisation of J

- Let us define $\boldsymbol{\delta\varphi}=\varepsilon\boldsymbol{\eta}$, a virtual displacement where

$$\varepsilon \in \mathbb{R} \text{ and } \boldsymbol{\eta} \in \Omega$$

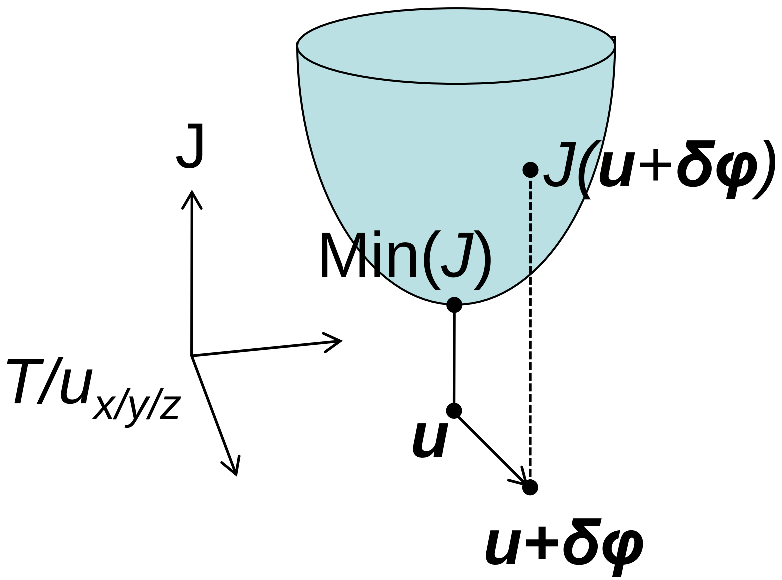
(with $\varepsilon=0$ on $\partial\Omega_d$)

- Minimisation:

$\boldsymbol{u}$ minimises J

$$\Leftrightarrow$$

$$\left.\frac{dJ(\boldsymbol{u}+\varepsilon\boldsymbol{\eta})}{d\varepsilon}\right|_{\varepsilon=0} = 0, \forall \boldsymbol{\eta} \in \Omega$$



Euler-Lagrange Equations (1/2)

$$\left.\frac{dJ(\boldsymbol{u}+\varepsilon\boldsymbol{\eta})}{d\varepsilon}\right|_{\varepsilon=0} = 0, \forall\boldsymbol{\eta}\in\Omega$$

$$\Leftrightarrow$$

$$\frac{d}{d\varepsilon}\left[\iiint_\Omega F(\boldsymbol{X},\boldsymbol{u}+\varepsilon\boldsymbol{\eta},\boldsymbol{\nabla}(\boldsymbol{u}+\varepsilon\boldsymbol{\eta}))dV - \iint_{\partial\Omega}\Phi(\boldsymbol{X},\boldsymbol{u}+\varepsilon\boldsymbol{\eta})dS\right]\Bigg|_{\varepsilon=0} = 0, \forall\boldsymbol{\eta}\in\Omega$$

$$\Leftrightarrow$$

$$\left[\iiint_\Omega\left(\frac{\partial F}{\partial u_i}(\boldsymbol{X},\boldsymbol{u}+\varepsilon\boldsymbol{\eta},\boldsymbol{\nabla}(\boldsymbol{u}+\varepsilon\boldsymbol{\eta}))\frac{d(u_i+\varepsilon\eta_i)}{d\varepsilon}+\frac{\partial F}{\partial u_{i,j}}(\boldsymbol{X},\boldsymbol{u}+\varepsilon\boldsymbol{\eta},\boldsymbol{\nabla}(\boldsymbol{u}+\varepsilon\boldsymbol{\eta}))\frac{d\left(u_{i,j}+\varepsilon\eta_{i,j}\right)}{d\varepsilon}\right)dV\right.$$
$$\left.-\iint_{\partial\Omega}\frac{\partial\Phi}{\partial u_i}(\boldsymbol{X},\boldsymbol{u}+\varepsilon\boldsymbol{\eta})\frac{d(u_i+\varepsilon\eta_i)}{d\varepsilon}dS\right]\Bigg|_{\varepsilon=0} = 0, \forall\boldsymbol{\eta}\in\Omega$$

$$\Leftrightarrow$$

$$\iiint_\Omega\left(\frac{\partial F}{\partial u_i}\eta_i+\frac{\partial F}{\partial u_{i,j}}\eta_{i,j}\right)dV - \iint_{\partial\Omega}\frac{\partial\Phi}{\partial u_i}\eta_i dS = 0, \forall\boldsymbol{\eta}\in\Omega$$

Euler-Lagrange Equations (2/2)

By integration by part:

$$\iiint_{\Omega}\left(\frac{\partial F}{\partial u_i}\eta_i+\frac{\partial F}{\partial u_{i,j}}\eta_{i,j}\right)dV-\iint_{\partial\Omega}\frac{\partial \Phi}{\partial u_i}\eta_i dS=0,\forall\boldsymbol{\eta}\in\Omega$$

$$\Leftrightarrow$$

$$\iiint_{\Omega}\left[\frac{\partial F}{\partial u_i}-\left(\frac{\partial F}{\partial u_{i,j}}\right)_{,j}\right]\eta_i dV+\iint_{\partial\Omega}\left(\frac{\partial F}{\partial u_{i,j}}n_j-\frac{\partial \Phi}{\partial u_i}\right)\eta_i dS=0,\forall\boldsymbol{\eta}\in\Omega$$

(Weak form)

$$\Updownarrow ?$$

$$\begin{cases}\dfrac{\partial F}{\partial u_i}-\left(\dfrac{\partial F}{\partial u_{i,j}}\right)_{,j}=0 \ in\ \Omega\\[2ex] \dfrac{\partial F}{\partial u_{i,j}}n_j-\dfrac{\partial \Phi}{\partial u_i}=0\ on\ \partial\Omega\end{cases}$$

(Strong form)