

Searching: Balanced Search Trees

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Acknowledgments: Material mainly based on the textbook Algorithms by Robert Sedgewick and Kevin Wayne (Chapters 3.3, 6), Prof. Mhaskar's course slides

Symbol Table Review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	$\frac{1}{2} N$	N	$\frac{1}{2} N$		<code>equals()</code>
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✓	<code>compareTo()</code>
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	\sqrt{N}	✓	<code>compareTo()</code>
goal	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✓	<code>compareTo()</code>

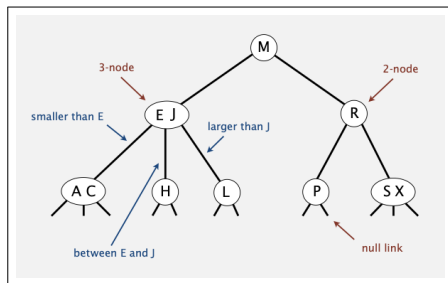
The further a Binary Search Tree departs from being fully filled, the worse our runtime is.

- In the worst case, BSTs have no advantage over unsorted arrays!
- Thus, we commonly employ **tree balancing algorithms** to improve our worst-case runtime.

2-3 Search Trees

Our goal is to modify our tree so that it can't be unbalanced by insert operations. One way of doing this is the **2-3 Tree**.

- 2-nodes have one key and two children.
- 3-nodes have two keys and three children.

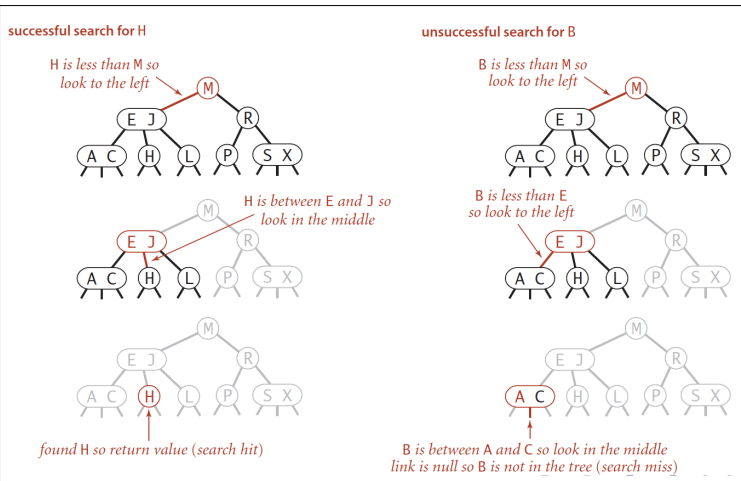


The two values in a 3-node are the upper and lower bounds of a range. All elements in that range fall into the center branch. Otherwise, 3-nodes function similarly to a normal BST.

- A BST is a 2-3 Tree containing only 2-nodes!

2-3 Search Tree: Search

Searching goes exactly as you might expect.



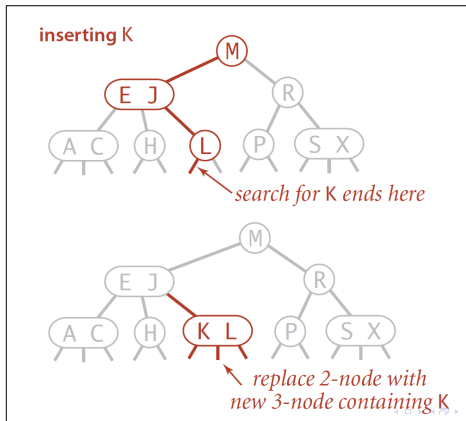
Insertion Philosophy

All paths through a 2-3 Tree are of the same length *by construction!*

- The reason that BSTs create unequal branch lengths is that we insert nodes by simply appending them to branches.
- The insertion procedure for 2-3 Trees avoids this by expanding the tree **from the root**, rather than from individual branches.
- We are still, however, required to add our values at the ends of branches to maintain sortedness.

Contradiction? Not hardly! Since nodes can hold more than one value now, we can insert into nodes as well as creating new ones!

2-3 Search Tree: Insert I



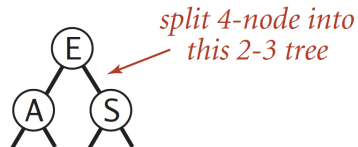
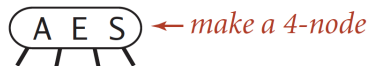
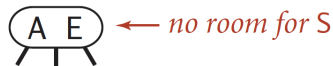
- We still proceed by finding the branch we would insert into under normal BST rules.
- If the leaf node we wish to add to is a 2-node, we add the value to this node (in the correct position) and change it into a 3-node!
- Therefore, we avoid creating a new node and extending the tree's branches, thus maintaining perfect balance.

2-3 Search Tree: Insert II

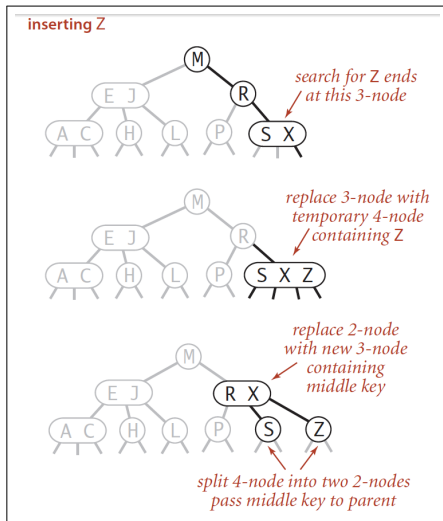
But professor! What if the node you're adding to already has two values!?

- You add it anyways, creating an unstable, **temporary 4-node**
 - 4-nodes have 3 values and 4 children.
- 4-nodes are quickly decomposed into three 2-nodes, with the middle value as parent and the other two as children.
- If we stopped here, we would still see uneven branch growth.

inserting S

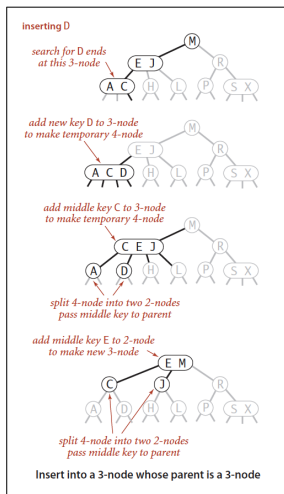


2-3 Search Tree: Insert a single 3-node whose parent is 2-node



- The generated parent node is quickly added to it's own parent.
- This may make the parent unstable as well, so this procedure follows the tree upwards until it hits the root node.
- If the root node decomposes into three 2-nodes, there is no parent to merge into. Therefore, the new branches add to the depth of the tree.
- Thus, 2-3 Trees can only expand at the root, maintaining balance!

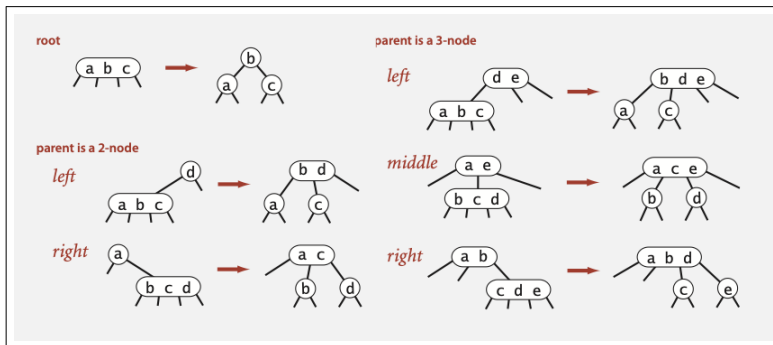
2-3 Search Tree: Insert a single 3-node whose parent is 3-node





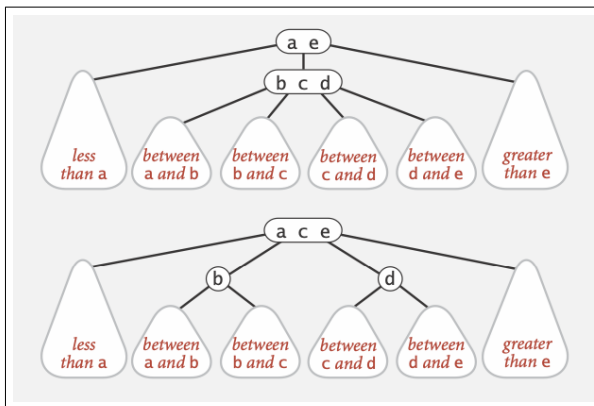
2-3 tree Analysis: Local transformations in a 2-3 Tree

At each level of the tree, splitting a 4-node requires a constant number of operations, as it involves one of six transformations:



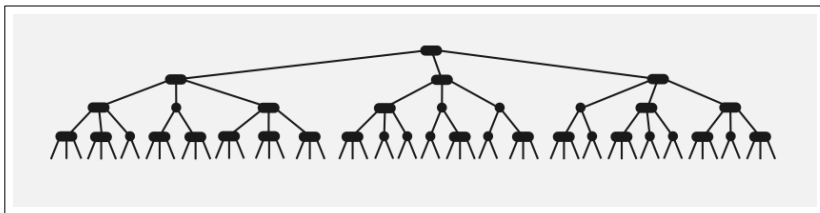
2-3 tree Analysis: Global properties in a 2-3 tree

Each transformation maintains symmetric order and perfect balance.



2-3 tree: performance

Every path from root to null link has same length.



Tree height.

- Worst case: $\log_2 N$ – all two nodes in the tree
- Best case: $\log_3 N \approx .631 \log_2 N$ – all three nodes in the tree

Either way, we are guaranteed **logarithmic** performance for search and insert.

ST implementation

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
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2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✓	compareTo()

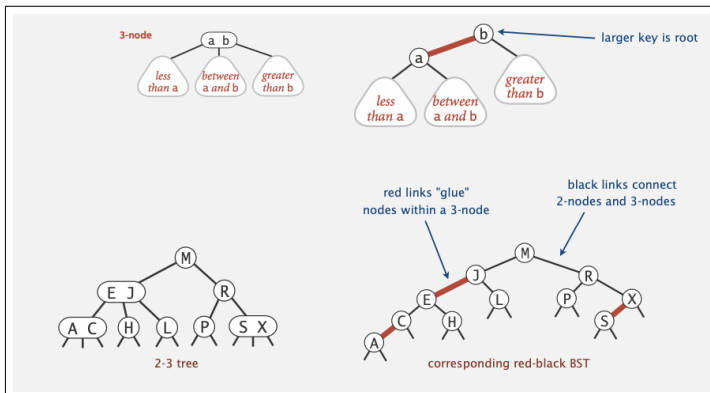


constant c depend upon implementation

Red-Black Trees

Left-leaning red-black BSTs

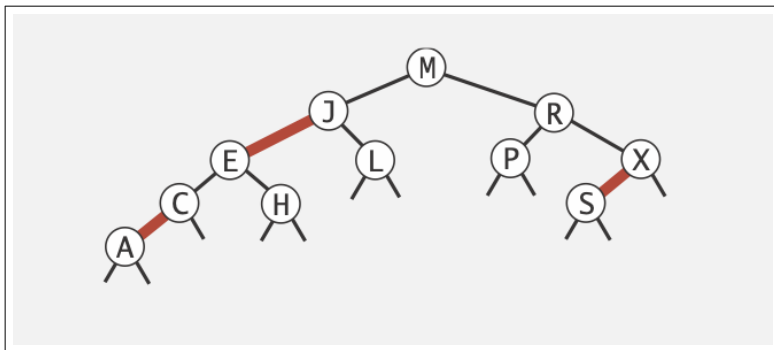
- Red-Black Trees are a way to encode 2-3 trees without needing to keep track of multiple node types and a variable number of values per node.
- Black links are the same in red-black and 2-3 trees.
- Red links connect values that are in the same nodes in 2-3 trees.



An equivalent definition of Red-Black Trees

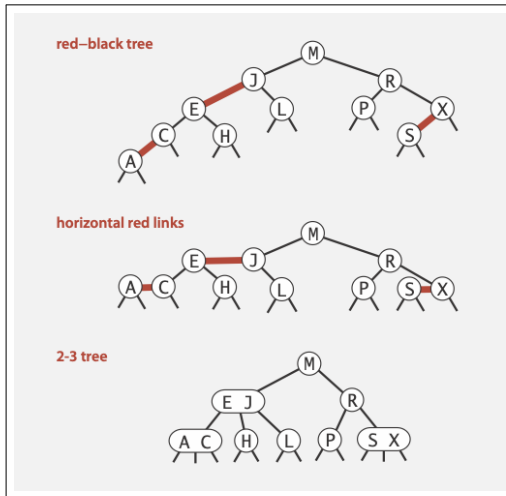
A BST such that:

- 1 No node has two red links connected to it.
- 2 Every path from root to null link has the same number of black links. – “Perfect Black Balance”
- 3 Red links lean left.



Like Checkers!

Key property. 2-3 and LLRB trees exactly correspond.



Red-black BST representation

- Each node has only one parent, so we can encode color of links in nodes (use Boolean variable *color*).
 - True = Red, False = Black
- The color of a node is the color of its parent link.

```
private static final boolean RED = true;
private static final boolean BLACK = false;
```

```
private class Node
{
    Key key;
    Value val;
    Node left, right;
    boolean color; // color of parent link
}
```

```
private boolean isRed(Node x)
{
    if (x == null) return false;
    return x.color == RED;
}
```

null links are black

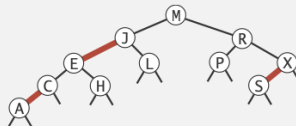


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
because of better balance

```
public Val get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```



Remark. Most other ops (e.g., floor, iteration, selection) are also identical.

Insertion in a LLRB tree: overview

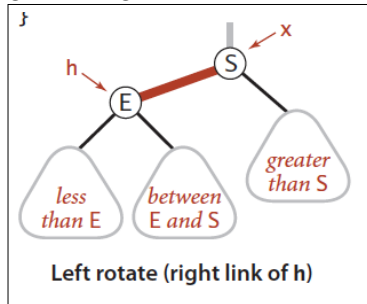
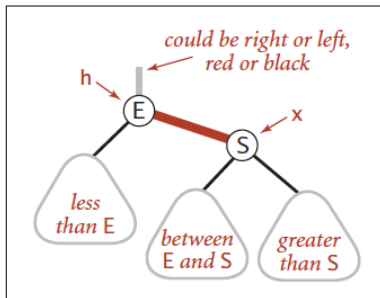
In general, we wish to maintain our one-to-one correspondence with 2-3 trees.

- In order to achieve this we will need to introduce two additional sub-operations
 - Rotation
 - Colour flip

Elementary red-black BST operations: Left Rotation

In the same way that 2-3 trees had unstable 4-nodes during some operations, left leaning red-black trees may have temporary right leaning links.

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

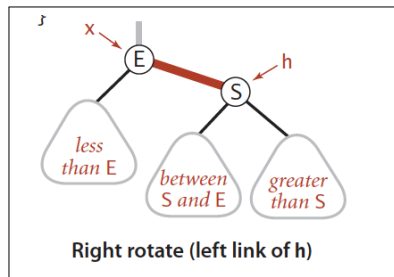
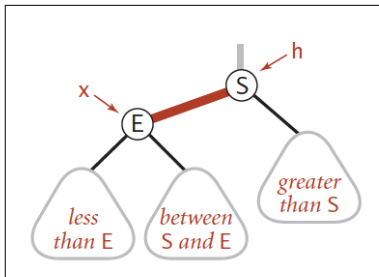


Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations: Right Rotation

We will also require the ability to rotate our left leaning red link right temporarily.

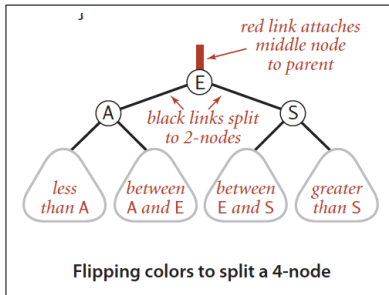
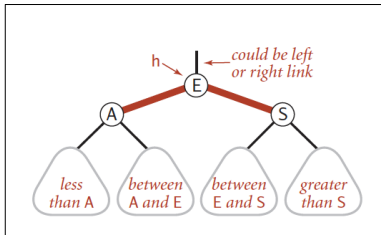
Right rotation. Orient a left-leaning red link to (temporarily) lean right.



Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations: Color Flip

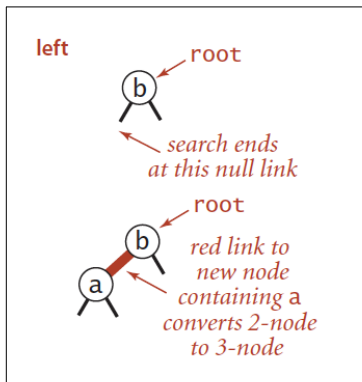
Color Flip. Recolor to split a (temporary) 4-node.



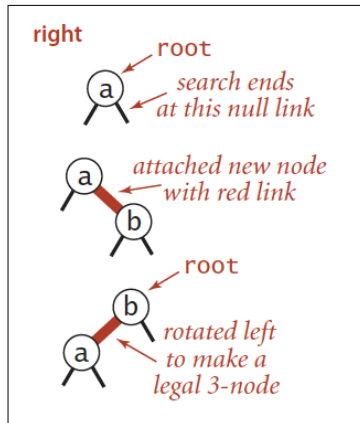
Invariants. Maintains symmetric order and perfect black balance.

Insertion in a LLRB tree: Insert into a single 2-node tree

Warmup 1. Insert into a tree with exactly one 2-node.



Case 1

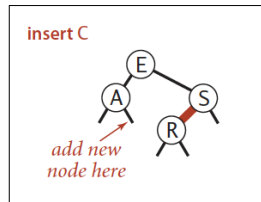


Case 2

Insertion in a LLRB tree: Insert into a 2-node at the bottom - I

Case 1. Insert into a 2-node at the bottom.

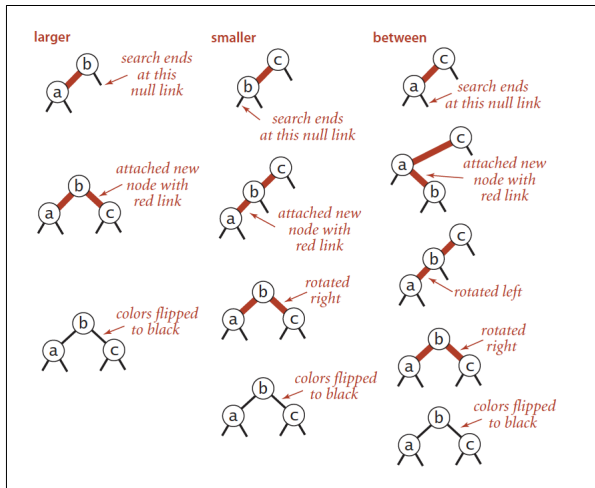
- Insert keys into a red-black BST normally, adding a new node at the bottom with a red link.
- If the parent is a 2-node, then the same two cases just discussed are effective.
- **In particular, if the new node is attached to the left link, the parent simply becomes a 3-node!**
- If the new red link is a right link, we simply left-rotate it.



Case 1

Insert a node into a two node in a (single 3-node tree)

Insert a node into a two node tree (equivalent to a single 3-node 2-3 tree)



Insert into a tree: Insert a node at the bottom - I

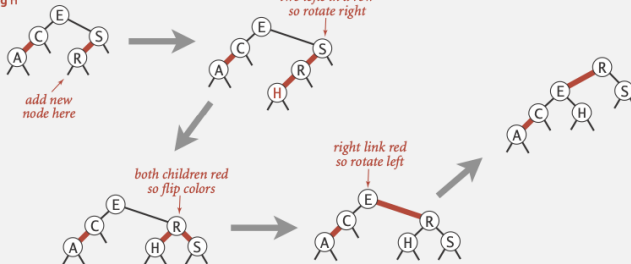
Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).

to maintain symmetric order
and perfect black balance

to fix color invariants

inserting H



Insert into a tree: Insert a node at the bottom - II

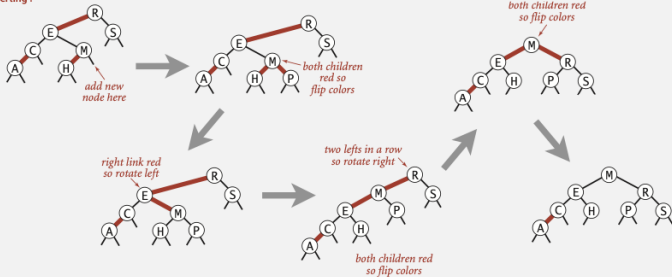
Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

to maintain symmetric order
and perfect black balance

to fix color invariants

inserting P



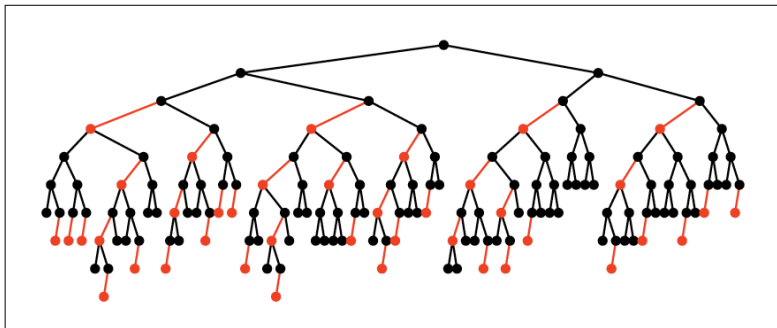
Keeping the root black

- A red root implies that the root is part of a 3-node, but that is not the case, so we color the root of the tree black after each insertion.
- Note that the **black height** of the tree increases by 1 whenever the color of the root is flipped from black to red.

RBT Analysis

Proposition. Height of tree is $\leq 2 \log_2 N$ in the worst case – why?

Below is the typical red-black BST built from random keys (null links omitted)



Property. Height of tree is $\sim 1.0 \log_2 N$ in typical applications.

ST implementations: summary

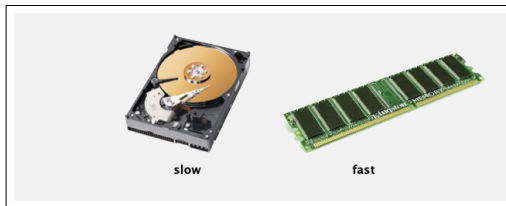
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2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✓	compareTo()
red-black BST	$2 \lg N$	$2 \lg N$	$2 \lg N$	$1.0 \lg N^*$	$1.0 \lg N^*$	$1.0 \lg N^*$	✓	compareTo()
* exact value of coefficient unknown but extremely close to 1								

B-Trees

File System Model

A **Page** is a contiguous block of data (e.g., a file or 1024 or 4,096-byte chunk).

A **Probe** is the first access of a page (e.g., from disk to memory).



- In general, the time required for a probe is much larger than time to access data within a page.
- When studying algorithms for external searching, we count the number of probes, or page accesses.
- We therefore wish to minimize the number of probes.

Cost model. Goal. Access data using minimum number of probes.

B-Trees

The **B-Tree** generalizes the 2-3 tree, and is a multi-way balanced search trees.

- B-Trees are used particularly for searching external filesystems.

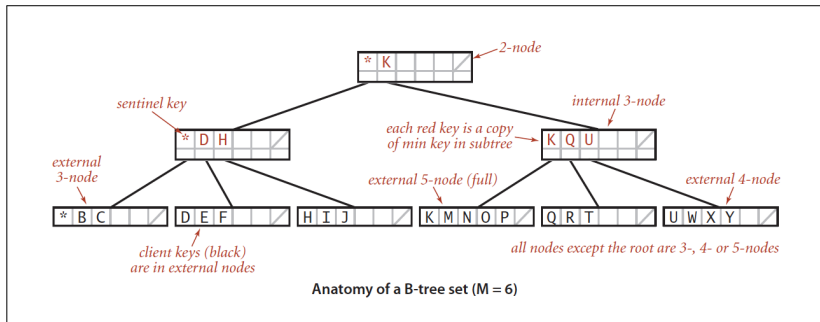
A **B-tree of order M**

- Allows up to $M - 1$ key-link pairs per node, where a link is the address of a page, rather than a symbol table value.
- We can choose a very large M , though it must be an even number.
 - For example, $M = 1024$
- We maintain at least 2 key-link pairs at root.
- At least $M/2$ key-link pairs in other nodes.
- External nodes contain client keys, and have references to actual data.
- Internal nodes contain copies of keys to guide search.

Example: In a B-tree of order 4, each node has at most 3 and at least 2 key-link pairs.

B-Tree Example - I

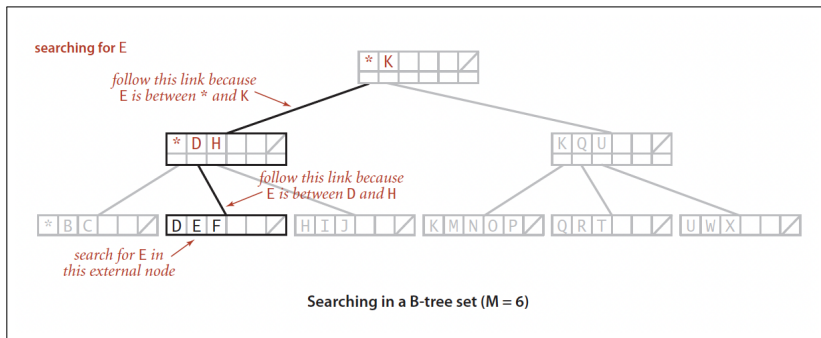
A special key (*), known as a sentinel, helps implement the B-Tree. Sentinels are a kind of wild-card entry that is defined as less than all other keys.



Search in a B-tree

Search in a B-tree

- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.

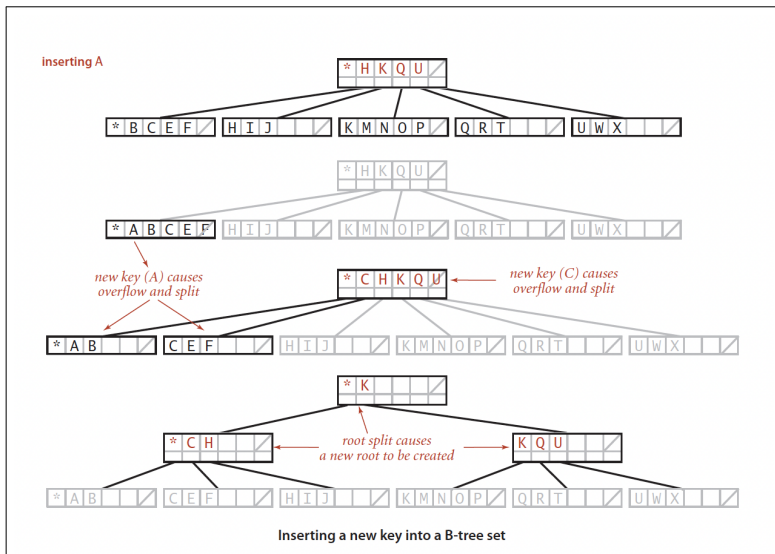


Insert in a B-tree - I

Insert in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.

Insert in a B-tree - II



Balance in B-Trees

Proposition A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Proof Sketch All internal nodes (besides root) have between $M/2$ and $M - 1$ links.

In practice Number of probes is at most 4. For instance, $M = 1024$ and $N = 62$ billion, $\log_{M/2} N \leq 4$.

Optimization Always keep root page in memory.

Balanced Trees Uses

Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: `map`, `multimap`, `multiset`.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.

B-tree variants. B+ tree, B* tree, B# tree, ...

B-trees (and variants) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.