Overview. My research interest lies in *representation theory* and is motivated by the connections between *algebraic groups* and *combinatorics*. A representation of a group is a collection of invertible linear transformations of a vector space (more generally known as a module) that multiply together in the same way as the group element. Surprisingly, the collection of linear transformations thus establishes a pattern of symmetry of the vector space which is encoded by the group [1]. For instance, consider the regular triangle inside the vector space $\mathbb{R}^2$ with vertices $(1, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Hence, the symmetric group $\mathfrak{S}_3$ permutes the vertices of this triangle which induces a homomorphism between $\mathfrak{S}_3$ and the group of invertible linear transformations $\mathrm{GL}(\mathbb{R}^2)$. Since symmetry is observed and understood so widely, there are a ton of applications of representation theory emerging in different fields of mathematics as well as in other disciplines. On the other hand, *algebraic geometry* itself is a vast and increasingly vibrant area of research nowadays. The genesis of it begins with the set of solutions to a system of equations gives rise to the Zariski topology, often called a variety [2]. Moreover, if we endow a variety with a group structure such that its operations are morphism, it is known as an *algebraic group*. In order to understand the representations of an algebraic group G, it turns out the strategy consists of breaking it down. The unipotent radical $R_u(G)$ of algebraic group G is the unique maximal closed connected solvable normal subgroup of G whose all elements are unipotent. When $R_u(G) = \mathrm{id}$, we say G is *reductive* (e.g., the general group GL_n). Two key players in this theory are the torus $\mathbf{T}$ ($\cong$ the n-dimensional multiplicative group $\mathbb{G}_m$ e.g., the diagonal matrices D_n w.r.t GL_n) and the Borel $\mathbf{B}$ (a maximal closed connected solvable subgroup e.g., the upper triangular matrices B_n w.r.t GL_n). It turns out that reductive groups has a rich representation theory. For instance, G is reductive if and only if every (finite-dimensional) representation is semisimple [3]. An irreducible G-variety Y is *spherical* if Y is normal and contains an open $\mathbf{B}$-orbit (e.g., all partial *flag* varieties, *homogeneous symmetric spaces* and *toric* varieties are spherical). There are other meaningful equivalent definitions of being spherical. Namely, Y is spherical if and only if $c(Y) = 0$ where the *complexity* $c(Y)$ is the minimal codimension of a $\mathbf{B}$-orbit. In our running example, the homogeneous space $\mathrm{GL}_n(\mathbb{C})/B_n(\mathbb{C})$ is isomorphic to the flag $\mathcal{F}\ell(\mathbb{C}^n)$. The torus $\mathbf{T}$ acts on $\mathcal{F}\ell(\mathbb{C}^n)$ by the left action, and the set of $\mathbf{T}$-fixed points is identified with $\mathfrak{S}_n$. Namely, each element $w \in \mathfrak{S}_n$ corresponds to a coordinate flag given by $(\{0\} \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset V_n = \mathbb{C}^n)$, where $\langle e_1, ..., e_n \rangle$ is the standard basis in $\mathbb{C}^n$. Furthermore, since $\mathcal{F}\ell(\mathbb{C}^n)$ has a Bruhat decomposition, every B_n-orbit BwB/B is isomorphic to $\mathbb{A}_{\mathbb{C}}^{\ell(w)}$ and called a *Schubert cell* and $\ell(w)$ stands for the length of w. The closure of BwB/B is the *Schubert variety* Z_w [4]. Finally, since in [5] they found out all Schubert varieties Z in $\mathrm{GL}_n/\mathbf{B}$ such that the action $\mathbf{T}$ on Z is of complexity one, and in my joint research with Dr. Mahir Can we are working to prove that these Schubert varieties are indeed spherical [6]. You might be wondering, "when does *combinatoric* show up?" it turns out the combinatoric gadgets will appear throughout the proof of our investigation. For example, we use the Billey-Postnikov decomposition [7].

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